House Markets with Matroid and Knapsack Constraints*

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Abstract

Classical online bipartite matching problem and its generalizations are central algorithmic optimization problems. The second related line of research is in the area of algorithmic mechanism design, referring to the broad class of house allocation or assignment problems. We introduce a single framework that unifies and generalizes these two streams of models. Our generalizations allow for arbitrary matroid constraints or knapsack constraints at every object in the allocation problem. We design and analyze approximation algorithms and truthful mechanisms for this framework. Our algorithms have best possible approximation guarantees for most of the special instantiations of this framework, and are strong generalizations of the previous known results.

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1 Introduction

Classic online bipartite matching is one of the central algorithmic optimization problems. Since the seminal paper of Karp, Vazirani and Vazirani [21], there have been new developments and generalizations of this model [4, 14, 20]. A related line of research is within algorithmic mechanism design for a broad class of house allocation/assignment problems [1, 7, 18, 19, 27]. House Allocation (HA) problem is a model of assigning indivisible objects to agents, where each agent with a preference order over a subset of objects requires at most one object and payments are not allowed. HA and its generalizations have wide real life applications such as Campus Housing Allocation [12], Student-project Allocation [2], Machine-job Assignment [9]. Our goal is to unify these two streams of models into a single algorithmic framework.

In classic HA, agents' preferences over objects are strict. In many situations, it is more suitable to allow the agent to express indifferences or ties among objects [25]. For example, in Campus Housing Allocation, each student may provide some features (e.g., separate bathroom) of dormitories, and he is indifferent between dormitories with same features. Besides, other optimality criteria for the allocation are also desired, like Pareto optimality. An allocation μ is Pareto optimal if there is no other allocation μ' such that no agent is worse off in μ' and at least one agent is strictly better in μ' compared to μ. Moreover, the agents

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are strategic (selfish) players who may not reveal their preference orders truthfully unless they can not get better by misreports. Thus, designing truthful mechanisms is a central issue in mechanism design for HA. A mechanism is truthful (in dominant strategies) if no agent can get better off by misreporting his preference, independent from other agents' reports.

In real world applications, not all the agents can be guaranteed to be matched since the set of their acceptable objects is a subset of all the objects and there may not be enough objects. Thus, designing algorithms to match as many agents as possible is also a much desired objective. This can be quantified by the approximation ratio between the sizes of the optimal matching and the matching obtained by the algorithm, preserving truthfulness and Pareto optimality. In addition, different agents may have different weights or priorities, and in this case, the ratio is between the weights of the optimal weighted matching and the output matching. Krysta et al. [22] initiated the study of designing truthful and Pareto optimal mechanisms for HA with good approximation ratios. They present a tight deterministic 2-approximation, truthful and Pareto optimal mechanism, and a randomized $\frac{e}{e-1}$-approximate, universally truthful and Pareto optimal mechanism. In their setting, each object can be allocated to at most one agent. Can we generalize their results to more general settings?

A simple generalization, called generalized HA, allows each object a to be allocated to at most a given number, $b_a \in \mathbb{N}$, of agents. We observe that all the results in [22] can easily be applied to this generalization. Let us create b_a copies of each object a, and replace a in each agent's preference order by these b_a copies in the same indifference class as a. By this modification, called modified HA, each copy of each object is allowed to be allocated to at most one agent. Each feasible allocation of the generalized HA can be modified to a feasible solution to the modified HA, and vice versa. The mechanism in [22], which is Pareto optimal and truthful for the modified HA, is also Pareto optimal and truthful for the generalized HA.

We study here as our framework, two further natural generalizations of the HA problem: Matroid House Allocation Problem (MHA) and Knapsack House Allocation Problem (KHA). Consider the following more realistic application of HA to the Campus Housing Allocation problem. A university has many dormitories, and each student submits a preference order over dormitories to the university. Besides this, some students may have further requirements about their dormitories which should be fulfilled, e.g., one may require that the room should have an independent bathroom or be far from the kitchen. Thus, for each dormitory a, viewed as an object, and each student i, the rooms in a satisfying the requirements of i are a subset of all the rooms in a. Hence, all the feasible sets of students allocated to a form a transversal matroid (see definition in Section 2) of dormitory a. Our first generalization of HA is based on this application, where each object in HA is associated with a (possibly different) matroid and can be allocated to any number of agents in the ground set (which is the set of agents) of the matroid. The set of agents allocated to each object forms an independent set of its matroid. We call this generalization the Matroid House Allocation Problem (MHA). We can show that MHA includes as special cases all the online bipartite matching models [4, 14, 20, 21] ([20] if fixed vertices are used at most once) mentioned earlier.

The second generalization is as follows: each agent (or job) i is associated with a capacity c_{ia} for each object (machine) a, and each object a is associated with a capacity C_a. As before, each object can be allocated to multiple agents, but the total capacity of the agents allocated to a does not exceed its capacity (i.e., $\sum_i c_{ia} \leq C_a$). We call this generalization the Knapsack House Allocation Problem (KHA). An interesting application of KHA is Job Scheduling Markets [9]. Our main results are the following:

1. We present a tight, deterministic, truthful and Pareto optimal mechanism (TMHA) for MHA with the approximation ratio of 2, where agents have weights and ties.

2. Based on TMHA, a randomized, universally truthful and Pareto optimal mechanism (RTMHA) is presented. We prove that RTMHA is an $\frac{e}{e-1}$-approximation mechanism for MHA where agents have weights and ties. Note, truthful $O(1)$-approximation is impossible if each agent has different weights over objects [17, 3]. 1. and 2. are strong generalizations of the results in [22], and are obtained with completely new techniques.
3. A universally truthful, Pareto optimal mechanism for KHA with a $(4, 2)$-approximation (4-approximation ratio, violating each knapsack constraint by a factor 2) for parallel machines (objects) ($c_{ia} = c_i$, for any object a) when weights and ties are allowed ($(4, 1)$-approximation without ties). For general KHA with weights and ties, we observe that [11] implies a universally truthful $O(1)$-approximate mechanism (but not Pareto optimal).
4. Our mechanism RTMHA applied to the online weighted matroid bipartite matching problem and the online matroid job recruitment problem, implies tight $\frac{e}{e-1}$-approximations.

Technical Contributions. We present a generic approach to design deterministic truthful and Pareto optimal mechanisms for MHA and KHA when agents' preferences include ties, by reducing mechanism design to algorithm design of the underlying graph (i.e., calling the algorithm of finding maximum cardinality matching on an auxiliary graph as a black box). The mechanism RTMHA for MHA shares in spirit the same idea as Random SDMT mechanism in [22]. The main difficulty lies in the analysis of its $\frac{e}{e-1}$-approximation. Our new setting is much more general than [22], thus a completely different analysis is needed. We split the analysis of RTMHA into three parts, and its starting point is based on the charging map method (extended to matroid constraints), widely used for matching problems [4, 10].

First, for unweighted agents without ties, we show that the symmetric difference between a matching μ^τ output by TMHA under an order τ of all agents and the matching $\mu^{\tau_{-i}}$ output by TMHA under order τ_{-i} with agent i absent, is an alternating path starting from i. This characterization implies that, for each object $a \in A$, the agents $S_a(\tau)$ matched to a under μ^τ and agents $S_a(\tau_{-i})$ matched to a under $\mu^{\tau_{-i}}$ satisfy a nice exchange property: if some agent can be added to $S_a(\tau)$ then he can also be added to $S_a(\tau_{-i})$ obeying the matroid constraint.

