Exponential Time Paradigms Through the Polynomial Time Lens*

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Abstract

We propose a general approach to modelling algorithmic paradigms for the exact solution of NP-hard problems. Our approach is based on polynomial time reductions to succinct versions of problems solvable in polynomial time. We use this viewpoint to explore and compare the power of paradigms such as branching and dynamic programming, and to shed light on the true complexity of various problems.

As one instantiation, we model branching using the notion of witness compression, i.e., reducibility to the circuit satisfiability problem parameterized by the number of variables of the circuit. We show this is equivalent to the previously studied notion of 'OPP-algorithms', and provide a technique for proving conditional lower bounds for witness compressions via a constructive variant of AND-composition, which is a notion previously studied in theory of preprocessing. In the context of parameterized complexity we use this to show that problems such as Pathwidth and Treewidth and Independent Set parameterized by pathwidth do not have witness compression, assuming $NP \not\subseteq coNP/poly$. Since these problems admit fast fixed parameter tractable algorithms via dynamic programming, this shows that dynamic programming can be stronger than branching, under a standard complexity hypothesis. Our approach has applications outside parameterized complexity as well: for example, we show if a polynomial time algorithm outputs a maximum independent set of a given planar graph on n vertices with probability $\exp(-n^{1-\epsilon})$ for some $\epsilon > 0$, then $NP \subseteq coNP/poly$. This negative result dims the prospects for one very natural approach to sub-exponential time algorithms for problems on planar graphs.

As two other illustrations (more exploratory) of our approach, we model algorithms based on inclusion-exclusion or group algebras via the notion of "parity compression", and we model a subclass of dynamic programming algorithms with the notion of "disjunctive dynamic programming". These models give us a way to naturally classify various parameterized problems with FPT algorithms. In the case of the dynamic programming model, we show that Independent Set parameterized by pathwidth is complete for this model.

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1 Introduction

The successes of theoretical computer science have often been driven by simple but general algorithmic approaches, or *paradigms*, leading to efficient algorithms in many different application domains. Indeed, paradigms such as divide-and-conquer, branching, dynamic programming and linear programming have been applied over and over to design algorithms.

A natural question that arises is to *quantify* the power and limitations of a given algorithmic paradigm. Doing so may have several benefits. It can help us understand what makes the paradigm effective. It can make algorithm design and analysis less ad hoc, with greater clarity about when and for which problems the paradigm is relevant. It can also enable us to compare various algorithmic paradigms with each other in terms of their power and usefulness. A crucial challenge in studying the power of algorithmic paradigms is the *modelling* question. We need a modelling framework which is rich enough to capture existing, successful algorithms within the paradigm. On the other hand, we need the modelling framework to be meaningfully restricted, so that we can prove interesting things about these models and the limits of their power. These goals are often in tension.

We aim to model exponential time algorithms. Understanding what can be computed in exponential time seems to be harder than understanding what can be computed in polynomial time, and less is known. In particular, showing general exponential-time lower bounds based on standard hypotheses about *polynomial-time* computation (for example, the hypotheses that $P \neq NP$ or that the Polynomial Hierarchy is infinite) seems out of reach. We propose to bypass this issue by arguing that several specific, established algorithmic paradigms can be modelled as polynomial-time reductions to (succinct) problems, so that limitations to their power may follow from these kinds of traditional hypotheses.

Approaches to algorithmic modelling can be broadly classified into syntactic and semantic approaches. Syntactic approaches attempt to faithfully represent the step-by-step operation of algorithms conforming to the method. Examples include the modelling of 1. DPLL algorithms by proof systems such as Resolution, 2. backtracking and dynamic programming by certain kinds of branching programs [1], 3. dynamic programming by feasible dominance relations [16], 4. linear programming by extended formulations [5]. These approaches, though natural, suffer from some drawbacks. The first is their lack of flexibility—they can fail to capture simple-looking variants of the method, e.g., the failure of proof systems to capture randomization. Second, in the search for accuracy, the models produced by such approaches can get quite complicated, which makes them hard to analyze.

Our models, in contrast, are semantic—we try to capture broad features of the algorithmic method rather than trying to model it in a step-by-step fashion. In particular, we allow arbitrary polynomial-time computations as constituent subroutines. This allows the model to flexibly accommodate preprocessing and natural variants of the method, and makes sense for the intended applications to exponential time algorithms. Our use of parameterization enables us to distinguish between algorithmic methods in a way that a traditional complexity-theoretic approach cannot. Although our approach is coarser than most syntactic approaches, it is more uniform, applying to a variety of algorithmic methods at once, and enables us to get useful information about the relative power of these methods.

