Spy-Game on Graphs*

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Abstract

We define and study the following two-player game on a graph G. Let $k \in \mathbb{N}^*$. A set of k *guards* is occupying some vertices of G while one *spy* is standing at some node. At each turn, first the spy may move along at most s edges, where $s \in \mathbb{N}^*$ is his *speed*. Then, each guard may move along one edge. The spy and the guards may occupy same vertices. The spy has to escape the surveillance of the guards, i.e., must reach a vertex at distance more than $d \in \mathbb{N}$ (a predefined distance) from every guard. Can the spy win against k guards? Similarly, what is the minimum distance d such that k guards may ensure that at least one of them remains at distance at most d from the spy? This game generalizes two well-studied games: Cops and robber games (when $s = 1$) and Eternal Dominating Set (when s is unbounded).

We consider the computational complexity of the problem, showing that it is NP-hard and that it is PSPACE-hard in DAGs. Then, we establish tight tradeoffs between the number of guards and the required distance d when G is a path or a cycle. Our main result is that there exists $\beta > 0$ such that $\Omega(n^{1+\beta})$ guards are required to win in any $n \times n$ grid.

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1 Introduction

We consider the following two-player game on a graph G, called *Spy-game*. Let $k, d, s \in \mathbb{N}$ be three integers such that $k > 0$ and $s > 0$. One player uses a set of k *guards* occupying some vertices of G while the other player plays with one *spy* initially standing at some node. This is a full information game, thus any player has full knowledge of the positions and previous moves of the other player. Note that several guards and even the spy could occupy a same vertex.

Initially, the spy is placed at some vertex of G. Then, the k guards are placed at some vertices of G. Then, the game proceeds turn-by-turn. At each turn, first the spy may move

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along at most s edges (s is the *speed* of the spy). Then, each guard may move along one edge. The spy wins if, after a finite number of turns (after the guards' move), it reaches a vertex at distance greater than d from every guard. The guards win otherwise, in which case we say that the guards *control* the spy at distance d, i.e. that there is always at least one guard at distance at most d from the spy.

Given a graph G and two integers $d, s \in \mathbb{N}$, $s > 0$, let the *guard-number*, denoted by $gn_{s,d}(G)$, be the minimum number of guards required to control a spy with speed s at distance d, against any strategy from the spy. We also define the following dual notion. Given a graph G and two integers $k, s \in \mathbb{N}$, $s > 0$, $k > 0$, let $d_{s,k}(G)$, be the minimum distance d such that k guards can control a spy with speed s at distance d, whatever be the strategy of the spy.

1.1 Preliminary remarks

We could define the game by placing the guards first. In that case, since the spy could choose its initial vertex at distance greater than d from any guard, we need to slightly modify the rules of the game. If the guards are placed first, they win if, after a finite number of turns, they ensure that the spy always remains at distance at most d from at least one guard. Equivalently, the spy wins if it can reach infinitely often a vertex at distance greater than d from every guard. We show that both versions of the game are closely related. In what follows, we consider the spy-game against a spy with speed s that must be controlled at distance d for any fixed integers $s > 0$ and d.

▶ **Claim 1.** *If the spy wins in the game when it starts first, then it wins in the game when it is placed after the guards.*

Proof of the claim. Assume that the spy has a winning strategy $\mathcal{S}$ when it is placed first. In particular, there is a vertex $v_0 \in V(G)$ such that, starting from v_0 and whatever be the strategy of the guards, the spy can reach a vertex at distance $> d$ from every guard. If the spy is placed after the guards, its strategy first consists in reaching v_0, and then in applying the strategy $\mathcal{S}$ until it is at distance $> d$ from every guard. The spy repeats this process infinitively often. ◀

The converse is not necessary true, however we can prove a slightly weaker result which is actually tight. For this purpose, let us recall the definition of the well known *Cops and robber* game [15, 4]. In this game, first k cops occupy some vertices of the graph. Then, one robber occupies a vertex. Turn-by-turn, each player may move its token (the cops first and then the robber) along an edge. The cops win if one of them reach the same vertex as the robber after a finite number of turns. The robber wins otherwise. The *cop-number* $cn(G)$ of a graph G is the minimum number of cops required to win in G [1].

▶ **Claim 2.** *If k guards win in the game when the spy is placed first in a graph G, then $k + cn(G) - 1$ guards win the game when they are placed first.*

Proof of the claim. Assume that k guards have a winning strategy when the spy is placed first. Such a strategy $\mathcal{S}$ is defined as follows. For any position $v \in V(G)$ of the spy, each guard g_i ($1 \leq i \leq k$) is assigned a vertex $pos(i, v)$, such that, for any vertex $w \in V(G)$ at distance at most s from v and for any $i \leq k$, $pos(i, w) \in N[pos(i, v)]$ where $N[x]$ denote the set of vertices at distance at most one from $x \in V$. Moreover, for any $v \in V(G)$, there exists $i \leq k$ such that the distance between v and $pos(i, v)$ is at most d.

Now, let us assume that $k+cn(G)-1$ guards are placed first. We show that after a finite number of turns, when the spy occupies some vertex v, the vertices $pos(i,v)$ are occupied for all $1 \leq i \leq k$ and then the guards occupying these vertices can follow $\mathcal{S}$ and so win.

Let $0 \leq j < k$ and assume that the vertices $pos(i,v)$ are occupied for all $1 \leq i \leq j$ ($j=0$ means no such vertex is occupied). The guards occupying the vertices $pos(1,v), \cdots, pos(j,v)$ follow the strategy $\mathcal{S}$. There remains $k+cn(G)-1-j \geq cn(G)$ "free" guards. A team of $cn(G)$ of free guards will target the position $pos(j+1,v)$ (which acts as a robber moving at speed one in G). Therefore, after a finite number of steps, one free guard reaches $pos(j+1,v)$ (where v is the position of the spy at this step). Continuing this way, the vertices $pos(i,v)$ are occupied for all $1 \leq i \leq k$ after a finite number of steps which concludes the proof. ◀

The bound of the previous claim is tight. Indeed, for any graph G, $gn_{1,0}(G)=1$ since one guard can be placed at the initial position of the spy and then follows it. On the other hand, if the guards are placed first, the game (for $s=1$ and $d=0$) is equivalent to the classical Cops and robber game and, therefore, $cn(G)$ guards are required.

1.2 Related work

Further relationship with Cops and robber games

The Cops and robber game has been generalized in many ways [3, 8, 2, 5, 9]. In [3], Bonato *et al.* proposed a variant with *radius of capture*. That is , the cops win if one of them reaches a vertex at distance at most d (a fixed integer) from the robber. The version of our game when the guards are placed first and for $s=1$ is equivalent to Cops and robber with radius of capture. Indeed, when the spy is not faster than the guards, capturing the spy (at any distance d) is equivalent to controling it at such distance: once a guard is at distance at most d from the spy, it can always maintain this distance (by following a shortest path toward the spy).

This equivalence is not true anymore as soon as $s>1$. Indeed, one cop is always sufficient to capture one robber in any tree, whatever be the speed of the robber or the radius of capture. On the other hand, we prove below that $\Theta(n)$ cops are necessary to control a spy with speed at least 2 at some distance d in any n-node path. This is mainly due to the fact that, in the spy-game, the spy may cross (or even occupy) a vertex occupied by a guard. Therefore, in what follows, we only consider the case $s \geq 2$.

Note that the Cops and robber games when the robber is faster than the cops is far from being well understood. For instance, the exact number of cops with speed one required to capture a robber with speed two is unknown in 2-dimensional grids [7]. One of our hopes when introducing the Spy-game is that it will lead us to a new approach to tackle this problem.

