Bounds on the Voter Model in Dynamic Networks*

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Abstract

In the *voter model*, each node of a graph has an opinion, and in every round each node chooses independently a random neighbour and adopts its opinion. We are interested in the *consensus time*, which is the first point in time where all nodes have the same opinion. We consider dynamic graphs in which the edges are rewired in every round (by an adversary) giving rise to the graph sequence $G_1, G_2, \ldots$, where we assume that G_i has conductance at least ϕ_i. We assume that the degrees of nodes don't change over time as one can show that the consensus time can become super-exponential otherwise. In the case of a sequence of d-regular graphs, we obtain asymptotically tight results. Even for some static graphs, such as the cycle, our results improve the state of the art. Here we show that the expected number of rounds until all nodes have the same opinion is bounded by $O(m/(d_{min} \cdot \phi))$, for any graph with m edges, conductance ϕ, and degrees at least d_{min}. In addition, we consider a *biased* dynamic voter model, where each opinion i is associated with a probability P_i, and when a node chooses a neighbour with that opinion, it adopts opinion i with probability P_i (otherwise the node keeps its current opinion). We show for any regular dynamic graph, that if there is an $\epsilon > 0$ difference between the highest and second highest opinion probabilities, and at least $\Omega(\log n)$ nodes have initially the opinion with the highest probability, then all nodes adopt w.h.p. that opinion. We obtain a bound on the convergence time, which becomes $O(\log n/\phi)$ for static graphs.

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1 Introduction

In this paper, we investigate the spread of opinions in a connected and undirected graph using the *voter model*. The standard voter model works in synchronous rounds and is defined as follows. At the beginning, every node has one opinion from the set $\{0, \ldots, n-1\}$, and

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in every round, each node chooses one of its neighbours uniformly at random and adopts its opinion. In this model, one is usually interested in the *consensus time* and the *fixation probability*. The consensus time is the number of rounds it takes until all nodes have the same opinion. The fixation probability of opinion i is the probability that this opinion prevails, meaning that all other opinions vanish. This probability is known to be proportional to the sum of the degrees of the nodes starting with opinion i [13, 22].

The voter model is the dual of the *coalescing random walk model* which can be described as follows. Initially, there is a pebble on every node of the graph. In every round, every pebble chooses a neighbour uniformly at random and moves to that node. Whenever two or more pebbles meet at the same node, they are merged into a single pebble which continues performing a random walk. The process terminates when only one pebble remains. The time it takes until only one pebble remains is called *coalescing time*. It is known that the coalescing time for a graph G equals the consensus time of the voter model on G when initially each node has a distinct opinion [2, 19].

In this paper we consider the voter model and a *biased* variant where the opinions have different popularity. We express the consensus time as a function of the graph *conductance* ϕ.

We assume a dynamic graph model where the edges of the graph can be rewired by an adversary in every round, as long as the adversary respects the given degree sequence and the given conductance for all generated graphs. We show that consensus is reached with constant probability after τ rounds, where τ is the first round such that the sum of conductances up to round τ is at least m/d_{min}, where m is the number of edges. For static graphs the above bound simplifies to $O(m/(d_{min} \cdot \phi))$, where d_{min} is the minimum degree.

For the biased model we assume a *regular* dynamic graph G. Similar to [19, 16] the opinions have a *popularity*, which is expressed as a probability with which nodes adopt opinions. Again, every node chooses one of its neighbours uniformly at random, but this time it adopts the neighbour's opinion with a probability that equals the popularity of this opinion (otherwise the node keeps its current opinion). We assume that the popularity of the most popular opinion is 1, and every other opinion has a popularity of at most $1 - \epsilon$ (for an arbitrarily small but constant $\epsilon > 0$). We also assume that at least $\Omega(\log n)$ nodes start with the most popular opinion. Then we show that the most popular opinion prevails w.h.p.[1] after τ rounds, where τ is the first round such that the sum of conductances up to round τ is of order $O(\log n)$. For static graphs the above bound simplifies as follows: the most popular opinion prevails w.h.p. in $O(\log n/\phi)$ rounds, if at least $\Omega(\log n)$ nodes start with that opinion.

1.1 Related work

A sequential version of the voter model was introduced in [14] and can be described as follows. In every round, a single node is chosen uniformly at random and this node changes its opinion to that of a random neighbour. The authors of [14] study infinite grid graphs. This was generalised to arbitrary graphs in [9] where it is shown among other things that the probability for opinion i to prevail is proportional to the sum of the degrees of the nodes having opinion i at the beginning of the process.

The standard voter model was first analysed in [13]. The authors of [13] bound the expected coalescing time (and thus the expected consensus time) in terms of the expected meeting time t_{meet} of two random walks and show a bound of $O(t_{meet} \cdot \log n) = O(n^3 \log n)$.

[1] An event happens *with high probability (w.h.p.)* if its probability is at least $1 - 1/n$.

Note that the meeting time is an obvious lower bound on the coalescing time, and thus a lower bound on the consensus time when all nodes have distinct opinions initially. The authors of [4] provide an improved upper bound of $O\big(\frac{1}{1-\lambda_2}(\log^4 n + \rho)\big)$ on the expected coalescing time for any graph G, where λ_2 is the second eigenvalue of the transition matrix of a random walk on G, and $\rho = \big(\sum_{u\in V(G)} d(u)\big)^2 / \sum_{u\in V(G)} d^2(u)$ is the ratio of the square of the sum of node degrees over the sum of the squared degrees. The value of ρ ranges from $\Theta(1)$, for the star graph, to n, for regular graphs.

The authors of [20, 2, 19] investigate coalescing random walks in a continuous setting where the movement of the pebbles are modelled by independent Poisson processes with a rate of 1. In [2], it is shown a lower bound of $\Omega(m/d_{max})$ and an upper bound of $O(t_{hit}\cdot\log n)$ for the expected coalescing time. Here m is the number of edges in the graph, d_{max} is the maximum degree, and t_{hit} is the (expected) hitting time. In [23], it is shown that the expected coalescing time is bounded by $O(t_{hit})$.

In[19] the authors consider the biased voter model in the continuous setting and two opinions. They show that for d-dimensional lattices the probability for the less popular opinion to prevail is exponentially small. In [16], it is shown that in this setting the expected consensus time is exponential for the line.

The authors of [5] consider a modification of the standard voter model with two opinions, which they call *two-sample voting*. In every round, each node chooses two of its neighbours randomly and adopts their opinion only if they both agree. For regular graphs and random regular graphs, it is shown that two-sample voting has a consensus time of $O(\log n)$ if the initial imbalance between the nodes having the two opinions is large enough. There are several other works on the setting where every node contacts in every round two or more neighbours before adapting its opinion [1, 7, 6, 10].

