Pricing Toll Roads under Uncertainty

Trivikram Dokka[1], Alain Zemkoho[2], Sonali Sen Gupta[3], and Fabrice Talla Nobibon[4]

1 **Department of Management Science, Lancaster University**
t.dokka@lancaster.ac.uk
2 **Department of Mathematics, University of Southampton**
a.b.zemkoho@soton.ac.uk
3 **Department of Economics, Lancaster University**
s.sengupta@lancaster.ac.uk
4 **Fedex Europe**
tallanob@gmail.com

Abstract

We study the toll pricing problem when the non-toll costs on the network are not fixed and can vary over time. We assume that users who take their decisions, after the tolls are fixed, have full information of all costs before making their decision. Toll-setter, on the other hand, do not have any information of the future costs on the network. The only information toll-setter have is historical information (sample) of the network costs. In this work we study this problem on parallel networks and networks with few number of paths in single origin-destination setting. We formulate toll-setting problem in this setting as a distributionally robust optimization problem and propose a method to solve to it. We illustrate the usefulness of our approach by doing numerical experiments using a parallel network.

1998 ACM Subject Classification H.4.2 Decision Support; G.1.6 Optimization

Keywords and phrases Conditional value at risk, robust optimization, toll pricing.

Digital Object Identifier 10.4230/OASIcs.ATMOS.2016.4

1 Introduction

Public-Private partnerships and private investment are becoming more popular than ever in infrastructure projects. For example more roads are now built by private companies as against the tradition of governments building roads. Typically these projects employ build-operate-transfer model. Here the investing company enters in a contract with government to build a road/highway. In return of the investment, the company is allowed to collect tolls for an agreed period of time before the transfer of ownership to the government. In fact, in recent years, tolls have become a primary way to encourage private investment in public infrastructure [4]. There are both successes and failures of this model. One of the notable examples is M6 toll between Cannock and Coleshill in the United Kingdom, which opened in 2003. According to a BBC News Report, "the company operating M6 toll made a 1 million pound loss in the year 2012","drivers have said the road is underused because of its prices". Therefore, a key element to the success of this model is the revenue generated from tolls. The investor company's main objective is to maximize the revenue from tolls. Therefore, toll pricing can be the defining factor to the success of the project and the key to a successful revenue maximization pricing mechanism lies in understanding the network users options compared to the toll roads. In [12], a bilevel model is proposed to capture the situation where the toll-setter anticipates the network user's reaction to his decisions. In a

16th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS'16).
Editors: Marc Goerigk and Renato Werneck; Article No. 4; pp. 4:1–4:14
Open Access Series in Informatics
OASICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

full information situation, it is assumed that costs of travel on the network are fixed and known to both toll-setter and users. However, cost of travel is rarely constant over time in a real-world transportation network. Modern technology enables users to estimate the travel costs (or times) more accurately than before, which implies users can change their decisions over time depending upon the costs in the network. The toll-setter, however, suffers from the disadvantage that (more often in practice) he is not allowed to change the toll very frequently due to policy regulations and other constraints. In most cases, the toll is required to be fixed for a minimum period. Even after the minimum period, changing the toll price, especially, increasing it usually has a negative impact on the user's beliefs and may end up resulting in reduced revenues. Naturally, in such a situation the toll-setter has to make his decisions under uncertainty about the user's future options. On the other hand, users have full information before they make their decisions. In this work, we study a robust toll-pricing mechanism which aims to minimize the risk of the toll-setter against this uncertainty. In doing so, we use the ideas from robust optimization literature and show that our approach is very near to the conditional value-at-risk approach used in portfolio optimization and other problems, see [15] and [16].

Profit and revenue maximization problems over a transportation network are given much attention in pricing literature, see for example, [17, 3, 11] to name a few. Within the huge body of papers, many have studied the application of the bilevel programming paradigm to pricing problems, such as [12] and many subsequent papers, [5, 2, 10, 14] considered different application areas. The deterministic version of the problem that we study in this paper has been well investigated, see [13] and references therein. However, the stochastic extensions of the problem have gained more interest only in recent years. Two different extensions of the model in [12] have been studied in [7] and [1]. In [7], authors study the logit pricing problem and [1] studies the two-stage stochastic problem with recourse extension of the deterministic toll pricing problem. In this paper we study the toll pricing problem under uncertainty and on single commodity parallel networks and networks with polynomial number of paths. The deterministic pricing problem on such networks can be easily solved by enumerating all paths and finding the least cost non-toll path. To the best of our knowledge, there is no work on robust pricing in the presence of uncertainty even in such basic networks, and as we will show, the pricing problem in these networks is quite complex. There, however, are two studies where robust optimization framework is applied to pricing problems, in [18] and [6]. In both of these works, the models considered are different from our model and problem setting. Understanding the pricing problem in parallel networks will provide useful insights into the complexity of pricing for more general networks involving more commodities and with variable demands. As we will show that the ideas we propose in this work will provide a basis for solving toll-pricing problem in more general networks.

