An ω-Algebra for Real-Time Energy Problems

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Abstract

We develop a *-continuous Kleene ω-algebra of real-time energy functions. Together with corresponding automata, these can be used to model systems which can consume and regain energy (or other types of resources) depending on available time. Using recent results on *-continuous Kleene ω-algebras and computability of certain manipulations on real-time energy functions, it follows that reachability and Büchi acceptance in real-time energy automata can be decided in a static way which only involves manipulations of real-time energy functions.

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1 Introduction

Energy and *resource management* problems are important in areas such as embedded systems or autonomous systems. They are concerned with the following types of questions:

- *Can the system reach a designated state without running out of energy before?*
- *Can the system reach a designated state within a specified time limit without running out of energy?*
- *Can the system repeatedly accomplish certain designated tasks without ever running out of energy?*

Instead of energy, these questions can also be asked using other resources, for example money or fuel.

As an example, imagine a satellite like in Fig. 1 which is being sent up into space. In its initial state when it has arrived at its orbit, its solar panels are still folded, hence no (electrical) energy is generated. Now it needs to unfold its solar panels and rotate itself and its panels into a position orthogonal to the sun's rays (for maximum energy yield). These operations require energy which hence must be provided by a battery, and there may be some operational requirements which state that they have to be completed within a given time limit. To minimize weight, one will generally be interested to use a battery which is as little as possible.

Figure 2 shows a simple toy model of such a satellite's initial operations. We assume that it opens its solar panels in two steps; after the first step they are half open and afterwards fully open, and that it can rotate into orthogonal position at any time. The numbers within the states signify energy gain per time unit, so that for example in the half-open state, the satellite gains 2 energy units per time unit before rotation and 4 after rotation. The (negative) numbers at transitions signify the energy cost for taking that transition, hence it costs 20 energy units to open the solar panels and 10 to rotate.

Now if the satellite battery has sufficient energy, then we can follow any path from the initial to the final state without spending time in intermediate states. A simple inspection reveals that a battery level of 50 energy units is required for this. On the other hand, if

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Figure 1 GPS Block II-F satellite (artist's conception; public domain).

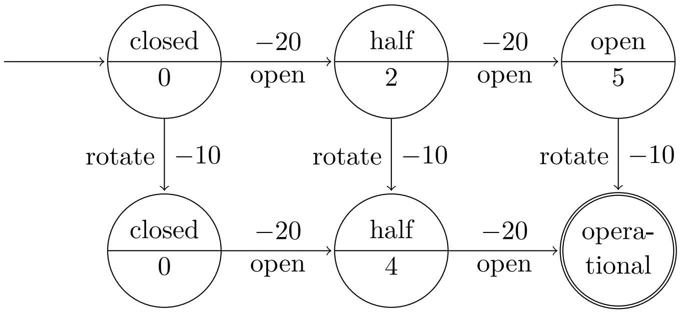


Figure 2 Toy model of the satellite in Fig. 1.

battery level is strictly below 20, then no path is available to the final state. With initial energy level between these two values, the device has to regain energy by spending time in an intermediate state before proceeding to the next one. The optimal path then depends on the available time and the initial energy. For an initial energy level of at least 40, the fastest strategy consists in first opening the panels and then spending 2 time units in state (open|5) to regain enough energy to reach the final state. With the smallest possible battery, storing 20 energy units, 5 time units have to be spent in state (half|2) before passing to (half|4) and spending another 5 time units there.

In this paper we will be concerned with models for such systems which, as in the example, allow to spend time in states to regain energy, some of which has to be spent when taking transitions between states. (Instead of energy, other resource types could be modeled, but we will from now think of it as energy.) We call these models *real-time energy automata*. Their behavior depends, thus, on both the initial energy and the time available; as we have seen in the example, this interplay between time and energy means that even simple models can have rather complicated behaviors. As in the example, we will be concerned with the *reachability* problem for such models, but also with *Büchi acceptance*: whether there exists an infinite run which visits certain designated states infinitely often.

Our methodology is strictly algebraic, using the theory of semiring-weighted automata [8] and extensions developed in [11, 10]. We view the finite behavior of a real-time energy automaton as a function $f(x_0, t)$ which maps initial energy x_0 and available time t to a final energy level, intuitively corresponding to the highest output energy the system can achieve when run with these parameters. We define a composition operator on such *real-time* *energy functions* which corresponds to concatenation of real-time energy automata and show that with this composition and maximum as operators, the set of real-time energy functions forms a **-continuous Kleene algebra* [19]. This implies that reachability in real-time energy automata can be decided in a static way which only involves manipulations of real-time energy functions.

To be able to decide Büchi acceptance, we extend the algebraic setting to also encompass real-time energy functions which model infinite behavior. These take as input an initial energy x_0 and time t, as before, but now the output is Boolean: true if these parameters permit an infinite run, false if they do not. We show that both types of real-time energy functions can be organized into a **-continuous Kleene ω-algebra* as defined in [11, 10]. This entails that also Büchi acceptance for real-time energy automata can be decided in a static way which only involves manipulations of real-time energy functions.

The most technically demanding part of the paper is to show that real-time energy functions form a *locally closed semiring* [8, 9]; generalizing some arguments in [9, 10], it then follows that they form a *-continuous Kleene ω-algebra. We conjecture that reachability and Büchi acceptance in real-time energy automata can be decided in exponential time.

