One-way Definability of Sweeping Transducers[*]

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Abstract

Two-way finite-state transducers on words are strictly more expressive than one-way transducers. It has been shown recently how to decide if a two-way functional transducer has an equivalent one-way transducer, and the complexity of the algorithm is non-elementary. We propose an alternative and simpler characterization for sweeping functional transducers, namely, for transducers that can only reverse their head direction at the extremities of the input. Our algorithm works in 2EXPSPACE and, in the positive case, produces an equivalent one-way transducer of doubly exponential size. We also show that the bound on the size of the transducer is tight, and that the one-way definability problem is undecidable for (sweeping) non-functional transducers.

1998 ACM Subject Classification F. Theory of Computation, F.4.3 Formal Languages

Keywords and phrases Regular word transductions, sweeping transducers, one-way definability

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2015.178

1 Introduction

Regular word languages form the best understood class of languages. They enjoy several characterizations, in particular by different kinds of finite-state automata. For instance, two-way finite-state automata have the same expressive power as one-way automata. This result has been established independently by Rabin and Scott [9] and Shepherdson [10]. Besides automata, regular languages have logical and algebraic characterizations, namely through monadic second-order logic and congruences of finite index.

Transducers extend automata by producing outputs with each transition. A run generates an output word by concatenating the words produced by its transitions. A transducer thus defines a relation over words. It is called functional when this relation is a function. For finite-state transducers, expressiveness is different than for finite-state automata. As an example, two-way transducers are strictly more expressive than one-way transducers. For instance, the function that maps a word to its mirror image can be done by a back-and-forth pass over the input, but no one-way transducer can do it.

As seen above, we lose some robustness when going from automata to transducers. On the other hand, some of the classical characterizations of regular languages generalize well to transducers. An important result is the equivalence of functional two-way transducers and Ehrenfeucht-Courcelle's monadic-second order transductions [5] over words. Another characterization of two-way transducers was provided through a new model called streaming string transducers [1, 2], that process the input one-way and store the output in write-only

[*] This work was partially supported by the ExStream project (ANR-13-JS02-0010) and the Technische Universität München – Institute for Advanced Study, funded by the German Excellence Initiative and the European Union Seventh Framework Programme under grant agreement n° 291763.

registers. Finally, first-order transductions are known to be equivalent to aperiodic streaming transducers [7] and to aperiodic two-way transducers [4].

The question whether a functional two-way word transducer is equivalent to a one-way transducer has been solved recently in [6]. The algorithm proposed by [6] takes a two-way transducer $\mathcal{S}$ and builds a one-way transducer $\mathcal{T}$ that is "maximal" in the following sense: (1) all accepting runs of $\mathcal{T}$ produce correct outputs, and (2) all runs of $\mathcal{S}$ that can be performed one-way are realized by $\mathcal{T}$. As a consequence, the two-way transducer $\mathcal{S}$ has an equivalent one-way transducer if and only if the constructed one-way transducer $\mathcal{T}$ has the same domain as $\mathcal{S}$, which is a decidable problem. The problem is that the upper bound on both the decision procedure and the size of the constructed one-way transducer is non-elementary.

The main contribution of this paper is an elementary procedure for deciding whether a functional two-way word transducer is equivalent to a one-way transducer, for the particular class of sweeping transducers. While two-way transducers can reverse their head direction at any position of the input, sweeping transducers can only reverse it at the first and last position. Unsurprisingly, sweeping transducers are strictly less expressive than two-way transducers, and the following example shows the difference: on input $u_1\ a\ u_2\ a\ \dots\ a\ u_{n-1}\ a\ u_n$, where the words u_i contain no occurrence of a, the two-way transducer produces as output $u_n\ a\ u_{n-1}\ a\ \dots\ a\ u_2\ a\ u_1$ (we assume that the alphabet contains at least two letters).

Our decision procedure works in doubly exponential space and, when it succeeds, it produces an equivalent one-way transducer of doubly exponential size. We show that the bound on the size of the transducer is tight for any decision algorithm producing an equivalent one-way transducer from a sweeping transducer. This improves the PSpace lower bound from [6]. The non-elementary procedure described in [6] relies on Rabin-Scott's construction for automata, and works by eliminating basic zigzags in runs. Our procedure is closer to the textbook approach (due to Shepherdson) and uses crossing sequences. This requires a decomposition of runs which is incomparable with the zigzag decomposition of [6]. Finally, we show that the one-way definability problem becomes undecidable for non-functional transducers.

Overview

Section 2 defines transducers and related concepts. Section 3 defines decomposition of runs and gives the construction of a one-way transducer based on such decompositions. In Section 4 we show that all one-way-definable runs admit a decomposition. Section 5 provides the lower bound and the undecidability result. A long version of the paper can be found on `http://www.labri.fr/perso/anca/Publications/fsttcs15.pdf`

2 Preliminaries

Transducers

A *two-way transducer* is a tuple $(\Sigma, \Delta, Q, I, F, \delta)$, where Σ (resp., Δ) is a finite input (resp., output) alphabet, Q is a finite set of states, I (resp., F) is a subset of Q representing the initial (resp., final) states, and $\delta \subseteq Q \times \Sigma \times \Delta^\star \times Q \times \{\mathsf{left}, \mathsf{right}\}$ is a finite set of transition rules describing, for each state and input symbol, the possible output string, target state, and direction of movement. We talk of a *one-way* transducer whenever $\delta \subseteq Q \times \Sigma \times \Delta^\star \times Q \times \{\mathsf{right}\}$. The *size* of a transducer is its number of states.

According to standard practice, the states of *one-way* automata and transducers are usually located between the letters of the input word $u = a_1 \dots a_n$. For this it is convenient to introduce $n+1$ *positions* $0, 1, \ldots, n$ and think of each position $i > 0$ (resp., 0) as a placeholder between the i-th and the $i+1$-th symbols (resp., just before the first symbol a_1). Moreover, since here we deal with *two-way* devices, a single position can be visited several times along a run. Thus, to describe a run of a two-way transducer on input $u = a_1 \ldots a_n$, we will associate states with *locations*, namely, with pairs (x, y) where x is a position among $0, 1, \ldots, n$ and y is an integer representing the number of reversals performed up to a certain point – for short, we call this number y the *level* of the location.

