Finding Global Optimum for Truth Discovery: Entropy Based Geometric Variance*

Hu Ding[1], Jing Gao[2], and Jinhui Xu[3]

1 **Department of Computer Science and Engineering, Michigan State University, East Lansing, USA**
huding@msu.edu

2 **Department of Computer Science and Engineering, State University of New York, Buffalo, USA**
jing@buffalo.edu

3 **Department of Computer Science and Engineering, State University of New York, Buffalo, USA**
jinhui@buffalo.edu

Abstract

Truth Discovery is an important problem arising in data analytics related fields such as data mining, database, and big data. It concerns about finding the most trustworthy information from a dataset acquired from a number of unreliable sources. Due to its importance, the problem has been extensively studied in recent years and a number techniques have already been proposed. However, all of them are of heuristic nature and do not have any quality guarantee. In this paper, we formulate the problem as a high dimensional geometric optimization problem, called *Entropy based Geometric Variance*. Relying on a number of novel geometric techniques (such as Log-Partition and Modified Simplex Lemma), we further discover new insights to this problem. We show, for the first time, that the truth discovery problem can be solved with guaranteed quality of solution. Particularly, we show that it is possible to achieve a $(1+\epsilon)$-approximation within nearly linear time under some reasonable assumptions. We expect that our algorithm will be useful for other data related applications.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems - Geometrical problems and computations

Keywords and phrases geometric optimization, data mining, high dimension, entropy

Digital Object Identifier 10.4230/LIPIcs.SoCG.2016.34

1 Introduction

Truth discovery is an emerging topic in data analytics which has received a great deal of attentions in recent years [3, 6, 11, 13, 12, 16, 18, 19]. Despite its extensive studies in the fields of data mining, machine learning, database, and big data, it has yet to be seriously considered by the theory community (to our best knowledge). The problem arises in scenarios where data are acquired from multiple sources which may contain false or inconsistent information, and the truth discovery problem is to find the most trustworthy information from these sources. The problem finds many applications. For example, in online social networks, a user's information can be recorded by multiple websites which may not be always consistent;

* This work was partially supported by NSF through grants CCF-1422324, IIS-1422591, and CNS-1547167. Part of the work was done when the first author was in IIIS, Tsinghua University.

32nd International Symposium on Computational Geometry (SoCG 2016).
Editors: Sándor Fekete and Anna Lubiw; Article No. 34; pp. 34:1–34:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

	George Washington	Abraham Lincoln	Mahatma Gandhi	John Kennedy	Barack Obama	Franklin Roosevelt
Source 1	Virginia	Illinois	Delhi	Texas	Kenya	Georgia
Source 2	Virginia	Kentucky	Porbandar	Massachusetts	Hawaii	New York
Source 3	Maryland	Kentucky	Mumbai	Massachusetts	Kenya	New York
Majority Voting	Virginia	Kentucky	Delhi	Massachusetts	Kenya	New York
Truth Discovery	Virginia	Kentucky	Porbandar	Massachusetts	Hawaii	New York

Figure 1 Three sources are providing the birthplaces of 6 politicians. For *Mahatma Gandhi*, each source has an individual answer (*i.e.,* a tie case), and majority voting can only randomly pick one. More importantly, for *Barack Obama*, voting provides a totally wrong answer. However, truth discovery tries to distinguish reliable and unreliable sources and thus provide the right answer. In this example, the algorithm[6] finds that source 2 has a higher reliability than the other two.

thus it is desirable to find the most trustworthy information for each user. Similar problem also occurs in other areas, such as in healthcare where medical records of a patient may be acquired by multiple hospitals or laboratories.

The main challenge of truth discovery comes from its unsupervised nature, *i.e.,* the level of reliability of each source is unknown in advance. A straightforward way for solving the problem is to take the average if the data are continuous or conduct majority voting if the data are categorical. Such approaches are implicitly based on the assumption that all sources are equally reliable. However, in many applications the level of reliability of each source could be quite different which may make the yielded solution significantly different from the truth, due to the neglect of “the wisdom of minority" [6, 11]. See the example in Fig. 1 from [6]. Thus, estimating the reliability of each source should be taken into account when building the optimization model for truth discovery. In general, the two components, **reliability estimation** and **truth finding**, are tightly coupled and thus are expected to be solved simultaneously, where the truth should be closer to the source with higher reliability, and as a feedback, the source providing closer information should have a higher reliability. Another challenge of truth discovery is how to handle the large number of unreliable sources in the big data era. For example, in the case of monitoring human health, there may be tens of thousands of wearable devices distributed among a population, and each device represents a source which records various kinds of health data of a person for a possibly long period of time (e.g., months or years).

1.1 Problem Formulation and Related Works

We first introduce the problem formulation of truth discovery used in the data mining community, and then convert it to a new geometric optimization problem, called **entropy based geometric variance**.

To model the truth discovery problem, the data from each source can be represented as a (possibly high dimensional) vector, where each dimension corresponds to one attribute/property (*e.g.,* age, income, or temperature). For categorical data, we can reduce them to continuous data as follows [6]. Suppose that one attribute has t categories; then it can be represented as a t-dimensional binary sub-vector, where each dimension indicates the membership of one category. We can finally embed all these sub-vectors (corresponding to the categorical attributes) into one unified vector in higher dimensional space[1]. Furthermore, we need a variable to represent the reliability of each source.

▶ **Definition 1** (Truth Discovery[13, 6])**.** Let $P = \{p_1, p_2, \cdots, p_n\}$ be a set of vectors in $\mathbb{R}^d$ space with each p_i representing the data from the i-th source (among a set of n sources). The truth discovery problem is to find the truth vector $p^* \in \mathbb{R}^d$ and the reliability (weight) w_i for each i-th source, such that the following objective function is minimized,

$$\sum_{i=1}^{n} w_i ||p^* - p_i||^2, \quad s.t. \quad \sum_{i=1}^{n} e^{-w_i} = 1. \tag{1}$$

In the above optimization problem (1), both p^* and the weights are variables. It is easy to see that when each w_i is fixed, p^* is simply the weighted mean, *i.e.,* $\frac{1}{\sum_{i=1}^{n} w_i} \sum_{i=1}^{n} w_i p_i$. This means that the higher the weight of p_i, the closer it is to p^*, which is consistent with the principle of truth discovery.

Weight normalization function. In the above optimization problem, equation $\sum_{i=1}^{n} e^{-w_i} = 1$ is used to normalize the source weights. This way of normalization was initially introduced in [13] (with no justification) and has demonstrated experimentally its superior performance. To understand the rationale behind this, below we give a theoretical justification. Firstly, we notice that some straightforward ways, such as $\sum_{i=1}^{n} w_i^p = 1$ for some $p > 0$, are inappropriate for weight normalization [13], since otherwise, p^* can trivially choose any p_l as its solution and set $w_l = 1$ and $w_i = 0$ for all $i \neq l$ (in this way the objective value will always be equal to the smallest possible value 0). By using equation $\sum_{i=1}^{n} e^{-w_i} = 1$, we can easily avoid this issue. Secondly, this exponential normalization function ensures that the resulting solution minimizes the **entropy**, which implies that the solution contains more information from the input according to Shannon's information theory [17]. To see this, we first borrow the following lemma from [13], which can be shown by using the *Lagrange multipliers* method.