There are several other models which are related to the voter model, most notably the *Moran process* and *rumor spreading* in the phone call model. In the case of the Moran process, a population resides on the vertices of a graph. The initial population consists of one mutant with fitness r and the rest of the nodes are non-mutants with fitness 1. In every round, a node is chosen at random with probability proportional to its fitness. This node then reproduces by placing a copy of itself on a randomly chosen neighbour, replacing the individual that was there. The main quantities of interest are the probability that the mutant occupies the whole graph (fixation) or vanishes (extinction), together with the time before either of the two states is reached (absorption time). There are several publications considering the fixation probabilities [15, 21, 8].

Rumor spreading in the phone call model works as follows. Every node v opens a channel to a randomly chosen neighbour u. The channel can be used for transmissions in both directions. A transmission from v to u is called *push* transmission and a transmission from u to v is called *pull*. There is a vast amount of papers analysing rumor spreading on different graphs. The result that is most relevant to ours is that broadcasting of a message in the whole network is completed in $O(\log n/\phi)$ rounds w.h.p, where ϕ is the conductance (see Section 1.2 for a definition) of the network. In [12], the authors study rumor spreading in dynamic networks, where the edges in every round are distributed by an adaptive adversary. They show that broadcasting terminates w.h.p. in a round t if the sum of conductances up to round t is of order $\log n$. Here, the sequence of graphs $G_1, G_2, \ldots$ have the same vertex set of size n, but possibly distinct edge sets. The authors assume that the degrees and the conductance may change over time. We refer the reader to the next section for a discussion of the differences. Dynamic graphs have received ample attention in various areas [3, 17, 24, 18].

1.2 Model and New Results

In this paper we show results for the standard voter model and biased voter model in dynamic graphs. Our protocols work in synchronous steps. The consensus time T is defined at the first time step at which all nodes have the same opinion.

Standard Voter Model

Our first result concerns the standard voter model in dynamic graphs. Our protocol works as follows. In every synchronous time step every node chooses a neighbour u.a.r. and adopts its opinion with probability $1/2$.[2]

We assume that the dynamic graphs $\mathscr{G} = G_1, G_2, \ldots$ are generated by an adversary. We assume that each graph has n nodes and the nodes are numbered from 1 to n. The sequence of conductances $\phi_1, \phi_2, \ldots$ is given in advance, as well as a degree sequence $d_1, d_2, \ldots, d_n$. The adversary is now allowed to create every graph G_i by redistributing the edges of the graph. The constraints are that each graph G_i has to have conductance ϕ_i and node j has to have degree d_j (the degrees of the nodes do not change over time). Note that the sequence of the conductances is fixed and, hence, cannot be regarded as a random variable in the following. For the redistribution of the edges we assume that the adversary knows the distribution of all opinions during all previous rounds.

Note that our model for dynamic graphs is motivated by the model presented in [12]. They allow the adversary to determine the edge set at every round, without having to respect the node degrees and conductances.

We show (Observation 1) that, allowing the adversary to change the node degrees over time can results in super-exponential voting time. Since this changes the behaviour significantly, we assume that the degrees of nodes are fixed. Furthermore, in contrary to [12], we assume that (bounds on) the conductance of (the graph at any time step) are fixed/given beforehand. Whether one can obtain the same results, if the conductance of the graph is determined by an adaptive adversary remains an open question. The reason we consider an adversarial dynamic graph model is in order to understand how the voting time can be influenced in the worst-case. Another interesting model would be to assume that in every round the nodes are connected to random neighbours. One obstacle to such a model seems to be to guarantee that neighbours are chosen u.a.r. and the degrees of nodes do not change. For the case of regular random dynamic graphs our techniques easily carry over since the graph will have constant conductance w.h.p. in any such round since the graph is essentially a random regular graph in every round.

For the (adversarial) dynamic model we show the following result bounding the consensus time T.

▶ **Theorem 1** (upper bound). *Consider the Standard Voter model and in the dynamic graph model. Assume $\kappa \leq n$ opinions are arbitrarily distributed over the nodes of G_1. Let ϕ_t be a lower bound on the conductance at time step t. Let $b > 0$ be a suitable chosen constant. Then, with a probability of $1/2$ we have that $T \leq \min\{\tau, \tau'\}$, where*

(i) *τ is the first round so that $\sum_{t=1}^{\tau} \phi_t \geq b \cdot m/d_{min}$.* (part 1)

(ii) *τ' is the first round so that $\sum_{t=1}^{\tau'} \phi_t^2 \geq b \cdot n \log n$.* (part 2)

For static graphs ($G_{i+1} = G_i$ for all i), we have $T \leq \min\{m/(d_{min} \cdot \phi), n \log n/\phi^2\}$.

[2] The factor of $1/2$ ensures that the process converges on bipartite graphs.

For *static* d-regular graphs, where the graph doesn't change over time, the above bound becomes $O(n/\phi)$, which is tight when either ϕ or d are constants (see Observation 2). Theorem 1 gives the first tight bounds for cycles and circulant graphs C_n^k (node i is adjacent to the nodes $i \pm 1, \ldots, i \pm k \mod n$) with degree $2k$ (k constant). For these graphs the consensus time is $\Theta(n^2)$, which matches our upper bound from Theorem 1.[3] For a comparison with the results of [4] note that $\phi^2 \leq 1 - \lambda_2 \leq 2\phi$. In particular, for the cycle $\phi = 1/n$ and $1/(1-\lambda_2) = \Theta(1/n^2)$. Hence, for this graph, our bound is by a factor of n smaller. Note that, due to the duality between the voter model and coalescing random walks, the result also holds for the coalescing time. In contrast to [4, 5], the above result is shown using a potential function argument, whereas the authors of [4, 5] show their results for coalescing random walks and fixed graphs. The advantage of analysing the process directly is, that our techniques allow us to obtain the results for the dynamic setting.

The next result shows that the bound of Theorem 1 is asymptotically tight if the adversary is allowed to change the node degrees over time.

▶ **Theorem 2** (lower bound). *Consider the Standard Voter model in the dynamic graph model. Assume that $\kappa \leq n$ opinions are arbitrarily distributed over the nodes of G_1. Let ϕ_t be an upper bound on the conductance at time step t. Let $b > 0$ be a suitable constant and assume τ'' is the first round such that $\sum_{t=1}^{\tau''} \phi_t \geq bn$. Then, with a probability of at least $1/2$, there are still nodes with different opinions in $G_{\tau''}$.*

Biased Voter Model

In the *biased* voter model we again assume that there are $\kappa \leq n$ distinct opinions initially. For $0 \leq i \leq \kappa - 1$, opinion i has popularity α_i and we assume that $\alpha_0 = 1 > \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{\kappa-1}$. We call opinion 0 the *preferred opinion*. The process works as follows. In every round, every node chooses a neighbour uniformly at random and adopts its opinion i with probability α_i.

We assume that the dynamic d-regular graphs $\mathscr{G} = G_1, G_2, \ldots$ are generated by an adversary. We assume that the sequence of ϕ_t is given in advance, where ϕ_i is a lower bound on the conductance of G_i. The adversary is now allowed to create the sequence of graphs by redistributing the edges of the graph in every step. The constraints are that each graph G_i has n nodes and has to have conductance at least ϕ_i. Note that we assume that the sequence of the conductances is fixed and, hence, it is not a random variable in the following.

