Report from Dagstuhl Seminar 16241

Graph Polynomials: Towards a Comparative Theory

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 16241 "Graph Polynomials: Towards a Comparative Theory".

The area of graph polynomials has become in recent years incredibly active, with new applications and new graph polynomials being discovered each year. However, the resulting plethora of techniques and results now urgently requires synthesis. Beyond catalogues and classifications we need a comparative theory. The intent of this 5-day Seminar was to further a general theory of graph polynomials.

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1 Executive Summary

Jo Ellis-Monaghan
Andrew Goodall
Johann A. Makowsky
Iain Moffatt

The intent of this 5-day Seminar was to develop a general theory of graph polynomials. Graph polynomials have played a key role in combinatorics and its applications, having effected breakthroughs in conceptual understanding and brought together different strands of scientific thought. The characteristic and matching polynomials advanced graph-theoretical techniques in chemistry; the Tutte polynomial married combinatorics and statistical physics, and helped resolve long-standing problems in knot theory. The area of graph polynomials is incredibly active, with new applications and new graph polynomials being discovered each year. However, the resulting plethora of techniques and results now urgently requires synthesis. Beyond catalogues and classifications we need a comparative theory.

There is a long history in this area of results in one field leading to breakthroughs in another when techniques are transferred, and this Seminar leveraged that paradigm. More critically, experts in the field have recently begun noticing strong resonances in both results

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Figure 1 *Above left*: J. Ellis-Monaghan, I. Moffatt, J. A. Makowsky. *Above middle*: A. Goodall, I. Moffatt. *Above right*: E. Gioan, B. Courcelle, B. Bollobás, J. Oxley, L. Kauffman, S. Backman. *Below left*: A. De Mier, N. Jonoska, L. McMahon. *Below middle*: J. Nešetřil, , K. Morgan, A. Goodall, I. Moffatt. *Below right*: The audience at large. Pictures courtesy J. A. Makowsky.

and proof techniques among the various polynomials. The species and genera of graph polynomials are diverse, but there are strong interconnections: the Seminar initiated work on creating a general theory that will bring them together under one family. The process of developing such a theory of graph polynomials should expose deeper connections, giving great impetus to both theory and applications. This has immense and exciting potential for all those fields of science where combinatorial information needs to be extracted and interpreted.

The Seminar provided conditions ripe for cross-fertilization of ideas among researchers in graph theory and topological graph theory, in logic and finite model theory, and in current biocomputing and statistical mechanics applications. During the Seminar the participants were offered a conspectus of the broad area of graph polynomials. The view was confirmed that a synthetic approach is needed in order to see the wood for the trees. The discussions and collaborations initiated at the workshop promise well for the development of a unified theory of graph polynomials. This Seminar represented a convincing beginning, and, hopefully, similar meetings in future will further the envisaged project.

In the light of our stated goals, the Seminar provided ample time for discussion groups and tutorials. The participants (44) of the Seminar included some of the leading experts in combinatorics, knot theory, matroid theory and graph polynomials from Europe, the Americas, Asia and Australia. The composition of participants was both age and gender balanced with a quarter of the participants being women. The younger researchers (more than a quarter of the participants) profited from intense contacts and discussions with their more experienced colleagues. An inspiring problem session brought about particular directions for further research.

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Open problems

3 Overview of Talks

3.1 Fourientations and the Tutte Polynomial

Spencer Backman (Universität Bonn, DE)

A fourientation of a graph G is a choice for each edge of the graph whether to orient that edge in either direction, leave it unoriented, or biorient it. I will describe a 12 variable expansion of the Tutte polynomial in terms of fourientation activities due to myself, Sam Hopkins, and Lorenzo Traldi, which specializes to known subgraph and orientation expansions. Time permitting, I will explain applications of this expression to the theory of zonotopes, hyperplane arrangements, chip-firing, and the reliability polynomial.

3.2 Which Graph Polynomials Have the Difficult Point Property?

Markus Bläser (Universität des Saarlandes, DE)

A class of graph polynomial has the difficult point property if for all graph polynomials G from this class, the following holds: If G is #P-hard to evaluate at one single point, then it is #P-hard to evaluate at (Zariski) almost all points. We present some rather general classes with the difficult point property and review some techniques how to prove that a class has the difficult point property.

3.3 Bill Tutte and His Polynomial

Béla Bollobás (University of Cambridge, GB & University of Memphis, US)

The influence of W.T. Tutte on our lives and on modern mathematics is hard to overestimate. In my brief talk I shall say a few words about Tutte's work during WWII, and his contribution to mathematics, with emphasis on his polynomial. In addition, I shall point out some of the many important extensions of this polynomial.

