

How to Solve the Cake-Cutting Problem in Sublinear Time

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Abstract

The cake-cutting problem refers to the issue of dividing a cake into pieces and distributing them to players who have different value measures related to the cake, and who feel that their portions should be “fair.” The fairness criterion specifies that in situations where n is the number of players, each player should receive his/her portion with at least $1/n$ of the cake value in his/her measure. In this paper, we show algorithms for solving the cake-cutting problem in sublinear-time. More specifically, we preassign fair portions to $o(n)$ players in $o(n)$ -time, and minimize the damage to the rest of the players. All currently known algorithms require $\Omega(n)$ -time, even when assigning a portion to just one player, and it is nontrivial to revise these algorithms to run in $o(n)$ -time since many of the remaining players, who have not been asked any queries, may not be satisfied with the remaining cake. To challenge this problem, we begin by providing a framework for solving the cake-cutting problem in sublinear-time. Generally speaking, solving a problem in sublinear-time requires the use of approximations. However, in our framework, we introduce the concept of “ ϵn -victims,” which means that ϵn players (victims) may not get fair portions, where $0 < \epsilon \leq 1$ is an arbitrary constant. In our framework, an algorithm consists of the following two parts: In the first (Preassigning) part, it distributes fair portions to $r < n$ players in $o(n)$ -time. In the second (Completion) part, it distributes fair portions to the remaining $n - r$ players except for the ϵn victims in $\text{poly}(n)$ -time. There are two variations on the r players in the first part. Specifically, whether they can or cannot be designated. We will then present algorithms in this framework. In particular, an $O(r/\epsilon)$ -time algorithm for $r \leq \epsilon n/127$ undesignated players with ϵn -victims, and an $\tilde{O}(r^2/\epsilon)$ -time algorithm for $r \leq \epsilon e^{\sqrt{\ln n}/7}$ designated players and $\epsilon \leq 1/e$ with ϵn -victims are presented.

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1 Introduction

1.1 What is a sublinear-time algorithm for the cake-cutting problem?

This paper reports the first results on sublinear-time algorithms for solving the cake-cutting problem, in which it is necessary to divide a given cake into pieces and to distribute those pieces to players in a way that ensures that all players are “satisfied,” more specifically, in

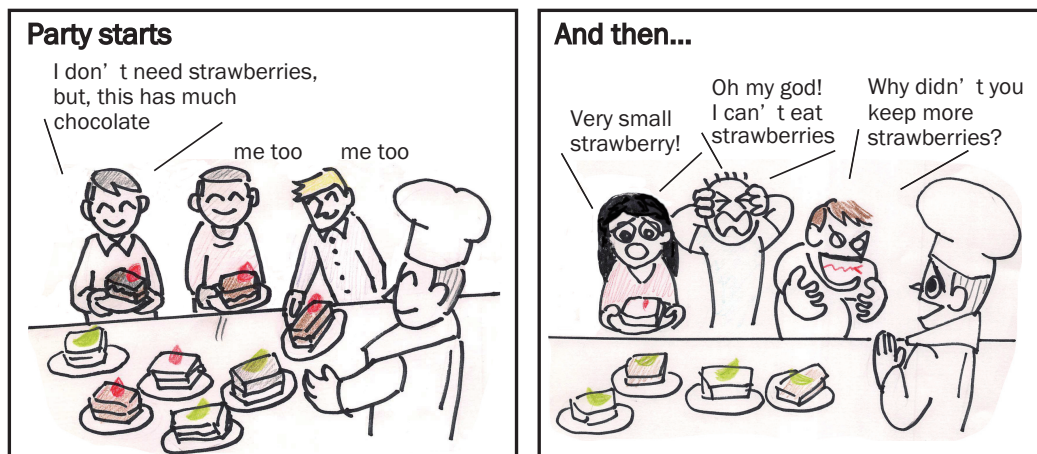


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■ **Figure 1** Participants who came late should be satisfied also.

a way that ensures every player believes that his/her portion has at least $1/n$ value of the whole cake, where n is the number of the players. It has previously been known that an $O(n \log n)$ -time algorithm [5] and that a $\Theta(n \log n)$ -time lower-bound exists for deterministic algorithms [3]. For approximation, a linear-time $O(1)$ approximation algorithm has been given [3].

However, when we divide a property and assign portions to persons, sometimes situations arise in which it is impossible to meet the requirements of all interested persons simultaneously. In such cases, it may be necessary to assign acceptable portions to some persons without asking for approval from all other interested persons. For example, in a reception party of a conference, a people who comes early takes a piece of cake that he likes, but it's better to leave "fair" pieces for participants who will come later (Fig. 1). This is the motivation behind our desire to develop algorithms for solving the cake-cutting problem in sublinear time.

Herein, we consider ways to *preassign portions to a number of* ($r = o(n)$) *players in* $o(n)$ -*time*. However, since all the known algorithms need $\Omega(n)$ -time, even for assigning a portion to just one player, the problem is nontrivial. In fact, it is a difficult matter to satisfy just one player. Moreover, even if r players can be satisfied, if the other $n - r$ players are unsatisfied, the solution is clearly suboptimal in many cases. Thus, it is better to be able to satisfy the remaining players by distributing the remaining cake appropriately. However, it is often very hard (or even impossible) to completely satisfy all the other players since we have already assigned a portion of the cake after giving queries to only a sublinear number of players. Thus, we need to make some approximations. With this in mind, we hereby introduce the concept of "*en-victims*," which means that *we can give up trying to satisfy at most en players (victims)*.

Recently, it has been learned that many problems can be approximated in sublinear time [1, 6, 7, 8, 9, 10, 11, 12, 13, 16]. To solve a problem in sublinear time, it is necessary to introduce some approximations by using a parameter $0 < \epsilon \leq 1$. There are two types of approximations. In the first, which is for decision problems, an edit distance between an instance and an objective property is defined and algorithms distinguish between instances satisfying the property and those ϵ -far from the property with high probability. The second type, which is for optimizing problems, provides ϵn -approximation solutions for objective functions [6].

All known sublinear-time algorithms use one of these two approximation types. Our approximation, ϵn -victims, can be seen as the latter approximation type. Thus, if we stipulate that the goal when solving the cake-cutting problem is to maximize the number of satisfied players, a solution with ϵn -victims is an ϵn -approximation solution.

