Popular Half-Integral Matchings

Telikepalli Kavitha*

Tata Institute of Fundamental Research, Mumbai, India
kavitha@tcs.tifr.res.in

Abstract

In an instance $G = (A \cup B, E)$ of the stable marriage problem with strict and possibly incomplete preference lists, a matching M is popular if there is no matching M' where the vertices that prefer M' to M outnumber those that prefer M to M'. All stable matchings are popular and there is a simple linear time algorithm to compute a maximum-size popular matching. More generally, what we seek is a *min-cost* popular matching where we assume there is a cost function $c : E \to \mathbb{Q}$. However there is no polynomial time algorithm currently known for solving this problem. Here we consider the following generalization of a popular matching called a popular *half-integral* matching: this is a fractional matching $\vec{x} = (M_1 + M_2)/2$, where M_1 and M_2 are the 0-1 edge incidence vectors of matchings in G, such that $\vec{x}$ satisfies popularity constraints. We show that every popular half-integral matching is equivalent to a stable matching in a larger graph G^*. This allows us to solve the min-cost popular half-integral matching problem in polynomial time.

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1 Introduction

Let $G = (A \cup B, E)$ be an instance of the stable marriage problem on n vertices and m edges. Each vertex has a strict preference list ranking its neighbors. A matching M is stable if M admits no *blocking edge*, i.e., an edge (a, b) such that both a and b prefer each other to their respective assignments in M. The existence of stable matchings in G and the Gale-Shapley algorithm [7] to find one are classical results in graph algorithms.

Stability is a very strict condition and here we consider a relaxation of this called *popularity*. This notion was introduced by Gärdenfors [9] in 1975. We say a vertex $u \in A \cup B$ *prefers* matching M to matching M' if u is matched in M and unmatched in M' or it is matched in both and $M(u)$ ranks better than $M'(u)$ in u's preference list. For any two matchings M and M' in G, let $\phi(M, M')$ be the number of vertices that prefer M to M'.

▶ **Definition 1.** A matching M is *popular* if $\phi(M, M') \geq \phi(M', M)$ for every matching M' in G, i.e., $\Delta(M, M') \geq 0$ where $\Delta(M, M') = \phi(M, M') - \phi(M', M)$.

Every stable matching is popular [9]. In fact, it is known that every stable matching is a minimum-size popular matching [10]. In applications such as matching students to projects or applicants to posts, it may be useful to consider a weaker notion (such as popularity) than the total absence of blocking edges for the sake of obtaining larger-sized matchings. Popularity provides "global stability" since a popular matching never loses an election to another matching; by relaxing stability to popularity, we have a larger pool of candidate matchings to choose from in such an application.

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When there is a cost function $c : E \to \mathbb{Q}$, what we seek is a min-cost popular matching. There are several polynomial time algorithms known [11, 5, 6, 16, 14, 15] for computing a min-cost stable matching in G. However, while a maximum-size popular matching can be computed in linear time [12], no polynomial time algorithm is currently known for computing a min-cost popular matching in an instance $G = (A \cup B, E)$ with strict preference lists, except when preference lists are complete [4].

A fractional matching $\vec{p}$ is a convex combination of matchings, i.e., $\vec{p} = \sum_i p_i \cdot I(M_i)$ where $\sum_i p_i = 1$, $p_i \geq 0$ for all i, M_i's are matchings in G, and $I(M)$ is the 0-1 edge incidence vector of M. The fractional matching $\vec{p}$ is *popular* if $\Delta(\vec{p}, M) \geq 0$ for all matchings M in G where $\Delta(\vec{p}, M) = \sum_i p_i \cdot \Delta(M_i, M)$ (see Definition 1). It follows by linearity that if $\vec{p}$ is a popular fractional matching then $\Delta(\vec{p}, \vec{q}) \geq 0$ for all fractional matchings $\vec{q}$.

Let $\mathcal{P}$ be the polytope defined by the constraints that $\vec{p}$ belongs to the matching polytope of G and $\Delta(\vec{p}, M) \geq 0$ for all matchings M in G. A simple description of $\mathcal{P}$ was given in [13]. Thus a min-cost popular *fractional* matching can be computed in polynomial time.

Our results and techniques. Our main result is a polynomial time algorithm to compute a min-cost popular *half-integral* matching in G. A popular half-integral matching is a vector $\vec{x} \in \{0, \frac{1}{2}, 1\}^m \cap \mathcal{P}$. For any two popular matchings M_1 and M_2 in G, the half-integral matching $(I(M_1) + I(M_2))/2$ is popular. However not every popular half-integral matching is a convex combination of popular matchings – we show such an example in Section 2. Thus if $\mathcal{Q}$ is the convex hull of popular half-integral matchings in G, then $\mathcal{Q}$ need not be integral.

We show that every extreme point of $\mathcal{Q}$ is a stable matching in a new (larger) graph G^* that we construct here. Thus the min-cost popular half-integral matching problem in G becomes the min-cost stable matching problem in G^* which can be solved in polynomial time. This also gives us a simple description of the polytope $\mathcal{Q}$ via the stable matching polytope of G^* (i.e., the convex hull of stable matchings in G^*).

The main tool that we use here is the description of the polytope $\mathcal{P}$ from [13]. We first show that every stable matching S in the new graph G^* can be mapped to a half-integral matching in G whose incidence vector belongs to $\mathcal{P}$. We then show that every extreme point $\vec{p}$ of the convex hull $\mathcal{Q}$ of popular half-integral matchings in G can be realized as a stable matching in G^*. We use the fact that $\vec{p} \in \mathcal{P}$ along with the fact that G is bipartite to show a "helpful witness" $(\alpha_u)_{u \in A \cup B} \in \{\pm 1, 0\}^n$. This witness will guide us in building a stable matching S in G^* that corresponds to $\vec{p}$.

A graph G', similar to the graph G^* used here, was recently used in [4] to show that any stable matching in G' maps to a maximum-size popular matching M in G. However every maximum-size popular matching in G need not be obtained as a stable matching in G'. In the special case when preference lists are complete (i.e., G is $K_{|A|,|B|}$), all popular matchings in G can be realized as stable matchings in G'. The method used in [4] is similar to the method used in previous algorithms to compute maximum-size popular matchings [10, 12] – these show that there is no *popularity-improving* alternating path or cycle with respect to the matching returned. In contrast, our technique here is based on linear programming.