The aim of this paper is to understand the toll-pricing problem faced by a risk-averse toll-setter when there is uncertainty on non-toll costs. We use the framework of distributional robustness which is very useful in making optimal decisions under limited or imprecise information, see [8] for recent developments on distributionally robust optimization. Our main contribution in this work is the study the toll-pricing problem in parallel networks and its modeling as a distributionally robust optimization problem, see Section 3.1. We propose an algorithm and a heuristic to solve the problem, see Section 3.1 and Section 4. We assess the heuristic performance using computational experiments in Section 5. Finally we conclude the work by giving possible extensions to solve the robust pricing problem on general networks in Section 6.

2 Problem definition

We will first describe the deterministic pricing model as used in [12]. We consider a single-commodity transportation network with a single origin and single destination, $G = (N, A)$, where N (of cardinality n) denotes the set of nodes, and A (of cardinality m) the set of arcs. The arc set A of the network G is partitioned into two subsets A_1 and A_2, where A_2 denotes the set of roads which are toll-free (*public roads*), and A_1 the set of roads which are owned by a toll-setter (*toll roads*). There can be more than one parallel roads between any two nodes in G.

With each toll arc a in A_1, we associate a generalized travel cost composed of two parts: toll (r_a) - set by the toll-setter expressed in time units, and non-toll cost (c_a) - which can vary over time (discretized into unit intervals). An arc $a \in A_2$ only bears the non-toll cost c_a. Once the toll is set on arcs in A_1, it cannot changed for T consecutive time periods. We will refer to the T consecutive time periods in which the toll is fixed as *tolling period.* We denote by $b \in R^n$ the fixed demand, with the assumption that all nodes except origin and destination nodes have a demand equal to 0. Assuming fixed demand and neglecting congestion implies users choose shortest paths between the origin and destination. Further we assume that when faced with two alternatives, a user will choose the one which maximizes the revenue of the toll-setter. Another key assumption in our model is that it allows conversion from time to money and assume this be to uniform throughout the users. Under this setting, when the non-toll costs are known to both toll-setter and users, the question that the toll-setter faces is:

How to set prices which maximizes the total toll revenue when the network user chooses the shortest paths to minimize his cost?

In the absence of uncertainty on non-toll costs, the deterministic toll-pricing problem is modeled as the following bilevel optimization problem, see [12]:

$$
\begin{aligned}
\text{(TOP)} \quad & \max_{R,X} \quad F(R,X) := \sum_{a\in A_1} r_a x_a \\
& \text{s.t.} \quad \min_{X} \sum_{a\in A_1} (c_a + r_a)\, x_a + \sum_{a\in A_2} c_a x_a \qquad\qquad (1)\\
& \qquad\qquad \sum_{a\in i^+} x_a - \sum_{a\in i^-} x_a = b_i, \qquad \forall i \in N \\
& \qquad\qquad x_a \ge 0, \qquad \forall a \in A.
\end{aligned}
$$

Here, X is the collection of decision variables in the lower level problem which in this setting is a shortest path problem. For each node i, i^+ (resp. i^-) corresponds to the leaving (resp. entering) flow arcs, and $b_i = 1$, if i is the origin while $b_i = -1$, if i is the destination.

We will now extend the above deterministic model to the case where there is uncertainty on non-toll costs c_a. Our uncertainty model and corresponding assumptions can be described follows:

- the toll-setter has the historical information encoded in the form of previously observed states. A state s corresponds to an observed state of the network in a single time period. In other words, in each state s, the non-toll cost on each arc $a \in A$ is fixed denoted as c_a^s. The advantage of modeling uncertainty in this way is that the correlations are captured in the states.
- The number of states is equal to $\#H \times T$. That is, the toll-setter observes $\#H$ tolling periods.

- We assume the variability on each arc is bounded, that is, the variance-to-mean ratio for the toll period is bounded by a constant which is unknown to the toll-setter. This is usually the case in real world networks.
- The cost distribution (unknown to toll-setter) of each arc is assumed to be fixed and belongs to a set of non-negative distributions D with support in $\Omega = [q, Q]$. Given the bounded variability assumption it is reasonable to assume fixed support. One can also consider different supports for different arcs, however, we see Ω as the aggregated support set. We will denote integers in Ω as $\bar{\Omega}$.

Given this setting, our aim is to answer the following question faced by a toll-setter:

How to set toll prices under uncertainty of non-toll costs, when users will have full knowledge of future non-toll costs based on which they choose shortest paths?

In the next section, we propose a robust model to answer this question and also propose the methods to solve it.

3 Robust model and solution method

3.1 Network with two parallel arcs

Consider a network with just two parallel arcs connecting the origin and destination. Let one of these arcs be the toll arc and the other arc be the non-toll arc whose costs are not known. We assume for the ease of exposition that the non-toll costs on the toll arc are zero or negligible. We will later remove this assumption and show that the method can be extended to such a case. As mentioned in the previous section, the toll-setter has a sample of costs of $\#H$ tolling periods from the recent history. Using this sample, the toll-setter wishes to calculate the toll, referred hereafter as r_a, on the toll arc. In the rest of the section we will drop the suffix a for the ease of notation and readability. If the toll-setter knows the distribution F (we denote the density of F with $\mathbf{F}$) of c, then to fix the toll which maximizes his expected revenue, he solves the following optimization problem which maximizes his expected revenue:

$$\max_{r\in\Omega} \int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc}. \tag{2}$$

