The Schützenberger Product for Syntactic Spaces*†

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Abstract

Starting from Boolean algebras of languages closed under quotients and using duality theoretic insights, we derive the notion of *Boolean spaces with internal monoids* as recognisers for arbitrary formal languages of finite words over finite alphabets. This leads to recognisers and syntactic spaces in a setting that is well-suited for applying tools from Stone duality as applied in semantics.

The main focus of the paper is the development of topo-algebraic constructions pertinent to the treatment of languages given by logic formulas. In particular, using the standard semantic view of quantification as projection, we derive a notion of *Schützenberger product* for Boolean spaces with internal monoids. This makes heavy use of the Vietoris construction – and its dual functor – which is central to the coalgebraic treatment of classical modal logic.

We show that the unary Schützenberger product for spaces yields a recogniser for the language of all models of the formula $\exists x.\Phi(x)$, when applied to a recogniser for the language of all models of $\Phi(x)$. Further, we generalise global and local versions of the theorems of Schützenberger and Reutenauer characterising the languages recognised by the binary Schützenberger product.

Finally, we provide an equational characterisation of Boolean algebras obtained by local Schützenberger product with the one element space based on an Egli-Milner type condition on generalised factorisations of ultrafilters on words.

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1 Introduction

This contribution lies at the interface of two distinct areas: one in semantics concerned with modelling binding of variables, and the other in the theory of formal languages and the search for separation results for complexity classes based on a generalisation of the algebraic theory of regular languages [22, 12]. In semantics of propositional and modal logics, Stone duality and coalgebraic logic have had great success, but in the presence of quantifiers more

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general categorical semantics is required. Quantifiers change the set of free variables in a formula, leading to a notion of indexing formulas by their contexts of free variables. In the theory of regular languages, classes of models indexed by finite alphabets have long been studied in the form of varieties of languages [5]. There, one considers Boolean algebras of languages closed under quotients over a category of finite alphabets with monoid morphisms between the corresponding finitely generated monoids. This paper is intended as a first step towards establishing a connection between categorical semantics of logics and fibrational approaches in language theory.

We follow the line set by [7, 8] and [9], which exploits the connection between the algebraic theory of formal languages and Stone duality, see also [2, 1]. In this paper we are interested in the effect that first-order quantifiers have at the level of the algebraic recognisers. This is well understood in the regular case, where a plethora of powerful tools, in the form of Schützenberger, Maltsev, and block products of finite (and profinite) monoids, is used. Beyond the regular setting, we take as a departure point classes of languages equipped with actions of the free monoid over a finite set and the standard view of existential quantification as projection, and we derive – via Stone duality – our notion of recognisers and of unary Schützenberger product. Our analysis arrives at an extension of the Schützenberger product, which was originally introduced in [19] as a means of studying the concatenation product of regular languages and was further extended in [21] and [16] to arbitrary arity and to ordered monoids, respectively. Reutenauer [18], and Pin [15] in the ordered setting, have provided exact characterisations of the regular languages accepted by the Schützenberger product.

In the setting of regular languages, equations have played an essential rôle in providing decidability results for varieties of languages and various generalisations thereof. For classes of arbitrary languages decidability is not to be expected and separation of classes is the main focus. For this reason soundness becomes more important than completeness per se. However, complete axiomatisations are useful for obtaining decidability results for the class of regular languages within a fragment. See [9] for an example and for further motivation relative to the study of circuit complexity classes.

Contributions and Structure. After some preliminaries on Stone duality and actions by monoids, Section 3 introduces our notion of recognisers and main objects of study, the *Boolean spaces with internal monoids*. In Section 4 we analyse the relation between recognisers for a language L_Φ, corresponding to a formula Φ with one free first-order variable x, and recognisers for the existentially quantified language $L_{\exists x.\Phi}$. To this end, in Section 4.1 we introduce a unary version of the Schützenberger product, $\Diamond M$, for a discrete monoid M and prove that if M recognises L_Φ, then $\Diamond M$ recognises $L_{\exists x.\Phi}$. In Section 4.2 we extend the unary Schützenberger product, and the results in Section 4.1, to Boolean spaces with internal monoids (noting this can be done for semigroups as well). We end the section with a characterisation of the languages recognised by the unary Schützenberger product $(\Diamond X, \Diamond S)$ of a Boolean space with an internal semigroup (X, S) (see Theorem 14). In Section 5 we introduce the binary Schützenberger product of Boolean spaces with internal monoids. Theorems 16 and 18 extend results of Reutenauer in the regular setting and establish the connection with concatenation product for arbitrary languages. Finally, in Section 6 we provide a completeness result for the Boolean algebra recognised by the local version of the Schützenberger product of a space with the one element space.

2 Preliminaries

2.1 Stone duality for Boolean algebras

Let $(\mathcal{B}, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Recall that a subset $\mu \subseteq \mathcal{B}$ is a *filter* of $\mathcal{B}$ if it satisfies the following conditions:

- non-emptiness: $1 \in \mu$,
- upward closure: if $L \in \mu$ and $N \in \mathcal{B}$ satisfies $L \leq N$, then $N \in \mu$,
- closure under finite meets: if $L, N \in \mu$, then $L \wedge N \in \mu$.

A filter $\mu \subseteq \mathcal{B}$ is *proper* if $\mu \neq \mathcal{B}$. *Ultrafilters* are those for which $L \in \mu$ or $\neg L \in \mu$ for each $L \in \mathcal{B}$. In the Boolean algebra $\mathcal{P}(S)$, an example of an ultrafilter is given, for each $s \in S$, by the *principal ultrafilter* associated with the element s, namely

$$\uparrow s := \{b \in \mathcal{P}(S) \mid s \in b\}. \tag{1}$$

Let $X_{\mathcal{B}}$ be the collection of all the ultrafilters of $\mathcal{B}$. The fundamental insight of Stone is that, equipped with an appropriate topology, one may recover $\mathcal{B}$ from $X_{\mathcal{B}}$. For $L \in \mathcal{B}$ set