Related work. Real-time energy problems have been considered in [20, 5, 4, 6, 15]. These are generally defined on *priced timed automata* [1, 2], a formalism which is more general than ours: it allows for time to be reset and admits several independent time variables (or *clocks*) which can be constrained at transitions. All known decidability results apply to priced timed automata with only *one* time variable; in [6] it is shown that with four time variables, it is undecidable whether there exists an infinite run.

The work which is closest to ours is [4]. Their models are priced timed automata with one time variable and energy updates on transitions, hence a generalization of ours. Using a sequence of complicated ad-hoc reductions, they show that reachability and existence of infinite runs is decidable for their models; whether their techniques apply to general Büchi acceptance is unclear.

Our work is part of a program to make methods from semiring-weighted automata available for energy problems. Starting with [12], we have developed a general theory of *-continuous Kleene ω-algebras [11, 10] and shown that it applies to so-called *energy automata*, which are finite (untimed) automata which allow for rather general *energy transformations* as transition updates. The contribution of this paper is to show that these algebraic techniques can be applied to a real-time setting.

Note that the application of Kleene algebra to real-time and hybrid systems is not a new subject, see for example [17, 7]. However, the work in these papers is based on *trajectories* and *interval predicates*, respectively, whereas our work is on real-time energy *automata*, *i.e.,* at a different level. A more thorough comparison of our work to [17, 7] would be interesting future work.

We acknowledge insightful discussions with Zoltán Ésik on the subject of this paper.

2 Real-Time Energy Automata

Let $\mathbb{R}_{\geq 0} = [0, \infty[$ denote the set of non-negative real numbers, $[0, \infty]$ the set $\mathbb{R}_{\geq 0}$ extended with infinity, and $\mathbb{R}_{\leq 0} =]-\infty, 0]$ the set of non-positive real numbers.

▶ **Definition 1.** A *real-time energy automaton* (RTEA) (S, s_0, F, T, r) consists of a finite set S of *states*, with *initial state* $s_0 \in S$, a subset $F \subseteq S$ of *accepting* states, a finite set $T \subseteq S \times \mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0} \times S$ of *transitions*, and a mapping $r : S \to \mathbb{R}_{\geq 0}$ assigning *rates* to states.

A transition (s, p, b, s') is written $s \xrightarrow[b]{p} s'$, p is called its *price* and b its *bound*. We assume $b \geq -p$ for all transitions $s \xrightarrow[b]{p} s'$.

An RTEA is *computable* if all its rates, prices and bounds are computable real numbers.

A *configuration* of an RTEA $A = (S, s_0, F, T, r)$ is an element $(s, x, t) \in C = S \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Let $\rightsquigarrow \subseteq C \times C$ be the relation given by $(s, x, t) \rightsquigarrow (s', x', t')$ iff $t' \leq t$ and there is a transition $s \xrightarrow[b]{p} s'$ such that $x + (t - t')r(s) \geq b$ and $x' = x + (t - t')r(s) + p$. Hence $t - t'$ time units are spent in state s and afterwards a transition $s \xrightarrow[b]{p} s'$ is taken.

A *run* in A is a path in the infinite directed graph $(C, \rightsquigarrow)$, *i.e.*, a finite or infinite sequence $(s_1, x_1, t_1) \rightsquigarrow (s_2, x_2, t_2) \rightsquigarrow \cdots$. We are ready to state the decision problems for RTEAs with which we will be concerned. Let $A = (S, s_0, F, T, r)$ be a computable RTEA and $x_0, t, y \in [0, \infty]$ computable numbers.

▶ Problem 1 (State reachability). Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$?

▶ Problem 2 (Coverability). Does there exist a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ and $x \geq y$?

▶ Problem 3 (Büchi acceptance). Does there exist $s \in F$ and an infinite run $(s_0, x_0, t) \rightsquigarrow (s_1, x_1, t_1) \rightsquigarrow \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$?

Note that the coverability problem only asks for the final energy level x to be *above* y; as we are interested in *maximizing* energy, this is natural. Also, state reachability can be reduced to coverability by setting $y = 0$. As the Büchi acceptance problem asks for infinite runs, there is no notion of output energy for this problem.

Asking the Büchi acceptance question for a *finite* available time $t < \infty$ amounts to finding (accepting) *Zeno runs* in the given RTEA, *i.e.*, runs which make infinitely many transitions in finite time. Hence one will usually be interested in Büchi acceptance only for an infinite time horizon. On the other hand, for $t = \infty$, a positive answer to the state reachability problem 1 consists of a finite run $(s_0, t_0, \infty) \rightsquigarrow \cdots \rightsquigarrow (s, x, \infty)$, and as one can delay indefinitely in the state $s \in F$, this leads to an infinite run. Per our definition of $\rightsquigarrow$, such an infinite run will *not* be a positive answer to the Büchi acceptance problem.

3 Weighted Automata over *-Continuous Kleene ω-Algebras

We now turn our attention to the algebraic setting of *-continuous Kleene algebras and related structures and review some results on *-continuous Kleene algebras and *-continuous Kleene ω-algebras which we will need in the sequel.

3.1 *-Continuous Kleene Algebras

An *idempotent semiring* [16] $S = (S, \vee, \cdot, \perp, 1)$ consists of an idempotent commutative monoid $(S, \vee, \perp)$ and a monoid $(S, \cdot, 1)$ such that the distributive and zero laws

$$x(y \vee z) = xy \vee xz \qquad (y \vee z)x = yx \vee zx \qquad \perp x = \perp = x \perp$$

hold for all $x, y, z \in S$. It follows that the product operation distributes over all finite suprema. Each idempotent semiring S is partially ordered by the relation $x \leq y$ iff $x \vee y = y$, and then sum and product preserve the partial order and $\perp$ is the least element.

