

# Deterministic Time-Space Trade-Offs for $k$ -SUM

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## Abstract

Given a set of numbers, the  $k$ -SUM problem asks for a subset of  $k$  numbers that sums to zero. When the numbers are integers, the time and space complexity of  $k$ -SUM is generally studied in the word-RAM model; when the numbers are reals, the complexity is studied in the real-RAM model, and space is measured by the number of reals held in memory at any point.

We present a time and space efficient deterministic self-reduction for the  $k$ -SUM problem which holds for both models, and has many interesting consequences. To illustrate:

- 3-SUM is in deterministic time  $O(n^2 \lg \lg(n) / \lg(n))$  and space  $O\left(\sqrt{\frac{n \lg(n)}{\lg \lg(n)}}\right)$ . In general, any polylogarithmic-time improvement over quadratic time for 3-SUM can be converted into an algorithm with an identical time improvement but low space complexity as well.
- 3-SUM is in deterministic time  $O(n^2)$  and space  $O(\sqrt{n})$ , derandomizing an algorithm of Wang.
- A popular conjecture states that 3-SUM requires  $n^{2-o(1)}$  time on the word-RAM. We show that the 3-SUM Conjecture is in fact equivalent to the (seemingly weaker) conjecture that every  $O(n^{51})$ -space algorithm for 3-SUM requires at least  $n^{2-o(1)}$  time on the word-RAM.
- For  $k \geq 4$ ,  $k$ -SUM is in deterministic  $O(n^{k-2+2/k})$  time and  $O(\sqrt{n})$  space.

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## 1 Introduction

We consider the  $k$ -SUM problem: given a list  $S$  of  $n$  values, determine whether there are distinct  $a_1, \dots, a_k \in S$  such that  $\sum_{i=1}^k a_i = 0$ . This classic problem is a parameterized version of the Subset Sum problem, which is among Karp's original NP-Complete problems.<sup>1</sup>

The brute-force algorithm for  $k$ -SUM runs in  $O(n^k)$  time, and it is known [22] that an  $n^{o(k)}$  time algorithm (where the little- $o$  depends on  $k$ ) would violate the Exponential Time

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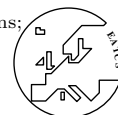
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<sup>1</sup> Karp's definition of the Knapsack problem is essentially Subset Sum [19].



Hypothesis [18]. A faster meet-in-the-middle algorithm reduces the  $k$ -SUM problem on  $n$  numbers to 2-SUM on  $O(n^{\lceil k/2 \rceil})$  numbers, which can then be solved by sorting and binary search in  $O(n^{\lceil k/2 \rceil} \log n)$  time. The belief that this meet-in-the-middle approach is essentially time-optimal is at the heart of many conditional 3-SUM-hardness results in computational geometry (e.g. [15]) and string matching (e.g. [5, 2]).

The space usage of the meet-in-the-middle approach is prohibitive: the  $O(n \log n)$  time solution for 2-SUM uses linear space, which causes the fast  $k$ -SUM algorithm to need  $\Omega(n^{\lceil k/2 \rceil})$  space. However, the brute-force algorithm needs only  $O(k \log n)$  space. This leads to the natural question: *how well can one trade off time and space in solving  $k$ -SUM?*

Schroeppe and Shamir [23] first studied time-space trade-off algorithms for Subset Sum. They showed how to reduce Subset Sum to an instance of  $k$ -SUM for any  $k \geq 2$ : split the elements into  $k$  sets of  $n/k$  elements each; for each set, compute  $2^{n/k}$  sums corresponding to the subsets of the set; this forms a  $k$ -SUM instance of size  $2^{n/k}$ . Since the  $k$ -SUM instance does not have to be explicitly stored, any time  $T(N)$ , space  $S(N)$  algorithm for  $k$ -SUM immediately implies a time  $T(2^{n/k})$ , space  $S(2^{n/k})$  algorithm for Subset Sum. Furthermore, Schroeppe and Shamir gave a deterministic  $\tilde{O}(n^2)$  time,  $\tilde{O}(n)$  space algorithm for 4-SUM, implying a  $O^*(2^{n/2})$  time,  $O^*(2^{n/4})$  space algorithm for Subset Sum.<sup>2</sup> They also generalized the algorithm to provide a smooth time-space trade-off curve, with extremal points at  $O^*(2^{n/2})$  time,  $O^*(2^{n/4})$  space and  $O^*(2^n)$  time,  $O^*(1)$  space.

A recent line of work leading up to Austrin et al. [6] has improved this long-standing trade-off curve for Subset Sum via *randomized* algorithms, resulting in a more complex curve. Wang [25] moved these gains to the  $k$ -SUM setting. In particular, for 3-SUM he obtains an  $\tilde{O}(n^2)$  time,  $\tilde{O}(\sqrt{n})$  space Las Vegas algorithm.

Despite the recent progress on the problem, all of the improved algorithms for the general case of  $k$ -SUM have heavily relied on randomization, either utilizing hashes or random prime moduli. These improvements also all rely heavily on the values in the lists being integers. For the general case of  $k$ -SUM, the previous best *deterministic*  $k$ -SUM results (even for integer inputs) are the brute-force algorithm, the meet-in-the-middle algorithm, and the Schroeppe and Shamir 4-SUM algorithm, and simple combinations thereof.