A run is a sequence of locations, labelled by states and connected by edges, called *transitions*. The state at location $\ell = (x, y)$ of a run ρ is denoted $\rho(\ell)$. The transitions must connect pairs of locations that are either at adjacent positions and on the same level, or at the same position and on adjacent levels. In addition, each transition is labelled with a pair a/v consisting of an input symbol a and an output v. There are four types of transitions:

$$\begin{tikzcd}
(i-1,2y+1) & (i,2y+1) \arrow[l, "a/v"'] & (i-1,2y) \arrow[r, "a/v"] & (i,2y) \\
& (i,2y+2) & (i-1,2y+1) & \\
& (i,2y+1) \arrow[u, bend left, "a/v"] & (i-1,2y) \arrow[u, bend right, "a/v"'] &
\end{tikzcd}$$

Note that the transitions between locations at even levels are all directed from left to right, while the transitions at odd levels are directed from right to left. More precisely, the upper left (resp., upper right) transition may occur in a run ρ on $u = a_1 \ldots a_n$ if $\big(\rho(i, 2y+1), a, v, \rho(i-1, 2y+1), \mathsf{left}\big)$ (resp., $\big(\rho(i-1, 2y), a, v, \rho(i, 2y), \mathsf{right}\big)$) is a valid transition rule of $\mathcal{T}$ and $a = a_i$. Similarly, the lower left (resp., lower right) transition may occur if $\big(\rho(i, 2y+1), a, v, \rho(i, 2y+2), \mathsf{right}\big)$ (resp., $\big(\rho(i-1, 2y), a, v, \rho(i-1, 2y+1), \mathsf{left}\big)$) is a valid transition rule of $\mathcal{T}$ and $a = a_i$. For technical reasons (namely, to enable distinguished transitions at the extremities of the input word), we will introduce the special fresh symbols $\rhd$ and $\lhd$ and allow the lower left (resp., lower right) transition also when $i = 0$ and $a = \rhd$ (resp., when $i = |u|$ and $a = \lhd$).

Given a sequence $x_1, \ldots, x_n$, a *factor* denotes any contiguous subsequence $x_i, \ldots, x_j$, for $1 \leq i \leq j \leq n$. A run on the input $u = a_1 \ldots a_n$ is said to be *successful* if it starts at the lower left location $(0, 0)$ with an initial state of $\mathcal{T}$ and ends at the upper right location $(|u|, y_{\mathsf{max}})$ in a final state of $\mathcal{T}$. The output produced by a run is the concatenation of the outputs of its transitions, and it is denoted by $\mathsf{out}(\rho)$. We denote by $\mathsf{dom}(\mathcal{T})$ the language of all words u that admit a successful run of $\mathcal{T}$. We order the locations along a run ρ by letting $\ell_1 < \ell_2$ if ℓ_2 is reachable from ℓ_1 following the transitions in ρ. Given two locations $\ell_1 < \ell_2$ of a run ρ, we denote by $\rho[\ell_1, \ell_2]$ the factor of the run that starts in ℓ_1 and ends in ℓ_2. Note that $\rho[\ell_1, \ell_2]$ is also a run, hence the notation $\mathsf{out}\big(\rho[\ell_1, \ell_2]\big)$ is consistent.

Further assumptions

We will mostly work with two-way transducers that are *sweeping*. This means that on every successful run, the head can change direction only at the extremities of the input. In other words, the lower right (resp., lower left) transition is possible only if $a = \lhd$ (resp., $a = \rhd$).

A transducer $\mathcal{T}$ is *functional* if, for each input word u, all successful runs on u produce the same output. In this case $\mathcal{T}(u)$ denotes the unique output produced on input u.

Unless otherwise stated, we will assume that all transducers are sweeping and functional. Note that functionality is a decidable property, as stated below. The proof is similar to the decidability proof of equivalence of streaming string transducers [1] and reduces the problem to the reachability of a 1-counter automaton of exponential size.

▶ **Proposition 1.** *Functionality of two-way transducers can be decided in polynomial space. This problem is* PSPACE*-hard even for sweeping transducers.*

Without loss of generality, we can also assume that the successful runs of a functional transducer are *normalized*, namely, they never visit two locations with the same position, the same state and both either at an even level or at an odd level. Indeed, if this were not the case, say if a successful run ρ visits two locations $\ell_1 = (x, y_1)$ and $\ell_2 = (x, y_2)$ such that $\rho(\ell_1) = \rho(\ell_2)$ and y_1, y_2 are both even or both odd, then the output produced by ρ between ℓ_1 and ℓ_2 is either empty – in which case we could remove $\rho[\ell_1, \ell_2]$ and obtain an equivalent successful run – or is non-empty – in which case, by repeating $\rho[\ell_1, \ell_2]$, we could obtain successful runs that produces different outputs on the same input, thus contradicting the assumption that the transducer is functional.

Crossing sequences

Consider a run ρ of a transducer on input $u = a_1 \dots a_n$. For each position $x \in \{0, 1 \dots, n\}$, we are interested in the sequence of states labelling the locations at position x. Formally, we define the *crossing sequence of* ρ *at* x as the tuple $\rho|x = \big(\rho(x, y_0), \dots, \rho(x, y_h)\big)$, where $y_0 < \dots < y_h$ are exactly the levels of the locations of ρ of the form (x, y), with $y \in \mathbb{N}$ (if the transducer is sweeping, we simply have $y_i = i$). If the considered run ρ is successful, then the bottom and top locations at position x have even levels, and the outgoing transitions move rightward. In particular, every crossing sequence of a successful run has odd length. Moreover, if we assume that the successful run is normalized, then every crossing sequence has length at most $2|Q| - 1$. The *crossing number* of a run is the maximal length of a crossing sequence of that run. The *crossing number* of a transducer is the maximal crossing number of any of its normalized runs – note that this value is bounded by $2|Q| - 1$.

Intercepted factors

An *interval* of positions has the form $I = [x_1, x_2]$, with $x_1 < x_2$. We say that an interval $I = [x_1, x_2]$ *contains* (resp., *strongly contains*) another interval $I' = [x'_1, x'_2]$ if $x_1 \leq x'_1 \leq x'_2 \leq x_2$ (resp., $x_1 < x'_1 \leq x'_2 < x_2$). We say that a factor of a run ρ is *intercepted* by an interval $I = [x_1, x_2]$ if it is maximal among the factors of ρ that visit only positions in I and that never make a reversal (recall that reversals in sweeping transducers can only occur at the extremities of the input word).

It is easy to see that distinct factors intercepted by the same interval I visit disjoint sets of locations. This means that a factor intercepted by I can be uniquely identified by specifying a location ℓ in it, e.g., the first or the last one. Accordingly, we will denote by $\rho \mid I/\ell$ the factor intercepted by I that visits the location ℓ (if this factor does not exist, we simply let $\rho \mid I/\ell = \varepsilon$).

A *loop* of a run ρ is an interval $L = [x_1, x_2]$ such that the crossing sequences at positions x_1 and x_2 are equal, that is, $\rho|x_1 = \rho|x_2$. Loops can be used to pump parts of runs, as explained below.