Related Previous Work

A large number of problems have been shown to be *Fixed Parameter Tractable (FPT)*, i.e., solvable in time $O^*(f(k))$, where k is a parameter provided with each input, $O^*(\cdot)$ suppresses factors polynomial in the input size, and $f(\cdot)$ is some computable function. For

many problems we now know essentially the optimal running time: there is an $O^*(f(k))$ time algorithm and an $O^*(g(k))$ time algorithm for any $g(k) < f(o(k))$ contradicts the Exponential Time Hypothesis (ETH). For a few problems we even know that $O^*(f(k))$ time algorithms cannot be improved to $O^*(f(k)^{1-\Omega(1)})$ time algorithm under stronger hypotheses as the Strong ETH. In this work we are mostly interested in problems for which $f(k) = 2^{\mathrm{poly}(k)}$ - this is the case for most natural FPT parameterizations of NP-complete problems.

Kernelization. A natural paradigm to prove a problem is solvable in $O^*(2^{\mathrm{poly}(k)})$ time is preprocessing plus brute force: given an instance (x, k) of a parameterized problem, transform it in polynomial time to an instance (x', k') of the same problem where $|x'|, k'$ are polynomial in k (this part is called the *polynomial kernel*), and then solve the smaller instance using brute-force search.[1] The power of polynomial kernelization has been extensively investigated, and is by now fairly well understood. For many parameterized problems, we have either found a polynomial kernel, or showed they do not exist unless $NP \subseteq coNP/poly$; the latter is proved by providing an *(OR* or *AND)-composition*, and appealing to results in [3, 13, 11] and related works. This fits as an excellent starting point for our study since it gives a lower bound for a class of exponential-time algorithms modelled via polynomial-time reductions, and is conditional on an hypothesis concerning polynomial-time computation.

Branching. Another heavily used paradigm to solve a problem in $O^*(2^{\mathrm{poly}(k)})$ time is that of *branching*, or *bounded search trees*. A natural model for this paradigm is the model of *One-sided Probabilistic Polynomial (OPP) algorithms* proposed by Paturi and Pudlak [24] in their study of algorithms for satisfiability. OPP algorithms are polynomial-time algorithms with one-sided error which never accept no-instances but only detect yes-instances with small but non-trivial probability (called the *success probability*). An OPP algorithm with success probability $f(n)$ can be converted to a bounded-error randomized algorithm running in time $\mathrm{poly}(n)/f(n)$ just by taking the OR of $f(n)$ independent trials. On the other hand if an exponential-time algorithm can be thought of as traversing an exponential-size recursion tree which performs polynomial-time checks at leaves and returns true if at some leaf true is returned, then we can cast this as an OPP algorithm provided we are able to sample leaves of the branching tree in an efficient, nearly uniform way (in [24], this observation was attributed to Eppstein [12]). We would like to remark that OPP is more powerful than one might think at first sight as it also directly captures, for example, Schöning's algorithm [27].

Concerning lower bounds, Paturi and Pudlak [24] showed that OPP algorithms with success probability significantly better than 2^{-n} for circuit satisfiability on n variables would have unlikely consequences. Particularly relevant for our work is work by Drucker [10] showing a $2^{-n^{1-\epsilon}}$ upper bound of OPP algorithms' success probability for 3-CNF-SAT (for any $\epsilon > 0$), assuming $NP \not\subseteq coNP/poly$.

Several closely-related formalisms of branching algorithms have been proposed in the literature [4, 26, 31]. In the context of parameterized complexity, Marx proposed a study of branching [20, 21] using a model 'BFPT' of branching FPT algorithms.[2] Also relevant is work of Dantsin and Hirsch [8], which discusses a notion closely related to our notion of witness compression in the context of exact algorithms for Satisfiability, and provides lower bounds conditioned on ETH.

[1] That is, try all bit-strings and see if a certificate arises.
[2] That turns out to be equivalent to OPP algorithms with success probability $2^{-O(k)}$.

Our Contribution

In this work, we argue that many contemporary exponential-time algorithms can be rewritten as polynomial-time reductions to succinct version of problems in P, and we also give several concrete results on the applicability of specific algorithmic paradigms to different problems. We outline these results next.