## 1.1 Our Results

We consider new ways of trading time and space in solving  $k$ -SUM, on both integer and real inputs (on the word-RAM and real-RAM respectively), without the use of randomization. Our improvements for  $k$ -SUM naturally extend to improvements to Subset Sum as well.

Our main result is a deterministic self-reduction for  $k$ -SUM. Informally, we show how to deterministically decompose a list of  $n$  numbers into a small collection of shorter lists, such that the  $k$ -SUM solution is preserved. This result is shown for  $k = 3$  in Section 4. It is shown for general  $k$  in Section 5.

► **Theorem 1.** *Let  $g$  be any integer between 1 and  $n$ .  $k$ -SUM on  $n$  numbers can be reduced to  $O(kg^{k-1})$  instances of  $k$ -SUM on  $n/g$  numbers. The reduction uses  $O(ng^{k-1})$  additional time and  $O(n/g)$  additional words of space.*

Theorem 1 has several interesting applications. First, it leads to more efficient  $k$ -SUM algorithms. For example, Gold and Sharir, building on other recent advances, report a deterministic algorithm for 3-SUM that works in both the word-RAM and real-RAM models

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<sup>2</sup> The notation  $\tilde{O}$  suppresses polylogarithmic factors in  $n$ , and  $O^*$  suppresses polynomial factors in  $n$ .

and which runs in time  $O(n^2 \lg \lg(n) / \lg(n))$  [16]. However, this algorithm uses a considerable amount of space to store a table of permutations. Applying Theorem 1 in multiple ways and calling their algorithm, we recover the *same* asymptotic running time but with drastically better space usage:

► **Theorem 2.** *There is an  $O(n^2 \lg \lg(n) / \lg(n))$  time deterministic algorithm for 3-SUM that stores at  $O(\sqrt{\frac{n \lg(n)}{\lg \lg(n)}})$  numbers in memory at point. (An analogous statement holds for 3-SUM over the integers.)*

Theorem 1 also directly leads to a derandomization of Wang’s space-efficient algorithm for 3-SUM:

► **Theorem 3.** *For all  $s \in [0, 1/2]$  there is a deterministic time  $O(n^{3-2s})$ , algorithm which uses  $O(n^s)$  words of space for 3-SUM.*

From Theorem 1 we can also derive a more space-efficient algorithm for 4-SUM, and lift it to a new algorithm for  $k$ -SUM:

► **Theorem 4.** *For  $k \geq 4$ ,  $k$ -SUM is solvable in deterministic  $O(n^{k-2+2/(k-3)})$  time and  $O(\sqrt{n})$  space in terms of words.*

### A more plausible 3-SUM conjecture

A rather popular algorithmic conjecture is the *3-SUM Conjecture* that 3-SUM on  $n$  integers requires  $n^{2-o(1)}$  time on a word-RAM with  $O(\log n)$  bit words. This conjecture has been used to derive conditional lower bounds for a variety of problems [15, 5, 2, 20, 3], and appears to be central to our understanding of lower bounds in low-polynomial time. To refute the conjecture, one could conceivably construct an algorithm that runs in  $O(n^{1.99})$  time, but utilizes  $\Omega(n^{1.99})$  space in some clever way. Here we consider a seemingly weaker (and thus more plausible) conjecture:

► **Conjecture 5** (The Small-Space 3-SUM Conjecture). *On a word-RAM with  $O(\log n)$ -bit words, there exists an  $\epsilon > 0$  such that every algorithm that solves 3-SUM in  $O(n^{1/2+\epsilon})$  space must take at least  $n^{2-o(1)}$  time.*

This conjecture looks weaker than the original 3-SUM Conjecture, because we only have to prove a quadratic-time lower bound for all algorithms that use slightly more than  $\sqrt{n}$  space. Proving time lower bounds is generally much easier when space is severely restricted (e.g. [9, 14, 12, 26, 8]).

Our self-reduction for 3-SUM yields the intriguing consequence that the original 3-SUM Conjecture is *equivalent* to the Small-Space 3-SUM conjecture! That is, the non-existence of a truly subquadratic-time 3-SUM algorithm is equivalent to the non-existence of a truly subquadratic-time  $n^{0.51}$ -space 3-SUM algorithm, even though the latter appears to be a more plausible lower bound. We prove:

► **Theorem 6.** *If 3-SUM is solvable in time  $O(n^{2-\epsilon})$  time, then for every  $\alpha > 0$  there is a  $\delta > 0$  such that 3-SUM is solvable in  $O(n^{2-\delta})$  time and space  $O(n^{1/2+\alpha})$  in terms of words.*

Theorem 6 is interesting, regardless of the veracity of the 3-SUM conjecture. On the one hand, the theorem reduces the difficulty of proving the 3-SUM Conjecture if it is true, because we only have to rule out small-space sub-quadratic time algorithms. On the other hand, the theorem means that refuting the 3-SUM conjecture immediately implies a truly-subquadratic time algorithm for 3-SUM using small space as well, which would be an algorithmic improvement.

## 2 Preliminaries

### 2.1 k-SUM and Selection

We will use the following version of the  $k$ -SUM problem:

► **Definition 7.** In the  $k$ -SUM problem, we are given an unsorted list  $L$  of  $n$  values (over  $\mathbb{Z}$  or  $\mathbb{R}$ ) and want to determine if there are  $a_1, \dots, a_k \in L$  such that  $\sum_{i=1}^k a_i = 0$ .