Pumping

Given a loop $L = [x_1, x_2]$ of a run ρ on u and a number $m \in \mathbb{N}$, we can replicate m times the factor $u[x_1, x_2]$ of the input and simultaneously, on the run, we replicate m times the loop L. Formally, with $\beta_0, \dots, \beta_h$ denoting the factors intercepted by L, we define the run obtained by replicating L as a sequence of the form

$$\mathsf{pump}_L^m(\rho) \;=\; \underbrace{\alpha_0\,\boldsymbol{\beta_0^m}\,\gamma_0}_{\text{forward}}\;\underbrace{\gamma_1\,\boldsymbol{\beta_1^m}\,\alpha_1}_{\text{backward}}\;\underbrace{\alpha_2\,\boldsymbol{\beta_2^m}\,\gamma_2}_{\text{forward}}\;\cdots\;\underbrace{\alpha_{y_{\max}}\,\boldsymbol{\beta_h^m}\,\gamma_{y_{\max}}}_{\text{forward}} \tag{1}$$

where $h < 2|Q| - 1$ is the maximum level visited by ρ, and α_y (resp., β_y, γ_y) is the factor of ρ at level y that is intercepted by the interval $[0, x_1]$ (resp., $L = [x_1, x_2]$, $[x_2, |u|]$). Note that each $\alpha_y\,\beta_y\,\gamma_y$ (resp., $\gamma_y\,\beta_y\,\alpha_y$) is a maximal factor of the run ρ that is forward-oriented (resp., backward-oriented). We also define

$$\mathsf{pump}_L^m(u) \;=\; u[1, x_1] \cdot \big(\boldsymbol{u[x_1 + 1, x_2]}\big)^{\boldsymbol{m}} \cdot u[x_2 + 1, |u|]$$

and we observe that $\mathsf{pump}_L^m(\rho)$ is a valid run on $\mathsf{pump}_L^m(u)$. We also remark that the above definition of pumped run works only for sweeping transducers, as for arbitrary two-way transducers we would need to take into account the possible reversals within a loop L and combine the intercepted factors in a more complex way.

3 Decompositions of one-way definable runs

The problem we consider in this paper is the *one-way definability* of functional, sweeping transducers: given a transducer $\mathcal{S}$, we ask if there exists an equivalent one-way transducer $\mathcal{T}$, namely, one such that $\mathcal{T}(u) = \mathcal{S}(u)$ for all $u \in \Sigma^\star$. In the answer is "yes", then we also want to compute an equivalent one-way transducer. Of course, there are sweeping transducers $\mathcal{S}$, like $\mathcal{S}(u) = u \cdot u$, that are not equivalent to any one-way transducer (assuming that the alphabet Σ is not unary).

Before introducing some technical concepts, let us consider an example that highlights the main idea of the proof. Fix a regular language R (not containing the empty word) and consider the transduction that maps a word on the mirror of the rightmost maximal factor belonging to R. That is, $f(u\,v\,w) = \mathsf{mirror}(v)$ whenever (1) $v \in R$, (2) there is no $v' \in R$ such that v' is prefix of vw, and (3) w has no factor in R. It can be easily seen that f can be realized by a two-way transducer, but not by a one-way transducer. However, f *can* be realized by a one-way transducer for particular regular languages R, like the periodic language $R = (ab)^+$: we simply guess v and output $\mathsf{mirror}(v) \in (ba)^+$ from left to right, then we check w. This example shows that periodicities play an important role in deciding whether a given transduction can be realized by a one-way transducer.

We introduce in the following some notations and concepts that will help us to state a sufficient condition for the one-way definability of sweeping transducers. For simplicity, we fix for the remaining of the paper a functional, sweeping transducer $\mathcal{S}$ as input. We first introduce some constants: $h_{\mathcal{S}}$ is the maximum number of levels visited by the normalized runs of $\mathcal{S}$, $c_{\mathcal{S}}$ is the maximum number of symbols produced by a single transition of $\mathcal{S}$, and $e_{\mathcal{S}} = c_{\mathcal{S}} \cdot |Q|^{2|Q|}$, where Q is the state space of $\mathcal{S}$. The constant $e_{\mathcal{S}}$ will be used to bound the lengths or the periods of certain parts of the output produced by $\mathcal{S}$, and is related to the number of crossing sequences of $\mathcal{S}$ (see also Lemma 6 in Section 4).

A word v is said to have *period* p if $v \in w^*\,w'$ for some word w of length p and some prefix w' of w. For example, $v = abc\,abc\,ab$ has period $p = 3$. Similarly, we say that v is *almost periodic with bound* p if $v = w_0\,w_1\,w_2$ for some words w_0, w_2 of length at most p and some non-empty word w_1 of period at most p.

We will also need to identify sub-sequences of a run ρ of $\mathcal{S}$ that are induced by particular sets of locations. Recall that $\rho[\ell_1, \ell_2]$ denotes the factor of ρ delimited by two locations ℓ_1 and ℓ_2. Similarly, we denote by $\rho \mid Z$ the sub-sequence of ρ induced by a set Z of locations – note that Z does not need to be an interval and, even though $\rho \mid Z$ might not be a valid run of $\mathcal{S}$, we can still refer to the number of transitions and the size of the output.

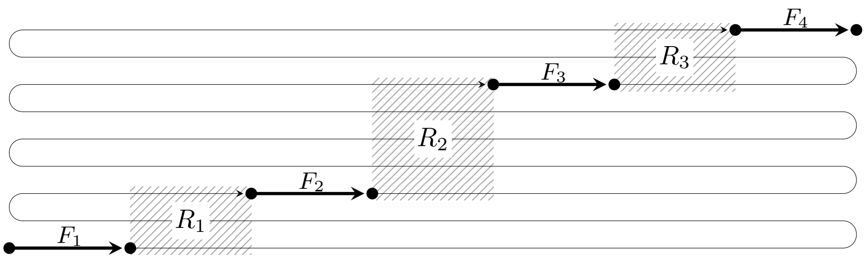
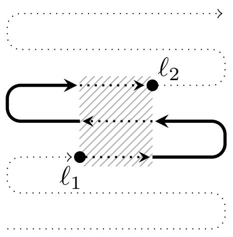


Figure 1 Decomposition of a run into floors and ramps.

▶ **Definition 2.** Let ρ be a run of $\mathcal{S}$ on u. We define two types of pairs of locations of ρ:

- A *floor* is a pair of locations (ℓ_1, ℓ_2) such that $\ell_1 \leq \ell_2$ are on the same *even* level.
- A *ramp* is a pair of locations (ℓ_1, ℓ_2), with $\ell_1 = (x_1, y_1)$ and $\ell_2 = (x_2, y_2)$, such that (i) $x_1 \leq x_2$, (ii) $y_1 < y_2$, (iii) both y_1 and y_2 are even, (iv) the output produced by $\rho[\ell_1, \ell_2]$ has length at most $(y_2 - y_1 + 1) \cdot e_{\mathcal{S}}$ or it is almost periodic with bound $2 \cdot e_{\mathcal{S}}$, and (v) the output produced by the sub-sequence $\rho \mid Z$, where $Z = [\ell_1, \ell_2] \setminus [x_1, x_2] \times [y_1, y_2]$, has length at most $2(y_2 - y_1) \cdot e_{\mathcal{S}}$.

