# A Complexity Dichotomy for Poset Constraint Satisfaction

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**Abstract**

We determine the complexity of all constraint satisfaction problems over partial orders, in particular we show that every such problem is NP-complete or can be solved in polynomial time. This result generalises the complexity dichotomy for temporal constraint satisfaction problems by Bodirsky and Kára. We apply the so called universal-algebraic approach together with tools from model theory and Ramsey theory to prove our result. In the course of this analysis we also establish a structural dichotomy regarding the model theoretic properties of the reducts of the random partial order.

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## 1 Motivation and the result

Reasoning about temporal knowledge is a common task in various areas of computer science, including artificial intelligence, scheduling, computational linguistics and operations research. Usually temporal constraints are expressed as collections of relations between time points or time intervals. A typical computational problem is then to determine whether such a collection is satisfiable or not.

A lot of research in this area concerns only linear models of time. In particular there exists a complete complexity classification of satisfiability problems for linear temporal constraints in [8]. However, many times more complex time models are helpful, for instance in the description of distributed and concurrent systems. In his influential paper [21] Lamport suggested to model time or, to be more precise, the precedence relation in such systems by a partial order. Since then a lot research on distributed computing is based on his approach (e.g. [22], [1], [17], [28]). Thus studying the satisfiability of constraints over partial orders is not only of theoretical, but also of practical interest.

The complexity of a small subclass of these computational problems has already been studied in [14]. We will provide a complete classification, showing that every constraint

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satisfaction problems over partial orders is either solvable in polynomial time or is NP-complete. In the following we give a more formal definition:

Let $\Phi$ be a finite set of quantifier-free formulas in the language consisting of a binary relation symbol $\leq$. We define Poset-SAT($\Phi$) as the following computational problem:

**Poset-SAT($\Phi$):**
INSTANCE: Variables $\{x_1, \ldots, x_n\}$ and a conjunction $\psi(x_1, \ldots, x_n)$ of formulas that are obtained from formulas $\phi \in \Phi$, by substituting the free variables of $\phi$ by variables from $\{x_1, \ldots, x_n\}$;
QUESTION: Is there a partial order $(A; \leq)$ such that $\psi(x_1, \ldots, x_n)$ is satisfied in $A$, i.e., $\psi(a_1, \ldots, a_n)$ holds for some elements $a_1, \ldots, a_n \in A$?

Our main result then states as the following complexity dichotomy:

▶ **Theorem 1.** *Let $\Phi$ be a finite set of quantifier-free $\leq$-formulas. Then the computational problem* Poset-SAT($\Phi$) *is in P or NP-complete.*

For the sake of illustration, we give two examples of Poset-SAT problems:

▶ **Example 2.** Let $x < y := x \leq y \wedge \neg(y \leq x)$. An instance of Poset-SAT($\{<\}$) consists of a set of variables $\{x_1, \ldots, x_n\}$ and a conjunction $\psi$ of formulas $x_i < x_j$. An instance is a yes-instance if and only if $\psi$ does not contain formulas $x_{i_1} < x_{i_2}, \ldots, x_{i_{n-1}} < x_{i_n}$, $x_{i_n} < x_{i_1}$. The existence of such cycles in $\psi$ can be verified in polynomial time. So Poset-SAT($\{<\}$) is in P.

It is not hard to see that an instance of Poset-SAT($\{<\}$) is satisfied in a partial order if and only if it is satisfied in any extension of it to a linear order. However, this is not true for general Poset-SAT problems; let us in the following denote by Temp-SAT($\Phi$) the problem that asks, if there is a *linear* order satisfying a given instance of Poset-SAT($\Phi$). The Temp-SAT problems were completely classified in [8] under the name of *temporal CSPs*. The complexities of Temp-SAT($\Phi$) and Poset-SAT($\Phi$) can be different even for very simple $\Phi$ as the following example shows:

▶ **Example 3.** For short let $x \perp y$ denote the formula $\neg(x \leq y \vee y \leq x)$ that defines the incomparability relation on partial orders, and let $x \not\perp y$ denote its negation. By our results, in particular Theorem 17, Poset-SAT($\{\perp, \not\perp\}$) is an NP-complete problem. On the other hand it is easy to see that Temp-SAT($\{\perp, \not\perp\}$) is solvable in polynomial time.