Branching. Our main technical contributions address branching algorithms as modelled by OPP algorithms or equivalently witness compressions (defined below). Building on machinery developed by Drucker [10] we give lower bounds for constructive OPP algorithms. For instance:

▶ **Theorem 1.1.** *If there is a polynomial time algorithm that given a planar graph outputs a maximum independent set of n vertices with probability $\exp(-O(n^{1-\epsilon}))$ for some $\epsilon > 0$, then $NP \subseteq coNP/poly$.*

Note that $\exp(O(\sqrt{n}))$ time algorithms are known (e.g. [18]), so this indicates that a rich class of branching algorithms is incapable of exploiting planarity for solving independent set. We also give a simple OPP algorithm that actually establishes success probability $\exp(-O(n/\sqrt{\log(n)}))$.

Following a hashing lemma from [24], we observe that having an OPP algorithm with success probability $f(k)$ is equivalent to having a polynomial-time Monte Carlo reduction from the problem at hand to CKT-SAT[3] with $1/\log(f(k))$ input gates. Thus in the generic context sketched in this paper, the succinct problem corresponding to our model of branching is CKT-SAT. If $f(k) = 2^{-\mathrm{poly}(k)}$, there are witnesses for the problem of size $\mathrm{poly}(k)$ and we will refer to the polynomial time Monte Carlo reduction as a *polynomial witness compression* since a satisfying solution of the circuit that the reduction outputs can be seen as a witness for the original instance to be a yes-instance. We call a witness compression *Levin* or *constructive* if we can determine a solution of the original problem given a satisfying assignment of the circuit.

We define a type of reduction we call 'constructive AND-composition' that is closely related to AND-compositions from kernelization theory, and show that assuming $NP \not\subseteq coNP/poly$ no parameterized problem can both have a constructive AND-composition and a Levin polynomial witness compression. As one particular application, we use this to *separate* dynamic programming from branching (as modelled via OPP algorithms). Specifically, we show that Independent Set parameterized by pathwidth,[4] which is known to be FPT via a dynamic programming algorithm, does not have Levin polynomial witness compressions unless $NP \not\subseteq coNP/poly$. An important question[5] is how fast this problem can be solved using only polynomial space. In [19], the authors provide an $O^*(2^{O(\mathtt{pw}^2)})$-time and polynomial-space algorithm based on a tradeoff between dynamic programming and Savitch's theorem, but the folklore dynamic programming algorithm uses $O^*(2^{\mathtt{pw}})$ time and space. Our results thus indicate that branching algorithms of the OPP type, a very natural class of polynomial space algorithms, will not be useful here.

We emphasize that the model of OPP algorithms and witness compressions are powerful by observing that problems such as Steiner Tree, Long Path and Directed Feedback

[3] Refer to Section 2 for a definition.
[4] That is, we assume a path decomposition of width $\mathtt{pw}$ is given as input.
[5] This question first appeared in print in [19], but was explicitly asked before at least in [22].

Vertex Set (DFVS) do have polynomial witness compressions as a consequence of methods from previous works.

Kernelization. The above results on branching have a number of consequences for kernelization theory. To explain these, let us first stress that it seems that if a problem has an AND-composition it seems very likely it also has an constructive AND-composition since all known AND-compositions are known to be constructive.

There has been interest recently in relaxed versions of kernelization, such as OR-kernels, where rather than computing one small instance from the initial instance, we compute a list of instances, at least one of which is in the language if and only if the original instance was. It is easy to see that a polynomial witness compression is a far reaching generalization of OR-kernelization: if a problem has a OR-kernel the witness would indicate which output of the OR-kernel is a yes-instance along with a certificate of this instance being a YES. On the other hand, a problem as CKT-SAT with k input variables is known to not have polynomial kernelization assuming $NP \not\subseteq coNP/poly$ (see e.g., [9]) but trivially has a polynomial witness compression. Our observation thus implies that problems cannot have both constructive AND-compositions and OR-kernelizations simultaneously unless $NP \subseteq coNP/poly$.

Our connection between constructive AND-composition and witness compressions combined with the polynomial witness compressions for Steiner Tree, Long Path and DFVS implies that these problems do not have constructive AND-compositions, which is a clear indication that they do not admit AND-compositions as studied in kernelization theory. We feel this is a useful insight especially for DFVS because the existence of a polynomial compression for this is a major open problem [6], and since we currently only know how to exclude polynomial compressions via AND- and OR-compressions our connection indicates we probably should not look for AND-compressions.

Parity Compression. There are several other important paradigms that in many cases seem essential to known algorithms for various problems, especially to obtain the best known bounds on the function $f(k)$. In [24], the authors mention as examples the paradigms of exponential-time divide-and-conquer; inclusion-exclusion; dynamic programming; group algebra; and Voronoi cell decomposition; and they argue that 'OPP and its generalizations could serve as an excellent starting point for the study of exponential-time algorithms for NP-complete problems in general', although they leave such generalizations unspecified.