Before discussing how the above definitions are used, we give some intuition. The simplest concept is that of floor, which is essentially a forward-oriented factor of a run. Ramps connect consecutive floors. An important constraint in the definition of a ramp (ℓ_1, ℓ_2) is that the output of $\rho[\ell_1, \ell_2]$ is bounded or almost periodic with small bound. We will see later how this constraint eases the production of the output of $\rho[\ell_1, \ell_2]$ by a one-way transducer. The last constraint on a ramp (ℓ_1, ℓ_2) bounds the length of the output produced by the sub-sequence $\rho \mid Z$, where $Z = [\ell_1, \ell_2] \setminus [x_1, x_2] \times [y_1, y_2]$. As shown by the figure to the right, this sub-sequence (represented by bold arrows) can be obtained from the factor $\rho[\ell_1, \ell_2]$ by removing the factors intercepted by $[x_1, x_2]$ (represented by the hatched area). The constraint is used for those parts of the run that are not covered by floors or ramps. In particular, it guarantees that the size of the output *above* each floor is bounded by $2h_{\mathcal{S}} \cdot e_{\mathcal{S}}$.



The general idea for turning $\mathcal{S}$ into an equivalent one-way transducer will be to guess (and check) a decomposition of the run of $\mathcal{S}$ into factors that are floors or ramps.

▶ **Definition 3.** A *decomposition* of a run ρ is a factorization into floors and ramps.

Figure 1 gives an example of a decomposition. Note that, thanks to Definition 2, the number of symbols produced outside the segments $F_1, F_2, \ldots$ and the rectangles $R_1, R_2, \ldots$ is small (indeed, bounded by $2h_{\mathcal{S}} \cdot e_{\mathcal{S}}$); so most of the output is produced inside these segments and rectangles. We can now state our main result:

▶ **Theorem 4.** *A sweeping functional transducer $\mathcal{S}$ is one-way definable if and only if every input word has some successful run of $\mathcal{S}$ that admits a decomposition.*
Moreover, we can construct from $\mathcal{S}$ a one-way transducer $\mathcal{T}$ that maps u to v whenever there is a successful run of $\mathcal{S}$ on u that outputs v and admits a decomposition. The construction of $\mathcal{T}$ takes doubly exponential time in $|\mathcal{S}|$.
In particular, $\mathcal{S}$ is one-way definable if and only if $\mathsf{dom}(\mathcal{T}) = \mathsf{dom}(\mathcal{S})$. *The latter condition can be tested in polynomial space in $|\mathcal{S}|$ and $|\mathcal{T}|$, so in doubly exponential space in $|\mathcal{S}|$.*

The first claim of the theorem gives the main characterization, namely, it shows that the existence of decompositions of successful runs, for all possible inputs in the domain of $\mathcal{S}$, is a *sufficient* and *necessary* condition for the transduction to be one-way definable. The second claim shows a property that is slightly more general than sufficiency: it allows to compute a one-way transducer $\mathcal{T}$ that is somehow the "best one-way approximation" of $\mathcal{S}$, in the sense that the transduction computed by $\mathcal{T}$ is always contained in the transduction computed by $\mathcal{S}$ and it is equal when $\mathcal{S}$ is one-way definable. The last claim deals with the *effectiveness* of the characterization, showing that one-way definability is decidable in 2ExpSpace. For the sake of presentation, we divide the proof of the theorem into two parts. The first part, given below, deals with the *sufficiency* and the *effectiveness* of the characterization (i.e. the second and third claims of the theorem). The second part, which is the most technical one and is deferred to Section 4, deals with the *necessary* part of the characterization.

Proof of Theorem 4 (sufficiency and effectiveness). We build from $\mathcal{S}$ a one-way transducer $\mathcal{T}$ that simulates all successful runs of $\mathcal{S}$ that admit a decomposition. Consider one such run ρ. We begin by observing that a decomposition of ρ can be described by a sequence of locations $\ell_0 < \ell_1 < \ldots < \ell_t$, where $\ell_0 = (0,0)$, $\ell_t = (x_{\mathsf{max}}, y_{\mathsf{max}})$, and, for all $0 \le i < t$, (ℓ_i, ℓ_{i+1}) is a floor or a ramp. In particular, the one-way transducer $\mathcal{T}$ will guess the crossing sequences of ρ, together with a sequence of locations $(\ell_i)_{i \le t}$, which are intended to represent a decomposition of ρ. Below, we show how to check that the guessed sequence of locations represents a valid decomposition, and how to produce the corresponding output.

Traversing the floors of the decomposition does not pose particular problems, as these are forward-oriented factors of the run ρ, which can be directly simulated by $\mathcal{T}$ without reversing the head. Of course, we need to store the bounded output on the levels above the floor, and check that the output on the levels below the floor matches some stored output words. The interesting case happens when $\mathcal{T}$ simulates a ramp (ℓ_i, ℓ_{i+1}), with $\ell_i = (x_i, y_i)$ and $\ell_{i+1} = (x_{i+1}, y_{i+1})$. First of all, it is easy for $\mathcal{T}$ to verify the first three conditions of the definition of ramp, namely, that $x_i \le x_{i+1}$, $y_i < y_{i+1}$, and both y_i and y_{i+1} are even. Checking the remaining conditions is more difficult and requires storing some words for a total length that does not exceed $8h_{\mathcal{S}} \cdot e_{\mathcal{S}}$ – in particular, this explains the doubly exponential blowup of the state space of $\mathcal{T}$. More precisely, at the beginning of the computation, $\mathcal{T}$ guesses, for each ramp (ℓ_i, ℓ_{i+1}), with $\ell_i = (x_i, y_i)$ and $\ell_{i+1} = (x_{i+1}, y_{i+1})$:

- a word v_i that has length at most $(y_{i+1} - y_i + 1) \cdot e_{\mathcal{S}}$ or is almost periodic with bound $2 \cdot e_{\mathcal{S}}$ (in the latter case, in fact, the word is described by a prefix, a suffix, and a repeating pattern, each one of length at most $2 \cdot e_{\mathcal{S}}$),
- some words $\overleftarrow{v}_y$ and $\overrightarrow{v}_y$, that is, two words for each level $y \in \{y_i + 1, \ldots, y_{i+1}\}$, whose lengths sum up to at most $2h_{\mathcal{S}} \cdot e_{\mathcal{S}}$.

The idea is that each word v_i represents the output produced by the factor $\rho[\ell_i, \ell_{i+1}]$ (this output is bounded or is almost periodic, thanks to the fourth condition of the definition of ramp). Note that, by construction, there can be at most $h_{\mathcal{S}}$ ramps in the decomposition, and hence the sum of the lengths of the words used to represent the v_i's does not exceed $6 \cdot h_{\mathcal{S}} \cdot e_{\mathcal{S}}$. Similarly, each word $\overleftarrow{v}_y$ (resp., $\overrightarrow{v}_y$) represents the output produced by the factor of the run ρ that is at level y and to the left of the position x_i (resp., right of x_{i+1}), where i is the unique index such that $y_i < y \le y_{i+1}$ (resp., $y_i \le y < y_{i+1}$). The total length of these words is at most $2h_{\mathcal{S}} \cdot e_{\mathcal{S}}$. Overall, the sum of the lengths of all the words guessed by $\mathcal{T}$ never exceeds $8h_{\mathcal{S}} \cdot e_{\mathcal{S}}$.