▶ **Lemma 2** ([13])**.** *If the truth vector p^* is fixed, the following value for each weight w_l minimizes the the objective function (1),*

$$w_l = \log(\frac{\sum_{i=1}^{n} ||p^* - p_i||^2}{||p^* - p_l||^2}). \tag{2}$$

Let S denote the total squared distance to p^* (*i.e.,* $S = \sum_{i=1}^{n} ||p^* - p_i||^2$), and f_l denote the contribution of each p_l to S (*i.e.,* $f_l = \frac{||p^* - p_l||^2}{S}$). Then the induced entropy is

$$H = -\sum_{l=1}^{n} f_l \log f_l = -\sum_{l=1}^{n} \frac{||p^* - p_l||^2}{S} \log \frac{||p^* - p_l||^2}{S} = \frac{1}{S} \sum_{l=1}^{n} ||p^* - p_l||^2 \log \frac{S}{||p^* - p_l||^2}. \tag{3}$$

Below, we define the *Entropy based Geometric Variance.*

▶ **Definition 3.** Given a set of points P and a point p^* in $\mathbb{R}^d$, the entropy base geometric variance induced by p^* is $H \times S$, where H and S are respectively the entropy and variance defined in the above discussion.

From Lemma 2 and the formula (3), we know that the objective function (1) is equal to the multiplication of S and H, *i.e.,* the entropy based geometric variance.

[1] Note that this representation for categorical data may cause fractional memberships in the final solution, which is often acceptable in practice (*e.g.,* we may claim that one object belongs to class 1, 2, and 3 with probabilities of 70%, 20%, and 10%, respectively).

▶ **Theorem 4.** *The optimization problem (1) is equivalent to finding a point p^* to minimize the entropy based geometric variance.*

Generally speaking, S represents the total variance from the sources to the truth vector, and the entropy H indicates how disorder the system is, where the higher the entropy, the greater disorder the system is. Since we minimize both of them, this implicitly explains the better performance of using the exponential normalization function in Definition 1.

Non-convexity. As shown in [13], when the truth vector or the weights are fixed, the optimization problem (1) is convex. However, when both of them are variables, the problem is non-convex in general. To see this, consider the following simple example. Suppose $n = 2$. Then the objective value is 0 when p^* coincides with either p_1 or p_2, according to Lemma 2 (note $\lim_{x \to 0} x \log(1/x) = 0$). This means that it is possible to have multiple isolated local or global optimal solutions for truth discovery, implying that truth discovery is non-convex.

Existing Approaches. To the best of our knowledge, all existing methods for truth discovery are based on some heuristic ideas, which achieve only a local optimal solution and have no quality guarantee on global optimality. A commonly used strategy is alternating minimization [13, 12, 15], which alternatively fixes either the weights or the truth vector, and optimizes the other. The optimization problem becomes convex when one of the two types of variables is fixed. This means that such approaches are guaranteed to converge to some local optima. Other approaches [19] follow similar ideas. The reader is referred to a recent survey [6] for a comprehensive introduction to these approaches.

1.2 Preliminaries and Our Main Results

Different from existing approaches, our goal is to achieve a quality guaranteed solution for the truth discovery problem. In practice, we can assume that the number of sources n and the size of the data in each source d are both large. As a starting point, the following theorem suggests that it is easy to generate a 2-approximation in quadratic time (due to space limit, we omit some of the proofs in our paper).

▶ **Theorem 5.** *If one tries every point in $\{p_i \mid 1 \leq i \leq n\}$ as a candidate for the truth vector, at least one yields a 2-approximation for the objective function in (1), and the total running time is $O(n^2 d)$.*

Theorem 5 implies that any further improvement needs to decrease either the approximation ratio or the running time. In this paper, we aim to achieve a $(1 + \epsilon)$-approximation for truth discovery and also keep the time complexity as low as possible.

For ease of discussion, we use the following notations throughout the rest of this paper. Let $L_{\min} = \min\{||p_i - p_{i'}|| \mid 1 \leq i \neq i' \leq n\}$, $L_{\max} = \max\{||p_i - p_{i'}|| \mid 1 \leq i \neq i' \leq n\}$, and the spread ratio $\Delta = \frac{L_{\max}}{L_{\min}}$. To achieve a $(1 + \epsilon)$-approximation for the truth discovery problem for any given small value $1 > \epsilon > 0$, we consider the following two cases.

Case 1. $\min_{1 \leq i \leq n} ||p^* - p_i|| \leq \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$, *i.e.,* some p_i locates very close to p^*.

Case 2. $\min_{1 \leq i \leq n} ||p^* - p_i|| > \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$, *i.e.,* no p_i locates very close to p^*.

In following sections, we will present efficient algorithms to solve the two cases separately. For case 1, we show that the nearest point p_i to p^* is actually a $(1 + \epsilon)$-approximation in Section 2. For case 2, we first give a simple linear time algorithm with large approximation ratio in Section 3, based on an analysis on the distribution of the weights; then in Section 4, we reveal several new insights to the weights by using a novel *Log-Partition* technique, and perform a sequence of geometric operations to obtain a $(1+\epsilon)$-approximation. The time complexity depends on Δ. Finally, in Section 5 we show that when Δ is not too large, the time complexity for both cases can be improved to nearly linear ($O(nd \times poly(\log n))$); also through dimension reduction, the complexity can be further improved to linear ($O(nd)$). Note that spread ratio is commonly used as a parameter in many geometric algorithms and appears in the time complexity (such as [8]); it is usually not very large [4], especially in high dimensional space.

We introduce the following two folklore lemmas [2, 10] which are repeatedly used in our analysis. Let $Q = \{q_i \mid 1 \le i \le n\}$ be a set of n points in $\mathbb{R}^d$ with each q_i associated with a weight $w_i \ge 0$, $W = \sum_{i=1}^n w_i$, and $m(Q)$ be the weighted mean of Q *i.e.*, $m(Q) = \sum_{i=1}^n w_i q_i / W$.

▶ **Lemma 6.** *For an arbitrary point q, $\sum_{i=1}^n w_i ||q - q_i||^2 = W||q - m(Q)||^2 + \sum_{i=1}^n w_i ||m(Q) - q_i||^2$.*

▶ **Lemma 7.** *Let Q_1 be a subset of Q with a total weight of αW for some $0 < \alpha \le 1$. Let $m(Q_1)$ be the weighted mean point of Q_1. Then $||m(Q_1) - m(Q)|| \le \sqrt{\frac{1-\alpha}{\alpha}}\delta$, where $\delta^2 = \frac{1}{W}\sum_{i=1}^n w_i ||q_i - m(Q)||^2$.*

2 A $(1+\epsilon)$-Approximation for Case 1

In this section, we consider case 1. Without loss of generality, we assume that $||p^* - p_{i_0}|| \le \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$, *i.e.*, p_{i_0} is the point very close to p^*. Then, we have:

▶ **Lemma 8.** *For any $i \ne i_0$, $||p^* - p_i|| \ge (1 - \frac{\epsilon}{4})||p_{i_0} - p_i||$.*

Proof. Since p^* is the weighted mean $\frac{\sum_{i=1}^n w_i p_i}{\sum_{i=1}^n w_i}$, we know that for any $1 \le l \le n$,

$$||p^* - p_l|| = ||\frac{\sum_{i=1}^n w_i p_i}{\sum_{i=1}^n w_i} - p_l|| \le \sum_{i=1}^n (\frac{w_i}{\sum_{i=1}^n w_i})||p_i - p_l|| \le L_{\max}. \tag{4}$$

Thus, we have $S \le nL_{\max}^2$, and consequently

$$||p^* - p_{i_0}|| \le \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta} \le \frac{\epsilon}{4}\frac{L_{\max}}{\Delta} = \frac{\epsilon}{4}L_{\min}. \tag{5}$$

Furthermore, due to triangle inequality, we have

$$||p^* - p_i|| \ge ||p_{i_0} - p_i|| - ||p^* - p_{i_0}|| \ge ||p_{i_0} - p_i|| - \frac{\epsilon}{4}L_{\min} \ge (1 - \frac{\epsilon}{4})||p_{i_0} - p_i|| \tag{6}$$

for any $i \ne i_0$. ◀

Now we can obtain a $(1+\epsilon)$-approximation for case 1.