The results of this paper can be summarized as follows:

- Presenting a framework for solving the cake-cutting problem in sublinear-time.
- Presenting sublinear-time algorithms under this framework.

The framework presented here is as follows:

The proposed sublinear-time cake-cutting framework

- **First (Preassigning) Part:** First, we preassign portions to $r = o(n)$ players in $o(n)$ -time.
- **Second (Completion) Part:** Next, we assign portions to the remaining $n - r$ players except for the ϵn victims in $\text{poly}(n)$ -time.

Note that it is impossible to do the second part in sublinear-time, since it is necessary to assign one portion to each of the remaining $\Omega(n)$ players (except for the ϵn victims).

Next, we will consider sublinear-time cake cutting algorithms that obey this framework. These algorithms can be divided into two types: one in which **the preassigned players cannot be designated**, and the other in which **they can be designated**. We will then present algorithms for both types. More specifically, for the first (undesignated) type, we can preassign portions to the $r \leq \epsilon n/127$ undesignated players in $O(tr/\epsilon)$ -time and set the success probability to at least $1 - (\frac{1}{64})^{t/\epsilon} - \frac{8}{(2t-3)^{2r}}$. After that, we can assign portions to the remaining players except for the ϵn victims in $O(n \log n)$ -time, where $t \geq 1$ is an arbitrary real number. For the latter (designated) type, for any $0 < \epsilon \leq 1/e$, we can preassign portions to $r \leq \epsilon e^{\sqrt{\ln n}/7}$ designated players in $\tilde{O}(tr^2/\epsilon)$ -time and set the success probability to at least $1 - (\epsilon/r)^t \geq 1 - e^{-t}$. After that, we can then assign portions to the remaining players except for the ϵn victims in $O(rn \log rn)$ -time, where $t \geq 1$ is an arbitrary real number.

1.2 Definition of the cake-cutting problem

Let P be the set of n players. We assume that every algorithm for solving the cake-cutting problem knows n (which is the number of players)¹.

The cake is represented by the unit interval $C = [0, 1]$. The portion of each player is a set of disjoint subintervals of C . Every player $p \in P$ has his/her subjective nonnegative value function $\mu_p : 2^C \rightarrow [0, 1]$, which is defined on every measurable subset of C . Furthermore, μ_p is additive. In other words, the value of the portion of a player is the sum of the subinterval values of his/her portion. The value function is normalized, i.e., $\mu_p(C) = 1$ for every $p \in P$.

A portion $C_p \subseteq C$ of a player $p \in P$ is deemed to be *fair* if $\mu_p(C_p) \geq 1/n$. For any positive real $c \geq 1$, C_p is deemed to be *c-fair* if $\mu_p(C_p) \geq 1/cn$.

When evaluating cake-cutting algorithms, the Robertson-Webb model [14] is generally used. In this model, the two query types listed below are allowed, and the complexity of an algorithm is evaluated by the *query complexity*. In other words, the number of these queries made by the algorithm.

- *Cut query:* For a continuous piece of a cake $D = [a, b] \subseteq C$ ($0 \leq a < b \leq 1$), a player $p \in P$, and a positive real number $0 \leq \alpha \leq 1$, a query $\text{CUT}(D, p, \alpha)$ returns the smallest

¹ Although this assumption may seem trivial, it is an important consideration in sublinear-time algorithms since the algorithm cannot count the number of players in sublinear-time. Such an assumption is generally introduced when sublinear-time algorithms are investigated.

value $x \in [a, b]$ such that $\mu_p([a, x]) = \alpha$. If there is no such x (i.e., $\mu_p(D) < \alpha$), then it returns some predefined message.

- *Evaluation query:* For a continuous piece of a cake $D = [a, b] \subseteq C$ ($0 \leq a < b \leq 1$), a player $p \in P$, and a positive real number $x \in [a, b]$ (= a point on D), a query $\text{EVAL}(D, p, x)$ returns the value $0 \leq \alpha \leq 1$ such that $\mu_p([a, x]) = \alpha$. If $x \geq b$ (i.e., asking to evaluate whole of D), then $\text{EVAL}(D, p, x)$ is simply expressed as $\text{EVAL}(D, p)$.

1.3 Previous work

One well known method of handling the cake-cutting problem, involves an $O(n \log n)$ -time algorithm using the divide-and-conquer concept [5]. This algorithm divides a cake into two pieces and assigns players into two half-sized subsets. It begins by assigning one of the pieces to one of the subgroups, and the other piece to the other subgroup. Then, it recursively applies this separation until every subgroup becomes a singleton. Note that this algorithm requires $\Theta(n)$ queries even if it only assigns a portion to one player. We refer to this algorithm as $\text{DC}(P, C)$.

For the lower-bound results, it is known that the time-complexity is $\Theta(n \log n)$ for a deterministic algorithm [3]. In the same paper, they showed that, with some restrictions, this bound can be also applied to a randomized case. For approximations, Edmonds and Pruhs [4] showed that for c -fair division with $c > 32$, there is an $O(n)$ -time randomized algorithm.

The success probability of this algorithm is at least

$$1 - \frac{2^{13}}{c^2(c-32)} - \frac{1024}{c^3} - \frac{128}{c^2} \quad (1)$$

for $c > 32$. We refer to this algorithm as $\text{APPROXFAIR}(P, C, c)$. Our algorithms use these algorithms as subroutines. In addition to these two, numerous other algorithms have been presented [2, 15]. If it is necessary to distribute pieces to $\Omega(n)$ players fairly, clearly we need $\Omega(n)$ -time. However, none of the previously known algorithms have considered preassigning portions to $o(n)$ players, and they all need $\Omega(n)$ queries even when assigning a portion to just one player.

