

# Composition of Stochastic Transition Systems Based on Spans and Couplings\*

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## Abstract

Conventional approaches for parallel composition of stochastic systems relate probability measures of the individual components in terms of product measures. Such approaches rely on the assumption that components interact stochastically independent, which might be too rigid for modeling real world systems. In this paper, we introduce a parallel-composition operator for stochastic transition systems that is based on couplings of probability measures and does not impose any stochastic assumptions. When composing systems within our framework, the intended dependencies between components can be determined by providing so-called spans and span couplings. We present a congruence result for our operator with respect to a standard notion of bisimilarity and develop a general theory for spans, exploiting deep results from descriptive set theory. As an application of our general approach, we propose a model for stochastic hybrid systems called stochastic hybrid motion automata.

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## 1 Introduction

When modeling complex systems, compositional approaches enjoy many favorable properties compared to their monolithic counterparts. They allow for a systematic system design, facilitate the interchangeability and reusability of components, and thus also ease the maintainability. A major objective in defining compositional frameworks is to separate concerns into components – specifying the operational behavior – and composition operators – addressing the communication and interaction of the components. Within conventional approaches for stochastic systems, the composition operator relates probability distributions of the individual components in terms of product distributions. Therefore, such operators

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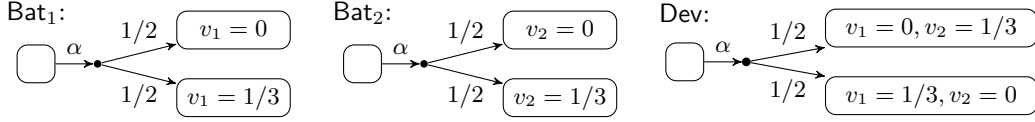
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are based on the assumption that the components interact stochastically independent, which is often not adequate. For instance, let us regard a systems composed of a device Dev and two batteries Bat<sub>1</sub> and Bat<sub>2</sub> providing the energy for Dev as detailed below:



In this example, the device provides the environmental context in which Bat<sub>1</sub> and Bat<sub>2</sub> are operating. Hence, Dev may, e.g., be the reason for common cause failures arising in the system. Let the variables  $v_1$  and  $v_2$  capture the amount of energy stored within Bat<sub>1</sub> and Bat<sub>2</sub>, respectively. The action label  $\alpha$  stands for the occurrence of a failure after which all the components will crash. As a consequence, the level of the stored energy of the batteries instantaneously drops to either 0 or 1/3 with probability 1/2, respectively. When considering the batteries in isolation, Bat<sub>1</sub> and Bat<sub>2</sub> appear stochastically independent in the first place and thus, product distributions in the parallel composition Bat<sub>1</sub> || Bat<sub>2</sub> seem to be adequate. However, additional dependencies can be imposed by Dev, influencing the interplay between the batteries. The assumption that Bat<sub>1</sub> and Bat<sub>2</sub> are stochastically independent is hence not adequate. Assume, e.g., that Dev uses Bat<sub>1</sub> as the default power supply and Bat<sub>2</sub> as a backup. Then, within a failure situation, Bat<sub>1</sub> is more likely to be affected than Bat<sub>2</sub>. The most likely case is that Bat<sub>1</sub> drops to 0 whereas Bat<sub>2</sub> drops to 1/3. Hence,  $v_1$  and  $v_2$  might be not independent in the composite system (Bat<sub>1</sub> || Bat<sub>2</sub>) || Dev.

Motivated by this example, we consider hybrid systems that combine discrete behaviors and continuous dynamics. In this setting, the most prominent modeling formalism are hybrid automata, which comprise a control graph with discrete jumps between (control) locations and flows that model the evolution of continuous variables over time. When time passes in a hybrid system, a flow starting from the current variable evaluation is selected non-deterministically and then the variables evolve according to the chosen flow. Besides the stochastic independence, additional aspects are relevant for the composition of hybrid systems. Let us assume that  $\alpha_1$  is a (local) action of Bat<sub>1</sub> which cannot be observed by Bat<sub>2</sub> or Dev. Particularly,  $\alpha_1$  does not affect the value of variable  $v_2$ . The hybrid automaton Bat<sub>1</sub> || Bat<sub>2</sub> has states of the form  $\langle s^1, s^2 \rangle$ . Suppose  $\langle s_1^1, s_1^2 \rangle \xrightarrow{t_1} \langle s_2^1, s_2^2 \rangle \xrightarrow{\alpha_1} \langle s_3^1, s_3^2 \rangle \xrightarrow{t_2} \langle s_4^1, s_4^2 \rangle$  is a finite path in Bat<sub>1</sub> || Bat<sub>2</sub>, comprising two timed transitions with time passages  $t_1$  and  $t_2$  and one jump transition involving action  $\alpha_1$ . As  $\alpha_1$  cannot be observed by Bat<sub>2</sub>, we expect  $s_1^2 \xrightarrow{t_1+t_2} s_4^2$  in Bat<sub>2</sub>. In particular, a faithful model for the composite system would allow for selecting a flow for  $v_1$  within time passage  $t_1$ , which is continued within the subsequent time passage. Thus, the adaption of the flow for variable  $v_2$  should only be possible when executing an action involving Bat<sub>2</sub> or Dev. This aspect is also crucial in the context of modeling controller strategies for hybrid systems. Typically, control decisions are made at distinct points and fixed until a next control decision is enabled. For instance, when considering a traffic alert and collision avoidance systems on aircraft, the advise of a corrective maneuver is determined when a critical situation occurs and fixed until sensor values exceed a threshold that indicates changes of the situation. A crucial point is to identify exactly those situations where adaptation of flows is allowed and required, as from a practical point of view it is important to minimize costs of adaptation and to keep the complexity of controllers manageable.

**Contribution.** We introduce a generic composition operator for stochastic transition systems (STs) [16] based on spans and span couplings. Our operator does not rely on the assumption that the STs to be composed are stochastically independent and covers standard composition

operators by dealing with specific spans. Spans provide a formal approach for introducing a universal notion of coupling probability measures. We develop an extensive theory for spans exploiting profound results known from descriptive set theory [34]. Based on a standard notion of bisimulation, we provide a congruence result with respect to our span composition. In the second part of the paper, we instantiate our general approach and introduce stochastic hybrid motion automata (SHMA) in which the progressing flow is recorded within states. We present a compositional framework for SHMA including an STS-semantics, a composition operator that does not rely on the assumption of stochastic independence, and where the adjustment of flows is always accompanied with an action. We show that the congruence result for STSs transfers to our SHMA framework.

Additional material and detailed proofs can be found in the technical report [25].

**Related Work.** We are not aware of a compositional modeling approach of stochastic systems which does not rely on the assumption that the components to be composed are stochastically independent. Our work thus addresses a fundamental challenge in the context of probabilistic operational models. The recent work [28] gives a comprehensive overview on compositional probabilistic modeling formalisms regarding expressive power and available analysis techniques. The concept of compositionality has its roots in the theory of process calculi [37, 32] and there are many fundamental contributions in the field of stochastic extensions of process calculi and probabilistic automata [39, 1, 2, 18, 13, 19]. Results on discrete systems have been extended to formalisms with continuous state spaces [16, 35]. The theory on non-deterministic labeled Markov processes (NLMPs) provide elegant notions and results on bisimulation and its logical characterization [21, 22, 17, 20, 6, 31]. Unfortunately, NLMPs are a priori not appropriate for our purposes as the class of NLMPs is not closed under the composition of stochastic transition systems [25]: Given two NLMPs, the transition function of their composition does not need to be measurable. When considering real-time systems, an important distinguishing aspect is the notion of residence time, which is the time spend in a state before moving to a successor state. In prominent compositional frameworks, timing behavior is modeled by clocks (timed automata) [4, 9, 38, 11] or one has exponential-distributed holding times (Markov automata and interactive Markov chains) [30, 23]. A general theory on compositionality and behavioral equivalences has been also achieved for probabilistic real-time systems modeled by interactive generalized semi-Markov processes [14, 12]. When adding flows to specify the evolution of continuous variables between jumps, one enters the field of hybrid systems [3, 29, 10]. The spirit of our work concerning hybrid systems is closest to the compositional frameworks developed for hybrid extensions of I/O-automata [36] and reactive modules [5] in the non-stochastic case. [36] studies parallel composition, simulation relations, and the receptiveness property and deals with prefix-, suffix- and concatenation-closed sets of flows on a syntactic level to obtain time-transitivity. Probabilistic hybrid automata [40, 26] extend classical hybrid automata by discrete probabilistic updates for the jumps. In [24, 27, 26], stochastic hybrid automata are considered where variables can be updated according to continuous distributions. Different from these hybrid automata, the change of flows in SHMA is only possible when some action is executed. Stochastic flows, i.e., where stochastic choices can be made continuously over time, are considered in [15, 33]. Our framework does not incorporate this kind of flows so far.