Given the distribution F, this is a trivial problem; F is however not known to the toll-setter. In the absence of this knowledge, a risk-averse toll-setter would prefer to insure his revenues by setting tolls such that the usage of toll roads is maximized. Now suppose that, the toll-setter first decides his toll and then nature, who plays adversary to the toll-setter, will decide on F. Then, the toll-setter wishes to calculate a robust toll price which maximizes his revenue by solving the following optimization problem:

$$\begin{aligned} \max_{r\in\Omega} \quad \min_{F\in D} \quad & \textstyle\int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc} \\ \text{s.t.} \quad & \underline{u} \le \mu_F(c) \le \overline{u}, \\ & \sigma_F^2(c) \le \kappa\mu_F(c), \\ & \kappa \le \overline{\kappa}. \end{aligned} \tag{3}$$

Here, the parameters $\underline{u}$, $\overline{u}$ are calculated as $(1-\alpha)\%$ confidence limits of mean; $\overline{\kappa}$ are calculated as $1 + \alpha \times \mathit{variance}/\mathit{mean}$ of the observed sample with $\alpha \in [0, 1]$. Note that the constraints in (3) correspond mainly to nature's problem, i.e., to find a distribution satisfying the mean (or moment) constraint. The second constraint limits the possible distributions by using the assumption of bounded variability. Such a situation with sufficiently high allowed variability gives too much power to adversarial nature forcing the toll-setter (to be too conservative) to set very low r if he chooses to be robust against all possible $F \in D$. To avoid such over-conservativeness, we assume that nature does not play such a role. Instead nature's objective is to minimize the overall expected cost of the network user, that is:

$$\int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc} + \int_{\mathbf{q}}^{\mathbf{r}} \mathbf{cF}(\mathbf{c})\mathbf{dc}, \tag{4}$$

where the first term is the expected cost of travel on the toll road and the second term is the expected cost on the non-toll road. The toll-setter then solves the following bi-level distributionally robust program to find the robust r:

$$\begin{array}{ll} \max\limits_{r\in\Omega} & \int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc} \\ \min\limits_{F\in D} & \int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc} + \int_{\mathbf{q}}^{\mathbf{r}} \mathbf{cF}(\mathbf{c})\mathbf{dc} \\ \text{s.t.} & \underline{u} \le \mu_F(c) \le \overline{u}, \\ & \sigma_F^2(c) \le \kappa\mu_F(c), \\ & \kappa \le \overline{\kappa}. \end{array} \tag{5}$$

Since we consider F with support in Ω, we can use $\int_r^Q \mathbf{F}(\mathbf{c})\mathbf{dc} + \int_{\mathbf{q}}^{\mathbf{r}} \mathbf{F}(\mathbf{c})\mathbf{dc} = \mathbf{1}$ and rewrite the nature's objective function as

$$\begin{aligned} \int_r^Q r\mathbf{F}(\mathbf{c})\mathbf{dc} + \int_{\mathbf{q}}^{\mathbf{r}} \mathbf{cF}(\mathbf{c})\mathbf{dc} &= r(1 - \int_q^r \mathbf{F}(\mathbf{c})\mathbf{dc}) + \int_{\mathbf{q}}^{\mathbf{r}} \mathbf{cF}(\mathbf{c})\mathbf{dc} \\ &= r - \int_q^r (r-c)\mathbf{F}(\mathbf{c})\mathbf{dc} \end{aligned}$$

We will now show that (5) can be written as a single level max-min optimization problem. For a fixed $\epsilon \in [0,1]$, consider the function

$$f_\epsilon(r,F) = \left[r - \frac{1}{(1-\epsilon)} \int_q^r (r-c)\mathbf{F}(\mathbf{c})\mathbf{dc} \right]$$

and observe that f is nothing but nature's objective function with an additional factor involving ϵ. We have the following property of f.

▶ **Lemma 1.** *For fixed $F \in D$ and $\epsilon \in [0,1]$, $f_\epsilon(r,F)$ is concave and continuously differentiable, and the maximum of f is attained at $r \in \Omega$ such that $\int_{c\le r} \mathbf{F}(\mathbf{c})\mathbf{dc} = \mathbf{1} - \epsilon$.*

As a corollary of this Lemma, we can infer that for fixed r and F, there exists an ϵ which maximizes the function $r \times \epsilon$ and is found by solving $1 - \frac{1}{(1-\epsilon)} \int_{c\le r} \mathbf{F}(\mathbf{c})\mathbf{dc} = \mathbf{0}$. Note that for a fixed F, if we put $\epsilon = \int_r^Q \mathbf{F}(\mathbf{c})\mathbf{dc}$, then maximizing $r\epsilon$ is nothing but maximizing the expected revenue of the toll-setter. One way of interpreting ϵ is that it can be seen as usage probability of a toll road. In other words, given F when the toll-setter decides the toll according to (2), he indirectly also chooses this probability. This implies that the bi-level problem in (5) can be written as a single level parametric problem with a max-min objective and $\epsilon \in [0,1]$ as a parameter:

$$\begin{array}{lll} \max\limits_{r\in\Omega} & \min\limits_{F\in D} & r - \frac{1}{1-\epsilon} \int_q^Q \max(r-c,0)\mathbf{F}(\mathbf{c})\mathbf{dc} \\ & \text{s.t.} & \underline{u} \le \mu_F(c) \le \overline{u}, \\ & & \sigma_F^2(c) \le \kappa\mu_F(c), \\ & & \kappa \le \overline{\kappa}. \end{array} \tag{6}$$

By choosing a value of the parameter ϵ, the toll-setter wishes to set the toll which ensures the expected probability of the toll road usage is at least ϵ with an expected revenue of at least $r\epsilon$.