1.4 Our results

In this subsection, we will explain the results we have obtained thus far. Throughout this paper, we assume that every player is honest, i.e., that he/she gives correct answers for every query². First we show a preliminary result, which can be obtained as a simple application of [3], as follows:

► **Proposition 1.** *For any $t \geq 64$ and any given subset $P_r \subseteq P$ of players with $|P_r| = r \leq n/t$, there is an $O(r)$ -time algorithm for assigning fair portions to all players in P_r with success probability at least $1 - 2^9/t^2$.*

The complexity of this algorithm is $O(r)$, and it is clearly the best possible because it matches with the trivial lower bound. Moreover, it also allows us to arbitrarily assign designated r players. However, one obvious flaw of this algorithm is that it may victimize all of the

² Even if there is a dishonest player, the honest players will get fair portions, while the dishonest player may not.

remaining players, so efforts should be made to reduce the number of victims. The following algorithm allows a maximum of ϵn victims for any given $0 < \epsilon \leq 1$:

► **Theorem 2.** *For any positive real number $0 < \epsilon \leq 1$, any positive integer $r \leq \epsilon n/127$, and any constant real number $t > 3/2$, there is an algorithm for preassigning fair portions to r players in $O(tr/\epsilon)$ -time, and then assigning fair portions to the remaining players except for the ϵn players (victims) in $O(n \log n)$ -time with success probability at least $1 - \frac{8}{(2t-3)^{2r}} - (\frac{1}{64})^{t/\epsilon}$.*

While in the algorithm of Theorem 2, preassigned r members cannot be designated, the following shows an algorithm in which they can:

► **Theorem 3.** *For any real numbers $0 < \epsilon \leq 1/e$ and $t \geq 1$, and any set of $r \leq \epsilon e^{\sqrt{\ln n}/7}$ players P_r , there is an algorithm for preassigning fair portions to all players in P_r in $O(\frac{tr^2}{\epsilon} (\log \frac{r}{\epsilon})^3)$ -time and then assigning fair portions to remaining players except for the ϵn players (victims) in $O(rn \log(rn))$ -time with success probability at least $1 - (\epsilon/r)^t \geq 1 - e^{-t}$.*

1.5 Organization

The remainder of this paper is organized as follows: In Section 2, we present a proof of Proposition 1. In Sections 3 and 4, we examine the undesignated version (Theorem 2) and the designated version (Theorem 3), respectively. In Section 5, we summarize our results and discuss future work.

2 Proof of Proposition 1

In this short section, we show the following proof of Proposition 1.

Proof of Proposition 1. It is sufficient to simply call $\text{APPROXFAIR}(r, P_r, C, t)$. It assigns t -fair portions to all players in P_r . In other words, they feel at least $\frac{1}{tr}$ value in their own portion. From the assumption $r \leq \frac{n}{t}$ it follows that $\frac{1}{tr} \geq \frac{1}{n}$. Therefore, they all get fair portions, and by considering (1) and $t \geq 64$, the probability of failing is at most

$$\frac{2^{13}}{t^2(t-32)} + \frac{2^{10}}{t^3} + \frac{2^7}{t^2} \leq \frac{2^{14}}{t^3} + \frac{2^{10}}{t^3} + \frac{2^7}{t^2} \leq \frac{2^8}{t^2} + \frac{2^4}{t^2} + \frac{2^7}{t^2} \leq \frac{2^9}{t^2}.$$

The time-complexity is clearly $O(r)$. ◀

3 Undesignated r players

In this section, we consider a case where P_r cannot be designated.

3.1 Algorithm for Theorem 2

When preassigned players are not designated, the algorithms can select P_r players arbitrarily. That is, players who feel a relatively high value in a specified part (e.g., the left-side part of the cake) are considered more suitable, such members can be selected at high probability levels by using asking cut-queries to a number ($\lceil tr/\epsilon \rceil$) of other players. Let P' be the set of selected players and let C' be a piece to which these players in P' have assigned high value ($128r/n$). Then, by applying APPROXFAIR to P' and C' with approximation parameter 128, the players in P' have a high probability of getting fair portions. This summarizes the preassigning part.

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For the completion part, it can be expected that a small number of the remaining players will feel that C' (the removed piece) has high value, and that the only way the remaining players can share the rest of the cake ($C - C'$) fairly is by removing the appropriate ϵn players (victims).

Before showing the details of this algorithm, we will first define a subroutine PCUT it uses. The objective of this subroutine is to get a set of $m \in \{0, \dots, n\}$ players from $Q \subseteq P$ who have a high probability of seeing relatively high value in the left-most part of the piece D .

procedure PCUT(Q, D, α, m)

Input: $Q \subseteq P$, $D \subseteq C$, real value $0 \leq \alpha \leq 1$, integer $0 \leq m \leq n$;

begin

1 **for** $p \in Q$ **do**

2 $x_p := \text{CUT}(D, p, \alpha)$

3 **enddo**

4 Let Q' be the set of players $p \in Q$ having the 1st, 2nd, \dots , and the m th smallest value x_p in Q ,

where ties are broken arbitrarily.

5 **Output** Q'

end.

The preassigning part of the algorithm used for proving Theorem 2 is as follows:

procedure PREASSIGNU(P, C, r, ϵ, t)

Input: The set P of n players, The cake $C = [0, 1]$, positive integers r and t , real value $0 < \epsilon \leq 1$;

begin

01 $P_0 := \emptyset$

02 **for** $\lceil \frac{tr}{\epsilon} \rceil$ times **do**

03 Select $p \in P$ UAR and $P_0 := P_0 \cup \{p\}$;

04 **enddo**

05 **if** $|P_0| < r$ **then output** “Failed” **and stop endif**;

06 $P' := \text{PCUT}(P_0, C, \frac{128r}{n}, r)$

07 $x := \max_{p \in P'} x_p$

08 $C' := [0, x]$

09 **for** $\lceil \frac{t}{\epsilon} \rceil$ times **do**

10 **call** APPROXFAIR($r, P', C', 128$)

11 **if** above APPROXFAIR succeeds **then**

12 **output** the assignment obtained in Line 10; **stop**;

13 **endif**;

14 **enddo**

15 **comment** all APPROXFAIR in Line 10 failed;

16 **output** “Failed”;

end.

By applying PREASSIGNU, providing it does not fail, all players in P' ($|P'| = r$) will have their own portions (which we will later prove are fair). The next important point is ensuring that the remaining players except for ϵn victims will be satisfied. We define the other terms used for treating this problem as follows:

► **Definition 4.** Let $Q \subseteq P$ and $D \subseteq C$ be a subset of players and a subset of the cake, respectively. A player $p \in Q$ is called *safe with respect to* (Q, D) if $\mu_p(D) \geq \frac{|Q|}{n}$, or *dangerous*

with respect to (Q, D) otherwise. We may omit “with respect to (Q, D) ” if (Q, D) is clear. If all players in Q are safe with respect to (Q, D) , we then say that Q is *safe with respect to D* , or *safe* in short, if D is clear. For $m \geq 0$, if there is a subset of $Q' \subset Q$ such that $|Q'| \leq m$ and $Q - Q'$ is safe, then Q is called *m -safe*.