## 2 Preliminaries

We suppose the reader is familiar with standard concepts from measure and probability theory [8]. We briefly summarize our notations used throughout this paper.

**Couplings.** Within our work we understand couplings as a “modeling tool”. Intuitively, couplings relate given measures in a product space by a measure with corresponding marginals.  $\text{Prob}(X)$  denotes the set of all probability measures on the measurable space  $X$ . Let  $X_1$  and  $X_2$  be measurable spaces. Given  $\mu_1 \in \text{Prob}(X_1)$  and  $\mu_2 \in \text{Prob}(X_2)$ ,  $\mu \in \text{Prob}(X_1 \times X_2)$  is called a *coupling of*  $(\mu_1, \mu_2)$  if  $\mu(M_1 \times X_2) = \mu_1(M_1)$  and  $\mu(X_1 \times M_2) = \mu_2(M_2)$  for all measurable  $M_1 \subseteq X_1$  and  $M_2 \subseteq X_2$ . The *independent coupling of*  $(\mu_1, \mu_2)$  is the *product measure* of  $\mu_1$  and  $\mu_2$  denoted by  $\mu_1 \otimes \mu_2$ . If  $\mu_1 = \text{Dirac}[x_1]$  for some  $x_1 \in X_1$ , then there is exactly one coupling of  $(\mu_1, \mu_2)$ , namely the independent one. Here,  $\text{Dirac}[x_1]$  denotes the probability measure where for all measurable  $M_1 \subseteq X_1$ ,  $\text{Dirac}[x_1](M_1) = 1$  iff  $x_1 \in M_1$ .

**Polish spaces.** A separable and completely metrizable topological space is called a *Polish space* [34]. If  $X$  is a Polish space, then  $\text{Prob}(X)$  is well equipped with the topology induced by the weak convergence of probability measures. To obtain a measurable space, Polish spaces are equipped with the Borel sigma algebra, i.e., the coarsest sigma-algebra where all open sets are measurable. We call a measurable space  $X$  *standard Borel* if there exists a Polish topology on  $X$  where the induced Borel sigma-algebra coincides with the given one. The Polish topology is in general not uniquely determined. We refer to measurable subsets of standard Borel spaces as *Borel sets*. Of course, every Polish space is standard Borel.

**Functions for probability measures.** Given a measurable function  $f: X_1 \rightarrow X_2$  between measurable spaces  $X_1$  and  $X_2$ , the *pushforward of*  $f$  is defined by  $f_{\#}: \text{Prob}(X_1) \rightarrow \text{Prob}(X_2)$ ,  $f_{\#}(\mu)(M_2) = \mu(f^{-1}(M_2))$ . Assuming Polish spaces  $X_1$  and  $X_2$ , a *Markov kernel* is a Borel function  $k: X_1 \rightarrow \text{Prob}(X_2)$ . Here, for every  $\mu_1 \in \text{Prob}(X_1)$  we define *semi-product measure*  $\mu_1 \bowtie k \in \text{Prob}(X_1 \times X_2)$ ,  $\mu_1 \bowtie k(M_1 \times M_2) = \int_{M_1} k(x_1)(M_2) d\mu_1(x_1)$ .

**Relations.** Let  $R \subseteq X_1 \times X_2$  be a binary relation over some sets  $X_1$  and  $X_2$ . We usually write  $x_1 R x_2$  instead of  $\langle x_1, x_2 \rangle \in R$ . Then,  $R$  is called *lr-total in*  $X_1 \times X_2$  if for all  $x_1 \in X_1$  there exists  $x_2 \in X_2$  such that  $x_1 R x_2$  and vice versa, i.e., also for all  $x_2 \in X_2$  there exists  $x_1 \in X_1$  where  $x_1 R x_2$ . Assume  $X_1$  and  $X_2$  constitute measurable spaces and let  $\mu_1 \in \text{Prob}(X_1)$  and  $\mu_2 \in \text{Prob}(X_2)$ . A *weight function for*  $(\mu_1, R, \mu_2)$  is a coupling  $W$  of  $(\mu_1, \mu_2)$  such that  $x_1 R x_2$  for  $W$ -almost all  $\langle x_1, x_2 \rangle \in X_1 \times X_2$ . We write  $\mu_1 R^w \mu_2$  if there exists a weight function for  $(\mu_1, R, \mu_2)$ . Notice,  $R^w$  constitutes a relation in  $\text{Prob}(X_1) \times \text{Prob}(X_2)$ . Notice, weight functions are also well-established in the discrete setting [39].

**Variables.** Let  $\text{Var}$  denote a countable set of variables and  $V \subseteq \text{Var}$ . We denote by  $\text{Ev}(V)$  the set of all *variable evaluations for*  $V$ , i.e., functions from  $V$  to  $\mathbb{R}$ . As the countable product of Polish spaces equipped with the product topology again yields a Polish space,  $\text{Ev}(V)$  constitutes a Polish space. Let  $e \in \text{Ev}(\text{Var})$  and  $\eta \in \text{Prob}(\text{Ev}(\text{Var}))$ . The projection  $e|_V \in \text{Ev}(V)$  is given by  $e|_V(v) = e(v)$  for all  $v \in V$ . As  $f: \text{Ev}(\text{Var}) \rightarrow \text{Ev}(V)$ ,  $f(e) = e|_V$  is measurable, we can safely define  $\eta|_V = f_{\#}(\eta)$ .  $\text{Cond}(\text{Var})$  denotes the set of all Boolean conditions over  $\text{Var}$  and we write  $e \models c$  if the variable evaluation  $e$  satisfies condition  $c$ . For instance,  $e \models (v \leq 3.14159) \wedge (v \geq 2.71828)$  iff  $e(v) \leq 3.14159$  and  $e(v) \geq 2.71828$ .

**Stochastic transition systems.** An *STS* is a triple  $\mathcal{T} = (S, \Gamma, \rightarrow)$  comprising a measurable space  $S$  of *states*, a set  $\Gamma$  of *labels*, and a relation  $\rightarrow \subseteq S \times \Gamma \times \text{Prob}(S)$  of *transitions*. If  $S$  is a standard Borel space, then  $\mathcal{T}$  is called *standard Borel*. Let  $\mathcal{T}_a = (S_a, \Gamma, \rightarrow_a)$  and  $\mathcal{T}_b = (S_b, \Gamma, \rightarrow_b)$  be STSs with the same sets of labels. A relation  $R \subseteq S_a \times S_b$  is a *bisimulation for*  $(\mathcal{T}_a, \mathcal{T}_b)$  if  $R$  is lr-total in  $S_a \times S_b$  and for all  $s_a R s_b$  and  $\gamma \in \Gamma$  it holds:

Given  $\mu_a \in \text{Prob}(S_a)$  where  $s_a \rightarrow_a^\gamma \mu_a$ , then there exists  $\mu_b \in \text{Prob}(S_b)$  such that  $s_b \rightarrow_b^\gamma \mu_b$  and  $\mu_a R^w \mu_b$ . Vice versa, given  $\mu_b \in \text{Prob}(S_b)$  with  $s_b \rightarrow_b^\gamma \mu_b$ , then there is  $\mu_a \in \text{Prob}(S_a)$  where  $s_a \rightarrow_a^\gamma \mu_a$  and  $\mu_a R^w \mu_b$ . We emphasize that a bisimulation is not required to be measurable. In the context of bisimulation an important question is how to lift a relation  $R \subseteq S_a \times S_b$  to probability measures. However, there are other approaches using  $R$ -stable pairs instead [22], closely related to the weight lifting [41, 39]. Given STSs  $\mathcal{T}_1 = (S_1, \Gamma_1, \rightarrow_1)$  and  $\mathcal{T}_2 = (S_2, \Gamma_2, \rightarrow_2)$  and a set of synchronization labels  $\text{Sync} \subseteq \Gamma_1 \cap \Gamma_2$ , their composition is the STS  $\mathcal{T}_1 \parallel_{\text{Sync}}^\otimes \mathcal{T}_2 = (S_1 \times S_2, \Gamma_1 \cup \Gamma_2, \rightarrow)$  with  $\langle s_1, s_2 \rangle \rightarrow^\gamma \mu_1 \otimes \mu_2$  iff the following holds [16]: If  $\gamma \in \Gamma_1 \setminus \text{Sync}$ , then  $s_1 \rightarrow^\gamma \mu_1$  and  $\mu_2 = \text{Dirac}[s_2]$ . If  $\gamma \in \Gamma_2 \setminus \text{Sync}$ , then  $\mu_1 = \text{Dirac}[s_1]$  and  $s_2 \rightarrow^\gamma \mu_2$ . If  $\gamma \in \text{Sync}$ , then  $s_1 \rightarrow^\gamma \mu_1$  and  $s_2 \rightarrow^\gamma \mu_2$ .

