A Preliminary Investigation of Satisfiability Problems Not Harder Than 1-In-3-SAT

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─── **Abstract** ───

The *parameterized satisfiability problem* over a set of Boolean relations Γ (SAT(Γ)) is the problem of determining whether a conjunctive formula over Γ has at least one model. Due to Schaefer's dichotomy theorem the computational complexity of SAT(Γ), modulo polynomial-time reductions, has been completely determined: SAT(Γ) is always either tractable or NP-complete. More recently, the problem of studying the relationship between the complexity of the NP-complete cases of SAT(Γ) with restricted notions of reductions has attracted attention. For example, Impagliazzo et al. studied the complexity of k-SAT and proved that the worst-case time complexity increases infinitely often for larger values of k, unless 3-SAT is solvable in subexponential time. In a similar line of research Jonsson et al. studied the complexity of SAT(Γ) with algebraic tools borrowed from clone theory and proved that there exists an NP-complete problem SAT($R_{1/3}^{\neq\neq\neq01}$) such that there cannot exist any NP-complete SAT(Γ) problem with strictly lower worst-case time complexity: the *easiest NP-complete* SAT(Γ) *problem*. In this paper we are interested in classifying the NP-complete SAT(Γ) problems whose worst-case time complexity is lower than 1-in-3-SAT but higher than the easiest problem SAT($R_{1/3}^{\neq\neq\neq01}$). Recently it was conjectured that there only exists three satisfiability problems of this form. We prove that this conjecture does not hold and that there is an infinite number of such SAT(Γ) problems. In the process we determine several algebraic properties of 1-in-3-SAT and related problems, which could be of independent interest for constructing exponential-time algorithms.

1   Introduction

The *parameterized satisfiability problem* (SAT(Γ)) is the computational decision problem of, given a conjunctive formula over a constraint language Γ, determining whether this formula is satisfiable. Some notable examples of problems that can be formulated as SAT(Γ) problems include 1-in-3-SAT, k-SAT, SAT, and not-all-equal-SAT. For example, if we let $R_{1/3} = \{(0,0,1),(0,1,0),(1,0,0)\}$ then SAT($\{R_{1/3}\}$) can be seen as an alternative formulation of the monotone 1-in-3-SAT problem, i.e., 1-in-3-SAT without negation. Hence, SAT(Γ) is in general NP-complete. It is also known that SAT(Γ) is either tractable, i.e., solvable in polynomial time, or NP-complete, for all choices of Γ [25]. Assume that we instead are interested in a more fine-grained analysis of the worst-case time complexity of all

$$\langle\{R_{1/3}\}\rangle_{\not\exists}$$
$$\uparrow$$
$$\langle\{R_{1/3}^{01}\}\rangle_{\not\exists}$$
$$\uparrow$$
$$\langle\{R_{1/3}^{\neq 01}\}\rangle_{\not\exists}$$
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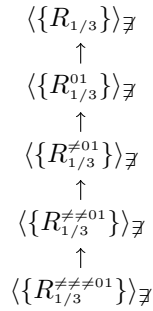
■ **Figure 1** The conjectured structure of weak partial co-clones below $\langle\{R_{1/3}\}\rangle_{\not\exists}$. A directed arrow $A \to B$ means that $A \subset B$.

NP-complete SAT(Γ) problems. Clearly, the fact that two problems SAT(Γ) and SAT(Δ) are both NP-complete does not reveal a great amount of information about their respective worst case time complexity, except that they are both unlikely to be solvable in polynomial time. For example, the monotone 1-in-3-SAT problem is solvable in $O(1.0984^n)$ time [30], where n denotes the number of variables in a given instance. On the other hand, 3-SAT is only known to be solvable in $O(1.308^n)$ time [11], and more generally it is known that the worst-case time complexity of k-SAT increases infinitely often for increasing values of k [13] – assuming 3-SAT is not solvable in $O(c^n)$ time for arbitrary $c > 1$. Hence, the family of NP-complete SAT(Γ) problems seems to contain members with wildly distinct worst-case time complexity, and it is safe to say that we currently cannot provide a complete explanation of this phenomena. More generally, say that SAT(Γ) is *easier* than SAT(Δ) if SAT(Γ) is solvable in $O(c^n)$ time whenever SAT(Δ) is solvable in $O(c^n)$ time, where n denotes the number of variables in a given instance. In symbols, we denote this by SAT(Δ) $\leq$ SAT(Γ). Jonsson et al. studied the complexity of SAT($\cdot$) viz a viz the ordering $\leq$ using *partial clone theory* [15]. The details of this approach is explained in greater detail in Section 2, but for the moment let us be content with the fact that there exists a lattice $\mathcal{X}$ such that every constraint language Γ can be mapped to an element $\langle\Gamma\rangle_{\not\exists} \in \mathcal{X}$, such that SAT($\Gamma$) $\leq$ SAT(Δ) if $\langle\Gamma\rangle_{\not\exists} \subseteq \langle\Delta\rangle_{\not\exists}$. An element $\langle\Gamma\rangle_{\not\exists} \in \mathcal{X}$ is usually referred to as a *weak system*, or a *weak partial co-clone*, and is a well-studied relational algebra known to consist of all relations definable by conjunctive logical formulas over Γ [23]. Hence, the lattice of weak partial co-clones can be used to compare SAT(Γ) problems with respect to worst-case time complexity. With the help of this algebraic approach Jonsson et al. gave a classification of the minimal element $\langle\{R_{1/3}^{\neq\neq\neq 01}\}\rangle_{\not\exists}$ of this lattice and proved that SAT($\{R_{1/3}^{\neq\neq\neq 01}\}$) results in the *easiest NP-complete* SAT($\cdot$) problem [15].

In this paper we continue the classification of NP-complete SAT(Γ) problems that in a certain precise sense are small elements in the ordering $\leq$. More specifically, we are interested in determining the structure of constraint languages resulting in NP-complete SAT($\cdot$) problems which are not computationally harder than monotone 1-in-3-SAT. In symbols, this can be rephrased as determining all constraint languages Γ such that SAT($\{R_{1/3}^{\neq\neq\neq 01}\}$) $\leq$ SAT(Γ) $\leq$ SAT($\{R_{1/3}\}$), or, in the language of clone theory, determining all constraint languages Γ satisfying $\langle\{R_{1/3}^{\neq\neq\neq 01}\}\rangle_{\not\exists} \subset \langle\Gamma\rangle_{\not\exists} \subset \langle\{R_{1/3}\}\rangle_{\not\exists}$. We begin by recapitulating the necessary technical prerequisites in Section 2, and give a brief introduction to the algebraic approach for studying the complexity of satisfiability problems. In Section 3 we introduce novel methods for better understanding the structure of algebras of the form $\langle\Gamma\rangle_{\not\exists}$, and in particular the structure of $\langle\{R_{1/3}\}\rangle_{\not\exists}$. This classification is then used in Section 4 where

we give a preliminary description of the satisfiability problems below $\mathrm{SAT}(\{R_{1/3}\})$ in the ordering $\leq$. We prove that this is a rich and complicated structure and that the cardinality of the set $\{\langle \Gamma \rangle_{\not\exists} \mid \langle \{R_{1/3}^{\neq\neq\neq01}\}\rangle_{\not\exists} \subset \langle \Gamma \rangle_{\not\exists} \subset \langle \{R_{1/3}\}\rangle_{\not\exists}\}$ is at least countably infinite. We remark that this contradicts a recent conjecture that this set consists of only three elements [20]. See Figure 1 for a visualization of the conjectured structure between $\langle \{R_{1/3}\}\rangle_{\not\exists}$ and $\langle \{R_{1/3}^{\neq\neq\neq01}\}\rangle_{\not\exists}$, and Section 2.4 for definitions of the involved relations.

From an algebraical point of view our results are a natural investigation of the largely unexplored lattice of weak partial co-clones. We remark that weak partial co-clones are useful not only for studying the exact complexity of problems [16, 19], but also for complexity classifications of optimisation problems and non-standard logical reasoning problems [3, 4, 27]. So far one of the limiting factors of this approach is the fact that very little is known of the relationship between weak partial co-clones and their dual objects, *partial polymorphisms*, which is in stark contrast to the status of the currently flourishing research program of classifying finite domain constraint satisfaction problems by properties of polymorphisms [2]. Similar observations have been made by for example Börner et al. [7], by Schölzel [28], and by Bulatov in the context of counting problems [8].