3.4 Introduction to Multimatroids and Their Polynomials

Robert Brijder (Hasselt University – Diepenbeek, BE)

Multimatroids have been introduced by Bouchet in a series of papers in the 1990s. Multimatroids generalize delta-matroids (which includes all matroids) and isotropic systems, and various interesting properties of these latter combinatorial structures carry over naturally to multimatroids. Unfortunately, the promising multimatroid theory has only scarcely been picked up by the community.

We give an introduction to multimatroids and give a general (multivariate) multimatroid polynomial that generalizes various well-known polynomials, such as the interlace polynomial, the Penrose polynomial, and the Tutte polynomial on the diagonal. We also show that various evaluations and recursive relations carry over to this general domain.

The multimatroid polynomial also generalizes the Martin polynomial of 4-regular graphs. We finally focus on the open problem of formulating a multimatroid version of the Martin polynomial of Eulerian graphs in general.

3.5 Tutorial: Aspects of the Characteristic Polynomial

Ada Sze Sze Chan (York University – Toronto, CA), Krystal Guo (University of Waterloo, CA), and Gordon Royle (The University of Western Australia – Crawley, AU)

In this session, we will discuss a number of our favourite techniques, results and open problems related to the characteristic polynomial.

Topics will include

(a) elementary spectral results relating the spectrum to graph properties (not covered in Monday's talk) and interlacing
(b) highly structured graphs such as strongly regular and distance regular graphs
(c) relationships between the characteristic polynomial and the walk-generating function of a graph

and others as determined by the intersection of the presenters' expertise and audience wishes.

3.6 An Introduction to the Theory of Matroids

Carolyn Chun (Brunel University London, GB) and James Oxley (Louisiana State University – Baton Rouge, US)

Matroids were introduced by Hassler Whitney in 1935 to provide a common framework for viewing dependence in linear algebra and graph theory. They arise naturally in optimisation as structures for which a greedy strategy always produces an optimal set. This talk will introduce matroids. It will discuss some well-known examples of these structures and some of their basic operations, and will conclude by addressing questions relating to matroids that are raised by the audience. No prior knowledge of matroid theory will be assumed.

3.7 Computations of Monadic Second-Order Definable Polynomials by Fly-Automata

Bruno Courcelle (University of Bordeaux, FR)

Most graph polynomials can be defined by monadic second-order (MSO) formulas. This is the case of the Tutte polynomial and the interlace polynomial. We take "defined" in a wide sense. In particular, if an integer value $f(G)$ attached to a graph is defined as the number of assignments (X, Y, Z) that satisfy an MSO formula phi(X,Y,Z) in graph G, it is "MSO-defined".

The MSO model-checking problem is FPT for tree-width, and even for clique-width in some cases, and so is the computation of $f(G)$ as above. Proofs can be given by constructions of finite automata that process algebraic terms describing the input graphs. However, these automata are inevitably huge and cannot be implemented by means of transitions tables. Fly-automata do not use such tables : their states are described (by finite words, according to some syntax) and their transitions are defined by "small" efficient programs. They overcome in many cases the "huge size problem". They can compute values, not only check membership. They can compute MSO definable graph polynomials.

The talk will show how these automata can be constructed from MSO formulas and report computer experiments.

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2 B. Courcelle, I. Durand: Automata for the verification of monadic second-order graph properties. J. Applied Logic 10(4): 368–409 (2012)
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3.8 Transition Polynomials: Definitions, Properties, and Interrelations

Jo Ellis-Monaghan (Saint Michael's College – Colchester, US)

As deletion-contraction reductions are to edges, so are transition systems to vertices. The Tutte polynomial is the universal object for deletion-contraction invariants, while the generalised transition polynomial is universal for the wide variety of polynomials that are defined via recursions that involve transition systems at a vertex. This tutorial will give an overview of the defining properties and supporting algebraic structures for transition polynomials. We will review several examples such as the Martin polynomial, Kauffman bracket, Penrose polynomial, and topological transition polynomial, showing them to be specialisations of a universal generalised transition polynomial. We will conclude with connections among transition polynomials, deletion/contraction invariants, and even the interlace polynomial. We wish we knew how the Characteristic polynomial fit in among these interrelations.

3.9 Graph Polynomials: Some Questions on the Edge

Graham Farr (Monash University – Clayton, AU)

Any general theory of graph polynomials will include some functions on graphs and leave others out. We consider some questions about graph polynomials that may lie somewhere near the "edge" of what can be covered by a general theory.