We further explore this direction, using our unifying perspective via succinct parameterized problems. Similar to witness compression, we define a notion of "parity compression" corresponding to reducibility to the problem $\oplus$CKT-SAT parameterized by the number of variables. The idea here is that algebraic and inclusion-exclusion based approaches to FPT algorithms often implicitly reduce the problem to a succinctly represented parity of exponentially many input bits, i.e, an instance of $\oplus$CKT-SAT. We illustrate this phenomenon by capturing the Long Path and K-Cycle problems in our model.

Disjunctive Dynamic Programming. We model a subclass of dynamic programming algorithms which we refer to as "disjunctive dynamic programming". Intuitively, this corresponds to dynamic programming tables whose entries are Boolean ORs of lexicographically-prior entries. We model this class via reducibility to the problem CNF-Reach, an instance of which is a directed graph succinctly encoded by a CNF, with the question being whether there is a source-sink path of a prescribed length in the graph. The parameter is the number

of variables of the CNF. Essentially, the existence of a path corresponds to a "trace" of a disjunctive dynamic programming algorithm with a YES answer.

More generally, one could study succinctness implemented by circuits rather than CNFs; however, the choice of CNFs has a nice benefit: it allows us to find natural complete problems for our model. Specifically, we show that Independent Set parameterized by pathwidth is complete for this model, thus in some sense dynamic programming is the "right" algorithmic technique for this problem. The completeness of Independent Set parameterized by pathwidth may also be interpreted as another signal that polynomial space algorithms for problems parameterized by pathwidth might be hard to find as they need to exploit the succinctness given by the CNF representation or otherwise need to improve over Savitch's theorem for short reachability. Let us remark that related research has been proposed earlier: the reduction as outlined here has been conjectured in the second author's PhD-thesis [23], and the aforementioned signal was remarked in [23, 2, 25]

Organization. This work is organized as follows: in Section 2 we provide a few preliminaries. Note that due to space constraints we do not cover basic definitions from parameterized complexity such as definitions of fixed-parameter tractability that provide context for our work; we refer the reader to a recent textbook [7]. Section 3 presents our main technical results, which are on branching algorithms. Section 4 introduces the model of parity compression, Section 5 introduces the model of disjunctive dynamic programming and in Section 6 we list a number of interesting directions for further research.

2 Preliminaries and Notation

For an integer p, $[p] := \{1, \ldots, p\}$, and $\binom{X}{p}$ denotes the family of size-p subsets of a set X.

Probabilistic Circuits. A *probabilistic circuit* is a (De Morgan) Boolean circuit $C(x, r)$ which, in addition to its input gates $x \in \{0,1\}^n$, has a designated set of "randomness gates" $r \in \{0,1\}^{\mathrm{poly}(n)}$. We say such a circuit computes a function $f(x)$ with success probability $p(n)$ if, for all $x \in \{0,1\}^n$, $\Pr_r[C(x,r) = f(x)] \geq p(n)$. Here the probability is taken over a uniform random setting to r. By Cook's transformation, any polynomial time randomized algorithm can be expressed as a (logspace-uniform) family of polynomial-size probabilistic circuits.

Problem Definitions. PC denotes the set of search problems whose solutions can be verified in polynomial time (following [14]). For $L \subseteq \{0,1\}^*$, χ_L denotes the characteristic vector of L. A *parameterized problem* is a set $Q \subseteq \{0,1\}^* \times \mathbb{N}$.

We use the following notation to define (parameterized) (search) problems in NP or PC: if k is some parameter of an unparameterized problem R, R/k denotes the associated problem parameterized by k. When a problem has a natural search version, we will use this to define it, as the decision version follows from the search version. We use L_Q to denote the decision version of a search problem Q. The following parameterized search problems will be important for this paper:

CKT-Sat **Parameter:** n

Instance: A Boolean circuit C on n variables.

Witness: An assignment $x \in \{0,1\}^n$ such that $C(x) = 1$.

(d)-CNF-SAT **Parameter:** n

Instance: A Boolean (d)-CNF-formula C on n variables.

Witness: An assignment $x \in \{0,1\}^n$ such that $C(x) = 1$.

For any search problem $R \in PC$, we define a search problem $AND(R)$ as follows:

$AND(R)$ **Parameter:** n

Instance: Instances $x_1, \ldots, x_t \in \{0,1\}^n$

Witness: $y_1, \ldots, y_t$ such that $(x_i, y_i) \in R$ for every i.