From a more pragmatic point of view, our results show that even for extremely simple constraint languages such as $\{R_{1/3}\}$, trying to fully characterize $\mathrm{SAT}(\Gamma)$ problems with a lower worst-case time complexity is an extremely difficult task. However, as we discuss in Section 5, even though we cannot hope to achieve a complete understanding of the complexity of $\mathrm{SAT}(\{R_{1/3}\})$ with our algebraic approach, we believe that similar studies are likely to open up new possibilities for studying the complexity of $\mathrm{SAT}(\{R_{1/3}\})$ and related problems.

2 Preliminaries

In this section we briefly review some basic concepts that will be needed later on, starting with a formal definition of the parameterized $\mathrm{SAT}(\cdot)$ problem and ending with universal algebra and partial clone theory.

2.1 The Parameterized SAT(·) Problem

Let $\mathbb{B} = \{0, 1\}$ and let $BR = \bigcup_{i=1} \mathbb{B}^i$ denote the set of all Boolean relations. Given $R \in \mathbb{B}^k$ we let $\mathrm{ar}(R) = k$. A *constraint language* is a set of relations $\Gamma \subseteq BR$. The *parameterized satisfiability problem* over a constraint language Γ ($\mathrm{SAT}(\Gamma)$) is defined as follows.

INSTANCE: A set V of variables and a set C of constraint applications $R(v_1, \ldots, v_k)$ where $R \in \Gamma$, $\mathrm{ar}(R) = k$, and $v_1, \ldots, v_k \in V$.
QUESTION: Is there a function $f : V \to \mathbb{B}$ such that $(f(v_1), \ldots, f(v_k)) \in R$ for each $R(v_1, \ldots, v_k)$ in C?

When $\Gamma = \{R\}$ we typically write $\mathrm{SAT}(R)$ instead of $\mathrm{SAT}(\{R\})$. As an example, if we let $R_{1/3} = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ then the problem $\mathrm{SAT}(R_{1/3})$ can be seen as an alternative formulation of monotone 1-in-3-SAT, i.e., the 1-in-3-SAT problem without negation.

2.2 Polymorphisms, Clones and Co-Clones

A Boolean function $f : \mathbb{B}^n \to \mathbb{B}$ is said to *preserve* a k-ary Boolean relation R if for every $t_1, \ldots, t_n \in R$ it holds that $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[k], \ldots, t_n[k])) \in R$. Here, $t_i[j]$ denotes the j-th element of the tuple t_i. If f preserves R we say that f is a *polymorphism* of R, and similarly we say that f is a polymorphism of a constraint language Γ if it preserves

each relation in Γ. Given a constraint language Γ we let $\mathrm{Pol}(\Gamma)$ denote the set of all polymorphisms of Γ. Sets of the form $\mathrm{Pol}(\Gamma)$ are usually referred to as *clones* and it is well-known that clones are (1) closed under functional composition and (2) contain all functions which projects one of its arguments. To be a bit more precise, the first condition means that if $f, g_1, \ldots, g_m \in \mathrm{Pol}(\Gamma)$, where the f has arity m and the functions $g_1, \ldots, g_m$ all have the same arity n, then the *composition* $f \circ (g_1, \ldots, g_m)$, the function defined as $f \circ (g_1, \ldots, g_m)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$ for all $x_1, \ldots, x_n \in \mathbb{B}$, is included in $\mathrm{Pol}(\Gamma)$. The second condition means that $\mathrm{Pol}(\Gamma)$ for each n and each $1 \leq i \leq n$ contains every function π_i^n defined as $\pi_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$. Functions of the form π_i^n are called *projection functions*. We let $\Pi_{\mathbb{B}}$ denote the set of all Boolean projection functions.

There is also a similar notion to clones on the relational side. Say that a k-ary relation R has a *primitive positive definition* (p.p. definition) over a constraint language Γ if there exists a conjunctive formula over k variables $x_1, \ldots, x_k$ over Γ, possibly making use of existential quantification and the equality relation $\mathrm{Eq} = \{(0,0),(1,1)\}$, such that R is the set of models of this formula. In symbols, we denote such a p.p. definition as $R(x_1, \ldots, x_k) \equiv \exists y_1, \ldots, y_{k'} . R_1(\mathbf{x_1}) \wedge \ldots \wedge R_m(\mathbf{x_m})$, where each $R_i \in \Gamma \cup \{\mathrm{Eq}\}$ and each $\mathbf{x_i}$ is an $\mathrm{ar}(R_i)$-ary tuple of variables over $x_1, \ldots, x_k, y_1, \ldots, y_{k'}$. If we let $\langle \Gamma \rangle$ be the smallest set of relations containing Γ which is closed under p.p. definitions we obtain a *relational clone*, or a *co-clone*. The relationship between clones and co-clones is given in the following theorem.

▶ **Theorem 1** ([5, 6, 9])**.** *Let Γ and Γ' be two constraint languages. Then $\Gamma \subseteq \langle \Gamma' \rangle$ if and only if $\mathrm{Pol}(\Gamma') \subseteq \mathrm{Pol}(\Gamma)$.*

This inverse relationship between two closure operators is in general known as a *Galois connection*, and using Theorem 1 it is not difficult to prove the following result.

▶ **Theorem 2** ([14])**.** *Let Γ and Γ' be two finite constraint languages. If $\mathrm{Pol}(\Gamma') \subseteq \mathrm{Pol}(\Gamma)$, then $\mathrm{SAT}(\Gamma)$ is polynomial-time many-one reducible to $\mathrm{SAT}(\Gamma')$.*

Hence, the clone of a constraint language determines the complexity of a satisfiability problem up to polynomial time reductions. Unfortunately, as noted in Section 1, the mere fact that two $\mathrm{SAT}(\cdot)$ problems are polynomial-time equivalent does not offer any insight into their worst-case time complexity. To study this we need a more fine-grained algebra, which in our case consists of *partial functions* instead of total functions.

2.3 Partial Polymorphisms, Strong Partial Clones and Weak Partial Co-Clones

In this section we investigate clones based on partial functions instead of total functions, and show that Theorem 2 can be significantly strengthened with these notions. First, an n-ary Boolean partial function f is a map $f : X \to \mathbb{B}$ where $X \subseteq \mathbb{B}^n$. In other words f is a function that is allowed to be undefined for one or more sequences of arguments. Given a partial function $f : X \to \mathbb{B}$, $X \subseteq D^n$, we let $\mathrm{dom}(f) = X$ and $\mathrm{ar}(f) = n$. If $u = (x_1, \ldots, x_n) \in \mathrm{dom}(f)$ we use the shorthand notation $f(u)$ instead of $f(x_1, \ldots, x_n)$. A partial function g is said to be a *subfunction* of a partial function f if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $g(u) = f(u)$ for all $u \in \mathrm{dom}(g)$. A set of partial functions is *strong* if it is closed under taking subfunctions. If f is an n-ary partial function and $X \subseteq \mathrm{dom}(f)$ we write $f_{|X}$ for the subfunction of f satisfying $\mathrm{dom}(f_{|X}) = X$. Composition of partial functions is defined similarly to the case of total functions. Hence, if f is an m-ary partial function and $g_1, \ldots, g_m$ are n-ary partial functions then the composition is defined as $f \circ (g_1, \ldots, g_m)(x_1, \ldots, x_n) =$

$f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$, and the resulting function is defined for every tuple $(x_1, \ldots, x_n) \in \bigcap_{i=1}^m \mathrm{dom}(g_i)$ such that $(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \in \mathrm{dom}(f)$.