3.10 Polynomials from Grassmannians

Alex Fink (Queen Mary University of London, GB)

Algebraic varieties provide a fertile source of "polynomial" invariants of matroids (and thus graphs), for instance by constructing their cohomology ring, or cohomology class in a larger object. We sketch a substantial part of a construction yielding the Tutte polynomial, while pointing out a few of the myriad variations on the approach.

3.11 Tutorial: Matroid Polytopes, Valuations, and Their Appearance in Algebraic Geometry

Alex Fink (Queen Mary University of London, GB)

Among the numberless axiom systems for matroids is one that presents them as certain 0-1 polytopes. I introduce these polytopes and their anatomy: several more familiar axiom systems are manifest in the polyhedral data. They make some generalisations of matroids natural, e.g. to polymatroids or delta-matroids. They also make it natural to consider a particular linear relation, "valuativity", which is quite obscured from more familiar points of view, but holds of the Tutte polynomial and at least a couple other polynomials from the literature. The universal invariant for valuativity is an understood object, and valuativity is also the property of these polytopes useful in my and Speyer's algebro-geometric construction of Tutte. As time permits I'll explain at least one of that construction and a more elementary one due to myself and Amanda Cameron based on lattice point enumeration.

3.12 On Six Tutte Polynomial Expressions for a Graph on a Linearly Ordered Set of Edges

Emeric Gioan (University of Montpellier & CNRS, FR)

I will present six interrelated general expressions for the Tutte polynomial of a graph, that are available as soon as the set of edges is linearly ordered, and that witness combinatorial properties of such a graph: the classical enumeration of spanning tree activities; its refinement into a four variable expression in terms of subset activities (that corresponds to the classical partition of the set of edge subsets into boolean intervals); the enumeration of orientation-activities for directed graphs; its refinement into a four variable expression in terms of subset orientation-activities (that corresponds to the partition of the set of orientations into active partition reversal classes); the convolution formula for the Tutte polynomial (that does not need the graph to be ordered); and an expression of the Tutte polynomial using only beta invariants of minors (that refines the above expressions). I will mention that these expressions are all interrelated by the underlying canonical active bijection between spanning trees and orientations, subject of a long-term joint work with Michel Las Vergnas.

3.13 Polynomial Graph and Matroid Invariants From Graph Homomorphisms

Andrew Goodall (Charles University – Prague, CZ)

The number of homomorphisms from a graph F to the complete graph K_n is the evaluation of the chromatic polynomial of F at n. Suitably scaled, this is the Tutte polynomial evaluation $T(F; 1-n, 0)$ and an invariant of the cycle matroid of F. Dual to colourings are flows. Tutte constructed his dichromate as a bivariate generalization of the chromatic polynomial and flow polynomial. The Tutte polynomial extends from graphs to matroids more generally.

Motivated by these observations I shall talk about the following questions, answering them in part and highlighting what remains open:

1. Which other graph polynomials arise from counting homomorphisms to the nth term of a sequence of graphs, like the chromatic polynomal from the sequence (K_n)?
2. Which of these yield a cycle matroid invariant? And which of these can be extended to a larger class of matroids closed under duality?

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3 D. Garijo, A.J. Goodall and J. Nešetřil, Polynomial graph invariants from homomorphism numbers, Discrete Math., 339 (2016), no. 4, 1315–1328. Preprint: arXiv: 1308.3999 [math.CO]

3.14 Non-Matroid Generalizations of the Tutte Polynomial

Gary P. Gordon (Lafayette College – Easton, US) and Liz McMahon (Lafayette College – Easton, US)

It is possible to imitate the corank-nullity definition of the Tutte polynomial to get a meaningful invariant for combinatorial structures that are not matroids. We explore these, concentrating on trees, rooted trees, and finite subsets of Euclidean space.

3.15 Algorithms for Computing the Tutte Polynomial

Thore Husfeldt (IT University of Copenhagen, DK)

I gave a brief survey of algorithms for computing the Tutte polynomial. The presentation was from the algorithmic perspective, so I focused the attention on computational complexity issues such as worst-case computation times. I sketched the constructions underlying total enumeration, deletion-contraction, and inclusion-exclusion algorithms and gave a brief analysis of their use for computational investigations. A brief connection was made to the current trend in computational complexity that attempts to establish a more fine-grained view of the hardness of NP-hard problems.

3.16 Graph Polynomials from DNA Rearrangements

Natasa Jonoska (University of South Florida – Tampa, US)

Joint work of Masahico Saito, Natasa Jonoska

Nucleotide rearrangements can be modelled by 4-regular rigid vertex graphs, called assembly graphs. They are closely related to double occurrence words, chord diagrams, and circle graphs. Edges of these graphs represent double-stranded DNA molecules, while vertices correspond to DNA recombination sites. Polynomial invariants related to the recombination processes of these assembly graphs are also invariants for circle graphs and chord diagrams. In addition we propose other variations of these invariants. These polynomial invariants are related to the possible products of the rearrangements modelled by the assembly graphs.