The following result shows that consensus is reached considerably faster in the biased voter model, as long as the bias $1 - \alpha_1$ is bounded away from 0, and at least a logarithmic number of nodes have the preferred opinion initially.

▶ **Theorem 3.** *Consider the Biased Voter model in the dynamic regular graph model. Assume $\kappa \leq n$ opinions are arbitrarily distributed over the nodes of G_1. Let ϕ_t be a lower bound on the conductance at time step t. Assume that $\alpha_1 \leq 1 - \epsilon$, for an arbitrary small constant $\epsilon > 0$. Assume the initial number of nodes with the preferred opinion is at least $c \log n$, for some constant $c = c(\alpha_1)$. Then the preferred opinion prevails w.h.p. in at most τ''' steps, where τ''' is the first round so that $\sum_{t=1}^{\tau'''} \phi_t \geq b \log n$, for some constant b. For static graphs ($G_{i+1} = G_i$ for all i), we have w.h.p. $T = O(\log n/\phi)$.*

[3] The lower bound of $\Omega(n^2)$ follows from the fact that two coalescing random walks starting on opposite sites of a cycle require in expectation time $\Omega(n^2)$ to meet.

The assumption on the initial size of the preferred opinion is crucial for the time bound $T = O(\log n/\phi)$, in the sense that there are instances where the expected consensus time is at least $T = \Omega(n/\phi)$ if the size of the preferred opinion is small.[4]

The rumor spreading process can be viewed as an instance of the biased voter model with two opinions having popularity 1 and 0, respectively. However, the techniques used for the analysis of rumor spreading do not extend to the voter model. This is due to the fact that rumor spreading is a progressive process, where nodes can change their opinion only once, from "uninformed" to "informed", whereas they can change their opinions over and over again in the case of the voter model. Note that the above bound is the same as the bound for rumor spreading of [11] (although the latter bound holds for general graphs, rather than just for regular ones). Hence, our above bound is tight for regular graphs with conductance ϕ, since the rumor spreading lower bound of $\Omega(\log n/\phi)$ is also a lower bound for biased voting in our model.

2 Analysis of the Voter Model

In this section we show the upper and lower bound for the standard voter model. We begin with some definitions. Let $G = (V, E)$. For a fixed set $S \subseteq V$ we define $cut(S, V \setminus S)$ to be the set of edges between the sets $S \subseteq V$ and $V \setminus S$ and let λ_u be the number of neighbours of u in $V \setminus S$. Let $\mathrm{vol}(S) = \sum_{u \in S} d_u$. The *conductance* of G is defined as

$$\phi = \phi(G) = \min \left\{ \frac{\sum_{u \in U} \lambda_u}{\mathrm{vol}(U)} : U \subset V \text{ with } 0 < \mathrm{vol}(U) \leq m \right\}.$$

We note $1/n^2 \leq \phi \leq 1$. We denote by $v_t^{(i)}$ the set of nodes that have opinion i after the first t rounds and $t \geq 0$. If we refer to the random variable we use $V_t^{(i)}$ instead.

First we show Theorem 1 for $\kappa = 2$ (two opinions), which we call 0 and 1 in the following. Then we generalise the result to an arbitrary number of opinions. We model the system with a Markov chain $M_{t \geq 0} = (V_t^{(0)}, V_t^{(1)})_{t \geq 0}$.

Let s_t denote the set having the smaller volume, i.e., $s_t = v_t^{(0)}$ if $\mathrm{vol}(v_t^{(0)}) \leq \mathrm{vol}(v_t^{(1)})$, and $s_t = v_t^{(1)}$ otherwise. Note that we use $s_t, v_t^{(0)}$ and $v_t^{(1)}$ whenever the state at time t is fixed, and $S_t, V_t^{(0)}$ and $V_t^{(1)}$ for the corresponding random variables. For $u \in v_t^{(0)}$, $\lambda_{u,t}$ is the number of neighbours of u in $V \setminus v^1(t)$ and for $u \in v_t^{(1)}$, $\lambda_{u,t}$ is the number of neighbours of u in $V \setminus v_t^{(0)}$; d_u is the degree of u (the degrees do not change over time).

To analyse the process we use a potential function. Simply using the volume of nodes sharing the same opinion as the potential function will not work. It is easy to calculate that the expected volume of nodes with a given opinion does not change in one step. Instead, we use a convex function on the number of nodes with the minority opinion. We define

$$\Psi(S_t) = \sqrt{\mathrm{vol}(S_t)}.$$

In Lemma 4 we first calculate the one-step potential drop of $\Psi(S_t)$. Then we show that every opinion either prevails or vanishes once the sum of conductances is proportional to the volume of nodes having that opinion (see Lemma 5), which we use later to prove Part 1 of Theorem 1. Additionally, Lemma 4 is used to prove Lemma 7 which allows us to prove Part 2 of Theorem 1.

[4] Consider a 3-regular graph and n opinions where all other $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1/2$. The preferred opinion vanishes with constant probability and the bound for the standard voter model of Observation 2 applies.

▶ **Lemma 4.** *Assume $s_t \neq \emptyset$ and $\kappa = 2$. Then*

$$\mathbf{E}[\Psi(S_{t+1}) \mid S_t = s_t] \leq \Psi(s_t) - \frac{\sum_{u \in V} \lambda_{u,t} \cdot d_u}{32 \cdot (\Psi(s_t))^3}.$$

Proof. W.l.o.g. we assume that opinion 0 is the minority opinion, i.e. $0 < \mathrm{vol}(V_t^{(0)}) \leq \mathrm{vol}(V_t^{(1)})$. To simplify the notation we omit the index t in this proof and write $v^{(0)}$ instead of $v_t(0)$, $v^{(1)}$ for $V \setminus v_t^{(0)}$, and λ_u instead of $\lambda_{u,t}$. Hence, $s_t = v^{(0)}$ and $\Psi(s_t) = \sqrt{\mathrm{vol}(v^{(0)})}$. Note that for $t = 0$ we have $\mathrm{vol}(v^{(0)}) = \Psi(s_t)^2$. Furthermore, we fix $S_t = s_t$ in the following (and condition on it). We define m as the number of edges. Then we have

$$\begin{aligned}
\mathbf{E}[\Psi(S_{t+1}) - \Psi(s_t) \mid S_t = s_t] &= \mathbf{E}[\sqrt{\mathrm{vol}(S_{t+1})} - \sqrt{\mathrm{vol}(s_t)}] \\
&= \mathbf{E}\left[\sqrt{\min\left\{\mathrm{vol}(V_{t+1}^{(0)}), m - \mathrm{vol}(V_{t+1}^{(0)})\right\}} - \sqrt{\mathrm{vol}(s_t)}\right] \\
&\leq \mathbf{E}\left[\sqrt{\mathrm{vol}(V_{t+1}^{(0)})} - \sqrt{\mathrm{vol}(v^{(0)})}\right] \qquad (1)
\end{aligned}$$