We now say that an n-ary partial function f is a *partial polymorphism* of a k-ary relation R if $(f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[k], \ldots, t_n[k])) \in R$ for all $t_1, \ldots, t_n \in R$ such that $\{(t_1[1], \ldots, t_n[1]), \ldots, (t_1[k], \ldots, t_n[k])\} \subseteq \mathrm{dom}(f)$. If we let $\mathrm{pPol}(\Gamma)$ denote the set of all partial polymorphisms of a constraint language Γ then the resulting set of partial functions is known as a *strong partial clone*. A strong partial clone $\mathrm{pPol}(\Gamma)$ is a set of partial functions which (1) is closed under composition of partial functions, i.e., if $f, g_1, \ldots, g_{\mathrm{ar}(f)} \in \mathrm{pPol}(\Gamma)$ then $f \circ (g_1, \ldots, g_{\mathrm{ar}(f)}) \in \mathrm{pPol}(\Gamma)$, (2) contains all projection functions, and (3) is closed under taking subfunctions. It is worth noting that the second and third conditions are equivalent to the condition that $\mathrm{pPol}(\Gamma)$ contains all partial projection functions, i.e., the total projection functions and all their possible subfunctions, and we let $\Pi_{\mathbb{B}}^p$ denote this set. Given a set of partial functions F we let $[F]_s$ denote the smallest strong partial clone which contains F. The set F is called a *base* of $[F]_s$. Similar to the relationship between clones and co-clones we can find a Galois connection between strong partial clones and sets of relation satisfying certain closure properties. In symbols, we say that a k-ary relation R has a *quantifier-free primitive positive definition* (q.f.p.p. definition) over a constraint language Γ if $R(x_1, \ldots, x_k) \equiv R_1(\mathbf{x_1}) \wedge \ldots \wedge R_m(\mathbf{x_m})$, where each $R_i \in \Gamma \cup \{\mathrm{Eq}\}$ and each $\mathbf{x_i}$ is an $\mathrm{ar}(R_i)$-ary tuple of variables over $x_1, \ldots, x_k$. We then let $\langle \Gamma \rangle_{\not\exists}$ denote the smallest set of relations containing Γ which is closed under q.f.p.p. definitions, and as usual we write $\langle R \rangle_{\not\exists}$ whenever $\Gamma = \{R\}$. Sets of the form $\langle \Gamma \rangle_{\not\exists}$ are known as *weak partial co-clones*, or *weak systems*. We have the following Galois connection.

▶ **Theorem 3** ([9, 24]). *Let Γ and Γ' be two constraint languages. Then $\Gamma \subseteq \langle \Gamma' \rangle_{\not\exists}$ if and only if $\mathrm{pPol}(\Gamma') \subseteq \mathrm{pPol}(\Gamma)$.*

Using this Galois connection Jonsson et al. [15] proved that the partial polymorphisms of a finite constraint language determines the complexity of the satisfiability problem up to $O(c^n)$ time complexity, where n denotes the number of variables in a given instance.

▶ **Theorem 4** ([15]). *Let Γ and Γ' be two finite constraint languages. If $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(\Gamma')$ and $\mathrm{SAT}(\Gamma)$ is solvable in $O(c^n)$ time, then $\mathrm{SAT}(\Gamma')$ is solvable in $O(c^n)$ time, too.*

Hence, a better understanding of the lattice of Boolean strong partial clones could lead to a better understanding of the worst-case time complexity of satisfiability problems. Unfortunately, the cardinality of this lattice is equal to the continuum [1], and besides some minor results [18, 26], the details of this structure is largely unknown. In particular, it is known that the set $\{\mathrm{pPol}(\Gamma) \mid \mathrm{SAT}(\Gamma) \text{ is NP-complete}\}$ is of uncountably infinite cardinality. With a reformulation of Schaefer's dichotomy theorem [25] we can state this result even more precisely as follows (where $\neg x$ denotes the unary function $\neg x = 1 - x$).

▶ **Theorem 5** ([29]). *The sets $\{\mathrm{pPol}(\Gamma) \mid \mathrm{Pol}(\Gamma) = \Pi_{\mathbb{B}}\}$ and $\{\mathrm{pPol}(\Gamma) \mid \mathrm{Pol}(\Gamma) = [\neg x]\}$ are of uncountably infinite cardinality.*

Note that this theorem immediately implies that there exists strong partial clones of the form $\mathrm{pPol}(\Gamma)$ which does not admit a finite base. Perhaps more surprising is the following theorem which states that a strong partial clone of the form $\mathrm{pPol}(\Gamma)$ cannot have a finite base whenever Γ is a finite constraint language such that $\mathrm{Pol}(\Gamma) \subseteq [\neg x]$.

▶ **Theorem 6** ([21]). *Let Γ be a finite constraint language such that $\mathrm{Pol}(\Gamma) \subseteq [\neg x]$. Then $\mathrm{pPol}(\Gamma)$ does not admit a finite base.*

2.4 The Easiest NP-complete SAT($\cdot$) Problem

In Jonsson et al [15] it is proven that the "easiest" NP-complete SAT($\cdot$) problem can be seen as a variant of 1-in-3-SAT where each variable occurring in a constraint has a complementary variable and each constraint contains two variables forced to constant values. We can represent this problem as SAT($R_{1/3}^{\neq\neq\neq01}$) where $R_{1/3}^{\neq\neq\neq01}(x_1, x_2, x_3, x_4, x_5, x_6, c_0, c_1) \equiv R_{1/3}(x_1, x_2, x_3) \wedge R_{1/3}(c_0, c_0, c_1) \wedge R_{1/3}(x_1, x_4, c_0) \wedge R_{1/3}(x_2, x_5, c_0) \wedge R_{1/3}(x_3, x_6, c_0)$, where we have choosen the variable names c_0 and c_1 to indicate that they are forced constant values. Hence, we have that $R_{1/3}^{\neq\neq\neq01} = \{(0, 0, 1, 1, 1, 0, 0, 1), (0, 1, 0, 1, 0, 1, 0, 1), (1, 0, 0, 0, 1, 1, 0, 1)\}$.

We are now interested in trying to determine the weak partial co-clones $\langle\Gamma\rangle_{\not\exists}$ such that $\langle R_{1/3}\rangle_{\not\exists} \supset \langle\Gamma\rangle_{\not\exists} \supset \langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$. In particular, is it possible to find a constraint language Γ such that $\langle\Gamma\rangle_{\not\exists}$ *covers* $\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$? By this we mean that there does not exist any Δ such that $\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists} \subset \langle\Delta\rangle_{\not\exists} \subset \langle\Gamma\rangle_{\not\exists}$. Since $R_{1/3}^{\neq\neq\neq01}$ has arity 8 and $R_{1/3}$ has arity 3, a reasonable first attempt to investigate this question is to gradually remove arguments from $R_{1/3}^{\neq\neq\neq01}$. Hence, let $R_{1/3}^{\neq\neq01}$, $R_{1/3}^{\neq01}$, and $R_{1/3}^{01}$ be the relations obtained from $R_{1/3}^{\neq\neq\neq01}$ by removing one, two, and three complemented arguments. That is, $R_{1/3}^{\neq\neq01} = \{(0, 0, 1, 1, 1, 0, 1), (0, 1, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0, 1)\}$, $R_{1/3}^{\neq01} = \{(0, 0, 1, 1, 0, 1), (0, 1, 0, 1, 0, 1), (1, 0, 0, 0, 0, 1)\}$, and $R_{1/3}^{01} = \{(0, 0, 1, 0, 1), (0, 1, 0, 0, 1), (1, 0, 0, 0, 1)\}$. We remark that $\mathrm{Pol}(R_{1/3}^{\neq\neq\neq01}) = \mathrm{Pol}(R_{1/3}^{\neq\neq01}) = \mathrm{Pol}(R_{1/3}^{\neq01}) = \mathrm{Pol}(R_{1/3}^{01}) = \mathrm{Pol}(R_{1/3}) = \Pi_{\mathbb{B}}$, i.e., that the only total polymorphisms of these relations are the projections. In Lagerkvist [20] it was proven that $\langle R_{1/3}\rangle_{\not\exists} \subset \langle R_{1/3}^{01}\rangle_{\not\exists} \subset \langle R_{1/3}^{\neq01}\rangle_{\not\exists} \subset \langle R_{1/3}^{\neq\neq01}\rangle_{\not\exists} \subset \langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$. Hence, the inclusions in Figure 1 are correct. However, the question of whether these weak partial co-clones also cover one another was left open. We will see in Section 4 that there in fact exist an infinite number of weak partial co-clones between $\langle R_{1/3}\rangle_{\not\exists}$ and $\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$.