$$\widehat{L} := \{\mu \in X_{\mathcal{B}} \mid L \in \mu\}. \tag{2}$$

Then the family $\{\widehat{L} \mid L \in \mathcal{B}\}$ forms a basis of open sets for a topology σ on $X_{\mathcal{B}}$, and the topological space $(X_{\mathcal{B}}, \sigma)$ is called the *dual space* of the Boolean algebra $\mathcal{B}$. The topology σ is compact, Hausdorff, and admits a basis of *clopen* sets (i.e. sets that are both open and closed) since the complement of $\widehat{L}$ is $\widehat{\neg L}$. Compact Hausdorff spaces that admit a basis of clopen sets are known as *Boolean* (or *Stone*) *spaces*. The collection of clopens of a Boolean space X (equipped with set-theoretic operations) constitutes a Boolean algebra, known as the *dual algebra* of X. These processes are, up to natural equivalence, inverse to each other. Given a morphism of Boolean algebras $h \colon \mathcal{A} \to \mathcal{B}$, the inverse image map on their power sets $h^{-1} \colon \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{A})$ sends ultrafilters to ultrafilters and provides the continuous map from the dual space of $\mathcal{B}$ to the dual space of $\mathcal{A}$. Similarly, the inverse image map of a continuous map $f \colon X \to Y$ provides the morphism from the dual algebra of Y to that of X. In this correspondence, quotient algebras correspond to embeddings as (closed) subspaces, and inclusions as subalgebras correspond to quotient spaces. In category-theoretic terms, this establishes a contravariant equivalence between the category of Boolean spaces and continuous maps, and the category of Boolean algebras and their morphisms. This is the content of the celebrated Stone duality for Boolean algebras [20, Theorems 67 and 68].

We end this section with an example of a Boolean algebra and its dual space which will play a key rôle in the sequel. Let S be a set. Then $\mathcal{P}(S)$ is a Boolean algebra and its dual space, denoted by $\beta(S)$, is known as the *Stone-Čech compactification* of the set S. We remark that the map $\iota \colon S \to \beta(S)$, mapping an element s to the principal ultrafilter $\uparrow s$ of (1), is injective and embeds S, with the discrete topology, as a dense subspace of $\beta(S)$. Henceforth, we will consider S as a subspace of $\beta(S)$, identifying $s \in S$ with $\uparrow s$, thus suppressing the embedding ι. The space $\beta(S)$ is characterised by the following *universal property*: if X is a compact Hausdorff space and $f \colon S \to X$ is any function, then there is a (unique) continuous function $g \colon \beta(S) \to X$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "f"'] & \beta(S) \arrow[d, "g"] \\ & X \end{tikzcd} \tag{3}$$

Consequently, if T is a discrete space, any function $f\colon S \to T$ can be extended to a continuous map $\beta(f)\colon \beta(S) \to \beta(T)$. Explicitly, the latter is given, for each $\mu \in \beta(S)$ and $L \in \mathcal{P}(T)$, by

$$L \in \beta(f)(\mu) \quad \text{if, and only if,} \quad f^{-1}(L) \in \mu.$$

2.2 Monoid actions

Let $(M, \cdot, 1)$ be a monoid, and X be a set. A function $\lambda\colon M \times X \to X$ is called a *left action* of M on X provided

- for all $x \in X$, $\lambda(1, x) = x$,
- for all $m, m' \in M$ and $x \in X$, $\lambda(m \cdot m', x) = \lambda(m, \lambda(m', x))$.

Similarly, one can define a *right action* $\rho\colon X \times M \to X$ of M on X. For each $m \in M$, we refer to the function $\lambda_m\colon X \to X$ given by $\lambda_m(x) := \lambda(m, x)$ (respectively to the function $\rho_m\colon X \to X$ given by $\rho_m(x) := \rho(x, m)$) as the *component* of the action λ at m (respectively, of the action ρ at m). A pair consisting of left and right actions λ, ρ of M on X is said to be *compatible* if, for all $m, m' \in M$, $\lambda_m \circ \rho_{m'} = \rho_{m'} \circ \lambda_m$. We call such a pair of compatible actions a *biaction* of M on X (or an M-*biaction* on X).

▶ **Example 1.** Any monoid M can be seen as acting on itself on the left and on the right. The component of the left action at $m \in M$ is the multiplication on the left by m, and the component of the right action is the multiplication on the right by m. The compatibility of the two actions amounts precisely to the associativity of the monoid operation.

▶ **Example 2.** Consider $\mathbb{N}$, the free monoid on one generator. As observed in Example 1, for each $n \in \mathbb{N}$ we have components $\lambda_n, \rho_n\colon \mathbb{N} \to \mathbb{N}$ of compatible left and right actions of $\mathbb{N}$ on itself. By the universal property (3) of the Stone-Čech compactification, we obtain continuous components $\beta(\lambda_n), \beta(\rho_n)\colon \beta(\mathbb{N}) \to \beta(\mathbb{N})$ of a biaction of $\mathbb{N}$ on $\beta(\mathbb{N})$. However the set $\beta(\mathbb{N})$ is not equipped with a continuous monoid operation, see [11, Chapter 4].

3 Recognition by spaces with dense monoids

We start by showing how our main objects of study (see Definition 3 below) arise naturally by considering duals of Boolean algebras of languages closed under certain operations known as quotients by words. Let Σ be a finite alphabet. Instantiating the monoid in Example 1 with the free monoid Σ^* on Σ, we obtain a biaction of Σ^* on itself. The components of the left and right actions are given by concatenation, and they will be denoted by

$$\lambda_w\colon \Sigma^* \to \Sigma^*,\ u \mapsto wu \quad \text{and} \quad \rho_w\colon \Sigma^* \to \Sigma^*,\ u \mapsto uw.$$

By discrete duality, i.e. by applying the contravariant power set functor, we obtain right and left Σ^*-actions on $\mathcal{P}(\Sigma^*)$ given by $\lambda_w^{-1}\colon \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$, and respectively, by $\rho_w^{-1}\colon \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$. These are the well-known *left quotients* and *right quotients* of language theory given, respectively, by

$$L \mapsto \{u \mid wu \in L\} =: w^{-1}L \quad \text{and} \quad L \mapsto \{u \mid uw \in L\} =: Lw^{-1}.$$

It is immediate that the λ_w^{-1} and ρ_w^{-1} are homomorphisms and compatible Σ^*-actions.