3.17 Introduction to Combinatorial Knot Polynomials

Louis H. Kauffman (University of Illinois – Chicago, US)

This talk is self-contained and will begin with an introduction to the bracket polynomial state sum model for the Jones polynomial. We will discuss how this combinatorial knot polynomial is related to the Tutte and to the dichromatic polynomial, the Temperley-Lieb algebra and the Potts model in statistical mechanics. We will discuss how the bracket state sum can be used to prove a number of results in knot theory such as the non-triviality of reduced alternating and adequate knots and links, and the existence of examples of non-trivial links with trivial Jones polynomial. We will then show how the bracket polynomial can be constructed as a state summation using solutions to the Yang-Baxter equation. This provides an entry into the general subject of quantum link invariants, knot polynomials constructed via solutions to the Yang-Baxter equation and via Hopf algebras.We will give a very quick introduction to Khovanov homology, based on the bracket polynomial. The talk will mention an important ancestor of these models – the Penrose state summation for counting colourings of planar graphs, and the speaker's solution to the problem of extending the Penrose structure to non-planar graphs. This will be sufficient material for a first talk. This will be a blackboard talk. It is intended as an introduction to these topics and to open problems related to them. Time permitting, we will discuss skein theory and the other skein polynomials and state sums related to them.

3.18 Tutorial: Khovanov Homology, Dichromatic Polynomial and the Potts Model

Louis H. Kauffman (University of Illinois – Chicago, US)

Khovavnov homology is a way of extracting topological information from the states of the bracket polynomial for a given knot or link via a chain complex associated with these states. If one regards the bracket polynomial as a relative of the Potts model (aka dichromatic polynomial with a physical parameter) (they live in the same parameter space of a generalized bracket that corresponds to a signed Tutte polynomial for the associated Tait graph for the link), then the loops in the states are boundaries of regions of constant spin in the Potts model. This suggests that geometric transitions between states obtained by site re-smoothing should be related to properties of the Potts partition function and that the Khovanov homology should have information relevant to the Potts models. This tutorial explores these questions. We show a direct correspondence with a quantum model and and indirect correspondence with the a Potts model at certain imaginary temperatures. The question is: How can we do better in understanding both the Potts model and the physical nature of Khovanov homology and its relationship with combinatorics.

3.19 Counting Walks and the Resulting Polynomial

Marsha Kleinbauer (TU Kaiserslautern, DE)

Counting the closed walks of length k in a graph G with n vertices is equivalent to finding the sum:

$$w_k \sum_{i=1}^{n} \lambda_i^k$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of G. It follows that $w_0 = n$, $w_1 = 0$, w_2 is two times the number of edges in G, and w_3 is six times the number of triangles in G. Extensions of these equations are presented. We present a method that uses generating functions to count certain types of closed walks in a graph.

3.20 4-Dimensional Discrete Ihara-Selberg Function and Binary Linear Codes

Martin Loebl (Charles University – Prague, CZ)

I will show how to write weight enumerator of a binary linear code as 4-dimension discrete Ihara-Selberg function.

3.21 Dichotomy Theorems for Generalized Chromatic Polynomials

Johann A. Makowsky (Technion – Haifa, IL)

Evaluation of the chromatic polynomial is easy on finitely many points, and $\#\mathbf{P}$ hard everywhere else. We call this the difficult point property **DPP**. Let F be a graph property and k be a positive integer. A function $f : V(G) \to [k]$ is an F-coloring if for every $i \in [k]$ the set $f^{-1}(i)$ induces a graph in F. The author and Boris Zilber have shown in 2006 that counting F-colorings with k colors is a polynomial $P_F(G; k)$ in k. We show infinitely many examples of properties F, where **DPP** holds for $P_F(G; k)$, and formulate several conjectures, including also multivariate graph polynomials.

References

1 J. A. Makowsky and T. Kotek and E. V. Ravve, A Computational Framework for the Study of Partition Functions and Graph Polynomials, *Proceedings of the 12th Asian Logic Conference '11*, World Scientific (2013), pages 210–230.

3.22 Some Pictures

Johann A. Makowsky (Technion – Haifa, IL)

The pictures in the report abstract were made by J. A. Makowsky

3.23 Where Do Topological Tutte Polynomials Come From?

Iain Moffatt (Royal Holloway University of London, GB)

I'll give a brief introduction, focussing on how they arise, to the three versions of the Tutte polynomial for graphs in surfaces due to M. Las Vergnas (1978), B. Bollobás and O. Riordan (2001), and V. Krushkal (2012). In particular, I will show how each of these polynomials arises naturally and canonically from attempts to extend the recursive deletion-contraction definition of the Tutte polynomial to graphs in surfaces.