3 The Partial Polymorphisms of $R_{1/3}$, $R_{1/3}^{01}$, $R_{1/3}^{\neq01}$, $R_{1/3}^{\neq\neq01}$ and $R_{1/3}^{\neq\neq\neq01}$

We now want to investigate the structure of weak partial co-clones between $\langle R_{1/3}\rangle_{\not\exists}$ and $\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$, and determine to which extent the conjectured structure in Figure 1 is complete. Since investigating this problem purely from a relational perspective appears to be complicated, we in this section tackle the problem of characterizing the partial polymorphisms of $R_{1/3}$, $R_{1/3}^{01}$, $R_{1/3}^{\neq01}$, $R_{1/3}^{\neq\neq01}$, and $R_{1/3}^{\neq\neq\neq01}$. As a shorthand, we let $\vec{0}^n$ denote an n-ary tuple consisting only of zeroes, and similarly for $\vec{1}^n$. We write $\bar{x}$ for the complement of a tuple x. Recall that if $t_1, \ldots, t_n$ are k-ary tuples and f an n-ary partial function we by $f(t_1, \ldots, t_n)$ denote the k-ary tuple resulting from applying f componentwise to the elements of $t_1, \ldots, t_n$. To be able to more conveniently refer to the n-ary tuples of the form $(t_1[i], \ldots, t_n[i])$ we let $Cols(t_1, \ldots, t_n) = ((t_1[1], \ldots, t_n[1]), \ldots, (t_n[k], \ldots, t_n[k]))$, and $ColsSet(t_1, \ldots, t_n) = \{(t_1[1], \ldots, t_n[1]), \ldots, (t_n[k], \ldots, t_n[k])\}$. For example, we have that $Cols((0, 0, 1, 0, 1), (0, 1, 0, 0, 1), (1, 0, 0, 0, 1)) = ((0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 0, 0), (1, 1, 1))$. We now introduce our method for describing domains of partial functions.

▶ **Definition 7.** We make the following definitions.

- A set $\{\omega_1, \ldots, \omega_k\} \subseteq \mathbb{B}^n$ is an *exact cover of* $R_{1/3}$ if (1) $2 \leq k \leq 3$ and (2) for every $i \in \{1, \ldots, n\}$ it holds that $\omega_1[i] + \ldots + \omega_k[i] = 1$.
- A set $C \cup \{\vec{0}^n, \vec{1}^n\} \subseteq \mathbb{B}^n$ is an *exact cover of* $R_{1/3}^{01}$ if C is an exact cover of $R_{1/3}$.
- A set $C \cup \{\omega, \vec{0}^n, \vec{1}^n\} \subseteq \mathbb{B}^n$ is an *exact cover of* $R_{1/3}^{\neq01}$ if (1) $\bar{\omega} \in C$ and (2) C is an exact cover of $R_{1/3}$.
- A set $C \cup \{\omega_1, \omega_2, \vec{0}^n, \vec{1}^n\} \subseteq \mathbb{B}^n$ is an *exact cover of* $R_{1/3}^{\neq\neq01}$ if (1) $\overline{\omega_1}, \overline{\omega_2} \in C$ and (2) C is an exact cover of $R_{1/3}$.
- A set $C \cup \{\omega_1, \omega_2, \omega_3, \vec{0}^n, \vec{1}^n\} \subseteq \mathbb{B}^n$ is an *exact cover of* $R_{1/3}^{\neq\neq\neq01}$ if (1) $\overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3} \in C$ and (2) C is an exact cover of $R_{1/3}$.

For each $R \in \{R_{1/3}, R_{1/3}^{01}, R_{1/3}^{\neq 01}, R_{1/3}^{\neq \neq 01}, R_{1/3}^{\neq \neq \neq 01}\}$ we say that an exact cover C of R is *maximal* if there is no $C' \supset C$ which is an exact cover of R. Given $T \subseteq \mathbb{B}^n$ we let $\mathrm{Cover}_R(T) = \{C \subseteq T \mid C$ is a maximal exact cover of $R\}$. We have the following link between exact covers and tuples from the relations $R_{1/3}, R_{1/3}^{01}, R_{1/3}^{\neq 01}, R_{1/3}^{\neq \neq 01}$, and $R_{1/3}^{\neq \neq \neq 01}$.

▶ **Lemma 8.** *Let $C \subseteq \mathbb{B}^n$. Then, for each $R \in \{R_{1/3}, R_{1/3}^{01}, R_{1/3}^{\neq 01}, R_{1/3}^{\neq \neq 01}, R_{1/3}^{\neq \neq \neq 01}\}$ it holds that C is maximal exact cover of R if and only if there exists $t_1, \ldots, t_n \in R$ such that $ColsSet(t_1, \ldots, t_n) = C$.*

Proof. We only prove the case when $R = R_{1/3}$ since the other relations can be proven with symmetrical arguments. Let $C \subseteq \mathbb{B}^n$ be a maximal exact cover of $R_{1/3}$. There are two cases to consider. First, assume that $|C| = 3$ and let $C = \{\omega_1, \omega_2, \omega_3\}$. For each $i \in \{1, \ldots, n\}$ we by definition have that $\omega_1[i] + \omega_2[i] + \omega_3[i] = 1$, and hence, $(\omega_1[i], \omega_2[i], \omega_3[i]) \in R_{1/3}$. Therefore, it is easy to find $t_1, \ldots, t_n \in R_{1/3}$ such that $ColsSet(t_1, \ldots, t_n) = C$. Second, assume that $|C| = 2$. In this case $C = \{\vec{0}^n, \vec{1}^n\}$, from which it folows that $ColsSet(t_1, \ldots, t_n) = \{\vec{0}^n, \vec{1}^n\}$ whenever $t_i = t_j$ for all $i, j \in \{1, \ldots, n\}$. Now assume that $t_1, \ldots, t_n \in R_{1/3}$. We must prove that $ColsSet(t_1, \ldots, t_n) = \{\omega_1, \ldots, \omega_k\}$ is an exact cover of $R_{1/3}$. But this is trivial since (1) $2 \leq |\{\omega_1, \ldots, \omega_k\}| \leq 3$ and (2) $\omega_1[i] \ldots + \omega_k[i] = 1$ for each $i \in \{1, \ldots, n\}$. ◀

We now have everything in place to state the main results of this section.

▶ **Theorem 9.** *Let f be an n-ary function. Then $f \in \mathrm{pPol}(R_{1/3})$ if and only if*
1. *$f_{|C} \in \Pi_{\mathbb{B}}^p$ for every $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$, or*
2. *$f(\vec{0}^n) = 1$, $\vec{1}^n \notin \mathrm{dom}(f)$, and for every $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ either (1) $f(\omega) = f(\overline{\omega}) = 0$ if $C = \{\vec{0}^n, \omega, \overline{\omega}\}$ or (2) $f_{|C} \in \Pi_{\mathbb{B}}^p$ if $\vec{0}^n \notin C$.*

Proof. We begin with the completeness part of the proof. Let $f \in \mathrm{pPol}(R_{1/3})$ be an n-ary partial function. We must prove that f satisfies condition (1) or condition (2). Assume first that there exists $\{\omega_1, \omega_2, \omega_3\} \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ such that $f_{|\{\omega_1, \omega_2, \omega_3\}}$ is not a partial projection function. Now assume that $\{\vec{0}^n, \omega, \overline{\omega}\} \notin \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$. Then there exists $t_1, \ldots, t_n \in R_{1/3}$ such that (1) $ColsSet(t_1, \ldots, t_n) = \{\omega_1, \omega_2, \omega_3\}$ and (2) $|\{t_1, \ldots, t_n\}| = 3$. This implies that $f_{|\{\omega_1, \omega_2, \omega_3\}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \notin R_{1/3}$. Since this contradicts the original assumption, it must be the case that $\vec{0}^n \in C$ for some $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$. Assume first that $f(\vec{0}^n) = 0$ and that $f(\omega) = f(\overline{\omega}) = 0$ for some $\{\vec{0}^n, \omega, \overline{\omega}\} \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$. In this case $f \notin \mathrm{pPol}(R_{1/3})$, and similarly when $f(\vec{0}^n) = 0$ and $f(\omega) = f(\overline{\omega}) = 1$. Last, assume that $f(\vec{0}^n) = 0$ and that $f(\omega) = \overline{f(\overline{\omega})}$. In this case $f_{|\{\vec{0}^n, \omega, \overline{\omega}\}}$ is a partial projection, and, furthermore, $f_{|C}$ must be a partial projection for every $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$, i.e., we are in case (1). Assume now instead that $\vec{0}^n \in C$ for some $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ and that $f(\vec{0}^n) = 1$. In this case one can easily verify that if there exists $\{\vec{0}^n, \omega, \overline{\omega}\} \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ such that either $f(\omega) = 1$ or $f(\overline{\omega}) = 1$ then $f \notin \mathrm{pPol}(R_{1/3})$. Hence, $f(\vec{0}^n) = 1$, $f(\omega) = f(\overline{\omega}) = 0$ for all $\{\vec{0}^n, \omega, \overline{\omega}\} \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$, and $f_{|C}$ is a partial projection for all $C \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ such that $\vec{0}^n \notin C$. This means that we are in case (2).