Reductions. For search problems[6] $Q, R \in PC$, a *Levin reduction* from Q to R consists of two polynomial time algorithms, A_1 and A_2, such that (i) $\exists y : (x,y) \in Q$ if and only if $\exists y' : (A_1(x), y') \in R$, (ii) if $(A_1(x), y') \in R$, then $(x, A_2(x,y')) \in Q$. A *Monte Carlo reduction* from language L to language L' is a randomized polynomial time algorithm that takes $x \in \{0,1\}^*$ as input and outputs $y \in \{0,1\}^*$ such that (i) if $x \notin L$ then $y \notin L'$, (ii) if $x \in L$ then $\Pr[y \in L'] \geq 1/4$. A *Levin Monte Carlo reduction* from search problem Q to search problem R is a pair of two randomized polynomial time algorithms A and B with the following properties: (i) A is a Monte Carlo reduction from L_Q to L_R mapping x to x', (ii) B takes as input x, x' and y', and if $(x', y') \in R$, then with probability $1/4$, B outputs y such that $(x, y) \in Q$.

Success Probability of Polynomial Time Algorithms. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$. We say that an algorithm *solves a parameterized problem Q with success probability f*, if given (x,k) it returns NO if $(x,k) \notin L_Q$ and YES with probability at least $f(|x|,k)$ if $(x,k) \in L_Q$. Moreover, if $Q \in PC$, *it finds solutions for Q* with probability at least f if given (x,k) it returns NO if $(x,k) \notin L_Q$ and it returns a certificate for $(x,k) \in L_Q$ with probability at least $f(|x|,k)$, otherwise. Note that an algorithm finding solutions for Q also solves Q.

By standard boosting arguments we see that if there is a polynomial time algorithm solving Q or finding solutions for Q with probability at least f, then for any polynomial p there is also a polynomial time algorithm solving Q or finding solutions for Q with probability at least $\min\{\frac{1}{2}, p(|x|)f(|x|,k)\}$. Therefore, if $f(|x|,k)$ is $1/(\text{poly}(|x|)f(k))$, we say it solves or finds solutions for Q with probability at least $f'(k)$ where $f'(k) = f(1,k)$.

Non-deterministic Direct Product Reductions. For a function $f : A \to B$ and integer t, we denote $f^{\otimes t} : A^t \to B^t$ to be the t-fold direct product of f, e.g., for $x_1, \ldots, x_t \in A$ we let $f^{\otimes t}(x_1, \ldots, x_t) = (f(x_1), \ldots, f(x_t))$. The following result will be crucial for this work:

▶ **Theorem 2.1** (Theorem 1.2 of [10]). *Let $f = \{f_N\}$ be a family of Boolean functions on N input bits, and suppose that $f \notin NP/poly \cap coNP/poly$. Let $100 \leq t(N) \leq \text{poly}(N)$ be a parameter and let $\{C_N\}_{N>0}$ be any family of polynomial-size probabilistic circuits outputting $t(N)$ bits. Then for infinitely many choices of N and $x \in \{0,1\}^{N \times t(N)}$,*

$$\Pr[C_N = f_N^{\otimes t(N)}(x)] < \exp(-\Omega(t(N)). \tag{2.1}$$

[6] In this work, we implicitly cast a parameterized (search) problem as a normal (search) problem by omitting the parameter where convenient.

3 Branching via OPP Algorithms and Witness Compressions

In this section we present our results on branching algorithms. We first formally define the notion of constructive AND-compositions and state how they exclude OPP algorithms. Then we formally introduce witness compressions and show their close relation with OPP algorithms. Subsequently, we point out implications to parameterized complexity.

Constructive AND-Compositions and Their Consequences.

▶ **Definition 3.1** (Constructive AND-composition)**.** Let L be a search problem, Q be a parameterized search problem and d be a constant. We say that a pair of algorithms (A,B) is a *constructive AND-composition of degree d from L into Q* if the following conditions hold:

1. A is given $x_1, x_2, \ldots, x_t$ and outputs an instance $(x, k) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $\sum_{i=1}^{t} |x_i|$ such that $k \leq \text{poly}(\max_i |x_i| \log(t))$ and $|x| \leq \text{poly}(\max_i |x_i| \log(t)) t^d$,
2. if for every i there exist y_i such that $(x_i, y_i) \in L$, then B does the following: B takes as input $x_1, x_2, \ldots, x_t$, the instance (x, k), and a certificate y such that $(x, k, y) \in Q$, and outputs y_i for every i such that

$$\Pr[\forall i : (x_i, y_i) \in L] \geq \exp\left(-\text{poly}\left(\max_i |x_i|\right) \log(t)\right).$$

This is closely related to AND-compositions as studied in kernelization complexity (see e.g. [7, Section 15.1.3]): it is more strict in the sense that the reduction needs to be Levin, but more general in the sense that we only need a weak probabilistic guarantee on the output. We will see that even constructive AND-compositions of degree 1 with trivial parameterizations have interesting consequences.