For a fixed F, the objective function is very similar to the concept of Conditional-Value-at-Risk, which has been applied to portfolio optimization problems in [15]. In fact it turns out that our problem formulation is similar to worst-case conditional value-at-risk studied in [19] and more recently in [16]. Hereafter we will assume that ϵ takes values with two significant digits after the decimal for numerical simplicity. Since we assume that time is discretized, we consider the discrete version of (6) which can be seen as nature optimizing over samples, C, drawn from distributions in D:

$$\begin{aligned} &\max_{r\in\Omega}\min_{C\in\Omega^T} \quad r - \frac{1}{(1-\epsilon)T}\sum_{i=1}^{T}\max(r-c_i,0) \\ &\text{s.t.} \quad \underline{u} \le \mu_F(c) \le \overline{u} \\ &\qquad \sigma_F^2(c) \le \kappa\mu_F(c) \\ &\qquad \kappa \le \overline{\kappa} \end{aligned} \tag{7}$$

For a fixed value of the toll-setter's decision, the problem (7), that is, the inner problem in (7), is a minimization problem with a concave objective function. Concave minimization problems are hard to solve; for some recent work on quasi-concave minimization over convex sets, see [9] and references therein. To solve the inner problem in (7) we reformulate the inner problem as the following non-convex integer programming problem by introducing additional variables:

$$\min_{C\in\Omega^T} \quad r - \frac{1}{(1-\epsilon)T}\sum_{i=1}^{T} z_i \tag{8}$$

$$\text{s.t.} \quad \underline{u} \le \mu(c) \le \overline{u}, \tag{9}$$

$$\sigma^2(c) \le \kappa\mu(c), \tag{10}$$

$$\kappa \le \overline{\kappa}, \tag{11}$$

$$c_i - r + z_i \ge 0 \quad i = 1,\dots,T, \tag{12}$$

$$r - c_i + My_i \ge 0 \quad i = 1,\dots,T, \tag{13}$$

$$z_i \le M(1-y_i) \quad i = 1,\dots,T, \tag{14}$$

$$z_i - (r-c_i)(1-y_i) \le 0 \quad i = 1,\dots,T, \tag{15}$$

$$Y \in \{0,1\},\ C, Z \ge 0. \tag{16}$$

Note that for M in the above formulation, any value greater than or equal to Q suffices; this leads to our next theorem:

▶ **Theorem 2.** *For fixed r and ϵ, (8)-(16) is a valid reformulation of the inner problem of (7).*

The only non-convex constraint apart from integrality constraints in the above formulation is (15). We linearize this by introducing two additional sets of variables as follows. Replace the product terms ry_i and c_iy_i in this constraint by variables u_i and v_i and then add constraints

(25)-(30). After doing this we get the following convex integer programming problem.

$$\min_{C \in \Omega^T} \quad r - \frac{1}{(1-\epsilon)T} \sum_{i=1}^{T} z_i \tag{17}$$

$$\text{s.t.} \quad \underline{u} \leq \mu(c) \leq \overline{u}, \tag{18}$$

$$\sigma^2(c) \leq \kappa\mu(c), \tag{19}$$

$$\kappa \leq \overline{\kappa}, \tag{20}$$

$$c_i - r + z_i \geq 0 \quad i = 1, \ldots, T, \tag{21}$$

$$r - c_i + My_i \geq 0 \quad i = 1, \ldots, T, \tag{22}$$

$$z_i \leq M(1 - y_i) \quad i = 1, \ldots, T, \tag{23}$$

$$z_i - r + c_i + u_i - v_i \leq 0 \quad i = 1, \ldots, T, \tag{24}$$

$$u_i \leq My_i \quad i = 1, \ldots, T, \tag{25}$$

$$v_i \leq My_i \quad i = 1, \ldots, T, \tag{26}$$

$$v_i \leq c_i \quad i = 1, \ldots, T, \tag{27}$$

$$u_i \leq r \quad i = 1, \ldots, T, \tag{28}$$

$$r - M(1 - y_i) \leq u \leq r \quad i = 1, \ldots, T, \tag{29}$$

$$Y \in \{0, 1\}; C, Z, U, V \geq 0. \tag{30}$$

▶ **Theorem 3.** *(17)-(30) is a valid reformulation of (8)-(16).*

For a fixed r and ϵ, (17) - (30) can be solved by using a state of the art commercial solver like CPLEX and more specialized algorithms are also conceivable owing to tremendous success and availability of techniques for solving convex quadratic integer programs from the last few years. However, (17) - (30) is still the inner problem of the main toll setting problem which takes r as input. Lemma 1 implies that we can use (17) - (30) and do a binary search over $r \in \Omega$ to find the robust toll. We formally give this in Algorithm 2, that we move to the appendix section due to space restriction.

