

Quantum Query Complexity of Subgraph Isomorphism and Homomorphism

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Abstract

Let H be a (non-empty) graph on n vertices, possibly containing isolated vertices. Let $f_H(G) = 1$ iff the input graph G on n vertices contains H as a (not necessarily induced) subgraph. Let α_H denote the cardinality of a maximum independent set of H . In this paper we show:

$$Q(f_H) = \Omega(\sqrt{\alpha_H \cdot n}),$$

where $Q(f_H)$ denotes the quantum query complexity of f_H .

As a consequence we obtain lower bounds for $Q(f_H)$ in terms of several other parameters of H such as the average degree, minimum vertex cover, chromatic number, and the critical probability.

We also use the above bound to show that $Q(f_H) = \Omega(n^{3/4})$ for any H , improving on the previously best known bound of $\Omega(n^{2/3})$ [16]. Until very recently, it was believed that the quantum query complexity is at least square root of the randomized one. Our $\Omega(n^{3/4})$ bound for $Q(f_H)$ matches the square root of the current best known bound for the randomized query complexity of f_H , which is $\Omega(n^{3/2})$ due to Gröger. Interestingly, the randomized bound of $\Omega(\alpha_H \cdot n)$ for f_H still remains open.

We also study the Subgraph Homomorphism Problem, denoted by $f_{[H]}$, and show that $Q(f_{[H]}) = \Omega(n)$.

Finally we extend our results to the 3-uniform hypergraphs. In particular, we show an $\Omega(n^{4/5})$ bound for quantum query complexity of the Subgraph Isomorphism, improving on the previously known $\Omega(n^{3/4})$ bound. For the Subgraph Homomorphism, we obtain an $\Omega(n^{3/2})$ bound for the same.

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1 Introduction

1.1 Classical and Quantum Query Complexity

The decision tree model (aka the query model), perhaps due to its simplicity and fundamental nature, has been extensively studied in the past and still remains a rich source of many



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fascinating investigations. In this paper we focus on Boolean functions, i.e., the functions of the form $f : \{0, 1\}^n \rightarrow \{0, 1\}$. A deterministic decision tree T_f for f takes $x = (x_1, \dots, x_n)$ as an input and determines the value of $f(x_1, \dots, x_n)$ using queries of the form “is $x_i = 1$?” Let $C(T_f, x)$ denote the cost of the computation, that is the number of queries made by T_f on an input x . The *deterministic decision tree complexity* (aka the deterministic query complexity) of f is defined as

$$D(f) = \min_{T_f} \max_x C(T_f, x).$$

We encourage the reader to see an excellent survey by Buhrman and de Wolf [7] on the decision tree complexity of Boolean functions.

A randomized decision tree \mathcal{T} is simply a probability distribution on the deterministic decision trees $\{T_1, T_2, \dots\}$ where the tree T_i occurs with probability p_i . We say that \mathcal{T} computes f correctly if for every input x : $\Pr_i[T_i(x) = f(x)] \geq 2/3$. The depth of \mathcal{T} is the maximum depth of a T_i . The *(bounded-error) randomized query complexity* of f , denoted by $R(f)$, is the minimum possible depth of a randomized tree computing f correctly on all inputs.

One can also define the quantum version of the decision tree model as follows: Start with an N -qubit state $|0\rangle$ consisting of all zeros. We can transform this state by applying an unitary transformation U_0 , then we can make a *quantum* query O , which essentially negates the amplitude of each basic state depending on whether the i th bit of the basic state is zero or one. A quantum algorithm with q queries looks like the following: $A = U_q O U_{q-1} \dots O U_1 O U_0$. Here U_i 's are fixed unitary transformation independent of the input x . The final state $A|0\rangle$ depends on the input x only via applications of O . We measure the final state outputting the rightmost qubit (WLOG there are no intermediate measurements). A bounded-error quantum query algorithm A computes f correctly if the final measurement gives the correct answer with probability at least $2/3$ for every input x . The *bounded-error quantum query complexity* of f , denoted by $Q(f)$, is the least q for which f admits a bounded-error quantum algorithm. We refer the reader to a survey by Buhrman and de Wolf [7] for more precise definition.