3.24 New Types of Chromatic Factorization

Kerri Morgan (Monash University – Clayton, AU)

The chromatic polynomial $P(G;\lambda)$ gives the number of ways a graph G can be coloured in at most λ colours. A graph G has a *chromatic factorisation* with *chromatic factors*, H_1 and H_2, if $P(G;\lambda) = P(H_1;\lambda) \times P(H_2;\lambda)/P(K_r;\lambda)$ where the chromatic factors have chromatic number at least r and K_r is the complete graph of order r. A graph is said to be *clique-separable* if it contains a clique whose removal disconnects that graph. It is well-known that any clique-separable graph has a chromatic factorisation. Morgan and Farr (2009) found graphs that are not clique-separable, nor *chromatically equivalent* to any clique-separable graphs, but factorised in the same way as clique-separable graphs. In all of these cases, the graphs have a factorisation that "behaves" like the factorisation of a clique-separable graph.

In this talk, we present new results on cases where the chromatic polynomial "factorises" but does not "behave" like the factorisation of clique-separable graphs. We give an infinite family of graphs that that have a chromatic factorisation that is "similar" to a clique-separable graph but one of the chromatic factors does **not** have chromatic number at least r. We also give examples of graphs that have chromatic polynomials $P(G;\lambda) = P(H_1;\lambda) \times P(H_2;\lambda)/P(D;\lambda)$ where D is **not** a complete graph.

3.25 Delta-Matroids

Steven Noble (Brunel University London, GB)

We describe Δ-matroids and their fundamental operations: minors, partial duality and loop complementation. We illustrate these concepts on ribbon graphs and binary Δ-matroids.

In particular for vf-safe Δ-matroids, we explain the 3 minor operations, twisted duality and their implications for Δ-matroid polynomials.

Finally we briefly mention a few new results such as chain and splitter theorems for 2 connected, even or vf-safe Δ-matroids, the 2-sum operation, characterising vf-safe Δ-matroids and counting labelled Δ-matroids.

3.26 Methods in the Study of Real Chromatic Roots

Thomas Perrett (Technical University of Denmark – Lyngby, DK)

Joint work of Carsten Thomassen, Thomas Perrett]

The chromatic polynomial is perhaps the best studied univariate graph polynomial, but many intriguing open problems remain unsolved. In particular the roots of the chromatic polynomial have attracted much attention, and the focus of many results in this subfield is to answer questions of the following type: For a class of graphs $\mathcal{G}$ and a set $D \subseteq \mathbb{C}$, let $R_D(\mathcal{G})$ denote the set of chromatic roots in D of graphs in $\mathcal{G}$. Can we characterise $\overline{R_D(\mathcal{G})}$? Such results are often attractive and surprising. Consider, for example, those of Sokal, Jackson, and Thomassen, which state that, if $\mathcal{G}$ denotes the family of all graphs, then $\overline{R_\mathbb{C}(\mathcal{G})} = \mathbb{C}$ and $\overline{R_\mathbb{R}(\mathcal{G})} = [32/27, \infty)$. On the other hand, $\overline{R_\mathbb{R}(\mathcal{P})}$ is still unknown if $\mathcal{P}$ denotes the planar graphs.

In this talk we promote a construction of Thomassen which, given a graph with certain properties, constructs a sequence of graphs with chromatic roots approaching a given real number. We show that this method is particularly easy to use if one is interested in minor-closed classes of graphs. Indeed, as an example, we show that $\overline{R_\mathbb{R}(\mathcal{P})}$ contains the interval $(3, 4)$, except for a tiny interval around $\tau + 2$, where $\tau \approx 1.618$ is the golden ratio. This constitutes a partial converse to a famous theorem of Tutte. Finally, we also discuss the limits of the construction and open problems for which it seems that a new technique is required.

3.27 Algebraic vs Graph Theoretic Properties of Graph Polynomials

Elena V. Ravve (ORT Braude College – Karmiel, IL)

Joint work of Johann A. Makowsky, Elena V. Ravve

Two graph polynomials are d.p.-equivalent (distinctive power equivalent) if they distinguish the same graphs. Graph theoretic properties are properties which are invariant under d.p.-equivalence. Algebraic properties are properties of the particular presentation of the graph polynomial and are not invariant under d.p.-equivalence. We exemplify this notion on the example of the location of the roots of a graph polynomial. Other properties are the unimodality of the coefficients, orthogonality of the polynomials for specific sequences of graphs, etc.