When preferences have ties, we say that two matchings are equivalent if the matched agents are the same and the matched objects of the same agent are in the same indifference class of that agent. The second part of our analysis, for unweighted agents with ties, shows a similar characterization between the equivalence class $\mathrm{CL}(\mu^\tau)$ (all matchings equivalent to matching μ^τ) and the class $\mathrm{CL}(\mu^{\tau_{-i}})$. That is, there exists an injective map from $\mathrm{CL}(\mu^\tau)$ to $\mathrm{CL}(\mu^{\tau_{-i}})$ such that the symmetric difference of each matching in $\mathrm{CL}(\mu^\tau)$ with its image in $\mathrm{CL}(\mu^{\tau_{-i}})$ is an alternating path. This lets us reduce the problem for agents with ties to that with agents without ties. We believe this technique will find further applications to analyze mechanisms for agents with ties in ordinal settings, which may simplify the proofs. For example, with this technique, we greatly simplify the proof for weighted agents with ties and objects with uniform matroid constraints in [22], where they use a different technique based on trading graphs. Based on this characterization we prove the important Injectivity Lemma; it can be used to show the $\frac{e}{e-1}$-approximation for unweighted agents with ties directly.

Thirdly, for the most general setting of weighted agents with ties, we carefully utilize the previous injectivity lemma, proving a strengthened version of such a lemma (see Lemma 13). Based on this lemma, we show that there exists an injective map from marginal 'bad' events to 'good' events, which suffices for the analysis of the final approximation ratio.

Related Work. Random Serial Dictator (RSD) [1] and Probabilistic Serial (PS) [7] are two paradigmatic randomized mechanisms for HA. Krysta et al. [22] generalize RSD mechanism to HA with ties and obtain a tight 2-approximate deterministic truthful and Pareto optimal mechanism for weighted agents and an $\frac{e}{e-1}$-approximate universally truthful, Pareto optimal mechanism. They show that no universally truthful, Pareto optimal mechanism can have approximation better than $\frac{18}{13}$, and if the mechanism is additionally non-bossy, the lower bound on the approximation ratio is $\frac{e}{e-1}$. Our random mechanism is non-bossy for agents with strict preferences and thus has best possible approximation ratio in this sense. Bogomolnaia and Moulin [8] show the same ratio $\frac{e}{e-1}$ also for PS. Recently, [17, 3] establish an $O(\sqrt{n})$ bound of RSD (w.r.t. the optimal social welfare) when agents have cardinal values (with arbitrary weights on the objects) over all the objects and show that no truthful in expectation mechanism can have approximation better than $\Omega(\sqrt{n})$. We obtain constant approximation ratios when the agents are weighted. This lower bound $\Omega(\sqrt{n})$ implies that we cannot obtain similar results when agents have different weights over objects. Tight deterministic truthful mechanisms for weighted matching markets were proposed by Dughmi and Ghosh [15].

(M)HA is related to online bipartite matching problem (OBM) as follows. If each agent in HA ranks his desired objects in the order that precisely follows the arrival order of objects in the OBM, the two problems are equivalent, as emphasized in [6, 22]. Karp et al. [21] initialize the study of OBM and provide a RANKING algorithm with a tight approximation ratio $\frac{e}{e-1}$. Aggarwal et al. [4] are the first to study the weighted version of OBM (WOBM) when the fixed vertices have weights (or priorities). They use the charging map approach and prove that the ratio $\frac{e}{e-1}$ also holds. Recently, the analysis of WOBM has been unified into the primal-dual framework by Devanur et al. [14]. There are further generalizations of the online bipartite matching model, e.g., [10, 23, 24]. Matroid HA has an online interpretation, which is loosely related to the Matroid Secretary Problem introduced by Babaioff et al. [5].

Organization. Section 2 contains preliminaries, and Section 3 presents our deterministic mechanism for MHA. We develop a randomized mechanism for MHA in Section 4. Its analysis is divided into three subsections: 4.1 for unweighted agents without ties, 4.2 for unweighted agents with ties, 4.3 for weighted agents with ties. Applications of our mechanisms for MHA and our results for KHA will be published in the the full version of the paper.

2 Preliminaries

Let $N = \{1, 2, \cdots, n_1\}$ be a set of n_1 *agents* and A a set of n_2 *objects*; $n = n_1 + n_2$. Let $[i] = \{1, 2, \cdots, i\}$. Each agent $i \in N$ finds a subset of objects acceptable and has a preference ordering, not necessarily strict, over these objects.

We write $a_t \succ_i a_s$ if agent i *strictly prefers* object a_t to object a_s, and $a_t \simeq_i a_s$ if i is *indifferent between* a_t and a_s. We use $a_t \succeq_i a_s$ to denote that i *weakly prefers* a_t to a_s, i.e., either $a_t \succ_i a_s$ or $a_t \simeq_i a_s$. In some cases a weight w_i is associated with each agent i, representing i's priority or importance; let $W = (w_1, w_2, \ldots, w_{n_1})$. If we are in an unweighted setting then $w_i = 1$ for each i.

Each agent's acceptable objects are divided into *indifference classes*: he is indifferent between the objects in the same class and has a strict preference ordering over these classes. For each agent i, let C^i_k, $1 \le k \le n_2$, be the kth indifference class (*tie*), of i; assume that $C^i_q = \emptyset$, $\forall q, l \le q \le n_2$, if $C^i_l = \emptyset$ for some $l \in [n_2]$. Let $L(i) = (C^i_1 \succ_i C^i_2 \succ_i \cdots \succ_i C^i_{n_2})$ be the preference list of i. We write $a \in L(i)$ if a appears in $L(i)$ (i finds a acceptable). Let $L = (L(1), L(2), \cdots, L(n_1))$ and $L(-i) = (L(1), \ldots, L(i-1), L(i+1), \ldots, L(n_1))$.

A matroid is a pair $\mathcal{M} = (X, \mathcal{I})$, where X is a ground set and $\mathcal{I} \subseteq 2^X$ (each $Y \in \mathcal{I}$ is called independent set) satisfying properties: (1) [**non-emptiness**] $\emptyset \in \mathcal{I}$; (2) [**heredity**]

If $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$; (3) [**exchange**] If $Z, Y \in \mathcal{I}$ and $|Z| > |Y|$, then there is $z \in Z \backslash Y$ such that $Y \cup \{z\} \in \mathcal{I}$. A base of $\mathcal{M}$ is an independent set with maximum size. By (3), any maximal independent set is a base. For any $S \subseteq X$, let $r_{\mathcal{M}}(S)$ be the maximum size of an independent set contained in S; $r_{\mathcal{M}}(X)$ is called the rank of $\mathcal{M}$. A matroid $\mathcal{M} = (X, \mathcal{I})$ is called an ℓ-uniform matroid if $Y \subseteq X$ is independent if and only if $|Y| \leq \ell$. A direct sum of uniform matroids (not necessarily with the same rank) is called a partition matroid. Given a set X and a family $\mathcal{F} = \{X_i\}_{i=1}^{s}$ of subsets of X, a set $T \subseteq X$ is called a transversal set of X if there is an injective map $f\colon T \to \mathcal{F}$ s.t. $x \in f(x)$, for any $x \in T$. The matroid $(X, \mathcal{I})$, with $\mathcal{I} = \{T \,|\, T \text{ is a transversal set of } X\}$, is called a transversal matroid.