A *Kleene algebra* [19] is an idempotent semiring $S = (S, \vee, \cdot, \perp, 1)$ equipped with an operation $^* : S \to S$ such that for all $x, y \in S$, yx^* is the least solution of the fixed point equation $z = zx \vee y$ and x^*y is the least solution of the fixed point equation $z = xz \vee y$ with respect to the order $\leq$.

A **-continuous Kleene algebra* [19] is a Kleene algebra $S = (S, \vee, \cdot, ^*, \bot, 1)$ in which the infinite suprema $\bigvee\{x^n \mid n \geq 0\}$ exist for all $x \in S$, $x^* = \bigvee\{x^n \mid n \geq 0\}$ for every $x \in S$, and product preserves such suprema: for all $x, y \in S$,

$$y\big(\bigvee_{n\geq 0} x^n\big) = \bigvee_{n\geq 0} yx^n \quad \text{and} \quad \big(\bigvee_{n\geq 0} x^n\big)y = \bigvee_{n\geq 0} x^n y\,.$$

An idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$ is said to be *locally closed* [9] if it holds that for every $x \in S$, there exists $N \geq 0$ so that $\bigvee_{n=0}^{N} x^n = \bigvee_{n=0}^{N+1} x^n$. In any locally closed idempotent semiring, we may define a *-operation by $x^* = \bigvee_{n\geq 0} x^n$.

▶ **Lemma 2.** *Any locally closed idempotent semiring is a *-continuous Kleene algebra.*

3.2 *-Continuous Kleene ω-Algebras

An *idempotent semiring-semimodule pair* [14, 3] (S, V) consists of an idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$ and a commutative idempotent monoid $V = (V, \vee, \bot)$ which is equipped with a left S-action $S \times V \to V$, $(s, v) \mapsto sv$, satisfying

$$\begin{aligned} (s \vee s')v &= sv \vee s'v & \qquad s(v \vee v') &= sv \vee sv' & \qquad \bot s &= \bot \\ (ss')v &= s(s'v) & s\bot &= \bot & 1v &= v \end{aligned}$$

for all $s, s' \in S$ and $v \in V$. In that case, we also call V a *(left) S-semimodule.*

A *generalized *-continuous Kleene algebra* [11] is an idempotent semiring-semimodule pair (S, V) where $S = (S, \vee, \cdot, ^*, \bot, 1)$ is a *-continuous Kleene algebra such that for all $x, y \in S$ and for all $v \in V$,

$$xy^*v = \bigvee_{n\geq 0} xy^n v$$

A **-continuous Kleene ω-algebra* [11] consists of a generalized *-continuous Kleene algebra (S, V) together with an infinite product operation $S^\omega \to V$ which maps every infinite sequence $x_0, x_1, \dots$ in S to an element $\prod_{n\geq 0} x_n$ of V. The infinite product is subject to the following conditions (see [11] for motivation):

- For all $x_0, x_1, \ldots \in S$, $\displaystyle\prod_{n\geq 0} x_n = x_0 \prod_{n\geq 0} x_{n+1}$. (C1)
- Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \leq \cdots$ a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\displaystyle\prod_{n\geq 0} x_n = \prod_{k\geq 0} y_k$. (C2)
- For all $x_0, x_1, \ldots, y, z \in S$, $\displaystyle\prod_{n\geq 0} (x_n(y \vee z)) = \bigvee_{x'_0, x'_1, \ldots \in \{y,z\}} \prod_{n\geq 0} x_n x'_n$. (C3)
- For all $x, y_0, y_1, \ldots \in S$, $\displaystyle\prod_{n\geq 0} x^* y_n = \bigvee_{k_0, k_1, \ldots \geq 0} \prod_{n\geq 0} x^{k_n} y_n$. (C4)

3.3 Matrix Semiring-Semimodule Pairs

For any idempotent semiring S and $n \geq 1$, we can form the matrix semiring $S^{n\times n}$ whose elements are $n \times n$-matrices of elements of S and whose sum and product are given as the usual matrix sum and product. It is known [18] that when S is a *-continuous Kleene algebra, then $S^{n\times n}$ is also a *-continuous Kleene algebra, with the *-operation defined by

$$M^*_{i,j} = \bigvee_{m\ge 0} \bigvee \{M_{k_1,k_2} M_{k_2,k_3} \cdots M_{k_{m-1},k_m} \mid 1 \le k_1,\dots,k_m \le n, k_1 = i, k_m = j\} \tag{1}$$

for all $M \in S^{n\times n}$ and $1 \le i,j \le n$. Also, if $n \ge 2$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension less than n, then

$$M^* = \begin{pmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{pmatrix} \tag{2}$$

For any idempotent semiring-semimodule pair (S,V) and $n \ge 1$, we can form the matrix semiring-semimodule pair $(S^{n\times n}, V^n)$ whose elements are $n \times n$-matrices of elements of S and n-dimensional (column) vectors of elements of V, with the action of $S^{n\times n}$ on V^n given by the usual matrix-vector product.