3.28 Deterministic Approximation Algorithms for the Tutte Polynomial, the Independence Polynomial and Partition Functions

Guus Regts (University of Amsterdam, NL)

Joint work of Alexander Barvinok, Viresh Patel, Guus Regts

In this talk I will discuss a general method that yields deterministic polynomial time approximation algorithms for evaluations of the Tutte and independence polynomial on bounded degree graphs as well as for partition functions of vertex- and edge-coloring models on bounded degree graphs. Ingredients of the method include: zero-free regions on bounded degree graphs, low order Taylor approximations of the logarithm of a polynomial and computations of coefficients of graph polynomials.

3.29 The Characteristic Polynomial of a Graph

Gordon Royle (The University of Western Australia – Crawley, AU)

The characteristic polynomial of a graph G is the characteristic polynomial of its adjacency matrix. While there are many different graph polynomials (chromatic, Tutte, matching etc), the characteristic polynomial is perhaps the most heavily studied of all, primaril y because the roots of the characteristic polynomial (i.e. the eigenvalues of its adjacency matrix) carry so much information about s tructure of the graph and its subgraphs. Indeed, a large proportion of the entire field of algebraic graph theory can be viewed as exp loring exactly which properties of graphs are, or are not, reflected in its spectrum.

In this talk, I will outline some of the main properties of the characteristic polynomial of a graph, but also introduce some of the i nteresting open questions that remain. As an example, it is not currently known whether or not almost all graphs are determined up to isomorphism by their characteristic polynomials.

3.30 Introduction to the Bipartition Polynomial and Its Relatives

Peter Tittmann (Hochschule Mittweida, DE)

Let $G = (V, E)$ be a finite simple undirected graph. The *open neighborhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices that are adjacent to v in G. The *closed neighborhood* of v is $N_G(v) \cup \{v\}$. Analogously, we define

$$N_G(W) = \bigcup_{v \in W} N_G(v) \setminus W$$

and $N_G[W] = N_G(W) \cup W$ for any vertex subset $W \subseteq V$. For a given vertex subset $W \subseteq V$, let ∂W be the set of all edges of G with exactly one of their end vertices in W, i.e.

$$\partial W = \{\{u, v\} \in E \mid u \in W, v \in V \setminus W\}.$$

The *bipartition polynomial* of G, introduced in [2], is

$$B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} z^{|F|}.$$

We give different representations of this polynomial and show its relations to other graph polynomials, including the domination, Ising, cut, independence, Eulerian subgraph, and matching polynomial.

We will discuss the role of linear orderings of the edge set, a modified version of external activity and provide some open problems.

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3.31 Recurrence Relations for Independence Polynomials in Hypergraphs

Martin Trinks (Nankai University – Tianjin, CN)

The independence polynomial of a hypergraph is the generating function for its independent (vertex) sets with respect to their cardinality. This talk aims to discuss several recurrence relations for the independence polynomial using some vertex and edge operations. Further, an extension of the well-known recurrence relation for simple graphs to hypergraphs is proven and other novel recurrence relations are given.

4 Open problems

4.1 The Open Problem Session

Andrew Goodall (Charles University – Prague, CZ) and Iain Moffatt (Royal Holloway University of London, GB)

Problems from the problem session of *Graph polynomials: towards a comparative theory*, Dagstuhl, Monday 13 June to Friday 17 June, 2016. Also from talks given by speakers during the week and submitted by other workshop participants.

4.2 Number of acyclic orientations and its relation to the size of the automorphism group

Spencer Backman (Universität Bonn, DE(

The following is conjectured:

Let G be a vertex-transitive simple graph. Then the number of acyclic orientations of G is at least equal to the size of the automorphism group of G, i.e.,

$$T(G; 2, 0) \geq |\mathrm{Aut}(G)|.$$

Furthermore, equality holds if and only if $G \cong K_n$.

4.3 Smallest ideal of graph mapping polynomial

Joanna Ellis-Monaghan (Saint Michael's College – Colchester, US)

Given a polynomial P mapping graphs to a commutative ring with unity R, what is the smallest ideal I of R such that graphs $\to R \to R/I$ is tractable to compute?

4.4 4-Colourability of matroid dual graphs (Hassler Whitney, 1993)

Graham Farr (Monash University– Clayton, AU)

A problem of Hassler Whitney from 1932.

Let G be a graph such that there exists a graph H with the property that $T(G^*; x, y) = T(H; x, y)$, where G^* is the matroid dual of G (a cographic matroid).

Are such graphs G 4-colourable?