In Matroid HA (MHA), we have a matroid $\mathcal{M}_a = (N, \mathcal{I}_a)$ on agent set N, for each object $a \in [n_2]$. A (feasible) *matching* (or *b-matching*) μ is an allocation (subset of $N \times A$) assigning at most one object to each agent i[1] such that, for each object a, the set of agents who are assigned a is an independent set of $\mathcal{M}_a$ (multiple agents may be allocated the same object). If $(i, a) \in \mu$, agent i and object a are *matched together*. If $(i, a) \in \mu$ for some a, we say that i is *matched*, and *unmatched* otherwise. The definitions of *matched* (*unmatched*) for an object are analogous. If $i \in N$ is matched, $\mu(i)$ denotes the object matched to i. If object a is matched, $\mu^{-1}(a)$ denotes the set of agents matched to a. Two matchings μ, μ' are *equivalent* (denoted $\mu \simeq \mu'$) if in these two matchings, the matched agents are the same and the matched objects for each agent are in the same indifference class of this agent, i.e., $\mu(i) \simeq_i \mu'(i)$, allowing $\emptyset \simeq_i \emptyset$, for any $i \in N$. Let $\mathrm{CL}(\mu)$ be the class of all matchings equivalent to μ. For two sets B and C, we sometimes use $B + C$ ($B - C$, resp.) to denote $B \cup C$ ($B \backslash C$, resp.), and sometimes write a set $\{x\}$ simply as x. Let Π denote the set of all permutations of agents.

In what follows, we will consider the undirected graph $G = (V, E)$ where $V = (N \cup A)$ and $E = \{(i, a), i \in N, a \in L(i)\}$. We also use μ to denote a matching (a feasible matching for MHA) in G. The *size* of a matching μ is equal to the number of agents matched under μ. In the presence of weights, the *weight* of a matching is equal to the sum of the weights of the matched agents. For any subset $E' \subseteq E$, define $E'_v = \{u \,|\, (u, v) \in E'\}$, for any $v \in V$. An instance of MHA is denoted by $I = (N, A, L, (\mathcal{M}_a)_a, W)$. We drop W and $(\mathcal{M}_a)_a$ and write $I = (N, A, L, \mathcal{M})$ if agents are unweighted. Let $\mathcal{I}^{MHA}$ denote the set of all possible instances of MHA. For two given matchings μ_1, μ_2, we will use $\mu_1 \oplus \mu_2$ to denote the symmetric difference with respect to their sets of edges. An *alternating path* in G, w.r.t. a matching μ_1, is a path that consists of edges that alternately belong to μ_1 and do not belong to μ_1.

A matching μ is *Pareto optimal* if there is no other matching under which some agent is strictly better off while none is worse off, w.r.t. their preferences. Formally, μ is *Pareto optimal* if there is no other matching μ' such that (i) $\mu'(i) \succeq_i \mu(i)$ for all $i \in N$, and (ii) $\mu'(i') \succ_{i'} \mu(i')$ for some $i' \in N$. Given an order $\sigma \in \Pi$, a matching μ is strictly lexicographically σ-better than a matching μ' if there exists a $k \in [n_1]$ such that $\mu(\sigma(i)) \simeq_{\sigma(i)} \mu'(\sigma(i))$, $i \in [k-1]$ and $\mu(\sigma(k)) \succ_{\sigma(k)} \mu'(\sigma(k))$. A matching μ is called a lexicographically σ-maximal matching if there does not exist a matching μ' such that μ' is strictly lexicographically σ-better than matching μ. Note, if μ is lexicographically σ-maximal, then μ must be Pareto optimal.

Let $\mathcal{M}^{MHA}$ denote the set of all possible matchings. A *deterministic mechanism* ϕ maps an instance of MHA to a matching, i.e., $\phi : \mathcal{I}^{MHA} \to \mathcal{M}^{MHA}$. Let $R : \mathcal{M}^{MHA} \to [0, 1]$ be a probability distribution over possible matchings (a *random matching*), i.e., $\sum_{\mu \in \mathcal{M}^{MHA}} R(\mu) = 1$. A *randomized mechanism* ϕ is a mapping from $\mathcal{I}^{MHA}$ to a distribution over possible matchings, i.e., $\phi : \mathcal{I}^{MHA} \to Rand(\mathcal{M}^{MHA})$, where $Rand(\mathcal{M}^{MHA})$ is the set of all random

[1] Note that if agents are allowed to receive more than one object, the model becomes a many-to-many matching and the preference spaces of agents are much more complicated, which is not studied here.

matchings. A deterministic (randomized, resp.) mechanism is Pareto optimal if it returns a Pareto optimal matching (a distribution over Pareto optimal matchings, resp.).

Agents' preferences are private knowledge and they may prefer not to reveal their preferences truthfully if it is not in their best interests, for a given mechanism. A *deterministic* mechanism is *truthful* (*in dominant strategies*) if agents always find it in their best interests to declare their preferences truthfully, no matter what other agents declare, i.e., for every i and every possible declared list $L'(i)$ for i, $\phi(L(i), L(-i)) \succeq_i \phi(L'(i), L(-i))$, $\forall L(i), L(-i)$. A randomized mechanism ϕ is *universally truthful* if it is a probability distribution over deterministic truthful mechanisms. Let $w(\phi(I))$ be the (expected) weight of the (random) matching output by ϕ on instance $I \in \mathcal{I}^{MHA}$, and $w(I)$ be the weight of a maximum weight Pareto optimal matching in I. The *approximation ratio* of ϕ is defined as $\max_{I\in\mathcal{I}^{MHA}} \frac{w(I)}{w(\phi(I))}$.

For any agent i, define a matroid on agent i as $\mathcal{M}_i = (A, \mathcal{I}_i)$, with $\mathcal{I}_i = \{S \mid S \subseteq A \text{ and } |S| \leq 1\}$. We define two useful matroids on edge set E. Let $\mathcal{M} = (E, \mathcal{I})$, where, for any $E' \subseteq E$, $E' \in \mathcal{I}$ if and only if $E'_a \in \mathcal{I}_a$, for any $a \in A$. Let $\mathcal{M}' = (E, \mathcal{I}')$, where, for any $E' \subseteq E$, $E' \in \mathcal{I}'$ if and only if $E'_i \in \mathcal{I}_i$, for any $i \in N$. Note, $\mathcal{M}'$ is a partition matroid. Also, an allocation $E' \subseteq E$ is feasible for MHA if and only if $E' \in \mathcal{I} \cap \mathcal{I}'$, meaning that E' is an independent set in both $\mathcal{M}'$ and $\mathcal{M}$. Hence, our objective is to design (universally) truthful, Pareto optimal mechanisms to find an edge set that is independent in both $\mathcal{M}'$ and $\mathcal{M}$, such that its size (or weight for weighted agents) is maximized. Edmonds [16] provides an algorithm to compute a maximum independent set that is independent for two matroids.

