Report from Dagstuhl Seminar 15301

The Constraint Satisfaction Problem: Complexity and Approximability

Edited by

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Abstract

During the past two decades, an impressive array of diverse methods from several different mathematical fields, including algebra, logic, mathematical programming, probability theory, graph theory, and combinatorics, have been used to analyze both the computational complexity and approximabilty of algorithmic tasks related to the constraint satisfaction problem (CSP), as well as the applicability/limitations of algorithmic techniques. This research direction develops at an impressive speed, regularly producing very strong and general results. The Dagstuhl Seminar 15301 "The Constraint Satisfaction Problem: Complexity and Approximability" was aimed at bringing together researchers using all the different techniques in the study of the CSP, so that they can share their insights obtained during the past three years. This report documents the material presented during the course of the seminar.

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1 Executive Summary

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The *constraint satisfaction problem*, or CSP in short, provides a unifying framework in which it is possible to express, in a natural way, a wide variety of computational problems dealing with mappings and assignments, including satisfiability, graph colorability, and systems of equations. The CSP framework originated 25–30 years ago independently in artificial intelligence, database theory, and graph theory, under three different guises, and

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it was realised only in the late 1990s that these are in fact different faces of the same fundamental problem. Nowadays, the CSP is extensively used in theoretical computer science, being a mathematical object with very rich structure that provides an excellent laboratory both for classification methods and for algorithmic techniques, while in AI and more applied areas of computer science this framework is widely regarded as a versatile and efficient way of modelling and solving a variety of real-world problems, such as planning and scheduling, software verification and natural language comprehension, to name just a few. An instance of CSP consists of a set of variables, a set of values for the variables, and a set of constraints that restrict the combinations of values that certain subsets of variables may take. Given such an instance, the possible questions include (a) deciding whether there is an assignment of values to the variables so that every constraint is satisfied, or optimising such assignments in various ways, (b) counting satisfying assignments, exactly or approximately, or (c) finding an assignment satisfying as many constraints as possible. There are many important modifications and extensions of this basic framework, e.g. those that deal with valued or global constraints.

Constraint satisfaction has always played a central role in computational complexity theory; appropriate versions of CSPs are classical complete problems for most standard complexity classes. CSPs constitute a very rich and yet sufficiently manageable class of problems to give a good perspective on general computational phenomena. For instance, they help to understand which mathematical properties make a computational problem tractable (in a wide sense, e.g. polynomial-time solvable or non-trivially approximable, fixed-parameter tractable or definable in a weak logic). It is only natural that CSPs play a role in many high-profile conjectures in complexity theory, exemplified by the Dichotomy Conjecture of Feder and Vardi and the Unique Games Conjecture of Khot.

The recent flurry of activity on the topic of the seminar is witnessed by three previous Dagstuhl seminars, titled "Complexity of constraints" (06401) and "The CSP: complexity and approximability" (09441, 12541), that were held in 2006, 2009, and 2012 respectively. This seminar was a follow-up to the 2009 and 2012 seminars. Indeed, the exchange of ideas at the 2009 and 2012 seminars has led to new ambitious research projects and to establishing regular communications channels, and there is a clear potential of a further systematic interaction that will keep on cross-fertilizing the areas and opening new research directions. The 2015 seminar brought together forty three researchers from different highly advanced areas of constraint satisfaction and involved many specialists who use universal-algebraic, combinatorial, geometric and probabilistic techniques to study CSP-related algorithmic problems. The participants presented, in 28 talks, their recent results on a number of important questions concerning the topic of the seminar. One particular feature of this seminar is a significant increase in the number of talks involving multiple subareas and approaches within its research direction – a definite sign of the growing synergy, which is one of the main goals of this series of seminars.

Concluding Remarks and Future Plans. The seminar was well received as witnessed by the high rate of accepted invitations and the great degree of involvement by the participants. Because of the multitude of impressive results reported during the seminar and the active discussions between researchers with different expertise areas, the organisers regard this seminar as a great success. With steadily increasing interactions between such researchers, we foresee a new seminar focussing on the interplay between different approaches to studying the complexity and approximability of the CSP. Finally, the organisers wish to express their gratitude to the Scientific Directors of the Dagstuhl Centre for their support of the seminar.