Dualising again, we see that the space $\beta(\Sigma^*)$ is equipped with (compatible and continuous) left and right Σ^*-actions given, for all $w \in \Sigma^*$, by $\beta(\lambda_w)$ and $\beta(\rho_w)$, respectively. By abuse of notation and for ease of readability, we will denote these actions again by λ_w, respectively ρ_w. We notice that the pair $(\beta(\Sigma^*), \Sigma^*)$ exhibits the following structure:

- a Boolean space $\beta(\Sigma^*)$,
- a dense subspace Σ^* equipped with a monoid structure,
- a biaction of Σ^* on $\beta(\Sigma^*)$ with continuous components extending that of Σ^* on itself.

Now, consider a Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(\Sigma^*)$ closed under left and right quotients by words. Then the maps λ_w^{-1} and ρ_w^{-1} restrict to Boolean algebra morphisms on $\mathcal{B}$, yielding the following commutative diagrams.

$$\begin{tikzcd} \mathcal{P}(\Sigma^*) \arrow[r, "\lambda_w^{-1}"] & \mathcal{P}(\Sigma^*) \\ \mathcal{B} \arrow[u, hook] \arrow[r, dashed, "\lambda_w^{-1}"] & \mathcal{B} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(\Sigma^*) \arrow[r, "\rho_w^{-1}"] & \mathcal{P}(\Sigma^*) \\ \mathcal{B} \arrow[u, hook] \arrow[r, dashed, "\rho_w^{-1}"] & \mathcal{B} \arrow[u, hook] \end{tikzcd} \tag{4}$$

Let $X_{\mathcal{B}}$ denote the dual space of the Boolean algebra $\mathcal{B}$. The embedding $\mathcal{B} \hookrightarrow \mathcal{P}(\Sigma^*)$ dually corresponds to a quotient $\tau\colon \beta(\Sigma^*) \twoheadrightarrow X_{\mathcal{B}}$. The space $X_{\mathcal{B}}$ also admits left and right Σ^*-actions induced by the duals of the maps λ_w^{-1}, respectively ρ_w^{-1}, from (4). We thus obtain

$$\begin{tikzcd} \beta(\Sigma^*) \arrow[r, "\lambda_w"] \arrow[d, two heads, "\tau"'] & \beta(\Sigma^*) \arrow[d, two heads, "\tau"] \\ X_{\mathcal{B}} \arrow[r, dashed, "\lambda_w"] & X_{\mathcal{B}} \end{tikzcd} \qquad \begin{tikzcd} \beta(\Sigma^*) \arrow[r, "\rho_w"] \arrow[d, two heads, "\tau"'] & \beta(\Sigma^*) \arrow[d, two heads, "\tau"] \\ X_{\mathcal{B}} \arrow[r, dashed, "\rho_w"] & X_{\mathcal{B}} \end{tikzcd} \tag{5}$$

Then $M := \tau[\Sigma^*]$ is a dense subspace of $X_{\mathcal{B}}$, and we have the following commutative diagram.

$$\begin{tikzcd} \beta(\Sigma^*) \arrow[r, two heads, "\tau"] & X_{\mathcal{B}} \\ \Sigma^* \arrow[u, hook] \arrow[r, two heads, "\tau"] & M \arrow[u, hook] \end{tikzcd} \tag{6}$$

We observe that the pair $(X_{\mathcal{B}}, M)$ exhibits the same kind of structure as $(\beta(\Sigma^*), \Sigma^*)$:

- a Boolean space $X_{\mathcal{B}}$,
- a dense subspace M equipped with a monoid structure,
- a biaction of M on $X_{\mathcal{B}}$ with continuous components extending the biaction of M on itself.

Indeed, recall that $X_{\mathcal{B}}$ is equipped with left and right Σ^*-actions which are preserved by the map τ by commutativity of (5). The Σ^*-actions on $X_{\mathcal{B}}$ restrict to Σ^*-actions on M, which are preserved by the restriction of τ. The monoid structure on M is then defined as follows. For any $m \in M$ pick $w_m \in \Sigma^*$ satisfying $\tau(w_m) = m$. Such an element exists because M is the image of Σ^* by τ. For $m, m' \in M$, set $m \cdot m' := \lambda_{w_m}(m')$. It is easily seen that the latter operation is well-defined and provides a monoid structure on M which makes the restriction of τ a monoid morphism.

As first introduced in [8], we will be using dual spaces equipped with actions as recognisers. The examples above motivate the following definition.

▶ **Definition 3.** A *Boolean space with an internal monoid* is a pair (X, M) consisting of

- a Boolean space X,
- a dense subspace M equipped with a monoid structure,
- a biaction of M on X with continuous components extending the biaction of M on itself.

▶ Remark. The recognisers introduced in [8] are monoids equipped with a uniform space structure, namely the Pervin uniformity given by a Boolean algebra of subsets of the monoid, so that the biaction of the monoid on itself has uniformly continuous components. Such an object was called a *semiuniform monoid*. One may show that the completion of a semiuniform monoid is a Boolean space with an internal monoid. Conversely, given a Boolean space with an internal monoid (X, M), the Pervin uniformity on M induced by the dual of X is a semiuniform monoid, and these two constructions are inverse to each other.

We are interested in maps between pairs (X, M) and (Y, N), i.e. continuous maps $X \to Y$ which preserve the additional structure.

▶ **Definition 4.** A *morphism* between two Boolean spaces with internal monoids (X, M) and (Y, N) is a continuous map $f\colon X \to Y$ such that f restricts to a monoid morphism $M \to N$.

Morphisms, as just defined, are in fact also biaction-preserving maps.

▶ **Lemma 5.** *Let $f\colon (X, M) \to (Y, N)$ be a morphism of Boolean spaces with internal monoids. Then f preserves the actions, i.e. for every $m \in M$*

$$f \circ \lambda_m = \lambda_{f(m)} \circ f \quad \textit{and} \quad f \circ \rho_m = \rho_{f(m)} \circ f.$$

▶ **Example 6.** The map $\tau\colon (\beta(\Sigma^*), \Sigma^*) \to (X_{\mathcal{B}}, M)$ of (6) is a morphism of Boolean spaces with internal monoids.

▶ Remark. The map $L \mapsto \widehat{L}$ of (2) establishes a one-to-one correspondence between the elements of $\mathcal{P}(\Sigma^*)$ and the clopens of $\beta(\Sigma^*)$. Thus, we will sometimes blur the distinction between recognition of a language L and recognition of the corresponding clopen $\widehat{L}$.