Generalization of Eternal Domination

A d-dominating set of a graph G is a set $D \subseteq V(G)$ of vertices such that any vertex $v \in V(G)$ is at distance at most d from a vertex in D. Let $\gamma_d(G)$ be the minimum size of a d-dominating set in G. Clearly, $gn_{s,d}(G) \leq \gamma_d(G)$ for any $s,d \in \mathbb{N}$. However these two parameters may differ arbitrary as shown by the following example. Let G be the graph obtained from a cycle C on n-vertices by adding a node x and, for any $v \in C$, adding a path of length $d+1$ between v and x. It is easy to check that $\gamma_d(G)=\Omega(n/d)$ while $gn_{s,d}(G)=2$ (the two guards moving on x and its neighbors).

In the *eternal domination* game [10, 11, 13, 14], a set of k *defenders* occupy some vertices of a graph G. At each turn, an *attacker* chooses a vertex $v \in V$ and the defenders may move to adjacent vertices in such a way that at least one defender is at distance at most d (a fixed predefined value) from v. Several variants of this game exist depending on whether exactly one or more defenders may move at each turn [11, 13, 14]. It is easy to see that the spy-game, when the spy has unbounded speed (equivalently, speed at least the diameter of the graph) is equivalent to the Eternal Domination game when all defenders may move at each turn.

1.3 Our contributions

In this paper, we initiate the study of the spy-game for $s \geq 2$. In Section 2, we study the computational complexity of the problem of deciding the guard-number of a graph. We prove that computing $gn_{3,1}(G)$ is NP-hard in the class of graph G with diameter at most 5. Then, we show the problem is PSPACE-complete in the case of DAGs (where guards and spy have to follow the orientation of arcs, but distances are in the underlying graph). Then, we consider particular graph classes. In Section 3, we precisely characterize the cases of paths and cycles. Precisely, for any $k \geq 1$, $s \geq 2$, we prove that

$$\left\lfloor \frac{n(s-1)}{2ks} \right\rfloor \leq d_{s,k}(P_n) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil$$

for any path P_n on n vertices, and

$$\left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor \leq d_{s,k}(C_n) \leq \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor$$

for any cycle C_n on n vertices. Our most interesting result concerns the case of grids. In Section 4, we prove that there exists $\beta > 0$ such that $gn_{s,d}(G_{n\times n}) = \Omega(n^{1+\beta})$ in any $n \times n$ grid $G_{n\times n}$. For this purpose, we actually prove a lower bound on the number of guards required in a *fractional relaxation* of the game (the formal definition is given in the corresponding section).

Notations

As usual, we consider connected simple graphs. Given a graph $G = (V, E)$ and $v \in V$, let $N(v) = \{w \mid vw \in E\}$ denote the set of neighbors of v and let $N[v] = N(v) \cup \{v\}$.

2 Complexity

2.1 NP-hardness

▶ **Theorem 3.** *Given a graph G with diameter at most 5 and an integer k as inputs, deciding whether $gn_{3,1}(G) \leq k$ is NP-hard.*

Proof. The result is obtained by reducing the classical Set Cover Problem. In the Set Cover Problem the input is a set of elements $\mathcal{U}$, a family $\mathcal{S}$ of subsets of $\mathcal{U}$ such that $\cup_{S\in\mathcal{S}} S = \mathcal{U}$ and an integer k. The question is whether there exists a set $C \subseteq \mathcal{S}$ such that $|C| \leq k$ and $\cup_{S\in C} S = \mathcal{U}$, the set C is called a cover of $\mathcal{U}$.

Let $(\mathcal{U} = \{u_1, \ldots, u_n\}, \mathcal{S} = \{S_1, \ldots, S_m\}, k)$ be an instance of the Set Cover Problem. Note that, for any $u_i \in \mathcal{U}$, there exists $S_j \in \mathcal{S}$ such that $u_i \in S_j$ (since $\cup_{S\in\mathcal{S}} S = \mathcal{U}$). We create a graph G such that there is a cover $C \subseteq \mathcal{S}$ of $\mathcal{U}$ with size at most k if and only if $g_1^3(G) \leq k$.

The graph G is constructed in the following way. Abusing the notation, let us identify the elements in $\mathcal{U} \cup \mathcal{S}$ with some vertices of G. Let $V(G) = \mathcal{S} \cup \mathcal{U} \cup \mathcal{V}$ with $\mathcal{V} = \{v_1, \cdots, v_n\}$. Start with a complete graph with set of vertices $\mathcal{S} = \{S_1, \cdots, S_m\}$ and, for any $1 \leq i \leq n$, add an edge $\{u_i, v_i\}$. Finally, for every $u_i \in \mathcal{U}$ and $S_j \in \mathcal{S}$ such that $u_i \in S_j$, let us add an edge $\{u_i, S_j\}$.

First, let us prove that, if $\mathcal{U}$ admits a cover C of size at most k, then $g_1^3(G) \leq k$. For this purpose, we give a strategy for the guards that ensure that the spy is always at distance at most 1 from at least one guard. When the spy occupies a vertex in $\mathcal{S} \cup \mathcal{U}$, the guards occupy all the vertices of C. When the spy occupies a vertex v_i for some $i \leq n$, let $j(i)$ be such that $u_i \in S_{j(i)} \in C$, then one guard occupies u_i and the other guards occupy the vertices of $C \setminus \{S_{j(i)}\}$. Because the speed of the spy is 3, from a vertex v_i, the spy can only reach a vertex in $\mathcal{S} \cup \mathcal{U}$. Therefore, whatever be the initial position of the spy and its moves, the guards can always ensure the previously defined positions.

Suppose now that there is no cover C of $\mathcal{U}$ with size k, we show that $g_1^3(G) > k$. Let us assume at most k guards are occupying vertices in G, let us consider the following strategy for the spy. The spy starts at S_1. If there exists $i \leq n$ such that no guards dominate u_i, i.e., no guards occupy a vertex of $N[u_i]$, the spy goes at v_i (note that any vertex in $\{v_1, \cdots, v_n\}$ is at distance at most 3 from S_1). Then, no guard can reach a vertex at distance at most 1 from v_i (since u_i is the only neighbor of v_i) and the spy wins.

Let us show that such a vertex u_i exists by reverse induction on the number ℓ of guards occupying vertices in $\{S_1, \cdots, S_m\}$. That is, let $\mathcal{O}$ be the set of vertices occupied by the guards (note that $|\mathcal{O}| = k$) and let $\ell = |\mathcal{O} \cap \mathcal{S}|$. We show that there exists $i \leq n$ such that $\mathcal{O} \cap N[u_i] = \emptyset$. If $\ell = k$, i.e., $\mathcal{O} \subseteq \mathcal{S}$, then the result holds since there is no cover of $\mathcal{U}$ of size at most k. If $\ell < k$, there exists $j \leq n$ such that a guard is occupying u_j or v_j, i.e., there exists $x \in \{u_j, v_j\}$ such that $x \in \mathcal{O}$. Let $z \leq m$ such that $u_j \in S_z$ and let $\mathcal{O}' = \mathcal{O} \cup \{S_z\} \setminus \{x\}$. By induction and because $|\mathcal{O}' \cap \mathcal{S}| = \ell + 1$, there exists $i \leq n$ such that $\mathcal{O}' \cap N[u_i] = \emptyset$. Since $\mathcal{O} \cap N[u_p] \subseteq \mathcal{O}' \cap N[u_p]$ for any $p \leq n$, the result follows. ◀

Note that the previous proof could be easily adapted for a speed $s > 2$ and distance $d = s - 2$ simply adjusting the size of the paths to $s - 1$. The question to generalize this result to any s and d is open. Moreover, since the set cover problem is not approximable within a factor of $(1 - o(1)) \ln n$ [6], our proof also implies the same result to the spy game.