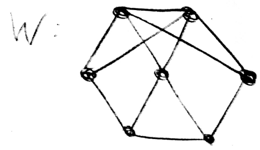
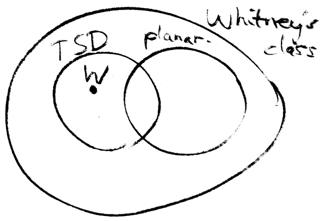


Figure 2 Whitney's example of a graph having a Tutte dual.



1. If G is planar then G^* is also a planar graph and we may take $H = G^*$.
2. A graph G is *Tutte self-dual* (TSD) if $T(G^*; x, y) = T(G; x, y)$, in which case we may take $H = G$.

4.5 Construction of a nice $\widehat{Q}(\Delta)$

Alex Fink (Queen Mary University of London, GB)

Let G be a connected bipartite graph and $V_e \amalg V_v$ its vertex set. A *hypertree* for G is the degree sequence in $Z^{|V_e|}$ of some spanning tree of G (these form a *hypergraphic polymatroid*). Define the bivariate polynomial $Q(G; t, u)$ so that, when t and u are naturals,

$$\begin{aligned} Q(G;t,u) = \#\{p \in Z^{|V_e|} : {} & p = a + b + c, \\ & a \text{ is a hypertree of } G, \\ & b_i \in Z_{\le 0}, \ \sum_i b_i = -t, \\ & c_i \in Z_{\ge 0}, \ \sum_i c_i = u\}. \end{aligned}$$

Ehrhart theory guarantees the existence of this polynomial. When all vertices in V_e are bipartite, then G is the barycentric subdivision of a graph H; in this case, hypertrees for G are in bijection with spanning trees for H, and $Q(G; t, u)$ contains the same information as $T(H; x, y)$. To wit, with Amanda Cameron we've shown that

$$\sum_{t,u \ge 0} Q(G;t,u)\alpha^t \beta^u = \frac{T\left(H; \frac{1-\alpha\beta}{1-\beta}, \frac{1-\alpha\beta}{1-\alpha}\right)}{(1-\alpha)^{|V(H)-1|}(1-\beta)^{|E(H)-V(H)+1|}(1-\alpha\beta)}.$$

Now let Δ be a three-coloured triangulation of the sphere. Then there are six ways to delete one colour class from Δ, leaving a bipartite graph G, and label the other two colour classes V_e and V_v. If V_e is colour i, V_v is colour j, and the deleted colour is k, let $Q_{ijk}(\Delta; t, u) = Q(G; t, u)$. These are interrelated. Firstly,

$$Q_{ijk}(\Delta; t, u) = Q_{ikj}(\Delta; u, t).$$

In the case where all vertices of colour i have degree 4, this is plane graph duality (in general, it's a polymatroid duality). Secondly, Kálmán and Postnikov have shown that

$$Q_{ijk}(\Delta; t, 0) = Q_{jik}(\Delta; t, 0).$$

This is all compatible with the existence of a trivariate polynomial $\widehat{Q}(\Delta; x_i, x_j, x_k)$ such that

$$\widehat{Q}(\Delta; 0, x_j, x_k) = Q_{ijk}(\Delta; x_k, x_j)$$

and such that permuting the colour classes of Δ permutes the variables of $\widehat{Q}(\Delta)$ in the corresponding way.
Problem: Construct a nice such $\widehat{Q}(\Delta)$.

4.6 $\hom(G, \mathrm{Cayley}(A_k, B_k))$

Andrew Goodall (Charles University – Prague, CZ)

Let A_k be an additive Abelian group and $B_k = -B_k \subseteq A_k$ an inverse-closed subset for each $k \in \mathbb{N}$. The graph $\mathrm{Cayley}(A_k, B_k)$ has vertices A_k and edges joining u and v precisely when $u - v \in B_k$.

- When $A_k = \mathbb{Z}_k$ and $B_k = \mathbb{Z}_k \setminus \{0\}$, $\mathrm{Cayley}(A_k, B_k) = K_k$ and $\hom(G, K_k) = P(G; k)$ is the chromatic polynomial of G evaluated at k.
- When $A_k = \mathbb{Z}_{sk}$ and $B_k = \{kr, kr+1, \ldots, k(s-r)\}$ then $\hom(G, \mathrm{Cayley}(A_k, B_k))$ is [1] a polynomial in k. (The minimum ratio s/r such that there exists a homomorphism from G to $\mathrm{Cayley}(\mathbb{Z}_s, \{r, r+1, \ldots, s-r\}$ is the *circular chromatic number* of G.)
- De la Harpe and Jaeger (1995) showed that when $A_k = \mathbb{Z}_k$ and $B_k = B/k\mathbb{Z}$ for some fixed $B = -B \subseteq \mathbb{Z}$ then $\hom(\mathbb{Z}_k, B_k)$ settles eventually to a fixed polynomial in k if and only if B is finite or cofinite. For example $\hom(G, C_k)$ ($B = \{-1, +1\}$) is a fixed polynomial in k only for $k > g(G)$.

