Modal Decomposition on Nondeterministic Probabilistic Processes

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Abstract

We propose a SOS-based method for decomposing modal formulae for nondeterministic probabilistic processes. The purpose is to reduce the satisfaction problem of a formula for a process to verifying whether its subprocesses satisfy certain formulae obtained from its decomposition. By our decomposition, we obtain (pre)congruence formats for probabilistic bisimilarity, ready similarity and similarity.

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1 Introduction

In this paper we provide a SOS [23] driven method for the decomposition of modal formulae for nondeterministic probabilistic transition systems (PTSs) [6, 24], which are a model in which nondeterminism and probability coexist. In essence, our target is to reduce the satisfaction problem of a modal formula for a process to the satisfaction of suitable formulae for its subprocesses, where these formulae are derived from the SOS transition rules.

In the non probabilistic setting, such a problem has been tackled in [2, 12–14, 22], by exploiting *ruloids* [3], which are SOS transition rules that are derived from the SOS specification and define the behavior of open processes in terms of the behavior of their variables. In [2,12,14] the decomposition of modal formulae is used to systematically derive expressive congruence formats for several behavioral equivalences and preorders from their modal characterizations. In [15] such an approach is applied to the reactive probabilistic model [21], which does not admit internal nondeterminism and is therefore less general than PTSs.

In the PTS model, processes perform actions and evolve to probability distributions over processes, i.e. an a-labeled transition is of the form $t \xrightarrow{a} \pi$, with t a process and π a distribution holding all information on the probabilistic behavior arising from this transition. All modal logics developed for the PTS model are equipped with modalities allowing for the specification of the quantitative properties of processes. In essence, this means that some modal formulae are (possibly indirectly) evaluated on distributions. In order to decompose this kind of formulae, we introduce a SOS machinery, called *distribution specification*, allowing us to infer transitions of the form $\pi \xrightarrow{q} t$ whenever the distribution π assigns probability

q to process t. Then we derive the *distribution ruloids*, which allow us to define the behavior of open distributions in terms of the behavior of their distribution variables. These distribution ruloids can support the decomposition of formulae in any modal logic for PTSs.

We present the decomposition of formulae from the two-sorted boolean-valued modal logic $\mathcal{L}$ of [7]. This is an expressive logic, which characterizes probabilistic bisimilarity [7] and bisimilarity metric [4]. We apply our decomposition method also to two subclasses of formulae in $\mathcal{L}$, denoted by $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{+}$, which we prove to characterize resp. probabilistic ready similarity and similarity. Finally, to show the robustness of our approach we apply it to derive the congruence theorem for probabilistic bisimilarity wrt. the PGSOS format [5] and the precongruence theorem for probabilistic ready similarity and similarity wrt. the PGSOS format and the positive PGSOS format, respectively. Summarizing:

1. We present new logical characterizations of probabilistic ready similarity and similarity obtained by means of two sublogics of $\mathcal{L}$, resp. $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{+}$.
2. We define a SOS machinery for the specification of the probabilistic behavior of processes, which can support the decomposition of any modal logic for PTSs.
3. We develop a method of decomposing formulae in $\mathcal{L}$ and in its sublogics $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{+}$.
4. We derive (pre)congruence formats for probabilistic bisimilarity, ready similarity and similarity by exploiting our decomposition method on the logics characterizing them.

2 Probabilistic Transition Systems

The PTS model. A *signature* Σ is a countable set of *operators*. We let $\mathfrak{n}$ range over the rank of the operators. We assume a countable set of (state) *variables* $\mathcal{V}_s$ disjoint from Σ. The set $\mathsf{T}(\Sigma, V)$ of *terms* over Σ and $V \subseteq \mathcal{V}_s$ is defined as usual. By $\mathcal{T}(\Sigma)$ we denote the set of the *closed terms* $\mathsf{T}(\Sigma, \emptyset)$. By $\mathbb{T}(\Sigma)$ we denote the set of the *open terms* $\mathsf{T}(\Sigma, \mathcal{V}_s)$.

Nondeterministic probabilistic transition systems (PTSs) [6,24] extend LTSs by allowing for probabilistic choices in the transitions. The state space is the set of the closed terms $\mathcal{T}(\Sigma)$. The *transitions* are of the form $t \xrightarrow{a} \pi$, with t a term in $\mathcal{T}(\Sigma)$, a an action label and π a probability distribution over $\mathcal{T}(\Sigma)$, i.e. a mapping $\pi\colon \mathcal{T}(\Sigma) \to [0,1]$ with $\sum_{t \in \mathcal{T}(\Sigma)} \pi(t) = 1$. By $\Delta(\mathcal{T}(\Sigma))$ we denote the set of all probability distributions over $\mathcal{T}(\Sigma)$.

▶ **Definition 1** (PTS, [6, 24])**.** A PTS is a triple $(\mathcal{T}(\Sigma), \mathcal{A}, \to)$, where:
(i) Σ is a signature,
(ii) $\mathcal{A}$ is a countable set of *actions*, and
(iii) $\to \subseteq \mathcal{T}(\Sigma) \times \mathcal{A} \times \Delta(\mathcal{T}(\Sigma))$ is a *transition relation*.

We say that a PTS $P = (\mathcal{T}(\Sigma), \mathcal{A}, \to)$ is *image finite* if each closed term in $\mathcal{T}(\Sigma)$ has finitely many outgoing a-labeled transitions for each $a \in \mathcal{A}$.

For $\pi \in \Delta(\mathcal{T}(\Sigma))$, $\mathsf{supp}(\pi) = \{t \in \mathcal{T}(\Sigma) \mid \pi(t) > 0\}$ is the *support* of π. For $t \in \mathcal{T}(\Sigma)$, δ_t is the *Dirac distribution* s.t. $\delta_t(t) = 1$ and $\delta_t(s) = 0$ for $s \neq t$. For $f \in \Sigma$ and $\pi_i \in \Delta(\mathcal{T}(\Sigma))$, $f(\pi_1, \ldots, \pi_{\mathfrak{n}})$ is the distribution defined by $f(\pi_1, \ldots, \pi_{\mathfrak{n}})(f(t_1, \ldots, t_{\mathfrak{n}})) = \prod_{i=1}^{\mathfrak{n}} \pi_i(t_i)$. The convex combination $\sum_{i \in I} p_i \pi_i$ of a family of distributions $\{\pi_i\}_{i \in I} \subseteq \Delta(\mathcal{T}(\Sigma))$ with $p_i \in (0,1]$ and $\sum_{i \in I} p_i = 1$ is defined by $(\sum_{i \in I} p_i \pi_i)(t) = \sum_{i \in I} (p_i \pi_i(t))$ for all $t \in \mathcal{T}(\Sigma)$.

Bisimulation. A (probabilistic) bisimulation is an equivalence relation over $\mathcal{T}(\Sigma)$ equating two terms if they can mimic each other's transitions and evolve to distributions related by the same bisimulation. To formalize this, we need to lift relations over terms to distributions.

▶ **Definition 2** (Relation lifting, [8]). The *lifting* of a relation $\mathcal{R} \subseteq \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ is the relation $\mathcal{R}^\dagger \subseteq \Delta(\mathcal{T}(\Sigma)) \times \Delta(\mathcal{T}(\Sigma))$ with $\pi \,\mathcal{R}^\dagger\, \pi'$ whenever there is a countable set of indexes I s.t.:

(i) $\pi = \sum_{i\in I} p_i \delta_{s_i}$,
(ii) $\pi' = \sum_{i\in I} p_i \delta_{t_i}$, and
(iii) $s_i \;\mathcal{R}\; t_i$ for all $i \in I$.

▶ **Definition 3** (Probabilistic (bi)simulations, [21, 24]). Assume a PTS $(\mathcal{T}(\Sigma), \mathcal{A}, \to)$.

1. A binary relation $\mathcal{R} \subseteq \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ is a *simulation* if, whenever $s \,\mathcal{R}\, t$, if $s \xrightarrow{a} \pi_s$ then there is a transition $t \xrightarrow{a} \pi_t$ s.t. $\pi_s \,\mathcal{R}^\dagger \pi_t$.
2. A simulation $\mathcal{R}$ is a *ready simulation* if, whenever $s \,\mathcal{R}\, t$, if $s \xrightarrow{a}\!\!\!\!\!\!/\;$ then $t \xrightarrow{a}\!\!\!\!\!\!/\;$.
3. A *bisimulation* is a symmetric simulation.

The union of all simulations (resp.: ready simulations, bisimulations) is the greatest simulation (resp.: ready simulation, bisimulation), denoted $\sqsubseteq$ (resp.: $\sqsubseteq_{\mathrm{r}}$, $\sim$), called *similarity* (resp.: *ready similarity*, *bisimilarity*), and is a preorder (resp.: preorder, equivalence).