Description of the Topics of the Seminar

Classical computational complexity of CSPs. Despite the provable existence of intermediate (say, between P and NP-complete, assuming P ≠ NP) problems, research in computational complexity has produced a widely known informal thesis that "natural problems are almost always complete for standard complexity classes". CSPs have been actively used to support and refine this thesis. More precisely, several restricted forms of CSP have been investigated in depth. One of the main types of restrictions is the *constraint language* restriction, i.e., a restriction on the available types of constraints. By choosing an appropriate constraint language, one can obtain many well-known computational problems from graph theory, logic, and algebra. The study of the constraint language restriction is driven by the CSP *Dichotomy Conjecture* of Feder and Vardi which states that, for each fixed constraint language, the corresponding CSP is either in P or NP-complete. There are similar dichotomy conjectures concerning other complexity classes (e.g. L and NL). Recent breakthroughs in the complexity of CSP have been made possible by the introduction of the universal-algebraic approach, which extracts algebraic structure from the constraint language and uses it to analyse problem instances. The above conjectures have algebraic versions which also predict in algebraic terms where the boundary between harder problems and easier problems lies. The algebraic approach has been applied to prove the Dichotomy Conjecture in many important special cases (e.g. Bulatov's dichotomy theorems for 3-valued and conservative CSPs), but the general problem remains open. Barto and Willard described the current state-of-the-art in proving this conjecture, gave insights into the main stumbling blocks (notably, the convoluted ways in which systems of linear equations appear in constraint problems), and outlined avenues of attack on those obstacles. Kozik gave a new simplified algorithm for CSPs solvable by local consistency methods, confirming an earlier conjecture. Brown-Cohen presented new results leading to closer interchange of ideas between algebraic and probabilistic approaches to CSPs.

Valued CSP is a significant generalisation of CSP that involves both feasibility and optimisation aspects. The complexity of language-based restriction for VCSPs was considered in the talks by Kolmogorov, Thapper, and Živný. Very strong result in this direction were reported, especially the full description of tractable cases modulo CSP, which closes a sequence of strong and unexpected results on VCSPs obtained during last five years.

The complexity of counting solutions for CSPs, with many results, was investigated by Goldberg, Jerrum, and Richerby.

Along with the constraint language restriction on CSP, the other main type is the structural restriction (i.e. restriction on the immediate interaction between variables in instances). Structural restrictions leading to tractability are well-understood, by results of Grohe and Marx. The so-called "hybrid" tractability in CSP, which is tractability that cannot be attributed to a constraint language restriction or to a structural restriction alone, has not received a great deal of attention yet, and is one of the possible avenues of future work. Rolínek, Scarcello, and Živný described recent results on hybrid tractability for CSPs and VCSPs, including counting problems.

Approximability of CSPs. The use of approximation algorithms is one of the most fruitful approaches to coping with NP-hardness. Hard optimization problems, however, exhibit different behavior with respect to approximability, making it an exciting, and by now, well-developed but far from fully understood, research area. The CSP has always played an important role in the study of approximability. For example, it is well known that the famous PCP theorem has an equivalent reformulation in terms of inapproximability of a certain CSP; moreover, the recent combinatorial proof of this theorem by Dinur in 2006 deals entirely with CSPs. The first optimal inapproximability results by Håstad in 2001 were about certain CSPs, and they led to the study of a new hardness notion called *approximation resistance* (which, intuitively, means that a problem cannot be approximated beyond the approximation ratio given by picking an assignment uniformly at random, even on almost satisfiable instances). Many CSPs have been classified as to whether they are approximation resistant but there is not even a reasonable conjecture for a full classification. Lee and Tulsiani presented new results on approximation resistance.

Many approximation algorithms for CSPs are based on the Sum-of-Squares method, Linear Programming and Semidefinite Programming. Recent developments in proving lower bounds for such algorithms were presented by Chan and Steurer.

Improved approximation algorithms for certain infinite-domain CSPs related to correlation clustering were given by K. Makarychev.

New applications of algebraic approach to investigate approximability of CSPs were given by Austrin and Dalmau.

Parameterized complexity of CSPs. A different way to cope with NP-hardness is provided by parameterized complexity, which relaxes the notion of tractability as polynomial-time solvability to allow non-polynomial dependence on certain problem-specific parameters. A whole new set of interesting questions arises if we look at CSPs from this point of view. Most CSP dichotomy questions can be revisited by defining a parameterized version; so far, very little work was done in this direction compared to investigations in classical complexity. A new research direction (often called "parameterizing above the guaranteed bound") led to unexpected positive results for Max r-SAT by Alon *et al.* in 2010. In this direction, the basic question is to decide the fixed-parameter tractability of the following type of problems: if some easily computable estimate guarantees satisfaction at least E constraints, find an assignment that satisfies at least $E + k$ constraints. Y. Makarychev presented recent results, including approximation issues, in this direction that concern the so-called ordering CSP. Wahlström and Yoshida described how algorithms for this problem, also for VCSP, can be designed when the estimate is given by the Linear Programming relaxation.

Logic and the complexity of CSP. Starting from earlier work by Kolaitis ad Vardi, concepts and techniques from logic have provided unifying explanations for many tractable CSPs. This has led to the pursuit of classifications of CSP with respect to *descriptive complexity*, i.e. definability in a given logic. Logics considered in this context include first order logic and its extensions, finite-variable logics, the logic programming language Datalog and its fragments. Kazda presented his recent results on the two most important open problems on descriptive complexity of CSPs, where he showed that one of these problems reduces to the other. These results are also related to dichotomy questions for complexity classes L and NL.