A min-cost popular half-integral popular matching has applications – consider the problem of assigning projects to students where each project can be split into two half-projects. Each half-project can be assigned to a distinct student and a student can be assigned two half-projects. A min-cost popular half-integral matching is a feasible assignment here that is popular and has the least cost. While fractional matchings, in general, may not be feasible in typical applications, half-integral matchings are more natural and suitable to applications.

Background. Algorithms for computing popular matchings [1] were first considered in the one-sided preference lists model where it is only vertices in A that have preferences and cast votes while vertices in B have no preferences. Popular matchings need not always exist in this model, however it was shown in [13] that popular fractional matchings always exist and using the description of $\mathcal{P}$, such a fractional matching can be found in polynomial time (via linear programming).

In the two-sided preference lists model, when preference lists have ties, $G = (A \cup B, E)$ need not always admit a popular matching and it is known that determining if G admits a popular matching or not is an NP-complete problem [2, 3]. When preference lists are strict, every stable matching is popular. The min-cost stable matching problem in an instance $G = (A \cup B, E)$ with strict preference lists is well-studied and descriptions of the stable matching polytope were given by Vande Vate [16], Rothblum [14], and Teo and Sethuraman [15].

We discuss preliminaries in Section 2. Section 3 describes the graph G^* and shows that every stable matching in G^* is a popular half-integral matching in G. Section 4 shows how every popular half-integral matching in G that is an extreme point of $\mathcal{Q}$ (the popular half-integral matching polytope) can be obtained as a stable matching in G^*.

2 Preliminaries

For any vertex $u \in A \cup B$ and neighbors v and w, we will use the following function to show u's preference for v vs w: $\mathsf{vote}_u(v, w) = 1$ if u prefers v to w, it is -1 if u prefers w to v, else (i.e., when $v = w$) it is 0. We will be using this function in the description of the popular fractional matching polytope $\mathcal{P}$.

Recall that a popular fractional matching is a point $\vec{x} = (x_e)_{e \in E}$ in the matching polytope of G such that $\Delta(\vec{x}, M) \geq 0$ for all matchings M in G. It will be convenient to assume that each vertex $u \in A \cup B$ is completely matched in every fractional matching $\vec{x}$ in G. So we will revise $\vec{x}$ so that each vertex u gets matched to an artificial last-resort neighbor $\ell(u)$ (which is placed at the bottom of u's preference list) with weight $(1 - \sum_{e \in E(u)} x_e)$, where the sum is over all the edges e incident on u.

For convenience, we will continue to use $\vec{x}$ to denote the revised $\vec{x}$ in $[0,1]^{m+n}$. We use $\tilde{E}$ to denote the edge set $E \cup \{(u, \ell(u)) : u \in A \cup B\}$ and $\tilde{E}(u)$ is the set of edges in $\tilde{E}$ that are incident on u. The following simple description of $\mathcal{P}$ was given in [13]. In the constraints below, a variable α_u is associated with each $u \in A \cup B$ and not to last-resort neighbors.

$$\alpha_a + \alpha_b \;\geq\; \sum_{(a,b') \in \tilde{E}(a)} x_{(a,b')} \cdot \mathsf{vote}_a(b, b') \;+\; \sum_{(a',b) \in \tilde{E}(b)} x_{(a',b)} \cdot \mathsf{vote}_b(a, a') \quad \forall (a,b) \in \tilde{E}$$

$$\sum_{u \in A \cup B} \alpha_u \;=\; 0 \quad \text{and} \quad \sum_{e \in \tilde{E}(u)} x_e \;=\; 1 \quad \forall u \in A \cup B \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}.$$

The constraints above arise as the dual to the maximum weight matching problem in the graph $\tilde{G}_x$ which is G augmented with last-resort neighbors and with edge set $\tilde{E}$, where the weight of an edge (a, b) is $\sum_{(a,b') \in \tilde{E}(a)} x_{(a,b')} \cdot \mathsf{vote}_a(b, b') + \sum_{(a',b) \in \tilde{E}(b)} x_{(a',b)} \cdot \mathsf{vote}_b(a, a')$. The constraint $\sum_{u \in A \cup B} \alpha_u = 0$ is equivalent to saying that the maximum weight of a matching in $\tilde{G}_x$ is 0, in other words, $\vec{x}$ is popular. We refer the reader to Section 3 of [13] for all the details.

For any fractional matching $\vec{x}$, if there exists $\vec{\alpha} = (\alpha_u)_{u \in A \cup B}$ such that $\vec{x}$ and $\vec{\alpha}$ satisfy the above constraints, then we say $\vec{x} \in \mathcal{P}$. The vector $\vec{\alpha}$ will be called a *witness* to $\vec{x}$'s popularity.

a_0	v_1		
a_1	b_1	v_1	
a_2	b_1	b_2	
u_1	v_1	v_2	b_0
u_2	v_2	b_2	v_1

b_0	u_1			
b_1	a_2	a_1		
b_2	a_2	u_2		
v_1	u_2	a_1	u_1	a_0
v_2	u_1	u_2		

Figure 1 The above table describes the preference lists of all the men and women in G. Here a_0 has a single neighbor v_1 while a_1's top choice is b_1, second choice is v_1 and so on for each vertex.

$\mathcal{P}$ is not integral. We now show an example of a graph G and a fractional matching $\vec{p} \in \mathcal{P}$, however $\vec{p}$ is not a convex combination of popular matchings. Let $A = \{a_0, a_1, a_2, u_1, u_2\}$, $B = \{b_0, b_1, b_2, v_1, v_2\}$, and the preference lists of vertices are described in Figure 1.

Consider the half-integral matching $\vec{p}$ which has $p_{(a_1,b_1)} = p_{(a_2,b_2)} = 1$ and $p_e = \frac{1}{2}$ for $e \in \{(u_1, v_1), (u_2, v_2), (u_1, v_2), (u_2, v_1)\}$. For any other edge e, we have $p_e = 0$. This fractional matching belongs to $\mathcal{P}$ by using the following α values: $\alpha_{a_0} = \alpha_{b_0} = 0$; $\alpha_{a_2} = \alpha_{b_1} = 1$; $\alpha_{a_1} = \alpha_{b_2} = -1$; and $\alpha_w = 0$ for $w \in \{u_1, u_2, v_1, v_2\}$.