Logical characterization. As a logic expressing behavioral properties over terms, we consider the modal logic $\mathcal{L}$ of [7], which extends HML [19] with a probabilistic choice modality.

▶ **Definition 4** (Modal logic $\mathcal{L}$, [7]). The classes of *state formulae* $\mathcal{L}^{\mathrm{s}}$ and *distribution formulae* $\mathcal{L}^{\mathrm{d}}$ over $\mathcal{A}$ are defined by the following BNF-like grammar:

$$\mathcal{L}^{\mathrm{s}}\colon\quad \varphi ::= \quad \top \mid \neg\varphi \mid \textstyle\bigwedge_{j\in J} \varphi_j \mid \langle a\rangle\psi \qquad\qquad \mathcal{L}^{\mathrm{d}}\colon\quad \psi ::= \quad \bigoplus_{i\in I} r_i\varphi_i$$

where:

(i) φ ranges over $\mathcal{L}^{\mathrm{s}}$,
(ii) ψ ranges over $\mathcal{L}^{\mathrm{d}}$,
(iii) $a \in \mathcal{A}$,
(iv) I, J are at most countable sets of indexes with $I, J \neq \emptyset$, and
(v) $r_i \in (0, 1]$ for each $i \in I$ and $\sum_{i\in I} r_i = 1$.

We shall write $\langle a\rangle\varphi$ for $\langle a\rangle \bigoplus_{i\in I} r_i\varphi_i$ with $I = \{i\}$, $r_i = 1$ and $\varphi_i = \varphi$.

▶ **Definition 5** (Satisfaction relation, [7]). The *satisfaction relation* $\models \subseteq (\mathcal{T}(\Sigma) \times \mathcal{L}^{\mathrm{s}}) \cup (\Delta(\mathcal{T}(\Sigma)) \times \mathcal{L}^{\mathrm{d}})$ is defined by structural induction on formulae by

- $t \models \top$ always;
- $t \models \neg\varphi$ iff $t \models \varphi$ does not hold;
- $t \models \bigwedge_{j\in J} \varphi_j$ iff $t \models \varphi_j$ for all $j \in J$;
- $t \models \langle a\rangle\psi$ iff $t \xrightarrow{a} \pi$ for a distribution $\pi \in \Delta(\mathcal{T}(\Sigma))$ with $\pi \models \psi$;
- $\pi \models \bigoplus_{i\in I} r_i\varphi_i$ iff $\pi = \sum_{i\in I} r_i\pi_i$ for distributions π_i with $t \models \varphi_i$ for all $t \in \mathsf{supp}(\pi_i)$.

Dealing with $\mathcal{L}$ is motivated by its characterization of bisimilarity, proved in [7] (see Thm. 6 below), bisimilarity metric, proved in [4], and similarity and ready similarity, proved here (see Thm. 8 below).

▶ **Theorem 6** ([7]). *Assume an image finite PTS $(\mathcal{T}(\Sigma), \mathcal{A}, \to)$ and terms $s, t \in \mathcal{T}(\Sigma)$. Then, $s \sim t$ if and only if they satisfy the same formulae in $\mathcal{L}^{\mathrm{s}}$.*

The characterization of ready similarity and similarity requires two subclasses of $\mathcal{L}$.

▶ **Definition 7.** The classes of *ready formulae* $\mathcal{L}_{\mathrm{r}}$ and *positive formulae* $\mathcal{L}_{+}$ are defined as

$$\begin{array}{lll} \mathcal{L}_{\mathrm{r}}^{\mathrm{s}}: & \varphi ::= & \top \mid \bar{a} \mid \bigwedge_{j\in J} \varphi_j \mid \langle a\rangle\psi \\ \mathcal{L}_{+}^{\mathrm{s}}: & \varphi ::= & \top \mid \bigwedge_{j\in J} \varphi_j \mid \langle a\rangle\psi \end{array} \qquad \begin{array}{lll} \mathcal{L}_{\mathrm{r}}^{\mathrm{d}}: & \psi ::= & \bigoplus_{i\in I} r_i\varphi_i \\ \mathcal{L}_{+}^{\mathrm{d}}: & \psi ::= & \bigoplus_{i\in I} r_i\varphi_i \end{array}$$

where $\bar{a}$ stays for $\neg\langle a\rangle\top$.

The classes $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{+}$ are strict sublogics of the one proposed in [9] for the characterization of failure similarity and forward similarity [24]. In particular, the logic used in [9] allows for arbitrary formulae to occur after the diamond modality. We can show that our sublogics are powerful enough for the characterization of ready similarity and similarity.

▶ **Theorem 8.** *Assume an image finite PTS* $(\mathcal{T}(\Sigma), \mathcal{A}, \to)$ *and terms* $s, t \in \mathcal{T}(\Sigma)$*. Then:*

1. $s \sqsubseteq_{\mathrm{r}} t$ *iff for any formula* $\varphi \in \mathcal{L}_{\mathrm{r}}^{\mathrm{s}}$*,* $s \models \varphi$ *implies* $t \models \varphi$*.*
2. $s \sqsubseteq t$ *iff for any formula* $\varphi \in \mathcal{L}_{+}^{\mathrm{s}}$*,* $s \models \varphi$ *implies* $t \models \varphi$*.*

Probabilistic transition system specifications. PTSs are usually defined by means of SOS rules, which are syntax-driven inference rules allowing us to infer the behavior of terms inductively wrt. their structure. Here we consider rules in the probabilistic GSOS format [5] (examples in Ex. 10), which allow for specifying most of probabilistic process algebras [16,18].

In these rules we need syntactic expressions that denote probability distributions. We assume a countable set of *distribution variables* $\mathcal{V}_d$. We denote by $\mathcal{V}$ the set of state and distribution variables $\mathcal{V} = \mathcal{V}_s \cup \mathcal{V}_d$. We let $\mu, \nu, \ldots$ range over $\mathcal{V}_d$ and ζ range over $\mathcal{V}$. The set of *distribution terms* over Σ, $V_s \subseteq \mathcal{V}_s$ and $V_d \subseteq \mathcal{V}_d$, notation $\mathsf{DT}(\Sigma, V_s, V_d)$, is the least set satisfying:

(i) $\{\delta_t \mid t \in \mathsf{T}(\Sigma, V_s)\} \subseteq \mathsf{DT}(\Sigma, V_s, V_d)$,
(ii) $V_d \subseteq \mathsf{DT}(\Sigma, V_s, V_d)$,
(iii) $f(\Theta_1, \ldots, \Theta_n) \in \mathsf{DT}(\Sigma, V_s, V_d)$ whenever $f \in \Sigma$ and $\Theta_i \in \mathsf{DT}(\Sigma, V_s, V_d)$, and
(iv) $\sum_{i\in I} p_i\Theta_i \in \mathsf{DT}(\Sigma, V_s, V_d)$ whenever $\Theta_i \in \mathsf{DT}(\Sigma, V_s, V_d)$ and $p_i \in (0, 1]$ with $\sum_{i\in I} p_i = 1$.

We write $\mathbb{DT}(\Sigma)$ for $\mathsf{DT}(\Sigma, \mathcal{V}_s, \mathcal{V}_d)$, i.e. the set of all *open distribution terms*, and $\mathcal{DT}(\Sigma)$ for $\mathsf{DT}(\Sigma, \emptyset, \emptyset)$, i.e. the set of all *closed distribution terms*. Distribution terms have the following meaning. An *instantiable Dirac distribution* δ_t instantiates to $\delta_{t'}$ if t instantiates to t'. A *distribution variable* $\mu \in \mathcal{V}_d$ is a variable that takes values from $\Delta(\mathcal{T}(\Sigma))$. Case (3) lifts the structural inductive construction of terms to distribution terms. Case (4) allows us to construct convex combinations of distributions. By $\mathrm{var}(t)$ (resp. $\mathrm{var}(\Theta)$) we denote the set of the variables occurring in t (resp. Θ).

A *positive (resp. negative) literal* is an expression of the form $t \xrightarrow{a} \Theta$ (resp. $t \not\xrightarrow{a}$) with $t \in \mathbb{T}(\Sigma)$, $a \in \mathcal{A}$ and $\Theta \in \mathbb{DT}(\Sigma)$. The literals $t \xrightarrow{a} \Theta$ and $t \not\xrightarrow{a}$ are said to *deny* each other.