▶ **Proposition 1** ([16, 13]). *Given two matroids $\mathcal{M}^1 = (X, \mathcal{I}^1)$ and $\mathcal{M}^2 = (X, \mathcal{I}^2)$, there is a matroid intersection algorithm $\mathcal{A}^{MI}$ for computing a maximum cardinality common independent set $S \in \mathcal{I}^1 \cap \mathcal{I}^2$, terminating in polynomial time $\Gamma(|X|) \leq O(|S|^{\frac{3}{2}}|X|Q^{test})$, where Q^{test} is the time needed to test if a given set is independent.*

We assume for simplicity that the test time $Q^{test} = O(1)$, and we omit it in the rest of the paper. We also need a fact from exchange theory between two bases of a matroid.

▶ **Proposition 2** (Corollary 39.12a [26]). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and $B_1, B_2 \in \mathcal{I}$ with $|B_1| = |B_2|$, there exists a bijection $f\colon B_1 - B_2 \to B_2 - B_1$ s.t. $\forall x \in B_1 - B_2$, $B_1 - x + f(x) \in \mathcal{I}$.*

3 Deterministic Mechanism for MHA

We introduce TMHA, a truthful and Pareto optimal mechanism, that generalizes SDMT-1 [22] to the case where sets allocated to the objects are independent sets of matroids. Let $I = (N, A, L, \mathcal{M})$ be an instance of MHA, and $\sigma \in \Pi$ (w.l.o.g., $\sigma(i) = i$ for $i \in N$). TMHA, given in Algorithm 1, proceeds in n_1 phases, each phase corresponds to one iteration of the for loop. Notice, at any stage of TMHA, $(i, a) \in E$ if and only if either agent i is matched in μ and $a \simeq_i \mu(i)$ or TMHA is at phase i, examining the indifference class that contains a.

▶ **Observation 3.** *At the end of phase i of TMHA, if agent i is assigned no object then he will be assigned no object when TMHA terminates. If i is provisionally assigned an object a, then he will be allocated an object in the same indifference class as a in the final matching.*

By Observation 3, different tie breaking rules lead to equivalent produced matchings.

▶ **Theorem 4.** *Given an order $\sigma \in \Pi$ of agents, the matching generated by TMHA is a lexicographically σ-maximal matching, thus, also Pareto optimal.*

TMHA is truthful, no matter which matching is selected in each phase of the mechanism:

Algorithm 1: Truthful Mechanism for MHA (TMHA)

Input: Agents N; Objects A; Preference list profile L; Matroids $(\mathcal{M}_a)_a$; Order σ
Output: Matching μ

Let $G = (N \cup A, E)$, $E \leftarrow \emptyset$, $\mu \leftarrow \emptyset$.
for *each agent* $i \in N$ *in the order of* σ **do**
  Let $\ell \leftarrow 1$
  Step (*): **if** $C^i_\ell \neq \emptyset$ **then**
    $E \leftarrow E \cup \{(i,a) : a \in C^i_\ell\}$; // all new edges are non-matching edges
    Run $\mathcal{A}^{MI}$ on G and obtain a maximum cardinality matching μ'
    if $|\mu'| = |\mu| + 1$ **then**
      modify μ to μ'; // i must be provisionally allocated some $a \in C^i_\ell$ and (i,a) is now a matching edge
    else
      $E \leftarrow E \setminus \{(i,a) : a \in C^i_\ell\}$; $\ell \leftarrow \ell + 1$; Go to Step (*)
Return μ; //each matched agent is allocated his matched object

▶ **Theorem 5.** *The mechanism TMHA is truthful.*

We now show a bound on the time complexity of TMHA. Let γ denote the size of the largest indifference class for a given instance I.

▶ **Theorem 6.** *TMHA terminates in time $O(n_1^3\gamma)$.*

Any Pareto optimal matching is at least half the size of a maximum size such matching [9]. Thus, TMHA achieves approximation ratio 2 with respect to the maximum cardinality matching. We show that, when agents are assigned arbitrary weights, TMHA is 2-approximate (w.r.t. maximum weight Pareto optimal matching) if the order σ is by non-increasing agents' weights, breaking ties arbitrarily. The bound is tight since no deterministic truthful mechanism can achieve an approximation ratio better than 2 even if $r_{\mathcal{M}_a} = 1$, for any $a \in A$, see [22].

▶ **Theorem 7.** *TMHA achieves a 2-approximation w.r.t. the size of a maximum weight Pareto optimal matching, if agents are ordered in σ by non-increasing order of their weights.*

4 Randomized Mechanism for MHA

We now present a universally truthful, Pareto optimal mechanism for Matroid House Allocation Problem (see Algorithm 2, where $g(y) = e^{y-1}$). When the matroid for each object is a uniform matroid with rank one, Algorithm 2 reduces to SDMT-1 from [22]. The analysis of RTMHA will be gradually developed in the following three subsections for various settings. We will start with the simplest setting of unweighted agents without ties (Subsection 4.1), then proceed to the setting of unweighted agents with ties (Subsection 4.2), and, finally, weighted agents with ties (Subsection 4.3). The next subsection builds on the previous one.

If the weights are agents' private data with no over-bidding assumption, Algorithm 2 is universally truthful (w.r.t. preferences and weights) and Pareto optimal as well [22].

4.1 Unweighted Agents without Ties

We will show now that RTMHA achieves $\frac{e}{e-1}$-approximation for unweighted agents without ties. Note that the order of RTMHA for unweighted agents is just the uniform random order

Algorithm 2: Random Truthful Mechanism for MHA (RTMHA)

Input: Agents N; Objects A; Preference list profile L; Matroids $(\mathcal{M}_a)_a$; Weights W

Output: Matching

for *each agent* $i \in N$ **do**

  Pick $Y_i \in [0,1]$ uniformly at random;

Sort agents in decreasing order of $w_i(1-g(Y_i))$ (break ties in favor of smaller index);

Run TMHA according to above order;

Return the matching;

of agents. Our technique is based on a charging map method (extended by us to matroids), which is widely used in analyzing the approximation ratio for matching problems [4, 10].

We assume here that agents' preferences are strict: $|C^i_j| \le 1$, for $i \in [n_1]$, $j \in [n_2]$. Recall, each object a is associated with a matroid $\mathcal{M}_a = (N, \mathcal{I}_a)$. We present a characterization of $\mu^\tau \oplus \mu^{\tau_{-i}}$, where μ^τ and $\mu^{\tau_{-i}}$ are the matchings obtained by TMHA under the order τ of agents and τ_{-i} (τ with i absent), respectively. We first prove a useful lemma.

Lemma 8 shows exchange properties between two independent sets of a matroid and their switching sets. Let $T_0 = \{i'_1, i'_2, \cdots, i'_k\}$ and $T_k = \{i_1, i_2, \cdots, i_k\}$ be two sets of size k with different elements (orders of elements in T_0 and T_k are fixed). We define the *switching sets* T_ℓ, $\ell \in [k-1]$ between T_0 and T_k, as $T_\ell = \{i_1, i_2, \cdots, i_\ell, i'_{\ell+1}, \cdots, i'_k\}$, $\ell \in [k-1]$. (Lemma 8 is used in the proof of Lemma 10 with elements being agents.)