▶ **Theorem 9.** *For case 1, if one tries every point in $\{p_i \mid 1 \le i \le n\}$ as a candidate for the truth vector, at least one yields a $(1+\epsilon)$-approximation for the objective function in (1), and the total time complexity is $O(n^2 d)$.*

Proof. We prove this theorem by showing how large the objective value will increase if p^* is moved to p_{i_0}. Firstly, we suppose that the weights are fixed temporarily. Then, by Lemma 8 and the fact that $0 < \epsilon < 1$, we have

$$\frac{||p_{i_0} - p_i||^2}{||p^* - p_i||^2} \leq \frac{1}{(1-\epsilon/4)^2} \leq 1+\epsilon \tag{7}$$

for any i. This means that the objective value is increased by a factor no more than $1+\epsilon$. Once p^* is moved to p_{i_0}, we can further update the weights according to Lemma 2, and the objective value will not increase. Note that the contribution of p_{i_0} to the objective value will become 0 since $\lim_{x\to 0} x \log \frac{S}{x} = 0$.

Since we need to try every point to find out p_{i_0} (as the candidate for the truth vector) and each point takes $O(nd)$ time to evaluate the objective function, the total time complexity is thus $O(n^2 d)$. ◀

3 A Simple Linear Time Algorithm for Case 2

In this section, we present a simple linear time approximation algorithm for case 2. Although the approximation ratio is relatively large ($O(\log \frac{n\Delta}{\epsilon})$), the idea used in the algorithm sheds some lights on how to find a more refined solution, *e.g.,* $(1+\epsilon)$-approximation in Section 4. Additionally, we believe that this simple linear time algorithm is also of some independent interest.

We first estimate the range for the weights. Lemmas 10 and 11 provides the upper and lower bounds for each w_i, and Lemma 12 shows a lower bound on their summation.

▶ **Lemma 10.** *For case 2, each weight $w_i \leq 2\log\frac{n\Delta}{\epsilon}$.*

Lemma 10 can be easily obtained from the assumption $\min_{1\leq i\leq n} ||p^* - p_i|| > \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$ and Lemma 2. Note that we assume $n \geq 16$ here; otherwise, we just need to increase the front constant “2" (of $2\log\frac{n\Delta}{\epsilon}$) a little bit.

▶ **Lemma 11.** *For any constant $2 > c > 1$, at least one of the following two events happens:*

1. $\min_{1\leq i\leq n} w_i \geq \log c$*;*
2. *all weights except one are at least* $\log\frac{c}{c-1}$*.*

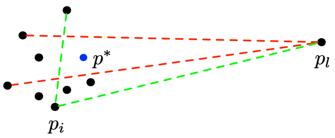
Proof. Suppose that the first event does not happen, *i.e.,* $\min_{1\leq i\leq n} w_i < \log c$. Then, by Lemma 2 we know that there exists a p_l such that $\frac{||p^*-p_l||^2}{\sum_{i=1}^n ||p^*-p_i||^2} > \frac{1}{c}$. Since $\frac{1}{c} > \frac{1}{2}$, there is at most one such p_l, and each of the other points should have a weight at least $\log\frac{1}{1-1/c} = \log\frac{c}{c-1}$, *i.e.,* the second event happens. Thus the lemma is true. ◀

▶ **Lemma 12.** *The sum of the weights $\sum_{i=1}^n w_i \geq n\log n$.*

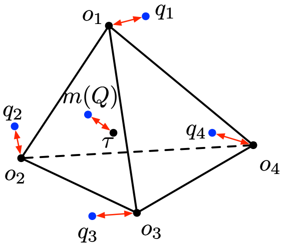
Proof. From Lemma 2, we know that $\sum_{i=1}^n w_i = \sum_{i=1}^n \log\frac{S}{||p^*-p_i||^2}$. It is easy to see that the function $f(x) = \log\frac{S}{x}$ is convex, since $f''(x) = \frac{1}{x^2} > 0$. By *Jensen's inequality*, we have

$$\sum_{i=1}^n \log\frac{S}{||p^*-p_i||^2} \geq n\log\frac{S}{\sum_{i=1}^n ||p^*-p_i||^2/n} = n\log n. \tag{8}$$

This completes the proof. ◀



■ **Figure 2** p_l and p_i are connected by dashed lines to their respective farthest and second farthest points; the blue point is p^*.



■ **Figure 3** An illustration for Lemma 18 with $k = 4$; each o_j has a bounded distance to q_j, the corresponding exact weighted mean of Q_j, and the distance between τ and $m(Q)$ is also bounded.

Algorithm 1 Linear time algorithm for case 2

```
Input: $P = \{p_i, | 1 \le i \le n\} \subset \mathbb{R}^d$
1. Compute the mean of $P$ , and denote it as $p_1^*$.
2. Arbitrarily pick one point from $P$ and compute the ratio of the largest and second
   largest distances from other points to it,
   a. remove the selected point from $P$ if the ratio is smaller than 1.618;
   b. or remove the farthest point to it otherwise.
3. Compute the mean of the remaining points, and denote it as $p_2^*$.
4. Take the one from $\{p_1^*, p_2^*\}$ with a smaller objective value as the truth vector.
```

Before introducing the algorithm, we first find a proper value for c in Lemma 11. For this purpose, we consider the second event in Lemma 11. If this event happens, we know that there exists one point, say p_l, with weight less than $\log c$, and all other weights are at least $\log \frac{c}{c-1}$. This implies that

$$||p^* - p_l||^2 > \frac{1}{c}S; \quad \textit{and} \quad ||p^* - p_i||^2 < (1 - \frac{1}{c})S \ \ \forall i \neq l. \tag{9}$$

This means that p_l is farther away from p^* than all other points, and the smaller c, the larger difference is. To differentiate p_l from others, we consider **the ratio of the largest over the second largest distances from other points to** p_l (see Fig. 2), which is smaller than

$$(\sqrt{\frac{1}{c}} + \sqrt{1 - \frac{1}{c}})/(\sqrt{\frac{1}{c}} - \sqrt{1 - \frac{1}{c}}) \tag{10}$$

due to triangle inequality and the fact that when $x > y > 0$, the function $f(x, y) = \frac{x+y}{x-y}$ is decreasing on x and increasing on y. Similarly, the ratio for any other p_i for $i \neq l$ is bigger than

$$(\sqrt{\frac{1}{c}} - \sqrt{1 - \frac{1}{c}})/(2\sqrt{1 - \frac{1}{c}}). \tag{11}$$

From the above two inequalities, we know that in order to make (11) larger than (10), we need to have $c < 10 - 4\sqrt{5} \approx 1.056$, and (11) = (10) ≈ 1.618 if $c = 10 - 4\sqrt{5}$. Consequently, we have the following lemma.