**Flows.** By  $\mathbb{T} = \mathbb{R}_{\geq 0}$  we denote the *time axis*. A *flow* is a function  $\vartheta: \mathbb{T} \rightarrow \text{Ev}(\text{Var})$  that has the càdlàg property, i.e.,  $\vartheta$  is right continuous and has left limits everywhere.  $\text{Flow}(\text{Var})$  denotes the set of all flows. Let  $\vartheta \oplus T(t) = \vartheta(T+t)$  denote the *shift of  $\vartheta$  at time  $T \in \mathbb{T}$  by time  $t \in \mathbb{T}$* . A subset  $F$  of  $\text{Flow}(\text{Var})$  is *shift invariant* if  $\vartheta \oplus T \in F$  for every  $\vartheta \in F$  and  $T \in \mathbb{T}$ . In the theory of stochastic processes, the càdlàg property is well established as, amongst others, there is a topology on  $\text{Flow}(\text{Var})$  such that  $\text{Flow}(\text{Var})$  becomes a Polish space [7]. The exact definition of this topology is not relevant for our purposes. If  $V \subseteq \text{Var}$  and  $\vartheta \in \text{Flow}(\text{Var})$ , then  $\vartheta|_V \in \text{Flow}(V)$  is given by  $\vartheta|_V(t) = \vartheta(t)|_V$  for all  $t \in \mathbb{T}$ . Given  $V_1, V_2 \subseteq \text{Var}$  where  $V_1 \cap V_2 = \emptyset$  and  $\vartheta_1 \in \text{Flow}(V_1)$  and  $\vartheta_2 \in \text{Flow}(V_2)$ , then  $\vartheta_1 \uplus \vartheta_2 \in \text{Flow}(V_1 \cup V_2)$  is the flow obtained by merging  $\vartheta_1$  and  $\vartheta_2$ .

### 3 Composition of stochastic transition systems

We develop our approach towards the composition of STSs. As a preparation, we introduce spans first and give some insights on our mathematical theory for those. After that, we present the main contribution of the paper, namely our composition operator for STSs. We then give a congruence theorem having a quite challenging proof. Section 4 presents an application of our framework in the context of stochastic hybrid systems.

#### 3.1 Spans

We will formalize dependencies for the composition of STSs using spans and span couplings, which is a generic and flexible formalism our framework benefits from in many occasions. The idea is to allow for arbitrary Polish spaces together with continuous functions that specify the relationships between the components. Various properties of spans then transfer to their probabilistic version, e.g., properness or the existence of inverses. This is an essential point in the context of stochastic models and hence also for STSs. We will then use spans within the definition of our composition in STS and later on also in the context of stochastic hybrid systems as a mathematical tool for our argumentation.

► **Definition 1.** A *span* is a tuple  $\mathcal{X} = (X, X_1, X_2, \iota_1, \iota_2)$  consisting of Polish spaces  $X$ ,  $X_1$ , and  $X_2$  and continuous functions  $\iota_1: X \rightarrow X_1$  and  $\iota_2: X \rightarrow X_2$ . We call  $\mathcal{X}$  *proper*, if  $\iota_1^{-1}(K_1) \cap \iota_2^{-1}(K_2)$  is compact in  $X$  for all compact sets  $K_1 \subseteq X_1$  and  $K_2 \subseteq X_2$ .

Intuitively,  $X$  denotes the joint state space of  $X_1$  and  $X_2$ , where  $\iota_1$  and  $\iota_2$  are projective functions from  $X$  to  $X_1$  and  $X_2$ , respectively. Properness connects topological aspects of the involved spaces. The following examples are natural instances of proper spans:

- $\mathcal{X}$  is a *Cartesian span* if  $X = X_1 \times X_2$  and  $\iota_1$  and  $\iota_2$  are the natural projections.

- $\mathcal{X}$  is a *variable span* if  $X_1 = \text{Ev}(\text{Var}_1)$ ,  $X_2 = \text{Ev}(\text{Var}_2)$ , and  $X = \text{Ev}(\text{Var}_1 \cup \text{Var}_2)$  for some sets of variables  $\text{Var}_1$  and  $\text{Var}_2$ , and  $\iota_1$  and  $\iota_2$  are the natural projections.
- $\mathcal{X}$  is a *identity span* if  $X = X_1 = X_2$  and  $\iota_1(x) = x$  and  $\iota_2(x) = x$  for all  $x \in X$ .

Span couplings are a crucial notion for our approach towards a composition operator in the next section. Given  $\mu_1 \in \text{Prob}(X_1)$  and  $\mu_2 \in \text{Prob}(X_2)$ , we call  $\mu \in \text{Prob}(X)$  a  $\mathcal{X}$ -*coupling* of  $(\mu_1, \mu_2)$  if  $(\iota_1)_\#(\mu) = \mu_1$  and  $(\iota_2)_\#(\mu) = \mu_2$ . Recall that  $(\iota_1)_\#$  and  $(\iota_2)_\#$  denote the pushforwards of  $\iota_1$  and  $\iota_2$ , respectively. A span coupling places two probability measures in the same probabilistic space specified by the span by exhibiting an adequate witness measure over pairs. Thus, the ordinary notion for couplings is generalized. For all  $x$  and  $\mu$  we use  $x|_1$ ,  $x|_2$ ,  $\mu|_1$ , and  $\mu|_2$  as shorthand notations for  $\iota_1(x)$ ,  $\iota_2(x)$ ,  $(\iota_1)_\#(\mu)$ , and  $(\iota_2)_\#(\mu)$  respectively. Given  $x_1 \in X_1$  and  $x_2 \in X_2$ , we then write  $x_1 \mathcal{X} x_2$  if there exists  $x \in X$  where  $x|_1 = x_1$  and  $x|_2 = x_2$ . Similarly, we write  $\mu_1 \mathcal{X}^c \mu_2$  if there is a  $\mathcal{X}$ -coupling of  $(\mu_1, \mu_2)$ . We sometimes drop the projection functions from the tuple and refer to  $(X, X_1, X_2)$  as a span.

**Probabilistic version.** There are various operations for spans that yield complex spans out of some given basic spans. The question whether the operation preserves properness is important for practical purposes. For instance, using Tychonoff's theorem, a countable product of proper spans yields a proper span again. Within stochastic models, the following operation is important: For a span  $\mathcal{X} = (X, X_1, X_2, \iota_1, \iota_2)$  its *probabilistic version* is given by the tuple  $\text{Prob}(\mathcal{X}) = (\text{Prob}(X), \text{Prob}(X_1), \text{Prob}(X_2), (\iota_1)_\#, (\iota_2)_\#)$ . Notice,  $\text{Prob}(\mathcal{X})$  involves all  $\mathcal{X}$ -couplings and  $\mu_1 \mathcal{X}^c \mu_2$  iff  $\mu_1 \text{Prob}(\mathcal{X}) \mu_2$  for all  $\mu_1 \in \text{Prob}(X_1)$  and  $\mu_2 \in \text{Prob}(X_2)$ .

► **Proposition 2.** *The probabilistic version of a span is a span. Moreover, the probabilistic version of a proper span is proper as well.*

The claim regarding properness follows from Prokhorov's theorem [7], which characterizes relatively compact subsets of  $\text{Prob}(X)$ : If  $P \subseteq \text{Prob}(X)$  is a set of probability measures, then  $P$  is relatively compact in  $\text{Prob}(X)$  iff  $P$  is tight in  $\text{Prob}(X)$ , i.e., for every  $\varepsilon \in \mathbb{R}_{>0}$  there is a compact set  $K \subseteq X$  where  $\mu(K) > 1 - \varepsilon$  for all  $\mu \in P$ .