▶ **Lemma 8.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid and $S \in \mathcal{I}$. Let $T_0 = \{i'_1, i'_2, , \cdots, i'_k\} \subseteq X$ and $T_k = \{i_1, i_2, \cdots, i_k\} \subseteq X$ and their switching sets $T_\ell = \{i_1, i_2, \cdots, i_\ell, i'_{\ell+1}, \cdots, i'_k\} \in \mathcal{I}$, $\ell \in [k-1]$. Suppose $S + T_\ell \in \mathcal{I}$, for any $\ell = 0, 1, \cdots, k$. We also have $S + \{i_1, \cdots, i_\ell, i'_\ell\} \notin \mathcal{I}$, for any $\ell \in [k]$. Then, for any element j (which is not in $S + T_0 + T_k$),*

(i) *if $S + T_k + j \in \mathcal{I}$ then $S + T_0 + j \in \mathcal{I}$;*

(ii) *if $S + T_0 + j \notin \mathcal{I}$ then $S + T_k + j \notin \mathcal{I}$ (contrapositive proposition of* (i)*).*

Proof. For (i), by the exchange property of matroids, there exists a $y \in T_k + j$ such that $S+T_0+y \in \mathcal{I}$. If $y \ne j$, suppose $y = i_\ell$ for some $\ell \in [k]$. Then we have $T = S+\{i_1, \cdots, i_\ell\} \in \mathcal{I}$ (from $S + T_k \in \mathcal{I}$) and $T' = S + \{i'_1, \cdots, i'_\ell, i_\ell\} \in \mathcal{I}$ (from $S + T_0 + y \in \mathcal{I}$). Then by the exchange property of matroids, there exists $y' \in T' - T$ such that $T + y' \in \mathcal{I}$. Let $y' = i'_{\ell'}$ for some $\ell' \le \ell$. Consequently, $S + \{i_1, i_2, \cdots, i_{\ell'}, i'_{\ell'}\} \in \mathcal{I}$, which leads to a contradiction. ◀

▶ Remark. Although in Lemma 8 we require two initial sets with the same cardinality, we can relax the requirement to allow them to have different cardinalities. Namely, let $T_0 = \{i'_1, i'_2, \cdots, i'_{k'}\}$ and $T_k = \{i_1, i_2, \cdots, i_k\}$ with $k' \le k$ (the orders of elements in two sets are fixed). The switching sets T_ℓ, $\ell \in [k']$ between T_0 and T_k are defined as: $T_\ell = \{i_1, i_2, \cdots, i_\ell, i'_{\ell+1}, \cdots, i'_{k'}\}$, $\ell \in [k'-1]$, and $T_{k'} = T_k$. Lemma 8 still holds.

Given $\tau \in \Pi$, let μ^τ be the matching obtained by TMHA under τ, and let $S_a(\tau)$ be the set of agents matched to $a \in A$ by TMHA. Let, for any $t \in [n_1]$, $S^t_a(\tau) \subseteq S_a(\tau)$ be the top t agents in $S_a(\tau)$, i.e., $S^t_a(\tau) = \{i \in S_a(\tau) \mid \tau^{-1}(i) \le t\}$. Then $S_a(\tau) = S^{n_1}_a(\tau)$. Let τ_{-i} be order τ after removing agent i from τ. Similarly, define $S^t_a(\tau_{-i})$ and $S_a(\tau_{-i}) = S^{n_1-1}_a(\tau_{-i})$. When we say "*under* τ", we mean the process of running TMHA under order τ. For any $a \in A$, let $F^t_a = S^t_a(\tau_{-i}) \cap S^t_a(\tau)$. $T^t_0 = S^t_a(\tau_{-i}) - S^t_a(\tau) = \{i'_1, i'_2, \cdots, i'_\ell, i'_{\ell+1}, \cdots, i'_{k'}\}$, $k' = |T^t_0|$. Let $T^t_k = S^t_a(\tau) - S^t_a(\tau_{-i}) = \{i_1, i_2, \cdots, i_\ell, i_{\ell+1}, \cdots, i_k\}$, $k = |T^t_k|$. We will show in the next lemma that $k \in \{k', k'+1\}$, for any $t \in [n_1]$ and $a \in [n_2]$. Note that this includes the case $k' = 0$ (in a sense that Property $\mathcal{P}$ (i)-(iv) below trivially holds). The orders in T^t_0 and T^t_k

both follow order τ. We have the switching sets $T_\ell^t = \{i_1, i_2, \cdots, i_\ell, i'_{\ell+1}, \cdots, i'_{k'}\}$ between T_0^t and T_k^t, $\ell \in [k'-1]$, and $T_{k'}^t = T_k^t$. We finally define Property $\mathcal{P}$ as follows.

▶ **Definition 9.** Let $E' = \{(i', \mu^\tau(i')), (i', \mu^{\tau_{-i}}(i')) : i' \in N, i'$ matched by μ^τ and $\mu^{\tau_{-i}}\}$. We say that Property $\mathcal{P}$ holds for μ^τ and $\mu^{\tau_{-i}}$ if: $C = ((\mu^\tau \oplus \mu^{\tau_{-i}}) \cap E') \cup \{(i, \mu^\tau(i))\}$ is an alternating path starting from agent i if it is non-empty, and for any $a \in A$, and $t \in [n_1]$:

- **(i)** $F_a^{n_1} + T_\ell^t \in \mathcal{I}_a$, for any $\ell \in [k']$;
- **(ii)** $F_a^t + \{i_1, \cdots, i_\ell, i'_\ell\} \notin \mathcal{I}_a$, $\ell \in [k']$;
- **(iii)** For any agent s, if $S_a^t(\tau) + s \in \mathcal{I}_a$, then $S_a^t(\tau_{-i}) + s \in \mathcal{I}_a$ (or equivalently, $S_a^t(\tau_{-i}) + s \notin \mathcal{I}_a$ implies that $S_a^t(\tau) + s \notin \mathcal{I}_a$).
- **(iv)** $\tau^{-1}(i_1) < \tau^{-1}(i'_1) < \tau^{-1}(i_2) < \tau^{-1}(i'_2) < \cdots < \tau^{-1}(i_{k'}) < \tau^{-1}(i'_{k'})$

We are ready to give our characterization lemma in the following.

▶ **Lemma 10.** *Property $\mathcal{P}$ holds for unweighted agents with strict preference orders.*

Lemma 10 can be used to directly prove that RTMHA is $\frac{e}{e-1} + o(1)$-approximate for unweighted agents without ties. Analysis in Subsection 4.2 builds on developments in this section.

4.2 Unweighted Agents with Ties

When agents have ties without weights, we will present a similar characterization in Lemma 11 (by alternating path) of symmetric differences between the matching obtained by TMHA under some order τ of all agents and the matching obtained by TMHA under τ_{-i} when an agent i is absent. We will prove that in any step of the two processes of TMHA under τ and under τ_{-i}, there is an injective map from the equivalence class of matchings generated by TMHA under τ to the class generated by TMHA under τ_{-i}. This injective map is such that each pair of the corresponding matchings from these two classes satisfies Property $\mathcal{P}$.

For any $t \in [n_1]$, let $\mathrm{CL}^t(\mu^\tau)$ denote the equivalence class of matchings equivalent to the matching found by TMHA under τ until tth iteration. Precisely, if $\mu^\tau_{\le t} = \{(i, \mu^\tau(i)) : \tau^{-1}(i) \le t\}$ is the matching μ^τ restricted to the first t agents of τ, then: $\mathrm{CL}^t(\mu^\tau) = \{\mu \in G \,|\, \mu \simeq \mu^\tau_{\le t}\}$. Let $\mathrm{CL}(\mu^\tau) = \mathrm{CL}^{n_1}(\mu^\tau)$. When we consider the process of running TMHA under τ_{-i}, to simplify the notation, we consider an equivalent process that runs TMHA under τ while imposing the condition that agent i is unmatched. Hence, in the following we suppose TMHA running on τ_{-i} is such an equivalent process. Thus, we have $\mathrm{CL}^t(\mu^{\tau_{-i}})$, for any $t \in [n_1]$.