One fundamental case is the 3-SUM problem. Sometimes 3-SUM is presented with three separate lists, which we denote as 3-SUM', but the two are reducible to each other in linear time, and with no impact on space usage.

► **Definition 8.** In the 3-SUM problem, we are given an unsorted list  $L$  of  $n$  values and want to know if there are  $a, b, c \in L$  such that  $a + b + c = 0$ . In the 3-SUM' problem, we are given three unsorted lists  $A, B,$  and  $C$  of values, where  $|A| = |B| = |C| = n$ , and want to know if there are  $a \in A, b \in B, c \in C$  such that  $a + b + c = 0$ .

As part of our  $k$ -SUM algorithms, the classical *Selection Problem* will also arise:

► **Definition 9.** In the  $s$ -SELECT problem, we are given an unsorted list  $L$  of  $n$  values and a natural number  $s$ , and want to determine the  $s^{\text{th}}$  smallest value in  $L$ .

### 2.2 Computational Model

As standard when discussing sub-linear space algorithms, the input is provided in read-only memory, and the algorithm works with auxiliary read/write memory which counts towards its space usage.

**Computation on Integers.** When the input values are integers, we work in the word-RAM model of computation: the machine has a word size  $w$ , and we assume all input numbers can be represented with  $w$  bits so that they fit in a word. Arithmetic operations ( $+, -, *$ ) and comparisons on two words are assumed to take  $O(1)$  time. Space is counted in terms of the number of words used.

**Computation on Reals.** When the input values are real numbers, we work in a natural *real-RAM* model of computation, which is often called the *comparison-addition model* (see, for example, [21]). Here, the machine has access to registers that can store arbitrary real numbers; addition of two numbers and comparisons on real numbers take  $O(1)$  time. Space is measured in terms of the number of reals stored.

**Time-Space Complexity Notation.** We say that  $k$ -SUM is solvable in  $\text{TISP}(T(n), S(n))$  if  $k$ -SUM on lists of length  $n$  can be solved by a single algorithm running in deterministic  $O(T(n))$  time and  $O(S(n))$  space simultaneously on the real-RAM (and if the lists contain integers, on the word-RAM).

### 2.3 Other Prior Work

Baran, Demaine and Patrascu [7] obtained randomized slightly subquadratic time algorithms for Integer 3-SUM in the word-RAM. Grønlund and Pettie [17] studied 3-SUM over the reals, presenting an  $O(n^2/(\log n/\log \log n))$  time randomized algorithm, as well as a deterministic algorithm running in  $O(n^2/(\log n/\log \log n)^{2/3})$  time. Recently, Gold and Sharir [16]

improved this deterministic running time to  $O(n^2/(\log n/\log \log n))$ . Abboud, Lewi and Williams [1] showed that Integer  $k$ -SUM is W[1]-complete under randomized FPT reductions (and under some plausible derandomization hypotheses, the reductions can be made deterministic). In the linear decision tree model of computation,  $k$ -SUM over the reals is known to require  $\Omega(n^{\lceil k/2 \rceil})$  depth  $k$ -linear decision trees [13, 4], but the problem can be solved with  $O(n^{k/2}\sqrt{\log n})$  depth  $(2k-2)$ -linear decision trees [17]. The randomized decision tree complexity was improved by Gold and Sharir [16] to  $O(n^{k/2})$ .

### 3 Building Blocks

In this section, we describe two tools we use to obtain our main self-reduction lemma for  $k$ -SUM and 3-SUM. The first tool helps us guarantee that we don't have to generate too many subproblems in our reduction; the second will allow us to find these subproblems in a time and space efficient way.

#### 3.1 Domination Lemma

Our deterministic self-reduction for  $k$ -SUM will split lists of size  $n$  into  $g$  sublists of size  $n/g$ , then solve subproblems made up of  $k$ -tuples of these sublists. Naively, this would generate  $g^k$  subproblems to enumerate all  $k$ -tuples. In this section, we show that we only need to consider  $O(kg^{k-1})$  subproblems.

First, we define a partial ordering on  $k$ -tuples on  $[n]^k$ . For  $t, t' \in [n]^k$ , we say that  $t \prec t'$  if  $t[i] < t'[i]$  for all  $i = 1, \dots, k$ . (Geometrically, the terminology is that  $t'$  dominates  $t$ .)

► **Lemma 10** (Domination Lemma). *Suppose all tuples in a subset  $S \subseteq [n]^k$  are incomparable with respect to  $\prec$ . Then  $|S| \leq kn^{k-1}$ .*

The Domination Lemma can be seen as an extension of a result in [24] (also used in [11] in a different context) which covers the  $k = 3$  case.

**Proof.** We will give a cover of all elements in  $[n]^k$  with few chains under  $\prec$ . Then by Dilworth's theorem, any set of incomparable elements under  $\prec$  can only have one element from each chain.

Take any  $k$ -tuple  $t \in [n]^k$  such that  $t[i] = 1$  for some  $i = 1, \dots, k$ . Letting  $\ell \in [n]$  be the largest element in  $t$ , we define the chain  $C(t) = \{t_0, t_1, \dots, t_{n-\ell}\}$ , where each  $t_j$  is given by  $t_j[i] = t[i] + j$  for all  $i = 1, \dots, k$ . Clearly  $C(t)$  forms a chain in  $[n]^k$  under  $\prec$ . Moreover these chains cover all elements of  $[n]^k$ : observe that the tuple  $t$  appears in the chain  $C(t')$  where  $t'[i] = t[i] - \min_j t[j] + 1$  for all  $i = 1, \dots, k$ .