For the soundness part of the proof, assume that f is an n-ary partial function fullfiling condition (1). Let $t_1, \ldots, t_n \in R_{1/3}$ and let $\{\omega_1, \omega_2, \omega_3\} = ColsSet(t_1, \ldots, t_n)$. We must prove that either $(f(\omega_1), f(\omega_2), f(\omega_3)) \in R_{1/3}$ or that $f(\omega_i)$ is undefined for some $i \in \{1, 2, 3\}$. Clearly, if $\{\omega_1, \omega_2, \omega_3\} \in \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$ then, by assumption, $(f(\omega_1), f(\omega_2), f(\omega_3)) \in R_{1/3}$ since $f_{|\{\omega_1, \omega_2, \omega_3\}}$ is a partial projection. Otherwise, if $\{\omega_1, \omega_2, \omega_3\} \notin \mathrm{Cover}_{R_{1/3}}(\mathrm{dom}(f))$, then $f(\omega_i)$ must be undefined for some $i \in \{1, 2, 3\}$. Now assume that f is an n-ary partial function fullfiling condition (2). Let $t_1, \ldots, t_n \in R_{1/3}$. Observe that if $\vec{0}^n \notin ColsSet(t_1, \ldots, t_n)$

then it directly follows that $f(t_1, \ldots, t_n) \in R_{1/3}$ or that $f(t_1, \ldots, t_n)$ is undefined, by recapitulating the proof of the preceding paragraph. Hence, assume that $ColsSet(t_1, \ldots, t_n) = \{\vec{0}^n, \omega, \overline{\omega}\}$. By assumption we have that either $f(t_1, \ldots, t_n) \in R_{1/3}$ (since $f(\vec{0}^n) = 1$ and $f(\omega) = f(\overline{\omega}) = 0$), or that $\omega \notin \mathrm{dom}(f)$ or that $\overline{\omega} \notin \mathrm{dom}(f)$.   ◄

Similarly we can characterize the partial polymorphisms of $R_{1/3}^{01}$, $R_{1/3}^{\neq 01}$, $R_{1/3}^{\neq\neq 01}$, and $R_{1/3}^{\neq\neq\neq 01}$.

▶ **Theorem 10.** *Let $R \in \{R_{1/3}^{01}, R_{1/3}^{\neq 01}, R_{1/3}^{\neq\neq 01}, R_{1/3}^{\neq\neq\neq 01}\}$. Let f be an n-ary partial function. Then $f \in \mathrm{pPol}(R)$ if and only if*

1. $\vec{0}^n \notin \mathrm{dom}(f)$, *or*
2. $\vec{1}^n \notin \mathrm{dom}(f)$, *or*
3. $\{\vec{0}^n, \vec{1}^n\} \subseteq \mathrm{dom}(f)$ *and* $f_{|C} \in \Pi_{\mathbb{B}}^p$ *for every* $C \in \mathrm{Cover}_R(\mathrm{dom}(f))$.

Proof. We only prove the case when $R = R_{1/3}^{\neq 01}$ since the other cases are entirely analogous. For soundness, assume that f is an n-ary partial function satisfying condition (1), (2), or (3). Let $t_1, \ldots, t_n \in R_{1/3}^{\neq 01}$. We have two cases to consider: either (1) $ColsSet(t_1, \ldots, t_n) \not\subseteq \mathrm{dom}(f)$ in which case $f(t_1, \ldots, t_n)$ is undefined; or (2) $ColsSet(t_1, \ldots, t_n) \subseteq \mathrm{dom}(f)$ in which case $ColsSet(t_1, \ldots, t_n) \in \mathrm{Cover}_{R_{1/3}^{\neq 01}}(\mathrm{dom}(f))$ and $f(t_1, \ldots, t_n) \in R_{1/3}^{\neq 01}$.

For completeness, let $f \in \mathrm{pPol}(R_{1/3}^{\neq 01})$ be an n-ary partial function such that $\{\vec{0}^n, \vec{1}^n\} \subseteq \mathrm{dom}(f)$. First, it is easy to verify that $f(\vec{0}^n) = 0$ and that $f(\vec{1}^n) = 1$, since otherwise $f \notin \mathrm{pPol}(R_{1/3}^{\neq 01})$. Let $C \in \mathrm{Cover}_{R_{1/3}^{\neq 01}}(\mathrm{dom}(f))$. We prove that $f_{|C}$ must be a partial projection function. There are a few different cases depending on the size of C. First note that the size of a maximum exact $R_{1/3}^{\neq 01}$-cover of $\mathrm{dom}(f)$ is always either 6, 4, or 2.

First, assume that $|C| = 6$ and let $C = \{\omega_1, \omega_2, \omega_3, \overline{\omega_1}, \vec{0}^n, \vec{1}^n\}$, where $\{\omega_1, \omega_2, \omega_3\}$ is an exact cover of $R_{1/3}$. According to Lemma 8, there exists $t_1, \ldots, t_n \in R_{1/3}^{\neq 01}$ such that $ColsSet(t_1, \ldots, t_n) = C$ and such that $\{t_1, \ldots t_n\} = R_{1/3}^{\neq 01}$. If $f_{|C}$ is not a projection on C then $f_{|C}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \notin R_{1/3}^{\neq 01}$.

Second, assume that $|C| = 4$ and let $C = \{\omega, \overline{\omega}, \vec{0}^n, \vec{1}^n\}$. Then, according to Lemma 8 there exists $t_1, \ldots, t_n \in R_{1/3}^{\neq 01}$ such that $ColsSet(t_1, \ldots, t_n) = C$. Now note that if $f_{|C}$ is not a projection, then either $f(\omega) = f(\overline{\omega}) = 0$ or that $f(\omega) = f(\overline{\omega}) = 1$, and in both these cases $f \notin \mathrm{pPol}(R_{1/3}^{\neq 01})$. Hence, $f(\omega) = \overline{f(\overline{\omega})}$, from which it follows that $f_{|C}$ is a partial projection.
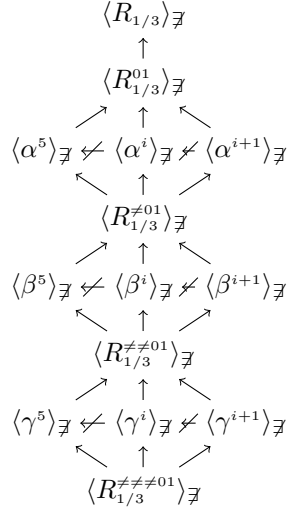
Third, assume that $|C| = 2$, and observe that this implies that $C = \{\vec{0}^n, \vec{1}^n\}$, due to the assumption that C is maximal. Since, by assumption, $f(\vec{0}^n) = 0$ and $f(\vec{1}^n) = 1$, it follows that $f_{|C}$ is a projection.   ◄

Hence, even though $\mathrm{pPol}(R_{1/3})$, $\mathrm{pPol}(R_{1/3}^{01})$, $\mathrm{pPol}(R_{1/3}^{\neq 01})$, $\mathrm{pPol}(R_{1/3}^{\neq\neq 01})$, and $\mathrm{pPol}(R_{1/3}^{\neq\neq\neq 01})$ cannot be described through finite bases (by Theorem 6), we could still obtain a complete understanding of the involved partial functions. We remark that the partial polymorphisms of 1-in-k-SAT has been described in earlier work [22], but in contrast to Theorem 9 and Theorem 10, the proposed classificaton only describes a finite subset of partial polymorphisms.