▶ **Definition 9** (PGSOS rules, [5])**.** A *PGSOS rule* r has the form:

$$\frac{\{x_i \xrightarrow{a_{i,m}} \mu_{i,m} \mid i \in I, m \in M_i\} \qquad \{x_i \xrightarrow{a_{i,n}} \not\rightarrow \mid i \in I, n \in N_i\}}{f(x_1, \ldots, x_{\mathfrak{n}}) \xrightarrow{a} \Theta}$$

with $f \in \Sigma$, $I = \{1, \ldots, \mathfrak{n}\}$, M_i, N_i finite indexes sets, $a_{i,m}, a_{i,n}, a \in \mathcal{A}$ actions, $x_i \in \mathcal{V}_s, \mu_{i,m} \in \mathcal{V}_d$ variables and $\Theta \in \mathbb{DT}(\Sigma)$ a distribution term. Furthermore, all $\mu_{i,m}$ for $i \in I$ and $m \in M_i$ are distinct, all $x_1, \ldots, x_{\mathfrak{n}}$ are distinct, and $\mathrm{var}(\Theta) \subseteq \{\mu_{i,m} \mid i \in I, m \in M_i\} \cup \{x_1, \ldots, x_{\mathfrak{n}}\}$.

We say that $P = (\Sigma, \mathcal{A}, R)$, with Σ a signature, $\mathcal{A}$ a countable set of actions and R a finite set of PGSOS rules, is a *PGSOS probabilistic transition system specification (PGSOS-PTSS).*

▶ **Example 10.** The operators of synchronous parallel composition $|$ and probabilistic alternative composition $+_p$, with $p \in (0,1]$, are specified by the following PGSOS rules:

$$\frac{x \xrightarrow{a} \mu \quad y \xrightarrow{a} \nu}{x|y \xrightarrow{a} \mu|\nu} \qquad \frac{x \xrightarrow{a} \mu \quad y \not\xrightarrow{a}}{x +_p y \xrightarrow{a} \mu} \qquad \frac{x \not\xrightarrow{a} \quad y \xrightarrow{a} \nu}{x +_p y \xrightarrow{a} \nu} \qquad \frac{x \xrightarrow{a} \mu \quad y \xrightarrow{a} \nu}{x +_p y \xrightarrow{a} p\mu + (1-p)\nu}\,.$$

For a PGSOS rule r, the positive (resp. negative) literals above the line are the *positive premises*, notation $\mathrm{pprem}(r)$ (resp. *negative premises*, notation $\mathrm{nprem}(r)$). The literal $f(x_1,\dots,x_{\mathtt{n}}) \xrightarrow{a} \Theta$ is called the *conclusion*, notation $\mathrm{conc}(r)$, the term $f(x_1,\dots,x_{\mathtt{n}})$ is called the *source* and the distribution term Θ is called the *target*.

A PGSOS rule r is said to be *positive* if $\mathrm{nprem}(r) = \emptyset$. Then we say that a PGSOS-PTSS $P = (\Sigma, \mathcal{A}, R)$ is *positive* if all the PGSOS rules in R are positive.

A PTS is derived from a PTSS through the notions of substitution and proof.

A *substitution* is a mapping $\sigma\colon \mathcal{V} \to \mathbb{T}(\Sigma) \cup \mathbb{DT}(\Sigma)$ s.t. $\sigma(x) \in \mathbb{T}(\Sigma)$ if $x \in \mathcal{V}_s$ and $\sigma(\mu) \in \mathbb{DT}(\Sigma)$ if $\mu \in \mathcal{V}_d$. It extends to terms, literals and rules by element-wise application. A substitution is *closed* if it maps variables to closed terms. A closed substitution instance of a literal (resp. PGSOS rule) is called a *closed literal* (resp. *closed PGSOS rule)*.

▶ **Definition 11** (Proof)**.** A *proof* from a PTSS P of a closed literal α is a well-founded, upwardly branching tree, with nodes labeled by closed literals, s.t. the root is labeled α and, if β is the label of a node $\mathfrak{q}$ and $\mathcal{K}$ is the set of labels of the nodes directly above $\mathfrak{q}$, then:

- either β is positive and $\mathcal{K}/\beta$ is a closed substitution instance of a rule in R,
- or β is negative and for each closed substitution instance of a rule in R whose conclusion denies β, a literal in $\mathcal{K}$ denies one of its premises.

A literal α is *provable* from P, notation $P \vdash \alpha$, if there exists a proof from P of α.

The set of literals provable from a PGSOS-PTSS P is unique and contains literals that do not deny each other [3]. The *model induced by* P is the PTS $(\mathcal{T}(\Sigma), \mathcal{A}, \to)$ whose transition relation $\to$ contains exactly the closed positive literals provable from P.

3 Distribution specifications

The decomposition of state formulae in Sec. 4 is based on a collection of rules extracted from the PTSS, called ruloids. To have a similar method for distribution formulae, we develop a SOS-like machinery allowing us to infer the expression $\Theta \xrightarrow{q} t$ whenever a closed distribution term Θ assigns probability weight q to a closed term t. Such a machinery can be exploited also to decompose formulae of any logic, and can be easily generalized to cover the case of sub-distributions that are used in models different from PTSs.

A *distribution literal* is of the form $\Theta \xrightarrow{q} t$, with $\Theta \in \mathbb{DT}(\Sigma)$, $q \in (0,1]$ and $t \in \mathbb{T}(\Sigma)$. A set of distribution literals $\{\Theta \xrightarrow{q_i} t_i \mid i \in I\}$ is a *distribution over terms* if $\sum_{i\in I} q_i = 1$ and all t_i are distinct. This expresses that Θ is the distribution over $\mathbb{T}(\Sigma)$ giving weight q_i to t_i.

To infer distributions over terms $\{\Theta \xrightarrow{q_i} t_i \mid i \in I\}$ inductively wrt. the structure of Θ, we introduce the Σ-*distribution rules*. Let $\delta_{\mathcal{V}_s} := \{\delta_x \mid x \in \mathcal{V}_s\}$ denote the set of all instantiable Dirac distributions with a variable as term, and $\vartheta, \vartheta_i, \dots$ denote distribution terms in $\mathbb{DT}(\Sigma)$ ranging over $\mathcal{V}_d \cup \delta_{\mathcal{V}_s}$. Then, for arbitrary sets $S_1,\dots,S_n$, we denote by $\bigtimes_{i=1}^{n} S_i$ the set of tuples $k = [s_1,\dots,s_n]$ with $s_i \in S_i$. The i-th element of k is denoted $k(i)$.

▶ **Definition 12** (Σ-distribution rules)**.** Assume a signature Σ. The set R_Σ of the Σ-*distribution rules* consists of the least set containing the following inference rules:

1. $\{\delta_x \xrightarrow{1} x\}$ for any state variable $x \in \mathcal{V}_s$;

2. $$\frac{\bigcup_{i=1,\dots,\mathfrak{n}} \left\{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i, \sum_{j\in J_i} q_{i,j} = 1\right\}}{\left\{f(\vartheta_1,\dots,\vartheta_\mathfrak{n}) \xrightarrow{q_k} f(x_{1,k(1)},\dots,x_{\mathfrak{n},k(\mathfrak{n})}) \mid q_k = \prod_{i=1,\dots,\mathfrak{n}} q_{i,k(i)},\, k \in \bigtimes_{i=1,\dots,\mathfrak{n}} J_i\right\}}$$
where $f \in \Sigma$, the distribution terms $\vartheta_i \in \mathcal{V}_d \cup \delta_{\mathcal{V}_s}$ are all distinct, and for each $i = 1,\dots,\mathfrak{n}$ the state variables $x_{i,j}$'s with $j \in J_i$ are pairwise distinct;

3. $$\frac{\bigcup_{i\in I} \left\{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i, \sum_{j\in J_i} q_{i,j} = 1\right\}}{\left\{\sum_{i\in I} p_i\vartheta_i \xrightarrow{q_x} x \mid q_x = \sum_{i\in I, j\in J_i \text{ s.t. } x_{i,j}=x} p_i \cdot q_{i,j} \text{ and } x \in \{x_{i,j} \mid j \in J_i, i \in I\}\right\}}$$
where I is an at most countable set of indexes, the distribution terms $\vartheta_i \in \mathcal{V}_d \cup \delta_{\mathcal{V}_s}$ are all distinct, and for each $i \in I$ the state variables $x_{i,j}$'s with $j \in J_i$ are pairwise distinct.

Then, the Σ-*distribution specification* (Σ-*DS*) is the pair $D_\Sigma = (\Sigma, R_\Sigma)$.

For each Σ-distribution rule r_D, all sets above the line are called *premises*, notation $\mathrm{prem}(r_\mathsf{D})$, and the set below the line is called *conclusion*, notation $\mathrm{conc}(r_\mathsf{D})$. It is not hard to see that all premises and the conclusion are distributions over terms.