1.2 Subgraph Isomorphism Problem

Let H be a (non-empty) graph on n vertices, possibly containing isolated vertices and let G be an unknown input graph (on n vertices) given by query access to its edges, i.e, queries of the form “Is $\{i, j\}$ an edge in G ?”. We say $H \leq G$ if G contains H as a (not necessarily induced) subgraph. Let $f_H : \{0, 1\}^{\binom{n}{2}} \rightarrow \{0, 1\}$ be defined as follows:

$$f_H(G) = \begin{cases} 1 & \text{if } H \leq G \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The well-known Graph Isomorphism Problem asks whether a graph H is isomorphic to another graph G . The Subgraph Isomorphism Problem is a generalization of the Graph Isomorphism Problem where one asks whether H is isomorphic to a subgraph of G . Several central computational problems for graphs such as containing a clique, containing a Hamiltonian cycle, containing a perfect matching can be formulated as the Subgraph Isomorphism Problem by fixing the H appropriately. Given the generality and importance of the problem, people have investigated various restricted special cases of this problem in different models of computation [1] [14]. In the context of query complexity, in 1992 Gröger [9] studied this problem in the randomized setting and showed that $R(f_H) = \Omega(n^{3/2})$, which is the best

known bound to this date. In this paper we investigate this problem in the quantum setting. To the best of our knowledge, quantum query complexity for the Subgraph Isomorphism Problem has not been noted prior to this work when H is allowed to be any graph on n vertices. A special case of this problem when H is of a constant size has been investigated before for obtaining upper bounds [20].

1.3 Subgraph Homomorphism Problem

We also investigate a closely related Subgraph Homomorphism Problem.

A homomorphism from a graph H into a graph G is a function $h : V(H) \rightarrow V(G)$ such that: if $(u, v) \in E(H)$ then $(h(u), h(v)) \in E(G)$.

Let $f_{[H]}$ be the function defined as follows: $f_{[H]}(G) = 1$ if and only if H admits a homomorphism into G .

Note that unlike the isomorphism, the homomorphism need not be an injective function from $V(H)$ to $V(G)$. We study the query complexity of the Subgraph Homomorphism Problem towards the end of this paper. In the next section, we review the relevant literature.

1.4 Related Work

Understanding the query complexity of monotone graph properties has a long history. In the deterministic setting the Aanderaa-Rosenberg-Karp Conjecture asserts that one must query all the $\binom{n}{2}$ edges in the worst-case. The randomized complexity of monotone graph properties is conjectured to be $\Omega(n^2)$. Yao [19] obtained the first super-linear lower bound in the randomized setting using the graph packing arguments. Subsequently his bound was improved by King [12] and later by Hajnal [11]. The current best known bound is $\Omega(n^{4/3} \sqrt{\log n})$ due to Chakrabarti and Khot [8]. Moreover, O'Donnell, Saks, Schramm, and Servedio [15] also obtained an $\Omega(n^{4/3})$ bound via a more generic approach for monotone transitive functions. Friedgut, Kahn, and Wigderson [10] obtain an $\Omega(n/p)$ bound where the p is the critical probability of the property. In the quantum setting, Buhrman, Cleve, de Wolf and Zalka [6] were the first to study quantum complexity of graph properties. Santha and Yao [16] obtain an $\Omega(n^{2/3})$ bound for general properties. Their proof follows along the lines of Hajnal's proof.

Gröger [9] obtained an $\Omega(n^{3/2})$ bound for the randomized query complexity of the Subgraph Isomorphism. This is currently the best known bound for the Subgraph Isomorphism Problem. Until very recently¹, it was believed that the quantum query complexity is at least square root of the randomized one. In this paper we address the quantum query complexity of the Subgraph Isomorphism Problem and obtain the square root of the current best randomized bound.

The main difference between the previous work and this one is that all the previous work, including that of Santha and Yao [16], obtained the lower bounds based on an embedding of a *tribe* function [5] on a large number of variables in monotone graph properties². Recall that the tribe function with parameters k and ℓ , is a function $T(k, \ell)$ on $k \cdot \ell$ variables defined as: $\bigvee_{i \in [k]} \bigwedge_{j \in [\ell]} x_{ij}$. This method yields a lower bound of $\Omega(k \cdot \ell)$ for the randomized query complexity and $\Omega(\sqrt{k \cdot \ell})$ for the quantum. We deviate from this line by embedding a threshold function T_n^t instead of a tribe. Recall that $T_n^t(z_1, \dots, z_n)$ is a function on n

¹ Very recently this has been falsified by Ben-David [22].