If Q is safe with respect to D , it is clear that all players in Q can get fair portions in D by using arbitrary cake-cutting-algorithms, such as $\text{DC}(Q, D)$ (Lemma 6, which will be shown later). Then, for proving the completion part following PREASSIGNU , we should show that $P - P'$ is ϵn -safe with respect to $C - C'$. The algorithm of the completion part is simple: It is sufficient to make a query $\text{EVAL}(C - C', p)$ for every player p in $P - P'$ and remove the lowest evaluating ϵn players. Pseudo code of this algorithm is shown below:

```

procedure COMPLETION( $P - P', C - C', n, \epsilon$ )
begin
1    $Q := \text{VICTIMIZE}(P - P', C - C', \lfloor \epsilon n \rfloor)$ 
2   call  $\text{DC}(Q, C - C')$ 
end.

```

```

procedure VICTIMIZE( $P'', D, m$ )

```

Input: Subset $P'' \subseteq P$ of players, subset $D \subseteq C$ of the cake, integer $m \geq 0$;

```

begin
1   for  $p \in P''$  do  $x_p := \text{EVAL}(D, p)$  enddo
2   Let  $Q_{\text{vict}} \subseteq P''$  be the set of  $m$  players having the 1st, 2nd, ...,  $m$ th smallest values of
    $x_p$ ,
   where ties are broken arbitrarily;
3   output  $Q := P'' - Q_{\text{vict}}$ ;
end.

```

3.2 Proof of Theorem 2

We prepare the following lemmas for showing the proof of Theorem 2.

► **Lemma 5.** *Let N be $\{1, 2, \dots, n\}$ and S be an $\lfloor \epsilon n \rfloor$ size subset of N for $0 < \epsilon \leq 1$. For real numbers $s, t > 1$ such that $(s - 1)(t - 1) > 1$ and a positive integer r such that $r \leq \epsilon n / s$, if we choose at least tr / ϵ elements from N uniformly at random (UAR), we then get at least r different elements in S with probability at least $1 - \frac{s^2}{((s-1)(t-1)-1)^2 r}$.*

Proof. Let X be the random variable of the number of chosen elements until we get r different elements in S from N . Further, let X_i be the random variable of the number of chosen elements until we get i -th different elements in S after $i - 1$ different elements were chosen from S . Clearly, $X = \sum_{i=1}^r X_i$.

Let p_i be the probability that we get a new element from S after we have gotten $i - 1$ different elements from S . The following inequalities hold:

$$p_i = \frac{\lfloor \epsilon n \rfloor - (i - 1)}{n} \geq \frac{\lfloor \epsilon n \rfloor - (r - 1)}{n} > \frac{\epsilon n - r}{n} \geq \frac{\epsilon n - \frac{\epsilon n}{s}}{n} = \frac{(s - 1)\epsilon}{s}.$$

Since the random variable X_i follows a geometric distribution, the expected value $E[X_i]$ and the variance $V[X_i]$ satisfy $E[X_i] = 1/p_i$ and $V[X_i] = (1 - p_i)/p_i^2$, respectively. By the linearity of expected value,

$$E[X] = \sum_{i=1}^r E[X_i] \leq \sum_{i=1}^r \frac{s}{(s - 1)\epsilon} = \frac{sr}{(s - 1)\epsilon}.$$

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Since each X_i is independent, the variance satisfies linearity, and thus

$$V[X] = \sum_{i=1}^r V[X_i] \leq \sum_{I=1}^r \frac{1 - \frac{(s-1)\epsilon}{s}}{\left(\frac{(s-1)\epsilon}{s}\right)^2} = \frac{(s - \epsilon s + \epsilon)sr}{(s-1)^2\epsilon^2}.$$

We compute the probability that we do not get at least r different elements in S when we choose tr/ϵ elements from N uniformly, at random, as follows:

$$\begin{aligned} \Pr\left[X > \frac{tr}{\epsilon}\right] &\leq \Pr\left[X \geq \frac{tr}{\epsilon}\right] \leq \Pr\left[\left|X - \frac{sr}{(s-1)\epsilon}\right| \geq \frac{tr}{\epsilon} - \frac{sr}{(s-1)\epsilon}\right] \\ &= \Pr\left[\left|X - \frac{sr}{(s-1)\epsilon}\right| \geq \frac{sr}{(s-1)\epsilon} \left(t\frac{s-1}{s} - 1\right)\right] \\ &\leq \Pr\left[|X - E[X]| \geq \frac{sr}{(s-1)\epsilon} \left(t\frac{s-1}{s} - 1\right)\right]. \\ &\quad (\text{From Chebyshev bound, } \Pr[|X - E[X]| \geq a] \leq \frac{V[X]}{a^2}, \forall a > 0) \\ &\leq \frac{V[X]}{\left(\frac{sr}{(s-1)\epsilon} \left(t\frac{s-1}{s} - 1\right)\right)^2} \\ &\leq \frac{\frac{(s - \epsilon s + \epsilon)sr}{(s-1)^2\epsilon^2}}{\left(\frac{sr}{(s-1)\epsilon} \left(t\frac{s-1}{s} - 1\right)\right)^2} = \frac{s - (s-1)\epsilon}{\left(t\frac{s-1}{s} - 1\right)^2 rs} \\ &\leq \frac{s}{\left(t\frac{s-1}{s} - 1\right)^2 rs} = \frac{s^2}{(st - s - t)^2 r} = \frac{s^2}{((s-1)(t-1) - 1)^2 r}. \end{aligned}$$

The desired inequality is obtained. \blacktriangleleft

► Lemma 6. For any $Q \subseteq P$ and $D \subseteq C$, if Q is safe with respect to D , then all players in Q can get fair portions in D by using arbitrary cake-cutting algorithms.