When (S,V) is a *-continuous Kleene ω-algebra, then $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra [11]. By [11, Lemma 17], there is an ω-operation on $S^{n\times n}$ defined by

$$M^\omega_i = \bigvee_{1 \le k_1, k_2, \dots \le n} M_{i,k_1} M_{k_1,k_2} \cdots$$

for all $M \in S^{n\times n}$ and $1 \le i \le n$. Also, if $n \ge 2$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension less than n, then

$$M^\omega = \begin{pmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^* ca^\omega \end{pmatrix} \tag{3}$$

It can be shown [13] that the number of semiring computations required in the computation of M^* and M^ω in (2) and (3) is $O(n^3)$ and $O(n^4)$, respectively.

3.4 Weighted automata

Let (S,V) be a *-continuous Kleene ω-algebra and $A \subseteq S$ a subset. We write $\langle A\rangle$ for the set of all finite suprema $a_1 \vee \cdots \vee a_m$ with $a_i \in A$ for each $i = 1,\dots,m$.

A *weighted automaton* [8] over A of dimension $n \ge 1$ is a tuple (α, M, k), where $\alpha \in \{\bot, 1\}^n$ is the initial vector, $M \in \langle A\rangle^{n\times n}$ is the transition matrix, and k is an integer $0 \le k \le n$. Combinatorially, this may be represented as a transition system whose set of states is $\{1,\dots,n\}$. For any pair of states i,j, the transitions from i to j are determined by the entry $M_{i,j}$ of the transition matrix: if $M_{i,j} = a_1 \vee \cdots \vee a_m$, then there are m transitions from i to j, respectively labeled $a_1,\dots,a_n$. The states i with $\alpha_i = 1$ are *initial*, and the states $\{1,\dots,k\}$ are *accepting*.

The *finite behavior* of a weighted automaton $A = (\alpha, M, k)$ is defined to be

$$|A| = \alpha M^* \kappa\,,$$

where $\kappa \in \{\bot, 1\}^n$ is the vector given by $\kappa_i = 1$ for $i \le k$ and $\kappa_i = \bot$ for $i > k$. (Note that α has to be used as a *row* vector for this multiplication to make sense.) It is clear by (1) that $|A|$ is the supremum of the products of the transition labels along all paths in A from any initial to any accepting state.

The *Büchi behavior* of a weighted automaton $A = (\alpha, M, k)$ is defined to be

$$\|A\| = \alpha \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix},$$

where $a \in \langle A \rangle^{k \times k}$, $b \in \langle A \rangle^{k \times (n-k)}$, $c \in \langle A \rangle^{(n-k) \times n}$ and $d \in \langle A \rangle^{(n-k) \times (n-k)}$ are such that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that M is split in submatrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ precisely so that a contains transitions between accepting states and d contains transitions between non-accepting states. By [11, Thm. 20], $\|A\|$ is the supremum of the products of the transition labels along all infinite paths in A from any initial state which infinitely often visit an accepting state.

4 Real-Time Energy Functions

Let $L = [0, \infty]_\perp$ denote the set of non-negative real numbers extended with a bottom element $\perp$ and a top element ∞. We use the standard order on L, *i.e.,* the one on $\mathbb{R}_{\geq 0}$ extended by declaring $\perp \leq x \leq \infty$ for all $x \in L$. L is a complete lattice, whose suprema we will denote by $\vee$ for binary and $\bigvee$ for general supremum. For convenience we also extend the addition on $\mathbb{R}_{\geq 0}$ to L by declaring that $\perp + x = x + \perp = \perp$ for all $x \in L$ and $\infty + x = x + \infty = \infty$ for all $x \in L \setminus \{\perp\}$. Note that $\perp + \infty = \infty + \perp = \perp$.

Let $\mathcal{F}$ denote the set of monotonic functions $f : L \times [0, \infty] \to L$ (with the product order on $L \times [0, \infty]$) for which $f(\perp, t) = \perp$ for all $t \in L$. We will frequently write such functions in curried form, using the equivalence $\langle L \times [0, \infty] \to L \rangle \approx \langle [0, \infty] \to L \to L \rangle$.

4.1 Linear Real-Time Energy Functions

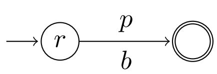
We will be considered with the subset of functions in $\mathcal{F}$ consisting of *real-time energy functions* (RTEFs). These correspond to functions expressed by RTEAs, and we will construct them inductively. We start with atomic RTEFs:

▶ **Definition 3.** Let $r, b, p \in \mathbb{R}$ with $r \geq 0$, $p \leq 0$ and $b \geq -p$. An *atomic real-time energy function* is an element f of $\mathcal{F}$ such that $f(\perp, t) = \perp$, $f(\infty, t) = \infty$, $f(x, \infty) = \infty$, and

$$f(x, t) = \begin{cases} x + rt + p & \text{if } x + rt \geq b\,, \\ \perp & \text{otherwise} \end{cases}$$

for all $x, t \in \mathbb{R}_{\geq 0}$. The numbers r, b and p are respectively called the *rate*, *bound* and *price* of f. We denote by $\mathcal{A} \subseteq \mathcal{F}$ the set of atomic real-time energy functions.

These functions arise from RTEAs with one transition:



Non-negativity of r ensures that atomic RTEFs are monotonic. In our examples, when the bound is not explicitly mentioned it corresponds to the lowest possible one: $b = -p$.

Atomic RTEFs are naturally combined along acyclic paths by means of a composition operator. Intuitively, a composition of two successive atomic RTEFs determines the optimal output energy one can get after spending some time in either one or both locations of the corresponding automaton. This notion of composition is naturally extended to all functions in $\mathcal{F}$, and formally defined as follows.