3.2 General networks

In this section we extend our method to more general networks.

3.2.1 Multiple parallel arcs

Let us first consider the immediate extension to a network where there are k non-toll arcs parallel to the toll arc between the origin and destination. Let $a_1, \ldots, a_k$ be the non-toll arcs. We input the mean and variance limits ($\overline{u}$, $\underline{u}$ and $\overline{\kappa}$) of the data obtained from taking the following minima, $\min_{i=1}^{k} c^s_{a_i}$ to Algorithm 3 to calculate the robust toll.

3.2.2 Multiple parallel arcs with positive non-toll costs on toll arc

Let us now consider the above network when the assumption that the non-toll costs on toll arc are not zero. Let a_{k+1} be the toll arc. To apply our method to this case, we calculate the mean and variance limits ($\overline{u}$, $\underline{u}$ and $\overline{\kappa}$) of

$$c^s_{a_{k+1}} - \min_{i=1}^{k} c^s_{a_i} \tag{31}$$

for all s and input these to Algorithm 3.

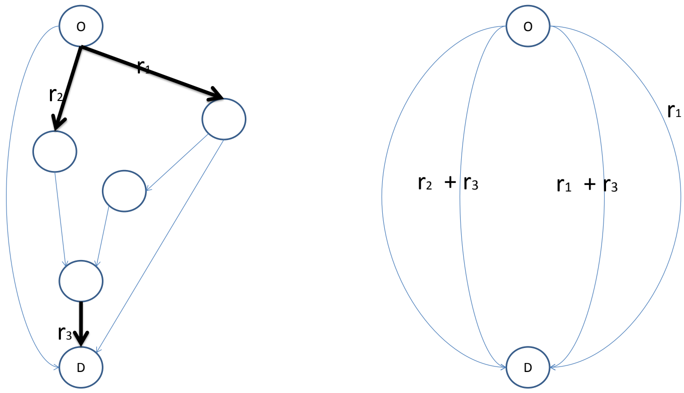

Figure 1 General network and an equivalent parallel network.

3.2.3 General networks with polynomial (in n) number of paths

Consider now a general single commodity network with multiple toll arcs but with few (polynomial) number of paths between origin and destination; for example the network on the left given in Figure 1. An equivalent parallel network is constructed as shown in the right side in Figure 1. For each path in this parallel network with toll arcs, we will calculate the quantities (state minima) given in (31) on each path involving toll arcs by ignoring all other paths with toll arcs. That is, we calculate the robust toll on each toll path as if that is the only path with toll arcs in the network. Using these quantities as input sample to Algorithm 3, we calculate an upper bound on the total toll on each path and then solve an integer programming problem to allocate the tolls to individual toll arcs. Suppose the upper bounds for the paths in the example network in Figure 1 are ς_1, ς_2, ς_3, respectively from left to right, then we solve the following optimization problem for prices of r_1, r_2, and r_3: $\max \;\; r_1 + r_2 + r_3 \,||\, \text{s.t} \;\; r_2 + r_3 \leq \varsigma_1, \;\; r_1 + r_2 \leq \varsigma_2, \;\; r_1 \leq \varsigma_3, \;\; r_i \in \mathbb{Z}$. We observe here that for general networks with not necessarily polynomial number of paths, such a procedure could still be used as a heuristic to calculate a robust toll on a subset of paths which can be obtained as shortest paths in the observed states. We omit details due to space restriction.

4 Two-point Heuristic

The formulation given in (17) can be hard to solve and can be time consuming when using a generic solver like CPLEX. Of course, one can derive efficient algorithms using branch and bound and/or other methodologies. In this section, however, we focus on constructing a simple approximate solution to (17). In our computational experience of solving (17) using CPLEX we found that in all cases, the solution found has two-point support. That is, the vector of costs returned by CPLEX has exactly two distinct values. If we restrict to the distributions with two-point support $\{\ell, u\}$ with probabilities $\{\frac{\lambda}{T}, 1 - \frac{\lambda}{T}\}$, assuming

Algorithm 1 Two Point Algorithm

$\lambda = T-1$, $\mu = \underline{u}$
while $\lambda \geq 1$ **do**
  $\ell = 0$
  while $\ell < \mu$ **do**
    $u = \frac{((\mu\times T)-(\lambda\times\ell))}{(T-\lambda)}$
    if $(\lambda\times\ell+(T-\lambda)\times u) \leq \overline{u}$ and $(\lambda\times\ell+(T-\lambda)\times u) \geq \underline{u}$ and $u \leq Q$ and $\lambda(\ell-\mu)^2+(T-\lambda)([\frac{\mu T-\lambda\ell}{(T-\lambda)}]-\mu)^2 \leq \overline{\kappa}\mu(T-1)$ **then**
      if $obj > (\lambda\times\ell+(T-\lambda)\times r)$ **then**
        $obj = (\lambda\times\ell+(T-\lambda)\times r)$
      end if
      break
    else
      $\ell = \ell+1$
    end if
  end while
  $\lambda = \lambda - 1$
end while