² Similar tribe-embeddings were used for obtaining lower bounds for metroidal Boolean functions in [13]

variables that evaluates to 1 if and only if at least t of the z_i 's are 1. Since the randomized complexity of T_n^t is $\Theta(n)$, this does not give us any advantage for obtaining super-linear randomized lower bounds. However, it *does* yield an advantage for the quantum lower bounds as the quantum query complexity of T_n^t is $\Theta(\sqrt{t(n-t)})$ [17], which can reach up to $\Omega(n)$ for large t . Since this technique works only in the quantum setting, the randomized versions of our bounds remain intriguingly open.

1.5 Our Results

Our main result is a lower bound on the quantum query complexity of the Subgraph Isomorphism Problem for H in terms of the maximum independence number of H .

► **Theorem 1.** *For any non-empty H ,*

$$Q(f_H) = \Omega(\sqrt{\alpha_H \cdot n}),$$

where α_H denotes the size of a maximum independent set of H .

► **Corollary 2.** *For any non-empty H ,*

1. $Q(f_H) = \Omega\left(\frac{n}{\sqrt{d_{\text{avg}}(H)}}\right),$
2. $Q(f_H) = \Omega\left(\frac{n}{\sqrt{\chi_H}}\right),$
3. $Q(f_H) = \Omega\left(\sqrt{\frac{n}{p}}\right),$

where $d_{\text{avg}}(H)$ denotes the average degree of the vertices of H , χ_H denotes the chromatic number of H , and p denotes the critical probability [10] of H .

In particular, we get an $\Omega(n)$ bound when the graph H is sparse ($|E(H)| = O(n)$), or H has a constant chromatic number, or the critical probability of H is $O(1/n)$. Friedgut, Kahn, and Wigderson [10] show an $\Omega(n/p)$ bound for the randomized query complexity of general monotone properties. Quantization of this bound remains open. General monotone properties can be thought of as the Subgraph Isomorphism for a *family* of minimal subgraphs. The item 3 above, gives a quantization of [10] in the case when the family contains only a single subgraph.

► **Corollary 3.** *For any non-empty H ,*

$$Q(f_H) = \Omega(n^{3/4}).$$

Prior to this work only an $\Omega(n^{2/3})$ bound was known from the work of Santha and Yao [16] on general monotone graph properties.

We extend our result to the 3-uniform hypergraphs. In particular, we show:

► **Theorem 4.** *Let H be a non-empty 3-uniform hypergraph on n vertices. Then,*

$$Q(f_H) = \Omega(n^{4/5}).$$

This improves the $\Omega(n^{3/4})$ bound obtained via the minimum certificate size.

The second part of this paper concerns the Subgraph Homomorphism Problem for H , denoted by $f_{[H]}$. Here we show the following two theorems:

► **Theorem 5.** *For any non-empty H ,*

$$Q(f_{[H]}) = \Omega(n).$$

► **Theorem 6.** *For any non-empty 3-uniform hypergraph H on n vertices:*

$$Q(f_{[H]}) = \Omega(n^{3/2}).$$

Our proofs crucially rely on the duality of monotone functions and appropriate embeddings of tribe and threshold functions. All our lower bounds hold for the approximate degree $\widetilde{\deg}(f)$, which is known to be strictly smaller than the quantum query complexity [4].

Organization

Section 2 contains some preliminaries. Section 3 and Section 4 deal with the Subgraph Isomorphism Problem. Section 3 contains the proofs of Theorem 1, Corollary 2 and Corollary 3. Then Section 4 contains the proof of Theorem 4. The next two sections (Section 5 and Section 6) involve the Subgraph Homomorphism Problem and contains the proof of Theorem 5 and Theorem 6. Finally Section 7 contains conclusion and some open ends.

2 Preliminaries

In this section, we review some preliminary concepts and restate some previously known results.

Let $[n]$ denote the set $\{1, \dots, n\}$.

► **Definition 7** (Dual of a Property). The dual \mathcal{P}^* , denoted by \mathcal{P}^* , is:

$$\mathcal{P}^*(x) := \neg\mathcal{P}(\neg x),$$

where $\neg x$ denotes the binary string obtained by flipping each bit in x .

Note that $\mathcal{P}^{**} = \mathcal{P}$ and $Q(\mathcal{P}) = Q(\mathcal{P}^*)$.

A property \mathcal{P} is said to be *monotone increasing* if for every $x \leq y$ we have $\mathcal{P}(x) \leq \mathcal{P}(y)$, where $x \leq y$ denotes $x_i \leq y_i$ for all i .