Now we define

$$X_u = \begin{cases} d_u & w.p.\ \frac{\lambda_u}{2 \cdot d_u}\ if\ u \in v^{(1)} \\ -d_u & w.p.\ \frac{\lambda_u}{2 \cdot d_u}\ if\ u \in v^{(0)} \\ 0 & otherwise \end{cases}$$

and $\Delta = \sum_{u \in V} X_u$. Note that we have $\Delta = \mathrm{vol}(V_{t+1}^{(0)}) - \mathrm{vol}(v^{(0)})$ and

$$\begin{aligned}
\mathbf{E}\left[\sqrt{\mathrm{vol}(V_{t+1}^{(0)})} - \sqrt{\mathrm{vol}(v^{(0)})}\right] &= \mathbf{E}\left[\sqrt{\mathrm{vol}(v^{(0)}) + \Delta} - \sqrt{\mathrm{vol}(v^{(0)})}\right] \\
&= \mathbf{E}\left[\sqrt{\mathrm{vol}(v^{(0)})}\left(\sqrt{1 + \tfrac{\Delta}{\mathrm{vol}(v^{(0)})}} - 1\right)\right] \\
&= \Psi(s_t) \cdot \mathbf{E}[\sqrt{1 + \Delta/\Psi(s_t)^2} - 1].
\end{aligned}$$

Unfortunately we cannot bound $\Psi(s_t) \cdot \mathbf{E}[\sqrt{1 + \Delta/\Psi(s_t)^2} - 1]$ directly. Instead, we define a family of random variables which is closely related to X_u.

$$Y_u = \begin{cases} \lambda_u & w.p.\ \frac{1}{2} \quad if\ u \in v^{(1)} \\ -d_u & w.p.\ \frac{\lambda_u}{2 \cdot d_u} \quad if\ u \in v^{(0)} \\ 0 & otherwise \end{cases}$$

Similarly, we define $\Delta' = \sum_{u \in V} Y(u)$. Note that $|E[Y_u]| = \lambda_u/2$ for both $u \in v^{(1)}$ and $u \in v^{(0)}$. In the full version we show that $\mathbf{E}[\sqrt{1 + \Delta/\Psi(s_t)^2}] \leq \mathbf{E}[\sqrt{1 + \Delta'/\Psi(s_t)^2}]$, which results in $\mathbf{E}[\Psi(S_{t+1}) - \Psi(s_t) \mid S_t = s_t] \leq \Psi(s_t) \cdot \mathbf{E}[\sqrt{1 + \Delta'/\Psi(s_t)^2} - 1]$ From the Taylor expansion $\sqrt{1+x} \leq 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$, $x \geq -1$ it follows that

$$\mathbf{E}[\Psi(S_{t+1}) - \Psi(s_t) \mid S_t = s_t] \quad \leq \quad \Psi(s_t) \cdot \mathbf{E}\left[\tfrac{\Delta'}{2\Psi(s_t)^2} - \tfrac{(\Delta')^2}{8\Psi(s_t)^4} + \tfrac{(\Delta')^3}{16\Psi(s_t)^6}\right].$$

It remains to bound $\mathbf{E}[\Delta']$, $\mathbf{E}[(\Delta')^2]$, and $\mathbf{E}[(\Delta')^3]$.

- $\mathbf{E}[\Delta']$: We have $\mathbf{E}[\Delta'] = \sum_{u \in V} E[Y_u] = \sum_{u \in v^{(1)}} \frac{\lambda_u}{2} - \sum_{v \in v^{(0)}} \frac{\lambda_v}{2} = 0$, where the last equality holds since $\sum_{u \in v^{(1)}} \lambda_u$ and $\sum_{u \in v^{(1)}} \lambda_u$ both count the number of edges crossing the cut between $v^{(0)}$ and $v^{(1)}$.

- $\mathbf{E}[(\Delta')^2]$: since $E[(Y_u)^2] = (\lambda_u)^2/2$ for $u \in v^{(1)}$ and $E[(Y_u)^2] = -d_u \cdot \lambda_u/2$ for $u \in v^{(0)}$ we have

$$\begin{aligned}\mathbf{E}[(\Delta')^2] &= \sum_{u\in V} \mathbf{Var}[Y_u] + (\mathbf{E}[Y_u])^2 = \sum_{u\in V} \mathbf{Var}[Y_u] + 0 = \sum_{u\in V} (\mathbf{E}[(Y_u)^2] - (\mathbf{E}[Y_u])^2) \\ &= \sum_{u\in v^{(0)}} (\mathbf{E}[(Y_u)^2] - (\mathbf{E}[Y_u])^2) + \sum_{u\in v^{(1)}} (\mathbf{E}[(Y_u)^2] - (\mathbf{E}[Y_u])^2) \\ &= \sum_{u\in v^{(0)}} \frac{\lambda_u d_u}{2} - \sum_{u\in v^{(0)}} \frac{\lambda_u^2}{4} + \sum_{u\in v^{(1)}} \frac{\lambda_u^2}{4} \geq \sum_{u\in v^{(0)}} \frac{\lambda_u d_u}{4}. \end{aligned} \tag{2}$$

- $\mathbf{E}[\Delta'^3]$: In the full version we show that $\mathbf{E}[\Delta'^3] = \sum_{u\in V} \left(\mathbf{E}[(Y_u)^3] - 3\,\mathbf{E}[(Y_u)^2] \cdot \mathbf{E}[Y_u] + 2\,\mathbf{E}[Y_u]^3\right)$. Note that $E[(Y_u)^3] = \frac{1}{2}(\lambda_u)^3$ for $u \in v^{(1)}$ and $E[(Y_u)^3] = -\frac{1}{2}\lambda_u \cdot (d_u)^2$ for $u \in v^{(0)}$. Hence,

$$\begin{aligned}\mathbf{E}[\Delta'^3] = &\sum_{u\in v^{(0)}} \left(-\frac{1}{2}\lambda_u \cdot (d_u)^2 + \frac{3}{4}(\lambda_u)^2 \cdot d_u - \frac{1}{4}\lambda_u^3\right) \\ &+ \sum_{u\in v^{(1)}} \left(\frac{1}{2}(\lambda_u)^3 - \frac{3}{4}(\lambda_u)^3 + \frac{1}{4}(\lambda_u)^3\right) \leq 0, \end{aligned} \tag{3}$$

  where the first sum is bounded by 0 because $\lambda_u \leq d_u$.

  Combining all the above estimations we get

$$\begin{aligned}\mathbf{E}[\Psi(S_{t+1}) - \Psi(s_t) \mid S_t = s_t] &\leq \Psi(s_t) \cdot \mathbf{E}\left[\frac{\Delta'}{2\Psi(s_t)^2} - \frac{\Delta'^2}{8\Psi(s_t)^4} + \frac{\Delta'^3}{16\Psi(s_t)^6}\right] \\ &\leq -\frac{\sum_{u\in v^{(0)}} \lambda_u d_u}{32\Psi(s_t)^3}.\end{aligned}$$

This completes the proof of Lemma 4. ◀

2.1 Part 1 of Theorem 1.

Using Lemma 4 we show that a given opinion either prevails or vanishes with constant probability as soon as the sum of ϕ_t is proportional to the volume of the nodes having that opinion.