$\ell \leq r \leq u$, the inner problem of (7) can be written as

$$\begin{aligned}\min_{\ell\in\Omega,u\in\Omega;\lambda\in[0,T]} \quad & rT-\lambda(r-\ell)\\ \text{s.t.}\quad & (T-\lambda)u+\lambda\ell=\mu T,\\ & \underline{u}\leq\mu\leq\overline{u},\\ & \lambda(\ell-\mu)^2+(T-\lambda)(u-\mu)^2\leq\kappa\mu(T-1),\\ & \kappa\leq\overline{\kappa}.\end{aligned}$$

Suppose now that we fix $\mu = \underline{u}$ and $\kappa = \overline{\kappa}$, we can write the problem of finding $\{\ell, u\}$ as

$$\begin{aligned}\min_{\ell\in\Omega,u\in\Omega;\lambda\in[0,T]} \quad & rT-\lambda(r-\ell)\\ \text{s.t.}\quad & (T-\lambda)u+\lambda\ell=\mu T,\\ & \lambda(\ell-\mu)^2+(T-\lambda)(u-\mu)^2\leq\overline{\kappa}\mu(T-1).\end{aligned}$$

Eliminating u, we get

$$\begin{aligned}\min_{\ell\in\Omega\lambda\in[0,T]} \quad & rT-\lambda(r-\ell)\\ \text{s.t.}\quad & \lambda(\ell-\mu)^2+(T-\lambda)([\frac{\mu T-\lambda\ell}{(T-\lambda)}]-\mu)^2\leq\overline{\kappa}\mu(T-1).\end{aligned} \tag{32}$$

For a fixed λ, the objective function in (32) is linear in ℓ with a positive slope. This implies that the optimal solution to (32) is simply the lowest value satisfying the inequality in (32) and $u \in \Omega$. Using this observation we now give a simple algorithm for finding a two-point solution to (17).

Algorithm 1 solves (32) by searching for all values of λ, where obj is the objective in (32). Note that we search for $\ell \in \bar{\Omega}$, this is again for numerical simplification and will only result in minor loss in terms of approximation. Note that the heuristic presented in Algorithm 1 is aimed mainly for a quick optimal solution for (17)–(30) with fixed $\mu = \underline{u}$ and $\kappa = \overline{\kappa}$. It

Table 1 Distributions and parameters used.

Distribution	First parameter	second parameter
Beta	[2,5]	[2,5]
Beta	[1,3]	[1,3]
Gamma	[1,3]	$[\frac{1}{3}, \frac{1}{5}]$
Normal	[90,110]	[10,30]
Lognormal	[0.1,0.3]	[0.1,0.3]

may be possible that for these value of μ and κ there is no solution to (32). In which case as mentioned in Algorithm (2), we choose $\underline{u}$ as the robust toll. However, this was never the case in our numerical experiments which we present in the next section.

5 Computational experiments

In this section, we report the performance of our approach with some numerical experiments. We have done experiments to assess the robustness of our procedure under two different experimental set-ups differing in the network structure and cost distributions. We explain them below.

- **First-Experiment**: We consider a parallel network with five parallel links connecting the origin to the destination. In this set-up we fix the distributions of the links to be same but allow the parameters to vary randomly within a given interval.
- **Second-Experiment**: We consider the same network as in the first but the distributions on each links can be different including parameters.

In both experiments, we would like to understand the robustness of the two-point toll. The distributions we use are Beta, Gamma, Normal and Lognormal. The parameters for each distribution are selected uniformly from an interval. The parameter intervals are given in Table 1.

We first created 50 samples (history sample), from each distribution which are used to calculate the robust tolls. We then created 5000 random samples from each distribution and computed optimal revenues generating tolls for each of these samples. We compare the revenues from optimal tolls in each of these 5000 samples with revenues when a robust toll is used which is calculated from a sample in history sample. To calculate an optimal toll for a given instance we try each integer in $\bar{\Omega}$ and select the toll which generates the most revenue. In total we compare robust tolls with optimal tolls on 250000 samples. We report the percentage relative regret from using the robust toll which is calculated as follows:

$$\text{relative regret}(\%) = \frac{\text{optimal revenue - robust toll revenue}}{\text{optimal revenue}} \tag{33}$$

From these experiments we want to understand the answers to the following two questions:

- how bad are revenues from robust tolls compared to the optimal revenues?
- how do robust tolls compare to optimal tolls?

In both experiments we used the two-point approximate algorithm to compute the robust tolls, and we set $T = 100$, $\#H = 1$, and $\alpha = 0$.

Table 2 Fixed case: average (%) relative regret.

Distribution	Robust toll Average	mean-variance toll Average	Robust toll Stdev	mean-variance toll Stdev
Beta	12.99	16.19	8.85	14.74
Gamma	13.35	21.93	11.65	21.22
Lognormal	6.61	29.63	5.44	19.36
Normal	9.06	22.96	6.25	15.11

Table 3 Exp 3: average (%) relative regret.