**Span inverse.** In a compositional setting, the states of the components determine the states of the composed system. Within our approach, a state as an element of  $X$  in the composed system is not required to be uniquely determined: Given a span  $\mathcal{X} = (X, X_1, X_2)$ ,  $x_1 \in X_1$ , and  $x_2 \in X_2$ , every  $x \in X$  where  $x|_1 = x_1$  and  $x|_2 = x_2$  stands for a state in the composed system resulting from the states  $x_1$  and  $x_2$  of the components. However, in applications later it is important to have a mapping with additional properties: Given a span  $\mathcal{X} = (X, X_1, X_2)$ , a Borel function  $f: X_1 \times X_2 \rightarrow X$  is called an  $\mathcal{X}$ -*inverse*, if for all  $x_1 \in X_1$  and  $x_2 \in X_2$ , if  $x_1 \mathcal{X} x_2$ , then  $f(x_1, x_2)|_1 = x_1$  and  $f(x_1, x_2)|_2 = x_2$ .

► **Theorem 3.** *Every proper span  $\mathcal{X}$  has an  $\mathcal{X}$ -inverse.*

It follows  $\mu_1 \mathcal{X}^c \mu_2$  iff  $\mu_1 \text{Rel}(\mathcal{X})^w \mu_2$  for all  $\mu_1 \in \text{Prob}(X_1)$  and  $\mu_2 \in \text{Prob}(X_2)$ , where  $\text{Rel}(\mathcal{X}) = \{\langle x|_1, x|_2 \rangle; x \in X\}$ . Our proof of Theorem 3 is an application of a *measurable selection theorem* [8]: Take some  $\hat{x} \in X$  and define  $\Phi: X_1 \times X_2 \rightarrow 2^X$ ,  $\Phi(x_1, x_2) = \{x \in X; x|_1 = x_1 \text{ and } x|_2 = x_2\}$ , if the set on the right-hand side is non-empty, and  $\Phi(x_1, x_2) = \{\hat{x}\}$ , otherwise. It suffices to argue that  $\Phi$  admits a measurable selection, i.e., there is a measurable function  $f: X_1 \times X_2 \rightarrow X$  where  $f(x_1, x_2) \in \Phi(x_1, x_2)$  for all  $x_1 \in X_1$  and  $x_2 \in X_2$ . To do so, we rely on results from descriptive set theory. Notice, together with Proposition 2, Theorem 3 yields an  $\text{Prob}(\mathcal{X})$ -inverse if  $\mathcal{X}$  is proper, which is an important observation for our discussions later. This is not obvious even for simple spans considering

for instance the probabilistic version of a variable span. We remark that there are spans  $\mathcal{X}$  that have no  $\mathcal{X}$ -inverses and thus, the properness assumption is important [25].

### 3.2 Composition

A major objective in defining compositional frameworks is to separate the concerns of components specifying the operational behavior and composition operators addressing their interaction or coordination. We start with two STSs  $\mathcal{T}_1 = (S_1, \Gamma_1, \rightarrow_1)$  and  $\mathcal{T}_2 = (S_2, \Gamma_2, \rightarrow_2)$ , where we assume  $S_1$  and  $S_2$  are Polish spaces. To declare the interactions between  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , we specify a set of synchronization labels  $\text{Sync} \subseteq \Gamma_1 \cap \Gamma_2$ , a span  $\mathcal{S} = (S, S_1, S_2)$  to characterizes the state space of the composition, and an so-called agreement  $\mathcal{G} = (LC_1, LC_2)$  between  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Here,  $LC_1$  and  $LC_2$  are so-called local constraints and for the moment, to present the central definition of this paper, it suffices to require  $LC_1, LC_2 \subseteq S \times \text{Prob}(S)$ . Intuitively, we use local constraints to specify the behavior of local variables within local transitions (see below).

► **Definition 4.** We define the STS  $\mathcal{T}_1 \parallel_{\mathcal{S}, \mathcal{G}, \text{Sync}} \mathcal{T}_2 = (S, \Gamma_1 \cup \Gamma_2, \rightarrow)$ , where for all  $s \in S$ ,  $\gamma \in \Gamma$ , and  $\mu \in \text{Prob}(S)$  it holds  $s \rightarrow^\gamma \mu$  iff one of the following three conditions hold:

- $\gamma \in \Gamma_1 \setminus \text{Sync}$  and  $s|_{S_1} \rightarrow_1^\gamma \mu|_{S_1}$  and  $s LC_2 \mu$ .
- $\gamma \in \Gamma_2 \setminus \text{Sync}$  and  $s LC_1 \mu$  and  $s|_{S_2} \rightarrow_2^\gamma \mu|_{S_2}$ .
- $\gamma \in \text{Sync}$  and  $s|_{S_1} \rightarrow_1^\gamma \mu|_{S_1}$  and  $s|_{S_2} \rightarrow_2^\gamma \mu|_{S_2}$ .

To illustrate the crux of our composition operator, we regard the case where  $\mathcal{S}$  is a Cartesian span, i.e.,  $S = S_1 \times S_2$ . Former approaches [39, 16] assume that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  behave stochastically independent in a synchronizing step, i.e., if  $s|_{S_1} \rightarrow_1^\gamma \mu|_{S_1}$  and  $s|_{S_2} \rightarrow_2^\gamma \mu|_{S_2}$ , then  $s \rightarrow^\gamma \mu|_{S_1} \otimes \mu|_{S_2}$  in  $\mathcal{T}_1 \parallel_H^\otimes \mathcal{T}_2$ . Our operator does not rely on any stochastic assumptions: Instead of considering only the independent coupling we take all the couplings into account, i.e., if  $s|_{S_1} \rightarrow_1^\gamma \mu|_{S_1}$  and  $s|_{S_2} \rightarrow_2^\gamma \mu|_{S_2}$ , then  $s \rightarrow^\gamma \mu$  for all couplings  $\mu$  of  $(\mu|_{S_1}, \mu|_{S_2})$ . During a discussion about the example from the introduction and SHMAs, we will see how additional stochastic information between the components can be incorporated within our general framework.

**Local constraints.** Our composition operator is indexed by a span, which determines the dependencies between the states of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . For instance, one can specify shared and local variables using the variable span. When composing STSs, one has to ensure that local transitions and variables of the components behave in a compatible way. Let us illustrate this and regard again the case where  $\mathcal{S}$  is a Cartesian span. If  $\mathcal{T}_1$  performs a local transition, i.e., a transition that is labeled by some  $\gamma \in \Gamma_1 \setminus \text{Sync}$ , then the current state of  $\mathcal{T}_2$  must not change. The properties of a local constraint should hence guarantee  $s LC_2 \mu$  iff  $\mu|_{S_2} = \text{Dirac}[s|_{S_2}]$ . It then follows that  $\langle s_1, s_2 \rangle \rightarrow^\gamma \mu|_{S_1} \otimes \text{Dirac}[s_2]$  for all  $s_2 \in S_2$  and  $s_1 \rightarrow_1^\gamma \mu|_{S_1}$  where  $\gamma \in \Gamma_1 \setminus \text{Sync}$ . Of course, the same discussion applies for  $\mathcal{T}_1$  and the local constraint  $LC_1$ . This leads to the following requirements for a local constraint  $LC_2 \subseteq S \times \text{Prob}(S)$ :

- For all  $s \in S$  and  $\mu \in \text{Prob}(S)$ , if  $\mu|_{S_2} = \text{Dirac}[s|_{S_2}]$ , then  $s LC_2 \mu$ .
- For all  $s LC_2 \mu$  and  $\mu' \in \text{Prob}(S)$ , if  $\mu|_{S_1} = \mu'|_{S_1}$  and  $\mu|_{S_2} = \mu'|_{S_2}$ , then  $s LC_2 \mu'$ .
- For all  $s LC_2 \mu$ , if  $\mu|_{S_1} \mathcal{S}^c \text{Dirac}[s|_{S_2}]$ , then  $\mu$  is a  $\mathcal{S}$ -coupling of  $(\mu|_{S_1}, \text{Dirac}[s|_{S_2}])$ .