Using the words v_i, $\overleftarrow{v}_y$, $\overrightarrow{v}_y$ and some additional pointers, the transducer $\mathcal{T}$ can verify that the guessed ramps satisfy the required conditions and that the decomposition is thus valid. In the same way, the words v_i, $\overleftarrow{v}_y$, $\overrightarrow{v}_y$ can be used to produce the output $\mathsf{out}(\rho[\ell_i, \ell_{i+1}])$

associated with each ramp (ℓ_i, ℓ_{i+1}). For this, it is sufficient to visit the positions of the ramp (ℓ_i, ℓ_{i+1}) and, at the same time, fetch blocks of symbols of appropriate length in the word v_i, so as to eventually match the length of the desired output $\mathsf{out}(\rho[\ell_i, \ell_{i+1}])$ – note that this requires taking into account also the words $\overleftarrow{v}_y$ and $\overrightarrow{v}_y$.

We just described informally a one-way transducer $\mathcal{T}$ that simulates any run ρ of $\mathcal{S}$ that admits a decomposition. The size of $\mathcal{T}$ is doubly exponential in the size of $\mathcal{S}$ and this proves the second claim of the theorem.

Assuming that the existence of decompositions of successful runs is also a necessary condition for the one-way definability of $\mathcal{S}$ (this will be proved in the next section), we can easily derive from the above constructions a 2ExpSpace decision procedure for testing one-way definability. More precisely, given a sweeping transducer $\mathcal{S}$, one constructs $\mathcal{T}$ as above in doubly exponential time, and then tests whether $\mathsf{dom}(\mathcal{S}) \subseteq \mathsf{dom}(\mathcal{T})$. The latter problem can be seen as a containment problem between a two-way non-deterministic finite-state automaton $\mathcal{A}$ and a one-way non-deterministic finite-state automaton $\mathcal{B}$. Using standard constructions, one can turn $\mathcal{A}$ into an equivalent one-way non-deterministic finite-state automaton $\mathcal{A}'$ (which is exponential in $\mathcal{S}$), and finally decide the containment $\mathcal{A}' \subseteq \mathcal{B}$ in polynomial space in $\mathcal{A}'$ and $\mathcal{B}$, that is, in doubly exponential space in $\mathcal{S}$. ◀

4 Characterizing one-way definability

In this section we prove the harder direction of the first claim of Theorem 4: if a sweeping functional transducer is one-way definable, then every accepted input has a successful run that can be decomposed into floors and ramps.

We begin by identifying some phenomena that prevent a transducer to be one-way definable. A first example is the mirror transduction, where a large number of symbols need to be generated from right to left. Another example is the doubling transduction $\mathcal{S}(u) = u \cdot u$. Here, we have an *inversion*, namely large parts of the input must be generated before other large parts that are located to their left. We give a formal definition of inversions below.

Fix a successful run ρ of $\mathcal{S}$ and consider a loop $L = [x_1, x_2]$ of ρ. A location ℓ_1 of ρ is called *entry point of* L if ℓ_1 is the first location of the factor intercepted by L at level y, for some y. Similarly, a location ℓ_2 is called an *exit point of* L if ℓ_2 is the last location of the factor intercepted by L at level k, for some k. Note that ℓ_1 belongs to $\{x_1\} \times (2\mathbb{N}) \cup \{x_2\} \times (2\mathbb{N}+1)$, and ℓ_2 belongs to $\{x_2\} \times (2\mathbb{N}) \cup \{x_1\} \times (2\mathbb{N}+1)$. Finally, we say that an intercepted factor $\rho \mid I/\ell$ is *captured* by a loop L if I contains L (possibly, $I = L$) and the subfactor intercepted by L on the same level as ℓ, has non-empty output.

▶ **Definition 5.** An *inversion* of the run ρ is a pair of locations ℓ_1 and ℓ_2 for which there exist two loops $L_1 = (x_1, x_1')$ and $L_2 = (x_2, x_2')$ such that:

- ℓ_1 is an entry point of L_1 and ℓ_2 is an exit point of L_2,
- $\ell_1 < \ell_2$ and $x_1 \geq x_2$ (namely, ℓ_2 strictly follows ℓ_1 along the run, but the left endpoint of L_2 precedes the left endpoint of L_1),
- for both $i = 1$ and $i = 2$, the intercepted factor $\rho \mid L_i/\ell_i$ is captured by the loop L_i, but it is not captured by any other loop strongly contained in L_i.

We say that the above loops L_1 and L_2 are *witnessing* the inversion (ℓ_1, ℓ_2).

The left-hand side of Figure 2 gives an example of an inversion, where the entry point ℓ_1 of L_1 and the exit point ℓ_2 of are represented by white circles.

The first lemma (proved in the appendix) can be used to bound the lengths of the outputs produced by the factors $\rho \mid L_1/\ell_1$ and $\rho \mid L_2/\ell_2$, where ℓ_1, ℓ_2, L_1, L_2 are as in Definition 5:

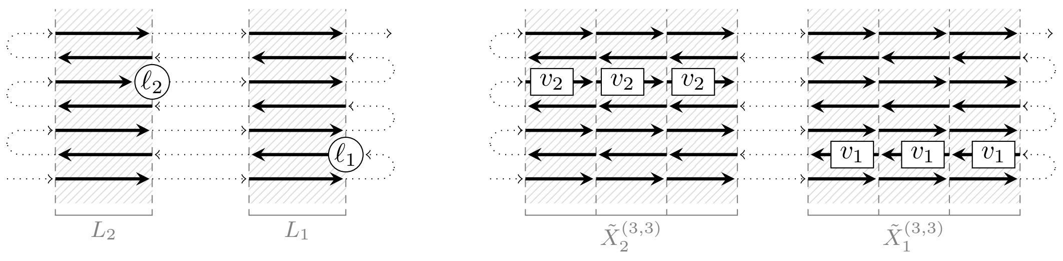


Figure 2 To the left: an entry point ℓ_1 of L_1 and an exit point ℓ_2 of L_2 forming an inversion. To the right: the run obtained by pumping the loops L_1 and L_2.

▶ **Lemma 6.** *If an intercepted factor $\rho \mid I/\ell$ is not captured by any loop L strongly contained in I, then the length of its output is at most $e_{\mathcal{S}}$.*
In particular, for every inversion (ℓ_1, ℓ_2) witnessed by some loops L_1 and L_2, we have $1 \le \big|\mathsf{out}(\rho \mid L_i/\ell_i)\big| \le e_{\mathcal{S}}$, for both $i = 1$ and $i = 2$.

The next proposition gives the crucial property for characterizing one-way definability, as it shows that the transducer $\mathcal{S}$ is one-way definable only if for every inversion (ℓ_1, ℓ_2), the output of $\rho[\ell_1, \ell_2]$ is periodic.