The CSP can be recast as the problem of deciding satisfiability of existential conjunctive formulas. Chen described recent results in this direction that also involve counting and parameterised complexity. Natural extension of this framework that also allows universal quantifiers is known as the Quantified CSP (QCSP). New results on the complexity of language-restricted QCSPs were presented by Martin. Zhuk gave a proof of an algebraic result that has direct strong consequences for complexity classification of QCSPs.

Bodirsky and Pinsker presented latest developments in infinite domain CSPs, obtained via a mixture of model-theoretic and algebraic methods.

Ochremiak investigated finite-domain CSPs on infinite instances definable by formulas in first-order logic.

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3 Overview of Talks

3.1 (2+ε)-SAT is NP-hard

Per Austrin (KTH Royal Institute of Technology, SE)

Joint work of Austrin, Per; Guruswami, Venkatesan; Håstad, Johan
Main reference P. Austrin, J. Håstad, V. Guruswami, "(2 + epsilon)-Sat Is NP-Hard," in Proc. of the 2014 IEEE 55th Annual Symp. on Foundations of Computer Science (FOCS'14), pp. 1–10, IEEE 2014.
URL http://dx.doi.org/10.1109/FOCS.2014.9

We prove the following hardness result for a natural promise variant of the SAT problem: given a CNF formula where each clause has width w and the guarantee that there exists an assignment satisfying at least $g = w/2 - 1$ literals in each clause, it is NP-hard to find a satisfying assignment to the formula (that sets at least one literal to true in each clause). On the other hand, when $g = w/2$, it is easy to find a satisfying assignment via simple generalizations of the algorithms for 2-SAT.

We also generalize this to prove strong NP-hardness for discrepancy problems with small size sets.

3.2 CSPs over hereditarily semisimple algebras

Libor Barto (Charles University – Prague, CZ)

In our latest joint effort with Marcin Kozik to resolve the CSP tractability conjecture we considered an inductive strategy to solve the problem. Most of the facts needed to make this strategy work turned out to be incorrect. Luckily, at least the base induction step works. This gives, for example, the following: $CSP(A)$ is tractable, whenever A is a Taylor algebra (ie, satisfies the necessary condition for tractability) and each subalgebra of A is semisimple.

3.3 Max-Closed Semilinear Constraints

Manuel Bodirsky (TU Dresden, DE)

A subset of Q^n is called semilinear if it can be defined by Boolean combinations of linear inequalities. Such a subset is called tropically convex if it is preserved by taking componentwise maximum and translations. We show that the Constraint Satisfaction Problem for tropically convex semilinear relations is in NP intersected coNP. This was previously only known for the substantially smaller class of max-atoms constraints.

3.4 Correlation Decay from Cyclic Polymorphisms

Jonah Brown-Cohen (University of California – Berkeley, US)

Joint work of Brown-Cohen, Jonah; Raghavendra, Prasad

In recent work with Prasad Raghavendra we prove that cyclic polymorphisms exhibit a correlation decay phenomenon. In particular, suppose D is a distribution on satisfying assignments to some constraint. If D has no perfect correlations between variables, then repeatedly applying a cyclic polymorphism to D will decay correlation until D becomes a product distribution. I will explain how this correlation decay theorem is proved, and give some toy examples of possible applications to algorithms for CSPs which admit cyclic polymorphisms.

3.5 Sum of Squares Lower Bounds from Pairwise Independence

Siu On Chan (The Chinese University of Hong Kong, HK)

Joint work of Barak, Boaz; Chan, Siu On; Kothari, Pravesh
Main reference B. Barak, S. O. Chan, P. Kothari, "Sum of Squares Lower Bounds from Pairwise Independence," arXiv:1501.00734v2 [cs.CC], 2015.
URL http://arxiv.org/abs/1501.00734v2

We prove that for every $\varepsilon > 0$ and k-ary predicate P that supports a pairwise independent distribution, there exists an instance I of the MaxP constraint satisfaction problem on n variables such that no assignment can satisfy more than a $|P^{-1}(1)|/2^k + \varepsilon$ fraction of I's constraints but the degree $\Omega(n)$ Sum of Squares semidefinite programming hierarchy cannot certify that I is unsatisfiable. Similar results were previously only known for weaker hierarchies.

3.6 The Parameterized Complexity Classification of – and the Logic of – Counting Answers to Existential Positive Queries

Hubie Chen (Universidad del País Vasco – Donostia, ES)

Joint work of Chen, Hubie; Mengel, Stefan

We consider the computational complexity of the problem of counting the number of answers to a logical formula on a finite structure.

We present two contributions.

First, in the setting of parameterized complexity, we present a classification theorem on classes of existential positive queries. In particular, we prove that (relative to the problem at hand) any class of existential positive formulas is interreducible with a class of primitive positive formulas. In the setting of bounded arity, this allows us to derive a trichotomy theorem indicating the complexity of any class of existential positive formulas, as we previously proved a trichotomy theorem on classes of primitive positive formulas (Chen and Mengel '15). This new trichotomy theorem generalizes and unifies a number of existing classification results in the literature, including the classifications on model checking primitive positive formulas (Grohe '07), model checking existential positive formulas (Chen '14), and counting homomorphisms (Dalmau and Jonsson '04).