The requirements for  $LC_1$  are similar. Intuitively, the first requirement for  $LC_2$  ensures that the STS  $\mathcal{T}_2$  cannot block a local transition of  $\mathcal{T}_1$  which is not critical from the view of  $\mathcal{T}_2$ , i.e., variables of  $\mathcal{T}_2$  are not affected within the transition of  $\mathcal{T}_1$ . Thus, such local transition of  $\mathcal{T}_1$  are independent of  $\mathcal{T}_2$  and can happen autonomously. Different couplings of given probability measures cannot be distinguished within local constraints imposed by the second property.

The third requirement intuitively demands that whenever  $\mathcal{T}_1$  performs a local transition where no local variables of  $\mathcal{T}_2$  are modified, the state of  $\mathcal{T}_2$  must not change. In case of a Cartesian span the above requirements yield

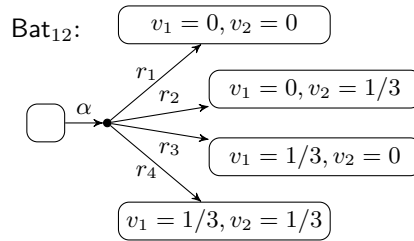
$$\begin{aligned} LC_2 &= \{ \langle \langle s_1, s_2 \rangle, \mu_1 \otimes \text{Dirac}[s_2] \rangle ; s_1 \in S_1 \text{ and } s_2 \in S_2 \text{ and } \mu_1 \in \text{Prob}(S_1) \} \quad \text{and} \\ LC_1 &= \{ \langle \langle s_1, s_2 \rangle, \text{Dirac}[s_1] \otimes \mu_2 \rangle ; s_1 \in S_1 \text{ and } s_2 \in S_2 \text{ and } \mu_2 \in \text{Prob}(S_2) \}. \end{aligned}$$

Hence, the agreement  $\mathcal{G}$  is uniquely determined by STSs  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . We thus simply write  $\mathcal{T}_1 \parallel_{\times, \text{Sync}} \mathcal{T}_2$  instead of  $\mathcal{T}_1 \parallel_{\mathcal{S}, \mathcal{G}, \text{Sync}} \mathcal{T}_2$ . Observe that  $\mathcal{T}_1 \parallel_{\times, \text{Sync}} \mathcal{T}_2$  and  $\mathcal{T}_1 \parallel_{\text{Sync}}^{\otimes} \mathcal{T}_2$  are not bisimilar in general. This is due to the fact that our composition operator does not incorporate any stochastic assumptions concerning the interaction of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . In case where  $\mathcal{S}$  is a variable span, i.e.,  $S_1 = \text{Ev}(\text{Var}_1)$ ,  $S_2 = \text{Ev}(\text{Var}_2)$ , and  $S = \text{Ev}(\text{Var}_1 \cup \text{Var}_2)$  for some sets of variables  $\text{Var}_1$  and  $\text{Var}_2$ , there are more possible local constraints:

$$\begin{aligned} LC'_2 &= \{ \langle e, \eta \rangle \in S \times \text{Prob}(S) ; \eta|_{\text{Var}_1} = \text{Dirac}[e|_{\text{Var}_1}] \text{ implies } \eta = \text{Dirac}[e] \}, \\ LC''_2 &= \{ \langle e, \eta \rangle \in S \times \text{Prob}(S) ; \eta|_{\text{Var}_2 \setminus \text{Var}_1} = \text{Dirac}[e|_{\text{Var}_2 \setminus \text{Var}_1}] \}, \quad \text{and} \\ LC'''_2 &= \{ \langle e, \eta \rangle \in S \times \text{Prob}(S) ; \eta|_{\text{Var}_2} = \text{Dirac}[e|_{\text{Var}_2}] \} \end{aligned}$$

all enjoy the requirements for a local constraint where  $LC'_2 \supseteq LC''_2 \supseteq LC'''_2$ . Considering for instance  $LC''_2$ , all the variables in  $\text{Var}_2 \setminus \text{Var}_1$  cannot be modified within a local transition of  $\mathcal{T}_1$ . Constraint  $LC'''_2$  is more restrictive: Here, all the variables in  $\text{Var}_2$  are controlled by  $\mathcal{T}_2$  and cannot be modified in a local transition of  $\mathcal{T}_1$ , i.e., variables in  $\text{Var}_1 \cap \text{Var}_2$  can be observed by  $\mathcal{T}_1$  only. It turns out  $LC'_2 \supseteq LC_2$  for every local constraint  $LC_2$ . Every local constraint hence enjoys the property that variables in  $\text{Var}_2 \setminus \text{Var}_1$  must not be adapted within a local transition of  $\mathcal{T}_1$  if the evaluations of the variables in  $\text{Var}_1$  remain the same.

**Example from the introduction.** We return to the introductory stochastic systems illustrated in Section 1. Of course,  $\text{Bat}_1$ ,  $\text{Bat}_2$ , and  $\text{Dev}$  can be seen as STSs with sets of states  $\text{Ev}(\{v_1\})$ ,  $\text{Ev}(\{v_2\})$ , and  $\text{Ev}(\{v_1, v_2\})$ , respectively. When composing them, we need not to worry about local constraints as there is only one synchronization action  $\alpha$ . In what follows, we rely on the obvious variable spans. The composition of  $\text{Bat}_1$  and  $\text{Bat}_2$  yields the STS  $\text{Bat}_{12}$  depicted below.



There are infinitely many transitions: Every solution of the linear equation system  $r_1 + r_2 = r_1 + r_3 = r_2 + r_4 = r_3 + r_4 = 1/2$  where  $r_1, r_2, r_3, r_4 \in [0, 1]$  represents a coupling of the involved measures. When composing  $\text{Bat}_{12}$  and  $\text{Dev}$ , the set of all couplings is refined. We are moreover able to handle more complex stochastic information that depend on the operational behavior of the components. To illustrate this, assume systems which result from  $\text{Bat}_1$  and  $\text{Bat}_2$  such that  $\alpha$  can be executed repeatedly (e.g., add some local transitions back to the blank state). An additional component might encode that, if the system has crashed repeatedly in the past, the event that the stored energy drops to 0 in both batteries



at the same time becomes more likely within an execution of  $\alpha$ . We emphasize that the ordinary composition of STSs can be expressed within our framework using an additional component [25].

### 3.3 Congruence

In the context of process calculi, an important issue of bisimulation is the compatibility with syntactic operators in the process calculus, such as parallel composition. We show that bisimulation is a congruence for our composition operator under reasonable side-constraints, i.e., our composition operator enjoys the substitution property with respect to bisimulation. Suppose STSs  $\mathcal{T}_{a1} = (S_{a1}, \Gamma_1, \rightarrow_{a1})$ ,  $\mathcal{T}_{a2} = (S_{a2}, \Gamma_2, \rightarrow_{a2})$ ,  $\mathcal{T}_{b1} = (S_{b1}, \Gamma_1, \rightarrow_{b1})$ , and  $\mathcal{T}_{b2} = (S_{b2}, \Gamma_2, \rightarrow_{b2})$  such that  $\mathcal{T}_{a1} \sim \mathcal{T}_{b1}$  and  $\mathcal{T}_{a2} \sim \mathcal{T}_{b2}$ . Define

$$\mathcal{T}_a = \mathcal{T}_{a1} \parallel_{S_a, \mathcal{G}_a, \text{Sync}} \mathcal{T}_{a2} \quad \text{and} \quad \mathcal{T}_b = \mathcal{T}_{b1} \parallel_{S_b, \mathcal{G}_b, \text{Sync}} \mathcal{T}_{b2},$$

where  $\text{Sync} \subseteq \Gamma_1 \cap \Gamma_2$ ,  $\mathcal{S}_a = (S_a, S_{a1}, S_{a2})$  and  $\mathcal{S}_b = (S_b, S_{b1}, S_{b2})$  are proper spans, and  $\mathcal{G}_a = (LC_{a1}, LC_{a2})$  and  $\mathcal{G}_b = (LC_{b1}, LC_{b2})$  are agreements. Assume  $R_1$  is a bisimulation for  $(\mathcal{T}_{a1}, \mathcal{T}_{b1})$  and  $R_2$  is a bisimulation for  $(\mathcal{T}_{a2}, \mathcal{T}_{b2})$  and define

$$R_1 \wedge R_2 = \{ \langle s_a, s_b \rangle \in S_a \times S_b ; s_{a|1} R_1 s_{b|1} \text{ and } s_{a|2} R_2 s_{b|2} \}.$$