▶ **Example 13.** An example of Σ-distribution rule with source $\mu|\nu$ is the following:

$$\frac{\{\mu \xrightarrow{1/4} x_1, \quad \mu \xrightarrow{3/4} x_2\} \quad \{\nu \xrightarrow{1/3} y_1, \quad \nu \xrightarrow{2/3} y_2\}}{\{\mu|\nu \xrightarrow{1/12} x_1|y_1, \quad \mu|\nu \xrightarrow{1/6} x_1|y_2, \quad \mu|\nu \xrightarrow{1/4} x_2|y_1, \quad \mu|\nu \xrightarrow{1/2} x_2|y_2\}}\,.$$

The following notion of reduction wrt. a substitution allows us to extend the notion of substitution to distributions over terms and, then, to Σ-distribution rules.

▶ **Definition 14** (Reduction wrt. a substitution)**.** Assume a substitution σ and a distribution over terms $L = \{\Theta \xrightarrow{q_i} t_i \mid i \in I\}$. We say that σ *reduces* L to the set of distribution literals $L' = \{\sigma(\Theta) \xrightarrow{q_j} t_j \mid j \in J\}$, or that L' is the *reduction wrt.* σ of L, notation $\sigma(L) = L'$, if:

- for each index $j \in J$ there is at least one index $i \in I$ with $\sigma(t_i) = t_j$;
- the terms $\{t_j \mid j \in J\}$ are pairwise distinct;
- for each index $j \in J$, we have $q_j = \sum_{\{i\in I \mid \sigma(t_i)=t_j\}} q_i$.

▶ **Proposition 15.** *For a substitution σ and a distribution over terms L, the set of distribution literals $\sigma(L)$ is a distribution over terms.*

▶ **Definition 16** (Reduced instance of a Σ-distribution rule)**.** The *reduced instance* of a Σ-distribution rule r_D wrt. a substitution σ is the inference rule $\sigma(r_\mathsf{D})$ s.t.:

1. If r_D is as in Def. 12.1, then $\sigma(r_\mathsf{D}) = \{\delta_{\sigma(x)} \xrightarrow{1} \sigma(x)\}$.
2. If r_D is as in Def. 12.2, then
$$\sigma(r_\mathsf{D}) = \frac{\bigcup_{i=1,\dots,\mathfrak{n}} \{\sigma(\vartheta_i) \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}}{\left\{f(\sigma(\vartheta_1),\dots,\sigma(\vartheta_\mathfrak{n})) \xrightarrow{q_\kappa} f(t_{1,\kappa(1)},\dots,t_{\mathfrak{n},\kappa(\mathfrak{n})}) \mid q_\kappa = \prod_{i=1,..,\mathfrak{n}} q_{i,\kappa(i)},\, \kappa \in \bigtimes_{i=1,..,\mathfrak{n}} H_i\right\}}$$
where $\{\sigma(\vartheta_i) \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\} = \sigma(\{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i\})$.
3. If r_D is as in Def. 12.3, then
$$\sigma(r_\mathsf{D}) = \frac{\bigcup_{i\in I} \{\sigma(\vartheta_i) \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}}{\left\{\sum_{i\in I} p_i\sigma(\vartheta_i) \xrightarrow{q_t} t \mid q_t = \sum_{i\in I, h\in H_i \text{ s.t. } t_{i,h}=t} p_i \cdot q_{i,h},\, t \in \{t_{i,h} \mid h \in H_i, i \in I\}\right\}}$$

where $\{\sigma(\vartheta_i) \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\} = \sigma(\{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i\})$.

Notice that Prop. 15 ensures that the premises of $\sigma(r_{\mathsf{D}})$ are distributions over terms. Moreover, it is not hard to see that also the conclusion of $\sigma(r_{\mathsf{D}})$ is a distribution over terms.

▶ **Definition 17** (Proof from the Σ-DS)**.** A *proof* from the Σ-DS D_Σ of a closed distribution over terms L is a well-founded, upwardly branching tree, whose nodes are labeled by closed distributions over terms, s.t. the root is labeled L, and, if β is the label of a node $\mathfrak{q}$ and $\mathcal{K}$ is the set of labels of the nodes directly above $\mathfrak{q}$, then $\mathcal{K}/\beta$ is a closed reduced instance of a Σ-distribution rule in R_Σ.

A closed distribution over terms L is *provable* from D_Σ, notation $D_\Sigma \vdash L$, if there exists a proof from D_Σ for L.

Since Σ-distribution rules have only positive premises, the set of the distribution over terms provable from the Σ-DS is unique. The following result confirms that all probability distributions over $\mathcal{T}(\Sigma)$ can be inferred through the Σ-DS.

▶ **Proposition 18.** *Assume a signature* Σ*. Let* $\pi \in \mathcal{DT}(\Sigma)$ *be a closed distribution term and* $\{t_m\}_{m\in M} \subseteq \mathcal{T}(\Sigma)$ *a set of pairwise distinct closed terms. Then*

$$D_\Sigma \vdash \{\pi \xrightarrow{q_m} t_m \mid m \in M\} \Leftrightarrow \textit{for all } m \in M \textit{ it holds } \pi(t_m) = q_m, \textit{ and } \sum_{m\in M} q_m = 1.$$

4 The decomposition method

In this section we present our method for decomposing formulae in $\mathcal{L}$, $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{+}$. Our aim is to reduce the satisfaction problem of a formula for a (distribution) term to the satisfaction problem of derived formulae for its subterms. In Sec. 4.1 we define *ruloids* and *distribution ruloids*, namely derived (distribution) rules allowing us to infer the behavior of any (distribution) term from the behavior of its variables. Both classes of ruloids are *sound and specifically witnessing* [3], i.e. a closed literal α (resp. a distribution over terms L) is provable from a PGSOS-PTSS (resp. the Σ-DS) iff α (resp. L) is an instance of the conclusion of a ruloid (resp. distribution ruloid) (Thm. 21 and Thm. 24). Then, in Sec. 4.2 we exploit the two classes of ruloids for the decomposition. The decomposition of state formulae follows [2, 12–15] and consists in assigning to each term $t \in \mathbb{T}(\Sigma)$ and formula $\varphi \in \mathcal{L}^{\mathrm{s}}$, a set of functions $\xi\colon \mathcal{V}_s \to \mathcal{L}^{\mathrm{s}}$, called *decomposition mappings*, assigning to each variable x in t a proper formula in $\mathcal{L}^{\mathrm{s}}$ s.t. for any closed substitution σ it holds that $\sigma(t) \models \varphi$ iff $\sigma(x) \models \xi(x)$ for each $x \in \mathrm{var}(t)$ (Thm. 28). Each mapping ξ is defined on a ruloid having t as source. The decomposition of distribution formulae consists in assigning to each distribution term $\Theta \in \mathbb{DT}(\Sigma)$ and distribution formula $\psi \in \mathcal{L}^{\mathrm{d}}$ a set of decomposition mappings $\eta\colon \mathcal{V} \to \mathcal{L}^{\mathrm{d}} \cup \mathcal{L}^{\mathrm{s}}$ s.t. for any closed substitution σ we get that $\sigma(\Theta) \models \psi$ iff $\sigma(\zeta) \models \eta(\zeta)$ for each $\zeta \in \mathrm{var}(\Theta)$ (Thm. 28).

4.1 Ruloids

Ruloids are defined by an inductive composition of PGSOS rules. All PGSOS rules are ruloids. Then, from a rule r and substitution σ, a ruloid ρ with conclusion $\sigma(\mathrm{conc}(r))$ is built as follows:

1. for each positive premise α in $\sigma(r)$, we take any ruloid having α as conclusion and we put its premises among the premises of ρ;

2. for each negative premise α in $\sigma(r)$ and for each ruloid ρ' having any literal denying α as conclusion, we select any premise β of ρ', we take any literal β' denying β, and we put β' among the premises of ρ.

For a PGSOS-PTSS $P = (\Sigma, \mathcal{A}, R)$, let $\mathrm{Lit}(P)$ denote the set of literals that can be built with terms in $\mathbb{T}(\Sigma) \cup \mathbb{DT}(\Sigma)$ and actions in $\mathcal{A}$.