Note that if \mathcal{P} is monotone, then so is \mathcal{P}^* .

A *minimal certificate* of size s for a monotone increasing property \mathcal{P} is an input z such that (a) The hamming weight of z , i.e, $|z|$, is s , (b) $\mathcal{P}(z) = 1$, and (c) for any y with $|y| < s$, $\mathcal{P}(y) = 0$. Every minimal certificate z can be uniquely associated with the subset $S_z := \{i \mid z_i = 1\}$.

► **Lemma 8** (Minimal Certificate [7]). *If \mathcal{P} has a minimal certificate of size s then*

$$Q(\mathcal{P}) \geq \Omega(\sqrt{s}).$$

We say that two minimal certificates z_1 and z_2 *pack* together, if $S_{z_1} \cap S_{z_2} = \emptyset$.

► **Lemma 9** (Packing Lemma [19]). *If z_1 is a minimal certificate of \mathcal{P} and z_2 is a minimal certificate of \mathcal{P}^* then z_1 and z_2 cannot be packed together.*

► **Lemma 10** (Turán [2]). *If the average degree of a graph G is d then G contains an independent set of size at least $\Omega(n/d)$.*

► **Lemma 11** (Extended Turán [2]). *If the average degree of a k -uniform hypergraph G is d then G contains an independent set of size at least $\Omega(n/d^{\frac{1}{k-1}})$.*

A Boolean function $f(x_1, \dots, x_n)$ is called *transitive* if there exists a group Γ acting transitively on the indices $\{1, \dots, n\}$ such that f is invariant under the action, i.e., for every $\sigma \in \Gamma$ we have $f(x_{\sigma_1}, \dots, x_{\sigma_n}) = f(x_1, \dots, x_n)$.

Note that graph properties and hypergraph properties are transitive functions.

► **Lemma 12** (Transitive Packing [18]). *Let f be a monotone transitive function on n variables. If f has a minimal certificate of size s then every certificate of f^* must have size at least n/s .*

A *Threshold function* $T_n^t(z_1, \dots, z_n)$ is a function on n variables such that T_n^t outputs 1 if and only if at least t variables are 1.

We are now ready to prove the quantum query complexity lower bound for the Subgraph Isomorphism Problem.

3 Subgraph Isomorphism for Graphs

Before proving Theorem 1 we first prove two lemmas.

Let S_d denote the star graph with d edges. Then f_{S_d} is the property of having a vertex of at least degree d . First we show:

► **Lemma 13.** *For $1 \leq d \leq n - 1$,*

$$Q(f_{S_d}) = \Omega(n).$$

Proof. We divide the proof into two cases:

Case 1: $d > n/2$. Fix a clique on the vertices $1, \dots, \lfloor n/2 \rfloor$ and fix an independent set on the vertices $\lfloor n/2 \rfloor + 1, \dots, n$. Note that we still have $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$ edge-variables that are not yet fixed. Now as soon as any vertex v from the clique has $(d - \lfloor n/2 \rfloor + 1)$ edges to the independent set present, we have a d -star. Thus f_{S_d} becomes an $OR_{\lfloor n/2 \rfloor} \circ T_{\lceil n/2 \rceil}^{(d - \lfloor n/2 \rfloor + 1)}$ function, which has a lower bound of $\Omega(n)$ via the Composition Theorem for quantum query complexity [21].

Case 2: $d \leq n/2$. A minimum certificate of f_{S_d} is a d -star. Now by the Lemma 9 we know that this d -star cannot be packed with any minimal certificate of the dual $f_{S_d}^*$. Thus every vertex in the dual $f_{S_d}^*$ must have degree $> n - d$. Hence the minimal certificate size is at least $\Omega(n^2)$ and $Q(f_{S_d}^*) = Q(f_{S_d}) = \Omega(n)$. ◀

Let t denote the smallest integer such that $f_H^*(K_t) = 1$, where K_t denotes the complete graph on t vertices. Note that $t = \alpha_H + 1$.

► **Lemma 14.**

$$Q(f_H) \geq \Omega(\sqrt{t(n-t)}).$$

Proof. We embed T_n^t in f_H^* (on inputs of Hamming weight $t - 1$ and t) via the following mapping: Let $x_{ij} := z_i \cdot z_j$ and let $f'(z_1, \dots, z_n) := f_H^*(\{x_{ij}\})$. Note that $f' \equiv T_n^t$. Also note³ that $Q(f_H) = Q(f_H^*)$ and $\widetilde{\deg}(f') \leq 2 \cdot \deg(f_H^*)$. Since $Q(f) \geq \widetilde{\deg}(f)$, it remains to prove the following:

³ Since $x_{ij} = z_i \cdot z_j$, every monomial of f_H^* of size d becomes a monomial of size at most $2d$ in f' .