The number of chains is exactly the number of  $k$ -tuples with a 1 in at least one coordinate. This number is less than  $k$  times the number of tuples that have a 1 in dimension  $i$ . The number of tuples with a 1 in dimension  $i$  is  $n^{k-1}$ . Thus, the total number of chains is  $\leq kn^{k-1}$ . ◀

The Domination Lemma can be applied to show that in any list of numbers, not too many  $k$ -SUM subproblems can have  $k$ -SUM solutions. In the following, let  $g$  divide  $n$  for simplicity. Given a list  $L$  of  $n$  numbers divided into  $g$  groups of size  $n/g$ , a *subproblem of  $L$*  is simply the union of a  $k$ -tuple of groups from  $L$ . Note that a subproblem contains at most  $kn/g$  numbers.

► **Corollary 11.** *Given a  $k$ -SUM instance  $L$ , suppose  $L$  is divided into  $g$  groups  $L_1, \dots, L_g$  where  $|L_i| = n/g$  for all  $i$ , and for all  $a \in L_i$  and  $b \in L_{i+1}$  we have  $a \leq b$ . Then there are*

$O(k \cdot g^{k-1})$  subproblems  $L'$  of  $L$  such that the smallest  $k$ -sum of  $L'$  is less than zero and the largest  $k$ -sum of  $L'$  is greater than zero. Furthermore, if some subproblem of  $L$  has its largest or smallest  $k$ -sum equal to 0, then the corresponding  $k$ -SUM solution can be found in  $O(g^k)$  time.

**Proof.** We associate each subproblem of  $L$  with a corresponding  $k$ -tuple  $(x_1, \dots, x_k) \in [g]^k$  corresponding to the  $k$  sublists  $(L_{x_1}, \dots, L_{x_k})$  of  $L$ .

Let  $m[i]$  be the element in position  $i \cdot (n/g)$  when  $L$  is in sorted order. Consider any subproblem with  $\sum_{i=1}^k m[x_i] > 0$  (smallest  $k$ -sum greater than zero) or  $\sum_{i=1}^k m[x_i + 1] < 0$  (largest  $k$ -sum less than zero). We call such a subproblem *trivial*, since it cannot contain  $k$ -SUM solutions.

In  $O(g^k)$  time, we can determine whether any subproblem has  $\sum_{i=1}^k m[x_i] = 0$ , and return the corresponding  $k$ -SUM solution if this is the case. Otherwise, we can assume that for each subproblem either it is trivial, or  $\sum_{i=1}^k m[x_i] < 0 < \sum_{i=1}^k m[x_i + 1]$ .

Consider the set of non-trivial subproblems. Because for all  $a \in L_i$  and  $b \in L_{i+1}$  we have  $a \leq b$ , if for two subproblem  $k$ -tuples we have  $t \prec t'$ , then the smallest  $k$ -sum of the subproblem  $t'$  is at least the largest  $k$ -sum of the subproblem  $t$ . This implies that at least one of the two subproblems must be trivial. In other words, the set of nontrivial problems corresponds to a set of incomparable  $k$ -tuples in  $[g]^k$ . Applying Lemma 10, the number of nontrivial subproblems is  $O(kg^k)$ . ◀

### 3.2 Bucket Retrieval and Space-Efficient Selection

A randomized algorithm for  $k$ -SUM can partition a list of numbers by choosing a hash function at random, then loop over the hash function range to partition a given list into smaller buckets. Given a hash and a bucket number, it is easy to retrieve the contents of that bucket by scanning the list.

To derandomize this process, we could try to create small “hash” buckets by grouping the  $n/g$  smallest elements together, then the next  $n/g$  smallest elements, and so on, *without actually sorting the list*. However, retrieving the contents of a bucket may now be difficult to do with small space: we need to know the smallest and largest elements of a bucket to retrieve its elements, and we may not be able to store all of these extrema. We require an efficient algorithm to compute the largest element of a bucket, given the smallest element and the bucket size.

This problem is equivalent to the *selection problem*, also known as  $s$ -SELECT, which asks for the  $s^{\text{th}}$  smallest element of a list, when we set  $s = n/g$ . To reduce from our problem to  $s$ -SELECT, pretend that every entry less than our smallest element is  $\infty$ . (To reduce from  $s$ -SELECT to our problem, we can pretend our smallest element is  $-\infty$ .)

The classic median-of-median algorithm can solve  $s$ -SELECT in  $O(n)$  time and  $O(n)$  space [10]. Since we care about space usage, we provide an algorithm below which has  $O(n)$  running time, but uses much less space. This algorithm turns out to be optimal for our purposes, since retrieving the bucket afterwards will already take  $O(n)$  time and  $O(s)$  space.

► **Lemma 12.**  $s$ -SELECT can be solved in  $O(n)$  time and  $O(s)$  space.

**Proof.** The plan is to scan through the elements of the list, inserting them to a data structure  $D$  which will allow us to track the smallest  $s$  elements. We perform  $n$  insertions, then query  $D$  to ask for the smallest  $s$  elements it contains. To get the claimed algorithm for selection, we give a data structure can handle these operations in  $O(1)$  amortized update time and  $O(s)$  query time, with a data structure using only  $O(s)$  space.