However, every Temp-SAT problem can be stated as a Poset-SAT problem: Let $\Phi$ be a finite set of quantifier-free order formulas that are, without loss of generality, in conjunctive normal form. Then, for every $\phi \in \Phi$, let $\phi'$ be the formula obtained by exchanging every negative literal $\neg(x \leq y)$ by $y < x$ and let $\Phi'$ be the collection of all such formulas $\phi'$. Thus all formulas in $\Phi'$ are positive in the language consisting of $<$ and $\leq$. It is not hard to prove that Temp-SAT($\Phi$) is the same problem as Poset-SAT($\Phi'$). Hence our complexity classification for Poset-SAT problems can be regarded as a strengthening of the complexity classification of temporal CSPs in [8].

Our complexity classification is based on the classification of reducts of the random partial order in [23] and the techniques from universal algebra and Ramsey theory that have been developed in [12].

An essential part of the proof is to determine all the model-complete cores of reducts of the random partial. This allows us to sift out those problems that are covered by already known complexity classifications. In other similar classification projects, such as the classification of phylogeny constraint satisfaction problems [6] or CSPs over Henson graphs [9], the authors only had to deal with one model-complete core after this step. In the case of the Poset-SAT problems, however, we saw us confronted with four such model-complete cores. We therefore expected many tedious case distinctions in our proof. But, quite surprisingly, we were able to handle them all in the same way. It turned out that the relation

$$\mathrm{Low}(x, y, z) := (x < y \wedge x \bot z \wedge y \bot z) \vee (x < z \wedge x \bot y \wedge y \bot z)$$

is in a certain sense the only source of NP-completeness in all these cases. The same strategy could be helpful to simplify the proof of future classification results. We will explain the details in Section 5.

# 2 Strategy and structural insight

## 2.1 Poset-SAT($\Phi$) as a constraint satisfaction problem

In this section we translate the Poset-SAT($\Phi$) problems to constraint satisfaction problems (CSPs) of certain relational structures. Let $\Gamma$ be a structure in a finite relational language $\tau$. A first order $\tau$-formula is called *primitive positive*, if it is of the form $\exists x_1, \ldots, x_n(\phi_1 \wedge \cdots \wedge \phi_k)$, where all $\phi_i$ are atomic $\tau$-formulas or equations. The *constraint satisfaction problem* of $\Gamma$, or short CSP($\Gamma$), is the problem of deciding whether a given primitive positive sentence is true in $\Gamma$. CSPs of finite structures are a well-studied topic in complexity theory. Feder and Vardi famously conjectured in [16] that every constraint satisfaction problem of a *finite* structure is either in P or NP-complete, which, under the assumption of P$\neq$NP would make finite CSPs to the biggest known class of problems in NP that avoid NP-intermediate problems. However we are going to consider CSPs of structures with *infinite* domain. For infinite CSPs the dichotomy does not hold; all possible complexities can appear, up to polynomial time [4]. However for Poset-SAT, the machinery developed by Bodirsky and Pinsker in [12] will allow us to prove out result.

Every Poset-SAT problem can be restated as a constraint satisfaction problem of a structure that is first-order definable in the *random partial order* $\mathbb{P} = (P; \leq)$, a well-known structure in model theory. The random partial order is the unique countable partial order that is both

- *homogeneous*, i.e., every isomorphism between finite substructures of $\mathbb{P}$ extends to an automorphism of $\mathbb{P}$, and
- *universal*, i.e., it contains an isomorphic copy of every finite partial order.

As a countable homogeneous structure in a finite relational language, $\mathbb{P}$ has several nice properties; in particular the theory of $\mathbb{P}$ has quantifier elimination and is $\omega$-categorical, i.e. it has a unique countable model up to isomorphism. For more model-theoretical background on homogeneous structures we refer to [18].

Let $\Phi$ now be a given set of quantifier-free $\leq$-formulas. We define $\Gamma_\Phi$ as the relational structure that has $P$ as a domain and contains for every $\phi_i \in \Phi$ a relation $R_i$, consisting of all the tuples in $\mathbb{P}$ satisfying $\phi_i$. So all the relations of $\Gamma_\Phi$ are first order definable in $\mathbb{P}$. Following an established convention [29, 11] we call $\Gamma_\Phi$ a *reduct* of $\mathbb{P}$. By the universality of $\mathbb{P}$ it is straightforward to see that Poset-SAT($\Phi$) is essentially the same problem as CSP($\Gamma_\Phi$).