There is only one way to express $\vec{p}$ as a convex combination of integral matchings, that is, $\vec{p} = (I(M_1) + I(M_2))/2$, where $M_1 = \{(a_1, b_1), (a_2, b_2), (u_1, v_1), (u_2, v_2)\}$ and $M_2 = \{(a_1, b_1), (a_2, b_2), (u_1, v_2), (u_2, v_1)\}$. We show below that neither M_1 nor M_2 is popular.

The matching $M_1' = \{(u_1, b_0), (a_1, v_1), (a_2, b_1), (u_2, v_2)\}$ is more popular than M_1 and the matching $M_2' = \{(a_0, v_1), (u_2, b_2), (a_2, b_1), (u_1, v_2)\}$ is more popular than M_2. Thus $\vec{p}$ is not in the convex hull of popular matchings in G.

The graph G'. Our input is a graph $G = (A \cup B, E)$ on n vertices and m edges. Note that there are no last-resort neighbors here – they were added only for the formulation of the polytope $\mathcal{P}$. Vertices in A and in B are usually referred to as men and women, respectively, and we follow the same convention here.

The construction of the following graph $G' = (A' \cup B', E')$, based on G, was shown in [4]. The set A' has two copies a_0 and a_1 of each man $a \in A$, the men in $\{a_0 : a \in A\}$ are called *level 0 men* of G' and those in $\{a_1 : a \in A\}$ are called *level 1 men* of G'. The set B' consists of all the women in B along with *dummy vertices* $\cup_{a \in A}\{d(a)\}$, where there is one dummy vertex per man in A. The preference lists of the vertices are as follows:

- each level 0 man a_0 has the same preference list as the corresponding man a in G except that the dummy vertex $d(a)$ occurs as his *least* preferred neighbor at the bottom of his preference list
- each level 1 man a_1 has the same preference list as the corresponding man a in G except that the dummy vertex $d(a)$ occurs as his *most* preferred neighbor at the top of his preference list
- each dummy vertex $d(a)$ has a_0 and a_1 as its neighbors: top choice is a_0, followed by a_1
- every woman $b \in B$ has the following preference list in G': all her level 1 neighbors (in the same order of preference as in G) followed by all her level 0 neighbors (in the same order of preference as in G).

We will be using this graph G' here; in fact, we will have two such graphs G' and G'' combining to form our new graph G^*. The graph G'' is analogous to the graph G' except that the roles of men and women (and also that of levels 0 and 1) are swapped here.

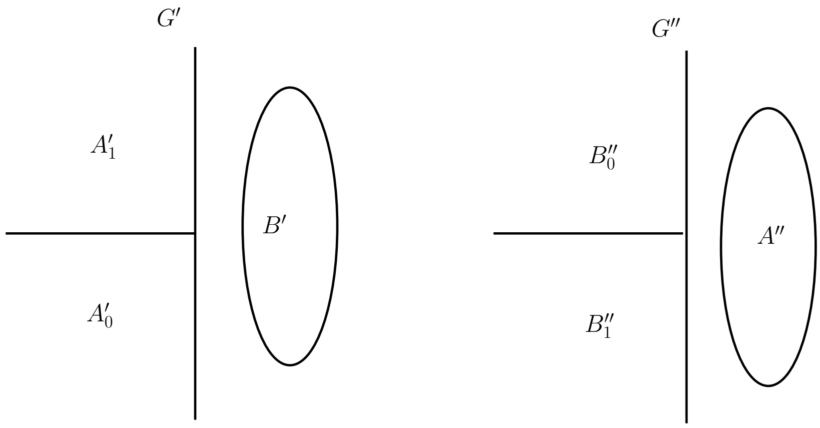

Figure 2 The graph G' on the left and the graph G'' on the right in G^*. For $i = 0, 1$, we use A'_i to refer to level i men in G' and we use B''_i to refer to level i women in G''.

3 The graph G^*

We define the graph G^* as follows: G^* consists of two vertex-disjoint subgraphs G' and G'' (see Figure 2). The graph G' was described in Section 2.

In the graph $G'' = (B'' \cup A'', E'')$, women are on the left side of G'' and men are on the right side – the set B'' has two copies b_0 and b_1 of each woman $b \in B$, the women in $\{b_0 : b \in B\}$ are called *level 0 women* of G'' and those in $\{b_1 : b \in B\}$ are called *level 1 women* of G''.

The set A'' consists of all the men in A along with new dummy vertices $\cup_{b \in B}\{d(b)\}$, where there is one dummy vertex per woman in B. The preference lists of the vertices are as follows:

- each level 0 woman b_0 has the same preference list as the corresponding woman b in G except that the dummy vertex $d(b)$ occurs as her *most* preferred neighbor at the top of her preference list
- each level 1 woman b_1 has the same preference list as the corresponding woman b in G except that the dummy vertex $d(b)$ occurs as her *least* preferred neighbor at the bottom of her preference list
- each dummy vertex $d(b)$ has only b_0 and b_1 as its neighbors: its top choice is b_1, followed by b_0
- every man $a \in A$ has the following preference list in G'': all his level 0 neighbors (in the same order of preference as in G) followed by all his level 1 neighbors (in the same order of preference as in G).

We want all stable matchings in G^* to be perfect matchings – note that all level 0 men in G' and all level 1 women in G'' will be matched in any stable matching in G^* since they are top-choice neighbors for their respective dummy neighbors. However the same cannot be said about level 1 men in G' and level 0 women in G''.

In order to take care of these vertices, we add the following "self-loop" edges to G^*: the edge (a_1, a) for each man a in A, where $a_1 \in A'_1$ and $a \in A''$, and the edge (b_0, b) for each woman b in B, where $b_0 \in B''_0$ and $b \in B'$. The vertex $a_1 \in A'_1$ regards $a \in A''$ as his worst ranked neighbor and similarly, $b_0 \in B''_0$ regards $b \in B'$ as her worst ranked neighbor.