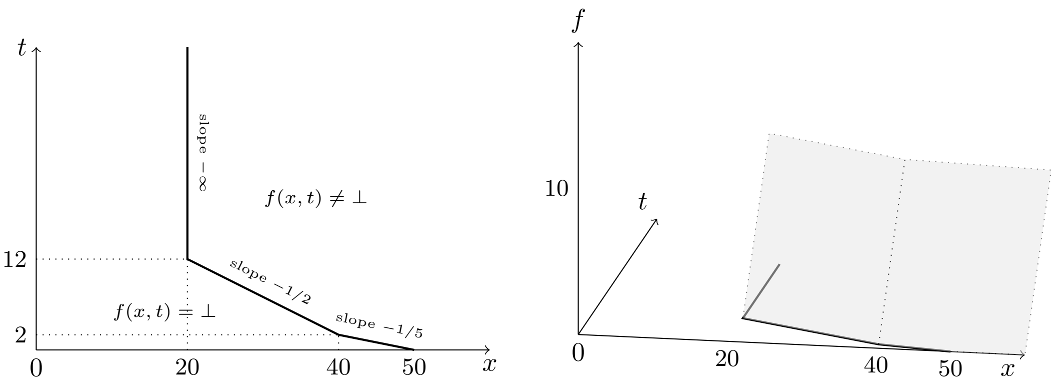


Figure 3 Graphical representation of the linear RTEF from Example 7.

▶ **Definition 4.** The *composition* of $f, g \in \mathcal{F}$ is the element $f \triangleright g$ of $\mathcal{F}$ such that

$$\forall t \in [0, \infty] : f \triangleright g(t) = \bigvee_{t_1+t_2=t} g(t_2) \circ f(t_1) \qquad (4)$$

Note that composition is written in diagrammatic order. Uncurrying the equation, we see that $f \triangleright g(x, t) = \bigvee_{t_1+t_2=t} g(f(x, t_1), t_2)$.

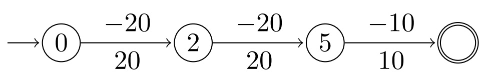
Let $\mathbf{1}, \bot \in \mathcal{F}$ be the functions defined by $\mathbf{1}(t)(x) = x$ and $\bot(t)(x) = \bot$ for all x, t.

▶ **Lemma 5.** *The $\triangleright$ operator is associative, with $\mathbf{1}$ as neutral and $\bot$ as absorbing elements.*

Compositions of atomic RTEFs along paths are called *linear* RTEFs:

▶ **Definition 6.** A *linear real-time energy function* is a finite composition $f_1 \triangleright f_2 \triangleright \cdots \triangleright f_n$ of atomic RTEFs.

▶ **Example 7.** As an example, and also to show that linear RTEFs can have quite complex behavior, we show the linear RTEF associated to one of the paths in the satellite example of the introduction. Consider the following (linear) RTEA:



Its linear RTEF f can be computed as follows:

$$f(x, t) = \begin{cases} \bot & \text{if } x < 20 \text{ or } (20 \leq x < 40 \text{ and } x + 2t < 44) \\ & \qquad\qquad \text{or } (x \geq 40 \text{ and } x + 5t < 50) \\ 2.5x + 5t - 110 & \text{if } 20 \leq x < 40 \text{ and } x + 2t \geq 44 \\ x + 5t - 50 & \text{if } x \geq 40 \text{ and } x + 5t \geq 50 \end{cases}$$

We show a graphical representation of f on Fig. 3. The left part of the figure shows the boundary between two regions in the (x, t) plane, corresponding to the minimal value 0 achieved by the function. Below this boundary, no path exists through the corresponding RTEA. Above, the function is linear in x and t. The coefficient of t corresponds to the maximal rate in the RTEA; the coefficient of x depends on the relative position of x with respect to the bounds b_i.

4.2 Normal Form

Next we need to see that all linear RTEFs can be converted to a *normal form*:

▶ **Definition 8.** A sequence $f_1, \dots, f_n$ of atomic RTEFs, with rates, bounds and prices $r_1, \dots, r_n$, $b_1, \dots, b_n$ and $p_1, \dots, p_n$, respectively, is in *normal form* if

- $r_1 < \cdots < r_n$,
- $b_1 \le \cdots \le b_n$, and
- $p_1 = \cdots = p_{n-1} = 0$.

▶ **Lemma 9.** *For any linear RTEF f there exists a sequence $f_1, \dots, f_n$ of atomic RTEFs in normal form such that $f = f_1 \triangleright \cdots \triangleright f_n$.*

Proof sketch. Let $f = f_1 \triangleright \cdots \triangleright f_n$, where $f_1, \dots, f_n$ are atomic RTEFs and assume $f_1, \dots, f_n$ is not in normal form. If there is an index $k \in \{1, \dots, n-1\}$ with $r_k \ge r_{k+1}$, then we can use the following transformation to remove the state with rate r_{k+1}:

$$\longrightarrow (r_k) \xrightarrow[b_k]{p_k} (r_{k+1}) \xrightarrow[b_{k+1}]{p_{k+1}} (r_{k+2}) \quad \underset{(r_k \ge r_{k+1})}{\longmapsto} \quad \longrightarrow (r_k) \xrightarrow[\max(b_k, b_{k+1} - p_k)]{p_k + p_{k+1}} (r_{k+2})$$

Informally, any run through the RTEA for $f_1 \triangleright \cdots \triangleright f_n$ which maximizes output energy will spend no time in the state with rate r_{i+1}, as this time may as well be spent in the state with rate r_i without lowering output energy.