Proof. By applying a cake-cutting algorithm, every player $p \in Q$ obtains a portion with value at least $\mu_p(D)/|Q|$. From that, Q is safe with respect to D , $\mu_p(D) \geq |Q|/n$ for $\forall p \in Q$. Thus, the value of the cake obtained by $\forall p \in Q$ is

$$\frac{\mu_p(D)}{|Q|} \geq \frac{1}{|Q|} \cdot \frac{|Q|}{n} = \frac{1}{n}.$$

\blacktriangleleft

Proof of Theorem 2. We will show the following facts:

- (i) All players in P' get fair portions with probability at least $1 - \frac{8}{(2t-3)^{2r}} - \left(\frac{1}{64}\right)^{t/\epsilon}$ after calling APPROXFAIR in line 10 of PREASSIGNU.
- (ii) Q (the output of VICTIMIZE($P - P'$, $C - C'$, $\lfloor \epsilon n \rfloor$) in line 01 of COMPLETION($P - P'$, $C - C'$, n, ϵ)) is safe with respect to $C - C'$.

In what follows, we show proofs for the above items.

- (i) First, we assume that $|P_0| \geq r$ in line 05 of PREASSIGNU and that at least one call of APPROXFAIR in line 10 of PREASSIGNU succeeds. Let C_p be the portion that player $p \in P'$ gets by this APPROXFAIR when it succeeds. From the property of APPROXFAIR,

C_p is at least 128-fair, i.e., $\mu_p(C_p) \geq \mu_p(C')/128r$. From the operations in lines 06–08 of PREASSIGNU, $\mu_p(C') \geq 128r/n$. It then follows that

$$\mu_p(C_p) \geq \frac{1}{128r} \cdot \frac{128r}{n} = \frac{1}{n},$$

i.e., each player in P' gets a fair portion.

Next, we estimate the probability that $|P_0| \geq r$ in line 05 of PREASSIGNU and that at least one call of APPROXFAIR in line 10 of PREASSIGNU succeeds. From Lemma 5 with regarding P and $\text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)$ as N and S , respectively³ and by letting $s = 127$, it follows that the probability that $|P_0 \cap \text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)| < r$ occurs is at most $\frac{127^2}{(126t-127)^2r}$. From the assumption of $t > 3/2$, this probability becomes

$$\frac{127^2}{(126t-127)^2r} < \frac{(127/126)^2}{(t-3/2)^2r} < \frac{8}{(2t-3)^2r}.$$

$|P_0 \cap \text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)| \geq r$ includes $|P_0| \geq r$ and $|P'| = r$.

From (1), the probability that one call of APPROXFAIR in line 10 of PREASSIGNU succeeds is at least

$$1 - \frac{2^{13}}{2^{14}(128-32)} - \frac{1024}{128^3} - \frac{128}{128^2} = 1 - \frac{83}{6144} > 1 - \frac{1}{64}.$$

Thus, the probability that all the calls of APPROXFAIR fail is at most $64^{-t/\epsilon}$.

Therefore, the success probability of this algorithm is at least

$$1 - \frac{8}{(2t-3)^2r} - \left(\frac{1}{64}\right)^{t/\epsilon}.$$

- (ii) Assume that $|P_0 \cap \text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)| \geq r$. From this, $P' \subseteq \text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)$ follows. This means that for every player $p \in P - \text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)$, $\mu_p(C - C') \geq \frac{|P| - 128r}{|P|}$. From the assumption of $r \leq \lfloor \epsilon n / 127 \rfloor$ ($\because r$ is an integer),

$$\mu_p(C - C') \geq \frac{|P| - 128r}{|P|} > \frac{|P| - \lfloor \epsilon n \rfloor - r}{|P|}.$$

It follows that $Q \subseteq P - \text{PCUT}(P, C, \frac{128r}{n}, \epsilon n)$ and $|Q| = n - \epsilon n - r$. Therefore, Q is safe with respect to $C - C'$.

From (ii) and Lemma 6, DC in line 02 of COMPLETION assigns fair portions to all players in $P - P'$. The query complexity of PREASSIGNU is clearly $O(tr/\epsilon)$. The query complexity of COMPLETION is $O(n \log n)$, since DC can be done in $O(n \log n)$. ◀

4 Designated r players

4.1 Algorithm for Theorem 3

In this section, we consider the case where P_r is given. The key to solving this problem is to find a piece C_p that a player $p \in P_r$ prefers. After finding C_p for all $p \in P_r$, if all C_p are disjoint, we then assign C_p to p . Otherwise, i.e., when some C_{p_1}, \dots, C_{p_k} are “connected”

³ The reason that $\text{PCUT}(P, C, \frac{128r}{n}, \lfloor \epsilon n \rfloor)$ is considered here is explained in (ii).

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(the definition is given later), we allot $C_{p_1} \cup \dots \cup C_{p_k}$ to $\{p_1, \dots, p_k\}$ by using a suitable cake-cutting algorithm, e.g., DC.

The basic strategy used to find C_p is as follows. In the beginning, $C_p := C$ (of course, it will be trimmed). We ask a randomly chosen constant number of players (let P_p be the set of chosen players) to evaluate C_p . If a small number of players evaluate it as high, then C_p is fixed. Otherwise (in the first iteration, this case must occur since $C_p = C$), we divide C_p into two pieces such that the half of players in P_p prefer one of the half pieces and the other players prefer the other piece, and let C_p be the half piece that p prefers. By iteratively applying the above operations some fixed number of times, we have a high probability of getting an appropriate C_p .

To show the details of the first (preassigning) part, we use the following concept. Let $\mathcal{C} = \{C_1, \dots, C_{|\mathcal{C}|}\}$ be a family of cake subsets. We define the *relation graph* $G_{\mathcal{C}} = (\mathcal{C}, E_{\mathcal{C}})$ with respect to \mathcal{C} as $(C_i, C_j) \in E_{\mathcal{C}}$ iff $C_i \cap C_j \neq \emptyset$ for $i, j \in \{1, \dots, |\mathcal{C}|\}$ and $i \neq j$.

procedure PREASSIGN(P, C, P_r, ϵ, t)

Input: The set P of n players, The cake $C = [0, 1]$, a subset of r players $P_r \subseteq P$, positive integer t , real value $0 < \epsilon \leq 1$;

begin

```

01  for all  $p \in P_r$  do
02     $C_p := \text{DEPOSIT}(p, P, C, \epsilon/r, t)$ 
03  enddo
04  Construct the relation graph  $G_{\mathcal{C}}$  with respect to  $\mathcal{C} := \{C_p \mid p \in P_r\}$ .
05  for all connected components  $\mathcal{C}'$  of  $G_{\mathcal{C}}$  do
06    Let  $C_{p_1}, \dots, C_{p_k}$  be the vertices (cake subsets) in  $\mathcal{C}'$ ;
07    call DC( $\{p_1, \dots, p_k\}, C_{p_1} \cup \dots \cup C_{p_k}$ )
08    Let  $C_{p_i}^*$  be the piece assigned by DC in Line 07 for  $i = 1, \dots, k$ ;
09  enddo
10  output  $C_p^*$  for every  $p \in P_r$ ;
end.