Our second contribution is to introduce and study an extension of first-order logic in which algorithms for the counting problem at hand can be naturally and conveniently expressed. In particular, we introduce a logic which we call #-logic where the evaluation of a so-called #-sentence on a structure yields a natural number, as opposed to just a propositional value (true or false) as in usual first-order logic. We discuss the width of a formula as a natural complexity measure and show that this measure is meaningful in #-logic and that there is an algorithm that minimizes width in the "existential positive fragment" of #-logic.

This is joint work with Stefan Mengel.

3.7 Approximating Bounded-Degree Boolean Counting CSPs

Leslie Ann Goldberg (University of Oxford, GB)

Joint work of Galanis, Andreas; Goldberg, Leslie Ann
Main reference A. Galanis, L. A. Goldberg, "The complexity of approximately counting in 2-spin systems on k-uniform bounded-degree hypergraphs," arXiv:1505.06146v3 [cs.CC], 2015.
URL http://arxiv.org/abs/1505.06146v3

The talk introduced some work on the complexity of approximately counting satisfying assignments of Boolean read-Δ CSPs (where each variable can be used at most Δ times). The work, which is joint with Andreas Galanis, is inspired by recent developments in the study of approximating partition functions. Here is an interesting connection which has been discovered recently by researchers in the area: for certain (weighted) Boolean binary constraints (called "anti-ferromagnetic 2-spin systems" in statistical physics) there are deep connections between the complexity of approximately counting satisfying assignments and a phenomenon known as "the uniqueness phase transition" which arises when the CSP is considered on the infinite Δ-regular tree.

I spent most of the talk explaining this connection. Our work is motivated by considering the extent to which the connection extends to a broader class of CSPs. We show that for every symmetric Boolean function f (apart from seven classes of trivial ones) there is a barrier Δ_0 such that for all $\Delta \geq \Delta_0$, it is NP-hard to approximate the partition function. The paper is available at http://arxiv.org/abs/1505.06146.

3.8 Approximately counting H-colourings is #BIS-hard

Mark R. Jerrum (Queen Mary University of London, GB)

Joint work of Galanis, Andreas; Goldberg, Leslie Ann; Jerrum, Mark
Main reference A. Galanis, L. A. Goldberg, M. Jerrum, "Approximately counting H-colourings is #BIS-Hard," arXiv:1502.01335v1 [cs.CC], 2015.
URL http://arxiv.org/abs/1502.01335v1

We consider the problem of counting H-colourings, i.e., homomorphisms from an instance graph G to a fixed target graph H. In the CSP setting, this corresponds to the case of a constraint language with one symmetric binary relation. The complexity of computing an exact solution is well understood even in massively more general situations, thanks to a sequence of papers, starting with Dyer and Greenhill [3], and culminating (for the moment) with Cai and Chen [1]. The complexity of computing approximate solutions is much less well understood.

We show that for any fixed graph H without trivial components, it is as hard to approximate the number of H-colourings of a graph as it is to approximately count independent sets in a bipartite graph. The latter problem, called #BIS, is a complete problem in an important complexity class for approximate counting, and is believed not to have an efficient approximation algorithm or FPRAS. If this is so, then our result shows that for every graph H without trivial components, the H-colouring counting problem has no FPRAS.

This problem was studied a decade ago by Goldberg, Kelk and Paterson [5]. They were able to show that approximately sampling H-colourings is #BIS-hard, but it was not known how to get the result for approximate counting. (For general H-colouring problems, there is a generic reduction from approximate counting to sampling due to Dyer, Goldberg and Jerrum [2], but still there is no known reduction in the opposite direction.)

The talk was based on the preprint [4], but the emphasis was on conveying the fundamentals of the complexity of approximate counting, and the flavour of the results, rather than on describing in detail the technicalities of the proof.

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2 M. E. Dyer, L. A. Goldberg, and M. Jerrum. Counting and sampling H-colourings. *Information and Computation*, 189(1):1–16, 2004.
3 M. E. Dyer and C. Greenhill. The complexity of counting graph homomorphisms. *Random Structures and Algorithms*, 17(3-4):260–289, 2000.
4 Andreas Galanis, Leslie Ann Goldberg and Mark Jerrum, Approximately counting H-colourings is #BIS-Hard. `arXiv:1502.01335`, February 2015.
5 L. A. Goldberg, S. Kelk, and M. Paterson. The complexity of choosing an H-coloring (nearly) uniformly at random. *SIAM J. Comput.*, 33(2):416–432, 2004.

3.9 Linear Datalog and n-permutability implies symmetric Datalog

Alexandr Kazda (IST Austria – Klosterneuburg, AT)

The classification of CSPs that can be solved by the Datalog programming language and its fragments, linear and symmetric Datalog, is useful to understand the finer complexity of CSPs that lie in P. The full Datalog corresponds to CSPs solvable by local consistency, and evaluating linear resp. symmetric Datalog lies in NL resp. L.