Distribution	Robust toll Average	mean-variance toll Average	Robust toll Stdev	mean-variance toll Stdev
Mixed	8.11%	12.03%	5.86%	10.82%

5.1 Fixed distributions

In this section we will evaluate the robustness of our two-point robust toll on the instances when all parallel arcs have same distributions but the parameters can be selected randomly in the intervals given in Table 1. Table 2 displays the average percentage relative regret for each of the four distributions when robust toll is used and when $(\mu(\#H) - 0.01\sigma^2(\#H))$ is used as toll. Here $\mu(\#H)$ and $\sigma^2(\#H)$ are mean and variance of the observed sample. From Table 2 we observe that the robust toll achieves a regret less than 14% in all distributions. On the other hand using a mean-variance toll can have regret as high as 30%. Our algorithm also performs well when comparing standard deviations of regrets with that of mean-variance toll. This suggests that revenues from robust toll compare well especially given the fact that the toll decision is taken with minimal knowledge about the network cost distributions.

As previously pointed out a measure of robustness of the toll is how it compares with the optimal revenue generating tolls. Figure 2 displays the comparison of minimum, maximum and average values of tolls over the 50 history samples, and optimal revenue generating tolls in each of the 5000 samples (sorted in increasing order). From Figure 2 we observe that it is possible that the Algorithm 2 can set the toll too high or too low especially seen in Gamma and Normal distributions. However, the average robust toll compares well with optimal tolls and roughly stands above the lower quartile of optimal tolls. Figures also suggest that with a higher $\#H$ the variability in robust tolls can be further reduced.

5.2 Mixed distributions

In this section we will evaluate the robustness of Algorithm 2 when arcs in the (same) network can have different distribution with parameters again chosen randomly from intervals given in Table 1. We observe from Table 3 that average regret from the robust toll is less than that in the case of fixed distribution case. This is also reflected in Figure 3 which again displays the comparison of minimum, maximum and average values of robust tolls with optimum revenue generating tolls. The average robust toll is set roughly around 40% mark of optimal curve indicating a better tradeoff between setting toll too high and setting it too low.

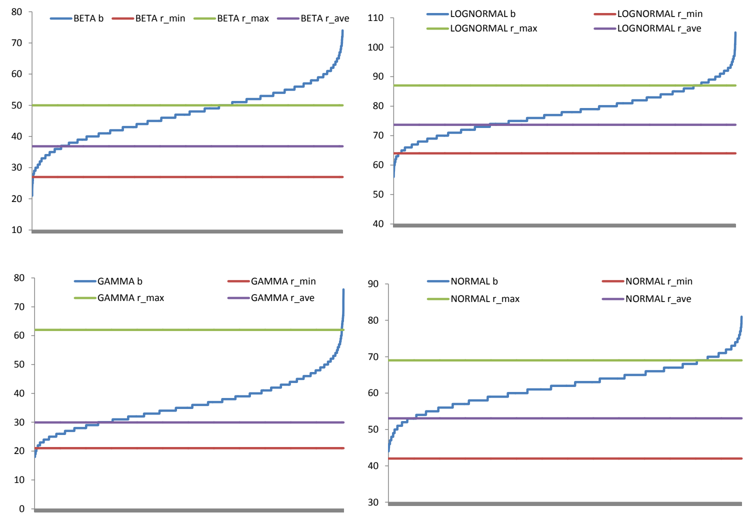

Figure 2 Exp. 2: Comparison of optimal tolls with robust tolls.

6 Conclusion

We have considered a basic version of the toll pricing problem in the presence of uncertainty. We formulate it as distributionally robust optimization problem and discuss its similarities with the concept of conditional value-at-risk. We present a two-point approximate algorithm to solve it and show by numerical experiments the robustness of the approach.

A number of questions remain to be studied even in the simple case considered in this work. Given the simplicity of the two-point algorithm and the experimental evidence it still remains to investigate that if two-point solutions are in fact among the optimal solutions to (17). Next immediate question to consider is to extend the approach to general single commodity networks where the number of paths are large. Here we note that the approach presented can be extended to such networks in designing an efficient heuristic. Finally, another direction we are currently working is to extend the pricing framework to dynamic setting as against the static setting considered in this paper.

Acknowledgements. We would like to thank Martine Labbé for an interesting discussion on the topic and also for pointing out references. We thank Marc Goerigk for spotting a mistake in an early version of the paper.

References

1 S.M. Alizadeh, P. Marcotte, and G. Savard. Two-stage stochastic bilevel programming over a transportation network. *Transportation Research Part B*, 58:92–105, 2013.
2 M. Bouhtou, G. Erbs, and M. Minoux. Joint optimization of pricing and resource allocation in competitive telecommunication networks. *Networks*, 50(1):37–49, 2007.
3 M. Bouhtou, S. van Hoesel, A. Van der Kraaij, and J. Lutton. Tarriff optimization in networks. *Informs Journal of Computing*, 19:458–469, 2007.
4 C. Brown. Financing transport infrastructure: For whom the road tolls. *Australian Economic Review*, 38:431–438, 2005.