```

procedure DEPOSIT(p, P, C, ϵ', t)

begin

```

01   $C' := C, h := \left\lceil \frac{2^{10}t}{\epsilon'} \ln \frac{1}{\epsilon'} \right\rceil$ 
02  from  $j = 1$  to  $54 \left(\ln \frac{1}{\epsilon'}\right)^2$  do
03    Choose a player from  $P$  UAR  $h$  times and let  $P_0$  be the multiset of the chosen
    players;
04    for all  $q \in P_0$  do
05       $\alpha_q := \text{EVAL}(C', q)$ 
06    enddo
07    Let  $P'$  be the multiset of the players  $q \in P_0$  such that  $\alpha_q \geq \epsilon'$ ;
08    if  $|P'| < 2^9 t \ln \frac{1}{\epsilon'}$  then
09      output  $C'$ ; return
10    endif
11    call CONDENSE( $p, P', C'$ )
12  enddo
13  return
end.

```

procedure CONDENSE($p, P', C' = [a, b]$)

begin

01 $x_L := a, x_R := b$

02 **for all** $q \in P'$ **do**

03 $\beta_q := \text{EVAL}([x_L, x_R], q)$

04 $x_q := \text{CUT}([x_L, x_R], q, \beta_q/2)$

05 **enddo**

06 Let q_0 be the player such that x_{q_0} is the median of multiset $\{x_q \mid q \in P'\}$;

07 $\alpha_L := \text{EVAL}([x_L, x_{q_0}], p)$

08 $\alpha_R := \text{EVAL}([x_{q_0}, x_R], p)$

09 **if** $\alpha_L > \alpha_R$ **then**

10 $C' := [x_L, x_{q_0}]$

11 **else**

12 $C' := [x_{q_0}, x_R]$

13 **endif**

14 **return**

end.

$\widehat{C} := \cup_{p \in P_r} C_p$. The completion part of the algorithm for Theorem 3 is simply applying COMPLETION($P - P_r, C - \widehat{C}, n, \epsilon$).

4.2 Proof of Theorem 3

Before showing the proof of Theorem 3, we show some lemmas, whose proofs are in Appendix. For $D \subseteq C$ and $0 \leq \alpha \leq 1$, we denote the set of players $p \in P$ such that $\text{EVAL}(D, p) \geq \alpha$ by $P(\alpha, D)$.

► **Lemma 7.** For $p \in P$, $D \subseteq C$, and real numbers $0 < \epsilon < 1/e$ and $t \geq 1$, we choose players from $P(\epsilon, D)$ uniformly at random at least $2^9 t / \epsilon$ times and let Q be the multiset of the chosen players. Let D' denote the output D of CONDENSE(p, Q, D). Then the following two conditions hold:

- $\text{EVAL}(D', p) \geq \text{EVAL}(D, p)/2$, and
- $\text{EVAL}(D', q) \leq \text{EVAL}(D, q)/2$ for at least $|P(\epsilon, D)|/3$ players $q \in P(\epsilon, D)$ with probability at least $1 - \epsilon^{16t}$.

Proof. The first item ($\text{EVAL}(D', p) \geq \text{EVAL}(D, p)/2$) is clear from the operations in Lines 09-13 of CONDENSE. Then, we prove the second item. Let $|P(\epsilon, D)| = m$. Define $P_L = \text{PCUT}(P(\epsilon, D), D, \text{EVAL}(D, q)/2, m/3)$ and $P_R = P - \text{PCUT}(P(\epsilon, D), D, \text{EVAL}(D, q)/2, 2m/3)$. Let Y_i^L (resp, Y_i^R) be a random variable such that it is 1 when the i th element of Q is included in P_L (rest., P_R) and 0 otherwise. $Y^L := \sum_{i=1}^{|Q|} Y_i^L$ and $Y^R := \sum_{i=1}^{|Q|} Y_i^R$. Clearly $E[Y^L] = E[Y^R] = |Q|/3 \geq 2^9 t / 3\epsilon$. Every Y_i^L and Y_i^R is an independent Bernoulli trial, and thus from Chernoff bound ($\Pr[X \geq (1 + \delta)E[X]] \leq e^{-\delta^2 E[X]/3}$), it follows that

$$\Pr \left[Y^L \geq \frac{|Q|}{2} \right] = \Pr \left[Y^L \geq \frac{3}{2} E[Y^L] \right] \leq e^{-E[Y^L]/12} \leq e^{-128t/9\epsilon}.$$

Here, by considering that for all real number x ,

$$x \ln \frac{1}{x} \leq \frac{1}{e} \leq \frac{4}{9},$$

we get

$$\Pr \left[Y^L \geq \frac{|Q|}{2} \right] \leq e^{-32t \ln(1/\epsilon)} = \epsilon^{32t}.$$

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Similarly, we also get $\Pr[Y^R \geq |Q|/2] \leq \epsilon^{32t}$. Let q_0 be the player in line 06 of CONDENSE(p, Q, D). Then, $\Pr[q_0 \in P_L \cup P_R] \leq 2\epsilon^{32t} \leq \epsilon^{16t}$. Therefore, for at least $m/3$ players q (i.e., players in P_L), $\text{EVAL}([x_{q_0}, x_R], q) \leq \text{EVAL}(D, q)/2$ and for at least $m/3$ players q' (i.e., players in P_R), $\text{EVAL}([x_L, x_{q_0}], q') \leq \text{EVAL}(D, q')/2$ with probability at least $1 - \epsilon^{16t}$. ◀