In this talk, we show that if $CSP(A)$ is solvable by linear Datalog and A satisfies the additional algebraic condition of congruence n-permutability (for some n), then $CSP(A)$ is solvable by symmetric Datalog, which is the weakest of the fragments of Datalog.

3.10 The Complexity of General-Valued CSPs

Vladimir Kolmogorov (IST Austria – Klosterneuburg, AT)

Joint work of Kolmogorov, Vladimir; Krokhin, Andrei; Rolinek, Michal
Main reference V. Kolmogorov, A. Krokhin, M. Rolinek, "The Complexity of General-Valued CSPs," arXiv:1502.07327v3 [cs.CC], 2015.
URL http://arxiv.org/abs/1502.07327v3

Consider a general valued language Γ. Kozik and Ochremiak recently showed that if the core of the language does not admit a cyclic fractional polymorphism of arity at least 2 then it is NP-hard. We prove that if this necessary condition is satisfied, and the underlying feasibility CSP is tractable, then Γ is tractable. The algorithm is a simple combination of the assumed algorithm for the feasibility CSP and the standard LP relaxation. As a corollary, we obtain that a dichotomy for ordinary CSPs would imply a dichotomy for general-valued CSPs.

Joint work with Andrei Krokhin and Michal Rolinek.

3.11 Congruence join semi-distributive varieties and constraint satisfaction problems

Marcin Kozik (Jagiellonian University – Kraków, PL)

I will present some new results on SD(∨) varieties: in particular a set of directed SD(∨) terms and an analogue of "graph absorbing" condition for congruence distributive varieties. I will discuss further applications of these results to solving CSPs for such algebras. Finally I will show examples of problematic algebras (in SD(∨)) which present new problems while solving CSPs (compared to the CD case).

3.12 Towards a Characterization of Approximation Resistance for Symmetric CSPs

Euiwoong Lee (Carnegie Mellon University, US)

Joint work of Lee, Euiwoong; Guruswami, Venkatesan
Main reference V. Guruswami, E. Lee, "Towards a Characterization of Approximation Resistance for Symmetric CSPs," ECCC, TR15-105, 2015.
URL http://eccc.hpi-web.de/report/2015/105/

A Boolean constraint satisfaction problem (CSP) is called approximation resistant if independently setting variables to 1 with some probability achieves the best possible approximation ratio for the fraction of constraints satisfied. We study approximation resistance of a natural subclass of CSPs that we call Symmetric Constraint Satisfaction Problems (SCSPs), where satisfaction of each constraint only depends on the number of true literals in its scope. Thus a SCSP of arity k can be described by a subset of allowed number of true literals.

For SCSPs without negation, we conjecture that a simple sufficient condition to be approximation resistant by Austrin and Hastad is indeed necessary. We show that this condition has a compact analytic representation in the case of symmetric CSPs (depending only on the gap between the largest and smallest numbers in S), and provide the rationale behind our conjecture. We prove two interesting special cases of the conjecture, (i) when S is an interval and (ii) when S is even. For SCSPs with negation, we prove that the analogous sufficient condition by Austrin and Mossel is necessary for the same two cases, though we do not pose an analogous conjecture in general.

3.13 Correlation Clustering on Complete Graphs

Konstantin Makarychev (Microsoft Corporation – Redmond, US)

We give new rounding schemes for the standard linear programming relaxation of the correlation clustering problem (which can be seen as a constraint satisfaction problem), achieving approximation factors almost matching the integrality gaps:

- For complete graphs our appoximation is $2.06 - \varepsilon$ for a fixed constant ε, which almost matches the previously known integrality gap of 2.
- For complete k-partite graphs our approximation is 3. We also show a matching integrality gap.
- For complete graphs with edge weights satisfying triangle inequalities and probability constraints, our approximation is 1.5, and we show an integrality gap of 1.2.

Our results improve a long line of work on approximation algorithms for correlation clustering in complete graphs, previously culminating in a ratio of 2.5 for the complete case by Ailon, Charikar and Newman (JACM'08). In the weighted complete case satisfying triangle inequalities and probability constraints, the same authors give a 2-approximation; for the bipartite case, Ailon, Avigdor-Elgrabli, Liberty and van Zuylen give a 4-approximation (SICOMP'12).

3.14 Satisfiability of Ordering CSPs Above Average Is Fixed-Parameter Tractable

Yury Makarychev (TTIC – Chicago, US)

We study the satisfiability of ordering constraint satisfaction problems (CSPs) above average. We prove the conjecture of Gutin, van Iersel, Mnich, and Yeo that the satisfiability above average of ordering CSPs of arity k is fixed-parameter tractable for every k. Previously, this was only known for $k = 2$ and $k = 3$. We also generalize this result to more general classes of CSPs, including CSPs with predicates defined by linear equations.

To obtain our results, we prove a new Bonami-type inequality for the Efron-Stein decomposition. The inequality applies to functions defined on arbitrary product probability spaces. In contrast to other variants of the Bonami Inequality, it does not depend on the mass of the smallest atom in the probability space. We believe that this inequality is of independent interest.