4   The Structure Between $\langle R_{1/3} \rangle_{\not\exists}$ and $\langle R_{1/3}^{\neq\neq\neq 01} \rangle_{\not\exists}$

In this section we use the results from Section 3 in order to investigate the structure of the weak partial co-clones between $\langle R_{1/3} \rangle_{\not\exists}$ and $\langle R_{1/3}^{\neq\neq\neq 01} \rangle_{\not\exists}$. Before delving deeper into the forthcoming proofs the reader is advised to first consult Figure 2 for a visualization of the main results. We concentrate on weak partial co-clones below $\langle R_{1/3}^{01} \rangle_{\not\exists}$ since it is readily seen that the problems $\mathrm{SAT}(R_{1/3})$ and $\mathrm{SAT}(R_{1/3}^{01})$ have the same worst-case time complexity. It is in fact not difficult to prove that there cannot exist any R such that $|R| \leq 3$ and such that $\langle R_{1/3}^{01} \rangle_{\not\exists} \subset \langle R \rangle_{\not\exists} \subset \langle R_{1/3}^{\neq 01} \rangle_{\not\exists}$, and the same also holds for weak partial co-clones between

$$\langle R_{1/3}\rangle_{\not\exists}$$
$$\uparrow$$
$$\langle R_{1/3}^{01}\rangle_{\not\exists}$$
$$\nearrow \quad \uparrow \quad \nwarrow$$
$$\langle\alpha^5\rangle_{\not\exists} \not\leftarrow \langle\alpha^i\rangle_{\not\exists} \not\leftarrow \langle\alpha^{i+1}\rangle_{\not\exists}$$
$$\nwarrow \quad \uparrow \quad \nearrow$$
$$\langle R_{1/3}^{\neq01}\rangle_{\not\exists}$$
$$\nearrow \quad \uparrow \quad \nwarrow$$
$$\langle\beta^5\rangle_{\not\exists} \not\leftarrow \langle\beta^i\rangle_{\not\exists} \not\leftarrow \langle\beta^{i+1}\rangle_{\not\exists}$$
$$\nwarrow \quad \uparrow \quad \nearrow$$
$$\langle R_{1/3}^{\neq\neq01}\rangle_{\not\exists}$$
$$\nearrow \quad \uparrow \quad \nwarrow$$
$$\langle\gamma^5\rangle_{\not\exists} \not\leftarrow \langle\gamma^i\rangle_{\not\exists} \not\leftarrow \langle\gamma^{i+1}\rangle_{\not\exists}$$
$$\nwarrow \quad \uparrow \quad \nearrow$$
$$\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists}$$

Figure 2 The structure of weak partial co-clones below $\langle R_{1/3}\rangle_{\not\exists}$. An arrow of the form $A \to B$ means that $A \subset B$. An arrow of the form $A \not\to B$ means that $A \not\subset B$.

all other cases. Hence, to find elements between we must consider relations of cardinality strictly larger than 3. With this as a guidance we define the following class of relations, where $\neq$ denotes the binary inequality relation $\{(0,1),(1,0)\}$ and $R_{1/k}$ denotes the k-ary relation $\{(x_1,\ldots,x_k) \in \mathbb{B}^k \mid \Sigma_{i=1}^k x_i = 1\}$.

▶ **Definition 11.** Let $k \geq 5$. The relation α^k is defined as

$$\alpha^k(x_1,\ldots,x_k,y_1,\ldots,y_{k-3},z_1,\ldots,z_{k-3},w_1,\ldots,w_{k-4},c_0,c_1) \equiv R_{1/k}(x_1,\ldots,x_k)\wedge$$

$$\bigwedge_{i=1}^{k-3} R_{1/i+2}(x_1,\ldots,x_{i+1},y_i) \wedge \bigwedge_{i=1}^{k-3} y_i \neq z_i \wedge \bigwedge_{i=3}^{k-2} R_{1/3}(x_1,x_i,w_{i-2}) \wedge R_{1/3}(c_0,c_0,c_1).$$

The relation β^k for $k \geq 5$ is defined similarly but with $k-2$ additional arguments which are the complement of $x_1, x_3, \ldots, x_{k-1}$. Hence, let

$$\beta^k(x_1,\ldots,x_k,y_1,\ldots,y_{k-3},z_1,\ldots,z_{k-3},w_1,\ldots,w_{k-4},v_1,\ldots,v_{k-2},c_0,c_1) \equiv$$

$$\alpha^k(x_1,\ldots,x_k,y_1,\ldots,y_{k-3},z_1,\ldots,z_{k-3},w_1,\ldots,w_{k-4},c_0,c_1) \wedge x_1 \neq z_1 \wedge \bigwedge_{i=3}^{k-1} x_i \neq v_{i-1}.$$

Finally, the relation γ^k for $k \geq 5$ can be defined as

$$\gamma^k(x_1,\ldots,x_k,y_1,\ldots,y_{k-3},z_1,\ldots,z_{k-3},w_1,\ldots,w_{k-4},v_1,\ldots,v_k,c_0,c_1) \equiv$$

$$\alpha^k(x_1,\ldots,x_k,y_1,\ldots,y_{k-3},z_1,\ldots,z_{k-3},w_1,\ldots,w_{k-4},c_0,c_1) \wedge \bigwedge_{i=1}^{k} x_i \neq v_i.$$

Later in this section we will see that $\langle R_{1/3}^{\neq\neq\neq01}\rangle_{\not\exists} \subset \langle\gamma^k\rangle_{\not\exists} \subset \langle R_{1/3}^{\neq\neq01}\rangle_{\not\exists} \subset \langle\beta^k\rangle_{\not\exists} \subset \langle R_{1/3}^{\neq01}\rangle_{\not\exists} \subset \langle\alpha^k\rangle_{\not\exists} \subset \langle R_{1/3}^{01}\rangle_{\not\exists}$ for each $k \geq 5$. The intuition behind the relation α^k is as follows.

- The k first arguments $x_1,\ldots,x_k$ encode a 1-in-k-constraint.
- Since $R_{1/k} \notin \langle R_{1/3}^{01}\rangle_{\not\exists}$ [20], we have to add arguments to make it q.f.p.p. definable by $R_{1/3}^{01}$.
- These arguments are $y_1,\ldots,y_{k-3}$, their complements $z_1,\ldots,z_{k-3}$, and the two constant arguments c_0 and c_1.

$\blacksquare$ To make sure that the resulting relation is not q.f.p.p. definable by $R_{1/3}^{\neq 01}$ we also need the additional arguments $w_1, \ldots, w_{k-4}$, which do not have any complementary arguments.

The relations β^k and γ^k can be understood in a similar way. We then have the following straightforward Lemma which states that the weak partial co-clones of the relations α^k, β^k and γ^k, are proper subsets of $\langle R_{1/3}^{01} \rangle_{\not\exists}$, $\langle R_{1/3}^{\neq 01} \rangle_{\not\exists}$, and $\langle R_{1/3}^{\neq\neq 01} \rangle_{\not\exists}$, respectively.

▶ **Lemma 12.** $\langle R_{1/3}^{01} \rangle_{\not\exists} \supset \langle \alpha^k \rangle_{\not\exists}$, $\langle R_{1/3}^{\neq 01} \rangle_{\not\exists} \supset \langle \beta^k \rangle_{\not\exists}$, and $\langle R_{1/3}^{\neq\neq 01} \rangle_{\not\exists} \supset \langle \gamma^k \rangle_{\not\exists}$ for each $k \geq 5$.

Proof. Let $k \geq 5$. We only consider the case $\langle R_{1/3}^{01} \rangle_{\not\exists} \supset \langle \alpha^k \rangle_{\not\exists}$ since the other cases are similar. We begin by proving that $\alpha^k \in \langle R_{1/3}^{01} \rangle_{\not\exists}$ which implies that $\langle R_{1/3}^{01} \rangle_{\not\exists} \supseteq \langle \alpha^k \rangle_{\not\exists}$. The base case when $k = 5$ is simple:

$$\alpha^5(x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, z_2, w_1, c_0, c_1) \equiv$$
$$R_{1/3}^{01}(x_1, x_2, y_1, c_0, c_1) \wedge R_{1/3}^{01}(x_3, y_2, z_1, c_0, c_1) \wedge$$
$$R_{1/3}^{01}(c_0, y_1, z_1, c_0, c_1) \wedge R_{1/3}^{01}(x_4, x_5, z_2, c_0, c_1) \wedge$$
$$R_{1/3}^{01}(c_0, y_2, z_2, c_0, c_1) \wedge R_{1/3}^{01}(x_1, x_3, w_1, c_0, c_1).$$

For the inductive step assume that $\alpha^{k-1} \in \langle R_{1/3}^{01} \rangle_{\not\exists}$. We can then implement α^k as follows.