2.2 PSPACE-hardness in the directed case

Then, we consider a variant of our game played on digraphs. In this variant, both the guards and the spy can move only by following the orientation of the arcs. However, the distances are the ones of the underlying undirected graph.

▶ **Theorem 4.** *The problem of computing $gn_{s,2}$ is PSPACE-hard in the class of DAGs, when the guards are placed first.*

The result is obtained by reducing the PSPACE-complete Quantified Boolean Formula in Conjunctive Normal Form (QBF) problem. Given a set of boolean variables $x_1, \ldots, x_n$ and a boolean formula $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where C_j is a disjunction of literals, the QBF problem asks whether the expression $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n F$ is true, where every Q_i is either $\forall$ or $\exists$.

Proof. For ease of readability, the proof below is given for $d = 2$ but can easily be adapted for any distance d.

Let ϕ be quantified boolean formula with n boolean variables. We construct a DAG D_ϕ such that ϕ is true if and only if n guards control a spy at distance 2 in D_ϕ after a finite number of turns.

For each $Q_i x_i$ of ϕ we construct a gadget digraph D_i. If $Q_i = \exists$ then $V(D_i) = \{w_{i-1}, z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \overline{x}_i, \overline{x}_i^*, y_i, v_i, v_i', w_i\}$, the arcs between the vertices are shown in Figure 1a. If $Q_i = \forall$ then $V(D_i) = \{w_{i-1}, z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \overline{x}_i, \overline{x}_i^*, y_i, \overline{y}_i, v_i, \overline{v}_i, v_i', w_i\}$ the arcs between the vertices are shown in Figure 1b.

Observe that the vertex w_i appears in both D_i and D_{i+1}. It remains to establish a relationship between each clause and the variables it contains. For each clause C_i we create a vertex c_i in D_ϕ and add an arc from w_n to c_i. We also add an arc from c_i to $x_i(\overline{x}_i)$ if clause C_i contains the literal $x_i(\overline{x}_i)$.

An example of the digraph D_ϕ for $\phi = \exists x_1 \forall x_2 (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2)$ is shown on Figure 1c.

It remains to prove that ϕ is true if and only if $\vec{g}_2(D_\phi) = n$.

First note that, for each gadget D_i, at least one guard have to pick a vertex from $S_i = \{z_i^1, z_i^2, z_i^3\}$ as his initial position, otherwise the spy would pick z_i^1 as his initial position and no guard could ever reach distance 2 from such vertex, therefore the spy would win. We will refer to the guard initially in S_i as p_i. Since D_ϕ has n such gadgets, then $\vec{g}_2(D_\phi) \geq n$. Furthermore, assuming that each guard p_i starts on z_i^1 he can only occupy the vertices on the set $R_i = \{z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \overline{x}_i, \overline{x}_i^*\}$ during the rest of the game.

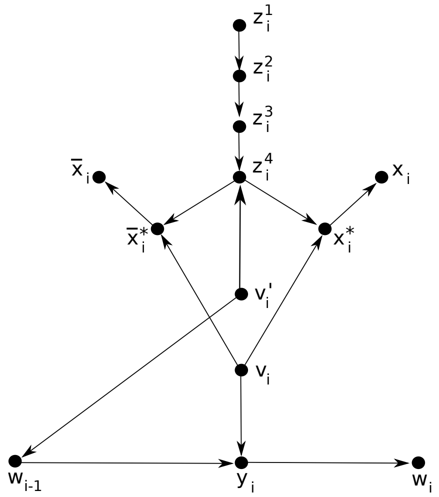
Suppose that $\phi = false$. We describe a winning strategy for the spy playing against n guards. Lets assume that there is exactly one guard in each set S_i, that is, the spy cannot win just initially positioning himself in one unprotected z_i^1. The spy starts on the vertex w_0.

Now, suppose that the spy is in some w_{i-1} of $D_i(\forall)$, then the only guard that can reach a vertex at distance at most 2 from w_{i-1} is p_i when he occupies the vertex z_i^4. The spy waits until the guard p_i moves to z_i^4, if the guard never do so the spy stays on w_{i-1} and wins the game. Therefore suppose that p_i eventually moves to z_i^4, then the spy chooses between moving to y_i or $\overline{y}_i$, depending the choice of the spy the guard p_i is then forced to move to x_i^* or to $\overline{x}_i^*$, because these are the only vertices that are reachable for any guard that are at distance at most 2 from y_i and $\overline{y}_i$ respectively. If p_i moves to x_i^* the corresponding variable x_i is set to $true$. Otherwise, if p_i moves to $\overline{x}_i^*$ then $x_i = false$. It means that for a quantified variable $\forall x_i$ the spy chooses the value of x_i.

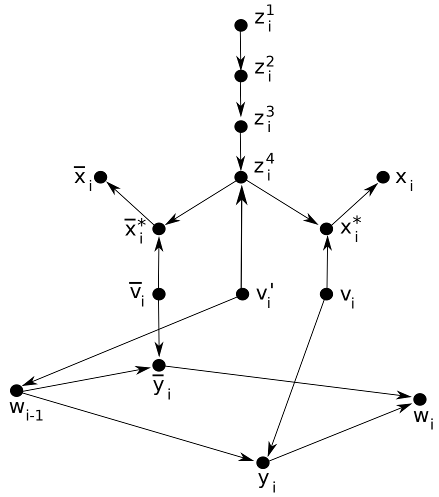
If the spy is in some w_{i-1} of $D_i(\exists)$, again, the only guard that can reach a vertex at distance at most 2 from the spy is p_i when he occupies the vertex z_i^4. The spy then waits until the guard p_i moves to z_i^4 and then moves to y_i, this time p_i is not forced to move to specifically x_i^* or to $\overline{x}_i^*$, but he still must choose one of them. Again, if p_i moves to x_i^* the corresponding variable x_i is set to $true$, otherwise, if p_i moves to $\overline{x}_i^*$ then $x_i = false$. It means that for a quantified variable $\exists x_i$ the guards choose the value of x_i.

When p_n moves to x_n^* or $\overline{x}_n^*$ each guard is on $x_i^*(\overline{x}_i^*)$ or $x_i(\overline{x}_i)$. Observe that each guard can only reach a safe distance from the vertices c_j corresponding to the clauses that contains the literal he set true. Since $\phi = false$ then the spy can choose between y_i and $\overline{y}_i$ on gadgets $D_i(\forall)$ in such a way that no matter how the guards choose x_i^* or $\overline{x}_i^*$ on gadgets $D_i(\exists)$ there is at least one vertex c_j that cannot be protected by any guard. Then the spy moves to such vertex, stays there and wins the game.