One first attempt might be to build a heap of  $s + 1$  elements, which throws away the largest element whenever it gets full. Since heaps have logarithmic update time and linear space usage, this results in  $O(\log s)$  update time,  $O(s)$  query time, and  $O(s)$  space.

We can improve the update time by batching when we throw out large elements. Suppose instead we keep an array which can hold up to  $2s$  elements. When the array gets full, we throw out the largest  $s$  elements. To do this, we first compute the  $(s+1)^{th}$  smallest element in the array. This can be done in  $O(s)$  time and  $O(s)$  space via the classical median-of-medians algorithm. We then do a linear scan of the array, and write all elements strictly less than the median to a new array. To handle ties, we write a copy of the median to the new array, until it has  $s$  elements. When we are given our final query, we again throw out large elements so that we only have  $s$  elements left, and then return those.

Updates now take amortized constant time: after  $s$  updates, we take  $O(s)$  time to clear out the large elements. The final query takes  $O(s)$  time, since we again need to throw out large elements. The space usage is  $O(s)$  since we store up to  $2s$  elements, and running median-of-medians takes  $O(s)$  space. This completes the proof. ◀

We will call the above algorithm `NEXTGROUP`. `NEXTGROUP` takes as input a value  $v$ , a natural number  $s$ , and a list of numbers  $L$ , and outputs the next  $s$  elements of  $L$  in sorted order after the value  $v$ .

## 4 Subquadratic 3-SUM implies Subquadratic small-space 3-SUM

We will begin by using our building blocks to prove a self reduction for 3-SUM. Then we will show three intriguing consequences of this self reduction. First, the self reduction can be used to show a general theorem that takes subquadratic algorithms for 3-SUM and produces subquadratic time algorithms that run in nearly  $\sqrt{n}$  space. Second, we show that algorithms for 3-SUM that are subquadratic by polylog factors can be used to obtain 3-SUM algorithms with the same asymptotic running time and simultaneously small space. Finally, we will prove that the Small-Space 3-SUM conjecture is equivalent to the 3-SUM conjecture.

### 4.1 3-SUM Self Reduction

We now proceed to solve 3-SUM using our bucket retrieval subroutine. We will use  $\max S$  and  $\min S$  to refer to the maximum and minimum elements of a list  $S$ , respectively.

As anticipated, we split the three arrays into groups of size  $n/g$ , and solve 3-SUM on subproblems of this size. Naively there are  $O(g^3)$  subproblems to solve, but we use Corollary 11 to argue we only get  $O(g^2)$  subproblems.

▶ **Theorem 13** (3-SUM Self-Reduction Theorem). *If 3-SUM is solvable in  $TISP(T(n), S(n))$  then for any  $g$ , 3-SUM can be solved in  $TISP(g^2(n + T(n/g)), n/g + S(n/g))$ .*

**Proof.** Consider the following algorithm.

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**Algorithm 1:** 3-SUM Algorithm
 