▶ **Proposition 7.** *Suppose that the sweeping transducer $\mathcal{S}$ is one-way definable. Then, for all inversions (ℓ_1, ℓ_2) of the run ρ witnessed by loops L_1, L_2 and for both $i = 1$ and $i = 2$, the output of $\rho[\ell_1, \ell_2]$ has period $\big|\mathsf{out}(\rho \mid L_i/\ell_i)\big|$, hence, in particular, at most $e_{\mathcal{S}}$.*

A key ingredient for the proof of the above proposition is Fine and Wilf's theorem [8]. In short, this theorem says that, whenever two periodic words w_1, w_2 share a sufficiently long factor, then they have the same periods. Below, we state a slightly stronger variant of Fine and Wilf's theorem, which contains an additional claim that shows how to align a common factor of the two words w_1, w_2 so as to form a third word containing a prefix of w_1 and a suffix of w_2. The additional claim will be exploited in the proof of Proposition 7 and Lemma 11.

▶ **Lemma 8** (Fine and Wilf). *If w_1 is a word with period p_1, w_2 is a word with period p_2, and w_1 and w_2 have a common factor of length at least $p_1 + p_2 - \gcd(p_1, p_2)$, then w_1 and w_2 have also period $\gcd(p_1, p_2)$. If in addition we have $w_1 = u_1\, w\, v_1$, $w_2 = u_2\, w\, v_2$, and $|w| \ge \gcd(p_1, p_2)$, then $w_3 = u_1\, w\, v_2$ has also period $\gcd(p_1, p_2)$.*

Proof of Proposition 7. Let $L_1 = (x_1, x_1')$, $L_2 = (x_2, x_2')$ be the loops witnessing the inversion (ℓ_1, ℓ_2). Note that the two loops L_1 and L_2 might not be ordered exactly as shown in Figure 2. In fact, two cases can arise: either $x_2 < x_2' \le x_1 < x_1'$ (that is, L_1 and L_2 are disjoint and L_1 is to the right of L_2) or $x_2 \le x_1 < x_2' \le x_1'$ (that is, L_1 overlaps to the right with L_2).

We begin by pumping the loops L_1 and L_2 (see the right-hand side of Figure 2). Formally, for all positive numbers m_1, m_2, we define

$$\rho^{(m_1,m_2)} \;=\; \mathsf{pump}_{L_2}^{m_2}(\mathsf{pump}_{L_1}^{m_1}(\rho)) \qquad \text{and} \qquad u^{(m_1,m_2)} \;=\; \mathsf{pump}_{L_2}^{m_2}(\mathsf{pump}_{L_1}^{m_1}(u)) \;.$$

We identify the positions of $u^{(m_1,m_2)}$ that mark the endpoints of the copies of the loops L_1 and L_2 in the pumped run $\rho^{(m_1,m_2)}$. Because L_2 precedes L_1 with respect to the ordering of positions, it is easier to define first the set of endpoints of the copies of L_2:

$$\tilde{X}_2^{(m_1,m_2)} \;=\; \{x_2 + i\Delta_2 \;\mid\; 0 \le i \le m_2\} \qquad\qquad \text{where} \;\; \Delta_2 = x_2' - x_2 \;.$$

The set of endpoints of the copies of L_1 is defined as

$$\tilde{X}_1^{(m_1,m_2)} \;=\; \{x_1 + i\Delta_1 + m_2\Delta_2 \;\mid\; 0 \le i \le m_1\} \qquad \text{where} \;\; \Delta_1 = x'_1 - x_1 \;.$$

We then exploit the hypothesis that $\mathcal{S}$ is one-way definable and assume that the one-way transducer $\mathcal{T}$ is equivalent to $\mathcal{S}$. In particular, $\mathcal{T}$ produces the same output as $\mathcal{S}$ on every input $u^{(m_1,m_2)}$. Let $\sigma^{(m_1,m_2)}$ be a successful run of $\mathcal{T}$ on $u^{(m_1,m_2)}$. Since $\mathcal{T}$ has finitely many states, we can find a large enough number $h > 0$ and two positions $\tilde{x}_1 < \tilde{x}'_1 \in \tilde{X}_1^{(h,h)}$ such that the crossing sequences of $\sigma^{(h,h)}$ at $\tilde{x}_1$ and $\tilde{x}'_1$ are the same. Similarly, we can find two positions $\tilde{x}_2 < \tilde{x}'_2 \in \tilde{X}_2^{(h,h)}$ such that the crossing sequences of $\sigma^{(h,h)}$ at $\tilde{x}_2$ and $\tilde{x}'_2$ are the same. This means that $\tilde{L}_1 = (\tilde{x}_1, \tilde{x}'_1)$ and $\tilde{L}_2 = (\tilde{x}_2, \tilde{x}'_2)$ can be equally seen as loops of $\rho^{(h,h)}$ or as loops of $\sigma^{(h,h)}$. In particular, there are constants $k_1, k_2 > 0$, $0 \le h_1 < k_1$, and $0 \le h_2 < k_2$ such that, for all positive numbers m_1, m_2:

$$u^{(k_1\cdot m_1+h_1,\,k_2\cdot m_2+h_2)} \;=\; \mathsf{pump}_{\tilde{L}_2}^{m_2}(\mathsf{pump}_{\tilde{L}_1}^{m_1}(u^{(h,h)}))$$

and the above word has a successful run in $\mathcal{T}$ of the form $\mathsf{pump}_{\tilde{L}_2}^{m_2}(\mathsf{pump}_{\tilde{L}_1}^{m_1}(\sigma^{(h,h)}))$. Consider the outputs v_1 and v_2 produced by the intercepted factors $\rho \mid L_1/\ell_1$ and $\rho \mid L_2/\ell_2$, respectively. By Lemma 6, both v_1 and v_2 are non-empty. Moreover, by definition of pumped run, the output produced by $\rho^{(k_1\cdot m_1+h_1,\,k_2\cdot m_2+h_2)}$ contains $k_1 \cdot m_1 + h_1$ consecutive occurrences of v_1 followed by $k_2 \cdot m_2 + h_2$ consecutive occurrences of v_2 (see again the right-hand side Figure 2). Formally, we can write

$$\mathsf{out}(\rho^{(k_1\cdot m_1+h_1,\,k_2\cdot m_2+h_2)}) \;=\; v_0(m_1,m_2)\cdot \boldsymbol{v_1^{k_1\cdot m_1+h_1}} \cdot v_3(m_1,m_2)\cdot \boldsymbol{v_2^{k_2\cdot m_2+h_2}} \cdot v_4(m_1,m_2) \tag{2}$$

for some words $v_0(m_1,m_2)$, $v_3(m_1,m_2)$, and $v_4(m_1,m_2)$ that may depend on m_1 and m_2 (we highlighted in bold the repeated occurrences of v_1 and v_2 and we observe that v_1 precedes v_2).