▶ **Definition 7.** Let Σ be a finite alphabet, and let $L \subseteq \Sigma^*$ be a language. We say that L (or $\widehat{L}$) is *recognised by the morphism* $f\colon (\beta(\Sigma^*), \Sigma^*) \to (X, M)$ if there is a clopen $C \subseteq X$ such that $\widehat{L} = f^{-1}(C)$. Moreover, the language L is *recognised by the space* (X, M) if there is a morphism $(\beta(\Sigma^*), \Sigma^*) \to (X, M)$ recognising L. Similarly, we say that a morphism (or a space) recognises a Boolean algebra if it recognises all its elements.

▶ Remark. In general, a morphism $(\beta(\Sigma^*), \Sigma^*) \to (X, M)$ with *infinite* M, recognises (in the sense of Definition 7) far less languages than the induced monoid morphism $\Sigma^* \to M$. On the other hand, a finite monoid M may be seen as a space with an internal monoid, in which the space component is the monoid itself, equipped with the discrete topology. A morphism $(\beta(\Sigma^*), \Sigma^*) \to (M, M)$ yields in particular a monoid morphism $\Sigma^* \to M$. Conversely, a monoid morphism $h\colon \Sigma^* \to M$ extends uniquely to a continuous map $\beta h\colon \beta(\Sigma^*) \to M$ whose restriction to Σ^* is a monoid morphism. Thus the notion of recognition introduced here extends the usual notion for regular languages, but is finer-grained in the non-regular setting.

4 A unary variant of the Schützenberger product

4.1 Logical motivation: existentially quantified languages

Consider the free monoid Σ^* over a finite alphabet Σ. A word $w \in \Sigma^*$ may be seen as a structure based on the set $\{0, \ldots, |w| - 1\}$,[1] equipped minimally with a unary predicate for each letter $a \in \Sigma$, which holds at i if and only if $w_i = a$. Now given a formula Φ (in a language interpretable over words as structures), assumed for simplicity to have only one free first-order variable x, we will see the set L_Φ of all words satisfying Φ as a language in the

[1] Here, as usual, $|w| \in \mathbb{N}$ denotes the length of the word $w = w_0 \cdots w_{|w|-1} \in \Sigma^*$.

extended alphabet $\Sigma \times 2$. In the terminology of [22], L_Φ consists of $\{x\}$-structures, which correspond to words in the subset $(\Sigma \times \{0\})^*(\Sigma \times \{1\})(\Sigma \times \{0\})^*$ of the free monoid $(\Sigma \times 2)^*$. An $\{x\}$-structure satisfies Φ provided the underlying word in the alphabet Σ satisfies Φ under the interpretation in which x points to the unique position marked with a 1. Notice that $(\Sigma \times \{0\})^*(\Sigma \times \{1\})(\Sigma \times \{0\})^*$ is isomorphic to the set $\Sigma^* \otimes \mathbb{N}$ of words in Σ^* with a marked spot defined by

$$\Sigma^* \otimes \mathbb{N} := \{(w, i) \in \Sigma^* \times \mathbb{N} \mid i < |w|\}.$$

Throughout this section we will make use of the following three maps

$$\gamma_0 \colon \Sigma^* \to (\Sigma \times 2)^*, \qquad \gamma_1 \colon \Sigma^* \otimes \mathbb{N} \to (\Sigma \times 2)^*, \qquad \pi \colon \Sigma^* \otimes \mathbb{N} \to \Sigma^*.$$

- The map $\gamma_0 \colon \Sigma^* \to (\Sigma \times 2)^*$ is the embedding given by $w \mapsto w^0$, where w^0 has the same length as w and

$$(w^0)_j := (w_j, 0) \quad \text{for each} \quad j < |w|.$$

- The map $\gamma_1 \colon \Sigma^* \otimes \mathbb{N} \to (\Sigma \times 2)^*$ is the embedding given by $(w, i) \mapsto w^{(i)}$, where $w^{(i)}$ has the same length as w and

$$(w^{(i)})_j := \begin{cases} (w_j, 0) & \text{if} \quad i \neq j < |w| \\ (w_i, 1) & \text{if} \quad i = j. \end{cases}$$

- The map $\pi \colon \Sigma^* \otimes \mathbb{N} \to \Sigma^*$ is the projection on the first coordinate.

▶ Remark. The language $L_{\exists x.\Phi}$ is obtained as $\pi[\gamma_1^{-1}(L_\Phi)]$. More generally, given a language $L \subseteq (\Sigma \times 2)^*$, we shall denote $\pi[\gamma_1^{-1}(L)] \subseteq \Sigma^*$ by $L_\exists$.

▶ Remark. Notice that, unlike γ_0, the maps γ_1 and π are not monoid morphisms. Indeed, $\Sigma^* \otimes \mathbb{N}$ does not have a suitable monoid structure. However, $\Sigma^* \otimes \mathbb{N}$ does have a Σ^*-biaction structure. For $v \in \Sigma^*$, the components of the left and right actions are given by

$$\lambda_v(w, i) := (vw, i + |v|),$$
$$\rho_v(w, i) := (wv, i).$$

It is clear that both γ_1 and π preserve the Σ^*-actions.

Assume that the language L_Φ is recognised by a monoid morphism $\tau \colon (\Sigma \times 2)^* \to M$. We have the following pair of functions[2] with domain $\Sigma^* \otimes \mathbb{N}$

$$\begin{tikzcd} & \Sigma^* \otimes \mathbb{N} \arrow[dl, "\pi"'] \arrow[dr, "\gamma_1"] & & \\ \Sigma^* & & (\Sigma \times 2)^* \arrow[dr, "\tau"] & \\ & & & M \end{tikzcd}$$

which gives rise to a relation $R \colon \Sigma^* \nrightarrow M$ given by

$$(w, m) \in R \quad \text{if, and only if,} \quad \exists (w, i) \in \pi^{-1}(w).\ (\tau \circ \gamma_1)(w, i) = m.$$

[2] Notice that this is not a relational morphism in the sense of Tilson's definition given in [5], since the domain $\Sigma^* \otimes \mathbb{N}$ does not have a compatible monoid structure.