To ensure the last two conditions of Definition 8, we use the following transformation:

$$\longrightarrow (r_k) \xrightarrow[b_k]{p_k} (r_{k+1}) \xrightarrow[b_{k+1}]{p_{k+1}} (r_{k+2}) \quad \longmapsto \quad \longrightarrow (r_k) \xrightarrow[b_k]{0} (r_{k+1}) \xrightarrow[\max(b_k, b_{k+1} - p_k)]{p_k + p_{k+1}} (r_{k+2})$$

Any run through the original RTEA can be copied to the other and vice versa, hence also this transformation does not change the values of f. ◀

▶ **Definition 10.** Let $f_1, \dots, f_n$ and $f'_1, \dots, f'_{n'}$ be normal-form sequences of atomic RTEFs with rate sequences $r_1 < \cdots < r_n$ and $r'_1 < \cdots < r'_{n'}$, respectively. Then $f_1, \dots, f_n$ is *not better than* $f'_1, \dots, f'_{n'}$, denoted $(f_1, \dots, f_n) \preceq (f'_1, \dots, f'_{n'})$, if $r_n \le r'_{n'}$.

Note that $(f_1, \dots, f_n) \preceq (f'_1, \dots, f'_{n'})$ does not imply $f_1 \triangleright \cdots \triangleright f_n \le f'_1 \triangleright \cdots \triangleright f'_{n'}$ even for very simple functions. For a counterexample, consider the two following linear RTEFs $f = f_1$, $f' = f'_1 \triangleright f'_2$ with corresponding RTEAs:

$$f: \quad \longrightarrow (4) \xrightarrow[0]{0} (\circledcirc) \qquad\qquad f': \quad \longrightarrow (1) \xrightarrow[1]{0} (5) \xrightarrow[2]{0} (\circledcirc)$$

We have $(f_1) \preceq (f'_1, f'_2)$, and for $x \ge 2$, $f(x,t) = x + 4t$ and $f'(x,t) = x + 5t$, hence $f(x,t) \le f'(x,t)$. But $f(0,1) = 4$, whereas $f'(0,1) = \bot$.

▶ **Lemma 11.** *If $f = f_1 \triangleright \cdots \triangleright f_n$ and $f' = f'_1 \triangleright \cdots \triangleright f'_{n'}$ are such that $(f_1, \dots, f_n) \preceq (f'_1, \dots, f'_{n'})$, then $f' \triangleright f \le f'$.*

Proof. Let $r_1 < \cdots < r_n$ and $r'_1 < \cdots < r'_{n'}$ be the corresponding rate sequences, then $r_n \le r'_{n'}$. The RTEAs for $f' \triangleright f$ and f' are as follows, where we have transformed the former to normal form using that for all indices i, $r_i \leq r_n \leq r'_{n'}$:

$$f' \rhd f : \longrightarrow \textcircled{r'_1} \xrightarrow[b'_1]{0} \cdots \longrightarrow \textcircled{$r'_{n'}$} \xrightarrow[\max(b'_{n'}, b_n - p')]{p + p'} \circledcirc$$

$$f' : \longrightarrow \textcircled{r'_1} \xrightarrow[b'_1]{0} \cdots \longrightarrow \textcircled{$r'_{n'}$} \xrightarrow[b'_{n'}]{p'} \circledcirc$$

As $p+p' \leq p'$ (because $p \leq 0$) and $\max(b'_{n'}, b_n - p') \geq b'_{n'}$, it is clear that $f' \rhd f(x,t) \leq f'(x,t)$ for all $x \in L$, $t \in [0,\infty]$. ◀

4.3 General Real-Time Energy Functions

We now consider all paths that may arise in a real-time energy automaton. When two locations of an automaton may be joined by two distinct paths, the optimal output energy is naturally obtained by taking the maximum over both paths. This gives rise to the following definition.

▶ **Definition 12.** Let $f, g \in \mathcal{F}$. The function $f \vee g$ is defined as the pointwise supremum:

$$\forall t \in [0,\infty] : (f \vee g)(t) = f(t) \vee g(t)$$

▶ **Lemma 13.** *With operations $\vee$ and $\rhd$, $\mathcal{F}$ forms a complete lattice and an idempotent semiring, with $\bot$ as unit for $\vee$ and $\mathbf{1}$ as unit for $\rhd$.*

Finally, a cycle in an RTEA results in a *-operation:

▶ **Definition 14.** Let $f \in \mathcal{F}$. The Kleene star of f is the function $f^* \in \mathcal{F}$ such that

$$\forall t \in [0,\infty] : f^*(t) = \bigvee_{n \geq 0} f^n(t)$$

Note that f^* is defined for all $f \in \mathcal{F}$ because $\mathcal{F}$ is a complete lattice. We can now define the set of general real-time energy functions, corresponding to general RTEAs:

▶ **Definition 15.** The set $\mathcal{E}$ of *real-time energy functions* is the subsemiring of $\mathcal{F}$ generated by $\mathcal{A}$, *i.e.*, the subset of $\mathcal{F}$ inductively defined by

- $\mathcal{A} \subseteq \mathcal{E}$,
- if $f, g \in \mathcal{E}$, then $f \rhd g \in \mathcal{E}$ and $f \vee g \in \mathcal{E}$.