▶ **Lemma 5.** *Assume that $s_{\hat{t}}$ is fixed for an arbitrary $(\hat{t} \geq 0)$ and $\kappa = 2$.*
Let $\tau^* = \min\left\{t' : \sum_{i=\hat{t}}^{t'} \phi_i \geq 129 \cdot vol(s_{\hat{t}})/d_{min}\right\}$. Then $\Pr\left(T \leq \tau^* + \hat{t}\right) \geq 1/2$.

In particular, if the graph is static with conductance ϕ, then $\Pr\left(T \leq \frac{129 \cdot vol(s_{\hat{t}})}{\phi \cdot d_{min}} + \hat{t}\right) \geq 1/2$.

Proof. From the definition of $\Psi(s_t)$ and ϕ_t it follows for all t that $\Psi(s_t)^2 = \sum_{u\in v^{(0)}} d_u = \mathrm{vol}(v^{(0)})$ and $\phi_t \leq \sum_{u\in v^{(0)}} \lambda_{u,t}/\mathrm{vol}(v^{(0)})$. Hence, $\Psi(s_t)^2 \cdot \phi_t \cdot d_{min} \leq \sum_{u\in v^{(0)}} \lambda_{u,t} \cdot d_u$. Together with Lemma 4 we derive for $s_t \neq \emptyset$

$$\mathbf{E}[\Psi(S_{t+1}) \mid S_t = s_t] \leq \Psi(s_t) - \frac{\sum_{u\in V} \lambda_{u,t} \cdot d_u}{32 \cdot (\Psi(s_t))^3} \leq \Psi(s_t) - \frac{d_{min} \cdot \phi_t}{32 \cdot \Psi(s_t)}. \tag{4}$$

Recall that $T = \min_t\{S_t = \emptyset\}$. In the following we use the expression $T > t$ to denote the event $s_t \neq \emptyset$. Using the law of total probability we get

$$\mathbf{E}[\Psi(S_{t+1}) | T > t] = \mathbf{E}\left[\Psi(S_t) - \frac{d_{min} \cdot \phi_t}{32 \cdot \Psi(S_t)} \Big| T > t\right]$$

and using Jensen's inequality we get

$$\begin{aligned}\mathbf{E}[\Psi(S_{t+1}) \mid T > t] &= \mathbf{E}[\Psi(S_t) \mid T > t] - \mathbf{E}\left[\frac{d_{min} \cdot \phi_t}{32 \cdot \Psi(S_t)} \mid T > t\right] \\ &\leq \mathbf{E}[\Psi(S_t) \mid T > t] - \frac{d_{min} \cdot \phi_t}{32 \cdot \mathbf{E}[\Psi(S_t) \cdot \mid T > t]}.\end{aligned}$$

Since $\mathbf{E}[\Psi(S_t) \mid T \leq t] = 0$ we have

$$\begin{aligned}\mathbf{E}[\Psi(S_t)] &= \mathbf{E}[\Psi(S_t) \mid T > t] \cdot \Pr[T > t] + \mathbf{E}[\Psi(S_t) \mid T \leq t] \cdot \Pr[T \leq t] \\ &= \mathbf{E}[\Psi(S_t) \mid T > t] \cdot \Pr[T > t] + 0.\end{aligned}$$

Hence,

$$\frac{\mathbf{E}[\Psi(S_{t+1})]}{\Pr(T > t)} \leq \frac{\mathbf{E}[\Psi(S_t)]}{\Pr(T > t)} - \frac{d_{min} \cdot \phi_t \cdot \Pr(T > t)}{32\,\mathbf{E}[\Psi(S_t)]}$$

and

$$\mathbf{E}[\Psi(S_{t+1})] \leq \mathbf{E}[\Psi(S_t)] - \frac{d_{min} \cdot \phi_t \cdot (\Pr(T > t))^2}{32\,\mathbf{E}[\Psi(S_t)]}.$$

Let $t^* = \min\{t\colon \Pr(T > t) < 1/2\}$. In the following we use contradiction to show

$$t^* \leq \max\{t : \sum_{\hat{t} \leq t < t^*} \phi_t \leq 128 \cdot \mathrm{vol}(s_{\hat{t}})/d_{min}\}.$$

Assume the inequality is not satisfied. With $t = t^* - 1$ we get

$$\mathbf{E}[\Psi(S_{t^*})] \leq \mathbf{E}[\Psi(S_{t^*-1})] - \frac{d_{min} \cdot \phi_t \cdot (\Pr(T > t^* - 1))^2}{32\,\mathbf{E}[\Psi(S_{t^*-1})]} \leq \mathbf{E}[\Psi(S_{t^*-1})] - \frac{d_{min} \cdot \phi_{t^*} \cdot (1/4)}{32\,\mathbf{E}[\Psi(S_{t^*-1})]}.$$

Applying this equation iteratively, we obtain

$$\mathbf{E}[\Psi(S_{t^*})] \leq \mathbf{E}[\Psi(S_{\hat{t}})] - \sum_{\hat{t} \leq t < t^*} \frac{d_{min} \cdot \phi_t \cdot 1/4}{32\,\mathbf{E}[\Psi(S_t)]} \leq \mathbf{E}[\Psi(S_{\hat{t}})] - \frac{d_{min} \cdot \sum_{\hat{t} \leq t < t^*} \phi_t}{128\,\mathbf{E}[\Psi(S_{\hat{t}})]}. \tag{5}$$

Using the definition of $\mathbf{E}[\Psi(S_{\hat{t}})] = \sqrt{\mathrm{vol}(s_{\hat{t}})}$ and the definition of t^* we get

$$\mathbf{E}[\Psi(S_{t^*})] < \sqrt{\mathrm{vol}(s_{\hat{t}})} - \frac{d_{min} \cdot 128 \cdot \mathrm{vol}(s_{\hat{t}})}{128 \cdot d_{min} \cdot \sqrt{\mathrm{vol}(s_{\hat{t}})}} = \sqrt{\mathrm{vol}(s_{\hat{t}})} - \frac{\mathrm{vol}(s_{\hat{t}})}{\sqrt{\mathrm{vol}(s_{\hat{t}})}} = 0.$$

This is a contradiction since $\mathbf{E}[\Psi(S_{t^*})]$ is non-negative.

From the definition of t^*, we obtain $\Pr\left(T > \tau^* + \hat{t}\right) < 1/2$, completing the proof of Lemma 5. ◀

Now we are ready to show the first part of the theorem.

Proof of Part 1 of Theorem 1. We divide the τ rounds into phases. Phase i starts at time $\tau_i = \min\{t : \sum_{j=1}^{t} \phi_j \geq 2i\}$ for $i \geq 0$ and ends at $\tau_{i+1} - 1$. Since $\phi_j \leq 1$ for all $j \geq 0$ we have $\tau_0 < \tau_1 < \ldots$ and $\sum_{j=\tau_i}^{\tau_{i+1}} \phi_j \geq 1$ for $i \geq 0$. Let ℓ_t be the number of distinct opinions at the beginning of phase t. Hence, $\ell_0 = \kappa$.