Though π is not injective, it does have *finite preimages.* As will be crucial in what follows, this allows us to represent R as a function (which, in general, is not a monoid morphism)

$$\xi_1 \colon \Sigma^* \to \mathcal{P}_{\mathit{fin}}(M), \quad w \mapsto \{\tau(w^{(i)}) \mid 0 \leq i < |w|\} \tag{7}$$

where $\mathcal{P}_{\mathit{fin}}(M)$ denotes the set of finite subsets of M. Consider the monoid structure on $\mathcal{P}_{\mathit{fin}}(M)$ with union as the multiplication, and the empty set as unit. Notice that the monoid M acts on $\mathcal{P}_{\mathit{fin}}(M)$ both to the left and to the right, and the two actions are compatible. The left action $M \times \mathcal{P}_{\mathit{fin}}(M) \to \mathcal{P}_{\mathit{fin}}(M)$ is given, for $m \in M$ and $S \in \mathcal{P}_{\mathit{fin}}(M)$, by $m \cdot S := \{m \cdot s \mid s \in S\}$. Similarly, the right action is given by $S \cdot m := \{s \cdot m \mid s \in S\}$.

▶ **Definition 8.** We define the *unary Schützenberger product* $\Diamond M$ of M as the bilateral semidirect product $\mathcal{P}_{\mathit{fin}}(M) * M$ of the monoids $(\mathcal{P}_{\mathit{fin}}(M), \cup)$ and $(M, \cdot)$. Explicitly, the underlying set of this monoid is the Cartesian product $\mathcal{P}_{\mathit{fin}}(M) \times M$, and the multiplication $*$ on $\mathcal{P}_{\mathit{fin}}(M) * M$ is given by

$$(S, m) * (T, n) := (S \cdot n \cup m \cdot T, m \cdot n).$$

Note that the projection onto the second coordinate, $\pi_2 \colon \Diamond M \to M$, is a monoid morphism.

▶ **Proposition 9.** *If $\tau \colon (\Sigma \times 2)^* \to M$ is a monoid morphism recognising L_Φ, then there exists a monoid morphism*

$$\xi \colon \Sigma^* \to \Diamond M$$

that recognises the language $L_{\exists x.\Phi}$ and makes the following diagram commute.

$$\begin{tikzcd} \Sigma^* \arrow[r, "\xi"] \arrow[d, "\gamma_0"'] & \Diamond M \arrow[d, "\pi_2"] \\ (\Sigma \times 2)^* \arrow[r, "\tau"] & M \end{tikzcd}$$

Proof idea. The map ξ is obtained by pairing $\xi_1 \colon \Sigma^* \to \mathcal{P}_{\mathit{fin}}(M)$ of (7) and $\tau \circ \gamma_0 \colon \Sigma^* \to M$. Explicitly,

$$w \mapsto (\{\tau(w^{(i)}) \mid 0 \leq i < |w|\}, \tau(w^0)).$$

One may show that the map ξ is a monoid morphism with respect to the concatenation on Σ^* and the multiplication $*$ on the semidirect product $\mathcal{P}_{\mathit{fin}}(M) * M$. Now let V be a subset of M such that $L_\Phi = \tau^{-1}(V)$, and consider the set $\diamond V \subseteq \mathcal{P}_{\mathit{fin}}(M)$ defined as $\{S \in \mathcal{P}_{\mathit{fin}}(M) \mid S \cap V \neq \emptyset\}$. Then $\xi^{-1}(\diamond V \times M)$ is precisely $L_{\exists x.\Phi}$. ◀

▶ Remark. In [21] Straubing generalised the Schützenberger product for any finite number of monoids. Using his construction, the unary Schützenberger product of M is simply M, and hence is different from $\Diamond M$ introduced above. For the connection between closure under concatenation product and first-order quantification in the regular setting, see [14].

▶ Remark. For lack of space, we have chosen to just 'pull Definition 8 (and consequently also the upcoming Definition 11) out of a hat'. However, by a careful analysis of how quotients in $\mathcal{P}(\Sigma^*)$ of languages $L_\exists$ are calculated, relative to corresponding calculations in $\mathcal{P}((\Sigma \times 2)^*)$, one may simply derive by duality that the operation given here is the right one.

4.2 The Schützenberger product for one space $\Diamond X$

In this section we assume that the language $L_\Phi \subseteq (\Sigma \times 2)^*$ is recognised by a morphism of Boolean spaces with internal monoids $\tau\colon (\beta(\Sigma \times 2)^*, (\Sigma \times 2)^*) \to (X, M)$. Notice that in this case we have a pair of continuous maps

$$\begin{tikzcd} & \beta(\Sigma^* \otimes \mathbb{N}) \arrow[dl, "\beta\pi"'] \arrow[dr, "\beta\gamma_1"] & & \\ \beta(\Sigma^*) & & \beta(\Sigma \times 2)^* \arrow[dr, "\tau"] & \\ & & & X \end{tikzcd} \tag{8}$$

which, as before, yields a relation $\beta(\Sigma^*) \nrightarrow X$. We would like to describe this relation as a continuous map on $\beta(\Sigma^*)$. To this end, we need an analogue for spaces of the finite power set construction. This is provided by the *Vietoris space construction* (see [10, Section B.1] for further details).

▶ **Definition 10.** Let X be a Boolean space. The *Vietoris space* $\mathcal{V}(X)$ is the Boolean space with underlying set $\{K \subseteq X \mid K \text{ is closed in } X\}$, and topology generated by the subbasis consisting of the sets, for V clopen in X, of the form

$$\Box V := \{K \in \mathcal{V}(X) \mid K \subseteq V\} \quad \text{and} \quad \Diamond V := \{K \in \mathcal{V}(X) \mid K \cap V \neq \emptyset\}.$$

Just as in the monoid case, diagram (8) yields a map

$$\xi_1\colon \beta(\Sigma^*) \to \mathcal{V}(X)$$

defined as the composition $\tau \circ \beta\gamma_1 \circ (\beta\pi)^{-1}$, or equivalently as the unique continuous extension of the map $\xi_1\colon \Sigma^* \to \mathcal{P}_{fin}(M)$ defined in (7).

▶ **Definition 11.** We define the *unary Schützenberger product* of a Boolean space with an internal monoid (X, M) as the pair $(\Diamond X, \Diamond M)$, where $\Diamond X$ is the space $\mathcal{V}(X) \times X$ equipped with the product topology and $\Diamond M$ is as in Definition 8.