▶ **Theorem 3.2.** *If there is a constructive AND-composition of degree d from a PC-hard search problem L into a parameterized search problem Q, then no polynomial time algorithm finds solutions for every instance (x, k) of Q with probability $\exp(-\text{poly}(k)|x|^{1/d-\Omega(1)})$, unless $NP \subseteq coNP/poly$.*

As one concrete application we obtain the Theorem as mentioned in the introduction:

▶ **Theorem 1.1** (restated)**.** *If there is a polynomial time algorithm that, given a planar graph, outputs a maximum independent set of n vertices with probability $\exp(-n^{1-\epsilon})$ for some $\epsilon > 0$, then $NP \subseteq coNP/poly$.*

Proof. Let L be the following search problem: given the adjacency list of a planar graph G and integer θ, find an independent set of G of size at least θ. The decision variant of this problem NP-complete and by inspecting the known reductions, the problem is also seen to be PC-complete. Let Q be L with a trivial parameterization (e.g., the parameter equals 1). We now give a constructive AND-composition of degree 1 from L to Q. Given instances $(G_1 = (V_1, E_1), \theta_1), \ldots, (G_t = (V_t, E_t), \theta_t)$, create an instance (G, θ^*) of Q where G is the disjoint union of $G_1, \ldots, G_t$ (i.e. it has each graph G_i as a connected component in it), and θ^* is picked uniformly at random from $\{1, \ldots, \sum_{i=1}^{t} |V_i|\}$. We see that with probability $1/\sum_{i=1}^{t} |V_i| \geq \exp\left(-\text{poly}\left(\max_i |x_i|\right) \log(t)\right)$, we have that θ^* equals the size of the maximum independent set. Moreover, if we are given a maximum independent set of G, its intersection with every component must be a maximum independent set in that component so if all instances are YES instances we find maximum independent sets of size at least θ_i in G_i for every i. Since (G, θ) is represented with $\text{poly}(\max_i |V_i|) t \log(t)$ bits, we therefore found a constructive AND-composition of degree 1, and no polynomial time algorithm finds solutions

for Q with probability $\exp(-|x|^{1-\Omega(1)})$ by Theorem 3.2. This implies the statement since $|x|$ is $n \log n$ for n-vertex graphs. ◀

We remark the naïve guessing procedure here is not optimal (the proof is postponed to the full version):

▶ **Theorem 3.3.** *There exists a polynomial time algorithm that outputs a maximum independent set of a planar graph on n vertices with probability* $\exp(-n/\sqrt{\log n})$.

Witness Compressions. We will now give an equivalent interpretation of OPP algorithms that paves the way for defining models of other paradigms in the next sections.

▶ **Definition 3.4** ((Levin) Witness Compression)**.** A *(Levin) $h(k, N)$-witness compression* for a parameterized (search) problem Q is a (Levin) Monte-Carlo reduction from Q to CKT-Sat that maps (x, k) with $|x| = N$ to (y, n) with $n \leq h(k, N)$.

Note that having a $h(k, N)$-witness compression is equivalent to having a $h(k, N + \lg(N))$-witness compression since we can brute-force over all assignments of $\lg(N)$ input bits in polynomial time. We say a (Levin) $h(k, N)$-witness compression is *polynomial* if $h(k, N) \leq \text{poly}(k)$ (or equivalently $\text{poly}(k) + \log(N)$). The following lemma shows the equivalence of witness compression and OPP algorithms.

▶ **Lemma 3.5.** *A parameterized (search) problem has a* (Levin) $h(k, N)$-witness compression *if and only if there is a polynomial time algorithm solving it (respectively, finding solutions) with success probability at least* $2^{-h(k,N)}$.

The proof is postponed to the full version. The forward direction in both variants is immediate. For the backward direction, we use the 'Hash-Down lemma' from [24] to prove both variants. Our proof of the equivalence takes advantage of the fact that we allow randomized reductions in the definition of witness compression. If we were to only allow deterministic reductions in the definition, the equivalence would still hold under a sufficiently strong derandomization hypothesis - we omit the details.

We emphasize the power of polynomial witness compression by revisiting a few FPT-algorithms and observing that they give rise to efficient witness compressions. Marx [21] observes that Vertex Cover and Feedback Vertex Set have a witness compression with $h(k)$ linear. Here, we add a few non-trivial witness compressions to this list with $h(k)$ quasi-linear. The relevant problem statements and proof of the following theorem can be found in the full version. All these results go via the connection from Theorem 3.5.