In a similar way, because the same output is also produced by the the one-way transducer $\mathcal{T}$, i.e. by the run $\mathsf{pump}_{\tilde{L}_2}^{m_2}(\mathsf{pump}_{\tilde{L}_1}^{m_1}(\sigma^{(h,h)}))$, and because the loop L_2 precedes the loop L_1 according to the natural ordering of positions, we have

$$\mathsf{out}(\rho^{(k_1\cdot m_1+h_1,\,k_2\cdot m_2+h_2)}) \;=\; w_0 \cdot \boldsymbol{w_2^{m_2}} \cdot w_3 \cdot \boldsymbol{w_1^{m_1}} \cdot w_4 \tag{3}$$

where w_1 (resp., w_2) is the output produced by the unique factor of $\sigma^{(h,h)}$ intercepted by $\tilde{L}_1$ (resp., $\tilde{L}_2$), and w_0, w_3, w_4 are the remaining parts of the output. Note that, differently from the previous equation, here the first repetition is produced during the loop $\tilde{L}_2$ and the remaining parts w_0, w_3, w_4 do not depend on m_1, m_2. We now consider the factor

$$v(m_1,m_2) \;=\; \boldsymbol{v_1^{k_1\cdot m_1+h_1}} \cdot v_3(m_1,m_2) \cdot \boldsymbol{v_2^{k_2\cdot m_2+h_2}}$$

of the output produced by $\mathcal{S}$. The following claim shows that this factor is periodic, with a small period that only depends on $\mathcal{S}$ (in particular, it does not depend on any of the indices $h, k_1, k_2, h_1, h_2, m_1, m_2$).

▶ **Claim.** For all numbers $m_1, m_2 > 0$, the word $v(m_1,m_2) = \boldsymbol{v_1^{k_1\cdot m_1+h_1}} \cdot v_3(m_1,m_2) \cdot \boldsymbol{v_2^{k_2\cdot m_2+h_2}}$ is periodic with period $\gcd(|v_1|,|v_2|)$.

The idea for the proof of the above claim, detailed in the appendix, is to let m_1 and m_2 grow independently. We exploit Equations (2) and (3) to show that $\boldsymbol{v_1^{k_1\cdot m_1+h_1}} \cdot v_3(m_1,m_2) \cdot \boldsymbol{v_2}$ has period $\gcd(|v_1|,|w_1|)$, and that $\boldsymbol{v_1} \cdot v_3(m_1,m_2) \cdot \boldsymbol{v_2^{k_2\cdot m_2+h_2}}$ has period $\gcd(|v_2|,|w_2|)$. A last application of Fine and Wilf's theorem (Lemma 8) gives the desired periodicity.

Recall that we aim at proving the periodicity of the output $\mathsf{out}(\rho[\ell_1,\ell_2])$ of the *original* run ρ of $\mathcal{S}$ between the locations ℓ_1 and ℓ_2. The previous arguments, however, concern the outputs $v(m_1,m_2)$, which are produced by factors of the *pumped* runs $\rho^{(k_1\cdot m_1+h_1,k_2\cdot m_2+h_2)}$. By Equation (1) in Section 2, $\mathsf{out}(\rho[\ell_1,\ell_2])$ can be obtained from any $v(m_1,m_2)$ by deleting some occurrences of non-empty words produced by factors intercepted by L_1 or L_2. More precisely, the words that need to be deleted in $v(m_1,m_2)$ to obtain $\mathsf{out}(\rho[\ell_1,\ell_2])$ are non-empty and of the form $\mathsf{out}(\rho \mid L_i/\ell'_i)$, for some $i \in \{1,2\}$ and some location ℓ'_i such that $\ell_1 \le \ell'_i \le \ell_2$. Let us denote by $v'_1,\dots,v'_m$ these words. Note that, as m_1 and m_2 get larger, $v(m_1,m_2)$ contains arbitrarily long repetitions of each word v'_i, and hence long factors of period $|v'_i|$, for $i=1,\dots,m$. Thus, by applying Lemma 8, we get that $v(m_1,m_2)$ has period $p = \gcd(|v_1|,|v_2|,|v'_1|,\dots,|v'_m|)$.

Towards a conclusion, we know that $\mathsf{out}(\rho[\ell_1,\ell_2])$ is obtained from $v(m_1,m_2)$ by removing occurrences of the words $v'_1,\dots,v'_m$ whose lengths are multiple of the period p of $v(m_1,m_2)$. This implies that $\mathsf{out}(\rho[\ell_1,\ell_2])$ is also periodic with period p, which divides $\big|\mathsf{out}(\rho \mid L_i/\ell_i)\big|$ for both $i=1$ and $i=2$. ◀

Recall that the proof of the remaining part of Theorem 4 (necessity of the condition characterizing one-way definability) amounts at constructing a decomposition of the successful run ρ under the assumption that $\mathcal{S}$ is one-way definable. We begin to construct a decomposition of ρ by identifying some ramps in it. Intuitively, such ramps are obtained by considering the classes of a suitable equivalence relation:

▶ **Definition 9.** Let $\leftrightsquigarrow$ be the relation that pairs every two locations ℓ,ℓ' along the run ρ whenever there is an inversion (ℓ_1,ℓ_2) of ρ such that $\ell_1 \le \ell,\ell' \le \ell_2$, namely, whenever ℓ and ℓ' occur within the same inversion. Let $\leftrightsquigarrow^\star$ be the reflexive and transitive closure of $\leftrightsquigarrow$.

It is easy to see that every equivalence class of $\leftrightsquigarrow^\star$ is a convex subset with respect to the natural ordering of locations of ρ. The following lemma shows that every *non-singleton* equivalence class of $\leftrightsquigarrow^\star$ is a union of a series of inversions that are two-by-two overlapping.

▶ **Lemma 10.** *If two locations $\ell \le \ell'$ of ρ belong to the same* non-singleton *equivalence class of $\leftrightsquigarrow^\star$, then there is a sequence of locations $\ell_1 \le \ell_3 \le \ell_4 \le \ldots \le \ell_{n-3} \le \ell_{n-2} \le \ell_n$, for some even number $n \ge 4$, such that*
- *$\ell_1 \le \ell \le \ell_4$ and $\ell_{n-3} \le \ell' \le \ell_n$,*
- *(ℓ_1,ℓ_4), (ℓ_3,ℓ_6), (ℓ_5,ℓ_8), $\ldots$, (ℓ_{n-5},ℓ_{n-2}), (ℓ_{n-3},ℓ_n) are inversions.*

The next result uses Lemma 6, Proposition 7, and Lemma 10 to show that the output produced inside a $\leftrightsquigarrow^\star$-equivalence class is also periodic with small period, provided that $\mathcal{S}$ is one-way definable.

▶ **Lemma 11.** *If $\mathcal{S}$ is one-way definable and $\ell \le \ell'$ are two locations of the run ρ such that $\ell \leftrightsquigarrow^\star \ell'$, then the output $\mathsf{out}(\rho[\ell,\ell'])$ produced between ℓ and ℓ' has period at most $e_{\mathcal{S}}$.*

Below, we introduce some "bounding boxes" of non-singleton $\leftrightsquigarrow^\star$-equivalence classes. Intuitively, these bounding boxes are the smallest possible rectangles that start and end at some even levels and that cover all the locations forming an inversion inside a non-singleton $\leftrightsquigarrow^\star$-equivalence class. Subsequently, in Lemma 13 we show that these bounding boxes can be given the status of ramps in a suitable decomposition of ρ.