► **Lemma 8.** *If $P(\epsilon', C') \geq \epsilon'n$ when an operation of Line 08 of DEPOSIT(p, P, C, ϵ', t) is done, then the probability that $|P'| < 2^9 t \ln \frac{1}{\epsilon'}$ occurs is at most ϵ'^{128t} .*

Proof. Let $P_0 = \{q_1, \dots, q_h\}$ be P_0 constructed in Line 03 of DEPOSIT. Let X_i ($i = 1, \dots, h$) be the random variable such that $X_i = 1$ if $q_i \in P(\epsilon', C')$ and $X_i = 0$ otherwise. Let X be the random variable representing $|P_0 \cap P(\epsilon', C')|$. Clearly, $X = \sum_{i=1}^h X_i$ and

$$E[X] = \frac{|P(\epsilon', C')|}{n} \cdot \frac{2^{10}t}{\epsilon'} \ln \frac{1}{\epsilon'} \geq 2^{10}t \ln \frac{1}{\epsilon'}.$$

From the Chernoff bound,

$$\begin{aligned} \Pr \left[X \leq 2^9 t \ln \frac{1}{\epsilon'} \right] &= \Pr \left[X \leq \left(1 - \frac{1}{2}\right) 2^{10} t \ln \frac{1}{\epsilon'} \right] \\ &\leq \Pr \left[X \leq \left(1 - \frac{1}{2}\right) E[X] \right] \\ &\leq e^{-E[X]/8} \\ &\leq e^{2^7 t \ln \frac{1}{\epsilon'}} \\ &= \epsilon'^{128t}. \end{aligned}$$

In our algorithm, we call CONDENSE(p, P', C') iteratively. Then, for distinguishing C' s in different calls, we number them such as $C^{(1)}, C^{(2)}, \dots$: $C^{(1)}$ is C' of the first call of CONDENSE(p, P', C') (i.e., $C^{(1)} = C$), and the output of CONDENSE($p, P', C^{(i)}$) is $C^{(i+1)}$ for $i \in \{1, 2, \dots\}$. We say a call CONDENSE($p, P', C^{(i)}$) is *good* if for at least $|P(\epsilon', C^{(i)})|/3$ players $q \in P(\epsilon', C^{(i)})$,

$$\text{EVAL}(C^{(i+1)}, q) \leq \text{EVAL}(C^{(i)}, q)/2. \quad (2)$$

From Lemma 7, a call CONDENSE($p, P', C^{(i)}$) is good with probability at least $1 - \epsilon'^{16t}$.

► **Lemma 9.** *Assume that $C^{(j)}$ is obtained from $C^{(i)}$ after at least $\frac{9}{2}(\ln_{1/2} \epsilon' + 1)$ good calls. Then $|P(\epsilon', C^{(j)})| \leq \frac{2}{3}|P(\epsilon', C^{(i)})|$.*

Proof. Assume that $|P(\epsilon', C^{(j)})| > \frac{2}{3}|P(\epsilon', C^{(i)})|$. It is clear that $C^{(j)} \subseteq C^{(j-1)} \subseteq \dots \subseteq C^{(i)}$. Let $m = |P(\epsilon', C^{(i)})|$. Then, for every $C^{(k)}$ ($k \in \{i, i+1, \dots, j\}$),

$$|P(\epsilon', C^{(k)})| > \frac{2}{3}m. \quad (3)$$

Here, assume that if (2) occurs for a player $q \in P(\epsilon', C^{(i)})$, then q gets a “stone.” If a player gets $\log_{1/2} \epsilon' + 1$ stones, then $\text{EVAL}(C', q) \leq \epsilon'$ and q is removed from $P(\epsilon', C')$. If CONDENSE($p, P', C^{(i)}$) is good, at least $|P(\epsilon', C^{(i)})|/3$ stones are distributed. By considering (3), after $\frac{9}{2}(\ln_{1/2} \epsilon' + 1)$ good calls, at least $\frac{2}{3}m \cdot \frac{1}{3} \cdot \frac{9}{2}(\ln_{1/2} \epsilon' + 1) = m(\ln_{1/2} \epsilon' + 1)$ stones are distributed. Since one player can get $\ln_{1/2} \epsilon' + 1$ stones at most, every player gets $\ln_{1/2} \epsilon' + 1$ stones and has been removed from the $P(\epsilon', C^{(j)})$, contradiction. ◀

► **Lemma 10.** Let C_p be the output of $\text{DEPOSIT}(p, P, C, \epsilon', t)$. Assume that $\epsilon' \leq 1/e$. Then

- (i) $\mu_p(C_p) \geq \left(\frac{1}{2}\right)^{54(\ln(1/\epsilon'))^2}$, and
- (ii) $|P(\epsilon', C_p)| \leq \epsilon'n$ with a probability of at least $1 - \epsilon'^{2t}$.

Proof. From Lemma 7, (i) is clear. Consider line 08 of $\text{DEPOSIT}(p, P, C, \epsilon', t)$. Assume that $|P(\epsilon', C')| \geq \epsilon'n$. Then, from Lemma 8, $\text{CONDENSE}(p, P', C')$ is called in probability at least $1 - \epsilon'^{128t}$. From Lemma 7, $\text{CONDENSE}(p, P', C')$ is good with probability at least $1 - \epsilon'^{16t}$

From Lemma 9, by the following number of good calls, we get $|P(\epsilon', C')| \leq \epsilon'n$.

$$\begin{aligned} & \frac{9}{2}(\ln_{1/2} \epsilon' + 1)(\ln_{2/3} \epsilon' + 1) = \frac{9}{2} \left(\frac{\ln \frac{1}{\epsilon'}}{\ln 2} + 1 \right) \left(\frac{\ln \frac{1}{\epsilon'}}{\ln \frac{3}{2}} + 1 \right) \\ & < \frac{9}{2} \left(2 \ln \frac{1}{\epsilon'} + 1 \right) \left(3 \ln \frac{1}{\epsilon'} + 1 \right) \quad (\because \ln 2 > 1/2 \text{ and } \ln 3/2 > 1/3) \\ & < \frac{9}{2} \left(3 \ln \frac{1}{\epsilon'} \right) \left(4 \ln \frac{1}{\epsilon'} \right) \quad (\because \text{from } \epsilon' \leq 1/e, \ln \frac{1}{\epsilon'} \geq 1) \\ & = 54 \left(\ln \frac{1}{\epsilon'} \right)^2. \end{aligned}$$