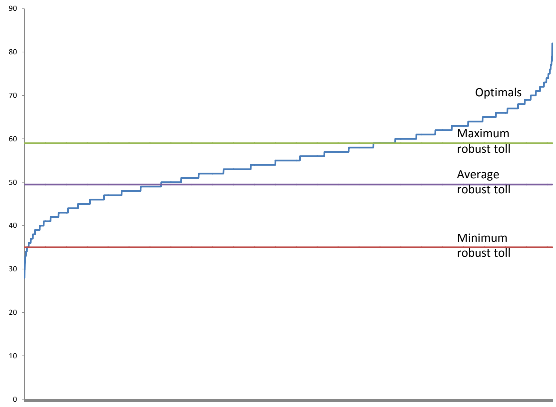

Figure 3 Exp. 3: Comparison of optimal tolls with robust tolls.

5 J.-P. Coté, P. Marcotte, and G. Savard. A bilevel modeling approach to pricing and fare optimization in the airline industry. *Journal of Revenue and Pricing Management*, 1:23–36, 2003.

6 L.M. Gardner, A. Unnikrishnan, and S.T. Waller. Solution methods for robust pricing of transportation networks under uncertain demand. *Transportation Research Part C*, 18:656–667, 2010.

7 F. Gilbert, P. Marcotte, and G. Savard. A numerical study of the logit network pricing problem. *Transportation Science*, 49(3):706–719, 2015.

8 J. Goh and M. Sim. Distributionally robust optimization and its tractable approximations. *Operations Research*, 58(4):902–917, 2010.

9 V. Goyal and R. Ravi. An fptas for minimizing a class of quasi-concave functions over a convex set. *Operations Research Letters*, 41(2):191–196, 2013.

10 G. Heilporn, M. Labbé, P. Marcotte, and G. Savard. A parallel between two classes of pricing problems in transportation and marketing. *Journal of Revenue and Pricing Management*, 9:110–125, 2010.

11 G. Karakostas and SG. Kolliopoulos. Edge pricing of multicommodity networks for heterogenous selfish users. *FOCS*, 2004.

12 M. Labbé, P. Marcotte, and G. Savard. A bilevel model of taxation and its application to optimal highway pricing. *Management Science*, 44:1608–1622, 1998.

13 M. Labbé and A. Violin. Bilevel programming and price setting problems. *4OR*, 11:1–30, 2013.

14 T.G.J. Myklebust, M.A. Sharpe, and L. Tuncel. Efficient heuristic algorithms for maximum utility product pricing problems. *Computers and Operations Research*, 69:25–39, 2016.

15 RT. Rockafeller and S. Uryasev. Optimization of conditional value-at-risk. *Journal of Risk*, 2(3):21–42, 2000.

16 Iakovos Toumazis and Changhyun Kwon. Worst-case conditional value-at-risk minimization for hazardous materials transportation. *Transportation Science*, http://dx.doi.org/10.1287/trsc.2015.0639:1–14, 2015.

17 S. van Hoesel. An overview of stackelberg pricing in networks. *European Journal of Operational Reseach*, 189:1393–1492, 2008.

18 A. Violin. Mathematical programming approaches to pricing problems. *PhD thesis, Universite Libre de Bruxelles*, 2014.

19 Shushang Zhu and Masao Fukushima. Worst-case conditional value-at-risk with application to robust portfolio management. *Operations Research*, 57(5):1155–1168, 2009.

A Robust Toll Algorithm

Algorithm 2 Robust Toll Algorithm

```
BinarySearch_{r̂∈Ω}(Core(r̂))
Output r̂ with maximum revenue.
if maximum revenue is 0 for every r̂ in BinarySearch(r̂ ∈ Ω) then
  Output u̲
end if
```

Algorithm 3 Core($\hat{r}$)

INPUT: $\hat{r}$, $\underline{u}$, $\overline{u}$, $\overline{\kappa}$
for $\epsilon \in [0, 1]$ **do**
  Solve (17) - (30) with $r = \hat{r}$ for all $\epsilon \in [0, 1]$,
  if $\frac{\sum_i y_i}{T} \geq \epsilon$ **then**
    revenue($\hat{r}$) $= \hat{r} \times \epsilon$
    break
  else
    revenue($\hat{r}$) $= 0$
  end if
end for
Output revenue($\hat{r}$).

B Proof of Lemma 1

Proof. Let $G(r) = \int_q^r (r - c)\mathbf{F}(\mathbf{c})\mathbf{dc}$. From Lemma 1 of [15] G is a convex continuously differentiable function. Using the fundamental of theorem of calculus and the differentiation by parts, we can derive $G'(r) = \int_{c \leq r} \mathbf{F}(\mathbf{c})\mathbf{dc}$. This implies $\frac{\partial f}{\partial r} = 1 - \frac{1}{(1-\epsilon)} \int_{c \leq r} \mathbf{F}(\mathbf{c})\mathbf{dc}$, which proves the statement. ◀

C Proof of Theorem 2

Proof. Constraints (12) - (14) ensure that $y_i = 0$ when $r > c_i$ and $y_i = 1$ otherwise, and (14)- (15) ensure $z_i = \max[r - c_i, 0]$. ◀

D Proof of Theorem 3

Proof. To see that this is true, note that for every solution to (8)-(16), we can create an equivalent solution to (17)-((30)) by taking the C, Z, Y values as they are and putting $u_i = r$ and $v_i = c_i$ for every i with $y_i = 1$ and 0 otherwise. ◀