We aim to show that  $R_1 \wedge R_2$  is a bisimulation for  $(\mathcal{T}_a, \mathcal{T}_b)$  and hence  $\mathcal{T}_a \sim \mathcal{T}_b$ . However, we cannot expect this result without any compatibility requirements for the involved spans and agreements, since important relationships concerning the communication of the components are determined within our composition operator. This motivates the following notions: We refer to the tuple  $\mathcal{C} = (S_a, S_b, R_1, R_2)$  as *span connection* and call  $\mathcal{C}$  *adequate* if for all  $\mu_{a1} R_1^w \mu_{b1}$  and  $\mu_{a2} R_2^w \mu_{b2}$  it holds  $\mu_{a1} S_a^c \mu_{a2}$  iff  $\mu_{b1} S_b^c \mu_{b2}$ . Intuitively, adequacy requires that the existence of span couplings is preserved by the relations  $R_1$  and  $R_2$ . Observe, if  $S_a$  and  $S_b$  are Cartesian spans, then  $\mathcal{C}$  is always adequate. The local constraints  $LC_{a2}$  and  $LC_{b2}$  are called  *$\mathcal{C}$ -bisimilar* if for all  $s_a (R_1 \wedge R_2) s_b$  holds:

- For all  $\mu_a \in \text{Prob}(S_a)$  and  $\mu_{b1} \in \text{Prob}(S_{b1})$ , if  $s_a LC_{a2} \mu_a$  and  $\mu_{a|1} R_1^w \mu_{b1}$ , then there is  $\mu_b \in \text{Prob}(S_b)$  where  $s_b LC_{b2} \mu_b$ ,  $\mu_{b|1} = \mu_{b1}$ , and  $\mu_{a|2} R_2^w \mu_{b|2}$ .
- For all  $\mu_b \in \text{Prob}(S_b)$  and  $\mu_{a1} \in \text{Prob}(S_{a1})$ , if  $s_b LC_{b2} \mu_b$  and  $\mu_{a1} R_1^w \mu_{b|1}$ , then there is  $\mu_a \in \text{Prob}(S_a)$  where  $s_a LC_{a2} \mu_a$ ,  $\mu_{a|1} = \mu_{a1}$ , and  $\mu_{a|2} R_2^w \mu_{b|2}$ .

$LC_{a1}$  and  $LC_{b1}$  are called  *$\mathcal{C}$ -bisimilar* if analogous properties are fulfilled. Observe that the stated requirement is motivated by the definition of bisimulation in the sense that each element of a local constraint  $LC_{a2}$  can be mimicked by  $LC_{b2}$  regarding the relations  $R_1$  and  $R_2$ . If  $LC_{a2}$  and  $LC_{b2}$  as well as  $LC_{a1}$  and  $LC_{b1}$  are  $\mathcal{C}$ -bisimilar, respectively, then we refer to  $\mathcal{G}_a$  and  $\mathcal{G}_b$  as  *$\mathcal{C}$ -bisimilar*.

► **Theorem 5.** *If the span connection  $\mathcal{C}$  is adequate and the agreements  $\mathcal{G}_a$  and  $\mathcal{G}_b$  are  $\mathcal{C}$ -bisimilar, then  $R_1 \wedge R_2$  is a bisimulation for  $(\mathcal{T}_a, \mathcal{T}_b)$ .*

The challenging part of the proof can be summarized by the following claim [25]: Let  $\mu_a \in \text{Prob}(S_a)$ ,  $\mu_{b1} \in \text{Prob}(S_{b1})$ , and  $\mu_{b2} \in \text{Prob}(S_{b2})$  where  $\mu_{a|1} R_1^w \mu_{b1}$  and  $\mu_{a|2} R_2^w \mu_{b2}$ . Then there is an  $S_b$ -coupling  $\mu_b$  of  $(\mu_{b1}, \mu_{b2})$  such that  $\mu_a (R_1 \wedge R_2)^w \mu_b$ . Our proof of this claim proceeds as follows. Assume  $W_1$  is a weight function for  $(\mu_{a|1}, R_1, \mu_{b1})$  and  $W_2$  is a weight function for  $(\mu_{a|2}, R_2, \mu_{b2})$ . Using disintegration of measures [34], there are Markov kernels  $k_1: S_{a1} \rightarrow \text{Prob}(S_{a2})$  and  $k_2: S_{b1} \rightarrow \text{Prob}(S_{b2})$  such that  $W_1 = \mu_{a|1} \times k_1$  and  $W_2 = \mu_{a|2} \times k_2$ . The crucial point is now to argue that there is a Markov kernel  $k: S_a \rightarrow \text{Prob}(S_b)$  where  $k(s_a)$

is an  $\mathcal{S}_b$ -coupling of  $(k_1(s_{a|1}), k_2(s_{a|2}))$  for  $\mu_a$ -almost all  $s_a \in S_a$ . Here, we make use of an  $\mathcal{S}_b$ -inverse (cf. Theorem 3). With this Markov kernel at hand, we define  $W \in \text{Prob}(S_a \times S_b)$  by  $W = \mu \times k$  and  $\mu_b \in \text{Prob}(S_b)$  by  $\mu_b(M_b) = W(S_a \times M_b)$ . It turns out that  $\mu_b$  is an appropriate  $\mathcal{S}_b$ -coupling. To summarize, we defined a potential weight function  $W$  out of the weight functions  $W_1$  and  $W_2$  and then introduced the measure  $\mu_b$  via  $W$ .

**Path measures.** When resolving the non-determinism in STSs using schedulers, one obtains a probability measure – the path measure – on the set of all infinite paths of the STS [16]. Besides our congruence result, we expect compatibility of path measures induced by schedulers in our compositional framework. To provide an intuition, assume STSs  $\mathcal{T}_1$  and  $\mathcal{T}_2$  and let  $\mathcal{T}$  be an STSs obtained by a composition involving  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Assume that  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  are schedulers for  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , respectively, and  $\mathfrak{S}$  is a scheduler for  $\mathcal{T}$ . If  $\mathfrak{S}$  satisfies certain compatibility requirements regarding  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ , one can show that the induced path measure for  $\mathcal{T}$  is a coupling of the corresponding path measures for  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Here, we consider a natural span that connects the sets of all infinite paths of  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}$ .

#### 4 Stochastic hybrid motion automata

We apply our general results of the preceding sections and develop a compositional modeling framework for stochastic hybrid systems. The formal definition of our model relies on a standard schema of hybrid automata [3, 29, 26], i.e., there are a discrete control structure consisting of locations and jumps in-between, and continuous variables whose values evolve according to a flow formalized by a motion function. Within a jump, the variables can be updated instantaneously. The novelty of our approach is that every jump is indexed by a set of those variables that are not affected in the corresponding discrete step. As a consequence, the adjustment of flows is always accompanied by a specific command.

**Syntax.** Every jump in our hybrid-automaton model is labeled by a command: Given a set  $\text{Var}$  of variables and a set  $\text{Act}$  of actions, a *command on*  $(\text{Var}, \text{Act})$  is a tuple  $\langle c, \alpha, V, \text{upd} \rangle$  consisting of a *guard*  $c \in \text{Cond}(\text{Var})$ , an *action*  $\alpha \in \text{Act}$ , a set of *disabled* variables  $V \subseteq \text{Var}$ , and an (*non-deterministic*) *update*  $\text{upd}: \text{Ev}(\text{Var}) \rightarrow 2^{\text{Prob}(\text{Ev}(\text{Var}))}$  where  $\eta|_V = \text{Dirac}[e|_V]$  for all  $\eta \in \text{upd}(e)$  and  $e \in \text{Ev}(\text{Var})$ .  $\text{Cmd}(\text{Var}, \text{Act})$  denotes the set of all commands on  $(\text{Var}, \text{Act})$ . Intuitively, a jump is enabled if the current variable evaluation satisfies the guard. The action name indicates whether the jump is an internal location switch or subject to an interaction with another component. The set of disabled variables specifies those variables which are not affected within the jump. This also clarifies the additional requirement for updates.

► **Definition 6.** An *SHMA* is a tuple  $(\text{Loc}, \text{Var}, \text{Act}, \text{Inv}, \text{Mot}, \rightarrow)$  where  $\text{Loc}$  is a finite set of *locations*,  $\text{Var}$  is a set of variables,  $\text{Act}$  is a set of actions,  $\text{Inv}: \text{Loc} \rightarrow \text{Cond}(\text{Var})$  is an *invariant function*,  $\text{Mot}: \text{Loc} \rightarrow 2^{\text{Flow}(\text{Var})}$  is a *motion function* which assigns a shift-invariant set of flows to every location, and  $\rightarrow \subseteq \text{Loc} \times \text{Cmd}(\text{Var}, \text{Act}) \times \text{Prob}(\text{Loc})$  is a *jump relation*.