Also the opposite direction holds: By the quantifier elimination of $\mathbb{P}$, every reduct $\Gamma$ of $\mathbb{P}$ can be defined by a set of quantifier-free formulas $\Phi_\Gamma$. It is not hard to see that CSP($\Gamma$) is the same problem as Poset-SAT($\Phi_\Gamma$). So the class of Poset-SAT problems corresponds exactly to the CSPs of reduct of $\mathbb{P}$.

Note that the reformulation of Poset-SAT as constraint satisfaction problem would also work with any other universal partial order. However, the choice of the random partial order is not arbitrary, since the concept of homogeneity plays a central role in infinite valued CSPs: On one hand the class of reducts of homogeneous structures is a wide generalisation of the class of finite structures and CSPs of such structures appear often as natural problems in many areas of computer science, e.g. in phylogenic analysis, computational linguistics, temporal and spatial reasoning and many others (see [4] for reference).

On the other hand homogeneity helps to transfer the algebraic techniques from finite-domain constraint satisfaction problems to infinite structures.

## 2.2 The universal algebraic approach

Our new aim thus is to classify all CSPs of reducts $\Gamma$ of the random partial order $\mathbb{P}$. We use the *universal-algebraic approach* to constraint satisfaction; in this section we explain its basic principles and give an outline of the proof.

A relation $R$ is *primitive positive definable* or *pp-definable* in $\Gamma$ if there is a primitive positive formula $\phi(x_1, \dots, x_n)$ in the language of $\Gamma$ such that $(a_1, \dots, a_n) \in R$ if and only if $\phi(a_1, \dots, a_n)$ holds in $\Gamma$. If $R$ is pp-definable in $\Gamma$, the CSP induced by the extended structure $(\Gamma, R)$ reduces to $\mathrm{CSP}(\Gamma)$ in polynomial time, as Jeavons observed in [19]. So we have to study the reducts of $\mathbb{P}$ only up to pp-interdefinability.

We say a function $f : P^n \to P$ *preserves* a relation $R \subseteq P^k$ if for all $\bar{t}_1, \dots, \bar{t}_n \in R$ also the component-wise computed tuple $f(\bar{t}_1, \dots, \bar{t}_n)$ lies in $R$. Otherwise we say $f$ *violates* $R$. By the *polymorphism clone* $\mathrm{Pol}(\Gamma)$ we denote the set of all finitary functions that preserve all relations of $\Gamma$. This set is closed under composition and contains all projections. Furthermore it is closed with respect to the *topology of pointwise convergence*, i.e. the topology on $\mathrm{Pol}(\Gamma)$ given by basis open neighbourhoods of the form $\{g \colon P^n \to P : g|_{A^n} = f|_{A^n}\}$ of $f \colon P^n \to P$, where $A$ is a finite set. It is well-known that for $\omega$-categorical $\Gamma$ the relations preserved by all elements of $\mathrm{Pol}(\Gamma)$ are exactly those that are primitive positive definable in $\Gamma$ (see e.g. [10]). So reducts with the same polymorphism clone induce CSPs of the same complexity. Hence our aim is to understand the polymorphism clones of reducts of $\mathbb{P}$.

The tools that we use to study those clones were invented by Bodirsky and Pinsker and used to classify the phylogeny CSPs [6], the CSPs of reducts of the random graph [12], and of all other homogeneous graphs [9]. A key component is the use of Ramsey theory and the concept of *canonical functions*. A short introduction can be found in [26], an extended survey in [4]; we will discuss this usage of Ramsey theory only very briefly in our paper.

We start in Section 3, by classifying the endomorphism monoids of reducts $\Gamma$ of $\mathbb{P}$ (in other words we study the unary part of $\mathrm{Pol}(\Gamma)$). This step generalises the main result of [23], which says that there are exactly five automorphism groups of reducts of $\mathbb{P}$. By our proof we also identify all the cases in which the complexity is already known from the classification of temporal CSPs in [8]. After that, only CSPs of reducts whose automorphism group lies dense in the endomorphism monoid are left unclassified.