► **Claim 1.** $\widetilde{\text{deg}}(f') = \Omega(\sqrt{t(n-t)})$

We need the following lemma due to Paturi [17]:

► **Lemma 15.** *Let g be a function on n variables such that $g(z) = 0$ for all z with $|z| = t - 1$ and $g(z) = 1$ for all z with $|z| = t$. Then: $\widetilde{\text{deg}}(g) = \Omega(\sqrt{t(n-t)})$.*

Proof of Claim 1. Note that f' ($\equiv T_n^t$) satisfies the condition of the Lemma 15. ◀

This finishes the proof of the Lemma 14. ◀

Now we are in a position to prove the Theorem 1.

Proof of Theorem 1. Recall that t denotes the smallest integer such that $f_H^*(K_t) = 1$. We divide the proof into two cases:

Case 1: $t > n/2$. In this case, we reduce the f_H to f_{S_p} for some $p = \Omega(n)$. Let ν_H denote the minimum vertex cover size of H . Since $t > n/2$, we have $\nu_H \leq n/2$. When $\nu_H = 1$ the property is trivially a star property and from the Lemma 13 we already get $Q(f_H) = \Omega(n)$. Otherwise we restrict f_H by picking a clique on $\nu_H - 1$ vertices and joining all the other $n - \nu_H + 1$ remaining vertices to each vertex in this clique. The resulting function takes a graph on $p = n - \nu_H + 1$ vertices as input. Let's denote these vertices by S .

As the clique on $\nu_H - 1$ vertices cannot accommodate all the vertices in the minimum vertex cover of H , in order to satisfy the property f_H at least one vertex v in the vertex cover must occur among S . This vertex v may have some edges incident on the vertices of the clique and some edges incident on the vertices of S . In the restriction all the possible edges to the clique are already present. Thus as soon as we have the remaining edges to the vertices of S the property f_H is satisfied.

Hence the property is now reduced to finding a star graph with d edges, f_{S_d} where d is defined as follows: Let C be a vertex cover. Furthermore let $d_{out}(v)$ denote the number of neighbors of a vertex v in C that are outside C and $d_{out}(C)$ be the minimum over all such vertices v in C . Then d is the minimum $d_{out}(C)$ of a minimum vertex cover C of H (minimized over all the minimum vertex covers). Thus as soon as we have the star graph with d edges, our original restricted f_H is satisfied.

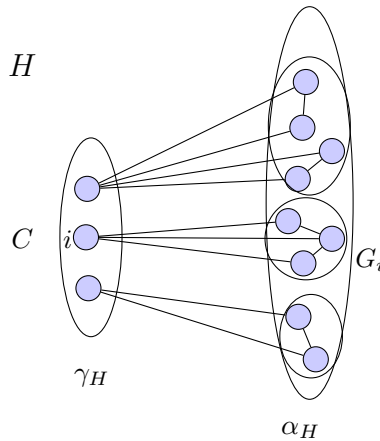
Now from the Lemma 13 we get $Q(f_H) = \Omega(n)$.

Case 2: $t \leq n/2$. Note that $t > \alpha_H$. And since $t \leq n/2$, we have $n - t = \Omega(n)$. Hence from the Lemma 14 we get the bound of $\Omega(\sqrt{t(n-t)})$, which is $\Omega(\sqrt{\alpha_H \cdot n})$. ◀

Proof of Corollary 2.

1. From Turán's theorem, we have: $\alpha_H \geq n/(2 \cdot d_{avg}(H))$.
2. Since $\alpha_H \cdot \chi_H \geq n$ we have $\alpha_H \geq n/\chi_H$.
3. Since the critical probability of H is p , the average degree of H is at most pn . Hence from Corollary 2(1), we get the $\Omega(n/p)$ bound. ◀

Proof of Corollary 3. When $d_{avg}(H) \geq \sqrt{n}$ the Lemma 8 gives an $\Omega(n^{3/4})$ bound. Otherwise when $d_{avg}(H) < \sqrt{n}$ we use the Corollary 2(1), which gives the same bound. ◀



■ **Figure 1** Structure of H .