▶ **Theorem 3.6.** Steiner Tree *and* Long Path *have Levin $O(k \log k)$-witness compressions, and* DFVS *has a Levin $O(k \log^3(k))$-witness compression.*

Implications to Parameterized Complexity. As mentioned in the introduction, polynomial witness compression appears significantly more powerful than polynomial kernelization. Indeed if a problem has an OR-kernelization,[7] it is easily seen we have a polynomial witness compression:

[7] Where rather than computing one small instance from the initial instance as in kernelization, we compute a list of instances at least one of which is in the language if and only if the original instance was, see e.g.[17] where the name 'disjunctive kernelization' was used.

▶ **Observation 3.7.** If Q admits a polynomial (Levin) OR-kernelization to a problem in NP, it admits a polynomial (Levin) witness compression.

On the other hand, let us remark here that there may be problems in NP that admit polynomial compressions[8] but no polynomial witness compression: Wahlström [30] gives an interesting polynomial compression of the K-cycle problem (see the full version for the problem definition) to a language that is not in NP, and remarks that this seems to separate polynomial kernelization from polynomial compression since it is not clear whether K-cycle has polynomial witness compressions.

The above connection is relevant for kernelization complexity because Theorem 3.8 suggests that parameterized problems with AND-compositions have no OR-kernelizations. Another interesting consequence obtained by combining Theorem 3.2 and Theorem 3.6 is (since the problems at hand are easily seen to be PC-complete):

▶ **Theorem 3.8.** Steiner Tree, Long Path, *and* DFVS *do not admit constructive AND-compositions unless* $NP \subseteq coNP/poly$.

As mentioned before, this is a useful fact especially for DFVS because the existence of a polynomial compressions for this is a big open problem [6], and we currently only know how to exclude polynomial compressions via AND- and OR-compressions Theorem 3.8. So this indicates researchers attacking this open problem probably should not look for AND-compressions.

Another useful implication concerns the following parameterized problem:

Independent Set (IS/pw) **Parameter:** pw

Instance: A graph G, path decomposition of G of width pw, integer θ.

Witness: An independent set of G of size at least θ.

As mentioned in the introduction, it is an important open question how fast this problem can be solved using only polynomial space. We show that branching algorithms (which is a subset of all polynomial space algorithms) are not useful here:

▶ **Theorem 3.9.** *Suppose a polynomial time algorithm takes as input a path decomposition of width* pw *of a graph G on n vertices, and outputs with probability* $\exp(-\text{poly}(\texttt{pw})n^{1-\Omega(1)})$ *a maximum independent set of G, then $NP \subseteq coNP/poly$.*

Since the proof is very similar to the proof of Theorem 1.1, it is postponed to the full version. Let us remark that several other interesting graph problems admit constructive AND-compositions. For example, following Lemma 7 from [3] we have that

▶ **Observation 3.10.** Let L be a parameterized graph search problem such that for any pair of graphs G_1 and G_2, and integer $k \in \mathbb{N}$, $(G_1, k) \in L \wedge (G_2, k) \in L \leftrightarrow (G_1 \cup G_2, k)$ where $G_1 \cup G_2$ is the disjoint union of G_1 and G_2. Then L admits a constructive AND-composition of degree 1.

Similar as in [3], this implies hardness for several problems. We refer to [3] for the definitions of these problems since our only goal is to point out the applicability of our framework.

[8] A compression is a kernelization where the target problem might be different (and crucially here, not even in NP).

▶ **Theorem 3.11.** *No polynomial time algorithm finds solutions for any instance* (x, k) *of* Cutwidth, Modified Cutwidth, Pathwidth, Branchwidth, Search Number, Gate Matrix Layout, *and* Front Size *with probability* $\exp(-\text{poly}(k)|x|^{1-\Omega(1)})$ *unless* $NP \not\subseteq coNP/poly$.

Proof. Following [3], we have by Observation 3.10 that all the above problems admit constructive AND-compositions. By inspection it can be seen that the reductions from these problems to CKT-Sat (which exist since all the above problems are NP-complete) are all Levin reductions, thus all problems are PC-complete. The claim follows from Theorem 3.2. ◀

4 Parity Compression

As mentioned before, witness compression tightly captures a large part of contemporary FPT-algorithms, but still far from all of them. Motivated by this, we propose the following natural generalization of witness compression, based on the definition of witness compression as a reduction to CKT-Sat. A *parity compression* is a polynomial time Monte Carlo reduction from the problem at hand to the $\oplus$CKT-Sat problem, defined as follows:

$\oplus$CKT-Sat **Parameter:** n

Instance: A Boolean circuit C on n variables

Asked: Whether the parity of the size of the set $\{x \in \{0,1\}^n : C(x) = 1\}$ is odd.