▶ **Lemma 13.** *It is possible to find the point p_l with the smallest weight in $O(nd)$ time, if the second event in Lemma 11 happens with $1 < c < 10 - 4\sqrt{5}$.*

Proof. To prove this lemma, we can arbitrarily pick one point from the input and compute the ratio of the largest over the second largest distances from other points to it. From the above analysis, we know that if the ratio is smaller than 1.618, this point is p_l; otherwise the farthest point to it is p_l. Obviously, the total time of the above procedure is linear, *i.e.,* $O(nd)$. ◀

Now we are ready to present our algorithm (see Algorithm 1).

▶ **Theorem 14.** *Algorithm 1 yields a* $(2\log\frac{n\Delta}{\epsilon}/\log c)$*-approximation for case 2, where* $c \approx 1.056$ *and the time complexity is* $O(nd)$*. In short, the approximation ratio is* $O(\log\frac{n}{\epsilon})$ *if* Δ *is a polynomial of* n*, or* $O(\log\frac{\Delta}{\epsilon})$ *otherwise.*

Proof. To prove this theorem, we consider the two events in Lemma 11 separately.

If the first event happens, we have $w_i \geq \log c$ for any i. Since p^*_1 is the mean of P, we have $\sum_{i=1}^{n}||p^*_1 - p_i||^2 \leq \sum_{i=1}^{n}||p^* - p_i||^2$. Consequently, if we fix the weights and move p^* to p^*_1 (note that we can use Lemma 2 to update the weights and further reduce the objective value), the objective value of (1) will be

$$\begin{aligned}\sum_{i=1}^{n} w_i||p^*_1 - p_i||^2 &\leq 2\log\frac{n\Delta}{\epsilon}\sum_{i=1}^{n}||p^*_1 - p_i||^2\\ &\leq 2\log\frac{n\Delta}{\epsilon}\sum_{i=1}^{n}||p^* - p_i||^2\\ &\leq 2\log\frac{n\Delta}{\epsilon}\sum_{i=1}^{n}\frac{w_i}{\log c}||p^* - p_i||^2 = (2\log\frac{n\Delta}{\epsilon}/\log c)\sum_{i=1}^{n} w_i||p^* - p_i||^2 \quad (12)\end{aligned}$$

based on Lemma 10 & 11, which implies that p^*_1 is a $(2\log\frac{n\Delta}{\epsilon}/\log c)$-approximation.

Now we consider the second event. Let p_l denote the point removed in step 2(a) or 2(b). From the proof of Lemma 13, we know that p_l has the smallest weight. Let $\tilde{p}^*$ be the weighted mean of $P \setminus \{p_l\}$. Suppose that the total weight of $P \setminus \{p_l\}$ is $\alpha\sum_{i=1}^{n} w_i$, then from Lemma 12 and the fact that $w_l \leq \log c < 1$, we have $\alpha > \frac{n\log n - 1}{n\log n}$. As a consequence, by Lemma 7 we have

$$||\tilde{p}^* - p^*||^2 < \frac{1}{n\log n - 1}\frac{\sum_{i=1}^{n} w_i||p^* - p_i||^2}{\sum_{i=1}^{n} w_i}. \quad (13)$$

Then applying Lemma 6 in Section 1.2, we get

$$\sum_{i=1}^{n} w_i||\tilde{p}^* - p_i||^2 \leq \frac{n\log n}{n\log n - 1}\sum_{i=1}^{n} w_i||p^* - p_i||^2. \quad (14)$$

(14) indicates that $\tilde{p}^*$ can replace p^* without causing much increase on the objective value. If we continue to move $\tilde{p}^*$ to p^*_2, the objective value becomes

$$\sum_{i=1}^{n} w_i||p^*_2 - p_i||^2 = \sum_{i\neq l} w_i||p^*_2 - p_i||^2 + w_l||p^*_2 - p_l||^2. \quad (15)$$

For the first term in the right hand side of (15), by a similar calculation to (12), we know that $\sum_{i\neq l} w_i||p^*_2 - p_i||^2 < (2\log\frac{n\Delta}{\epsilon}/\log\frac{c}{c-1})\sum_{i\neq l} w_i||\tilde{p}^* - p_i||^2$. For the second term in the right hand side of (15), by an estimation similar to (10), we have $\frac{||p^*_2 - p_l||}{||\tilde{p}^* - p_l||} \leq (\sqrt{\frac{1}{c}} + \sqrt{1 - \frac{1}{c}})/(\sqrt{\frac{1}{c}} - \sqrt{1 - \frac{1}{c}}) \approx 1.618$. (Note that both p^*_2 and $\tilde{p}^*$ are a convex combination of

$P \setminus p_l$). Putting (14) and (15) together, we know that p_2^* is a solution with approximation ratio

$$\frac{n\log n}{n\log n - 1} \times \max\{2\log\frac{n\Delta}{\epsilon}/\log\frac{c}{c-1}, 1.618\} \le 2\log\frac{n\Delta}{\epsilon}/\log c. \tag{16}$$

Finally, it is easy to know that the time complexity is $O(nd)$. ◀

4 A $(1+\epsilon)$-Approximation for Case 2

In this section, we present a $(1+\epsilon)$-approximation for case 2. In Theorem 14, we consider only two groups of the points, *i.e.*, the point with the smallest weight and all others. In this section we show that by further partitioning the input points into more groups, it is possible to obtain a much better solution.

▶ **Definition 15** (Log-Partition)**.** In case 2, let p_l and $p_{l'}$ be the points with the smallest and the second smallest weights, respectively. Then the log-partition is to divide the points in $\{p_i \mid 1 \le i \le n\}$ into $k = \lceil \log_{1+\beta} \frac{2\log(n\Delta/\epsilon)}{w_{l'}} \rceil + 1$ (where β is a small positive number that will be determined later) groups as follows:

- $\mathcal{G}_1 = \{p_l\}$.
- $\mathcal{G}_j = \{p_i \mid (1+\beta)^{j-2} w_{l'} \le w_i < (1+\beta)^{j-1} w_{l'}\}$ for $j \ge 2$.

Note that we cannot explicitly obtain the log-partition since we do not know the weights in advance. We can only assume that such a partition exists, which will be useful in the following analysis.

From Lemmas 10 and 11 and the fact that $\log(1+\beta) \approx \beta$ when β is a small positive number, we can easily have the following lemma.

▶ **Lemma 16.** *In the log-partition, $k = O(\frac{1}{\beta}\log\log\frac{n\Delta}{\epsilon})$.*

In each $\mathcal{G}_j$, their weight difference is no more than a factor of $(1+\beta)$; as a consequence, their weighted mean and weighted standard deviation are very close to their mean and standard deviation respectively. In the remaining parts, we denote the mean and weighted mean of each $\mathcal{G}_j$ by $\hat{m}_j$ and m_j, the standard deviation and weighted standard deviation by $\hat{\delta}_j$ and δ_j, [2] respectively.

▶ **Lemma 17.** *For each $\mathcal{G}_j$ in the log-partition, $||\hat{m}_j - m_j|| \le \beta\sqrt{1+\beta}\,\delta_j$, and $\delta_j \in [\frac{1}{\sqrt{1+\beta}}\hat{\delta}_j, \sqrt{1+\beta}\,\hat{\delta}_j]$.*

Using Lemma 17, we can obtain a $(1+\epsilon)$-approximation algorithm for case 2. Below is the sketch of our idea.