▶ **Definition 19** (Ruloids). Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS. The set of *P-ruloids* $\Re^P$ is the smallest set s.t.:

- $\dfrac{x \xrightarrow{a} \mu}{x \xrightarrow{a} \mu}$ is a P-ruloid for all $x \in \mathcal{V}_s$, $a \in \mathcal{A}$ and $\mu \in \mathcal{V}_d$;
- $\dfrac{\bigcup_{i \in I} \left(\bigcup_{m \in M_i} \mathcal{H}_{i,m} \cup \bigcup_{n \in N_i} \mathcal{H}_{i,n} \right)}{f(t_1, \ldots, t_{\mathfrak{n}}) \xrightarrow{a} \Theta}$ is a P-ruloid if there is a PGSOS rule $r \in R$

$$\frac{\{x_i \xrightarrow{a_{i,m}} \mu_{i,m} \mid i \in I, m \in M_i\} \qquad \{x_i \xrightarrow{a_{i,n}} \not\rightarrow \mid i \in I, n \in N_i\}}{f(x_1, \ldots, x_{\mathfrak{n}}) \xrightarrow{a} \Theta'}$$

together with a substitution σ, with $\sigma(x_i) = t_i$ for $i = 1, \ldots, \mathfrak{n}$ and $\sigma(\Theta') = \Theta$, s.t.:

- for every positive premise $x_i \xrightarrow{a_{i,m}} \mu_{i,m}$ of r
  - either $\sigma(x_i)$ is a variable and $\mathcal{H}_{i,m} = \{\sigma(x_i) \xrightarrow{a_{i,m}} \sigma(\mu_{i,m})\}$,
  - or there is a P-ruloid $\rho_{i,m} = \mathcal{H}_{i,m} / \sigma(x_i) \xrightarrow{a_{i,m}} \sigma(\mu_{i,m})$;
- for every negative premise $x_i \xrightarrow{a_{i,n}} \not\rightarrow$ of r
  - either $\sigma(x_i)$ is a variable and $\mathcal{H}_{i,n} = \{\sigma(x_i) \xrightarrow{a_{i,n}} \not\rightarrow\}$,
  - or $\mathcal{H}_{i,n} = \mathrm{opp}(\mathrm{pick}(\Re^P_{(a_{i,n})}))$, where:
    - **(i)** define $\Re^P_{(a_{i,n})} \in \mathcal{P}(\mathcal{P}(\mathrm{Lit}(P)))$ as the set containing the sets of the premises of all P-ruloids with conclusion $\sigma(x_i) \xrightarrow{a_{i,n}} \theta$, formally

$$\Re^P_{(a_{i,n})} = \{\mathrm{prem}(\rho) \mid \rho \in \Re^P \text{ and } \mathrm{conc}(\rho) = \sigma(x_i) \xrightarrow{a_{i,n}} \theta \text{ for some } \theta \in \mathbb{DT}(\Sigma)\},$$

    - **(ii)** define any mapping $\mathrm{pick} \colon \mathcal{P}(\mathcal{P}(\mathrm{Lit}(P))) \to \mathcal{P}(\mathrm{Lit}(P))$ s.t. for any set of literals L_k with $k \in K$, $\mathrm{pick}(\{L_k \mid k \in K\}) = \{l_k \mid k \in K \wedge l_k \in L_k\}$,
    - **(iii)** define any mapping $\mathrm{opp} \colon \mathcal{P}(\mathrm{Lit}(P)) \to \mathcal{P}(\mathrm{Lit}(P))$ satisfying $\mathrm{opp}(L) = \{\mathrm{opp}(l) \mid l \in L\}$ for all set of literals L, where $\mathrm{opp}(t' \xrightarrow{a} \theta) = t' \xrightarrow{a} \not\rightarrow$, and $\mathrm{opp}(t' \xrightarrow{a} \not\rightarrow) = t' \xrightarrow{a} \theta$ for some fresh distribution term θ;
- right hand side variables $\mathrm{rhs}(\rho_{i,m})$ are all pairwise disjoint.

▶ **Example 20.** From rules in Ex. 10, we can build the following ruloids for term $x +_p (y|z)$:

$$\frac{x \xrightarrow{a} \mu \quad y \xrightarrow{a} \not\rightarrow}{x +_p (y|z) \xrightarrow{a} \mu} \quad \frac{x \xrightarrow{a} \mu \quad z \xrightarrow{a} \not\rightarrow}{x +_p (y|z) \xrightarrow{a} \mu} \quad \frac{x \xrightarrow{a} \not\rightarrow \quad y \xrightarrow{a} \nu \quad z \xrightarrow{a} \upsilon}{x +_p (y|z) \xrightarrow{a} \nu|\upsilon} \quad \frac{x \xrightarrow{a} \mu \quad y \xrightarrow{a} \nu \quad z \xrightarrow{a} \upsilon}{x +_p (y|z) \xrightarrow{a} p\mu + (1-p)(\nu|\upsilon)}.$$

We describe the construction of the first ruloid: $\dfrac{\dfrac{x \xrightarrow{a} \mu}{x \xrightarrow{a} \mu} \qquad \dfrac{y \xrightarrow{a} \not\rightarrow}{y|z \xrightarrow{a} \not\rightarrow}}{x +_p (y|z) \xrightarrow{a} \mu}$.

It is not hard to see that if the PTSS is positive then also the derived ruloids are positive.

▶ **Theorem 21** (Ruloid theorem). *Assume a PGSOS-PTSS P and a closed substitution σ. Then $P \vdash \sigma(t) \xrightarrow{a} \Theta'$ for $t \in \mathbb{T}(\Sigma)$ and $\Theta' \in \mathcal{DT}(\Sigma)$ if and only if there are a P-ruloid $\dfrac{\mathcal{H}}{t \xrightarrow{a} \Theta}$ and a closed substitution σ' with $P \vdash \sigma'(\mathcal{H})$, $\sigma'(t) = \sigma(t)$ and $\sigma'(\Theta) = \Theta'$.*

As the Σ-DS is positive, the definition of Σ-*distribution ruloids* results technically simpler.

▶ **Definition 22** (Distribution ruloids). Let $D_\Sigma = (\Sigma, R_\Sigma)$ be the Σ-DS. The set of Σ-*distribution ruloids* $\Re^\Sigma$ is the smallest set s.t.:

- $\dfrac{\{\delta_x \xrightarrow{1} x\}}{\{\delta_x \xrightarrow{1} x\}}$ is a Σ-distribution ruloid for any $x \in \mathcal{V}_s$;
- $\dfrac{\{\mu \xrightarrow{q_i} x_i \mid \sum_{i\in I} q_i = 1\}}{\{\mu \xrightarrow{q_i} x_i \mid i \in I\}}$ is a Σ-distribution ruloid for any $\mu \in \mathcal{V}_d$;
- $\dfrac{\bigcup_{i=1,\dots,\mathfrak{n}} \mathcal{H}_i}{\left\{f(\Theta_1,\dots,\Theta_\mathfrak{n}) \xrightarrow{Q_m} f(t_{1,m},\dots,t_{\mathfrak{n},m}) \mid m \in M\right\}}$ is a Σ-distribution ruloid if there is a Σ-distribution rule $r_\mathsf{D} \in R_\Sigma$ of the form

$$\frac{\bigcup_{i=1,\dots,\mathfrak{n}} \{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i, \sum_{j\in J_i} q_{i,j} = 1\}}{\left\{f(\vartheta_1,\dots,\vartheta_\mathfrak{n}) \xrightarrow{q_k} f(x_{1,k(1)},\dots,x_{\mathfrak{n},k(\mathfrak{n})}) \mid q_k = \prod_{i=1,\dots,\mathfrak{n}} q_{i,k(i)},\ k \in \bigtimes_{i=1,\dots,\mathfrak{n}} J_i\right\}}$$

together with a substitution σ, with $\sigma(\vartheta_i) = \Theta_i$ for $i = 1,\dots,n$, s.t.:

- $\sigma(r_\mathsf{D}) = \dfrac{\bigcup_{i=1,\dots,\mathfrak{n}} \{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i, \sum_{h\in H_i} q_{i,h} = 1\}}{\left\{f(\Theta_1,\dots,\Theta_\mathfrak{n}) \xrightarrow{q_\kappa} f(t_{1,\kappa(1)},..,t_{\mathfrak{n},\kappa(\mathfrak{n})}) \mid q_\kappa = \prod_{i=1,..,\mathfrak{n}} q_{i,\kappa(i)},\ \kappa \in \bigtimes_{i=1,..,\mathfrak{n}} H_i\right\}}$,
- there is a bijection $\mathfrak{f}\colon \bigtimes_{i=1}^{\mathfrak{n}} H_i \to M$ such that $t_{i,\kappa(i)} = t_{i,\mathfrak{f}(\kappa)}$ and $q_\kappa = Q_{\mathfrak{f}(\kappa)}$,
- for every Θ_i with $i = 1,\dots,\mathfrak{n}$ we have that:
  - either Θ_i is a variable or a Dirac distribution and $\mathcal{H}_i = \{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}$,
  - or there is a Σ-distribution ruloid $\rho_i^\mathsf{D} = \mathcal{H}_i / \{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}$;
- $\dfrac{\bigcup_{i\in I} \mathcal{H}_i}{\left\{\sum_{i\in I} p_i\Theta_i \xrightarrow{Q_m} t_m \mid m \in M\right\}}$ is a Σ-distribution ruloid if there is a Σ-distribution rule $r_\mathsf{D} \in R_\Sigma$ of the form

$$\frac{\bigcup_{i\in I}\{\vartheta_i \xrightarrow{q_{i,j}} x_{i,j} \mid j \in J_i, \sum_{j\in J_i} q_{i,j} = 1\}}{\left\{\sum_{i\in I} p_i\vartheta_i \xrightarrow{q_x} x \mid q_x = \sum_{i\in I, j\in J_i \text{ s.t. } x_{i,j}=x} p_i \cdot q_{i,j},\ x \in \{x_{i,j} \mid j \in J_i, i \in I\}\right\}}$$

together with a substitution σ, with $\sigma(\vartheta_i) = \Theta_i$ for $i \in I$, s.t.:

- $\sigma(r_\mathsf{D}) = \dfrac{\bigcup_{i\in I}\{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i, \sum_{h\in H_i} q_{i,h} = 1\}}{\left\{\sum_{i\in I} p_i\Theta_i \xrightarrow{q_u} u \mid q_u = \sum_{i\in I, h\in H_i \text{ s.t. } t_{i,h}=u} p_i \cdot q_{i,h},\ u \in \{t_{i,h} \mid h \in H_i, i \in I\}\right\}}$
- there is a bijection $\mathfrak{f}\colon \{t_{i,h} \mid h \in H_i, i \in I\} \to M$ s.t. $u = t_{\mathfrak{f}(u)}$ and $q_u = Q_{\mathfrak{f}(u)}$,
- for every Θ_i with $i \in I$ we have that:
  - either Θ_i is a variable or a Dirac distribution and $\mathcal{H}_i = \{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}$,
  - or there is a Σ-distribution ruloid $\rho_i^\mathsf{D} = \mathcal{H}_i / \{\Theta_i \xrightarrow{q_{i,h}} t_{i,h} \mid h \in H_i\}$.

▶ **Example 23.** Consider the distribution term $\frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon)$, which is an instance of the target of the fourth ruloid in Ex. 20. Then, we can build the following Σ-distribution ruloid:

$$\dfrac{\dfrac{\{\mu \xrightarrow{1/4} x_1 \quad \mu \xrightarrow{3/4} x_2\}}{\{\mu \xrightarrow{1/4} x_1 \quad \mu \xrightarrow{3/4} x_2\}} \qquad \dfrac{\{\nu \xrightarrow{1/3} y_1, \quad \nu \xrightarrow{2/3} y_2\} \quad \{\upsilon \xrightarrow{1} w\}}{\{\nu|\upsilon \xrightarrow{1/3} y_1|w \quad \nu|\upsilon \xrightarrow{2/3} y_2|w\}}}{\left\{\frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon) \xrightarrow{\frac{1}{10}} x_1, \frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon) \xrightarrow{\frac{3}{10}} x_2, \frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon) \xrightarrow{\frac{1}{5}} y_1|w, \frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon) \xrightarrow{\frac{2}{5}} y_2|w\right\}}$$

▶ **Theorem 24** (Distribution ruloid theorem)**.** *Assume the Σ-DS D_Σ and a closed substitution σ. Then $D_\Sigma \vdash \{\sigma(\Theta) \xrightarrow{q_m} t_m \mid m \in M\}$ for $\Theta \in \mathbb{DT}(\Sigma)$ and $t_m \in \mathcal{T}(\Sigma)$ pairwise distinct if and only if there are a Σ-distribution ruloid $\dfrac{\mathcal{H}}{\{\Theta \xrightarrow{q_m} u_m \mid m \in M\}}$ and a closed substitution σ' with $D_\Sigma \vdash \sigma'(\mathcal{H})$, $\sigma'(\Theta) = \sigma(\Theta)$ and $\sigma'(u_m) = t_m$ for each $m \in M$.*

Although the construction of our ruloids resembles that in [15], the two classes are quite different. [15] bases on the rule format of [20] instead of the PGSOS format of [5], deals with reactive systems, which are less expressive than PTSs since they do not admit internal nondeterminism, and considers transitions of the form $t \xrightarrow{a,p} t'$, denoting that t evolves by a to t' with probability p. Informally, our ruloids generalize those in [15] in the same way PTSs generalize reactive systems. In fact, to deal with $t \xrightarrow{a,p} t'$, ruloids in [15] are defined by keeping track of rules and ruloids used in their construction, in order to obtain a partitioning over ruloids ensuring that the probabilities of all a-labeled transitions from a term t sum up to either 0 or 1. Here we do not need this technicality, since, given a term t, all ruloids in one partition for t of [15] are captured by one of our ruloids and one Σ-distribution ruloid. Our ruloid captures all the requirements that the subterms of t must satisfy to derive the transition to the desired distribution over terms. The proper probability weights are then assigned by the Σ-distribution ruloid.

4.2 Decomposition of modal formulae

First we need to introduce the notion of *matching* for a distribution term, seen as a probability distribution over terms, and a distribution formula, which can be viewed as a probability distribution over state formulae [4, 7].

▶ **Definition 25** (Matching)**.** Assume $\Theta \in \mathbb{DT}(\Sigma)$, a Σ-distribution ruloid $\mathcal{H}/\{\Theta \xrightarrow{q_m} t_m \mid m \in M\}$ and a distribution formula $\psi = \bigoplus_{i\in I} r_i\varphi_i \in \mathcal{L}^{\mathrm{d}}$. Then a *matching* for Θ and ψ is a distribution over the product space $\mathfrak{w} \in \Delta(\mathbb{T}(\Sigma) \times \mathcal{L}^{\mathrm{s}})$ having Θ and ψ as left and right marginals, that is $\sum_{i\in I} \mathfrak{w}(t_m, \varphi_i) = q_m$ for all $m \in M$ and $\sum_{m\in M} \mathfrak{w}(t_m, \varphi_i) = r_i$ for all $i \in I$. We denote by $\mathfrak{W}(\Theta, \psi)$ the set of all matchings for Θ and ψ.

▶ **Definition 26** (Decomposition of $\mathcal{L}$)**.** Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS and let D_Σ be the Σ-DS. We define the mapping $\cdot^{-1}\colon \mathbb{T}(\Sigma) \to (\mathcal{L}^{\mathrm{s}} \to \mathcal{P}(\mathcal{V}_s \to \mathcal{L}^{\mathrm{s}}))$ as the function that for each $t \in \mathbb{T}(\Sigma)$ and $\varphi \in \mathcal{L}^{\mathrm{s}}$ returns the set $t^{-1}(\varphi) \in \mathcal{P}(\mathcal{V}_s \to \mathcal{L}^{\mathrm{s}})$ of *decomposition mappings* $\xi\colon \mathcal{V}_s \to \mathcal{L}^{\mathrm{s}}$ generated as follows. Let t denote an univariate term. Then:

1. $\xi \in t^{-1}(\top)$ iff $\xi(x) = \top$ for all $x \in \mathcal{V}_s$;
2. $\xi \in t^{-1}(\neg\varphi)$ iff there is a function $\mathfrak{f}\colon t^{-1}(\varphi) \to \mathrm{var}(t)$ s.t.

$$\xi(x) = \bigwedge_{\xi' \in \mathfrak{f}^{-1}(x)} \neg\xi'(x), \text{ if } x \in \mathrm{var}(t), \text{ and } \xi(x) = \top, \text{ otherwise;}$$

3. $\xi \in t^{-1}(\bigwedge_{j\in J} \varphi_j)$ iff there exist decomposition mappings $\xi_j \in t^{-1}(\varphi_j)$, for $j \in J$, s.t.

$$\xi(x) = \bigwedge_{j\in J} \xi_j(x) \text{ for all } x \in \mathcal{V}_s;$$

4. $\xi \in t^{-1}(\langle a \rangle \psi)$ iff there are a P-ruloid $\frac{\mathcal{H}}{t \xrightarrow{a} \Theta}$ and a decomposition mapping $\eta \in \Theta^{-1}(\psi)$ s.t.

$$\xi(x) = \bigwedge_{x \xrightarrow{b} \mu \in \mathcal{H}} \langle b \rangle \eta(\mu) \wedge \bigwedge_{x \xrightarrow{c} \not\in \mathcal{H}} \neg \langle c \rangle \top \wedge \eta(x), \text{ if } x \in \mathrm{var}(t), \text{ and } \xi(x) = \top, \text{ otherwise;}$$

5. $\xi \in (\sigma(t))^{-1}(\varphi)$ for a non injective substitution $\sigma \colon \mathrm{var}(t) \to \mathcal{V}_s$ iff there is a decomposition mapping $\xi' \in t^{-1}(\varphi)$ s.t.

$$\xi(x) = \bigwedge_{y \in \sigma^{-1}(x)} \xi'(y) \quad \text{for all } x \in \mathcal{V}_s.$$