For any man $a \in A''$, the vertex a_1 is in the middle of his preference list, sandwiched between all his level 0 neighbors and all his level 1 neighbors as shown in (1) below. More precisely, a_1 is sandwiched between b''_0 and b'_1, where $b' > \cdots > b''$ is a's preference list in G. Thus b''_0 is a's worst level 0 neighbor and b'_1 is a's best level 1 neighbor.

$$a: \; b'_0 > \cdots > b''_0 > \underline{a_1} > b'_1 > \cdots > b''_1; \qquad b: \; a'_1 > \cdots > a''_1 > \underline{b_0} > a'_0 > \cdots > a''_0. \quad (1)$$

Similarly, for any woman $b \in B'$, the vertex b_0 is in the middle of her preference list, sandwiched between all her level 1 neighbors and all her level 0 neighbors as shown in (1). More precisely, b_0 is sandwiched between a''_1 and a'_0, where $a' > \cdots > a''$ is b's preference list in G. Using the fact that all stable matchings in G^* match the same set of vertices [8], it can be shown that every stable matching in G^* is perfect.

The function f. We now define a function f : {stable matchings in G^*} $\rightarrow$ {half-integral matchings in G}. Observe that every stable matching in G^* has to match all dummy vertices since each of these is a top-choice neighbor for someone. Thus out of a_0 and a_1 in A', only one is matched to a non-dummy neighbor and similarly, out of b_0 and b_1 in B'', only one is matched to a non-dummy neighbor.

Let S be any stable matching in G^*. By removing all self-loops that occur in S and those edges in S that contain a dummy vertex, the resulting matching is the union of two matchings S' and S'' in G. We define $f(S)$ to be $(I(S') + I(S''))/2$, where $I(M) \in \{0,1\}^m$ is the 0-1 edge incidence vector of M. So $f(S)$ is a valid half-integral matching in G.

▶ **Theorem 2.** *For any stable matching S in G^*, the half-integral matching $f(S)$ is popular in G.*

Proof. We are given a stable matching S in G^*. Recall that we pruned all edges that contain a dummy vertex and all self-loops from S to define $f(S)$. We now prune all dummy vertices, their partners in S, and self-loops from G^* also – let H^* denote the pruned graph G^*. Let H' denote the pruned subgraph G' and let H'' denote the pruned subgraph G''.

The men in the graph H' consist of one copy of each $a \in A$ – some of these are in level 0 and the rest are in level 1. The women in H' are exactly those in B. The women in H'' consist of one copy of each $b \in B$ – some of these are in level 0 and the rest are in level 1. The men in H'' are exactly those in A. Thus H' and H'' are two copies of the graph G.

Let S' be the pruned matching (resulting from S) restricted to H' and let S'' be the pruned matching (resulting from S) restricted to H''. Let $\tilde{A}'_i$ denote the set of level i men in H', for $i = 0, 1$ (see Figure 3). Let $\tilde{B}'_i$ consist of women matched in S' to men in $\tilde{A}'_i$, for $i = 0, 1$. Women unmatched in S' are added to $\tilde{B}'_1$.

Similarly, $\tilde{B}''_i$ consists of level i women in the H'' part of H^* and $\tilde{A}''_i$ denotes the set of men matched in S'' to women in $\tilde{B}''_i$, for $i = 0, 1$. Men unmatched in S'' are added to $\tilde{A}''_0$.

For each edge $e = (a, b) \in H'$, define the function $w'(e)$ as follows: $w'(e) = \mathsf{vote}_a(b, S'(a)) + \mathsf{vote}_b(a, S'(b))$. If $S'(u)$ is undefined for any vertex u, then $\mathsf{vote}_u(v, S'(u)) = 1$ for any neighbor v of u since every vertex prefers being matched to being unmatched. Note that if $(a, b) \in S'$ then $w'(e) = 0$.

Similarly, for each edge $e = (a, b) \in H''$, define the function $w''(e)$ as follows: $w''(e) = \mathsf{vote}_a(b, S''(a)) + \mathsf{vote}_b(a, S''(b))$. For any vertex u that is unmatched in S'', we take $\mathsf{vote}_u(v, S''(u)) = 1$, for any neighbor v of u. Note that $w'(e)$ and $w''(e)$ always take values in $\{-2, 0, 2\}$. Due to the stability of the matching S in G^*, the following observations hold:

- Every edge $e \in \tilde{A}'_1 \times \tilde{B}'_0$ has to satisfy $w'(e) = -2$. Similarly, every edge $e \in \tilde{A}''_1 \times \tilde{B}''_0$ has to satisfy $w''(e) = -2$.

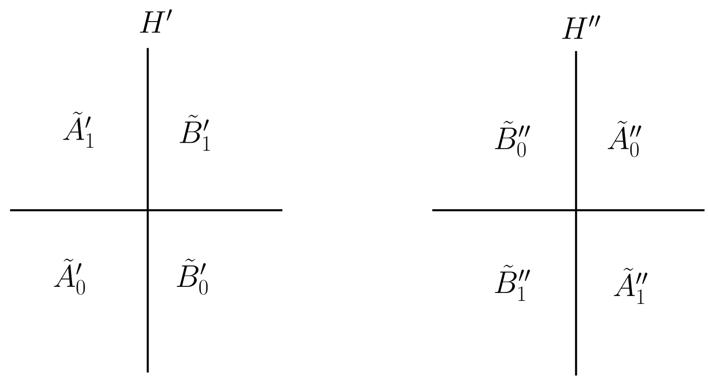

Figure 3 The graph H' on the left and the graph H'' on the right in the graph H^*.

Consider an edge (a_1, b) in $\tilde{A}'_1 \times \tilde{B}'_0$. It follows from the definition of preference lists of women in G' that the woman b prefers a_1 (a level 1 man) to her partner $S'(b)$ (a level 0 man). Since S is stable, it follows that a_1 prefers his partner $S'(a_1)$ to b. Moreover, a_0 prefers b to $S'(a_0) = d(a)$, since $d(a)$ is a_0's last choice. Thus b prefers her partner $S'(b)$ to a_0. So $\mathsf{vote}_a(b, S'(a)) = \mathsf{vote}_b(a, S'(b)) = -1$. A similar proof holds for any edge $e \in \tilde{A}''_1 \times \tilde{B}''_0$.

- Every edge e such that $w'(e) = 2$ has to be in $\tilde{A}'_0 \times \tilde{B}'_1$. Similarly, every edge e such that $w''(e) = 2$ has to be in $\tilde{A}''_0 \times \tilde{B}''_1$.