3.15 Algebra and the Complexity of Quantified Constraints

Barnaby Martin (Middlesex University, GB)

We survey connections between algebra and the complexity of Quantified Constraint Satisfaction Problems over finite templates. The complexity of such problems is well-known to hinge on their surjective polymorphisms, and early results studied the complexity divide between P and NP-hard. More recent work focuses on the gap between NP and PSPACE-hard and the connections to the so-called Polynomial Generate Powers property of the respective algebra's direct powers.

3.16 Deciding FO-definable CSP instances

Joanna Ochremiak (University of Warsaw, PL)

Joint work of Klin, Bartek; Kopczynski, Eryk; Ochremiak, Joanna; Toruńczyk, Szymon
Main reference B. Klin. E. Kopczynski, J. Ochremiak, S. Toruńczyk, "Locally Finite Constraint Satisfaction Problems," in Proc. of the 30th Annual ACM/IEEE Symp. on Logic in Computer Science (LICS'15), pp. 475–486, IEEE, 2015; pre-print available from author's webpage.
URL http://dx.doi.org/10.1109/LICS.2015.51
URL http://www.mimuw.edu.pl/~ochremiak/papers/lics15.pdf

In this talk I showed how to decide infinite CSP instances that exhibit high level of symmetry. I considered instances founded upon a fixed infinite relational structure, and defined by finitely many FO formulas. The number of elements and constraints in such instances is usually infinite, but thanks to the finite presentation they can be treated as an input for algorithms. The decidability proof is based on results in topological dynamics. Moreover, I showed tight complexity bounds for this problem.

3.17 Constraint Satisfaction Problems on K_n-free graphs

Michael Pinsker (University Paris-Diderot, FR)

We present a dichotomy result for constraint satisfaction problems over K_n-free graphs, i.e., graphs which do not contain any clique of size n, where $n \geq 3$ is a fixed natural number. In these problems, the input consists of variables and constraints about them which have to be taken from a fixed finite set of quantifier-free formulas in the language of graphs. The decision problem is whether the variables can be assigned vertices in a K_n-free graph so that all constraints are satisfied.

Using methods from infinite constraint satisfaction, in particular Ramsey theory, we show that all such problems are either in P or NP-complete.

This result is a (incomparable) variant of Schaefer's theorem for graphs proven by Bodirsky + Pinsker in 2011, and we will compare the proof of that theorem with the one of the theorem we present.

3.18 Counting Matrix Partitions of Graphs

David Richerby (Oxford University, GB)

Joint work of Dyer, Martin; Goebel, Andreas; Goldberg, Leslie Ann; McQuillan, Colin; Yamakami, Tomoyuki

A matrix partition of a graph G is a partition of its vertices into parts $V_1, \ldots, V_k$ whose connections to one another are specified by a symmetric k-by-k $\{0, 1, \star\}$ matrix M according to the following rules:

- if $M_{i,j} = 1$, there must be an edge in G between every pair of distinct vertices $x \in V_i$ and $y \in V_j$;
- if $M_{i,j} = 0$, there must be no edge in G between any vertex in V_i and any in V_j
- if $M_{i,j} = \star$, there may be any pattern of edges and non-edges between V_i and V_j.

If M contains no 1s, M-partitions are equivalent to graph homomorphisms. General matrix problems can also be formulated as CSPs with restrictions on the input.

Matrix partitions naturally encode many classes of graphs, such as split graphs and (a, b)-graphs and graph structures such as k-colourings, skew cutsets and various generalizations of clique-cross partitions.

I will discuss the complexity of counting matrix partitions of graphs. This is joint work with Martin Dyer, Andreas Goebel, Leslie Ann Goldberg, Colin McQuillan and Tomoyuki Yamakami.

3.19 Effectiveness of structural restrictions for hybrid CSPs

Michal Rolinek (IST Austria – Klosterneuburg, AT)

Joint work of Kolmogorov, Vladimir; Rolinek, Michal; Takhanov, Rustem
Main reference V. Kolmogorov, M. Rolinek, R. Takhanov, "Effectiveness of Structural Restrictions for Hybrid CSPs," arXiv:1504.07067v3 [cs.CC], 2015.
URL http://arxiv.org/abs/1504.07067v3

We focus on the hybrid setting of the CSP that restricts both sides of the homomorphism problem simultaneously. It assumes that the input belongs to a certain class of relational structures (called a structural restriction in this paper). We study which structural restrictions are effective, i.e. there exists a fixed template (from a certain class of languages) for which the problem is tractable when the input is restricted, and NP-hard otherwise. We provide a characterization for structural restrictions that are closed under inverse homomorphisms and show its implications (for example for minor-closed families, or ordered CSP). We extend the results to certain Valued CSPs (namely conservative valued languages).