We will show below that $\mathcal{E}$ is locally closed, which entails that for each $f \in \mathcal{E}$, also $f^* \in \mathcal{E}$, hence $\mathcal{E}$ indeed encompasses all RTEFs.

▶ **Lemma 16.** *For every $f \in \mathcal{E}$ there exists $N \geq 0$ so that $f^* = \bigvee_{n=0}^N f^n$.*

Proof. By distributivity, we can write f as a finite supremum $f = \bigvee_{k=1}^m f_k$ of linear energy functions $f_1, \dots, f_m$. For each $k = 1, \dots, m$, let $f_k = f_{k,1} \rhd \cdots \rhd f_{k,n_k}$ be a normal-form representation. By re-ordering the f_k if necessary, we can assume that $(f_{k,1}, \dots, f_{k,n_k}) \preceq (f_{k+1,1}, \dots, f_{k+1,n_{k+1}})$ for every $k = 1, \dots, n-1$.

We first show that $f^* \leq \bigvee_{0 \leq n_1, \dots, n_m \leq 1} f_1^{n_1} \rhd \cdots \rhd f_m^{n_m}$: The expansion of $f^* = (\bigvee_{k=1}^m f_k)^*$ is an infinite supremum of finite compositions $f_{i_1} \rhd \cdots \rhd f_{i_p}$. By Lemma 11, any occurrence of $f_{i_j} \rhd f_{i_{j+1}}$ in such compositions with $i_j \geq i_{j+1}$ can be replaced by $f_{i_{j+1}}$. The compositions which are left have $i_j < i_{j+1}$ for every j, so the claim follows.

Now $\bigvee_{0 \leq n_1, \dots, n_m \leq 1} f_1^{n_1} \rhd \cdots \rhd f_m^{n_m} \leq \bigvee_{n=0}^m \big(\bigvee_{k=1}^m f_k\big)^n = \bigvee_{n=0}^m f^n \leq f^*$, so with $N = m$ the proof is complete. ◀

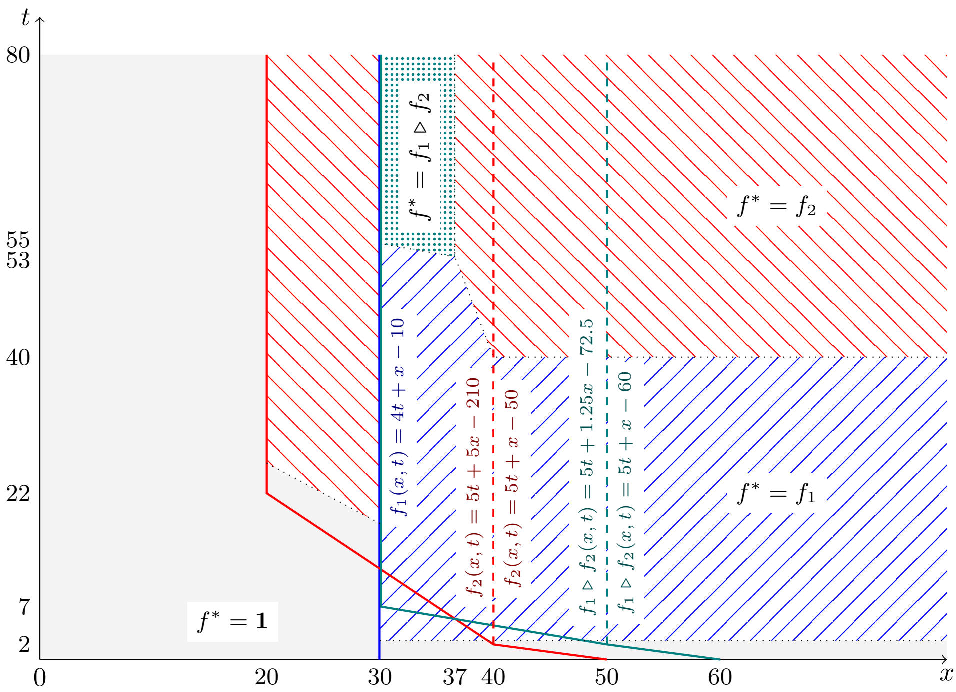
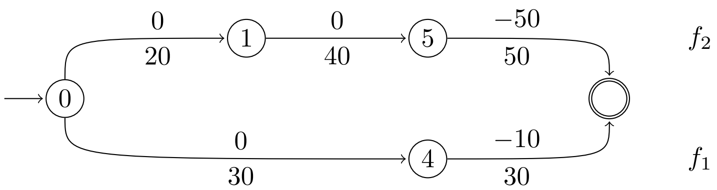


Figure 4 Computation of f^* from Example 18.

▶ **Corollary 17.** *$\mathcal{E}$ is locally closed, hence a *-continuous Kleene algebra.*

Proof. For every $f \in \mathcal{E}$ there is $N \geq 0$ so that $f^* = \bigvee_{n=0}^{N} f^n$ (Lemma 16), hence $\bigvee_{n=0}^{N} f^n = \bigvee_{n=0}^{N+1} f^n$. Thus $\mathcal{E}$ is locally closed, and by Lemma 2, a *-continuous Kleene algebra. ◀

▶ **Example 18.** To illustrate, we compute the Kleene star of the supremum $f = f_1 \vee f_2$ of two linear RTEFs as below. These are slight modifications of some RTEFs from the satellite example, modified to make the example more interesting:



These functions are in normal form and $f_1 \preceq f_2$. Lemma 16 and its proof allow us to conclude that $f^* = \mathbf{1} \vee f_1 \vee f_2 \vee f_1 \triangleright f_2$. Figure 4 shows the boundaries of definition of these functions and the regions in the (x, t) plane where each of them dominates.