If e is an edge in H' such that $w'(e) = 2$, then $e \notin \tilde{A}'_i \times \tilde{B}'_i$ (for $i = 0, 1$) as such an edge would block S. We have already seen that any edge $e \in \tilde{A}'_1 \times \tilde{B}'_0$ satisfies $w'(e) = -2$. Thus any edge e such that $w'(e) = 2$ has to be in $\tilde{A}'_0 \times \tilde{B}'_1$. We can similarly show that any edge e in H'' such that $w''(e) = 2$ has to be in $\tilde{A}''_0 \times \tilde{B}''_1$.

We will now show that $f(S) \in \mathcal{P}$ by assigning appropriate α_u values for all u in $A \cup B$. We first define α'_u and α''_u:

- let $\alpha'_u = -1$ if $u \in \tilde{A}'_1 \cup \tilde{B}'_0$ and let $\alpha'_u = 1$ if $u \in \tilde{A}'_0 \cup \tilde{B}'_1$.
- let $\alpha''_u = -1$ if $u \in \tilde{A}''_1 \cup \tilde{B}''_0$ and let $\alpha''_u = 1$ if $u \in \tilde{A}''_0 \cup \tilde{B}''_1$.

The following is an immediate corollary of the above observations and the definitions of α'_u and α''_u: $\alpha'_a + \alpha'_b \geq w'(a, b)$ and $\alpha''_a + \alpha''_b \geq w''(a, b)$ for all edges (a, b). Also for any vertex u that is unmatched in S' and S'', we have $\alpha'_u + \alpha''_u = 0$.

Define $\alpha_u = (\alpha'_u + \alpha''_u)/2$ for all $u \in A \cup B$. Observe that $\sum_{u:A\cup B} \alpha_u = 0$. The above constraints imply that $(\alpha_u)_{u \in A\cup B}$ and the incidence vector of $f(S)$ satisfy the constraints of the polytope $\mathcal{P}$. Thus $f(S)$ is a popular half-integral matching. ◀

4 Constructing a stable matching in G^*

We showed in the previous section that f maps stable matchings in G^* to popular half-integral matchings in G. In fact, $f(S)$ is what we will call a *full* half-integral matching, i.e., for every vertex $u \in A \cup B$, either u is fully matched in $f(S)$ or it is fully unmatched in $f(S)$. Let $\vec{p} \in \{0, \frac{1}{2}, 1\}^m$ be a full half-integral matching that is popular. Since $\vec{p} \in \mathcal{P}$, there exists a witness $(\alpha_u)_{u \in A\cup B}$ to $\vec{p}$'s popularity. The following lemma will be useful to us.

▶ **Lemma 3.** *There exists a witness $(\alpha_u)_{u \in A\cup B}$ to $\vec{p}$'s popularity such that $\alpha_u \in \{\pm 1, 0\}$, for each $u \in A \cup B$.*

Proof. In order to show such a witness, we will consider the following linear program:

$$\text{minimize} \sum_{u \in A \cup B} \alpha_u \qquad \text{(LP1)}$$

subject to

$$\alpha_a + \alpha_b \;\geq\; \textstyle\sum_{(a,b') \in \tilde{E}(a)} p_{(a,b')} \cdot \mathsf{vote}_a(b, b') + \sum_{(a',b) \in \tilde{E}(b)} p_{(a',b)} \cdot \mathsf{vote}_b(a, a') \;\forall\, (a,b) \in \tilde{E}$$

Recall that $\tilde{E}$ is the set $E \cup \{(u, \ell(u)) : u \in A \cup B\}$, where $\ell(u)$ is the artificial last-resort neighbor of vertex u. In the above constraints, let us denote the right hand side quantity corresponding to edge e by $\mathsf{value}_p(e)$. Since $\vec{p}$ is a full half-integral matching, it is easy to see that $\mathsf{value}_p(e)$ is integral for all edges e.

Consider the polyhedron defined by the above constraints $N \cdot \vec{\alpha} \geq \vec{c}$, where N is the above $(m + n) \times n$ constraint matrix, $\vec{\alpha}$ is the column of unknowns α_u, for $u \in A \cup B$, and $\vec{c}$ is the column vector of $\mathsf{value}_p(\cdot)$ values. The top $m \times n$ sub-matrix of N is the edge-vertex incidence matrix U of the graph G and the bottom $n \times n$ matrix is the identity matrix I. Since the graph G is bipartite, the matrix U is totally unimodular and hence the matrix N is totally unimodular. Since $\vec{c}$ is an integral vector, it follows that all the vertices of $N \cdot \vec{\alpha} \geq \vec{c}$ are integral.

Thus there is an integral optimal solution to (LP1), call it $\vec{\alpha}^*$. We need to now show that $\vec{\alpha}^* \in \{\pm 1, 0\}^n$. It follows from the constraints corresponding to the edges $(u, \ell(u))$ that $\alpha^*_u \geq -1$ if u is matched in $\vec{p}$ and $\alpha^*_u \geq 0$ for u unmatched in $\vec{p}$. We now show the following claim.

▶ **Claim 4.** *Let $e = (a, b)$ be any edge such that $p_e > 0$. Then the constraint in (LP1) corresponding to e is tight, i.e., $\alpha^*_a + \alpha^*_b = \mathsf{value}_p(e)$.*

Proof. Consider the dual program of (LP1): it is the maximum weight matching problem in the graph G augmented with last-resort neighbors and with edge set $\tilde{E}$, where the weight of edge e is $\mathsf{value}_p(e)$. A maximum weight matching in this graph has weight 0 (because $\vec{p}$ is popular). Since $\Delta(\vec{p}, \vec{p}) = 0$, the fractional matching $\vec{p}$ is an optimal dual solution. It follows from complementary slackness conditions that if $p_{(a,b)} > 0$, then the constraint in (LP1) for edge (a, b) is tight. ◀

Observe that for any vertex u, there has to be an edge e incident on it with $p_e > 0$ and either $\mathsf{value}_p(e) = 0$ or $\mathsf{value}_p(e) = -1$ (the edge e between u and its worse partner v in $\vec{p}$). Using Claim 4 and the fact that $\alpha^*_v \geq -1$, we can now conclude that $\alpha^*_u \leq 1$. ◀

We will use the above lemma to show the following theorem in this section.