▶ **Definition 12.** Let K be a non-singleton $\leftrightsquigarrow^\star$-equivalence class and let H be the subset of K that contains all the locations $\ell,\ell' \in K$ forming an inversion (ℓ,ℓ').
We define $[K]$ to be the pair of locations $\ell_1 = (x_1,y_1)$ and $\ell_2 = (x_2,y_2)$ such that

- x_1 (resp., x_2) is the position of the leftmost (resp., rightmost) location $\ell \in H$,
- y_1 (resp., y_2) is the highest (resp., lowest) even level such that $y_1 \leq y$ (resp., $y_2 \geq y$) for all locations $\ell = (x, y) \in H$.

▶ **Lemma 13.** *If K is a non-singleton $\rightleftharpoons^\star$-equivalence class, then $[K]$ is a ramp.*

For the sake of brevity, we call $\rightleftharpoons^\star$-*ramp* any ramp of the form $[K]$, where K is a non-singleton $\rightleftharpoons^\star$-equivalence class. The results obtained so far imply that every location of the run ρ covered by an inversion is also covered by a $\rightleftharpoons^\star$-ramp. To complete the decomposition of ρ, we need to consider the locations that are not strictly covered by $\rightleftharpoons^\star$-ramps, formally, the set $B = \{\ell \mid \nexists\,(\ell_1, \ell_2)\ \rightleftharpoons^\star\text{-ramp s.t. } \ell_1 < \ell < \ell_2\}$. We equip B with the natural ordering of locations induced by ρ. We now consider some maximal convex subset C of B. Note that the left/right endpoint of C coincides with the first/last location of the run ρ or with the right/left endpoint of some $\rightleftharpoons^\star$-ramp. Below, we show how to decompose the sub-run $\rho \mid C$ into a series of floors and ramps. After this, we will be able to get a full decomposition of ρ by interleaving the $\rightleftharpoons^\star$-ramps that we defined above with the floors and the ramps that decompose each sub-run $\rho \mid C$.

Let D_C be the set of locations $\ell = (x, y)$ of C such that there is some loop $L = [x, x']$, with $x' \geq x$, whose intercepted factor $\rho \mid L/\ell$ lies entirely inside C and produces non-empty output. We remark that the set D_C may be non-empty. To see this, one can imagine the existence of two consecutive $\rightleftharpoons^\star$-ramps (e.g. R_1 and R_2 in Figure 1) and a loop between them that produces non-empty output (e.g. the factor F_2). In a more general scenario, one can find several loops between two consecutive $\rightleftharpoons^\star$-ramps that span across different levels. We can observe however that all the locations in D_C are on even levels. Indeed, if this were not the case for some $\ell = (x, y) \in D_C$, then we could select a minimal loop $L = [x_1, x_2]$ such that $x_1 \leq x \leq x_2$ and $\mathsf{out}(\rho \mid L/\ell) \neq \varepsilon$. Since y is odd, $\ell_1 = (x_2, y)$ is an entry point of L and $\ell_2 = (x_1, y)$ is an exit point of L, and hence (ℓ_1, ℓ_2) is an inversion. Since $\ell_1 \leq \ell \leq \ell_2$ and all inversions are covered by $\rightleftharpoons^\star$-ramps, there is a $\rightleftharpoons^\star$-ramp (ℓ'_1, ℓ'_2) such that $\ell'_1 \leq \ell \leq \ell'_2$. However, as ℓ'_1 and ℓ'_2 are at even levels, ℓ must be different from these two locations, and this would contradict the definition of D_C. Using similar arguments, one can also show that the locations in D_C are arranged along a "rising diagonal", from lower left to upper right.

The above properties suggest that the locations in D_C identify some floors and ramps that form a decomposition of $\rho \mid C$. The following lemma shows that this is indeed the case, namely, that any two consecutive locations in D_C form a floor or a ramp.

▶ **Lemma 14.** *Let $\ell_1 = (x_1, y_1)$ and $\ell_2 = (x_2, y_2)$ be two consecutive locations of D_C. Then, $x_1 \leq x_2$ and $y_1 \leq y_2$ and the pair (ℓ_1, ℓ_2) is a floor or a ramp, depending on whether $y_1 = y_2$ or $y_1 < y_2$.*

We have just shown how to construct a decomposition of the entire run ρ, assuming that the sweeping transducer $\mathcal{S}$ is one-way definable. This completes the proof of the only-if direction of the first claim of Theorem 4.

5 Lower bound and undecidability

We show now that the doubly exponential blow-up in size stated by Theorem 4, cannot be avoided.

▶ **Proposition 15.** *There is a family $(f_n)_n$ of functions from $\{0,1\}^*$ to $\{0,1\}^*$ such that:*

- *f_n can be computed by a sweeping transducer of size quadratic in n,*
- *f_n can be computed by a one-way transducer,*
- *any one-way transducer computing f_n has at least $2^{2^{n-1}}$ states.*

We exhibit such a family of function by defining the domain of f_n to be the set of words of the form

$$a_0 \ \mathsf{bin}_0 \ a_1 \ \mathsf{bin}_1 \ a_2 \ \ldots \ a_{2^n-1} \ \mathsf{bin}_{2^n-1} \ a_{2^n}$$

where $a_i \in \{0,1\}$ for all $i \in \{0,\ldots,2^n\}$. On those words, we define f_n as follows:

$$f_n\big(a_0 \ \mathsf{bin}_0 \ a_1 \ \mathsf{bin}_1 \ a_2 \ \ldots \ a_{2^n-1} \ \mathsf{bin}_{2^n-1} \ a_{2^n}\big) \ = \ w \cdot w \qquad \text{where } w \ = \ a_0 \, a_1 \ldots a_{2^n} \ .$$

We conclude the section by showing that the one-way definability problem becomes undecidable for relations computed by sweeping *non-functional* transducers. Note that ε-transitions are needed in order to capture the class of one-way definable relations. On the other hand, for one-way definable functions, ε-transitions can be excluded.

▶ **Proposition 16.** *The problem of testing whether a sweeping* non-functional *transducer is one-way definable is undecidable.*

6 Conclusion

In this paper we proposed a new algorithm that decides whether a sweeping transducer is equivalent to a one-way transducer. Our decision algorithm works in doubly exponential space and produces one-way transducers of doubly exponential size. The latter bound is shown to be optimal. An open question is whether the decision problem has lower complexity if we do not build the one-way transducer.

The main open question is whether our algorithm can be extended to two-way functional transducers that are not necessarily sweeping. We conjecture that this is the case and that a similar characterization based on decompositions of runs into floors and ramps can be obtained. The main difficulty is that (de)pumping loops is more complicated because of permutations.

One-way definability is also a special case of the following open problem: given an integer k and a two-way transducer, decide if there is an equivalent k-crossing two-way transducer. Finally, note that the problem that we considered here becomes much simpler in the origin semantics of [3]: there, the output of a transducer also includes the origin of each symbol, i.e., the input position where the symbol was generated. In the origin semantics, the one-way definability problem is PSpace-complete, and an equivalent one-way transducer has exponential size.

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