4 Subgraph Isomorphism for 3-Uniform Hypergraphs

In this section we extend the $\Omega(n^{3/4})$ bound for the Subgraph Isomorphism for graphs to the 3-uniform hypergraphs. In particular, we obtain an $\Omega(n^{4/5})$ bound for the Subgraph Isomorphism for 3-uniform hypergraphs, improving upon the $\Omega(n^{3/4})$ bound obtained via the minimal certificate size.

Before going to the proof of Theorem 4, we extend the Lemma 14 to the 3-uniform hypergraphs. Let t be the smallest such that $f_H^*(K_t) = 1$. Note that $t = \alpha_H + 1$.

► **Lemma 16.** *Let H be a 3-uniform hypergraph on n vertices. Then:*

$$Q(f_H) \geq \Omega(\sqrt{t(n-t)}).$$

Proof. Let $T_n^t(z_1, \dots, z_n)$ denote the threshold function on n variables that outputs 1 if and only if at least t variables are 1. We embed a T_n^t in f_H^* (on inputs of Hamming weight $t-1$ and t) via the following mapping: Let $x_{ijk} := z_i \cdot z_j \cdot z_k$. Let $f'(z_1, \dots, z_n) := f_H^*(\{x_{ijk}\})$. Note that $f' \equiv T_n^t$. Also note⁴ that the $\widehat{\deg}(f') \leq 3 \cdot \widehat{\deg}(f_H^*)$. Since $Q(f) \geq \widehat{\deg}(f)$, it remains to prove that $\widehat{\deg}(f') = \Omega(\sqrt{t(n-t)})$, which follows from the Lemma 15. ◀

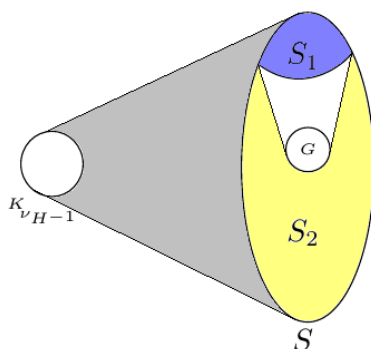
Now we give a proof of the Theorem 4.

Proof of Theorem 4. We divide the proof into two main cases.

Case 1: $\alpha_H > n/2$. Let H be a 3-uniform hypergraph on n vertices. Let C denote a minimal vertex cover of H . Let $|C| = \nu_H$. Note that the hypergraph induced on $V - C$ is empty. For a vertex $i \in C$ let G_i denote the projection graph of the neighbors of i on $V - C$, i.e., $(i, u, v) \in E(H)$ (see Figure 1).

Let \mathcal{P}_H denote the restriction of the f_H defined as follows: set the hyper-clique on $\nu_H - 1$ vertices to be present and add all the hyper-edges incident on the vertices of this clique. Let S denote the set of remaining $n - \nu_H + 1$ vertices. The hyper-edges among S are still undetermined. Note that \mathcal{P}_H is a non-trivial property of $n - \nu_H + 1$ vertex hypergraphs,

⁴ Since $x_{ijk} = z_i \cdot z_j \cdot z_k$, every monomial of f_H^* of size d becomes a monomial of size at most $3d$ in f' .



■ **Figure 2** The Restriction \mathcal{P}'' : the hyper-clique K_{ν_H-1} is present, all the hyper-edges in the gray area are present. All the hyper-edges in blue region are present, all the hyper-edges in yellow region are absent. G is fixed. White region symbolizes the hyper-edges with two-end points in S_1 and one in S_2 to be absent and one end point in S_1 and two end points in S_2 to be undetermined.

since H cannot be contained in the $\nu_H - 1$ hyper-clique and edges incident on it as the minimum vertex cover size of H is ν_H .

► **Lemma 17.** *If $\exists C, \exists i : |E(G_i)| = O(n^{7/5})$, then $Q(f_H) = \Omega(n^{4/5})$.*

Proof. In this case \mathcal{P}_H has a certificate of size $O(n^{7/5})$. Hence from Lemma 12 the certificate size of \mathcal{P}_H^* is $\Omega(\frac{n^3}{n^{7/5}}) = \Omega(n^{8/5})$. Now from the Lemma 8 we get $Q(f_H) = \Omega(n^{4/5})$. ◀

Hence from now onwards we assume that for all $i, |E(G_i)| = \Omega(n^{7/5})$. Moreover, we may also note that $\nu_H = O(n^{1/5})$, if not we have a minimal certificate for \mathcal{P}_H of size $\Omega(n^{8/5})$. And hence from the Lemma 8 we already get the desired bound of $Q(f_H) = \Omega(n^{4/5})$.