Analogous to witness compressions, we can interpret parity compressions as exponential time algorithms by solving the resulting $\oplus$CKT-Sat instance in time $2^n|x|^{O(1)}$ (the analogue of witness compressions was to solve the CKT-Sat instance by simple brute-force enumeration). By an easy application of the Isolation Lemma of [29], there is a polynomial time Monte Carlo reduction from CKT-Sat to $\oplus$CKT-Sat that increases the number of input variables by $O(\text{polylog}(|C|))$. Thus parity compression is a generalization of witness compression. No polynomial-time reduction in the reverse direction is known, and such a reduction (even randomized) would imply a collapse of PH in light of Toda's theorem [28].

While we are not yet able to show lower bounds for parity compressions since its study is still in its infancy, we do argue in the full version that several interesting contemporary algorithms (mainly, ones using inclusion/exclusion or group algebra) are exponential parities. This motivates a very interesting future research direction:

▶ **Open Problem 1.** Find non-trivial evidence against a polynomial time Monte Carlo reduction from CKT-Sat on n-variable circuits to $\oplus$CKT-Sat on n' circuits where $n' << n$.

Another goal would be to further exclude more polynomial space paradigms that are able to solve Independent Set parameterized by the pathwidth:

▶ **Open Problem 2.** Find evidence against a polynomial time Monte Carlo reduction from IS/pw to $\oplus$CKT-Sat where $n = o(\texttt{pw}^2)$.

5 Disjunctive Dynamic Programming

One natural other algorithmic paradigm unaddressed so far (as highlighted in Theorem 3.9 and Open Problem 2) is dynamic programming. We focus in this work on a subclass of dynamic programming algorithms which we call 'disjunctive dynamic programming' - this corresponds to dynamic programming tables where the entries are Boolean ORs of previous

entries. Specifically, we say a disjunctive dynamic programming algorithm is a polynomial time parameter reduction to the following problem:

CNF-Reach **Parameter:** n

Instance: A CNF-formula $\varphi : \{0,1\}^n \to \{0,1\}$ with m clauses and n even, integer $\ell = \text{poly}(n)$.

Witness: $x^1, \ldots, x^\ell \in \{0,1\}^{n/2}$ with $x^1 = 0 \cdots 0$,$x^\ell = 1 \cdots 1$ and $\varphi(x^i x^{i+1}) = 1$ for every $0 \leq i \leq \ell - 1$.

In the full version we show that IS/pw is almost equivalent to CNF-Reach[9] by giving almost tight reductions between the two problems. Thus IS/pw can be seen as complete for the class of problems efficiently solvable with disjunctive dynamic programming. We feel such a reduction expresses the hardness of a problem typically solved with dynamic programming better than e.g., a reduction to CNF-Sat or even CKT-Sat since these problem do have small witnesses and polynomial-space algorithms. Next to Theorem 3.9, this may be seen as additional evidence that finding fast space-efficient algorithms for IS/pw might be very hard (e.g., we either need to exploit the succinct representation via CNF-formula's or find new algorithms for the directed reachability problem).

We also show that an algorithm for Set Cover is a disjunctive dynamic programming algorithm: we reduce Set Cover to CNF-Reach in the full version.

6 Directions for Further Research

We conclude this paper with a few open questions. First, for several problems, the existence of polynomial witness compression is open (see the full version for missing problem definitions):

▶ **Open Problem 3.** Do Subset Sum, Knapsack, Knapsack/Weight-Value, K-Cycle or Disjoint Paths have polynomial witness compressions?

Note that currently, it is not clear whether there exists a parameterized problem that has a polynomial compression but no polynomial witness compression, although as suggested in [30] the K-Cycle would be a good candidate for such a problem.

One algorithmic paradigm not addressed is exponential time divide and conquer [15], which is also closely related to applications of Savitch's Theorem as used by [19]:

▶ **Open Problem 4.** Is there a good model of exponential time divide and conquer based on reductions to a succinct version of a natural problem? Can it solve IS/pw in $O^*(2^{o(\mathtt{pw}^2)})$ time?

Ambitiously, having finer-grained lower bounds would be very insightful:

▶ **Open Problem 5.** Can we rule out linear witness compressions for some problems with quasilinear witness compressions under standard assumptions?

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