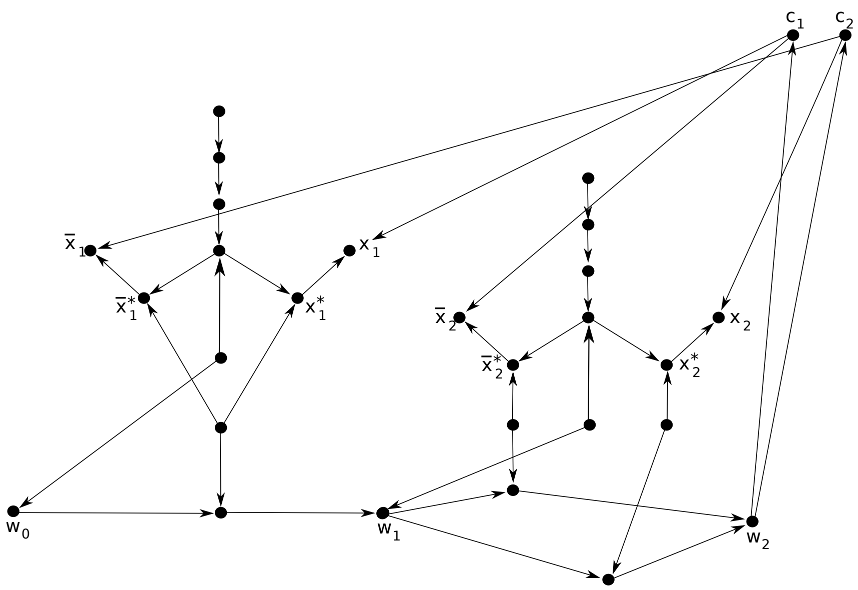
Suppose that $\phi = true$. Each guard p_i, $i = 1, ..., n$, will choose z_i^3 as his initial position. If the spy choose as his initial position $z_i^1, z_i^2, z_i^3, z_i^4, x_i^*$ or $\overline{x}_i^*$ the guard p_i do not need to move since the spy is at distance at most 2 from z_i^3. The only vertices that the spy can go from these initial positions that are not under the protection of p_i are x_i or $\overline{x}_i$. If he goes to any of them the guard p_i just moves to z_i^4. Since the spy cannot move anymore and is at



(a) gadget $D_i(\exists)$ for existential quantifier.



(b) Gadget $D_i(\forall)$ for universal quantifier.



(c) Example of the graph D_ϕ for the formula $\phi = \exists x_1 \forall x_2 (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2)$.

Figure 1

distance at most 2 from a guard, the guards win the game. If the spy starts on some $v_i, \overline{v}_i$ or v_i' then p_i moves to z_i^4, after that, if the spy goes to $x_i^*, \overline{x}_i^*$ or z_i^4 then p_i follows the same strategy from above. Therefore the spy, independent of his initial position, must eventually move to a vertex $w_i, y_i, \overline{y}_i$ or some clause vertex c_j, otherwise he loses.

Suppose that the spy is in some vertex w_{i-1} of $D_i(\forall)$ then the guard p_i moves to z_i^4 and prevents the spy from communicating. The spy must move to y_i or $\overline{y}_i$ forcing p_i to move to x_i^* or $\overline{x}_i^*$ accordingly. Again, for a quantified variable $\forall x_i$ the spy chooses the value of x_i. After the spy moves from $y_i(\overline{y}_i)$ the cop moves to $x_i(\overline{x}_i)$ and stays there forever.

Similarly, if the spy is in some vertex w_{i-1} of $D_i(\exists)$ then the guard p_i moves to z_i^4 and prevents the spy from communicating. The spy must move to y_i, this time p_i is not forced to move to specifically x_i^* or to $\overline{x}_i^*$, but he still must choose one of them. Therefore, for a quantified variable $\exists x_i$ the guards choose the value of x_i. After the spy moves from y_i the cop moves to x_i or $\overline{x}_i$ depending of his previous movement and stays on that vertex forever.

Observe that after the spy moves from y_n or $\overline{y}_n$ every guard is at distance 2 from w_n at distance 1 from each clause vertex that contains the literal he chose to set true and at distance 2 from each of the other literals of these clauses. Since $\phi = true$ then the guards can choose between y_i and $\overline{y}_i$ on gadgets $D_i(\exists)$ in such a way that no matter how the spy chooses x_i^* or $\overline{x}_i^*$ on gadgets $D_i(\forall)$ all clause vertices are at distance 1 from at least one guard. Therefore the only vertices reachable for the spy are at distance at most 2 from the guards. ◀

The question of the complexity of the spy game in undirected graphs is left open. Is it PSPACE-hard, or more probably EXPTIME-complete as Cops and Robber games [12]? The question of parameterized complexity is also open.

3 Case of paths and rings

In this section, we characterize optimal strategies in the case of two simple topologies: the path and the ring. For ease of readability, some proofs are given in the case $s = 2$. The general proofs (for any $s \geq 2$) are similar.

3.1 Paths

The following theorem directly follows from Lemmas 6 and 7.

▶ **Theorem 5.** *For any path P with $n+1$ nodes and for any $k \geq 1$ and $s \geq 2$,*

$$\left\lfloor \frac{n(s-1)}{2ks} \right\rfloor \leq d_{s,k}(P_n) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil .$$

▶ **Lemma 6.** *For any path P with $n+1$ nodes and for any $k \geq 1$ and $s \geq 2$,*

$$d_{s,k}(P) \geq \left\lfloor \frac{n(s-1)}{2ks} \right\rfloor .$$

Proof. For ease of readability, we prove the lemma in the case $\frac{2d-1}{s-1} \in \mathbb{N}$.

Let $P = (v_0, v_1, \cdots, v_n)$. Let $d = \lfloor \frac{n(s-1)}{2ks} \rfloor$. We show that a spy with speed s playing against at most k guards can reach a vertex at distance at least d from any guard. Intuitively, the strategy of the spy simply consists in starting from one end of P and running at full speed toward the other end. We show that there must be a turn when the spy is at distance at least d from every guard and therefore $d_{s,k}(P) \geq d$.

More formally, let the strategy for the spy be the following. Initially, the spy is occupying an end of the path, say vertex v_0. Then, at each turn $i \geq 1$, the spy moves from $v_{i(s-1)}$ to v_{is}.

We prove by induction on $1 \leq i \leq k$, after turn $i\frac{2d-1}{s-1}$ (when the spy occupies $v_{si\frac{2d-1}{s-1}}$), either at least i guards are occupying vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}-d}\}$, or there is turn $0 \leq j < i\frac{2d-1}{s-1}$ such that, after Turn j, the distance between the spy and all guards was at least d.

Initially, there must be at least one guard, call g_1, occupying some vertex in $\{v_0, \cdots, v_{d-1}\}$ because otherwise all guards are at distance at least d from the spy at Turn 0. Therefore, after Turn $\frac{2d-1}{s-1}$, Guard g_1 is occupying a vertex in $\{v_0, \cdots, v_{\frac{2d-1}{s-1}+d-1}\} = \{v_0, \cdots, v_{s\frac{2d-1}{s-1}-d}\}$ and the spy is occupying $v_{s\frac{2d-1}{s-1}}$. Hence, the induction hypothesis holds for $i = 1$. Note that the spy is at distance at least d from g_1.

Let $1 \leq i \leq k$ and let us assume by induction that, after Turn $i\frac{2d-1}{s-1}$, there are at least i guards occupying vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}-d}\}$. Moreover, by definition of the spy's strategy, the spy is occupying $v_{si\frac{2d-1}{s-1}}$. Note that, all these i guards are at distance at least d from the spy.