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```

Set  $preva = -\infty$ ;
for  $i \in [0, g - 1]$  do
  Set  $A' = \text{NEXTGROUP}(A, preva, n/g + 1)$ ;
  Set  $prevb = -\infty$ ;
  for  $j \in [0, g - 1]$  do
    Set  $B' = \text{NEXTGROUP}(B, prevb, n/g + 1)$ ;
    Set  $C' = \text{NEXTGROUP}(C, -\max A' - \max B', n/g + 1)$ ;
    while  $\min C' \leq -\min A' - \min B'$  do
      if  $3\text{-SUM}(A', B', C')$  returns true then
        return true;
      Set  $C' = \text{NEXTGROUP}(C, \max C', n/g + 1)$ ;
    Set  $prevb = \max B'$ ;
  Set  $preva = \max A'$ ;
return false;

```

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Algorithm 1 is correct because we consider all possible elements of  $C$  where the sum of elements from  $A'$  and  $B'$  could land, and the choices of  $A'$  and the choices of  $B'$  cover all of  $A$  and  $B$ , respectively. If there are multiple copies of a value in a list we will fail to list all copies only if it already appeared in a previous sublist. This will not affect correctness because the value will have already been analyzed.

It's easy to see that the algorithm calls `NEXTGROUP`  $O(g)$  times for  $A'$ ,  $O(g^2)$  times for  $B'$ . We claim that we also only call it  $O(g^2)$  times for  $C'$ . To show this, we want to apply Corollary 11. Unfortunately, the groups of  $C$  that we extract don't always line up with our ideal  $n/g$  division; since we start at  $-\max A' - \max B'$ , we may not align at the endpoints of blocks. Fortunately, we've only introduced an extra  $O(1)$  possibilities of  $C'$  for every  $(A', B')$  pair, or  $O(g^2)$  extras total. Hence we still only make  $O(g^2)$  calls to `NEXTGROUP`. By Lemma 12, these calls will require  $O(ng^2)$  time and  $O(n/g)$  space.

Our algorithm also calls the `TISP( $T(n), S(n)$ )` algorithm for 3-SUM  $O(g^2)$  times on instances of size  $O(n/g)$ , which requires  $O(g^2T(n/g))$  time and  $O(S(n/g))$  space.

We have shown Algorithm 1 is correct and has the desired runtime and space usage, so this completes the proof.  $\blacktriangleleft$

## 4.2 General Theorem for Space Reduction

Our self-reduction for 3-SUM yields the following intriguing consequence: *subquadratic-time algorithms for 3-SUM imply subquadratic-time small-space algorithms for 3-SUM*. Plugging this connection into known 3-SUM algorithms, we can automatically obtain more space-efficient 3-SUM algorithms for free. From a complexity-theoretic point of view, the consequence is perhaps even more intriguing: it means that the 3-SUM Conjecture is *equivalent* to the statement that there is no subquadratic-time  $n^{0.51}$ -space 3-SUM algorithm, even though the latter appears to be a more plausible lower bound(!).

We begin by stating our generic space reduction theorem.

$\blacktriangleright$  **Theorem 14** (3-SUM Space Reduction). *Suppose 3-SUM is solvable in  $n^2/f(n)$  time, where  $1 \leq f(n) \leq n$ . Then 3-SUM is solvable by an algorithm running in  $O(n^2/f(n/g))$  time and*



$O(n/h)$  space simultaneously, where  $g(n), h(n) \in [1, n]$  satisfy the relations

$$g(n) \geq \Omega\left(\sqrt{\frac{n}{f(n/g(n))}}\right) \quad \text{and} \quad h^2 + \frac{n}{f(n/(hg(n/h)))} \leq O\left(\frac{n}{f(n/g(n))}\right).$$

**Proof.** We will apply our Self-Reduction Theorem for 3-SUM (Theorem 13) in two different ways. First, we will use the self-reduction (and the constraint on  $g(n)$ ) to convert our 3-SUM algorithm into a linear-space algorithm, with a modest increase in running time (if at all). Pushing the linear-space algorithm through the self-reduction once more will reduce the space bound further, without increasing the running time asymptotically (using the constraint on  $h(n)$ ).

Let  $T(n) := n^2/f(n)$ . Set the parameter  $g(n) \geq 1$  to satisfy

$$T(n/g) = \frac{n^2/g^2}{f(n/g)} = O(n); \quad \text{or, equivalently} \quad g = \Omega\left(\sqrt{\frac{n}{f(n/g)}}\right). \quad (1)$$

Assuming  $g$  satisfies (1), applying the 3-SUM Self-Reduction (Theorem 13) with  $T(n) = S(n)$  and  $g$ , we can then solve 3-SUM in

$$\text{TISP}\left(g^2(n + T(n/g)), n/g + T(n/g)\right) = \text{TISP}\left(\frac{n^2}{f(n/g)}, n\right). \quad (2)$$

Now, set new time and space bounds  $T(n) := n^2/f(n/g(n))$ ,  $S(n) = n$  from (2). Then, applying the 3-SUM Self-Reduction (Theorem 13) with the new  $T(n)$ ,  $S(n)$  and some parameter  $h$ , we can then solve 3-SUM in  $\text{TISP}(h^2(n + T(n/h)), n/h + S(n/h)) =$

$$\text{TISP}\left(h^2\left(n + \frac{n^2/h^2}{f(n/(hg(n/h)))}\right), n/h\right) \subseteq \text{TISP}\left(\frac{n^2}{f(n/g)}, n/h\right),$$

by our hypothesis on  $h$ . ◀

### 4.3 Space-Efficient Fast 3-SUM

When we apply Theorem 14 directly to known algorithms, we obtain immediate space improvements with negligible loss in running time. Very recently, Gold and Sharir [16] have given a faster 3-SUM algorithm in the real-RAM model, building on the work of Gronlund and Pettie [17]:

► **Theorem 15** (Gold and Sharir [16]). *3-SUM can be solved in  $O(n^2 \lg \lg(n)/\lg(n))$  time over the reals and integers.*