Synopsis. The essential task of truth discovery is to find the weighted mean without knowing the weights in advance. Using log-partition, we can first divide the input points implicitly into k groups, and Lemma 17 enables us to ignore the weights inside each group. Then by applying random sampling techniques, we can estimate the weighed mean of each group, and find the weighted mean of the whole input using *simplex lemma*. We elaborate our ideas in the following subsections.

[2] $\hat{\delta}_j = \sqrt{\frac{1}{|\mathcal{G}_j|}\sum_{p_i\in\mathcal{G}_j} ||p_i - \hat{m}_j||^2}$ and $\delta_j = \sqrt{\frac{1}{\sum_{p_i\in\mathcal{G}_j} w_i}\sum_{p_i\in\mathcal{G}_j} w_i||p_i - m_j||^2}$.

4.1 Modified Simplex Lemma

In [2], Ding and Xu introduced a simplex lemma for solving a large class of constrained clustering problems in high dimensional space. In this subsection, we show that despite developed for a different purpose, the simplex lemma is still applicable to our truth discovery problem.

▶ **Lemma 18** (Simplex Lemma [2]). *Given an unknown weighted point-set $Q \subset \mathbb{R}^d$, which is implicitly divided into k mutually exclusive groups $\{Q_j \mid 1 \leq j \leq k\}$, and k points $\{o_j \mid 1 \leq j \leq k\}$ satisfying the condition that for each j, the distance between o_j and the weighted mean of the unknown Q_j is no more than a fixed value $L \geq 0$, it is possible to construct a grid of size $((8k/\epsilon)^k)$ inside the simplex determined by $\{o_j \mid 1 \leq j \leq k\}$ such that at least one grid point τ satisfies the following inequality.*

$$||\tau - m(Q)|| \leq \sqrt{\epsilon}\delta(Q) + (1+\epsilon)L, \tag{17}$$

where $m(Q)$ and $\delta(Q)$ are the weighted mean and standard deviation of Q, respectively.

Simplex lemma shows that it is possible to find an approximate weighted mean of an unknown point-set. The only known information is the approximate weighted mean of each unknown subset. L is a slack parameter to control the error bound in (17). See Fig. 3. Also, a nice feature of the simplex lemma is that it needs to consider only a low dimensional subspace determined by the simplex ($k \ll d$), and thus can be applied to problems in high dimensional space.

It is easy to see that the simplex lemma is immediately applicable to the truth discovery problem for finding the weighted mean, if we are able to obtain the weighted mean (or only the mean due to Lemma 17) of each $\mathcal{G}_j$. The difficulty is that since some $\mathcal{G}_j$ could be quite small in its cardinality, it is extremely challenging to estimate the mean by using random sampling techniques. The following modified simplex lemma shows that it is actually possible to ignore such small-size groups.

▶ **Lemma 19** (Modified Simplex Lemma). *Let Q, Q_j, ϵ, δ, and k be defined as in Lemma 18, and $\Gamma = \{j \mid \frac{w(Q_j)}{w(Q)} \geq \frac{\epsilon}{k}\}$, where $w(\cdot)$ is the total weight of a point-set. Then it is possible to construct a grid of size $((8k/\epsilon)^k)$ inside the simplex determined by $\{o_j \mid j \in \Gamma\}$ such that at least one grid point τ satisfies the following*

$$||\tau - m(Q)|| \leq 2\sqrt{\frac{\epsilon}{1-\epsilon}}\delta(Q) + (1+\epsilon)L. \tag{18}$$

Proof. Let $Q_\Gamma = \cup_{j\in\Gamma} Q_j$. Then by Lemma 18, we immediately have the following inequality.

$$||\tau - m(Q_\Gamma)|| \leq \sqrt{\epsilon}\delta(Q_\Gamma) + (1+\epsilon)L, \tag{19}$$

where Q is simply replaced by Q_Γ. Now, we consider the differences between $m(Q)$, $\delta(Q)$ and $m(Q_\Gamma)$, $\delta(Q_\Gamma)$, respectively. Similar to (13) in Theorem 14 for proving the distance between $\tilde{p}^*$ and p^*, based on Lemma 7 we have

$$||m(Q_\Gamma) - m(Q)||^2 \leq \frac{w(Q \setminus Q_\Gamma)}{w(Q_\Gamma)}\delta^2(Q) \leq \frac{\epsilon}{1-\epsilon}\delta^2(Q), \tag{20}$$

where the last inequality comes from the facts that $w(Q \setminus Q_\Gamma) \leq k \times \frac{\epsilon}{k} w(Q)$ and $w(Q_\Gamma) \geq (1-\epsilon)w(Q)$. Furthermore, since $w(Q)\delta^2(Q) \geq w(Q_\Gamma)\delta^2(Q_\Gamma)$, we have

$$\delta^2(Q_\Gamma) \leq \frac{w(Q)}{w(Q_\Gamma)}\delta^2(Q) \leq \frac{1}{1-\epsilon}\delta^2(Q). \tag{21}$$

Plugging (20) and (21) into (19), we have

$$\begin{aligned} ||\tau - m(Q)|| &\leq ||\tau - m(Q_\Gamma)|| + ||m(Q_\Gamma) - m(Q)|| \\ &\leq \sqrt{\epsilon}\delta(Q_\Gamma) + (1+\epsilon)L + \sqrt{\frac{\epsilon}{1-\epsilon}}\delta(Q) \\ &\leq \sqrt{\epsilon}\frac{1}{\sqrt{1-\epsilon}}\delta(Q) + (1+\epsilon)L + \sqrt{\frac{\epsilon}{1-\epsilon}}\delta(Q) \\ &= 2\sqrt{\frac{\epsilon}{1-\epsilon}}\delta(Q) + (1+\epsilon)L. \qquad (22) \end{aligned}$$

This completes the proof. ◀

4.2 The Algorithm Using Modified Simplex Lemma

The following two lemmas are commonly used random sampling techniques in Euclidean space. Lemma 20 shows that in order to estimate the mean of a point-set, one just needs to take the mean of a randomly selected sample. Lemma 21 further shows how to sample points in order to ensure that there are enough number of points in the sample from a hidden subset.

▶ **Lemma 20** ([5]). *Let T be a set of n points in $\mathbb{R}^d$ space, T' be a randomly selected subset of size t from T, and $\hat{m}(T)$, $\hat{m}(T')$ be the mean points of T and T' respectively. With probability $1-\eta$, $||\hat{m}(T) - \hat{m}(T')||^2 < \frac{1}{\eta t}\hat{\delta}^2(T)$, where $\hat{\delta}^2(T) = \frac{1}{n}\sum_{s\in T}||s - m(T)||^2$ and $0 < \eta < 1$.*

▶ **Lemma 21** ([2]). *Let Ω be a set of elements, and T be a subset of Ω with $\frac{|T|}{|\Omega|} = \alpha$ for some $\alpha \in (0,1)$. If randomly select $\frac{t\log\frac{t}{\eta}}{\log(1+\alpha)} = O(\frac{t}{\alpha}\log\frac{t}{\eta})$ elements from Ω, then with probability at least $1-\eta$, the sample contains t or more elements from T for $0 < \eta < 1$ and $t \in \mathbb{Z}^+$.*

By Lemma 19, we know that only those groups with large enough weight need to be considered. The following lemma further shows that each of such groups contains a significant fraction of the input points. This means that we can directly apply Lemmas 20 and 21 to estimate their means.