Now we obtain a restriction \mathcal{P}' of \mathcal{P}_H as follows: divide S into two parts say S_1 and S_2 of size n_1 and n_2 respectively, where we choose $n_1 = \Theta(n^{1/5})$ and $n_2 = \Theta(n)$. Set all the hyper-edges within S_1 to be present and set all the hyper-edges within S_2 to be absent. Also set all the hyper-edges with two endpoints in S_1 and one in S_2 to be absent. Only possible undetermined hyper-edges are with one endpoint in S_1 and two in S_2 . Note that even after setting all hyper edges in S_1 to be present we can safely assume that the property remains non-trivial. Otherwise we would have a certificate for \mathcal{P}_H of size $O(n^{3/5})$, hence the dual will have large ($\gg \Omega(n^{8/5})$) certificates.

Let G be a projection graph among all the G_i 's containing the least number of edges inside S_2 . We further obtain a restriction \mathcal{P}'' by fixing a copy of G inside S_2 and allowing only potential hyper-edges with one endpoint in S_1 and the other two endpoints forming an edge of G (see Figure 2).

Let C be a vertex cover of H of minimum cardinality. Note that in order to satisfy \mathcal{P}_H , at least one of the vertices from C must move to S . Let us call a vertex of C that moves to S as pivot. Let k be the largest integer such that \mathcal{P}_H has a minimal certificate with k pivots. Note that from Lemma 17 each pivot has $\Omega(n^{7/5})$ edges incident on it. Therefore if $k > n_1/2$ then we already have a minimal certificate whose size is $\Omega(n^{8/5})$. Otherwise: $k \leq n_1/2$. First we argue that any pivot must belong to S_1 . If on the contrary, it were in S_2 then the only possible edges incident on such a pivot v are of the form (v, u, w) where $u \in S_1$ and $w \in S_2$. But there can be at most $O(n^{6/5})$ such edges, which contradicts the fact that any pivot supports at least $\Omega(n^{7/5})$ edges. Let the degree of a pivot be the number of edges inside S_2 that are adjacent to it. Next we choose a certificate for \mathcal{P}_H with at most $k \leq n_1/2$

pivots such that the degree of the minimum degree pivot is minimum possible. Then we leave aside the minimum degree pivot in this certificate and fix the $k - 1$ other pivots and their projection on S_2 . From each of the remaining $n_1 - k + 1$ vertices we keep the projection of the minimum degree pivot on S_2 as the only possible edges.

Now from minimality of our choice at least one of these vertices must have all these $\Omega(n^{7/5})$ edges in order for the original graph to contain H . Thus we get an $\bigvee_{\Omega(n^{1/5})} \bigwedge_{\Omega(n^{7/5})}$ function as the restriction.

Since an $OR \circ AND$ on m variables admits an $\Omega(\sqrt{m})$ lower bound on the quantum query complexity we get $Q(f_H) = \Omega(n^{4/5})$.

Case 2: $\alpha_H \leq n/2$. In this case we use Lemma 16. Since $n - \alpha_H \geq n/2$, we immediately get $Q(f_H) = \Omega(\sqrt{\alpha_H \cdot n})$.

Now in order to prove Theorem 4, we need to show that the above bound always yields an $\Omega(n^{4/5})$ bound. Thus we further consider two cases based on the average degree. And in fact this gives us a larger $\Omega(n^{5/6})$ bound for the case 2.

Let d denote the average degree of H .

Case 2a: $d > n^{2/3}$. In this case $|E(H)| > \Omega(n^{5/3})$. Hence from Lemma 8 we get an $\Omega(n^{5/6})$ bound.

Case 2b: $d \leq n^{2/3}$. Here we use the extension of Turán's Theorem (see Lemma 11) to 3-uniform hypergraphs. Since the average degree is $O(n^{2/3})$, we get $\alpha_H \geq \Omega(n^{2/3})$. Therefore from Lemma 16 we get $Q(f_H) = \Omega(n^{5/6})$.

This completes the proof of Theorem 4. ◀

In the following two sections we study the Subgraph Homomorphism Problem. We first prove the quantum query complexity lower bounds for graphs and then for 3-uniform hypergraphs.