▶ **Lemma 12.** *The unary Schützenberger product $(\Diamond X, \Diamond M)$ of (X, M) is a Boolean space with an internal monoid.*

Proof Idea. Recall that M is a dense subspace of X. It follows by [13, Theorem 4, p. 163] that $\mathcal{P}_{fin}(M)$ is a dense subspace of $\mathcal{V}(X)$. Thus the monoid $\Diamond M$ is a dense subspace of $\Diamond X$. Next we define the actions of $\Diamond M$ on $\Diamond X$ as follows:

$$l_{(S,m)}(T, x) := (\{\lambda_s(x) \mid s \in S\} \cup \lambda_m[T], \lambda_m(x)),$$
$$r_{(S,m)}(T, x) := (\{\rho_s(x) \mid s \in S\} \cup \rho_m[T], \rho_m(x)).$$

It is not difficult to see that the above maps are the unique continuous extensions to $\Diamond X$ of the multiplication by (S, m), to the left and to the right, on $\Diamond M$. ◀

The projection $\pi_2\colon \Diamond X \to X$ is a morphism of Boolean spaces with internal monoids.

▶ **Proposition 13.** *If $\tau\colon (\beta(\Sigma \times 2)^*, (\Sigma \times 2)^*) \to (X, M)$ is a morphism of Boolean spaces with internal monoids recognising L_Φ, then there is a morphism $\xi\colon (\beta(\Sigma^*), \Sigma^*) \to (\Diamond X, \Diamond M)$ recognising $L_{\exists x.\Phi}$ and such that the following diagram commutes.*

$$\begin{tikzcd} \beta(\Sigma^*) \arrow[r, "\xi"] \arrow[d, "\beta\gamma_0"'] & \Diamond X \arrow[d, "\pi_2"] \\ \beta(\Sigma \times 2)^* \arrow[r, "\tau"] & X \end{tikzcd}$$

All the constructions introduced so far can be carried out for semigroups. In particular, we can consider Boolean spaces with internal semigroups as recognisers of languages in $\mathcal{P}(\Sigma^+)$. Along the lines of Definition 8, we introduce the unary Schützenberger product $\Diamond S$ of a semigroup S as the bilateral semidirect product of the semigroups $(\mathcal{P}^+_{\mathit{fin}}(S), \cup)$ and $(S, \cdot)$, where $\mathcal{P}^+_{\mathit{fin}}(S)$ denotes the family of finite non-empty subsets of S. Similarly, at the level of spaces, in the Vietoris construction we will consider only non-empty closed subsets.

Now, write $\mathcal{B}(X, \Sigma)$ for the Boolean algebra of languages in $\mathcal{P}(\Sigma^+)$ recognised by the Boolean space with an internal semigroup (X, S), and note that the latter Boolean algebra is always closed under quotients. Moreover, given a language $L \subseteq (\Sigma \times 2)^+$, recall that $L_\exists$ denotes the language $\pi[\gamma_1^{-1}(L)]$.

▶ **Theorem 14.** *Let (X, S) be a Boolean space with an internal semigroup, and let $\mathcal{B}(X, \Sigma\times 2)_\exists$ denote the Boolean subalgebra closed under quotients of $\mathcal{P}(\Sigma^+)$ generated by the family $\{L_\exists \mid L \in \mathcal{B}(X, \Sigma \times 2)\}$. Then $\mathcal{B}(\Diamond X, \Sigma)$ coincides with the Boolean algebra generated by the union of $\mathcal{B}(X, \Sigma)$ and $\mathcal{B}(X, \Sigma \times 2)_\exists$.*

The proof of this theorem hinges on the fact that the first components of the recognising morphisms evaluate to non-empty subsets. An analogous statement can be formulated for monoids, but we would have to restrict the recognising morphisms when defining $\mathcal{B}(\Diamond X, \Sigma)$.

5 A variant of the Schützenberger product for two spaces

Given two monoids $(M, \cdot), (N, \cdot)$, the Schützenberger product $\Diamond(M, N)$ can be defined as the monoid $\mathcal{P}_{\mathit{fin}}(M \times N) \times M \times N$ whose operation is given by

$$(S, m_1, n_1) \cdot (T, m_2, n_2) := (m_1 \cdot T \cup S \cdot n_2, m_1 \cdot m_2, n_1 \cdot n_2).$$

Now, consider two Boolean spaces with internal monoids (X, M) and (Y, N). We define the space $\Diamond(X, Y)$ as the product $\mathcal{V}(X \times Y) \times X \times Y$. It is clear that the monoid $\Diamond(M, N)$ is dense in $\Diamond(X, Y)$. Moreover, the left action of $\Diamond(M, N)$ on itself can be extended to $\Diamond(X, Y)$ by setting, for any $(S, m_1, n_1) \in \Diamond(M, N)$,

$$\lambda_{(S,m_1,n_1)} \colon \Diamond(X, Y) \to \Diamond(X, Y),\ (Z, x, y) \mapsto (m_1 Z \cup Sy, \lambda_{m_1}(x), \lambda_{n_1}(y)), \tag{9}$$

where

$$m_1 Z := \{(\lambda_{m_1}(x), y) \in X \times Y \mid (x, y) \in Z\} \ \text{ and } \ Sy := \{(m, \lambda_n(y)) \in X \times Y \mid (m, n) \in S\}.$$

Similarly, the right action can be defined by

$$\rho_{(S,m_1,n_1)} \colon \Diamond(X, Y) \to \Diamond(X, Y),\ (Z, x, y) \mapsto (Zn_1 \cup xS, \rho_{m_1}(x), \rho_{n_1}(y)), \tag{10}$$

where

$$Zn_1 := \{(x, \rho_{n_1}(y)) \in X \times Y \mid (x, y) \in Z\} \ \text{ and } \ xS := \{(\rho_m(x), n) \in X \times Y \mid (m, n) \in S\}.$$

It is easy to see that we obtain a biaction of $\Diamond(M, N)$ on $\Diamond(X, Y)$. Furthermore,

▶ **Lemma 15.** *The biaction of $\Diamond(M, N)$ on $\Diamond(X, Y)$ defined in (9) and (10) has continuous components. Thus $(\Diamond(X, Y), \Diamond(M, N))$ is a Boolean space with an internal monoid.*

The next three results establish the connection between concatenation of possibly non-regular languages and the Schützenberger product of Boolean spaces with internal monoids. We thus extend the theorems of Schützenberger [19] and Reutenauer [18].