As discussed in the introduction, their novel approach uses quite a bit of space. Applying Theorem 14, we can reduce the space usage to only  $O\left(\sqrt{n \lg(n)/\lg \lg(n)}\right)$ , with the same asymptotic running time of Gold and Sharir.

► **Corollary 16** (Space-Efficient 3-SUM Algorithm). *3-SUM is in  $\text{TISP}\left(n^2 \frac{\lg \lg(n)}{\lg(n)}, \sqrt{\frac{n \lg(n)}{\lg \lg(n)}}\right)$ .*

**Proof.** We shall apply Theorem 14. First, set  $f(n) := \lg(n)/\lg \lg(n)$ , so that 3-SUM is solvable in  $O(n^2/f(n))$  time by Theorem 15.

Set  $g(n) := \sqrt{\frac{n \lg \lg(n)}{\lg(n)}}$  and  $h(n) := \sqrt{\frac{n \lg \lg(n)}{\lg(n)}}$ . By our choice of  $f(n)$  and basic properties of logarithms, observe that

$$f(n/g) = f(\tilde{O}(\sqrt{n})) = \Theta(f(n)), \quad (3)$$

and furthermore

$$f(n/(hg(n/h))) = f\left(\tilde{O}(\sqrt{n})/\tilde{O}(n^{1/4})\right) = \Theta(f(n)). \quad (4)$$

By (3), we have

$$g = \sqrt{\frac{n \lg \lg(n)}{\lg(n)}} \geq \Omega\left(\sqrt{\frac{n}{f(n/g)}}\right), \text{ so the first constraint of Theorem 14 is satisfied.}$$

Moreover, by (4) we have

$$h^2 + \frac{n}{f(n/(hg(n/h)))} = \frac{n \lg \lg(n)}{\lg(n)} + \frac{n}{\Theta(f(n))}, \text{ which is } O\left(\frac{n}{f(n/g)}\right) \text{ by (3).}$$

Therefore the second constraint of Theorem 14 is also satisfied, and 3-SUM is solvable by an algorithm running in  $O(n^2/f(n))$  time and  $O(\sqrt{nf(n)})$  space simultaneously. ◀

In general, Theorem 14 provides a generic *reduction* from faster 3-SUM algorithms to faster space-efficient 3-SUM algorithms. To illustrate:

► **Corollary 17.** *If 3-SUM is solvable in  $O(n^2/\lg^a(n))$  time for some constant  $a > 0$ , then 3-SUM is in  $\text{TISP}(n^2/\lg^a(n), \sqrt{n}\lg^{a/2}(n))$ .*

**Proof.** We apply Theorem 14. By assumption we have 3-SUM in  $O(n^2/f(n))$  time, where  $f(n) = \lg^a n$ . Set  $g(n) := \sqrt{n}/\lg^{a/2}(n)$ , and  $h(n) := \sqrt{n}/\lg^{a/2}(n)$ . Note that  $f(n/g(n)) = \Theta(\lg^a(n))$  and  $f(n/(h(n) \cdot g(n/h(n)))) = \Theta(\lg^a(n))$ , similar to Corollary 16. Therefore

$$g(n) = \sqrt{n}/\lg^{a/2}(n) \geq \Omega\left(\sqrt{\frac{n}{f(n/g(n))}}\right)$$

and

$$h^2 + \frac{n}{f(n/(hg(n/h)))} \leq O\left(\frac{n}{\lg^a n}\right) \leq O\left(\frac{n}{f(n/g(n))}\right).$$

Hence Theorem 14 applies to these settings of the parameters, and 3-SUM is in  $O(n^2/f(n/g)) = O(n^2/\lg^a(n))$  time and  $O(n/h) = O(\sqrt{n}\lg^{a/2}(n))$  space. ◀

#### 4.4 The 3-SUM Conjecture and Small Space

Finally, we use the Space Reduction Theorem (Theorem 14) to show that the 3-SUM conjecture is false, then it is also false with respect to small-space algorithms.

► **Lemma 18.** *If 3-SUM is in  $O(n^{2-\epsilon})$  time for some  $\epsilon > 0$ , then for every  $\alpha > 0$ , there is a  $\delta > 0$  such that 3-SUM is solvable in  $O(n^{2-\delta})$  time and  $O(n^{1/2+\alpha})$  space, simultaneously.*

**Proof.** The proof of Theorem 14 applies the 3-SUM Self Reduction (Theorem 13) twice. We will basically perform the first part of the proof of Theorem 14, but instead of applying the second part of the proof, we have to choose a different setting of parameters, focused on minimizing the space usage instead of preserving running time.

Let  $T(n) := n^2/f(n)$  with  $f(n) = n^\epsilon$ . We first reduce the space usage of the algorithm to linear. To this end, set  $g(n) := n^{(1-\epsilon)/(2-\epsilon)}$ . Then, applying the 3-SUM Self-Reduction (Theorem 13) with  $T(n) = S(n)$  and  $g(n)$ , we can then solve 3-SUM in

$$\text{TISP}(g^2(n + T(n/g)), n/g + T(n/g)) = \text{TISP}\left(\frac{n^2}{f(n/g)}, n\right) = \text{TISP}\left(\frac{n^2}{n^{\epsilon/(2-\epsilon)}}, n\right).$$

Now reset  $f(n) := n^{\epsilon/(2-\epsilon)}$ , and reset  $g(n) := n^{1/2+\alpha}$  with  $\alpha \in (0, 1/2)$ . Applying the 3-SUM Self-Reduction (Theorem 13) with  $T(n) = n^2/f(n)$ ,  $S(n) = n$ , and  $g(n)$  as above, we find an algorithm for 3-SUM in

$$\text{TISP}\left(n^{2-2\alpha} + n^{2-(1/2-\alpha)\epsilon/(2-\epsilon)}, n^{1/2+\alpha}\right).$$

Note that for all  $\epsilon > 0$  and  $\alpha \in (0, 1/2)$ , the running time bound is truly subquadratic. Further note that for any  $\alpha \geq 1/2$ , we only have more space to work with, so we clearly obtain  $O(n^{2-\delta})$  time and  $O(n^{1/2+\alpha})$  space (for some  $\delta > 0$ ) in that case as well. ◀

This lemma can be applied to show that the 3-SUM Conjecture is equivalent to seemingly much weaker statement:

**The Small-Space 3-SUM Conjecture (Conjecture 5).** *On a word-RAM with  $O(\log n)$ -bit words, there exists an  $\epsilon > 0$  such that every algorithm that solves 3-SUM in  $O(n^{1/2+\epsilon})$  space must take at least  $n^{2-o(1)}$  time.*

► **Theorem 19.** *The Small-Space 3-SUM Conjecture is equivalent to the 3-SUM Conjecture.*