Question: When is $\hom(G, \mathrm{Cayley}(A_k, B_k))$ a polynomial in $|A_k|$ and $|B_k|$? (The polynomial here depends on G, but must not depend on k.)

A motivation for this problem is that for each such graph polynomial counting B_k-tensions in A_k (equivalently, vertex A_k-colourings such that adjacent colours differ by an element in B_k) there is a dual graph polynomial counting B_k-flows, which together may be unified in a bivariate polynomial akin to the Tutte polynomial. Interest may then lie in what other combinatorial information is encapsulated in this bivariate polynomial. Also, is there a reduction formula analogous to the deletion-contraction recurrence for the chromatic polynomial?

References

1 A. J. Goodall, J. Nešetřil and P. Ossona de Mendez, Strongly polynomial sequences as interpretations of trivial structures, J. Appl. Logic, to appear. Preprint: arXiv:1405.2449 [math.CO]

4.7 Tutte polynomials for matroids and their relationship to other graph polynomials

Joseph Kung (University of North Texas – Denton, US)

By an (n, r)-matroid we mean a rank r matroid with groundset of size n. The Tutte polynomial of an (n, r)-matroid has degree r as a polynomial in x and degree $n - r$ as a polynomial in y. The Tutte polynomials of such matroids span a subspace if $\mathbb{C}[x, y]$ and an upper bound for

$$\dim\langle\, T(M; x, y) : M \text{ an } (n, r)\text{-matroid}\,\rangle$$

is $(r + 1)(n - r + 1)$.

The dimension is in fact equal to $r(n - r) + 1$.

Problem: Answer the same question for other graph polynomials.

The dimension of the subspace spanned by the graph polynomial for graphs of given order and size serves as a measure of information contained in a graph polynomial: how useful is this way of measuring the combinatorial information contained in a given polynomial graph invariant?

4.8 The Frustration Conjecture

Martin Loebl (Charles University – Prague, CZ)

The Frustration Conjecture. Let $G = (V, E)$ be a graph, $w : E \to \{0, 1, -1\}$, and let there be given g disjoint pairs of edges $p_1, \ldots, p_g$. A perfect matching $M \subseteq E$ is a *parity matching* if $|M \cap p_i|$ is even for all i.

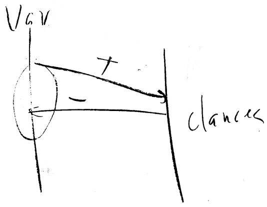
Is it true that there is no algorithm for finding a maximum weight maximum parity matching that has complexity better than 2^g?

Reference: M. Loebl, Parity matching, preprint, 2016. Available at: http://kam.mff.cuni.cz/~loebl/clanky/parityMatching0416.pdf

4.9 A polynomial related to the SAT-problem

Johann A. Makowsky (Technion – Haifa, IL)

Let C be a set of clauses and (V, C, R) a directed graph in which arcs (R) join variables (V) to clauses (C) with direction according to whether the variable is negated or not in the clause. (See figure, right.)



For $A \subseteq V$, define $\mathrm{SAT}(A) = \{c \in C : A \text{ satisfies } c\}$ and the SAT-polynomial in indeterminate X by

$$\sum_{A \subseteq V} \prod_{c \in \mathrm{SAT}(A)} X = \sum_{A \subseteq V} X^{|\mathrm{SAT}(A)|}.$$

Question: Is this polynomial useful to study the satisfiability problem SAT?

4.10 Independence polynomial of G (Alan Sokal, 2001)

Guus Regts (University of Amsterdam, NL)

A problem of Alan Sokal from 2001. Let $\Delta \in \mathbb{N}$ and let $Z_G(\lambda)$ denote the independence polynomial of G.

For $\epsilon > 0$ is there $\delta > 0$ such that the following holds? For each graph G of maximum degree at most Δ it holds that $Z_G(\lambda) \neq 0$ for λ satisfying

$$\mathrm{Im}(\lambda) < \delta,$$

$$0 \leq \mathrm{Re}(\lambda) \leq (1-\epsilon)\frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}.$$

Note. It is known that $Z_G(\lambda) \neq 0$ if $|\lambda| \leq \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$.

4.11 Tutte polynomials of medial graphs

William Whistler (Durham University, GB)

Are the Tutte polynomials of the medial graphs of different plane embeddings of a given planar graph identical?

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