▶ **Theorem 5.** *Let $\vec{p} \in \{0, \frac{1}{2}, 1\}^m$ be a full half-integral matching that is popular. Then $\vec{p} = f(S)$ for some stable matching S in G^*.*

We will now build a stable matching S in the graph G^* such that $f(S) = \vec{p}$. For every edge $e = (a, b)$ such that $p_e > 0$, we need to decide which of the edges $(a_0, b), (a_1, b), (b_0, a), (b_1, a)$ will get included in S. In order to make this decision, we will build a graph H^*. The graph H^* consists of two copies H' and H'' of the input graph G.

Every vertex $u \in A \cup B$ gets assigned a level, denoted by $\mathsf{level}'(u)$, in H'. For $a \in A$, $\mathsf{level}'(a) = i$ fixes $a_i \in \{a_0, a_1\}$ to be the one that will be matched to a woman (i.e., a non-dummy vertex) in S. For $b \in B$, we say $\mathsf{level}'(b) = i$ to fix b getting matched to some level i man in H'. We will say u is in level i in H' to mean $\mathsf{level}'(u) = i$.

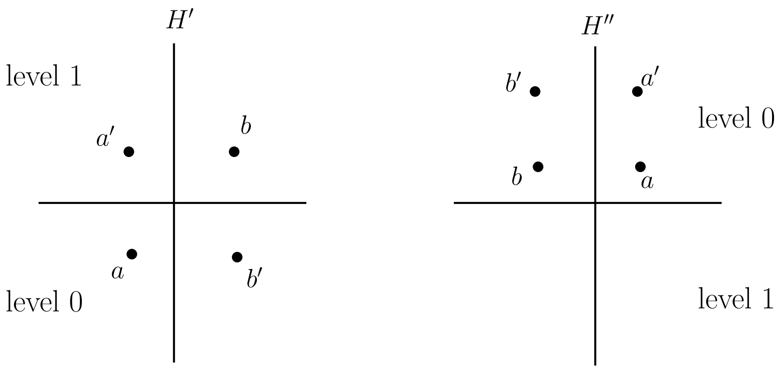

Figure 4 Since $\alpha^*_a = 1$ and $\alpha^*_{b'} = -1$, we have $\mathsf{level}'(a) = \mathsf{level}'(b') = 0$ and similarly, $\mathsf{level}''(a) = \mathsf{level}''(b') = 0$. Since $\alpha^*_b = \alpha^*_{a'} = 0$, we have $\mathsf{level}'(b) = \mathsf{level}'(a') = 1$ and similarly, $\mathsf{level}''(b) = \mathsf{level}''(a') = 0$. So we place a and b' in level 0 in both H and H' and we place a' and b in level 1 in H' and in level 0 in H''.

Similarly, every vertex $u \in A \cup B$ gets assigned a level, denoted by $\mathsf{level}''(u)$, in H''. For $b \in B$, $\mathsf{level}''(b) = j$ fixes $b_j \in \{b_0, b_1\}$ to be the one that will be matched to a man (i.e., a non-dummy vertex) in S. For $a \in A$, we say $\mathsf{level}''(a) = j$ to fix a getting matched to some level j woman in H''. We will say u is in level j in H'' to mean $\mathsf{level}''(u) = j$.

Since $\vec{p}$ is a full half-integral matching that is popular, we know from Lemma 3 that there exists a witness $\vec{\alpha}^* = (\alpha^*_u)_{u \in A \cup B}$ in $\{-1, 0, 1\}^n$ to the popularity of $\vec{p}$. We will use $\vec{\alpha}^*$ to fix $\mathsf{level}'(u)$ and $\mathsf{level}''(u)$ for each vertex u as follows.

- $\alpha^*_u = -1$: If $u \in A$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 1$. If $u \in B$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 0$.
- $\alpha^*_u = 1$: If $u \in A$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 0$. If $u \in B$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 1$.
- $\alpha^*_u = 0$: For all $u \in A \cup B$, $\mathsf{level}'(u) = 1$ and $\mathsf{level}''(u) = 0$.

As an example, consider the 4-cycle G on 2 men a, a' and 2 women b, b' where both a and a' prefer b to b' and both b and b' prefer a to a'. Let $\vec{p}$ be the half-integral matching with $p_e = 1/2$ for each edge e. This is popular and $\alpha^*_a = 1$, $\alpha^*_b = \alpha^*_{a'} = 0$, and $\alpha^*_{b'} = -1$ is a witness to $\vec{p}$'s popularity. Figure 4 shows how these vertices get placed in H' and in H''.

For any vertex u, let v and v' be its neighbors in G such that $\vec{p}$ has positive support on (u, v) and (u, v'). We will refer to v and v' as *partners* of u in $\vec{p}$. We need to show that either (i) $\mathsf{level}'(u) = \mathsf{level}'(v)$ and $\mathsf{level}''(u) = \mathsf{level}''(v')$, or (ii) $\mathsf{level}'(u) = \mathsf{level}'(v')$ and $\mathsf{level}''(u) = \mathsf{level}''(v)$. In other words, we need to show that u, v are level-compatible in one of H', H'' and u, v' are level-compatible in the other graph in H', H''.

We will now show that our allocation of levels to men and women based on their α^*-values ensures this. If $v = v'$ then $p_{(u,v)} = 1$ and the (tight) constraint for edge (u, v) in the description of $\mathcal{P}$ is $\alpha^*_u + \alpha^*_v = 0$. Thus (α^*_u, α^*_v) has to be one of $(1, -1), (0, 0), (-1, 1)$: in all three cases we have level-compatibility in both H' and H''. The following lemma shows that even when u has two distinct partners v and v' in $\vec{p}$, there is level-compatibility.

▶ **Lemma 6.** *Every vertex that has two distinct partners in $\vec{p}$ is level-compatible in H' with one partner and is level-compatible in H'' with another partner.*

Proof. We will show this lemma for any vertex $b \in B$. An analogous proof holds for any vertex in A. Let $a \neq a'$ be the partners of b in $\vec{p}$ and let b prefer a to a'. We know that $p_{(a,b)} = p_{(a',b)} = 1/2$. Since $\vec{p}$ is a full half-integral matching, a (similarly, a') has another neighbor $r(a)$ (resp., $r(a')$) with positive support in $\vec{p}$. We have four cases depending on how a and a' rank b versus $r(a)$ and $r(a')$, respectively.