▶ **Lemma 22.** *In the log-partition for case 2, if a group $\mathcal{G}_j$ has a total weight no less than $\frac{\epsilon}{k}\sum_{i=1}^n w_i$, it contains at least $\frac{\epsilon \log n}{2k\log(n\Delta/\epsilon)}n$ points,* i.e., $|\mathcal{G}_j|/|P| \geq \frac{\epsilon\log n}{2k\log(n\Delta/\epsilon)}$.

Proof. Let $|\mathcal{G}_j|$ and $w(\mathcal{G}_j)$ denote the number of points and the total weight in $\mathcal{G}_j$, respectively. From previous discussion (*i.e.,* Lemmas 10 and 12), we know that

$$w(\mathcal{G}_j) \leq 2\log\frac{n\Delta}{\epsilon}|\mathcal{G}_j|; \qquad \sum_{i=1}^n w_i \geq n\log n. \qquad (23)$$

With the assumption $w(\mathcal{G}_j) \geq \frac{\epsilon}{k}\sum_{i=1}^n w_i$, we have $|\mathcal{G}_j| \geq (\frac{\epsilon}{k}\sum_{i=1}^n w_i)/(2\log\frac{n\Delta}{\epsilon}) \geq \frac{\epsilon\log n}{2k\log(n\Delta/\epsilon)}n$. ◀

Now we are ready to present our refined algorithm for truth discovery. Firstly, we use Lemmas 21 and 22 to sample an enough number of points from each group with large enough weight. Then, we apply Lemma 20 to obtain their approximate means. Finally, we use the modified simplex lemma (*i.e.,* Lemma 19) to obtain the desired $(1+\epsilon)$-approximation. See Algorithm 2. Below, we analyze the correctness of the algorithm. For convenience, we denote by $\delta(P)$ the weighted standard deviation induced by p^*, *i.e.,* $\sqrt{\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n w_i||p^* - p_i||^2}$.

Algorithm 2 $(1+\epsilon)$-algorithm for case 2

```
Input: $P = \{p_i, | 1 \leq i \leq n\} \subset \mathbb{R}^d$, and $\alpha = \frac{\epsilon^3 \log n}{2(\log\log(n\Delta/\epsilon))^2 \log(n\Delta/\epsilon)}$.
1. Randomly take a sample $N$ from the input with size $\frac{4k}{\alpha\beta^2} \log \frac{16k^2}{\beta^2}$.
2. Enumerate all the subsets having $4k/\beta^2$ points from $N$, compute their means, and put
   them into a set $U$.
3. For any $k'$-tuple from $U$, where $k'$ is enumerated from $\{1, 2, \cdots, k\}$, apply Lemma 19
   to build a grid inside the simplex determined by the $k'$-tuple.
4. Try all the grid points, and output the one with the smallest objective value of (1) in
   Definition 1.
```

A key step for analyzing the correctness of the algorithm is to determine the value of β for log-partition. When applying the modified simplex lemma, we have to keep the value of L to be roughly $O(\sqrt{\epsilon}\delta(P))$, such that the obtained grid point τ can result in a $(1+O(1)\epsilon)$-approximation solution by Lemma 6. Note that the value of L depends on two factors, the distance between m_j and $\hat{m}_j$ (Lemma 17), and the error for estimating the position of $\hat{m}_j$ (Lemma 20). For simplicity, we only consider the first factor; the following analysis will show that the first factor actually dominates the value of L. Firstly, when $j \in \Gamma$ (see Lemma 19), we have the upper bound of $||m_j - \hat{m}_j||$,

$$\beta\sqrt{1+\beta}\delta_j < 2\beta\delta_j \leq 2\beta\sqrt{\frac{k}{\epsilon}}\delta(P) \tag{24}$$

by Lemma 17. Meanwhile, we know that $k = O(\frac{1}{\beta} \log\log \frac{n\Delta}{\epsilon})$ by Lemma 16. Thus, we need to set

$$\beta = \frac{\epsilon^2}{\log\log \frac{n\Delta}{\epsilon}} \tag{25}$$

to guarantee that $L = O(\sqrt{\epsilon}\delta(P))$. As a consequence, we have

$$k = \frac{1}{\epsilon^2}(\log\log \frac{n\Delta}{\epsilon})^2. \tag{26}$$

(26) together with Lemma 22 implies that $|\mathcal{G}_j|/|P| \geq \frac{\epsilon^3 \log n}{2(\log\log(n\Delta/\epsilon))^2 \log(n\Delta/\epsilon)}$ for each $j \in \Gamma$. By simple calculations and Lemmas 21, we know that with probability $(1-\frac{1}{4k})^k \geq 1 - 1/4 = 3/4$ the sample N contains at least $4k/\beta^2$ points from each of such group $\mathcal{G}_j$. From Lemma 20, we know that with probability $(1-\frac{1}{4k})^k \geq 3/4$, for each of such $\mathcal{G}_j$ the mean of the corresponding $4k/\beta^2$ points has a distance no more than $\beta\hat{\delta}_j \leq \beta\sqrt{1+\beta}\delta_j = O(\sqrt{\epsilon}\delta(P))$ to its mean (the inequality comes from Lemma 17). In total, L is bounded by $O(\sqrt{\epsilon}\delta(P))$, and we have a $(1+O(1)\epsilon)$-approximation (by Lemma 6).

As for the running time, we note that $k = \frac{1}{\epsilon^2}(\log\log \frac{n\Delta}{\epsilon})^2$. The total number of grid points is

$$(|N|^{4k/\beta^2})^k (8k/\epsilon)^k = 2^{O(\frac{1}{\epsilon^8}(\log\log n\Delta)^7)}. \tag{27}$$

Since $\frac{1}{\epsilon^8}(\log\log n\Delta)^7 < \sigma \log n\Delta$ for any small positive σ if ϵ is fixed and $n\Delta$ is large enough, the time complexity is $O(2^{\sigma \log n\Delta} nd) = O((n\Delta)^\sigma nd)$.

From the above analysis, we have the following theorem.

▶ **Theorem 23.** *With probability $9/16$, Algorithm 2 outputs a $(1+\epsilon)$-approximation for case 2, and the time complexity is $O((n\Delta)^\sigma nd)$, where σ is a small positive number. In short, the time complexity is $O(n^{1+\sigma}d)$ if Δ is a polynomial of n, or $O(\Delta^\sigma nd)$ otherwise.*

5 Improving the Time Complexity

A common strategy adopted by the $(1+\epsilon)$-approximation algorithms in Section 2 and 4 for the two cases is to first identify a set of candidates for the truth vector, then compute the objective value for each candidate, and finally output the candidate with the smallest objective value. Since computing the objective value for each candidate costs $O(nd)$ time, the total time is thus $O(z \times nd)$, where z is the number of candidates (*i.e.*, $z = n$ for case 1 and $z = (n\Delta)^{\sigma}$ for case 2). In this section, we show that when the spread ratio Δ is not too large, the **amortized time complexity** for computing all the objective values of the candidates can be reduced to sub-linear, and consequently the overall time complexity is nearly linear.

Recall that Theorem 4 tells us that the objective value is equal to the entropy based geometric variance $S \times H$ induced by p^*. In order to reduce the time complexity for computing the objective value, below we show how to efficiently compute S and H, respectively.