Then, after Turn $i\frac{2d-1}{s-1}$, there must be at least one guard, call it g_{i+1}, occupying some vertex in $\{v_{si\frac{2d-1}{s-1}-d+1}, \cdots, v_{si\frac{2d-1}{s-1}+d-1}\}$ because otherwise all guards are at distance at least d from the spy at Turn i. Therefore, after Turn $(i+1)\frac{2d-1}{s-1}$, Guard g_{i+1} is occupying a vertex in $\{v_0, \cdots, v_{(si+1)\frac{2d-1}{s-1}+d-1}\}$, that is in $\{v_0, \cdots, v_{s(i+1)\frac{2d-1}{s-1}-d}\}$, and the spy is occupying $v_{(i+1)s\frac{2d-1}{s-1}}$. Similarly, all the i guards that were occupying some vertices in $\{v_0, \cdots, v_{si\frac{2d-1}{s-1}}\}$ after Turn $i\frac{2d-1}{s-1}$ must occupy vertices in $\{v_0, \cdots, v_{s(i+1)\frac{2d-1}{s-1}-d}\}$ after Turn $(i+1)\frac{2d-1}{s-1}$. Hence, the induction hypothesis holds for $i + 1$.

Therefore, after Turn $k\frac{2d-1}{s-1}$, either there has been a previous turn when the spy was at distance at least d from all guards, or all the k guards are occupying vertices in $\{v_0, \cdots, v_{sk\frac{2d-1}{s-1}-d}\}$ while the spy occupies $v_{ks\frac{2d-1}{s-1}}$ (note that this vertex exists since $ks\frac{2d-1}{s-1} \leq n$ by definition of d). In the latter case, the spy is at distance at least d from all guards at this turn. ◀

▶ **Lemma 7.** *For any path P with $n + 1$ nodes and any $k \geq 1$, $s \geq 2$,*

$$d_{s,k}(P) \leq \left\lceil \frac{(n+1)(s-1)}{2ks} \right\rceil.$$

Proof. For ease of readability, we prove the lemma for $s = 2$.

It is clearly sufficient to prove the result in the case $d = \frac{n+1}{4k} \in \mathbb{N}$. Let $P = (v_0, \cdots, v_n)$ and, for any $1 \leq i \leq k$, let $P_i = (v_{4(i-1)d}, \cdots, v_{4di})$.

We design a strategy ensuring that k guards may maintain the spy at distance at most d from at least one guard. The i^{th} guard is assigned to the subpath P_i (it moves only in P_i). Moreover, a guard i will move at some turn only if the move of the spy at this turn is along an edge of P_i (note that the subpaths P_i are edge-disjoint).

Let $i \leq k$ be such that the spy occupies the node $x = v_{(4i-2)d+\ell}$ with $-2d \leq \ell \leq 2d$. That is, $x \in P_i$. Let us assume that

- for any $1 \leq j < i$, the j^{th} guard occupies $v_{(4j-1)d}$;
- for any $i < j \leq k$, the j^{th} guard occupies $v_{(4j-3)d}$;
- the i^{th} guard occupies $v_{(4i-2)d+\lfloor \ell/2 \rfloor}$ if $\ell \geq 0$ and $v_{(4i-2)d+\lceil \ell/2 \rceil}$ if $\ell \leq 0$.

Clearly, if these conditions are satisfied, the spy is at distance at most $\lceil |\ell|/2 \rceil \leq d$ from the i^{th} guard. Moreover, such positions can be chosen by the guards once the spy has chosen its initial position.

We next show that, whatever be the move of the spy, we can maintain these conditions. Let y be the next vertex to be occupied by the spy. Note that $y = v_{(4i-2)d+\ell+a}$ with $a \in \{-2,-1,0,+1,+2\}$.

We start with the case when x and y are not in the same subpath P_i. It may happen in only two cases: either $x = v_{4id-1}$ and $y = v_{4id+1}$ ($\ell = 2d-1$ and $a = +2$) or $x = v_{4(i-1)d+1}$ and $y = v_{4(i-1)d-1}$ ($\ell = -2d+1$ and $a = -2$). In the first case, the i^{th} guard goes from $v_{(4i-1)d-1}$ to $v_{(4i-1)d}$ and the $(i+1)^{th}$ guard goes from $v_{(4(i+1)-3)d} = v_{(4i+1)d}$ to $v_{(4i+1)d+1}$. In the latter case, the i^{th} guard goes from $v_{(4i-3)d+1}$ to $v_{(4i-3)d}$ and the $(i-1)^{th}$ guard goes from $v_{(4(i-1)-1)d}$ to $v_{(4(i-1)-1)d-1}$. In both cases, the conditions remain valid.

From now on, let us assume that x and y belong to P_i. In that case, only the i^{th} guard may move. There are several cases depending on the value of $a \in \{-2,-1,0,+1,+2\}$ and ℓ,

- if $\ell \geq 0$ and $\ell + a \geq 0$, then
$v_{(4i-2)d+\lfloor(\ell+a)/2\rfloor} \in \{v_{(4i-2)d+\lfloor \ell/2\rfloor-1}, v_{(4i-2)d+\lfloor \ell/2\rfloor}; v_{(4i-2)d+\lfloor \ell/2\rfloor+1}\}$.
Hence, whatever be the move of the spy, the i^{th} guard can go from $v_{(4i-2)d+\lfloor \ell/2\rfloor}$ to $v_{(4i-2)d+\lfloor(\ell+a)/2\rfloor}$ either moving to one of its neighbor or staying idle.
- if $\ell \leq 0$ and $\ell + a \leq 0$ then
$v_{(4i-2)d+\lceil(\ell+a)/2\rceil} \in \{v_{(4i-2)d+\lceil \ell/2\rceil-1}, v_{(4i-2)d+\lceil \ell/2\rceil}; v_{(4i-2)d+\lceil \ell/2\rceil+1}\}$.
Hence, whatever be the move of the spy, the i^{th} guard can go from $v_{(4i-2)d+\lceil \ell/2\rceil}$ to $v_{(4i-2)d+\lceil(\ell+a)/2\rceil}$ either moving to one of its neighbor or staying idle.
- finally, if $\ell * (\ell + a) < 0$, then $(\ell, a) = (-1, 2)$ or $(\ell, a) = (1, -2)$. In that case, the i^{th} guard remains on $v_{(4i-2)d}$.

In all cases, all properties are satisfied after the move of the guards. ◀

3.2 Cycles

We then consider the case of cycles. The following theorem directly follows from Lemmas 9 and 10.

▶ **Theorem 8.** *For any cycle C with $n+1$ nodes and any $k \geq 1$,*

$$\left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor \leq d_{s,k}(C_n) \leq \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor.$$

▶ **Lemma 9.** *For any cycle C with $n+1$ nodes and any $k \geq 1$, $s \geq 2$,*

$$d_{s,k}(C) \geq \left\lfloor \frac{(n-1)(s-1)}{k(2s+2)-4} \right\rfloor.$$

Proof. Again, the proof is given in the case $s = 2$ for ease of readability.

Let $C = (v_0, v_1, \cdots, v_n)$. Let $d = \lfloor \frac{n-1}{6k-4} \rfloor$. Let the strategy for the spy be the following. Initially, the spy is occupying v_0 and one guard, denoted by g_0, occupies v_{-d} or v_{-d-1} or v_{-d-2} after the guard's turn (the indices of the vertices must be understood modulo $n+1$). Note that such intial position can always be achieved (up to renaming the nodes): the spy goes at distance $d+1$ from the guard g_0 and after the guards' turn, g_0 is at distance $d, d+1$ or $d+2$ from the spy. Then, at each turn $i \geq 1$, the spy moves from v_{2i-2} to v_{2i}.

We prove by induction on $1 \leq i < k$, after Turn $2id$, either at least $i+1$ guards are occupying vertices in $\{v_{-d-2id-2}, \cdots, v_{(4i-1)d-1}\}$, or there is turn $0 \leq j \leq i$ such that, after Turn j, the distance between the spy and all guards was at least d.