$$\alpha^k(x_1, \ldots, x_k, y_1, y_2, \ldots, y_{k-4}, y_{k-3}, z_1, z_2, \ldots, z_{k-4}, z_{k-3}, w_1, \ldots, w_{k-5}, w_{k-4}, c_0, c_1) \equiv$$
$$\alpha^{k-1}(x_1, \ldots, x_{k-2}, y_{k-3}, y_1, \ldots, y_{k-4}, z_1, \ldots, z_{k-4}, w_1, \ldots, w_{k-5}, c_0, c_1) \wedge$$
$$R_{1/3}^{01}(z_{k-3}, x_{k-1}, x_k, c_0, c_1) \wedge R_{1/3}^{01}(c_0, y_{k-3}, z_{k-3}, c_0, c_1) \wedge R_{1/3}^{01}(x_1, x_{k-2}, w_{k-4}, c_0, c_1).$$

To prove the proper inclusion $\langle R_{1/3}^{01} \rangle_{\not\exists} \supset \langle \alpha^k \rangle_{\not\exists}$ we show that there exists $f \in \mathrm{pPol}(\alpha^k)$ such that $f \notin \mathrm{pPol}(R_{1/3}^{01})$. Consider the ternary partial function f defined such that $f(0, 0, 0) = 0$, $f(1, 1, 1) = 1$, and $f(0, 0, 1) = f(0, 1, 0) = f(1, 0, 0) = 0$. By Theorem 10 it follows that $f \notin \mathrm{pPol}(R_{1/3}^{01})$, but it is not difficult to verify that for any sequence of three tuples $t_1, t_2, t_3 \in \alpha^k$, the set $ColsSet(t_1, t_2, t_3)$ will either be of the form $\{(0, 0, 0), (1, 1, 1)\}$, or it will contain the complement of $(0, 0, 1)$, $(0, 1, 0)$, or $(1, 0, 0)$. Hence, $f(t_1, t_2, t_3)$ is either a projection or undefined, from which it follows that $f \in \mathrm{pPol}(\alpha^k)$.   ◀

We now need to prove that it cannot be the case that $\langle \alpha^k \rangle_{\not\exists} = \langle \alpha^{k'} \rangle_{\not\exists}$ whenever $k \neq k'$, and similarly for the relations β^k and γ^k. Before proving this we need a slight generalisation of the concept of exact covers from Definition 7. A set $\{\omega_1, \ldots, \omega_{k'}\} \subseteq \mathbb{B}^n$ is a *exact k'-cover of* $R_{1/k}$ if (1) $2 \leq k' \leq k$ and (2) for every $i \in \{1, \ldots, n\}$ it holds that $\omega_1[i] + \ldots + \omega_{k'}[i] = 1$. From Lagerkvist et al. [22] it follows that $t_1, \ldots, t_{k'} \in R_{1/k}$ if and only if $ColsSet(t_1, \ldots, t_{k'})$ is an exact k'-cover of $R_{1/k}$. This implies that $t_1, \ldots, t_{k'} \in \alpha^k$ if and only if $\{(t_1[1], \ldots, t_{k'}[1]), \ldots, (t_1[k], \ldots, t_{k'}[k])\}$ is an exact k'-cover of $R_{1/k}$.

▶ **Lemma 13.** *For each $k, k' \geq 5$ such that $k > k'$ there exists $f \in \mathrm{pPol}(\alpha^k)$, $f' \in \mathrm{pPol}(\beta^k)$, and $f'' \in \mathrm{pPol}(\gamma^k)$ such that $f \notin \mathrm{pPol}(\alpha^{k'})$, $f' \notin \mathrm{pPol}(\beta^{k'})$, and $f'' \notin \mathrm{pPol}(\gamma^{k'})$.*

Proof. We only consider the relations α^k and $\alpha^{k'}$ since the other cases are similar. We provide a partial function $f^{k'}$ such that $f^{k'} \in \mathrm{pPol}(\alpha^k)$ but $f^{k'} \notin \mathrm{pPol}(\alpha^{k'})$. Let $j = \mathrm{ar}(\alpha^{k'})$, let $\{t_1, \ldots, t_{k'}\} = \alpha^{k'}$, and define $f^{k'}$ such that $\mathrm{dom}(f^{k'}) = ColsSet(t_1, \ldots, t_{k'})$ and such that $f^{k'}(t_1, \ldots, t_{k'}) = (t_1[1], \ldots, t_1[j - 3], c, 0, 1)$, where $c = t_1[1] \oplus t_1[k' - 3]$. Note that, by definition, $f^{k'}(t_1, \ldots, t_{k'}) \notin \alpha^{k'}$ since any tuple $t \in \alpha^{k'}$ satisfies $t[1] + t[k' - 3] + t[j - 2] = 1$. Hence, $f^{k'} \notin \mathrm{pPol}(\alpha^{k'})$. Given a tuple $t \in \{0, 1\}^k$ we let $\Sigma t = \Sigma_{i=1}^k t[i]$. Let $\mathbf{x} = (t_1[j - 3], \ldots, t_{k'}[j - 3])$, and note that due to the constraint $R_{1/3}^{01}(x_1, x_{k-2}, w_{k-4}, c_0, c_1)$ in Definition 11, $\Sigma\overline{\mathbf{x}} = 2$, i.e., the sequence $\mathbf{x}$ contains exactly two zeroes. We now prove that $f^{k'} \in \mathrm{pPol}(\alpha^k)$. Let $u_1, \ldots, u_{k'} \in \alpha^k$ and assume, with the aim of reaching a contradiction,

that $f^{k'}(u_1, \ldots, u_{k'}) \notin \alpha^k$. Since $f^{k'}(\mathbf{y}) = \pi_1^{k'}(\mathbf{y})$ for all $\mathbf{y} \in \mathrm{dom}(f^{k'}) \setminus \{\mathbf{x}\}$, it follows that $\mathbf{x}$ is included in the sequence $Cols(u_1, \ldots, u_{k'})$. There are now only three possible distinct cases to consider depending on where the sequence $\mathbf{x}$ occurs in $Cols(u_1, \ldots, u_{k'})$. We will show that for each of these cases $f^{k'}(u_1, \ldots, u_{k'})$ must be undefined, contradicting the assumption that $f^{k'}(u_1, \ldots, u_{k'}) \notin \alpha^k$. Let

$$(a_1, \ldots, a_k, b_1, \ldots, b_{k-3}, c_1, \ldots, c_{k-3}, d_1, \ldots, d_{k-4}, \vec{0}^{k'}, \vec{1}^{k'}) = Cols(u_1, \ldots, u_{k'}).$$

First assume that $\mathbf{x} = a_i$ for some $1 \le i \le k$. Since $\{a_1, \ldots, a_k\}$ must form an exact k'-cover and since $\Sigma \mathbf{x} = \Sigma a_i = k' - 2$ it follows that there exists a_j and $a_{j'}$ such that $\{a_i, a_j, a_{j'}\}$ is an exact k'-cover and that all other a_l sequences are equal to $\vec{0}^{k'}$. If $|\{a_i, a_j, a_{j'}\}| = 2$ then $\overline{a_j} = \overline{a_{j'}} = a_i$, which is a contradiction since $\overline{a_j} = \overline{\mathbf{x}} \notin \mathrm{dom}(f^{k'})$. Assume that $|\{a_i, a_j, a_{j'}\}| = 3$. This implies that $\Sigma a_j = \Sigma a_{j'} = 1$ and that there exists a tuple $u \in ColsSet(u_1, \ldots, u_{k'})$ such that either $u = \overline{a_i}, u = \overline{a_j}$, or $u = \overline{a_{j'}}$. In each of these cases it follows that $u \notin \mathrm{dom}(f)$, and we reach a contradiction.

Second, assume that $\mathbf{x} = b_i$ for some $i \in \{1, \ldots, k-3\}$. Then $c_i = \overline{\mathbf{x}}$. This is a contradiction since $\overline{\mathbf{x}} \notin \mathrm{dom}(f^{k'})$. The case when $\mathbf{x} = c_i$ for some $i \in \{1, \ldots, k-3\}$ is entirely analogous.