We show in Lemma 6 below that the expected number of phases before the number of opinions drops by a factor of 5/6 is bounded by $6c \cdot \mathrm{vol}(V)/(\ell_t \cdot d_{min})$. For $i \geq 1$ let T_i be the number of phases needed so that the number of opinions drops to $(5/6)^i \cdot \ell_0$. Then only

one opinion remains after $\log_{6/5}\kappa$ many of these meta-phases. Then, for a suitably chosen constant b,

$$\mathbf{E}[T] \quad = \quad \sum_{j=1}^{\log_{6/5}\kappa} E[T_j] \leq \sum_{j=1}^{\log_{6/5}\kappa - 1} \frac{6c \cdot \mathrm{vol}(V)}{\ell_j \cdot d_{min}} \leq \sum_{j=1}^{\log_{6/5}\kappa} \frac{6c\,\mathrm{vol}(V)}{(5/6)^j \cdot \ell_0 \cdot d_{min}} = \frac{b \cdot m}{4 \cdot d_{min}}.$$

By Markov inequality, consensus is reached w.p. at least $1/2$ after $b \cdot m/(2d_{min})$ phases. By definition of τ and the definition of the phases, we have that the number of phases up to time step τ is at least $b \cdot m/(2d_{min})$. Thus, consensus is reached w.p. at least $1/2$ after τ time steps, which finishes the proof. ◀

▶ **Lemma 6.** *Fix a phase t and assume $c = 129$ and $\ell_t > 1$. The expected number of phases before the number of opinions drops to $5/6 \cdot \ell_t$ is bounded by $6c \cdot vol(V)/(\ell_t \cdot d_{min})$.*

Proof. Consider a point when there are ℓ' opinions left, with $5/6 \cdot \ell < \ell' \leq \ell$. Among those ℓ' opinions, there are at least $\ell' - \ell/3$ opinions i such that the volume of nodes with opinion i is at most $3 \cdot \mathrm{vol}(V)/\ell$. Let S denote the set of these opinions and let Z_i be an indicator variable which is 1 if opinion $i \in S$ vanished after $s = 3c \cdot \mathrm{vol}(V)/(\ell \cdot d_{min})$ phases and $Z_i = 0$ if it prevails. To estimate Z_i we consider the process where we have two opinions only. All nodes with opinion i retain their opinion and all other nodes have opinion 0. It is easy to see that in both processes the set of nodes with opinion i remains exactly the same. Hence, we can use Lemma 5 to show that with probability at least $1/2$, after s phases opinion i either vanishes or prevails. Hence,

$$\mathbf{E}\left[\Sigma_{j \in S} Z_j\right] = \Sigma_{j \in S}\, \mathbf{E}[Z_j] \geq |S|/2 \geq (\ell' - \ell/3)/2.$$

Using Markov's inequality we get that with probability $1/2$ at least $(\ell' - \ell/3)/4$ opinions vanish within s phases, and the number of opinions remaining is at most $\ell' - (\ell' - \ell/3)/4 = 3/4 \cdot \ell' + \ell/12 \leq 5/6 \cdot \ell$. The expected number of phases until $5/6 \cdot \ell$ opinions can be bounded by $\sum_{i=1}^{\infty} 2^{-i} \cdot s \leq 2s = \frac{6c \cdot \mathrm{vol}(V)}{\ell \cdot d_{min}}$. ◀

2.2 Part 2 of Theorem 1

We first bound the expected potential drop in round $t+1$, i.e., we bound $\mathbf{E}[\Psi(S_{t+1}) - \Psi(s_t) \mid S_t = s_t]$. This time however, we express the drop as a function which is linear in $\Psi(s_t)$. This allows us to bound the expected size of the potential at time τ', i.e., $\mathbf{E}[\Psi(S_{\tau'})]$, directly. From the expected size of the potential at time τ' we derive the desired bound on $Pr\,(T \leq \tau')$. The proof can be found in the full version.

▶ **Lemma 7.** *Assume $\kappa = 2$. We have $\Pr(T \leq \tau') \geq 1/n^2$. In particular, if the graph is static with conductance ϕ, then $\Pr\left(T \leq \frac{96 \cdot n \log n}{\phi^2}\right) \geq 1 - 1/n^2$.*

We now prove Part 2 of Theorem 1 which generalises to $\kappa > 2$.

Proof of Part 2 of Theorem 1. We define a parameterized version of the consensus time T. We define $T(\kappa) = \min\{t \colon \Psi(S_t) = 0 : \text{the number of different opinions at time } t \text{ is } \kappa\}$ for $\kappa \leq n$. We want to show that $\Pr(T(\kappa) \leq \tau') \geq 1 - 1/n$. From Lemma 7 we have that, that $\Pr(T(2) \leq \tau') \geq 1 - 1/n^2$. We define the 0/1 random variable Z_i to be one if opinion i vanishes or is the only remaining opinion after τ' rounds and $Z_i = 0$ otherwise. We have that $\Pr(Z_i = 1) \geq 1 - 1/n^2$ for all $i \leq \kappa$. We derive $\Pr(T(\kappa) \leq \tau') = \Pr(\wedge_{i \leq \kappa} Z_i) \geq 1 - 1/n$, by union bound. This yields the claim. ◀

2.3 Lower Bounds

In this section, we give the intuition behind the proof of Theorem 2 and state two additional observations. Recall that Theorem 2 states that our bound for regular graphs is tight for the adaptive adversary, even for $k = 2$. The first observation shows that the expected consensus time can be super-exponential if the adversary is allowed to change the degree sequence. The second observation can be regarded as a (weaker) counter part of Theorem 2 showing a lower bound of $\Omega(n/\phi)$ for static graphs, assuming that either d or ϕ is constant.

We now give the intuition behind the proof of Theorem 2 and refer the reader to the full version for the actual proof. The high level approach is as follows. For every step t we define an adaptive adversary that chooses G_{t+1} after observing $V_t^{(0)}$ and $V_t^{(1)}$. The adversary chooses G_{t+1} such that the cut between $V_t^{(0)}$ and $V_t^{(1)}$ is of order of $\Theta(\phi_t \cdot dn)$. We show that such a graph exists when the number of nodes in both $V_t^{(0)}$ and $V_t^{(1)}$ is at least of linear size (in n). By this choice the adversary ensures that the expected potential drop of $\Psi(S_{t+1})$ at most $-c\phi_t d/\Psi(s_t)$ for some constant c. Then we use the expected potential drop, together with the optional stopping theorem, to derive our lower bound.

In the following we observe that if the adversary is allowed to change the degrees, then the expected consensus time is super-exponential. A proof sketch can be found in the full version.