We write  $l \text{--}[cmd] \rightarrow \lambda$  instead of  $\langle l, cmd, \lambda \rangle \in \rightarrow$ . The behavior in a location  $l$  depends on the current variable evaluation  $e$ . In a discrete step, a jump  $l \text{--}[c, \alpha, V, \text{upd}] \rightarrow \lambda$  where  $e \models c$  is chosen non-deterministically. Then, action  $\alpha$  is executed and a successor location is sampled according to  $\lambda$ . The evaluation of the variables changes according to a non-deterministically chosen probability measure contained in  $\text{upd}(e)$ . Entering a location  $l'$ , a flow in  $\text{Mot}(l')$  is also chosen non-deterministically and the variables then evolve according to this flow.

**Semantics.** Every SHMA  $\mathcal{H} = (\text{Loc}, \text{Var}, \text{Act}, \text{Inv}, \text{Mot}, \rightarrow)$  can be interpreted as an STS resulting from unfolding. In what follows,  $S = \text{Loc} \times \text{Flow}(\text{Var})$  denotes the set of states. Notice,  $S$  constitutes a Polish space as  $\text{Flow}(\text{Var})$  is known to be a Polish space [7]. Intuitively, a state  $\langle l, \vartheta \rangle$  represents the actual location  $l$  and the current active flow  $\vartheta$ , i.e.,  $\vartheta$  corresponds to the flow chosen in the preceding jump. Moreover,  $\vartheta(0)$  stands for the present variable evaluation. We call  $\langle l, \vartheta \rangle$  *well-formed* if  $\vartheta \in \text{Mot}(l)$  and  $\vartheta(0) \models \text{Inv}(l)$ .

There are two kinds of transitions within our STS for  $\mathcal{H}$ , namely transitions where time passes and transitions corresponding to a jump. Time can pass in a location  $l$  as long as the flow does not violate the invariant  $\text{Inv}(l)$ . Transitions for jumps are more intricate. Assume  $l \dashv [c, \alpha, V, \text{upd}] \mapsto \lambda$  is enabled in state  $\langle l, \vartheta \rangle$ , i.e.,  $e \models c$  where  $e = \vartheta(0)$ . Basically, jumps in SHMAs proceed in two phases: First, a successor location and a variable evaluation are sampled according to  $\lambda$  and some  $\eta \in \text{upd}(e)$ , respectively. In the second phase, a flow is chosen non-deterministically for those variables which are not disabled, i.e., the variables in  $\text{Var} \setminus V$ . This is formalized as follows: A *flow adapter for*  $(\vartheta, V)$  is a Borel function  $\chi: \text{Loc} \times \text{Ev}(\text{Var}) \rightarrow \text{Flow}(\text{Var})$  such that for all  $l' \in \text{Loc}$  and  $e', \tilde{e}' \in \text{Ev}(\text{Var})$ :

$$\chi(l', e')|_V = \vartheta|_V \quad \text{and} \quad e'|_{\text{Var} \setminus V} = \tilde{e}'|_{\text{Var} \setminus V} \text{ implies } \chi(l', e')|_{\text{Var} \setminus V} = \chi(l', \tilde{e}')|_{\text{Var} \setminus V}.$$

Intuitively, if state  $\langle l', e' \rangle$  is sampled within the first phase of a jump, then  $\chi(l', e')$  represents the new flow, i.e., the flow which determines the evolution of variables in a subsequent time passage. The first condition for a flow adapter requires that the flow for disabled variables is not allowed to change. The required implication ensures that a flow is chosen independently of the disabled variables. This is important for our compositional approach, as we want to make sure that the choice of a new flow in an SHMA obtained by composition does not depend on the local variables of the respective communication partners. If  $\chi$  is a flow adapter, then we define the auxiliary function  $\hat{\chi}: \text{Loc} \times \text{Ev}(\text{Var}) \rightarrow S$ ,  $\hat{\chi}(l, e) = \langle l, \chi(l, e) \rangle$ .

► **Definition 7.** The *semantics of*  $\mathcal{H}$  is given by the STS  $\llbracket \mathcal{H} \rrbracket = (S, \mathbb{T} \cup \text{Act}, \rightarrow)$ , where  $\rightarrow$  is the smallest relation satisfying the following requirements for all well-formed states  $s = \langle l, \vartheta \rangle$ :

- For all  $T \in \mathbb{T}$ , if  $\vartheta(t) \models \text{Inv}(l)$  for every  $t \in [0, T]$ , then  $s \rightarrow^t \text{Dirac}[\langle l, \vartheta \oplus T \rangle]$ .
- For all  $l \dashv [c, \alpha, V, \text{upd}] \mapsto \lambda$ ,  $\eta \in \text{upd}(e)$ , couplings  $\nu$  of  $(\lambda, \eta)$ , and flow adapter  $\chi$  for  $(\vartheta, V)$ , if  $e \models c$  and for  $\nu$ -almost all  $\langle l', e' \rangle \in \text{Loc} \times \text{Ev}(\text{Var})$  the state  $\hat{\chi}(l', e')$  is well-formed, then  $s \rightarrow^\alpha \hat{\chi}_\#(\nu)$ . Here, we abbreviate  $e = \vartheta(0)$ .

An SHMA almost surely enters a well-formed state, i.e., if  $s \rightarrow^\gamma \mu$  where  $\gamma \in \mathbb{T} \cup \text{Act}$ , then  $s'$  is well-formed for  $\mu$ -almost all  $s' \in S$ . We emphasize that for our approach concerning the adaption of flows it is crucial that the current flow is part of a state. Otherwise, it would be not possible to ensure that the flow for disabled variables is not allowed to change.

**Composition.** We now introduce a composition operator for SHMAs. For  $i \in \{1, 2\}$  let  $\mathcal{H}_i = (\text{Loc}_i, \text{Var}_i, \text{Act}_i, \text{Inv}_i, \text{Mot}_i, \rightarrow_i)$  be SHMAs. When running  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in parallel,  $\mathcal{H}_1$  and  $\mathcal{H}_2$  synchronize on all actions contained in  $\text{Act}_1 \cap \text{Act}_2$  and the variables in  $\text{Var}_1 \cap \text{Var}_2$  are shared, i.e.,  $\text{Var}_1 \setminus \text{Var}_2$  and  $\text{Var}_2 \setminus \text{Var}_1$  represent the sets of the respective local variables. Abbreviate  $\text{Loc} = \text{Loc}_1 \times \text{Loc}_2$ ,  $\text{Var} = \text{Var}_1 \cup \text{Var}_2$ , and  $\text{Act} = \text{Act}_1 \cup \text{Act}_2$ . Let  $\text{upd}_1$  and  $\text{upd}_2$  be updates for  $\text{Var}_1$  and  $\text{Var}_2$ , respectively. The *Var-lifting of*  $(\text{upd}_1, \text{upd}_2)$  is the update  $\text{upd}$  for  $\text{Var}$  such that for all  $e \in \text{Ev}(\text{Var})$ ,  $\text{upd}(e)$  consists of all  $\eta \in \text{Prob}(\text{Ev}(\text{Var}))$  where  $\eta|_{\text{Var}_1} = \eta_1$  and  $\eta|_{\text{Var}_2} = \eta_2$  for some  $\eta_1 \in \text{upd}(e|_{\text{Var}_1})$  and  $\eta_2 \in \text{upd}(e|_{\text{Var}_2})$ . We define *Var-liftings* with respect to an update accordingly, i.e.,  $\text{upd}$  is a *Var-lifting of*  $\text{upd}_1$  if for all  $e \in \text{Ev}(\text{Var})$ ,  $\text{upd}(e)$  consists of all  $\eta \in \text{Prob}(\text{Ev}(\text{Var}))$  where  $\eta|_{\text{Var}_1} = \eta_1$  for some  $\eta_1 \in \text{upd}(e|_{\text{Var}_1})$  and  $\eta|_{\text{Var} \setminus \text{Var}_1} = \text{Dirac}[e|_{\text{Var} \setminus \text{Var}_1}]$ . Notice, the definition of *Var-liftings* involves couplings concerning a variable span, which provides a connection to the preceding sections.