5 Subgraph Homomorphism for Graphs

Proof of Theorem 5. Let $\chi(H)$ denote the chromatic number of H . Note that H has a homomorphism into K_t for $t = \chi(H)$, i.e., $f_{[H]}(K_{t-1}) = 0$ and $f_{[H]}(K_t) = 1$.

We consider the following two cases.

Case 1: $t \geq n/2$. As K_{t-1} is a no instance and K_t is an yes instance for the property $f_{[H]}$, the minimum certificate size, $m(f_{[H]}) = \Omega(t^2) = \Omega(n^2)$. Hence from Lemma 8 we get an $\Omega(n)$ lower bound on the quantum query complexity.

Case 2: $t < n/2$. Consider the following restriction: We set a clique K_{t-2} on $t - 2$ vertices to be present and we also set all the edges from the remaining $n - t + 2$ vertices to this clique to be present. Now notice that as soon as there is an edge between any two of the remaining $n - t + 2$ vertices, we have a K_t . Hence the property $f_{[H]}$ has become the property of containing an edge among the $n - t + 2$ vertices. Since $t < n/2$, this is an OR function on $\Omega(n^2)$ variables. Thus $Q(f_{[H]}) = \Omega(n)$. ◀

► **Remark.** Our proof in fact shows that the minimum certificate size of either $f_{[H]}$ or $f_{[H]}^*$ is $\Omega(n^2)$. Hence we also obtain

$$R(f_{[H]}) = \Omega(n^2)$$

We now proceed to prove the quantum query complexity lower bound of the Subgraph Homomorphism Problem for 3-uniform hypergraph.

6 Subgraph Homomorphism for 3-Uniform Hypergraphs

Proof of Theorem 6. Proof of this theorem is similar to proof of Theorem 5.

Let $\chi(H)$ denote the chromatic number of H . Note that H has a homomorphism into K_t for $t = \chi(H)$, i.e., $f_{[H]}(K_{t-1}) = 0$ and $f_{[H]}(K_t) = 1$.

We consider the following two cases.

Case 1: $t \geq n/2$. Unlike the graph homomorphism case, we cannot claim the presence of a K_t in this case. However we can still use the following fact:

► **Fact 1** (Alon [3]). *If H is a 3-uniform hypergraph which is not k colorable then*

$$|E(H)| = \Omega(k^3).$$

Therefore, the minimum certificate size $m(f_{[H]}) = \Omega(t^3) = \Omega(n^3)$. Hence from Lemma 8 we get an $\Omega(n^{3/2})$ lower bound on the quantum query complexity.

Case 2: $t < n/2$. Consider the following restriction: We set a clique K_{t-3} on $t-3$ vertices to be present and we also set all the edges from remaining $(n-t+3)$ vertices to this clique to be present. Now notice that as soon as there is an edge between any three of the remaining $(n-t+3)$ vertices, we have a K_t . Hence the property $f_{[H]}$ has become the property of containing an edge among the $n-t+3$ vertices. Since $t < n/2$, this is an OR function on $\Omega(n^3)$ variables. Thus $Q(f_{[H]}) = \Omega(n^{3/2})$. ◀

7 Conclusion & Open Ends

We obtained an $\Omega(n^{3/4})$ lower bound for the quantum query complexity of Subgraph Isomorphism Problem for graphs, improving upon previously known $\Omega(n^{2/3})$ bound for the same. We extend our result to the 3-uniform hypergraphs by exhibiting an $\Omega(n^{4/5})$ bound, which improves on previously known $\Omega(n^{3/4})$ bound. Besides the obvious question of settling the randomized and quantum query complexity of the Subgraph Isomorphism problem, there are a few interesting questions that might be approachable. We list some of them below:

► **Question 1.** *Is it true that for any n -vertex graph H we have:*

- (a) $R(f_H) = \Omega(\alpha_H \cdot n)$?
- (b) $R(f_H) = \Omega(n^2/d_{avg}^H)$?
- (c) $R(f_H) = \Omega(n^2/\chi_H)$?

► **Question 2.** *Is it true that for any 3-uniform hypergraph H we have:*

$$Q(f_H) = \Omega(n)?$$

Note that, in the proof of Theorem 4 we managed to get a slightly stronger $\Omega(n^{5/6})$ bound for the case 2. Thus an improved lower bound of $\Omega(n^{5/6})$ for the case 1 (when $\alpha_H > n/2$) would improve the overall bound.

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