Third, assume that $\mathbf{x} = d_i$ for some $i \in \{1, \ldots, k-4\}$. Then, due to the constraint $R_{1/3}(x_1, x_{i-2}, w_i)$ in Definition 11, it follows that $\Sigma a_1 = \Sigma a_{i-2} = 1$. Assume without loss of generality that $a_1[k'] = 1$. Now note that each constraint of the form $R_{1/3}(x_1, x_j, w_{j-2})$ for $j \in \{3, \ldots, k-2\}$ also implies that $\Sigma a_j \ge 1$. First, assume $\Sigma a_j = 1$ for $j \in \{1, \ldots, k-2\}$. Since $k' < k$ and since $\{a_1, \ldots, a_k\}$ must form an exact k'-cover, it follows that either $\Sigma a_{k-1} + \Sigma a_k = 1$ or that $\Sigma a_{k-1} + \Sigma a_k = 0$, and in both these cases $f^{k'}(u_1, \ldots, u_{k'})$ must be undefined. Assume that there exists some $a_{j'}, j' \in \{2, \ldots, i-1, i+1, \ldots, k\}$ such that $\Sigma a_{j'} > 1$. Since $\Sigma a_j \ge 1$ for each $j \in \{1, \ldots, k-2\}$ and since $\Sigma_{j=1}^k \Sigma a_j = k' < k$, it follows that $\Sigma a_{j'} = 2$, $k' = k-1$, and that $\Sigma a_j = 1$ for every $j \in \{1, \ldots, k-2\}$. This implies that $\Sigma a_{k-1} + \Sigma a_k = 1$, and, due to the constraint $R_{1/4}(x_{k-2}, x_{k-1}, x_k, z_{k-4})$, that $c_{k-4} = \overline{a_{k-2}}$. If $\Sigma a_{k-2} = 1$ then $\overline{a_{k-2}} \notin \mathrm{dom}(g)$. Hence, assume that $j' = k-2$ and that $\Sigma a_{k-2} = 2$. Due to the constraint $R_{1/3}(x_1, x_{k-2}, w_{k-4})$ it follows that $\Sigma d_{k-4} = k-3$, and since $a_1[k'] = 1$, we have that $a_{k-2} \notin \mathrm{dom}(f^{k'})$ or that $d_{k-4} \notin \mathrm{dom}(f^{k'})$. ◀

Last, to get the inclusion structure in Figure 2, we need to prove that $\langle \alpha^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq 01} \rangle_{\nexists}$, $\langle \beta^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq \neq 01} \rangle_{\nexists}$, and $\langle \gamma^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq \neq \neq 01} \rangle_{\nexists}$.

▶ **Lemma 14.** $\langle \alpha^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq 01} \rangle_{\nexists}$, $\langle \beta^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq \neq 01} \rangle_{\nexists}$, and $\langle \gamma^k \rangle_{\nexists} \supset \langle R_{1/3}^{\neq \neq \neq 01} \rangle_{\nexists}$, for each $k \ge 5$.

Proof. We only consider α^k since the other cases are similar. To prove that $\langle \alpha^k \rangle_{\nexists} \supseteq \langle R_{1/3}^{\neq 01} \rangle_{\nexists}$, we use the q.f.p.p. definition

$$R_{1/3}^{\neq 01}(x_1, x_2, x_3, x_4, c_0, c_1) \equiv$$
$$\alpha^k(\underbrace{c_0, \ldots, c_0}_{k-3}, x_1, x_2, x_3, \underbrace{c_1, \ldots, c_1}_{k-4}, x_4, \underbrace{c_0, \ldots c_0}_{k-4}, x_1, \underbrace{c_1, \ldots, c_1}_{k-5}, x_4, c_0, c_1).$$

For the proper inclusion, simply note that the function f^k in the proof of Lemma 13 does not preserve α^k. An application of Theorem 10 shows that $f^k \in \mathrm{pPol}(R_{1/3}^{\neq 01})$. ◀

By combining Lemma 12, Lemma 13 and Lemma 14 we have thus proved the main result of the paper.

▶ **Theorem 15.** *The cardinalities of the sets* $\{\langle \Gamma \rangle_{\nexists} \mid \langle R_{1/3}^{\neq 01} \rangle_{\nexists} \subset \langle \Gamma \rangle_{\nexists} \subset \langle R_{1/3}^{01} \rangle_{\nexists}\}$, $\{\langle \Gamma \rangle_{\nexists} \mid \langle R_{1/3}^{\neq \neq 01} \rangle_{\nexists} \subset \langle \Gamma \rangle_{\nexists} \subset \langle R_{1/3}^{\neq 01} \rangle_{\nexists}\}$, *and* $\{\langle \Gamma \rangle_{\nexists} \mid \langle R_{1/3}^{\neq \neq \neq 01} \rangle_{\nexists} \subset \langle \Gamma \rangle_{\nexists} \subset \langle R_{1/3}^{\neq \neq 01} \rangle_{\nexists}\}$ *are at least countably infinite.*

5   Concluding Remarks and Future Research

We have studied the structure of NP-complete satisfiability problems whose complexity is not higher than $\mathrm{SAT}(R_{1/3})$. By using partial clone theory we have proven that one can find an infinite number of such satisfiability problems, and in the process we have also obtained a complete description of the partial polymorphisms of $R_{1/3}$. These results raise two questions that we deem particularly interesting for future research.

Algorithms based on partial polymorphisms.   There exist many examples in the literature of polynomial-time algorithms based on properties of polymorphisms of constraint languages. For example, one can use polynomial-time algorithms based on Gaussian elimination to solve constraint satisfaction problems whenever the constraint language contains a so-called k-edge polymorphism [12]. By Theorem 4 we know that the partial polymorphisms of a constraint language correlates to the worst-case complexity of the corresponding satisfiability problem. Is it possible to exploit the information given by the partial polymorphisms to construct better exponential-time algorithms for satisfiability problems? In particular, can the classification in Theorem 9 be used to improve algorithms for 1-in-3-SAT? For a concrete example, consider the following strategy: it is known that the *inverse satisfiability problem* for $R_{1/3}$, Inv-$\mathrm{SAT}(R_{1/3})$, is co-NP-complete [17]. In our terminology this problem can be stated as determining whether a given relation R is included in $\langle R_{1/3} \rangle_{\not\exists}$, and can therefore be restated as whether $\mathrm{pPol}(R_{1/3}) \subseteq \mathrm{pPol}(R)$. Hence, to solve $\mathrm{SAT}(R_{1/3})$ we can utilize a Turing reduction to Inv-$\mathrm{SAT}(R_{1/3})$, which in turn can be solved by enumerating the partial polymorphisms of $R_{1/3}$ and checking if they preserve R. Would it be possible to transform this rather implicit algorithm into an efficient algorithm for 1-in-3-SAT?

Uncountably many weak partial co-clones?   We have proven that there exists at least a countably infinite number of weak partial co-clones below $\langle R_{1/3} \rangle_{\not\exists}$. Is it possible to strengthen this even further and prove that there exists an uncountably infinite number of such weak partial co-clones? A starting point for proving this is to first show that the converse of Lemma 13 also holds, i.e., that $\langle \alpha^k \rangle_{\not\exists}$ and $\langle \alpha^{k'} \rangle_{\not\exists}$ are always incomparable whenever $k \neq k'$.

Does $\langle R_{1/3} \rangle_{\not\exists}$ cover $\langle R^{01}_{1/3} \rangle_{\not\exists}$?   In this paper we restricted ourselves to study weak partial co-clones below $\langle R^{01}_{1/3} \rangle_{\not\exists}$ since the two problems $\mathrm{SAT}(R_{1/3})$ and $\mathrm{SAT}(R^{01}_{1/3})$ have the same worst-case time complexity. From an algebraical point of view, however, it would be interesting to prove or disprove that $\langle R_{1/3} \rangle_{\not\exists}$ covers $\langle R^{01}_{1/3} \rangle_{\not\exists}$, since only a handful of such results are known in the literature [10]. This question might not be as easy as one might believe at a first glance, since it is e.g. known that there exist an uncountably infinite number of weak partial co-clones between $\langle \mathrm{OR} \rangle_{\not\exists}$ and $\langle \mathrm{OR}^{01} \rangle_{\not\exists}$, where $\mathrm{OR} = \{(0,1),(1,0),(1,1)\}$ and $\mathrm{OR}^{01} = \{(0,1,0,1),(1,0,0,1),(1,1,0,1)\}$ [29]. Hence, even though the relations $R_{1/3}$ and $R^{01}_{1/3}$ might appear to be almost identical, it might indeed be very hard to prove that $\langle R_{1/3} \rangle_{\not\exists}$ covers $\langle R^{01}_{1/3} \rangle_{\not\exists}$.

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