Then we define the mapping $\cdot^{-1} \colon \mathbb{DT}(\Sigma) \to (\mathcal{L}^{\mathrm{d}} \to \mathcal{P}(\mathcal{V} \to \mathcal{L}))$ as the function that for each $\Theta \in \mathbb{DT}(\Sigma)$ and $\psi \in \mathcal{L}^{\mathrm{d}}$ returns the set $\Theta^{-1}(\psi) \in \mathcal{P}(\mathcal{V} \to \mathcal{L})$ of *decomposition mappings* $\eta \colon \mathcal{V} \to \mathcal{L}$ generated as follows. Let Θ denote an univariate distribution term. Then:

6. $\eta \in \Theta^{-1}(\bigoplus_{i \in I} r_i \varphi_i)$ iff there are a Σ-distribution ruloid $\frac{\mathcal{H}}{\{\Theta \xrightarrow{q_m} t_m \mid m \in M\}}$ and a matching $\mathfrak{w} \in \mathfrak{W}(\Theta, \bigoplus_{i \in I} r_i \varphi_i)$ s.t. for all $m \in M$ and $i \in I$ there is a decomposition mapping $\xi_{m,i}$ with $\xi_{m,i} \in t_m^{-1}(\varphi_i)$, if $\mathfrak{w}(t_m, \varphi_i) > 0$, and $\xi_{m,i} \in t_m^{-1}(\top)$, otherwise, s.t.:
   a. for $\mu \in \mathcal{V}_d$ we have $\eta(\mu) = \begin{cases} \bigoplus_{\{\mu \xrightarrow{q_j} x_j \mid \sum_{j \in J} q_j = 1\} \in \mathcal{H}} q_j \bigwedge_{\substack{i \in I \\ m \in M}} \xi_{m,i}(x_j) & \text{if } \mu \in \mathrm{var}(\Theta) \\ 1\top & \text{otherwise} \end{cases}$
   b. for $x \in \mathcal{V}_s$ we have $\eta(x) = \begin{cases} \bigwedge_{i \in I, m \in M} \xi_{m,i}(x) & \text{if } x \in \mathrm{var}(\Theta) \\ \top & \text{otherwise.} \end{cases}$
7. $\eta \in (\sigma(\Theta))^{-1}(\psi)$ for a non injective substitution $\sigma \colon \mathrm{var}(\Theta) \to \mathcal{V}$ iff there is a decomposition mapping $\eta' \in \Theta^{-1}(\psi)$ s.t. for $\zeta \in \mathrm{var}(\sigma(\Theta))$ it holds $\eta'(z) = \eta'(z')$ for all $z, z' \in \sigma^{-1}(\zeta)$ and

$$\eta(\zeta) = \eta'(\tilde{z}) \text{ if } \zeta \in \mathrm{var}(\sigma(\Theta)) \text{ and } \tilde{z} \in \sigma^{-1}(\zeta), \text{ and } \eta(\zeta) = \top \text{ if } \zeta \notin \mathrm{var}(\sigma(\Theta)).$$

We discuss only the decomposition of $\psi = \bigoplus_{i \in I} r_i \varphi_i \in \mathcal{L}^{\mathrm{d}}$. Let σ be a closed substitution and consider $\Theta \in \mathbb{DT}(\Sigma)$. We have $\sigma(\Theta) \models \psi$ iff $\sigma(\Theta) = \sum_{i \in I} r_i \pi_i$ with $t \models \varphi_i$ for all $t \in \mathsf{supp}(\pi_i)$. So, we need to identify which properties each $\sigma(\zeta)$ with $\zeta \in \mathrm{var}(\Theta)$ must satisfy to guarantee that $\sigma(\Theta)$ is such a distribution $\sum_{i \in I} r_i \pi_i$. Assume $\mathsf{supp}(\sigma(\Theta)) = \{t_m \mid m \in M\}$ and $\sigma(\Theta)(t_m) = q_m$. By Prop. 18, this is equivalent to have $D_\Sigma \vdash \{\sigma(\Theta) \xrightarrow{q_m} t_m \mid m \in M\}$. From Thm. 24, $D_\Sigma \vdash \{\sigma(\Theta) \xrightarrow{q_m} t_m \mid m \in M\}$ iff there are a Σ-distribution ruloid $\mathcal{H}/\{\Theta \xrightarrow{q_m} u_m \mid m \in M\}$ and a closed substitution σ' with $\sigma'(\Theta) = \sigma(\Theta)$, $\sigma'(u_m) = t_m$ and $\mathsf{D}_\Sigma \vdash \sigma'(\mathcal{H})$. Since the weights q_m are univocally determined by the distributions over terms in $\mathcal{H}$, we can define, for each $\mu \in \mathrm{var}(\Theta) \cap \mathcal{V}_d$, $\eta(\mu)$ using as weights the q_j in $\{\mu \xrightarrow{q_j} x_j \mid \sum_{j \in J} q_j = 1\} \in \mathcal{H}$. Finally, to ensure that if $\sigma'(u_m) \in \mathsf{supp}(\pi_i)$, then $\sigma'(u_m) \models \varphi_i$, we define $\mathfrak{w}(u_m, \varphi_i)$ positive if $\sigma'(u_m) \in \mathsf{supp}(\pi_i)$ so that we can assign the proper decomposed formula $\xi_{m,i}(x)$ to each $x \in \mathrm{var}(u_m)$. Since each $\sigma'(u_m)$ may occur in the support of more than one π_i, we impose that each $x \in \mathrm{var}(u_m)$ satisfies the conjunction of all the decomposed formulae $\xi_{m,i}(x)$.

▶ **Example 27.** We exemplify two mappings in $t^{-1}(\varphi)$ for $\varphi = \langle a \rangle \psi$, with $\psi = \frac{1}{2}\langle a \rangle \top \oplus \frac{1}{2} \neg \langle a \rangle \top$, and $t = x +_{2/5} (y|z)$, which is the term in Ex. 20 with $p = 2/5$. Let ρ be the last

Table 1 Derived decomposition mappings.

$x_1^{-1}(\langle a\rangle\top) = \{\xi_1\}$	$\xi_1(x_1) = \langle a\rangle\top$, $\xi_1(x) = \top$ for all other variables
$x_2^{-1}(\neg\langle a\rangle\top) = \{\xi_2\}$	$\xi_2(x_2) = \neg\langle a\rangle\top$, $\xi_2(x) = \top$ for all other variables
$(y_1\|w)^{-1}(\neg\langle a\rangle\top) = \{\xi_3, \xi_4\}$	$\xi_3(y_1) = \neg\langle a\rangle\top \quad \xi_3(w) = \top$, $\xi_3(x) = \top$ for all other variables $\xi_4(y_1) = \top \quad \xi_4(w) = \neg\langle a\rangle\top$, $\xi_4(x) = \top$ for all other variables
$(y_2\|w)^{-1}(\langle a\rangle\top) = \{\xi_5\}$	$\xi_5(y_2) = \langle a\rangle\top \quad \xi_5(w) = \langle a\rangle\top$, $\xi_5(x) = \top$ for all other variables

ruloid for t in Ex. 20, $\Theta = \frac{2}{5}\mu + \frac{3}{5}(\nu|\upsilon)$ denote its target, and ρ^D be the Σ-distribution ruloid for Θ in Ex. 23. By Def. 26.4, the decomposition mappings $\xi \in t^{-1}(\varphi)$ built over ρ are s.t.:

$$\xi(x) = \langle a\rangle\eta(\mu) \qquad \xi(y) = \langle a\rangle\eta(\nu) \qquad \xi(z) = \langle a\rangle\eta(\upsilon) \tag{1}$$

where $\eta \in \Theta^{-1}(\psi)$. Consider the matching $\mathfrak{w} \in \mathfrak{W}(\Theta, \psi)$ for Θ and ψ defined through ρ^D by

$$\mathfrak{w}(x_1, \langle a\rangle\top) = 1/10 \quad \mathfrak{w}(x_2, \neg\langle a\rangle\top) = 3/10 \quad \mathfrak{w}(y_1|w, \neg\langle a\rangle\top) = 1/5 \quad \mathfrak{w}(y_2|w, \langle a\rangle\top) = 2/5.$$

For the terms and the formulae to which $\mathfrak{w}$ gives a positive weight, we obtain the decomposition mappings in Tab. 1, where ξ_3 and ξ_4 derive from Def. 26.2.