3.20 Hybrid Tractability of the Counting Problem, with a case study from Game Theory

Francesco Scarcello (University of Calabria, IT)

Joint work of Scarcello, Francesco; Greco, Gianluigi
Main reference F. Scarcello, G. Greco, "Counting solutions to conjunctive queries: structural and hybrid tractability," in Proc. of the 33rd ACM SIGMOD/SIGACT/SIGART Symp. on Principles of Database Systems, pp. 132–143, ACM, 2014.
URL http://dx.doi.org/10.1145/2594538.2594559

Counting the number of solutions of CSPs with output variables is an intractable problem, formally #P-hard, even over classes of acyclic instances. We describe structural methods that allow us to identify large islands of tractability, such as the method based on #-generalized hypertree decompositions.

Based on this notion, a "hybrid" decomposition method is eventually conceived, where structural properties of the left-hand structure (i.e., properties of constraint scopes and output variables) are exploited in combination with properties of the right-hand structure (i.e., the values). Intuitively, such features may induce different structural properties that are not identified by the "worst-possible" perspective of purely structural methods.

We eventually describe a successful application of such CSP techniques to solve a game theory problem, namely, the polynomial-time computation of the Shapley value in allocations games. These are coalitional games defined in the literature as a way to analyze fair division problems of indivisible goods.

References

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2 G. Greco and F. Scarcello, *Mechanisms for Fair Allocation Problems: No-Punishment Payment Rules in Verifiable Settings*. J. Artif. Intell. Res. (JAIR) 49:403–449 (2014).
3 G. Greco and F. Scarcello, *Counting Solutions to Conjunctive Queries: Structural and Hybrid Tractability*, in Proceedings of the 33rd ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS'14), ACM, 2014.

3.21 Lower bounds for LP/SDP formulations of CSPs

David Steurer (Cornell University – Ithaca, US)

We introduce a method for proving lower bounds on the efficacy of semidefinite programming (SDP) relaxations for combinatorial problems.

3.22 The power of Sherali-Adams relaxations for valued CSPs

Johan Thapper (University Paris-Est – Marne-la-Vallée, FR)

Joint work of Thapper, Johan; Živný, Stanislav

We give an algebraic characterisation of when some bounded level of the Sherali-Adams hierarchy solves the valued constraint satisfaction problem (VCSP) parameterised by a valued constraint language to optimality. We also give examples of classes of VCSPs that are NP-hard unless they are solved by such an LP relaxation. Our result is fundamentally based on the algebraic characterisation of the notion of bounded relational width for ordinary decision CSPs by Larose/Zádori (2007) and Barto/Kozik (2014).

This is joint work with Stanislav Živný.

3.23 A Characterization of Strong Approximation Resistance

Madhur Tulsiani (TTIC – Chicago, US)

Joint work of Jones, Mark; Sheng, Bin

For a predicate $f : \{-1,1\}^k \to \{0,1\}$ with $E[f] = \rho$, we call the predicate strongly approximation resistant if given a near-satisfiable instance of $CSP(f)$, it is computationally hard to find an assignment such that the fraction of constraints satisfied is outside the range $(\rho - \varepsilon, \rho + \varepsilon)$ for every $\varepsilon > 0$.

We present a characterization of strongly approximation resistant predicates under the Unique Games Conjecture. We also present characterizations in the mixed linear and semi-definite programming hierarchy and the Sherali-Adams linear programming hierarchy. In the former case, the characterization coincides with the one based on UGC. Each of the two characterizations is in terms of existence of a probability measure on a natural convex polytope associated with the predicate.

The predicate is called approximation resistant if given a near-satisfiable instance of CSP(f), it is computationally hard to find an assignment such that the fraction of constraints satisfied is at least $\rho + \varepsilon$. When the predicate is odd, i.e. $f(-z) = 1 - f(z)$ for all z in $\{-1,1\}^k$, it is easily observed that the notion of approximation resistance coincides with that of strong approximation resistance. Hence for odd predicates, in all the above settings, our characterization of strong approximation resistance is also a characterization of approximation resistance.

3.24 Parameterized VCSPs, Euler digraphs and diamond lattices

Magnus Wahlström (Royal Holloway University of London, GB)

Joint work of Jones, Mark; Sheng, Bin; Wahlström, Magnus

VCSPs (Valued CSPs, an optimisation variant of CSP) is a general problem framework that encompasses many problems of independent interest. In particular, many of the success stories of parameterized complexity, and many of its still open problems, can be phrased as questions about the parameterized complexity properties (in particular, FPT algorithms) of particular VCSPs.

P vs NP dichotomies for VCSP are known (Thapper and Živný, FOCS 2012/STOC 2013, extended by Kolmogorov, Krokhin, and Rolinek, 2015), showing essentially that the P-time solvable cases coincide with a natural LP-relaxation of the problem. These results, and the LP-relaxation, have also shown to be useful for FPT algorithms – the LP-branching approach has been shown to imply some very powerful FPT algorithms (Wahlström, SODA 2014/Iwata, Wahlström, and Yoshida, 2014), and the results of Thapper and Živný are central to the correctness of these algorithms. More concretely, it was shown that so-called k-submodular functions, which were shown to be tractable by Thapper and Živný, have direct applications for VCSP FPT algorithms.