▶ **Theorem 16** (Reutenauer's theorem, global version). *Consider Boolean spaces with internal monoids* (X, M) *and* (Y, N)*. Let* $\mathcal{L}$ *be the Boolean algebra generated by all the* Σ^**-languages of the form* L_1, L_2 *and* $L_1 a L_2$*, where* L_1 *(respectively* L_2*) is recognised by* X *(respectively* Y*) and* $a \in \Sigma$*. Then a* Σ^**-language is recognised by* $X \Diamond Y$ *if, and only if, it belongs to* $\mathcal{L}$*.*

Proof Idea. Suppose the languages L_1, L_2 are recognised by morphisms $\phi_1\colon (\beta(\Sigma^*), \Sigma^*) \to (X, M)$ and $\phi_2\colon (\beta(\Sigma^*), \Sigma^*) \to (Y, N)$, respectively, and fix $a \in \Sigma$. By abuse of notation, call $\phi_1 \times \phi_2\colon \beta(\Sigma^* \times \{a\} \times \Sigma^*) \to X \times Y$ the unique continuous extension of the product map $\Sigma^* \times \{a\} \times \Sigma^* \to X \times Y$ whose components are $(w, a, w') \mapsto \phi_1(w)$ and $(w, a, w') \mapsto \phi_2(w')$. Let $\zeta_a\colon \beta(\Sigma^*) \to \mathcal{V}(X \times Y)$ be the continuous function induced by the diagram

$$\beta(\Sigma^*) \xleftarrow{\beta c} \beta(\Sigma^* \times \{a\} \times \Sigma^*) \xrightarrow{\phi_1 \times \phi_2} X \times Y \tag{11}$$

just as for diagram (8), where $c\colon \Sigma^* \times \{a\} \times \Sigma^* \to \Sigma^*$ is the concatenation map $(w, a, w') \mapsto waw'$. One can prove that the map ζ_a is a morphism recognising L_1, L_2 and $L_1 a L_2$.

Conversely, for any morphism $\langle \zeta, \phi_1, \phi_2 \rangle\colon (\beta(\Sigma^*), \Sigma^*) \to (X \Diamond Y, M \Diamond N)$ and clopens $C_1 \subseteq X$, $C_2 \subseteq Y$, we must prove that $\zeta^{-1}(\diamond(C_1 \times C_2)) \cap \Sigma^* \in \mathcal{L}$. One observes that each

$$L_{C_1 \times C_2, a} := \{w \in \Sigma^* \mid \exists u, v \in \Sigma^* \text{ s.t. } w = uav \text{ and } \phi_1(u)\zeta(a)\phi_2(v) \in \diamond(C_1 \times C_2)\}$$

is in the Boolean algebra $\mathcal{L}$. Then $\zeta^{-1}(\diamond(C_1 \times C_2)) \cap \Sigma^* = \bigcup_{a \in \Sigma} L_{C_1 \times C_2, a}$. ◀

The next corollary follows at once by Theorem 16, by noting that $L_1 L_2 = \bigcup_{a \in \Sigma} L_1 a(a^{-1} L_2)$ whenever L_2 does not contain the empty word and $L_1 L_2 = \bigcup_{a \in \Sigma} L_1 a(a^{-1} L_2) \cup L_1$ otherwise.

▶ **Corollary 17.** *The Boolean space with an internal monoid* $(\Diamond(X, Y), \Diamond(M, N))$ *recognises the concatention* $L_1 L_2$ *of languages* L_1*,* L_2 *recognised by* (X, M) *and* (Y, N)*, respectively.*

Finally, the following local statement is a direct consequence of the proof of Theorem 16.

▶ **Theorem 18** (Reutenauer's theorem, local version). *Consider morphisms* $\phi_1\colon (\beta(\Sigma^*), \Sigma^*) \to (X, M)$ *and* $\phi_2\colon (\beta(\Sigma^*), \Sigma^*) \to (Y, N)$*. Let* $\mathcal{L}$ *be the Boolean algebra generated by all the* Σ^**-languages of the form* L_1, L_2 *and* $L_1 a L_2$*, where* L_1 *(respectively* L_2*) is recognised by* ϕ_1 *(respectively* ϕ_2*) and* $a \in \Sigma$*. Then a* Σ^**-language is recognised by the morphism*

$$\langle \langle \zeta_a \rangle_{a \in \Sigma}, \phi_1, \phi_2 \rangle\colon \beta(\Sigma^*) \to \mathcal{V}(X \times Y)^\Sigma \times X \times Y$$

where $\zeta_a\colon \beta(\Sigma^*) \to \mathcal{V}(X \times Y)$ *is induced by diagram* (11) *if, and only if, it belongs to* $\mathcal{L}$*.*

6 Ultrafilter equations

Identifying simple equational bases for the Boolean algebras of languages recognised by Schützenberger products, in terms of the equational theories of the input Boolean algebras, is an important step in studying classes built up by repeated application of quantification or language concatenation. See e.g. [17, 3] for examples of such work in the regular setting.

As a proof-of-concept and first step, we provide a fairly easy completeness result for the Boolean algebra recognised by the local version of a Schützenberger product of a space with the one element space. First we introduce notation for the dual construction, see Theorem 18.

▶ **Definition 19.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Boolean algebras of Σ^*-languages closed under quotients. We define the *binary Schützenberger sum* of $\mathcal{B}_1$ and $\mathcal{B}_2$ to be the Boolean algebra of languages

$$\mathcal{B}_1 \diamondplus \mathcal{B}_2 := \langle \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{L_1 a L_2 \mid L_1 \in \mathcal{B}_1,\ L_2 \in \mathcal{B}_2,\ a \in \Sigma\}\rangle.$$

Note that this Boolean algebra is also closed under quotients.