▶ **Observation 1.** *There is a sequence $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, ... be a sequence of graphs with n nodes, where the edges $E_1, E_2, \ldots$ are distributed by an adaptive adversary, such that the expected consensus time is at least $\Omega((n/c)^{n/c})$ for some constant c.*

The bound of Theorem 1 for static regular graphs of $O(n/\phi)$ is tight for regular graphs if either the degree or the conductance is constant. A proof sketch can be found in the full version.

▶ **Observation 2.** *For every n, $d \geq 3$, and constant ϕ, there exists a d-regular graph G with n nodes and a constant conductance such that the expected consensus time on G is $\Omega(n)$. Furthermore, for every even n, $\phi > 1/n$, and constant d, there exists a (static) d-regular graph G with $\Theta(n)$ nodes and a conductance of $\Theta(\phi)$ such that the expected consensus time on G is $\Omega(n/\phi)$.*

3 Analysis of the Biased Voter Model

In this section, we prove Theorem 3. We show that the set S_t of nodes with the preferred opinion grows roughly at a rate of $1 + \Theta(\phi_t)$, as long as S_t has at least logarithmic size. For the analysis we break each round down into several steps, where exactly one node which has at least one neighbour in the opposite set is considered. Instead of analysing the growth of S_t for every round we consider larger time *intervals* consisting of a suitably chosen number of steps. We change the process slightly by assuming that there is always one node with the preferred opinion to allow for an easier analysis. If all other opinions vanish, then node 1 is set to opinion 1. Note that this will only increase the runtime of the process. We also assume that if the preferred opinion vanishes totally, node 1 is set back to the preferred opinion. This alters the process, but as we show later, this event does not happen w.h.p.

The proof unfolds in the following way. First, we define formally the *step sequence* $\mathscr{S}$. Second, we define (Definition 8) a step sequence $\mathscr{S}$ to be *good* if, intuitively speaking, the preferred opinion grows quickly enough in any sufficiently large subsequence of $\mathscr{S}$. Afterward, we show that if $\mathscr{S}$ is a good step sequence, then the preferred opinion prevails in at most

τ''' rounds (Lemma 10). Finally, we show that $\mathscr{S}$ is indeed a good step sequence w.h.p. (Lemma 11).

We now give some definitions. Again, we denote by S_t the random set of nodes that have the preferred opinion right after the first t rounds, and let $S'_t = V \setminus S_t$. For a fixed time step t we write s_t and s'_t. We define the *boundary* ∂s_t as the subset of nodes in s'_t which are adjacent to at least one node from s_t. We use the symmetric definition for $\partial s'_t$. For each $u \in V$, let $\lambda_{u,t}$ be the number of edges incident with u crossing the cut $cut(s_t, s'_t)$, or equivalently, the number of u's neighbours that have a different opinion than u's before round t.

We divide each round t into $|s_t| + |s'_t|$ *steps*, in every step a single node v from either ∂s_t or $\partial s'_t$ randomly chooses a neighbour u and adopts its opinion with the corresponding bias. Note that we assume that v sees u's opinion referring to the *beginning* of the round, even if u was considered before v and changed its opinion in the meantime. It is convenient to label the steps independently of the round in which they take place. Hence, step i denotes the i-th step counted from the *beginning* of the first round. Also u_i refers to the node considered in step i and $\lambda_i = \lambda_{u_i,t}$. We define the indicator variable o_i with $o_i = 1$ if u_i has the preferred opinion and $o_i = 0$ otherwise. Let

$$\Lambda(i) = \sum_{j=1}^{i} (1 - o_i) \cdot \lambda_i \qquad \text{and} \qquad \Lambda'(i) = \sum_{j=1}^{i} o_i \cdot \lambda_i \,.$$

Unfortunately, the order in which the nodes are considered in a round is important for our analysis and cannot be arbitrary. Note that such an ordering does not affect the outcome of the process since the probabilities for a node to switch its opinion only depends on the distribution of opinions at the beginning of the round.

Intuitively, we order the nodes in s_t and s'_t such that the sum of the degrees of nodes which are already considered from s_t and the sum of the degrees of nodes already considered from s'_t differs by at most d, i.e.,

$$|\Lambda_i - \Lambda'_i| \leq d. \tag{6}$$

The following rule determines the node to be considered in step $j+1$: if $\Lambda(j) \leq \Lambda'(j)$, then the (not yet considered) node $v \in \partial s_t$ is with smallest identifier is considered. Otherwise the node $v \in \partial s'_t$ with the smallest identifier is considered. Note that at the first step i of any round we have $\Lambda_i = \Lambda'_i$. This guarantees that (6) holds. The *step sequence* $\mathscr{S}$ is now defined as a sequence of tuples, i.e., $\mathscr{S} = (u_1, Z_1),\ (u_2, Z_2),\ \ldots$, where $Z_j = 1$ if u_j changed its opinion in step j and $Z_j = 0$ otherwise for all $j \geq 1$. Observe that when given the initial assignment and the sequence up to step i, then we know the *configuration* $\mathscr{C}_i$ of the system, i.e., the opinions of all nodes at step i and in which round step i occurred.

In our analysis we consider the increase in the number of nodes with the preferred opinion in time intervals which contain a sufficiently large number of steps, instead of considering one round after the other. The following definitions identify these intervals.

For all $i, k \geq 0$ where $\mathscr{C}_i$ is fixed, we define the random variable $S_{i,k} := \min\{j \colon \Lambda_j - \Lambda_i \geq k\}$, which is the first time step such that nodes with a degree-sum of at least k were considered. Let $I_{i,k} = [i+1, S_{i,k}]$ be the corresponding interval where we note that the length is a random variable. We proceed by showing an easy observation proven in the full version.

▶ **Observation 3.** *The number of steps in the interval $I_{i,k}$ is at most $2k + 2d$, i.e., $|I_{i,k}| \leq 2k + 2d$. Furthermore, $\Lambda'(S_{i,k}) - \Lambda'(i) \leq k + 2d$.*

Fix $\mathscr{C}_i$ and let $X_{i,k}$ be the total number of times during interval $I_{i,k}$ that a switch from a non-preferred opinion to the preferred one occurs; and define $X'_{i,k}$ similarly for the reverse switches. Finally, we define $Y_{i,k} = X_{i,k} - X'_{i,k}$; thus $Y_{i,k}$ is the increase in number of nodes that have the preferred opinion during the time interval $I_{i,k}$.

Define $\ell = \frac{132\beta \log n}{(1-\alpha_1)^2}$ and $\beta' = \frac{600d}{\alpha_1 \cdot (1-\alpha_1)^2}$. In the following we define a *good* sequence.

▶ **Definition 8.** We call the sequence $\mathscr{S}$ of steps *good* if it has all of the following properties for all $i \leq T' = 2\beta' \cdot n$. Consider the first T' steps of $\mathscr{S}$ (fix $\mathscr{C}_{T'}$). Then,

(a) $Y_{0,T'} \geq 2n$ (the preferred opinion prevails in at most T' steps).
(b) $Y_{0,i} + |S_0| > 1$ (the preferred opinion never vanishes).
(c) For any $1 \leq k \leq T'$, $Y_{i,k} \geq -\ell$ (the number of nodes with the preferred opinion never drops by ℓ).
(d) For any $\ell \leq \gamma \leq T'$, $Y_{i,k} > \gamma$, where $k = \gamma \cdot \beta'$ (the nodes with the preferred opinion increase).