Initially, some vertex in $\{v_{-d+1}, \cdots, v_{d-1}\}$ must be occupied by at least one guard, call it g_1, because otherwise the spy is at distance at least d from each guard. Note that g_0 and g_1 are different guards.

Therefore, after Turn $2d$, some vertices in $\{v_{-3d-2}, \cdots, v_{3d-1}\}$ are occupied by Guards g_0 and g_1, and the spy is occupying v_{4d}. Hence, the induction hypothesis holds for $i = 1$.

Let $1 \leq i < k-1$ and let us assume by induction that, after Turn $2id$, there are at least $i+1$ guards occupying vertices in $\{v_{-d-2id-2}, \cdots, v_{(4i-1)d-1}\}$. Moreover, by definition of the spy's strategy, the spy is occupying v_{4id}.

Then, after Turn $2id$, there must be at least one guard, call it g_{i+1}, occupying some vertex in $\{v_{(4i-1)d+1}, \cdots, v_{(4i+1)d-1}\}$ because otherwise all guards are at distance at least d from the spy at Turn i. Therefore, after Turn $2(i+1)d$, Guard g_{i+1} is occupying a vertex in $\{v_{(4i-3)*d+1}, \cdots, v_{(4i+3)d-1}\}$ and the spy is occupying $v_{4(i+1)d}$. Similarly, all the $i+1$ guards that were occupying some vertices in $\{v_{-d-2id-2}, \cdots, v_{(4i-1)d-1}\}$ after Turn $2id$ can only occupy vertices in $\{v_{-d-2(i+1)d-2}, \cdots, v_{(4i+1)d-1}\}$ after Turn $2(i+1)d$. Hence, the induction hypothesis holds for $i+1$: the guards $g_0, \cdots, g_{i+1}$ are occupying nodes in $\{v_{-d-2(i+1)d-2}, \cdots, v_{(4i+3)d-1}\}$.

Therefore, after Turn $2(k-1)d$, either there has been a previous turn when the spy was at distance at least d from all guards, or all the k guards are occupying vertices in $\{v_{-d-2(k-1)d-2}, \cdots, v_{(4k-5)d-1}\}$ while the spy occupies $v_{4(k-1)d}$.

In the latter case, if $v_{-d-2(k-1)d-2}$ is at distance at least d from $v_{4(k-1)d}$ and $v_{4(j-1)d} \notin \{v_{-d-2(k-1)d-2}, \cdots, v_{(4k-5)d-1}\}$ (in other words, if $4(k-1)d + d \leq -d - 2(k-1)d - 2 \mod (n+1)$), then the spy is at distance at least d from all guards at this turn. This is actually the case since $(6k-4)d < n$. ◀

▶ **Lemma 10.** *For any cycle C with $n+1$ nodes and any $k \geq 1$ and $s \geq 2$,*

$$d_{s,k}(C) \leq \left\lfloor \frac{(n+1)(s-1)}{k(2s+2)-4} \right\rfloor.$$

Proof. Again, the proof is given in the case $s = 2$.

It is clearly sufficient to prove the result in the case $d = \frac{n+1}{6k-4} \in \mathbb{N}$. Let $C = (v_0, \cdots, v_n)$. Note that, the indices of the vertices must be understood modulo $n+1$. We design a strategy ensuring that k guards may maintain the spy at distance at most d from at least one guard (note that, in the following strategy, the guard g_1 is at distance $\ell \leq d$ from the spy).

Initially, the spy is in v_h for some $0 \leq h \leq n$. We want to maintain the property that there exists $0 \leq \ell \leq d$ such that the configuration is the following. A guard g_1 is in $v_{\ell+h}$, a guard g_2 is in $v_{4d+3\ell+h}$, and a guard v_{-1} is in $v_{-4d+3\ell+h}$. Then, for any $3 \leq i \leq k-1$, a guard g_i is in $v_{4d+3\ell+6d(i-2)+h} = v_{6di+3\ell-8d+h}$. Note that, for $3 \leq i \leq k-1$, the guard g_i is at distance $6d$ from the guard g_{i-1}, and the guard g_{k-1} is at distance $6d$ from g_{-1}. We show how to maintain such a configuration whatever be the move of the spy.

Obviously, if the spy does not move, no guards move and we are done. If the spy moves along one edge clockwise (resp. anti-clockwise), all guards do the same move and the configuration is maintained. Hence, we only have to consider the cases when the spy moves along 2 edges.

Roughly, in each remaining case, the guard g_1 executes the same move as the spy, and all other guards do the opposite move.

- Case when the spy moves to v_{h+2} (i.e., clockwise) and $\ell \geq 1$. Then, g_1 moves clockwise and all other guards move anti-clockwise. We show that the properties hold for $0 \leq \ell' = \ell - 1 \leq d$ and $h' = h + 2 \mod n+1$. Indeed, g_1 moves from $v_{\ell+h}$ to $v_{\ell+h+1} = v_{\ell'+h'}$. The guard g_2 moves from $v_{4d+3\ell+h}$ to $v_{4d+3\ell+h-1} = v_{4d+3\ell'+h'}$. The guard g_{-1} moves

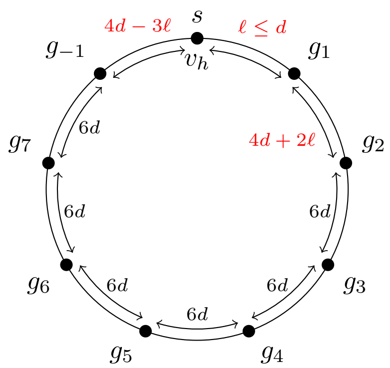


Figure 2 General position in the case $k = 8$, $s = 2$.

from $v_{-4d+3\ell+h}$ to $v_{-4d+3\ell+h-1} = v_{-4d+3\ell'+h'}$. Finally, for any $3 \leq i \leq k-1$, the guard g_i moves from $v_{6di+3\ell-8d+h}$ to $v_{6di+3\ell-8d+h-1} = v_{6di+3\ell'-8d+h'}$. Hence, the property is still valid after the guards' turn.

- Case when the spy moves to v_{h-2} (i.e., anti-clockwise) and $\ell \leq d-1$. Then, g_1 moves anti-clockwise and all other guards move clockwise. Similarly as the previous item, it can be checked that the property holds for $0 \leq \ell' = \ell + 1 \leq d$ and $h' = h - 2 \mod n+1$.
- Case $\ell = 0$. Let us assume that the spy goes anti-clockwise from v_h to v_{h-2} (the case when it goes to v_{h+2} is symmetric). Then, g_1 goes anti-clockwise to v_{-1}, and all other guards go clockwise. Similarly as the previous items, it can be checked that the property holds for $\ell' = 1$ and $h' = h - 2$.
- Case $\ell = d$. Let us assume that the spy goes clockwise from v_h to v_{h+2} (the case when it goes to v_{h-2} is symmetric, the guard g_{-1} playing the role of the guard g_1). Then, g_1 goes clockwise to v_{h+d+1}, and all other guards go anti-clockwise. Similarly as the previous items, it can be checked that the property holds for $\ell' = d - 1$ and $h' = h + 2$. ◀

4 Case of Grids

It is clear that, for any $n \times n$ grid G, $gn_{s,d}(G) = O(n^2)$. However, the exact order of magnitude of $gn_{s,d}(G)$ is not known. In this section, we prove that there exists $\beta > 0$, such that $\Omega(n^{1+\beta})$ guards are necessary to win against one spy in an $n \times n$-grid. Our lower bound actually holds for a relaxation of the game that we now define.