The probability that “ $\text{CONDENSE}(p, P', C')$ is called and the call is good” $54 \left(\ln \frac{1}{\epsilon'} \right)^2$ times in a row is at least

$$\begin{aligned} & 1 - (\epsilon'^{128t} + \epsilon'^{16t}) \cdot 54 \left(\ln \frac{1}{\epsilon'} \right)^2 \\ & \geq 1 - (\epsilon'^{8t}) (1/\epsilon')^4 (1/\epsilon')^2 \quad (\because 54 < e^4 \leq (1/\epsilon')^4 \text{ and } \ln 1/\epsilon' \leq 1/\epsilon') \\ & \geq 1 - \epsilon'^{2t}. \end{aligned}$$

Therefore (ii) is obtained. ◀

Proof of Theorem 3. We will show the following facts:

- (i) Each player in P_r gets a fair portion by $\text{PREASSIGNS}(P, C, P_r, \epsilon, t)$.
- (ii) Q (the output of $\text{VICTIMIZE}(P - P_r, C - \widehat{C}, \lfloor \epsilon n \rfloor)$ in line 01 of $\text{COMPLETIONU}(P - P_r, C - \widehat{C}, n, \epsilon)$) is safe with respect to $C - \widehat{C}$ with probability at least $1 - (\epsilon/r)^t \geq 1 - e^{-t}$.

In what follows, we show proofs of the above items. Note that it is sufficient to consider the case that $\epsilon' = \epsilon/r$ in Lemmas 8, 9, and 10.

- (i) From Lemma 10, $\mu_p(C_p) \geq (1/2)^{54(\ln(r/\epsilon))^2}$ for every player $p \in P_r$. Thus by PREASSIGNS , every player finally gets a portion having at least $(1/2)^{54(\ln(r/\epsilon))^2}/r$ value. We will show

$$(1/2)^{54(\ln(r/\epsilon))^2}/r \geq 1/n. \quad (4)$$

This inequality can be transformed as $\ln r + 54 \ln 2 \cdot \left(\ln \frac{r}{\epsilon} \right)^2 \leq \ln n$. Here, from $\ln r \leq \ln(r/\epsilon) \leq (\ln(r/\epsilon))^2$ ($\because 1/\epsilon \geq e$), the following inequalities hold:

$$\ln r + 54 \ln 2 \cdot \left(\ln \frac{r}{\epsilon} \right)^2 \leq (1 + 54 \ln 2) \left(\ln \frac{r}{\epsilon} \right)^2 \leq \left(7 \ln \frac{r}{\epsilon} \right)^2$$

Thus, if $(7 \ln(r/\epsilon))^2 \leq \ln n$, then (4) holds. This is equivalent to $r \leq \epsilon e^{\sqrt{\ln n}/7}$. That is, (4) holds.

- (ii) For $p \in P_r$, if $|P(\epsilon/r, C_p)| \leq (\epsilon/r)n$, then we say that p is *polite*. From Lemma 10, the probability that $p \in P_r$ is not polite is at most $(\epsilon/r)^{2t}$. Thus, the probability that at least one $p \in P_r$ is not polite is at most $r(\epsilon/r)^{2t} \leq (\epsilon/r)^t \leq e^{-t}$ (since $\epsilon/r \leq \epsilon \leq 1/e$). If all players in P_r are polite, then

$$|P(\epsilon, \widehat{C})| \leq \sum_{p \in P_r} |P(\epsilon/r, C_p)| \leq r \cdot \frac{\epsilon}{r} n = \epsilon n.$$

Since $|P(\epsilon, \widehat{C})|$ is an integer, $|P(\epsilon, \widehat{C})| \leq \lfloor \epsilon n \rfloor$. Thus, all players in $P(\epsilon, \widehat{C})$ are removed by VICTIMIZE with probability at least $1 - (\epsilon/r)^t \geq 1 - e^{-t}$.

It remains necessary to calculate the query complexity. In PREASSIGNS, DEPOSIT is called r times and needs

$$O\left(r \cdot \frac{rt}{\epsilon} \ln \frac{r}{\epsilon} \cdot \left(\ln \frac{r}{\epsilon}\right)^2\right) = O\left(\frac{r^2 t}{\epsilon} \left(\log \frac{r}{\epsilon}\right)^3\right)$$

time. DC for k players can be done in $O(k \log k)$ -time if a cake is continuous. However, DC($\{p_1, \dots, p_k\}, C_{p_1} \cup \dots \cup C_{p_k}$) in line 07 of PREASSIGNS treats $C_{p_1} \cup \dots \cup C_{p_k}$, which may be separated into at most r continuous pieces. One query on a cake consisting of k continuous pieces is simulated by k queries on the continuous parts. Hence, the query complexity of this DC is $O(r^2 \log(r^2)) = O(r^2 \log r)$. Therefore, the time-complexity of PREASSIGNS is $O((r^2 t/\epsilon)(\log(r/\epsilon))^3 + r^2 \log r) = O((r^2 t/\epsilon)(\log(r/\epsilon))^3)$.

For the completion part, DC($Q, C - \widehat{C}$) in COMPLETION is dominant. $C - \widehat{C}$ may be separated into at most $r + 1$ continuous parts. Thus, the query complexity of DC (and the completion part) is $O(rn \log(rn))$. ◀

5 Summary

Herein, we considered a way to solve the cake-cutting problem in sublinear time. For this purpose, we introduced the concept of “ ϵn victims,” and presented the following framework. In the first (preassigning) part, we preassign portions to $r = o(n)$ players in $o(n)$ time. Then, in the second (completion) part, we assign portions to the remaining $n - r$ players except for the ϵn victims in polynomial-time. (Note that the second part clearly requires $\Omega(n)$ -time.) Within this framework, we presented two types of algorithms. In the first, the preassigned players cannot be designated, while in the second, they can be.

For our future work, it remains necessary to show nontrivial lower-bounds. For example, we have not yet proven that only one victim is needed to preassign sublinear players in sublinear-time. Since numerous variations may be considered in our framework, the ability to make extended algorithms is also an attractive subject.

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