Let $\mathcal{B} \subseteq \mathcal{P}(\Sigma^*)$ be a Boolean algebra closed under quotients. We give equations for $\mathcal{B} \diamondplus 2$. Recall that an equation for a Boolean subalgebra of $\mathcal{P}(\Sigma^*)$ is a pair $\mu \approx \nu$, where $\mu, \nu \in \beta(\Sigma^*)$, and that $L \in \mathcal{P}(\Sigma^*)$ *satisfies the ultrafilter equation* $\mu \approx \nu$ provided

$$L \in \mu \quad \text{if, and only if,} \quad L \in \nu.$$

A Boolean subalgebra of $\mathcal{P}(\Sigma^*)$ satisfies an ultrafilter equation provided each of its elements satisfies it. For background and more details on equations see e.g. [7, 9, 6]. Now, set

$$f_a \colon \Sigma^* \otimes \mathbb{N} \to \Sigma^*,\ (w, i) \mapsto w(a@i) \quad \text{and} \quad f_r \colon \Sigma^* \otimes \mathbb{N} \to \Sigma^*,\ (w, i) \mapsto w_{|i} = w_0 \cdots w_{i-1}$$

where $a \in \Sigma$ and $w(a@i)$ denotes the word obtained by replacing the ith letter of the word $w = w_0 \cdots w_{|w|-1}$ by an a.

The intuition is that the extension βf_a will allow us to *factor* an ultrafilter at an occurrence of the letter a, whereas the extension βf_r gives us access to the prefix of this factorisation.

▶ **Definition 20.** Let $\mathcal{E}(\mathcal{B} \diamondplus 2)$ denote the set of all equations $\mu \approx \nu$ so that

- $\mu \approx \nu$ holds in $\mathcal{B}$;
- for each $\gamma \in \beta(\Sigma^* \otimes \mathbb{N})$ so that $\mu = \beta f_a(\gamma)$, there exists $\delta \in \beta(\Sigma^* \otimes \mathbb{N})$ such that $\nu = \beta f_a(\delta)$ and the equation $\beta f_r(\gamma) \approx \beta f_r(\delta)$ holds in $\mathcal{B}$;
- for each $\delta \in \beta(\Sigma^* \otimes \mathbb{N})$ so that $\nu = \beta f_a(\delta)$, there exists $\gamma \in \beta(\Sigma^* \otimes \mathbb{N})$ such that $\mu = \beta f_a(\gamma)$ and the equation $\beta f_r(\gamma) \approx \beta f_r(\delta)$ holds in $\mathcal{B}$.

▶ **Theorem 21.** *The ultrafilter equations in $\mathcal{E}(\mathcal{B} \diamondplus 2)$ characterise the Boolean algebra $\mathcal{B} \diamondplus 2$.*

The proof of Theorem 21 relies on the following two lemmas.

▶ **Lemma 22.** *Let $\gamma \in \beta(\Sigma^* \otimes \mathbb{N})$. If $\mu = \beta f_a(\gamma)$ and $L \in \beta f_r(\gamma)$, then $La\Sigma^* \in \mu$.*

▶ **Lemma 23.** *Let $\mathscr{F} \subseteq \mathcal{P}(\Sigma^*)$ be a proper filter, $\mu \in \beta(\Sigma^*)$ and $a \in \Sigma$. If $La\Sigma^* \in \mu$ for all $L \in \mathscr{F}$, then there exists $\gamma \in \beta(\Sigma^* \otimes \mathbb{N})$ such that $\mu = \beta f_a(\gamma)$ and $\mathscr{F} \subseteq \beta f_r(\gamma)$.*

Proof Idea for Theorem 21. Soundness follows easily from the lemmas. For completeness notice that, by repeated use of compactness, $K \in \mathcal{P}(\Sigma^*)$ belongs to $\mathcal{B} \diamondplus 2$ if and only if for each $\mu \in \widehat{K}$, the clopen $\widehat{K}$ extends the set

$$C_\mu := \bigcap \{\widehat{L} \mid L \in \mathcal{B},\ L \in \mu\} \cap \bigcap \{\widehat{La\Sigma^*} \mid a \in \Sigma, L \in \mathcal{B}, La\Sigma^* \in \mu\}$$
$$\cap \bigcap \{(\widehat{La\Sigma^*})^c \mid a \in \Sigma, L \in \mathcal{B}, La\Sigma^* \notin \mu\}.$$

Finally one shows, again using the lemmas, that $\mu \approx \nu \in \mathcal{E}(\mathcal{B} \diamondplus 2)$ for any $\nu \in C_\mu$. ◀

7 Conclusion

The concepts of recognition and of syntactic monoid, stemming from the algebraic theory of regular languages, inherently arise in the setting of Stone/Priestley duality for Boolean algebras and lattices with additional operations, see [7]. Reasoning by analogy, this led in

[8] to generalisations of recognition and syntactic objects for arbitrary languages of finite words. In *loc. cit.* this was achieved in the setting of monoids equipped with uniform space structures, the so called *semiuniform monoids.* In this paper we naturally arrive at an isomorphic notion of recogniser – Boolean spaces with internal monoids – which is however more amenable to existing tools from duality theory.

Our first contribution is setting up the right framework that allows us to extend to the non-regular setting algebraic constructions whose logical counterpart is adding a layer of quantifier depth. We should mention that both the Schützenberger and the block product are algebraic constructions that can be used for this purpose in the regular case. However, for technical reasons, extending the former to Boolean spaces with internal monoids is more natural. The unary Schützenberger product that we introduce (which actually does not appear in the (pro)finite monoid literature to the best of our knowledge) arises naturally via duality for the Boolean algebra with quotients generated by the languages $L_\exists$, for L coming from some Boolean algebra $\mathcal{B}$. Moreover, our framework can be easily extended to the case of bounded distributive lattices, one would just need to use the Vietoris functor on spectral spaces instead. A comparison between our unary Schützenberger product and the block product introduced in [12] for finitely typed monoids remains a topic for future investigation.

Furthermore, Theorem 14 of Section 4.2 and Theorem 16 of Section 5, provide characterisations of the languages accepted by our unary and binary Schützenberger products of Boolean spaces. Finally, in Section 6 we derive a preliminary result on equations. Theorem 21 on equational completeness is by no means the final word, but rather a first stepping stone in this direction. In the regular setting, as well as in the special cases treated in [9] and [4], much smaller subsets of $\mathcal{E}(\mathcal{B}\diamondsuit 2)$ have been shown to provide complete axiomatisations. We expect that a notion akin to the derived categories of profinite monoid theory [23] have to be developed, and we expect the remainder of the Stone-Čech compactification to play a key rôle in this.

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