▶ **Lemma 24.** *The value of S for all the z candidates can be computed in a total of $O((n+z)d)$ time,* i.e., *$O(\frac{nd}{z} + d)$ amortized time complexity for each candidate.*

Proof. Let $\hat{m}(P)$ be the (unit weighted) mean of the point-set P, *i.e.*, $\hat{m}(P) = \frac{1}{n}\sum_{i=1}^{n} p_i$. Then in $O(nd)$ time, we can compute the value of $\hat{S} = \sum_{i=1}^{n} ||\hat{m}(P) - p_i||^2$. For each candidate p^*, we know that its total variance $S = \sum_{i=1}^{n} ||p^* - p_i||^2 = n||p^* - \hat{m}(P)||^2 + \hat{S}$ (by Lemma 6 in Section 1.2). Clearly, the variance of p^* can be computed in $O(d)$ time by using the value of $\hat{S}$. This implies that the total time for computing the value of S for all z candidates is $O(nd + zd)$. ◀

From Lemma 24, we know that it is possible to compute the total variance S in an amortized sub-linear time. Below we discuss how to efficiently compute the entropy H. The following lemma comes from [7] for entropy estimation.

▶ **Lemma 25** ([7]). *Let $F = \{f_i \mid 1 \le i \le n\}$ be a discrete probabilistic distribution with the entropy $H = \sum_{i=1}^{n} -f_i \log f_i$, and two parameters $\epsilon, \delta \in (0,1)$. There exists an algorithm outputting a value $\tilde{H} \in [(1-\epsilon)H, (1+\epsilon)H]$ with probability $1-\delta$, which makes at most $O(\frac{1}{\epsilon^2 H} \log n \log(\frac{1}{\delta}))$ queries on F.*

To estimate H, the algorithm presented in [7] does not read all the values in F. Instead, it takes only a subset of $O(\frac{1}{\epsilon^2 H} \log n \log(\frac{1}{\delta}))$ samples (*i.e.*, queries) from F. From the above lemma, we know that if H is small, the number of needed queries could be quite large, and consequently the time complexity could be high. To avoid this issue, we show in the following lemma that H can actually be lower bounded in our problem if Δ is not too large. Also note that in our problem each query costs only $O(d)$ time, since it can be computed by equation $f_i = \frac{||p^* - p_i||^2}{S}$, where S is the total variance already obtained in Lemma 24.

▶ **Lemma 26.** *If $\Delta = \tilde{O}(\sqrt{n})$ $(= O(\sqrt{n} \times poly(\log n)))$, $H \ge \frac{1}{poly(\log n)}$.*

For simplicity, we let $l_i^2 = ||p^* - p_i||^2$. Then $H = \sum_{i=1}^{n} \frac{l_i^2}{S} \log \frac{S}{l_i^2}$. Since we only need to care about the ratio $\frac{l_i^2}{S}$, without loss of generality we can assume that $\min_{i<j} ||p_i - p_j||^2 = 1$ and $\max_{i<j} ||p_i - p_j||^2 = \Delta^2$. Before proving Lemma 26, we first have the following lemma.

▶ **Lemma 27.** *Except for the smallest value in $\{l_i^2 \mid 1 \le i \le n\}$, all other values are between $\frac{1}{4}$ and Δ^2. Furthermore, $S > \frac{1}{4}\Delta^2$.*

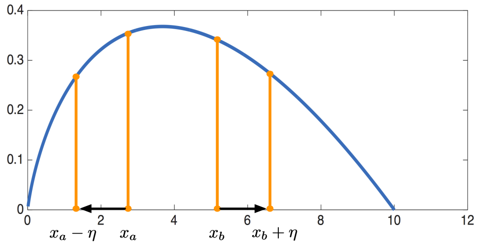


Figure 4 The curve of $g(x)$ with $S = 10$.

Proof of Lemma 26. Let $h = \min\{S, \Delta^2\}$, and $g(x) = \frac{x}{S}\log\frac{S}{x}$ for any $x \in [1/4, h]$, which is concave (as $g''(x) = -\frac{1}{Sx} < 0$). Considering two values $1/4 < x_a \leq x_b < h$, we know that

$$g(x_a) + g(x_b) > g(x_a - \eta) + g(x_b + \eta), \tag{28}$$

where $\eta = \min\{x_a - 1/4, h - x_b\}$ (see Fig. 4), and the sum of $x_a - \eta$ and $x_b + \eta$ is always $x_a + x_b$. This suggests that to find a lower bound of H for a fixed S, we can first identify two values $1/4 < l_{i_1}^2 \leq l_{i_2}^2 < h$, and then decrease $l_{i_1}^2$ and increase $l_{i_2}^2$ in the same speed until either $l_{i_1}^2 = 1/4$ or $l_{i_2}^2 = h$. After repeating the above operation at most $n-1$ times, we have at most one $l_i^2 \in (1/4, h)$, one smaller than $1/4$ (recall Lemma 27), and all the others are either $1/4$ or h. Suppose that t of them have a value of $1/4$ and $n-2-t$ of them have a value of h in $\{l_i^2 \mid 1 \leq i \leq n\}$, where $0 \leq t \leq n-2$. Then we have:

$$\begin{aligned} H &\geq \frac{t}{4S}\log(4S) + \frac{(n-2-t)h}{S}\log\frac{S}{h}; \qquad (29)\\ (t+1)\frac{1}{4} + (n-2-t)h \leq S &\leq (t+1)\frac{1}{4} + (n-1-t)h. \qquad (30) \end{aligned}$$

If $S \leq 2\Delta^2$, we know that at most two items equal to h (*i.e.,* $n-2-t \leq 2$). Consequently, we know that the right hand side of (29) is at least $\frac{t}{4S}\log(4S) \geq \frac{n-4}{4S}\log(4S)$. Also notice that $S \in (\Delta^2/4, 2\Delta^2]$ in this case (where $S > \Delta^2/4$ comes from Lemma 27). Thus, we have

$$H \geq \frac{n-4}{4S}\log(4S) \geq \frac{n-4}{8\Delta^2}\log\Delta^2. \tag{31}$$

To bound the right hand side of the above inequality, we can actually assume that $\Delta^2 \geq 2$. Otherwise, from Lemma 27 and some simple calculations, we know that $H = \sum_{i=1}^{n}\frac{l_i^2}{S}\log\frac{S}{l_i^2} = \Theta(\log n)$. Thus, (31) becomes $H \geq \frac{n-4}{8\Delta^2}\log 2 \geq \frac{1}{poly(\log n)}$ and the lemma is true.

Now, we consider the case of $S > 2\Delta^2$, which immediately implies that $h = \Delta^2$. Note that $t \in [0, n-2]$. From (29) and (30), we have

$$\begin{aligned} H &\geq \frac{t}{(t+1)+4(n-1-t)\Delta^2}\log(4S) + \frac{4(n-2-t)\Delta^2}{(t+1)+4(n-1-t)\Delta^2}\log\frac{S}{\Delta^2}\\ &\geq \left(\frac{t}{(t+1)+4(n-1-t)\Delta^2} + \frac{4(n-2-t)\Delta^2}{(t+1)+4(n-1-t)\Delta^2}\right)\log 2\\ &= \frac{t+4(n-2-t)\Delta^2}{(t+1)+4(n-1-t)\Delta^2}\log 2\\ &= \frac{t+4(n-2-t)\Delta^2}{t+4(n-2-t)\Delta^2+1+4\Delta^2}\log 2 \geq \frac{n-2}{n-2+1+4\Delta^2}\log 2 \geq \frac{1}{poly(\log n)}. \qquad (32) \end{aligned}$$

The second inequality follows from $S \geq 2\Delta^2$.