In this talk, we survey the above results and discuss an attempt at handling the directed versions of these problems. We show how certain digraph cut problems (in particular, directed multiway cut) can be recast as optimisation of functions which are submodular over a diamond lattice. Although we cannot yet recreate the existing FPT results in this way, we get the following results:

- A 2-approximation for directed multiway cut (previously shown by Naor and Zosin, SICOMP 2001);
- A polynomial-time algorithm for directed multiway cut in Euler digraphs (the existence of such an algorithm was obscure, but implied by results of Frank, 1989).

Thanks to the algebraic VCSP setting, the results also extend to fairly natural varations of problems where arcs are labelled by bijections over a set of labels (e.g., directed versions of the so-called unique label cover and group feedback arc set problems).

Joint work with Mark Jones and Bin Sheng.

3.25 Maltsev constraints revisited

Ross Willard (University of Waterloo, CA)

Once upon a time, the (finite) CSP Dichotomy Conjecture was a Big Deal. However, since 2009 there has been no significant progress in solving it. Why? Because we are stupid, and the conjecture is hard, and most sensible researchers have moved on, but mostly because we lack algorithms.

Only two general algorithms for finite CSPs are known:

1. local consistency checking and
2. a generalization, largely due to Bulatov and Dalmau, of the Feder-Vardi algorithm for subgroup CSPs.

Each algorithm has its own naturally defined scope, and while there was early hope on the part of algebraists that some combination of the two algorithms would magically solve the Dichotomy Conjecture, such a solution hasn't materialized.

The problem, as I see it, is that the Bulatov-Dalmau algorithm is too simple. Like the Feder-Vardi algorithm it generalizes, the Bulatov-Dalmau algorithm neatly finesses the need to analyze the convoluted ways by which constraint systems can encode linear equations. It is now time, I suggest, to undertake this analysis. In this lecture I will sketch my recent (so far unsuccessful) efforts to fully understand encoded linear systems in constraint systems over finite templates having a Maltsev polymorphism.

3.26 Half-Integrality, LP-Branching and FPT Algorithms

Yuichi Yoshida (National Institute of Informatics – Tokyo, JP)

Joint work of Yoichi Iwata; Magnus Wahlström; Yoshida, Yuichi

A recent trend in parameterized algorithms is the application of polytope tools to FPT algorithms (e.g., Cygan et al., 2011; Narayanaswamy et al., 2012). Although this approach has yielded significant speedups for a range of important problems, it requires the underlying polytope to have very restrictive properties, including half-integrality and Nemhauser-Trotter-style persistence properties. To date, these properties are essentially known to hold only for two classes of polytopes, covering the cases of Vertex Cover (Nemhauser and Trotter, 1975) and Node Multiway Cut (Garg et al., 1994).

Taking a slightly different approach, we view half-integrality as a discrete relaxation of a problem, e.g., a relaxation of the search space from $\{0,1\}^V$ to $\{0,1/2,1\}^V$ such that the new problem admits a polynomial-time exact solution. Using tools from CSP (in particular Thapper and Živný, 2012) to study the existence of such relaxations, we are able to provide a much broader class of half-integral polytopes with the required properties.

Our results unify and significantly extend the previously known cases, and yield a range of new and improved FPT algorithms, including an $O^\star(|\Sigma|^{2k})$-time algorithm for node-deletion Unique Label Cover and an $O^\star(4k)$-time algorithm for Group Feedback Vertex Set where the group is given by oracle access. The latter result also implies the first single-exponential time FPT algorithm for Subset Feedback Vertex Set, answering an open question of Cygan et al. (2012). Additionally, we propose a network-flow-based approach to solve several cases of the relaxation problem. This gives the first linear-time FPT algorithm to edge-deletion Unique Label Cover.

Joint work with Yoichi Iwata and Magnus Wahlström.

3.27 EGP/PGP dichotomy

Dmitriy Zhuk (Moscow State University, RU)

For each algebra A, we count the minimal number $g(n)$ of generators of A^n as a function of n. We show that if $g(n)$ is not bounded from above by a polynomial (PGP) then $g(n)$ is bounded from below by an exponential function (EGP).

3.28 Hybrid (V)CSPs

Stanislav Živný (Oxford University, GB)

Given two classes A and B of finite relational structures over the same signature, the constraint satisfaction problem $CSP(A, B)$ amounts to, given structures $I \in A$ and $J \in B$, deciding whether there is a homomorphism from I to J. While the computational complexity of $CSP(A, -)$ is well understood (here $-$ denotes the class of all structures), the computational complexity of $CSP(-, B)$ is still open even if B consists of a single structure; these non-uniform CSPs have attracted a lot of attention due to the success of the so-called algebraic approach.

In this talk I will survey known results (older and recent ones) on so-called hybrid CSPs, which restrict both A and B. I will also mention known results for hybrid Valued CSPs.

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