► **Definition 8.**  $\mathcal{H}_1 \parallel \mathcal{H}_2 = (\text{Loc}, \text{Var}, \text{Act}, \text{Inv}, \text{Mot}, \rightarrow)$  is the SHMA with  $\text{Inv}(l_1, l_2) = \text{Inv}(l_1) \wedge \text{Inv}(l_2)$  and  $\text{Mot}(l_1, l_2) = \{\vartheta \in \text{Flow}(\text{Var}) ; \vartheta|_{\text{Var}_1} \in \text{Mot}_1(l_1) \text{ and } \vartheta|_{\text{Var}_2} \in \text{Mot}_2(l_2)\}$  for all  $\langle l_1, l_2 \rangle \in \text{Loc}$  and  $\rightarrow$  is the smallest relation such that  $\langle l_1, l_2 \rangle \dashv\vdash [c, \alpha, V, \text{upd}] \mapsto \lambda$ , if  $\lambda$  is a coupling of  $\lambda_1 \in \text{Prob}(\text{Loc}_1)$  and  $\lambda_2 \in \text{Prob}(\text{Loc}_2)$  and one of the following holds:

- $\alpha \in \text{Act}_1 \setminus \text{Act}_2$ ,  $\lambda_2 = \text{Dirac}[l_2]$ , and there is  $l_1 \dashv\vdash [c_1, \alpha, V_1, \text{upd}_1] \mapsto_1 \lambda_1$  such that  $c = c_1$ ,  $V = V_1 \cup (\text{Var}_2 \setminus \text{Var}_1)$ , and  $\text{upd}$  is the Var-lifting of  $\text{upd}_1$ .
- $\alpha \in \text{Act}_2 \setminus \text{Act}_1$ ,  $\lambda_1 = \text{Dirac}[l_1]$ , and there is  $l_2 \dashv\vdash [c_2, \alpha, V_2, \text{upd}_2] \mapsto_2 \lambda_2$  such that  $c = c_2$ ,  $V = V_2 \cup (\text{Var}_1 \setminus \text{Var}_2)$ , and  $\text{upd}$  is the Var-lifting of  $\text{upd}_2$ .
- $\alpha \in \text{Act}_1 \cap \text{Act}_2$  and there are  $l_1 \dashv\vdash [c_1, \alpha, V_1, \text{upd}_1] \mapsto_1 \lambda_1$  and  $l_2 \dashv\vdash [c_2, \alpha, V_2, \text{upd}_2] \mapsto_2 \lambda_2$  where  $c = c_1 \wedge c_2$ ,  $V = V_1 \cup V_2$ , and  $\text{upd}$  is the Var-lifting of  $(\text{upd}_1, \text{upd}_2)$ .

When composing SHMAs, local variables of participating SHMAs become disabled for corresponding internal jumps. Within our semantics, flow adapters thus ensure that the adaption of flows in internal jumps in  $\mathcal{H}_1 \parallel \mathcal{H}_2$  are independent of the local variables of the respective communication partners. Moreover, flows for local variables of  $\mathcal{H}_2$  cannot be adapted within an internal jump of  $\mathcal{H}_1$  and vice versa. It is easy to see that the composition operator for SHMAs is commutative and associative.

**Congruence.** We aim for a congruence theorem for SHMAs relying on Theorem 5. For this, we relate the composition of SHMAs with our general approach towards a composition of STSs, i.e., we represent the STS  $\llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket$  as a composition involving the components  $\llbracket \mathcal{H}_1 \rrbracket$  and  $\llbracket \mathcal{H}_2 \rrbracket$ . Notice that sampling a successor location in  $\mathcal{H}_1 \parallel \mathcal{H}_2$  happens according to a coupling measure. This observation also applies when combining measures for locations and variable evaluations within our semantics of SHMAs. To this end, it is easy to define the corresponding span  $\mathcal{S}$  and agreement  $\mathcal{G}$  such that

$$\llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket = \llbracket \mathcal{H}_1 \rrbracket \parallel_{\mathcal{S}, \mathcal{G}, \text{Act}_1 \cap \text{Act}_2} \llbracket \mathcal{H}_2 \rrbracket.$$

More precisely,  $\mathcal{S}$  is a span arising from a Cartesian span for the locations and a span for the sets of flows. For the agreement  $\mathcal{G}$ , we regard local constraints where the shared variables can be modified by both involved systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The obtained representation of  $\llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket$  underpins again the flexibility of our composition operator for STS.

We rephrase Theorem 5 in the context SHMAs. Two SHMAs are bisimilar if their semantics in terms of STSs are bisimilar. Let  $\mathcal{H}_{a1}$  and  $\mathcal{H}_{b1}$  be SHMAs with the same sets of variables  $\text{Var}_1$  and actions  $\text{Act}_1$  and similar, let  $\mathcal{H}_{a2}$  and  $\mathcal{H}_{b2}$  be SHMAs with variables  $\text{Var}_2$  and actions  $\text{Act}_2$ . Abbreviate  $\text{LVar}_1 = \text{Var}_1 \setminus \text{Var}_2$ ,  $\text{LVar}_2 = \text{Var}_2 \setminus \text{Var}_1$ , and  $\text{SVar} = \text{Var}_1 \cap \text{Var}_2$ .

► **Theorem 9.** *Let  $R_1$  and  $R_2$  be bisimulations for  $(\mathcal{H}_{a1}, \mathcal{H}_{b1})$  and  $(\mathcal{H}_{a2}, \mathcal{H}_{b2})$ , respectively.  $\mathcal{H}_{a1} \parallel \mathcal{H}_{a2}$  and  $\mathcal{H}_{b1} \parallel \mathcal{H}_{b2}$  are bisimilar if  $R_1$  and  $R_2$  do not involve shared variables, i.e.,*

$$\begin{aligned} R_1 &= \{ \langle \langle l_{a1}, \vartheta_{a1}|_{\text{LVar}_1} \uplus \vartheta^S \rangle, \langle l_{b1}, \vartheta_{b1}|_{\text{LVar}_1} \uplus \vartheta^S \rangle \rangle ; \\ &\quad \langle l_{a1}, \vartheta_{a1} \rangle R_1 \langle l_{b1}, \vartheta_{b1} \rangle \text{ and } \vartheta^S \in \text{Flow}(\text{SVar}) \}, \\ R_2 &= \{ \langle \langle l_{a2}, \vartheta_{a2}|_{\text{LVar}_2} \uplus \vartheta^S \rangle, \langle l_{b2}, \vartheta_{b2}|_{\text{LVar}_2} \uplus \vartheta^S \rangle \rangle ; \\ &\quad \langle l_{a2}, \vartheta_{a2} \rangle R_2 \langle l_{b2}, \vartheta_{b2} \rangle \text{ and } \vartheta^S \in \text{Flow}(\text{SVar}) \}. \end{aligned}$$

Our requirement that  $R_1$  and  $R_2$  do not distinguish between shared variables yields the compatibility assumption required for Theorem 5. Our proof then simply exploits the representation of  $\llbracket \mathcal{H}_{a1} \parallel \mathcal{H}_{a2} \rrbracket$  and  $\llbracket \mathcal{H}_{b1} \parallel \mathcal{H}_{b2} \rrbracket$  in terms of a composition of STSs.

## 5 Concluding remarks

In this paper, we introduced a generic parallel-composition operator for STSs and SHMAs. The essential new feature that distinguishes the novel composition from previous ones is that it uses the mathematical concepts of spans and couplings to model the effect of composing (potentially dependent) stochastic behaviors. The latter is crucial for systems where the components communicate via shared variables. A further feature of the novel stochastic-hybrid-system model (SHMA) is that the adaption of flows depends on commands rather happening on arbitrary occasions. We proved important algebraic properties in the context of composition, e.g., congruence with respect to bisimulation. This shows that even within our generic operator one does not have to forgo desired properties of compositional systems. There is plenty room for further elaborations. Firstly, we are going to develop a mathematical theory for SHMA that also involves stochastic flows. Furthermore, we will work on a modeling language for couplings and spans in order to obtain a theoretical basis for practical tools. Also other kinds of models, where spans yield a powerful approach for compositional modeling, could be investigated. Moreover, our approach concerning couplings as a modeling formalism enables many new verification questions, e.g., for directly reasoning about the coordination between components.

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