This definition allows us to prove in a convenient way that a step sequence $\mathscr{S}$ is w.h.p. good: For each property, we simply consider each (sufficiently large) subsequence S separately and we show that w.h.p. S has the desired property. We achieve this by using a concentration bound on $Y_{i,k}$ which we establish in Lemma 9. Afterward, we take a union bound over all of these subsequences and properties. Using the union bound allows us to show the desired properties in all subsequences in spite of the emerging dependencies. This is done in Lemma 10.

We now show the concentration bounds on $Y_{i,k}$. These bounds rely on the Chernoff-type bound established in the full version. This Chernoff-type bound shows concentration for variables having the property that the sum of the conditional probabilities of the variables, given all previous variables, is always bounded (from above or below) by some b. The bound might be of general interest and its proof can be found in the full version.

▶ **Lemma 9.** *Fix configuration $\mathscr{C}_i$. Then,*

(a) *For $k = \gamma \frac{256d}{\alpha_1 \cdot (1-\alpha_1)^2}$ with $\gamma \geq 1$ it holds that* $\Pr\left(Y_{i,k} < \gamma\right) \leq \exp\left(-\gamma\right)$.
(b) *For $k \geq 0$, any $b' = \alpha_1 \cdot (k + 2d)/d$, and any $\delta > 0$ it holds that*

$$\Pr\left(Y_{i,k} < -(1+\delta)b'\right) \leq \exp\left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{b'}.$$

The following two lemmas are proven in the full version.

▶ **Lemma 10.** *Let $\mathscr{S}$ be a step sequence. Then $\mathscr{S}$ is good with high probability.*

▶ **Lemma 11.** *If $\mathscr{S}$ is a good step sequence, then in at most T' time steps, the preferred opinion prevails and the T' time steps occur before round τ'''.*

Proof of Theorem 3. The claim follows from Lemma 10 together with Lemma 11. ◀

References

1 Mohammed Amin Abdullah and Moez Draief. Global majority consensus by local majority polling on graphs of a given degree sequence. *Discrete Applied Mathematics*, 180:1–10, 2015.

2 D. Aldous and J. Fill. Reversible markov chains and random walks on graphs, 2002. Unpublished. `http://www.stat.berkeley.edu/~aldous/RWG/book.html`.

3 Chen Avin, Michal Koucký, and Zvi Lotker. How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Tack A: Algorithms, Automata, Complexity, and Games*, pages 121–132, 2008. doi:10.1007/978-3-540-70575-8_11.

4 C. Cooper, R. Elsässer, H. Ono, and T. Radzik. Coalescing random walks and voting on connected graphs. *SIAM J. Discrete Math.*, 27(4):1748–1758, 2013. doi:10.1137/120900368.

5 C. Cooper, R. Elsässer, and T. Radzik. The power of two choices in distributed voting. In *ICALP*, pages 435–446, 2014.

6 Colin Cooper, Robert Elsässer, Tomasz Radzik, Nicolás Rivera, and Takeharu Shiraga. Fast Consensus for Voting on General Expander Graphs. In *Proc. DISC '15*, pages 248–262, 2015.

7 James Cruise and Ayalvadi Ganesh. Probabilistic consensus via polling and majority rules. *Queueing Systems*, 78(2):99–120, 2014.

8 J. Díaz, L. Goldberg, G. Mertzios, D. Richerby, M. Serna, and P. Spirakis. Approximating fixation probabilities in the generalized moran process. *Algorithmica*, 69(1):78–91, 2014. doi:10.1007/s00453-012-9722-7.

9 P. Donnelly and D. Welsh. Finite particle systems and infection models. *Mathematical Proceedings of the Cambridge Philosophical Society*, 94:167–182, 1983.

10 R. Elsässer, T. Friedetzky, D. Kaaser, F. Mallmann-Trenn, and H. Trinker. Efficient k-Party Voting with Two Choices. *ArXiv e-prints*, February 2016. arXiv:1602.04667.

11 G. Giakkoupis. Tight bounds for rumor spreading in graphs of a given conductance. In *STACS*, pages 57–68, 2011.

12 G. Giakkoupis, T. Sauerwald, and A. Stauffer. Randomized rumor spreading in dynamic graphs. In *Automata, Languages, and Programming – 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, pages 495–507, 2014. doi:10.1007/978-3-662-43951-7_42.

13 Y. Hassin and D. Peleg. Distributed probabilistic polling and applications to proportionate agreement. *Information and Computation*, 171(2):248–268, 2001. doi:http://dx.doi.org/10.1006/inco.2001.3088.

14 R. Holley and T. Liggett. Ergodic theorems for weakly interacting infinite systems and the voter model. *The Annals of Probability*, 3(4):643–663, 1975. URL: http://www.jstor.org/stable/2959329.

15 B. Houchmandzadeh and M. Vallade. The fixation probability of a beneficial mutation in a geographically structured population. *New Journal of Physics*, 13(7):073020, 2011. URL: http://stacks.iop.org/1367-2630/13/i=7/a=073020.

16 M. Kearns and J. Tan. Biased voting and the democratic primary problem. In *WINE*, pages 639–652, 2008.

17 F. Kuhn, N. Lynch, and R. Oshman. Distributed computation in dynamic networks. In *Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010*, pages 513–522, 2010. doi:10.1145/1806689.1806760.

18 H. Lam, Z. Liu, M. Mitzenmacher, X. Sun, and Y. Wang. Information dissemination via random walks in d-dimensional space. In *Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA'12, pages 1612–1622. SIAM, 2012. URL: http://dl.acm.org/citation.cfm?id=2095116.2095244.

19 N. Lanchier and C. Neuhauser. Voter model and biased voter model in heterogeneous environments. *Journal of Applied Probability*, 44(3):770–787, 2007.

20 T. Liggett. *Interacting Particle Systems*. Springer Berlin Heidelberg, 1985. doi:10.1007/b138374.

21 G. Mertzios and P. Spirakis. Strong bounds for evolution in networks. In *ICALP*, pages 669–680, 2013.

22 T. Nakata, H. Imahayashi, and M. Yamashita. A probabilistic local majority polling game on weighted directed graphs with an application to the distributed agreement problem. *Networks*, 35(4):266–273, 2000. doi:10.1002/1097-0037(200007)35:4<266::AID-NET5>3.0.CO;2-4.

23 R. Oliveira. On the coalescence time of reversible random walks. *Trans. Am. Math. Soc.*, 364(4):2109–2128, 2012.

24 Y Peres, A. Sinclair, P. Sousi, and A. Stauffer. Mobile geometric graphs: detection, coverage and percolation. *Probability Theory and Related Fields*, 156(1-2):273–305, 2013. doi:10.1007/s00440-012-0428-1.