▶ **Lemma 11.** *For any t, there exists an injective map f from $\mathrm{CL}^t(\mu^\tau)$ to $\mathrm{CL}^t(\mu^{\tau_{-i}})$ such that for any $\mu \in \mathrm{CL}^t(\mu^\tau)$, μ and $f(\mu)$ satisfy Property $\mathcal{P}$.*

Proof. The proof is by induction on the iterations of the two processes of TMHA run under τ and under τ_{-i}. First, note that $\mathrm{CL}^s(\mu^\tau) = \mathrm{CL}^s(\mu^{\tau_{-i}})$, for any $s < \tau^{-1}(i)$. Thus each matching from $\mathrm{CL}^s(\mu^\tau)$ can be mapped to itself. This shows that if i is unmatched under τ, the lemma trivially holds. Hence, w.l.o.g., suppose agent i is matched under τ.

Second, for any matching $\mu^{s+1} \in \mathrm{CL}^{s+1}(\mu^\tau)$, there exists a unique matching $\mu^s \in \mathrm{CL}^s(\mu^\tau)$ such that $\mu^{s+1} = \mu^s \cup \{(\tau(s+1), \mu^{s+1}(\tau(s+1)))\}$, for any $s \le n_1 - 1$.

We now prove the induction base case for $s = \tau^{-1}(i)$. For any matching $\mu^s \in \mathrm{CL}^s(\mu^\tau)$, we define f as $f(\mu^s) = \mu^s_{<s} \in \mathrm{CL}^s(\mu^{\tau_{-i}})$. Now we can see that the symmetric difference of μ^s and $f(\mu^s)$ is a single edge and all parts of Property $\mathcal{P}$ hold. Hence, the base case is true.

Now suppose the lemma holds for $s = k-1$, where $k - 1 \ge \tau^{-1}(i)$, i.e., we have an injective function f: $\mathrm{CL}^{k-1}(\mu^\tau) \to \mathrm{CL}^{k-1}(\mu^{\tau_{-i}})$ such that for any $\mu \in \mathrm{CL}^{k-1}(\mu^\tau)$, μ and

$f(\mu)$ satisfy Property $\mathcal{P}$. Let's see the case $s = k \leq t$. Let $j = \tau(k)$ be the kth agent of τ. Consider this new agent j, and any $\mu \in \mathrm{CL}^k(\mu^{\tau_{-i}})$. Since for any a which is strictly better than $\mu(j)$, i.e., $a \succ_j \mu(j)$, agent j will not be matched to a under μ, which means $j + S_a^{k-1}(\mu) \notin \mathcal{I}_a$, by part (iii) of Property $\mathcal{P}$, $j + S_a^{k-1}(f^{-1}(\mu_{<k})) \notin \mathcal{I}_a$ if $f^{-1}(\mu_{<k})$ exists. Therefore, for any $\mu^{k,\tau} \in \mathrm{CL}^k(\mu^{\tau})$ and $\mu^{k,\tau_{-i}} \in \mathrm{CL}^k(\mu^{\tau_{-i}})$, we will have $\mu^{k,\tau}(j) \preceq_j \mu^{k,\tau_{-i}}(j)$. Let $b = \mu^{k,\tau}(j)$ and $c = \mu^{k,\tau_{-i}}(j)$. Note that for each matching $\mu^{k,\tau} \in \mathrm{CL}^k(\mu^{\tau})$, we have $\mu^{k,\tau}_{\leq k-1} \in \mathrm{CL}^{k-1}(\mu^{\tau})$; similar property holds for $\mathrm{CL}^k(\mu^{\tau_{-i}})$. Hence, we consider two cases:
Case(i). If $b \simeq_j c$, we define $f(\mu^{k,\tau}) = f(\mu^{k,\tau}_{<k}) \cup (j, b)$. We can see that f is well defined since j can be added into $S_b(\mu^{k,\tau}_{<k})$, it also can be added into $S_b(f(\mu^{k,\tau}_{<k}))$ by Property $\mathcal{P}$ (iii). Second, there is no change for alternating path after j is considered. Third, j will be added into the set $S_b(\mu^{k,\tau}) \cap f(S_b(\mu^{k,\tau}))$ and the matched object of the other agents will not change when compared their matchings under τ and under τ_{-i}. Thus, Property $\mathcal{P}$ holds.
Case (ii). If $b \prec_j c$, by induction hypothesis, Property $\mathcal{P}$ (ii), there exists an object $b' \simeq_j c$ such that $j + S_{b'}(\mu^{k,\tau}_{<k}) \notin \mathcal{I}_{b'}$ and $j + S_{b'}(f(\mu^{k,\tau}_{<k})) \in \mathcal{I}_{b'}$. We know the current alternating path (up-to iteration $k-1$) must be with b' as its another end point. Hence, b' is unique. Otherwise, let $d \neq b'$ be the end point of the alternating path. Then the switching agent, i.e., i'_ℓ in Property $\mathcal{P}$ (ii) for object b' is not j, and by Property $\mathcal{P}$ (i), $j + S_{b'}(\mu^{k,\tau}_{<k}) \in \mathcal{I}_{b'}$, a contradiction. Define $f(\mu^{k,\tau}) = f(\mu^{k,\tau}_{<k}) \cup (j, b')$. Now we see that two edges $(j, b') \in \mu^{k,\tau_{-i}}$ and $(j, b) \in \mu^{k,\tau}$ will be added to the alternating path; thus, Property $\mathcal{P}$ holds. The end point of the alternating path is now object b. The last part of the proof of the induction step (omitted here) is to show that sets of agents matched to objects b and b' satisfy Property $\mathcal{P}$ (note that sets of agents matched to other objects do not change). ◀

By Lemma 11, we can suppose w.l.o.g. that there exists a matching in the graph G such that all the agents are matched. (Otherwise, we can find a maximum matching in graph G and remove the unmatched agents in this matching and their adjacent edges. After this, Lemma 11 implies that the expected matching size of RTMHA can not increase (because $|\mu| \geq |f(\mu)|$) and the graph has a matching with all agents matched.) Let μ^* denote this matching in G. Given τ, let τ_i^t denote the order obtained from τ by first removing agent i from τ and then inserting him into the tth position of τ_{-i}, i.e., $\tau_i^t(i) = t$. Now fix $t \in [n_1]$. For any $\tau \in \Pi$, $a \in A$, let $U_a^t(\tau)$ be defined as $U_a^t(\tau) = \{i \in S_a(\mu^*) \,|\, i \,\text{is not matched by TMHA under}\, \tau_i^t\}$.

▶ **Lemma 12** (INJECTIVITY LEMMA). $|U_a^t(\tau)| \leq |S_a^t(\tau)|$.