**Proof.** It suffices to show that the 3-SUM Conjecture if true implies the Small-Space 3-SUM Conjecture and that the refutation of the 3-SUM Conjecture implies the Small-Space 3-SUM Conjecture. First, we observe that the 3-SUM Conjecture trivially implies the Small-Space 3-SUM Conjecture.

Suppose the 3-SUM Conjecture is false. Then a  $O(n^{2-\epsilon})$  time algorithm for 3-SUM exists, and Lemma 18 implies that for every  $\alpha > 0$ , there is a  $\delta > 0$  such that 3-SUM is solvable in  $O(n^{2-\delta})$  time and  $O(n^{1/2+\alpha})$  space, simultaneously. But this means that for any choice of  $\epsilon' > 0$  for the Small-Space 3-SUM Conjecture, we can find a truly-subquadratic 3-SUM algorithm that uses only  $O(n^{1/2+\epsilon'/2})$  space. This would falsify the Small-Space 3-SUM Conjecture. ◀

We conclude that, in order to prove the 3-SUM conjecture, it is sufficient to prove that no algorithm can solve 3-SUM in  $\text{TISP}(n^{2-\epsilon}, n^{0.51})$  for some  $\epsilon > 0$ .

## 5 k-SUM

### 5.1 k-SUM Self-Reduction

We now generalize from 3-SUM to  $k$ -SUM. Again, we plan to split the lists into  $g$  groups of size  $O(n/g)$ . By Corollary 11, we will have only  $O(g^{k-1})$  subproblems of size  $O(n/g)$ . Unlike 3-SUM, where we just used the naive algorithm to solve subproblems, in this section we use a general algorithm; we reduce from  $k$ -SUM to itself (albeit on smaller instances).

► **Theorem 20.** *Suppose real  $k$ -SUM can be solved in  $\text{TISP}(T(n), S(n))$ . Then for any  $g$ , it can also be solved in  $\text{TISP}(g^{k-1}(n + T(n/g)), n/g + S(n/g))$ .*

**Proof.** This follows from a generalized analysis of the proof of Theorem 13. We brute force over which groups the first  $k - 1$  elements are in. We then extract groups where the negative sum of elements from these first  $k - 1$  groups could land. By Corollary 11 and similar reasoning as before, there are only  $O(g^{k-1})$  tuples of blocks. For each tuple, we make a call to NEXTGROUP and to our input  $k$ -SUM algorithm on a subproblem of size  $O(n/g)$ . This gives the desired time and space, completing the proof. ◀

## 5.2 Applying our $k$ -SUM Self-Reduction

We want to apply the self-reduction on efficient deterministic algorithms. One of the best starting points is the Schroepel-Shamir 4-SUM algorithm, which we note is actually deterministic and works on reals because it simply uses priority queues and reduces to the classic 2-SUM algorithm, both of which only use comparisons.

► **Lemma 21** (From [23]). *Real 4-SUM is solvable in  $TISP(n^2, n)$ .*

Another useful fact observed by Wang is that an algorithm for  $k$ -SUM can be transformed into an algorithm for  $(k + 1)$ -SUM by brute-forcing one element:

► **Lemma 22** (From [25]). *If Real  $k$ -SUM is solvable in  $TISP(T(n), S(n))$  then real  $k + 1$ -SUM is solvable in  $TISP(nT(n), S(n) + 1)$ .*

Suppose we want to use our results to derive a linear-space algorithm for  $k$ -SUM. We will assume  $k$  is a multiple of 4, although Lemma 22 allows us to fill in for the other values of  $k$ . By writing down sums of  $k/4$  elements, we can transform  $k$ -SUM to 4-SUM, yielding a  $TISP(n^{k/2}, n^{k/4})$  algorithm. We can then apply Theorem 20 with  $g = n^{(k-4)/k}$  to get a  $TISP(n^{k-3+4/k}, n)$  algorithm. Notice that this algorithm runs significantly faster than  $O(n^k)$  time; we get  $O(n^{11/2})$  for 8-SUM and  $O(n^{28/3})$  for 12-SUM. As a coarse upper bound, we can apply Lemma 22 and round down our savings (to make things cleaner), compensating for  $k$  which are not a multiple of 4, we get:

► **Corollary 23.** *For  $k \geq 4$ ,  $k$ -SUM is solvable in  $TISP(n^{k-3+4/(k-3)}, n)$ .*

Suppose we wanted to use  $O(\sqrt{n})$  space instead. We get smaller subproblems by making more groups; choosing  $g = n^{(k-2)/k}$  instead yields a  $TISP(n^{k-2+2/k}, \sqrt{n})$ . Similarly applying Lemma 22 and round down our savings to compensate for  $k$  which are not a multiple of 4, we get another coarse upper bound:

► **Corollary 24.** *For  $k \geq 4$ ,  $k$ -SUM is solvable in  $TISP(n^{k-2+2/k}, \sqrt{n})$ .*

## 6 Future Work

We would like to extend these results to derandomize other known randomized algorithms for  $k$ -SUM. To do that, it seems we require a “deterministic simulation” of the hash functions used in those results. Baran, Demaine, and Patrascu use hashing to get subquadratic algorithms for 3-SUM [7]; Patrascu uses it to reduce 3-SUM to Convolution 3-SUM [20]; Wang uses it to produce a family of linear-space algorithms for  $k$ -SUM [25]. Which of these results, if any, can be derandomized?

The hash families involved have three crucial properties: load-balancing (the hash buckets are not “too large”), few subproblems (the number of  $k$ -tuples of hash buckets examined is “small”), and few false positives (there are few non- $k$ -SUM solutions mapped to  $k$ -tuples of hash buckets examined). Our  $s$ -SELECT algorithm (Lemma 12) and Domination Lemma (Lemma 10) are used to achieve the first two properties, without using randomization. Can the last property also be simulated deterministically? (Note that it’s not entirely clear what it would mean to simulate “few false positives” deterministically.) If so, it is likely that all these results can be derandomized efficiently.

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