Fractional relaxation

In the *fractional relaxation* of the game, each guard can be *split* at any time, i.e., the guards are not required to be integral entities at any time but can be "fractions" of guards. More formally, let us assume that some *amount* $\alpha \in \mathbb{R}^+$ of guards occupies some vertex v at some step t, and let $N(v) = \{v_1, \cdots, v_{deg(v)}\}$. Then, at their turn, the guards can choose any $deg(v)+1$ non-negative reals $\alpha_0, \cdots, \alpha_{deg(v)} \in \mathbb{R}^+$ such that $\sum_i \alpha_i = \alpha$, and move an amount α_i of guards toward v_i, for any $0 \leq i \leq deg(v)$ (where $v = v_0$). Then, the guards must ensure that, at any step, the sum of the amount of guards occupying the nodes at distance at most d from the spy is at least one. That is, let $c_t(v) \in \mathbb{R}^+$ be the amount of guards occupying vertex v at step t. The guards wins if, for any step t, $\sum_{v \in B(R_t,d)} c_t(v) \geq 1$, where $B(R_t, d)$ denotes the ball of radius d centered into the position R_t of the spy at step t.

Let $g^{frac}_{s,d}(G)$ be the infimum total amount of guards (i.e., $\sum_{v\in V} c_0(v)$) required to win the fractional game at distance d and against a spy with speed s. Since any *integral strategy* (i.e. when guards cannot be split) is a fractional strategy, we get:

▶ **Proposition 11.** *For any graph G and any integers d, s, $g^{frac}_{s,d}(G) \le gn_{s,d}(G)$.*

Conversely, a fractional strategy can be represented to some extent by a variation of an integral strategy. Let G be a graph and d, s be two integers. Let also t, k be any two integers. In what follows, t and k will be arbitrary large and can be some function of n, the number of vertices of G. Let $g^{k,t}_{s,d}(G)$ be the minimum number of (integral) guards necessary to maintain at least k guards at distance $\le d$ from a spy with speed s in G, during t turns. The next lemma will be used below to give a lower bound on $g^{frac}_{s,d}$.

▶ **Lemma 12.** *Let G be a graph with n vertices and $d, s, t, k \in \mathbb{N}$ (t and k may be given by any function of n). Then,*

$$g^{k,t}_{s,d}(G) \le k g^{frac}_{s,d}(G) + tn^2 \,.$$

Asymptotically, this yields a useful bound on $g^{frac}_{s,d}$: $\limsup_{k\to\infty} \frac{g^{k,t}_{s,d}(G)}{k} \le g^{frac}_{s,d}(G)$.

Proof. From a fractional strategy using an amount c of guards, we produce an integer strategy keeping $\ge k$ guards around the spy. Initially, each vertex which has an amount x of guards receives $\lfloor xk \rfloor + tn$ guards, for total number of $\le ck + tn^2$ guards.

We then ensure that, at step $i \in \{1, ..., t\}$, a vertex having an amount of x guards in the fractional strategy has $\ge xk + (t-i)n$ guards in the integer strategy. To this aim, whenever an amount x_{uv} of guards is to be transferred from u to v in the fractional strategy, we move $\lfloor x_{uv}k \rfloor + 1$ in the integer strategy.

As our invariant is preserved throughout the t steps, the spy which had an amount of ≥ 1 guards within distance d in the fractional strategy now has $\ge k$ guards around it, which proves the result. ◀

In what follows, we prove that $g^{frac}_{s,d}(G) = \Omega(n^{1+\beta})$ for some $\beta > 0$ in any $n \times n$-grid G. The next lemma is a key argument for this purpose.

▶ **Lemma 13.** *Let $G = (V, E)$ be a graph and $d, s \in \mathbb{N}$ ($s \ge 2$), with $g^{frac}_{s,d}(G) > c \in \mathbb{Q}^*$ and the spy wins in at most t steps against c guards starting from $v \in V(G)$. For any strategy using a total amount $k > 0$ of guards, there exists a strategy for the spy (with speed $\le s$) starting from $v \in V(G)$ such that after at most t steps, the amount of guards at distance at most d from the spy is less than k/c.*

Proof. For purpose of contradiction, assume that there is a strategy $\mathcal{S}$ using $k > 0$ guards that contradicts the lemma. Then consider the strategy $\mathcal{S}'$ obtained from $\mathcal{S}$ by multiplying the number of guards by c/k. That is, if $v \in V$ is initially occupied by $q > 0$ guards in $\mathcal{S}$, then $\mathcal{S}'$ places qc/k guards at v initially (note that $\mathcal{S}'$ uses a total amount of kc/k=c guards). Then, when $\mathcal{S}$ moves an amount q of guards along an edge $e \in E$, $\mathcal{S}'$ moves qc/k guards along e. Since $\mathcal{S}$ contradicts the lemma, at any step $\le t$, at least an amount k/c of guards is at distance at most d from the spy, whatever be the strategy of the spy. Therefore, $\mathcal{S}'$ ensures that an amount of at least 1 cop is at distance at most d from the spy during at least t steps. This contradicts that $g^{frac}_{s,d}(G) > c$ and that the spy wins after at most t steps. ◀

While it holds for any graph and its proof is very simple, we have not been able to prove a similar lemma in the classical (i.e., non-fractional) case.

The main technical lemma is the following. To prove it, we actually prove Lemma 18 which gives a lower bound on $g^{k,t}_{s,d}(G)$ in any grid G (this technical lemma is postponed at the end of the section). Then, it is sufficient to apply Lemmas 12 and 18 to obtain the following result.

▶ **Lemma 14.** *Let G be a $n \times n$-grid and $a \in \mathbb{N}^*$ such that $d = 2n/a \in \mathbb{N}$. There exists $\gamma > 0$ such that $g^{frac}_{s,d}(G) \geq \gamma a H(a)$, where H is the harmonic function. Moreover, the spy wins after at most $2n$ steps starting from a corner of G.*

From Lemmas 13 and 14, we get

▶ **Corollary 15.** *Let G be a $n \times n$-grid and $a \in \mathbb{N}^*$. For any strategy using a total amount of $k > 0$ guards, there exists a strategy for the spy (with speed $\leq s$) starting from a corner of G such that after at most $2n$ steps, the amount of guards at distance at most $2n/a$ from the spy is less than $k * (aH(a))^{-1}$.*

▶ **Theorem 16.** *$\exists \beta, \gamma > 0$ such that, for any $n \times n$-grid $G_{n\times n}$ and $s, d \in \mathbb{N}$ ($s \geq 2$), the spy (with speed $\leq s$) can win (for distance d) in at most $2n$ steps against $< \gamma n^{1+\beta}$ guards.*

Proof. We actually prove that $\exists \beta > 0$ such that $\Omega(n^{1+\beta}) = g^{frac}_{s,d}(G_{n\times n})$ in any $n \times n$-grid $G_{n\times n}$ and the result follows from Proposition 11.

Let $a_0 \in \mathbb{N}$ be such that $H(a_0)^{-1} \leq 1/2$. Since $g^{frac}_{s,d}(G_{n\times n})$ is non-decreasing as a function of n, it is sufficient to prove the lemma for $n = (a_0)^i$ for any $i \in \mathbb{N}^*$.

We prove the result by induction on i. It is clearly true for $i = 1$ since a_0 is a constant. Assume by induction that there exists $\gamma, \beta > 0$, such that, for $i \geq 1$ with $n = (a_0)^i$, the spy (with speed $\leq s$) can win (for distance d) in at most $2n$ steps against $\gamma a_0^{i(1+\beta)}$ guards in any $n \times n$ grid.