This completes the proof of Lemma 26. ◀

From previous discussion, we know that the time complexity can be improved to nearly linear if $\Delta = \tilde{O}(\sqrt{n})$. We can take the union of the candidates in both case 1 and case 2 (*i.e.*, $\min_{1\leq i\leq n} ||p^* - p_i|| \leq \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$ or $\min_{1\leq i\leq n} ||p^* - p_i|| > \frac{\epsilon\sqrt{S}}{4\sqrt{n}\Delta}$), and denote it as $\mathcal{Z}$, where $|\mathcal{Z}| = n + (n\Delta)^\sigma = O(n)$ since $\Delta = \tilde{O}(\sqrt{n})$. Note that in case 2, the candidates are obtained by using random sampling which takes sub-linear time (see Section 4.2). Consequently, finding such a set $\mathcal{Z}$ needs only $O(|\mathcal{Z}|d) = O(nd)$ time. By Lemmas 24, 25[3], and 26, we have the following theorem.

▶ **Theorem 28.** *Given an instance P of the truth discovery problem with $\Delta = \tilde{O}(\sqrt{n})$ and two parameters $\epsilon, \delta \in (0,1)$, there exists an algorithm yielding an $(1+\epsilon)$-approximation with success probability $\frac{9}{16}(1-\delta)$. The time complexity is $\tilde{O}(nd)$, where the hiding constant in the big-O notation depends on ϵ and δ.*

In some real applications, the dimensionality d (which is the maximum size of the data from each source) could be much larger than n. For this case, we can first apply the well known *JL-Lemma* [9] to reduce the dimensionality from d to $O(\frac{\log n}{\epsilon^2})$ and then run our algorithm. Note that this can only slightly increase the objective value, since p^* is the weighted mean and consequently

$$\sum_{i=1}^{n} w_i||p_i - p^*||^2 = \frac{1}{2\sum_{i=1}^{n} w_i}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ||p_i - p_j||^2 \tag{33}$$

where JL-Lemma based dimension reduction approximately preserves the pairwise distances. For the running time, we know that a straightforward approach is just multiplying the data matrix $A \in \mathbb{R}^{d\times n}$ to the random projection matrix $R \in \mathbb{R}^{O(\frac{\log n}{\epsilon^2})\times d}$ which costs $O(nd\log n/\epsilon^2)$ time in total. We can also let R be a rescaled random sign matrix [1] and use the technique in [14] to further reduce the time complexity to $O(nd\frac{\log n}{\epsilon^2 \log d})$.

▶ **Corollary 29.** *When $d = \Omega(n^c)$ with some constant $c > 0$, the time complexity in Theorem 28 can be improved to $O(nd\frac{\log n}{\epsilon^2\log d} + n \times poly(\log n)) = O(nd)$, where the hiding constant depends on c, ϵ and δ.*

References

1 D. Achlioptas, “Database-friendly Random Projections: Johnson-Lindenstrauss with Binary Coins,” *J. Comput. Syst. Sci.*, vol. 66, no. 4, pp. 671–687, 2003.

2 H. Ding and J. Xu, “A Unified Framework for Clustering Constrained Data without Locality Property", *Proc. 26th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '15)*, pp. 1471–1490, 2015.

3 X. L. Dong, L. Berti-Equille, and D. Srivastava, “Integrating conflicting data: The role of source dependence", *PVLDB*, 2(1):550–561, 2009.

4 H. Edelsbrunner, P. Valtr, and E. Welzl, “Cutting Dense Point Sets in Half", *Proc. of the Tenth Annual Symposium on Computational Geometry (SoCG'94)*, pp. 203–209, 1994.

5 M. Inaba, N. Katoh, and H. Imai, *Applications of Weighted Voronoi Diagrams and Randomization to Variance-Based k-Clustering,* Proc. 10th ACM Symposium on Computational Geometry (SoCG'94), pp. 332–339, 1994.

6 Y. Li, J. Gao, C. Meng, Q. Li, L. Su, B. Zhao, W. Fan, and J. Han, “A Survey on Truth Discovery", *CoRR* abs/1505.02463, 2015.

[3] When applying Lemma 25, we should replace the parameter δ by $O(\frac{\delta}{n})$ since we have to guarantee the success for all the $O(n)$ candidates. The number of queries increases only by a factor of $\log n$.

7 S. Guha, A. McGregor, and S. Venkatasubramanian, “Sublinear Estimation of Entropy and Information Distances", *ACM Transactions on Algorithms* 5(4) (2009).

8 P. Indyk, R. Motwani, and S. Venkatasubramanian, “Geometric Matching Under Noise: Combinatorial Bounds and Algorithms", *Proc. of the Tenth Annual ACM-SIAM Symposium on Discrete Algorithms(SODA'99)*, pp. 457–465, 1999.

9 W. Johnson and J. Lindenstrauss, “Extensions of Lipschitz mappings into a Hilbert space," *Conference in Modern Analysis and Probability (New Haven, Conn., 1982)*, Contemporary Mathematics 26, Providence, RI: American Mathematical Society, 189–06.

10 A. Kumar, Y. Sabharwal, and S. Sen, “Linear-time Approximation Schemes for Clustering Problems in Any Dimensions", *J. ACM*, 57(2), 2010.

11 H. Li, B. Zhao, and A. Fuxman, “The Wisdom of Minority: Discovering And Targeting The Right Group of Workers for Crowdsourcing", *Proc. of the International Conference on World Wide Web (WWW'14)*, pp. 165–176, 2014.

12 Q. Li, Y. Li, J. Gao, L. Su, B. Zhao, M. Demirbas, W. Fan, and J. Han, “A Confidence-Aware Approach for Truth Discovery on Long-Tail Data", *PVLDB* 8(4):425–436, 2014.

13 Q. Li, Y. Li, J. Gao, B. Zhao, W. Fan, and J. Han, “Resolving Conflicts in Heterogeneous Data by Truth Discovery and Source Reliability Estimation", *Proc. the 2014 ACM SIGMOD International Conference on Management of Data (SIGMOD'14)*, pp. 1187–1198, 2014.

14 E. Liberty and S. W. Zucker, '“The Mailman Algorithm: A Note on Matrix Vector Multiplication," *Inf. Process. Lett.*, vol. 109, no. 3, pp. 179–182, 2009.

15 L. Su, Q. Li, S. Hu, S. Wang, J. Gao, H. Liu, T. F. Abdelzaher, J. Han, X. Liu, Y. Gao, and L. M. Kaplan, “Generalized Decision Aggregation in Distributed Sensing Systems", *Proc. 35rd IEEE Real-Time Systems Symposium (RTSS'14)*, pp. 1-10, 2014.

16 J. Pasternack and D. Roth, “Knowing what to believe (when you already know something)", *Proc. of the International Conference on Computational Linguistics (COLING'10)*, pp. 877–885, 2010.

17 C. E. Shannon, “A Mathematical Theory of Communication", *Bell System Technical Journal*, 27, pp. 379–423 & 623-656, July & October, 1948.

18 J. Whitehill, P. Ruvolo, T. Wu, J. Bergsma, and J. Movellan, “Whose Vote Should Count More: Optimal Integration of Labelers of Unknown Expertise", *Advances in Neural Information Processing Systems (NIPS'09)*, pp. 2035–2043, 2009.

19 X. Yin, J. Han, and P. S. Yu, “Truth discovery with multiple conflicting information providers on the web", *Proc. of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD'07)*, pp. 1048–1052, 2007.