1. If both a and a' prefer $r(a)$ and $r(a')$ respectively to b, then $\mathsf{value}_p(a,b) = -\frac{1}{2} + \frac{1}{2} = 0$ and $\mathsf{value}_p(a',b) = -\frac{1}{2} - \frac{1}{2} = -1$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 0$ and $\alpha^*_{a'} + \alpha^*_b = -1$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is either $(1,-1,0)$ or $(0,0,-1)$.
   - In the former case $\mathsf{level}'(a) = \mathsf{level}'(b) = 0$ and $\mathsf{level}''(a') = \mathsf{level}''(b) = 0$.
   - In the latter case $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
2. If both a and a' prefer b to $r(a)$ and $r(a')$ respectively, then $\mathsf{value}_p(a,b) = \frac{1}{2} + \frac{1}{2} = 1$ and $\mathsf{value}_p(a',b) = \frac{1}{2} - \frac{1}{2} = 0$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 1$ and $\alpha^*_{a'} + \alpha^*_b = 0$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is either $(0,1,-1)$ or $(1,0,0)$.
   - In the former case $\mathsf{level}'(a) = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a') = \mathsf{level}''(b) = 1$.
   - In the latter case $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
3. If a prefers b to $r(a)$ while a' prefers $r(a')$ to b, then $\mathsf{value}_p(a,b) = \frac{1}{2} + \frac{1}{2} = 1$ and $\mathsf{value}_p(a',b) = -\frac{1}{2} - \frac{1}{2} = -1$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 1$ and $\alpha^*_{a'} + \alpha^*_b = -1$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is $(1,0,-1)$.
   - Here $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
4. If a prefers $r(a)$ to b while a' prefers b to $r(a')$, then $\mathsf{value}_p(a,b) = -\frac{1}{2} + \frac{1}{2} = 0$ and $\mathsf{value}_p(a',b) = \frac{1}{2} - \frac{1}{2} = 0$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 0$ and $\alpha^*_{a'} + \alpha^*_b = 0$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is $(1,-1,1)$ or $(0,0,0)$ or $(-1,1,-1)$.
   - In the first case, all three vertices a, b, and a' are in level 0 in both H' and H''.
   - In the second case, all three vertices are in level 1 in H' and in level 0 in H''.
   - In the third case, all three vertices are in level 1 in both H' and H''. ◀

For any vertex u with partners v and v' in $\vec{p}$, where u prefers v to v', we call v the *better partner* of u and v' the *worse partner* of u. If $p_{(a,b)} = 1$ for some edge (a,b), then we regard a as both the better partner and the worse partner of b.

We are now ready to describe the construction of our matching S. We give the following two pairing rules for any $b \in B$ (let a be b's better partner and a' be b's worse partner):

1. if $\alpha^*_b \in \{\pm 1\}$ then pair b with a in H' and with a' in H''.
2. if $\alpha^*_b = 0$ then pair b with a' in H' and with a in H''.

More precisely, if $\alpha^*_b = -1$ then we include (a_0, b) and (b_0, a') in S; if $\alpha^*_b = 1$ then we include (a_1, b) and (b_1, a') in S; and if $\alpha^*_b = 0$ then we include (a'_1, b) and (b_0, a) in S.

Note that the above rules for pairing vertices follow from the proof of Lemma 6. A woman b with $\alpha^*_b = -1$ (similarly, $\alpha^*_b = 1$) is level-compatible with her better partner in level 0 (resp., level 1) in H' and with her worse partner in level 0 (resp., level 1) in H''. Similarly, if $\alpha^*_b = 0$ then b is level-compatible with her worse partner in level 1 in H' and with her better partner in level 0 in H''.

Thus level-compatibility unambiguously fixes for us in which of H, H' a vertex gets paired with which partner till we are left with a set T of vertices forming a cycle: each vertex in T has both its partners in T, and all these vertices are in the same level in both H' and H''. We again know from the proof of Lemma 6 that this happens only when $(\alpha^*_a, \alpha^*_b) \in \{(1,-1),(0,0),(-1,1)\}$ for each edge (a,b) in this cycle. The cycle can be resolved as per the two rules above (which is what our algorithm for constructing S does). Thus rule 1 and rule 2 given above always work.

As the last step, we add the dummy vertices to H' and H''. We also add the *inactive* men and women (the ones who will get matched to dummy vertices in S). We now add to S the edges $(a_j, d(a))$ for all inactive men a_j and similarly, the edges $(b_j, d(b))$ for all inactive women b_j. We also add self-loops to match each unmatched vertex with its copy on the other side, i.e., we add the edges (a_1, a) for each $a \in A$ that is unmatched in $\vec{p}$ and the edges (b_0, b)

for each $b \in B$ that is unmatched in $\vec{p}$. Thus the final matching S is a perfect matching in the graph G^* and it follows from the construction of S that $f(S) = \vec{p}$.

In order to prove that the matching S is stable in G^*, we show in Lemmas 7 and 8 that S has no blocking edge in G'. We can similarly show that S admits no blocking edge in G''. Regarding the other edges in G^*, no self-loop (a_1, a) or (b_0, b) can be a blocking edge since a is the least preferred neighbor of a_1 and similarly, b is the least preferred neighbor of b_0. Similarly, since the dummy vertex $d(a)$ is the least preferred neighbor of a_0 and since a_1 is the least preferred neighbor of $d(a)$, no edge $(a_i, d(a))$ can block S. It is the same with edges $(b_i, d(b))$, for $i = 0, 1$. Hence S is a stable matching in G^* and Theorem 5 follows.

▶ **Lemma 7.** *Let $a \in A$ be in level 0 in H' and b be any neighbor of a in G. Neither edge (a_0, b) nor edge (a_1, b) in G' can block S.*

Proof. The following are the three cases that we need to consider here and show that none is a blocking edge to S:
1. the edge (a_1, b),
2. the edge (a_0, b) where b is in level 1 in H',
3. the edge (a_0, b) where b is in level 0 in H'.