Let G be a $n \times n$-grid with $n = (a_0)^{i+1}$. Let $k \leq \gamma n^{1+\beta}$. By Corollary 15, there exists a strategy for the spy (with speed $\leq s$) starting from a corner of G such that after $t \leq 2n$ steps, the amount of guards at distance at most $2n/a_0$ from the spy is less than $k * (a_0 H(a_0))^{-1} \leq k/(2a_0) \leq \gamma n^{1+\beta}/(2a_0)$.

Let v be the vertex reached by the spy at the step t of strategy $\mathcal{S}$. Let G' be any subgrid of G with side n/a_0 and corner G. By previous paragraph at most $\gamma n^{1+\beta}/(2a_0)$ can occupy the nodes at distance at most d from any node of G' during the next $2n/a_0$ steps of the strategy. So, by the induction hypothesis, the spy playing an optimal strategy in G' against at most $\gamma n^{1+\beta}/(2a_0)$ guards will win. ◀

▶ **Corollary 17.** *$\exists \beta > 0$ such that, for any $n \times n$-grid $G_{n\times n}$ and $s, d \in \mathbb{N}$ ($s \geq 2$),*

$$g_{s,d}(G_{n\times n}) = \Omega(n^{1+\beta}).$$

To conclude, it remains to prove Lemma 14. As announced above, we actually prove a lower bound on $g^{k,t}_{s,d}(G)$. Since $g^{k,t}_{s,d}(G)$ is an nondecreasing function of s, it is sufficient to prove it for $s = 2$.

▶ **Lemma 18.** *Let G be a $n\times n$ grid. $\exists \beta > 0$ such that for any $d, k > 0$, $g^{k,2n}_{2,d}(G) \geq \beta k \frac{n}{d} H(\frac{n}{d})$.*

Proof. Let G be a $n \times n$ grid and let us identify its vertices by their natural coordinates. That is, for any $(i_1, j_1), (i_2, j_2) \in [n]^2$, vertex (i_1, j_1) is adjacent to vertex (i_2, j_2) if $|i_1 - i_2| + |j_1 - j_2| = 1$.

In order to prove the result, we will consider a *family* of strategies for the spy. For every $r \in [n]$, the spy starts at position $(0, 0)$ and runs at full speed toward $(r, 0)$. Once there, it continues at full speed toward $(r, n-1)$. We name P_r the path it follows during this strategy, which is completed in $\lceil \frac{1}{2}(r+n-1) \rceil$ tops.

Let us assume that there exists a strategy using an amount q of guards that maintains at least k guards at distance at most d from the spy during at least $2n$ turns. Moreover, the spy only plays the strategies described above.

Assuming that the guards are labelled with integers in $[q]$, we can name at any time of strategy P_r the labels of k guards that are at distance $\leq d$ of the spy. In this way, we write $c(2r, 2j)$ this set of k guards that are at distance $\leq d$ from the spy, when the spy is at position $(2r, 2j)$.

▶ **Claim 19.** *If $|j_2 - j_1| > 2d$, then $c(2r, 2j_1)$ and $c(2r, 2j_2)$ are disjoint.*

Proof of the claim. Assuming $j_1 < j_2$, it takes $j_2 - j_1$ tops for the spy in strategy P_r to go from $(2r, 2j_1)$ to $c(2r, 2j_2)$. A cop cannot be at distance $\leq d$ from $(2r, 2j_1)$ and, $j_2 - j_1$ tops later, at distance $\leq d$ from $(2r, 2j_2)$. Indeed, to do so its speed must be $\geq 2(j_2 - j_1 - d)/(j_2 - j_1) > 1$, a contradiction. ◀

▶ **Claim 20.** *If $|r_2 - r_1| > 2d + 2\min(j_1, j_2)$, then $c(2r_1, 2j_1)$ and $c(2r_2, 2j_2)$ are disjoint.*

Proof of the claim. Assuming $r_1 < r_2$, note that strategies P_{2r_1} and P_{2r_2} are identical for the first r_1 tops. By that time, the spy is at position $(2r_1, 0)$. If $c(2r_1, 2j_1)$ intersects $c(2r_2, 2j_2)$, it means that at this instant some cop is simultaneously at distance $\leq d + j_1$ from $(2r_1, 2j_1)$ (strategy P_{2r_1}) and at distance $\leq d + |r_2 - r_1| + j_2$ from $(2r_2, 2j_2)$ (strategy P_{2r_2}). As those two points are at distance $2|r_2 - r_1| + 2|j_2 - j_1|$ from each other, we have:

$$\begin{aligned} 2|r_2 - r_1| + 2|j_2 - j_1| &\leq (d + j_1) + (d + |r_2 - r_1| + j_2) \\ |r_2 - r_1| + 2|j_2 - j_1| &\leq 2d + j_1 + j_2 \\ |r_2 - r_1| &\leq 2d + 2\min(j_1, j_2) \end{aligned}$$ ◀

We can now proceed to prove that the number of guards is sufficiently large. To do so, we define a graph H on a subset of $V(G)$ and relate the distribution of the guards (as captured by c) with the independent sets of H. It is defined over $V(H) = \{(2r, 4dj) : 2r \in [n], 4dj \in [n]\}$, where:

- $(2r, 4dj_1)$ is adjacent with $(2r, 4dj_2)$ for $j_1 \neq j_2$ (see Claim 19).
- $(2r_1, 4dj_1)$ is adjacent with $(2r_2, 4dj_2)$ if $|r_2 - r_1| > 4d(1 + \min(j_1, j_2))$ (see Claim 20).

By definition, c gives k colors to each vertex of H, and any set of vertices of H receiving a common color is an independent set of H. If we denote by $\#c^{-1}(x)$ the number of vertices which received color x, and by $\alpha_{(2r_1, 4dj_1)}(H)$ the maximum size of an independent set of H containing $(2r_1, 4dj_1)$, we have:

$$\begin{aligned} q &= \sum_{(2r_1, 4dj_1) \in V(H)} \sum_{x \in c(2r_1, 4dj_1)} \frac{1}{\#c^{-1}(x)} \\ &\geq \sum_{(2r_1, 4dj_1) \in V(H)} \frac{k}{\alpha_{((2r_1, 4dj_1))}(H)} \end{aligned}$$

It is easy, however, to approximate this lower bound.

▶ **Claim 21.** $\alpha_{((2r_1, 4dj_1))}(H) \leq 4d(j_1 + 1) + 1\,.$

Proof of the claim. An independent set $S \subseteq V(H)$ containing $(2r_1, 4dj_1)$ cannot contain two vertices with the same first coordinate. Furthermore, $(2r_1, 4dj_1)$ is adjacent with any vertex $(2r_2, 4dj_2)$ if $|r_2 - r_1| > 4d(1 + j_1)$. ◀

We can now finish the proof:

$$\begin{aligned} q &\geq \sum_{(2r_1,4dj_1)\in V(H)} \frac{k}{\alpha_{((2r_1,4dj_1))}(H)} \\ &\geq \sum_{(2r_1,4dj_1)\in V(H)} \frac{k}{4d(j_1+1)+1} \\ &\geq \frac{n}{2} \sum_{j_1\in\{0,\ldots,n/4d\}} \frac{k}{4d(j_1+1)+1} \\ &\geq \frac{kn}{16d} \sum_{j_1\in\{1,\ldots,n/4d+1\}} \frac{1}{j_1} \\ &\geq \frac{kn}{16d} H(n/4d) \end{aligned}$$ ◀

We leave the exact value of $gn_{s,d}$ in grids as an intriguing open problem.

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