Next, we construct the decomposition mappings for the variable ν in Θ wrt. ρ^{D} and $\mathfrak{w}$. By Def. 26.6a we consider the weights of the premises of ρ^D having ν as left-hand side, namely $\mathcal{H}_\nu = \{\nu \xrightarrow{1/3} y_1, \quad \nu \xrightarrow{2/3} y_2\}$, and use them as weights of the $\bigoplus$ operator. Then for the variables y_1, y_2 in the right side of $\mathcal{H}_\nu$, we consider the conjunction of the formulae assigned to y_1, y_2 by one mapping from each set in the first column of Tab. 1. The choice of ξ_3 or ξ_4 generates two different mappings in $\Theta^{-1}(\psi)$: by ξ_3 we obtain the mapping $\eta_1 \in \Theta^{-1}(\psi)$ with $\eta_1(\nu) = 1/3\neg\langle a\rangle\top \oplus 2/3\langle a\rangle\top$ and by ξ_4 we obtain the mapping $\eta_2 \in \Theta^{-1}(\psi)$ with $\eta_2(\nu) = 1/3\top \oplus 2/3\langle a\rangle\top$. By applying the same reasoning to μ and υ we obtain

$$\eta_1(\mu) = 1/4\langle a\rangle\top \oplus 3/4\neg\langle a\rangle\top \quad \eta_1(\nu) = 1/3\neg\langle a\rangle\top \oplus 2/3\langle a\rangle\top \quad \eta_1(\upsilon) = 1(\top \wedge \langle a\rangle\top)$$

$$\eta_2(\mu) = 1/4\langle a\rangle\top \oplus 3/4\neg\langle a\rangle\top \quad \eta_2(\nu) = 1/3\top \oplus 2/3\langle a\rangle\top \quad \eta_2(\upsilon) = 1(\neg\langle a\rangle\top \wedge \langle a\rangle\top)$$

where we have omitted multiple occurrences of the $\top$ formulae in conjunctions. Finally, we obtain two mappings in $t^{-1}(\varphi)$ by substituting η with either η_1 or η_2 in Eq. (1).

The following result confirms that our decomposition method is correct.

▶ **Theorem 28** (Decomposition theorem). *Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS and let D_Σ be the Σ-DS. For any $t \in \mathbb{T}(\Sigma)$, closed substitution σ and $\varphi \in \mathcal{L}^{\mathrm{s}}$ we have*

$$\sigma(t) \models \varphi \Leftrightarrow \exists \xi \in t^{-1}(\varphi) \text{ s.t. for all } x \in \mathrm{var}(t) \text{ it holds } \sigma(x) \models \xi(x)$$

and for any $\Theta \in \mathbb{DT}(\Sigma)$, closed substitution σ and $\psi \in \mathcal{L}^{\mathrm{d}}$ we have

$$\sigma(\Theta) \models \psi \Leftrightarrow \exists \eta \in \Theta^{-1}(\psi) \text{ s.t. for all } \zeta \in \mathrm{var}(\Theta) \text{ it holds } \sigma(\zeta) \models \eta(\zeta).$$

The decompositions of formulae in $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_+$ can be derived from the one for $\mathcal{L}$.

▶ **Definition 29** (Decomposition of $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_+$). Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS and D_Σ be the Σ-DS. The mappings $\cdot^{-1}\colon \mathbb{T}(\Sigma) \to (\mathcal{L}_{\mathrm{r}}^{\mathrm{s}} \to \mathcal{P}(\mathcal{V}_s \to \mathcal{L}_{\mathrm{r}}^{\mathrm{s}}))$ and $\cdot^{-1}\colon \mathbb{DT}(\Sigma) \to (\mathcal{L}_{\mathrm{r}}^{\mathrm{d}} \to \mathcal{P}(\mathcal{V} \to \mathcal{L}_{\mathrm{r}}))$ are obtained as in Def. 26 by rewriting Def. 26.2 and Def. 26.4, resp., by

2. $\xi \in t^{-1}(\bar{a})$ iff there is a function $\mathfrak{f}\colon t^{-1}(\langle a\rangle\top) \to \mathrm{var}(t)$ s.t.

$$\xi(x) = \bigwedge_{\xi' \in \mathfrak{f}^{-1}(x)} \neg\xi'(x), \text{ if } x \in \mathrm{var}(t), \text{ and } \xi(x) = \top, \text{ otherwise;}$$

4. $\xi \in t^{-1}(\langle a\rangle\psi)$ iff there are a ruloid $\frac{\mathcal{H}}{t \xrightarrow{a} \Theta}$ and a decomposition mapping $\eta \in \Theta^{-1}(\psi)$ s.t.

$$\xi(x) = \bigwedge_{x \xrightarrow{b} \mu \in \mathcal{H}} \langle b\rangle\eta(\mu) \;\wedge\; \bigwedge_{x \xrightarrow{c} \not\in \mathcal{H}} \bar{c} \;\wedge\; \eta(x), \text{ if } x \in \mathrm{var}(t), \text{ and } \xi(x) = \top, \text{ otherwise.}$$

If P is positive, the mappings $\cdot^{-1}\colon \mathbb{T}(\Sigma) \to (\mathcal{L}_+^{\mathrm{s}} \to \mathcal{P}(\mathcal{V}_s \to \mathcal{L}_+^{\mathrm{s}}))$ and $\cdot^{-1}\colon \mathbb{DT}(\Sigma) \to (\mathcal{L}_+^{\mathrm{d}} \to \mathcal{P}(\mathcal{V} \to \mathcal{L}_+))$ are obtained as in Def. 26 by removing Def. 26.2 and by rewriting Def. 26.4 by

4. $\xi \in t^{-1}(\langle a\rangle\psi)$ iff there are a positive P-ruloid $\frac{\mathcal{H}}{t \xrightarrow{a} \Theta}$ and a decomposition mapping $\eta \in \Theta^{-1}(\psi)$ s.t.

$$\xi(x) = \bigwedge_{x \xrightarrow{b} \mu \in \mathcal{H}} \langle b\rangle\eta(\mu) \;\wedge\; \eta(x), \text{ if } x \in \mathrm{var}(t), \text{ and } \xi(x) = \top, \text{ otherwise.}$$

Notice that by decomposing formulae in $\mathcal{L}_{\mathrm{r}}$ (resp. $\mathcal{L}_+$) we get formulae in $\mathcal{L}_{\mathrm{r}}$ (resp. $\mathcal{L}_+$).

▶ **Theorem 30** (Decomposition theorem II). *Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS and D_Σ be the Σ-DS. Assume the decomposition mappings as in Definition 29. Then:*

- *The results in Theorem 28 hold for $\varphi \in \mathcal{L}_{\mathrm{r}}^{\mathrm{s}}$ and $\psi \in \mathcal{L}_{\mathrm{r}}^{\mathrm{d}}$.*
- *Moreover, if P is positive, then the results in Theorem 28 hold for $\varphi \in \mathcal{L}_+^{\mathrm{s}}$ and $\psi \in \mathcal{L}_+^{\mathrm{d}}$.*

4.3 Probabilistic bisimilarity as a congruence

To support the compositional reasoning, the congruence (resp. precongruence) property is required for any behavioral equivalence (resp. preorder) $\mathcal{R}$. It consists in verifying whether $f(t_1, \ldots, t_{\mathtt{n}}) \;\mathcal{R}\; f(t'_1, \ldots, t'_{\mathtt{n}})$ whenever $t_i \,\mathcal{R}\, t'_i$ for $i = 1, \ldots, \mathtt{n}$. In [5] it is proved that probabilistic bisimilarity is a congruence for all operators defined by a PGSOS-PTSS. We can restate this result as a direct consequence of the characterization result of [7] (Thm. 6) combined with our first decomposition result in Thm. 28. Then, by our characterization results in Thm. 8 and our decomposition results in Thm. 30 we can derive precongruence formats for both ready similarity and similarity.

▶ **Theorem 31.** *Let $P = (\Sigma, \mathcal{A}, R)$ be a PGSOS-PTSS. Then:*

1. *Probabilistic bisimilarity is a congruence for all operators defined by P;*
2. *Probabilistic ready similarity is a precongruence for all operators defined by P;*
3. *If P is positive, probabilistic similarity is a precongruence for all operators defined by P.*

5 Conclusions

We proposed distribution ruloids as a powerful tool supporting the decomposition of modalities over the PTS model. This allowed us to define modular proof systems for modal properties of probabilistic systems (Thms. 28, 30), from which we also derived congruence formats (Thm. 31). Our approach can be easily adapted to models with subdistributions.

We will continue this line of research as follows. We will apply our decomposition method to derive congruence formats for testing and trace equivalences. Next, we will use our decomposition method to systematically derive formats for bisimilarity metric [10,25], weak

metric semantics [11] and metric variants of branching bisimulation equivalence [1]. These metric semantics provide notions of *distance* over processes, and the formats will guarantee that a small variance in the behavior of the subprocesses leads to a bounded small variance in the behavior of the composed processes (*uniform continuity*, [16–18]). Then, we will study decomposition methods for real-valued modal formulae.

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