Proof. Suppose that $|U_a^t(\tau)| > |S_a^t(\tau)|$. Then by matroid exchange property, there is an agent $i \in U_a^t(\tau)$ such that $S_a^t(\tau) + i \in \mathcal{I}_a$. Suppose i is unmatched by TMHA under τ. Since i is unmatched under τ_i^t, $S_a^t(\tau) = S^t(\tau_i^t)$. However $S_a^t(\tau_i^t) + i = S_a^t(\tau) + i \in \mathcal{I}_a$, implying agent i will be matched to an object at least as good as a by TMHA under τ_i^t, a contradiction. Suppose now that agent i is matched by TMHA under τ. Removing agent i from τ, we get τ_{-i}. Then inserting i into the tth position of τ_{-i} we get τ_i^t. Since agent i is unmatched by TMHA under τ_i^t, the processes of TMHA under τ_i^t and under τ_{-i} is the same. Let f be the injective function from $\mathrm{CL}^t(\mu^{\tau})$ to $\mathrm{CL}^t(\mu^{\tau_{-i}})$ in Lemma 11. By Property $\mathcal{P}$ (iii), $S_a^t(\tau) + i \in \mathcal{I}_a$, then $S_a^t(f(\tau)) + i \in \mathcal{I}_a$. That is $S_a^t(\tau_i^t) + i \in \mathcal{I}_a$, implying agent i will be matched to an object at least as good as a by TMHA under τ_i^t. This leads to a contradiction. ◀

Lemma 12 can be used to directly prove that RTMHA is $\frac{e}{e-1} + o(1)$-approximate for unweighted agents with ties. Analysis in Subsection 4.3 builds on developments in this section.

4.3 Weighted Agents with Ties

Building on Subsection 4.1 and 4.2 we give the proof of the $\frac{e}{e-1}$-approximation for weighted agents with ties. A careful utilization of Lemmas 11 and 12 is needed to obtain a strengthened version of injectivity lemma (Lemma 13). This lemma helps define an injective function from marginal 'bad' events to 'good' events, allowing to prove the ratio. Interestingly, Lemma 11 and primal-dual analysis [22], leads to a significantly simpler proof (compared to [22]) of the ratio $\frac{e}{e-1}$ when each associated matroid is uniform. The main technical ingredient of the proof is a strong version of the injectivity lemma similar to Lemma 12 to define an injective map from marginal set $\bigcup_{t\geq 1} M_t$ (defined later) to the set $2^{\bigcup_{t\geq 1} Q_t}$.

Because of weights, we consider the discrete version of the function $1-g(y)=1-e^{y-1}$ used in the sampling. For every $i\in N$, we will choose an integer $t=\tau(i)$ uniformly at random from $\{1,\cdots,\kappa\}$, where $\kappa\in\mathbb{N}_+$ is a parameter. Let $\psi(t)=1-(1-\frac{1}{\kappa})^{\kappa-t+1}$. The random order of agents follows the decreasing order of $w_i\psi(\tau(i))$, for any $i\in[n_1]$. Note, $\tau\in[\kappa]^{n_1}$ is different from previous subsections (where $\tau\in\Pi$), but we still call it a permutation and the discrete process is the same as RTMHA when $\kappa\to\infty$. For each $\tau\in[\kappa]^{n_1}$, there is a corresponding order of agents defined above. Hence, when we say 'under τ', we mean to run the above discrete RTMHA (called DRTMHA) under decreasing order of $w_i\psi(\tau(i))$, $i\in[n_1]$.

Let $\tau_i^t\in[\kappa]^{n_1}$ denote the same order of agents except we set i's τ value to t, i.e., $\tau_i^t(i)=t$ and $\tau_i^t(k)=\tau(k)$, for any agent $k\neq i$. The other notions have the same meaning as before unless explicitly redefined. Let μ^* be the maximum weighted matching on G, i.e., the optimal matching to our problem. Recall, $S_a(\mu^*)$ is the set of agents matched to a under μ^*. Let Q_t be the set of all the triples among permutations, τ values and agents such that the agent with τ value t at the current order is matched by TMHA. Precisely,

$$Q_t=\{(\tau,t,i)\,|\,i\ \text{is matched by TMHA under}\,\tau\,\text{and}\,\tau(i)=t,\,i\in N,\tau\in[\kappa]^{n_1}\}.$$

Let R_t be the set of all the triples among permutations, t values and agents such that the agent matched in μ^* with τ value at the current order is unmatched. That is:

$$R_t=\{(\tau,t,i)\,|\,i\ \text{is unmatched under}\,\tau\,\text{and}\,\tau(i)=t,\,\text{and}\,i\,\text{is matched in}\,\mu^*,\tau\in[\kappa]^{n_1}\}.$$

Our goal now is to define an injective map from R_t to $2^{\bigcup_{s\leq t} Q_t}$. Towards this aim, we define the marginal (loss) set M_t, which is a subset of R_t with marginal property, i.e., an agent in M_t will be matched after decreasing his τ value by one: $M_t=\{(\tau,t,i)\in R_t\,|\,i\ \text{is matched under}\,\tau_i^{t-1}\}$. Let $M_0=\emptyset$. Let $OPT=w(\mu^*)$ be the optimal weight and $B=\frac{OPT}{\kappa}$. Let $w(Q_t)$ denote the total weights of agents for triples in Q_t: $w(Q_t)=\sum_{(\tau,t,i)\in Q_t} w_i$. Similarly, define $w(R_t)$ and $w(M_t)$. Let $x_t=\frac{w(Q_t)}{\kappa^{n_1}}$ be the expected weights gained by DRTMHA on all agents with τ value t, and $y_t=\frac{w(M_t)}{\kappa^{n_1}}$ the expected marginal loss of all agents with τ value t. Since each agent appears with τ equal to $t\in[\kappa]$ in DRTMHA with equal probability $\frac{1}{\kappa}$, if $Q'_t\subseteq Q^t$ is the set of triples with agents matched in μ^*, then $\frac{w(Q'_t)}{\kappa^{n_1}}+\frac{w(R_t)}{\kappa^{n_1}}=B$, for any $t\in[\kappa]$. Therefore,

$$x_t+\frac{w(R_t)}{\kappa^{n_1}}\geq B. \tag{1}$$

So the expected weight of matching obtained by DRTMHA is $\sum_{t\in[n_1]} x_t$. There is a bijection $h:R_t\to\bigcup_{s\leq t} M_s$. Note, for any $(\tau,t,i)\in R_t$, if there is $s\in[t-1]$ s.t. $(\tau_i^s,s,i)\in Q_s$, then there is such s that is a unique maximal one. Note, in this case $(\tau_i^{s+1},s+1,i)\in M_{s+1}$. Define $h(\tau,t,i)=(\tau_i^{s+1},s+1,i)$. Otherwise, define $h(\tau,t,i)=(\tau_i^1,1,i)\in M_1$; note that

$M_0 = \emptyset$. (h is well defined.) For any $(\tau, s, i) \in M_s$, and some $s \le t$, agent i will not be matched under τ_i^t since it is unmatched under τ, by Lemma 11 (Property $\mathcal{P}$ (iii)). Then $(\tau_i^t, t, i) \in R_t$, which means h is surjective, i.e., $h(\tau_i^t, t, i) = (\tau, s, i)$. If $h(\tau, t, i) = h(\tau', t, i') = (\sigma, s, i'') \in M_s$, by definition $i = i' = i''$ and $\tau = \tau' = \sigma_{i''}^t$. Then h is also injective. Hence,

$$w(R_t) = \sum_{s \le t} w(M_s) \tag{2}$$

For any $\tau \in [\kappa]^{n_1}$, $a \in A$, let $S_a(\tau)$ be the set of agents matched to a by DRTMHA under τ. Let $U_a(\tau)$ be the set of agents in $S_a(\mu^*)$ such that changing the τ of any one of them, e.g., agent $i \in S_a(\mu^*)$ to some t_i, the new triple of τ together with his new value t_i and himself belongs to M_{t_i} (Note that for any agent $i \in S_a(\mu^*)$, if there exists a t_i such that $(\tau_i^{t_i}, t_i, i) \in M_{t_i}$, then t_i is unique for agent i). Precisely, $U_a(\tau) = \{i \in S_a(\mu^*) \mid (\tau_i^{t_i}, t_i, i) \in M_{t_i}\}$.