Consider Case 1. Since a is in level 0 in H', the vertex a_1 is matched to $d(a)$ in S. Since $d(a)$ is a_1's most preferred neighbor, it follows that the edge (a_1, b) cannot block S for any neighbor b.

Consider Case 2. The woman b is in level 1 and this implies that $S(b)$ is a level 1 vertex in H'. Since b prefers any level 1 neighbor to a level 0 neighbor in G', it follows that b prefers $S(b)$ to a_0, thus (a_0, b) cannot block S.

Consider Case 3. Since both a and b are in level 0 in H', we have $\alpha^*_a = 1$ and $\alpha^*_b = -1$. These α^*-values and $p_{(a,b)}$ satisfy the constraint corresponding to edge (a, b) in the description of the popular matching polytope $\mathcal{P}$. Thus we have $0 \geq \mathsf{value}_p(a, b)$, where $\mathsf{value}_p(a, b)$ is the right hand side of the constraint for (a, b) in $\mathcal{P}$. The following sub-cases can occur here (since $\mathsf{value}_p(a, b) \leq 0$):

(i) both the partners of a are better than b or both the partners of b are better than a

(ii) $p_{(a,b)} = 1/2$ and either a regards its other partner better than b or vice-versa

(iii) a has one partner better than b and the other worse than b and similarly, b has one partner better than a and the other worse than a

Sub-case (i) is straightforward and it is easy to see that (a_0, b) does not block S here. In sub-cases (ii) and (iii), we know that a woman b with $\alpha^*_b = -1$ gets matched to her better partner in H'. Thus in sub-case (ii) either b is matched to a (if a is b's better partner) or to a partner that b prefers to a. Similarly, in sub-case (iii), b gets matched to a neighbor that she prefers to a, thus (a_0, b) does not block S in any of these cases. This completes the proof of Lemma 7. ◀

▶ **Lemma 8.** *Let $a \in A$ be in level 1 in H' and b be any neighbor of a in G. Neither edge (a_0, b) nor edge (a_1, b) in G' can block S.*

Proof. The following are the three cases that we need to consider here and show that none is a blocking edge to S:
1. the edge (a_0, b),
2. the edge (a_1, b) where b is in level 0 in H',
3. the edge (a_1, b) where b is in level 1 in H'.

Consider Case 1. When b is in level 1 in H', she is matched to a level 1 man; since b prefers any level 1 neighbor to a level 0 neighbor in G', it follows that b prefers $S(b)$ to a_0, thus (a_0, b) cannot block S.

Let us consider the case when b is in level 0 in H'. So $\alpha^*_b = -1$. Since a is in level 1 in H', we have either $\alpha^*_a = -1$ or $\alpha^*_a = 0$. So $\mathsf{value}_p(a, b) \leq -1$. Hence b prefers her better partner to a and since b satisfies $\alpha^*_b = -1$, she gets matched to her better partner in H'. Thus (a_0, b) does not block S.

Consider Case 2. We will show that a_1 prefers his partner $S(a_1)$ to b. Either (i) $\alpha^*_a = -1$ in which case $\mathsf{value}_p(a, b) \leq -2$ or (ii) $\alpha^*_a = 0$ in which case $\mathsf{value}_p(a, b) \leq -1$.

In case (i), $\mathsf{vote}_a(b, S(a_1)) = -1$ and so a prefers $S(a_1)$ to b. In case (ii), $\mathsf{vote}_a(b, S(a_1)) \leq 0$ and so a prefers his better partner in $\vec{p}$ to b. It follows from the proof of Lemma 6 that if $\alpha^*_a = 0$, then the man a is matched to his better partner in H'. Thus (a_1, b) does not block S in either case.

Consider Case 3. There are four sub-cases here based on possible values of (α^*_a, α^*_b): (i) $(\alpha^*_a, \alpha^*_b) = (-1, 1)$, (ii) $(\alpha^*_a, \alpha^*_b) = (-1, 0)$, (iii) $(\alpha^*_a, \alpha^*_b) = (0, 1)$, and (iv) $(\alpha^*_a, \alpha^*_b) = (0, 0)$.

- Cases (i) and (iv) are analogous to case 3 in the proof of Lemma 7 since $\mathsf{value}_p(a, b)$ is at most 0 in both these cases and a similar proof holds here for both these cases.
- In case (ii) above, we have $\mathsf{value}_p(a, b) \leq -1$. So either (I) a prefers both his partners in $\vec{p}$ to b or vice-versa, in which case (a_1, b) does not block S or (II) $p_{(a,b)} = 1/2$ and both a and b prefer their other partners in $\vec{p}$ to each other, in which case $(a_1, b) \in S$.
- In case (iii) above, we know that both a and b get paired to their respective better partners in H' (since $\alpha^*_a = 0$ and $\alpha^*_b = 1$). We have $\mathsf{value}_p(a, b) \leq 1$ here. So either (I) a prefers its better partner in $\vec{p}$ to b or vice-versa (in which case (a_1, b) does not block S) or (II) $p_{(a,b)} = 1/2$ and both a and b prefer each other to their other partners in $\vec{p}$, in which case $(a_1, b) \in S$. Thus (a_1, b) does not block S in any of these cases. ◀

Thus we have shown that f is a surjective map from the set of stable matchings in G^* to the set of full half-integral matchings in G that are popular. It can be shown that if $\vec{p}$ is a popular half-integral matching that is *not* full, then the edge incidence vector of $\vec{p}$ is a convex combination of the edge incidence vectors of popular half-integral matchings that are full. Hence the extreme points of the convex hull $\mathcal{Q}$ of popular half-integral matchings are the full ones. Thus the description of $\mathcal{Q}$ can be obtained in a straightforward manner from the description of the stable matching polytope of G^*.

We have shown the following theorem.

▶ **Theorem 9.** *A min-cost popular half-integral matching in $G = (A \cup B, E)$ with strict preference lists and cost function $c : E \to \mathbb{Q}$ can be computed in polynomial time.*

Conclusions. We gave a simple description of the convex hull of popular half-integral matchings in a stable marriage instance $G = (A \cup B, E)$ with strict preference lists. This allowed us to solve the min-cost popular half-integral matching problem in G in polynomial time. The main open problem here is to settle the complexity of the min-cost popular matching in G.

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