4.4 Infinite Products

Let $\mathbb{B} = \{\mathbf{ff}, \mathbf{tt}\}$ denote the Boolean lattice with standard order $\mathbf{ff} < \mathbf{tt}$. Let $\mathcal{V}$ denote the set of monotonic functions $v : L \times [0, \infty] \to \mathbb{B}$ for which $v(\bot, t) = \bot$ for all $t \in L$. We define an infinite product operation $\mathcal{F}^\omega \to \mathcal{V}$:

▶ **Definition 19.** For an infinite sequence of functions $f_0, f_1, \ldots \in \mathcal{F}$, $\prod_{n \geq 0} f_n \in \mathcal{V}$ is the function defined for $x \in L$, $t \in [0, \infty]$ by $\prod_{n \geq 0} f_n(x, t) = \mathbf{tt}$ iff there is an infinite sequence $t_0, t_1, \ldots \in [0, \infty]$ such that $\sum_{n=0}^{\infty} t_n = t$ and for all $n \geq 0$, $f_n(t_n) \circ \cdots \circ f_0(t_0)(x) \neq \bot$.

Hence $\prod_{n\geq 0} f_n(x,t) = \mathbf{tt}$ iff in the infinite composition $f_0 \rhd f_1 \rhd \cdots (x,t)$, all finite prefixes have values $\neq \perp$. There is a (left) action of $\mathcal{F}$ on $\mathcal{V}$ given by $(f,v) \mapsto f \rhd v$, where the composition $f \rhd v$ is given by the same formula as composition $\rhd$ on $\mathcal{F}$. Let $\perp \in \mathcal{V}$ denote the function given by $\perp(x,t) = \mathbf{ff}$.

▶ **Lemma 20.** *With the $\mathcal{F}$-action $\rhd$, $\vee$ as addition, and $\perp$ as unit, $\mathcal{V}$ is an idempotent left $\mathcal{F}$-semimodule.*

Let $\mathcal{U} \subseteq \mathcal{V}$ be the $\mathcal{F}$-subsemimodule generated by $\mathcal{E} \subseteq \mathcal{F}$. Then $\mathcal{U}$ is an idempotent left $\mathcal{E}$-semimodule.

▶ **Proposition 21.** *$(\mathcal{E},\mathcal{U})$ forms a *-continuous Kleene ω-algebra.*

5 Decidability

We can now apply the results of Section 3.4 to see that our decision problems as stated at the end of Section 2 are decidable. Let Let $A = (S, s_0, F, T, r)$ be an RTEA, with matrix representation (α, M, K), and $x_0, t, y \in [0, \infty]$.

▶ **Theorem 22.** *There exists a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ iff $|A|(x_0, t) > \perp$.*

▶ **Theorem 23.** *There exists a finite run $(s_0, x_0, t) \rightsquigarrow \cdots \rightsquigarrow (s, x, t')$ in A with $s \in F$ and $x \geq y$ iff $|A|(x_0, t) \geq y$.*

▶ **Theorem 24.** *There exists $s \in F$ and an infinite run $(s_0, x_0, t) \rightsquigarrow (s_1, x_1, t_1) \rightsquigarrow \cdots$ in A in which $s_n = s$ for infinitely many $n \geq 0$ iff $\|A\|(x_0, t) = \top$.*

▶ **Theorem 25.** *Problems 1, 2 and 3 from Section 2 are decidable.*

Proof sketch. We have seen in the examples that RTEFs are *piecewise linear*, *i.e.*, composed of a finite number of (affine) linear functions which are defined on polygonal regions in the (x,t)-plane. Such functions can be represented using the (finitely many) corner points of these regions together with their values at these corner points. (In case some regions are not convex or disconnected, they have to be split into convex regions.)

It is clear that computable atomic RTEFs are computable piecewise linear (*i.e.*, all numbers in their finite representation are computable), and that compositions and suprema of computable piecewise linear are again computable piecewise linear. Using Lemma 16, we see that all functions in M^* are computable piecewise linear. ◀

6 Conclusion

We have developed an algebraic methodology for deciding reachability and Büchi problems on a class of weighted real-time models where the weights represent energy or similar quantities. The semantics of such systems is modeled by real-time energy functions which map initial energy of the system and available time to the maximal final energy level. We have shown that these real-time energy functions form a *-continuous Kleene ω-algebra, which entails that reachability and Büchi acceptance can be decided in a static way which only involves manipulations of energy functions.

We have seen that the necessary manipulations of real-time energy functions are computable, and in fact we conjecture that our method leads to an exponential-time algorithm for deciding reachability and Büchi acceptance in real-time energy automata. This is due to the fact that operations on real-time energy functions can be done in time linear in the size of their representation, and the representation size of compositions and suprema of real-time energy functions is a linear function of the representation size of the operands. In future work, we plan to do a careful complexity analysis which could confirm this result and to implement our algorithms to see how it fares in practice.

This paper constitutes the first application of methods from Kleene algebra to a timed-automata like formalism. In future work, we plan to lift some of the restrictions of the current model and extend it to allow for time constraints and resets à la timed automata. We also plan to extend this work with action labels, which algebraically means passing from the semiring of real-time energy functions to the one of formal power series over these functions. In applications, this means that instead of asking for existence of accepting runs, one is asking for controllability.

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