▶ **Lemma 13** (INJECTIVITY LEMMA). *There exists an injective function $g_{a\tau}\colon U_a(\tau) \to S_a(\tau)$ such that $w_i\psi(t_i) \le w_{i'}\psi(\tau(i'))$, for any $i \in U_a(\tau)$ and $i' = g_{a\tau}(i)$.*

Proof (Sketch). From map $h\colon R_t \to \bigcup_{s \le t} M_s$, we know $U_a^t(\tau)$ is well defined. Now, for any $i \in U_a^t(\tau)$, by definition, i is unmatched under $\tau_i^{t_i}$. Let ℓ_i denote the position where agent i sits in the order of agents of DRTMHA under $\tau_i^{t_i}$ (we call it a position value). We can partition $U_a^t(\tau)$ into different groups such that different groups have different position values ℓ_i's, and agents in the same group have the same position value. That is, partition set $U_a^t(\tau)$ into n_3 such groups $\{T_s\}_{s \in [n_3]}$, where T_s has position value ℓ_s.

Note that if i is unmatched when his position value is ℓ_i, then he will remain unmatched when he is in any position $\ell > \ell_i$, by Lemma 11 (part (iii) of Property $\mathcal{P}$). We now use Lemma 12 iteratively on set $\bigcup_{j \le s} T_j$, $s \in [n_3]$ to prove Lemma 13 as follows: suppose we have constructed a map $g_{a\tau}$ on set $\bigcup_{j \le s} T_j$ satisfying the property of Lemma 13. Then by Lemma 12, there exists an injective map $f_{a\tau}^{\ell_{s+1}}\colon \bigcup_{j \le s+1} T_j \to S_a^{\ell_{s+1}}(\tau)$, satisfying the property of the lemma. Combining the two maps $g_{a\tau}$ and $f_{a\tau}^{\ell_{s+1}}$, we obtain the map on $\bigcup_{j \le s+1} T_j$. ◀

Next we define a map H from $\bigcup_{t \in [\kappa]} M_t$ to $2^{\bigcup_{t \in [\kappa]} Q_t}$ to bound the marginal 'bad' events $\bigcup_{t \in [\kappa]} M_t$ by 'good' events $\bigcup_{t \in [\kappa]} Q_t$ as follows: For any $t \in [\kappa]$, $(\tau, t, i) \in M_t$,

$$H(\tau, t, i) = \{(\tau_i^s, p_i^s, g_{a\tau_i^s}(i)) \mid a = \mu^*(i) \,\text{and}\, p_i^s = \tau_i^s(g_{a\tau_i^s}(i)) \,\text{and}\, s \in [\kappa]\}.$$

Note that for any $(\tau, t, i) \in M_t$, $|H(\tau, t, i)| = \kappa$. Due to the injectivity of $g_{a\tau}$ (Lemma 13), we will show that the image sets of H are disjoint.

▶ **Lemma 14.** *For any $(\tau, t, i) \in M_t$ and $(\tau', t', i') \in M_{t'}$, if $(\tau, t, i) \ne (\tau', t', i')$, then $H(\tau, t, i) \cap H(\tau', t', i') = \emptyset$.*

Proof. Suppose there is a triple $(\sigma, s, \ell) \in H(\tau, t, i) \cap H(\tau', t', i')$. Then there exists $a \in A$ s.t. $\mu^*(i) = \mu^*(i') = a$, by the definition of $g_{a\sigma}$ (since $S_a(\sigma) \cap S_b(\sigma) = \emptyset$, for any $a \ne b$). By the definition of H, we know that $\tau = \sigma_i^t$ and $\tau' = \sigma_{i'}^{t'}$. Hence, $i, i' \in U_a(\sigma)$ since $(\tau, t, i) \in M_t$, $(\tau', t', i') \in M_{t'}$. As $g_{a\sigma}$ is an injective function from $U_a(\sigma)$ to $S_a(\sigma)$, $g_{a\sigma}(i) = g_{a\sigma}(i')$, implies $i = i'$. As $\tau = \sigma_i^t$, $\tau' = \sigma_i^{t'}$, there exists a unique t such that $(\sigma_i^t, t, i) \in M_t$, which gives that $t = t'$ and $\tau = \sigma_i^t = \sigma_i^{t'} = \tau'$. This shows $(\tau, t, i) = (\tau', t', i')$, and thus a contradiction. ◀

By Lemmas 13 and 14, we have for any $(\tau, t, i) \in M_t$, $t \in [k]$, and any $(\sigma, s, i') \in H(\tau, t, i)$, $w_i\psi(t) \le w_{i'}\psi(s)$ and $(\sigma, s, i') \in \bigcup_{\ell \le k} Q_\ell$. Summing over M_t: $\psi(t)w(M_t) =$

$\sum_{(\tau,t,i)\in M_t} w_i\psi(t) \leq \frac{\sum_{(\sigma,s,i')\in H(M_t)} w_{i'}\psi(s)}{\kappa}$. Therefore:

$$\begin{aligned}
\sum_{t\leq\kappa}\psi(t)w(M_t) &= \sum_{t\leq\kappa}\sum_{(\tau,t,i)\in M_t} w_i\psi(t)\\
&\leq \frac{\sum_{t\leq\kappa}\sum_{(\sigma,s,i')\in H(M_t)} w_{i'}\psi(s)}{\kappa}\\
&= \frac{\sum_{(\sigma,s,i')\in\bigcup_{t\leq\kappa} H(M_t)} w_{i'}\psi(s)}{\kappa}\\
&\leq \frac{\sum_{(\sigma,s,i')\in\bigcup_{t\leq\kappa} Q_t} w_{i'}\psi(s)}{\kappa}\\
&= \frac{\sum_{t\leq\kappa}\sum_{(\tau,t,i)\in Q_t} w_i\psi(t)}{\kappa}\\
&= \frac{\sum_{t\leq\kappa}\psi(t)w(Q_t)}{\kappa}.
\end{aligned}$$

Dividing both sides of this inequality by κ^{n_1} implies $\sum_{t\leq\kappa}\psi(t)y_t \leq \frac{\sum_{t\leq\kappa} x_t\psi(t)}{\kappa}$. Combining this inequality with (1) and (2), and following a step in the analysis from [4] implies.

▶ **Theorem 15.** *DRTMHA is universally truthful and has an approximation ratio* $\frac{e}{e-1}+O(\frac{1}{\kappa})$ *for weighted agents with ties and terminates in* $O(n_1^4\gamma\log\kappa)$ *time, for any* $\kappa\in\mathbb{N}_+$*.*

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