The next proof step is to study in every such case which NP-hard relations can appear, and to prove, with the help of the universal-algebraic approach, that these are the only ones. Since already the proof of [23] was quite involved, it would be reasonable to expect that also our proof splits up in many case distinctions. But quite surprisingly, in Section 5 we are able to reduce several cases to the one where the relations $<$ and $\perp$ are primitively positive definable. We can do so by fixing finitely many constants in the reduct and constructing a pp-interpretation of $(P; <, \perp)$ in this extended structure. This strategy could also be helpful in other CSP classifications.

As in previous results in [8], [12], [6] and [9], our complexity dichotomy corresponds to a structural dichotomy expressible in the language of model theory and universal algebra:

▶ **Theorem 4.** *Let* $\Gamma$ *be a reduct of the random partial order and let* $\Delta$ *be its model-complete core. Then exactly one of the following cases applies:*

- *There are polymorphisms* $f$ *and endomorphisms* $e_1, e_2$ *of* $\Delta$ *such that*

  $$e_1(f(x, y)) = e_2(f(y, x))$$

  *or there are polymorphisms* $f$ *and endomorphisms* $e_1, e_2, e_3$ *such that*

  $$e_1(f(x, x, y)) = e_2(f(x, y, x)) = e_3(f(y, x, x))$$

  *holds for all* $x, y \in \Delta$*. In this case* CSP($\Gamma$) *is in P.*
- *An extension of* $\Delta$ *by finitely many constants pp-interprets all finite structures. In this case* CSP($\Gamma$) *is NP-complete.*

The concept of *pp-interpretability* mentioned in Theorem 4 is a natural generalisation of pp-definability: A *pp-interpretation* of a structure $\Delta$ in $\Gamma$ is a surjective partial map $I : \Gamma^n \to \Delta$ such that the domain of $I$ and the preimage of every relation of $\Delta$ (including equality) are pp-definable in $\Gamma$. If there is a pp-interpretation of $\Delta$ in $\Gamma$, then CSP($\Delta$) reduces to CSP($\Gamma$) in polynomial time.

By Theorem 4 our result is also in accordance with the algebraic dichotomy conjecture for CSPs over finitely bounded homogeneous structure (see [2]). We will give a proof sketch of Theorem 4 in the following sections; the complete proof can be found in the extended version of this paper, which is available as arXiv preprint [20]. The sections in the extended version correspond to the sections of the same name in the extended paper.

## 3 A preclassification by model-complete cores

By Aut($\Gamma$) we denote the automorphism group and by End($\Gamma$) the endomorphism monoid of a structure $\Gamma$. Both are also topological objects with respect to the topology of pointwise convergence. A countable structure $\Gamma$ is called a *model-complete core* if Aut($\Gamma$) is dense in End($\Gamma$). Every countable $\omega$-categorical structure $\Gamma$ is *homomorphically equivalent* to a unique model-complete core $\Delta$, meaning that there exists a homomorphism from $\Gamma$ to $\Delta$ and vice-versa [3]. Since the CSPs of homomorphically equivalent structures are equal, one can work with model-complete cores instead. In many complexity classification projects it has been proven that working with model-complete cores is more manageable than non-model-complete core structures. Moreover recently a dichotomy complexity conjecture for infinite-domain CSPs has been stated in [2] that, as for finite CSPs, talks about the model-complete cores of a structure.

In this section we sketch the proof of the following proposition, which essentially calculates the model-complete cores of the reducts of the random partial order.

▶ **Proposition 5.** *Let* $\Gamma$ *be a reduct of* $\mathbb{P}$*. Then at least one of the following cases applies:*

1. End($\Gamma$) *contains a constant function,*
2. End($\Gamma$) *contains an injection* $g_<$ *that preserves* $<$ *and maps* $P$ *onto a chain,*
3. End($\Gamma$) *contains an injection* $g_\perp$ *that preserves* $\perp$ *and maps* $P$ *onto an antichain,*
4. *The automorphism group* Aut($\Gamma$) *is dense in* End($\Gamma$).

In the first case the model-complete core of $\Gamma$ contains only one element and $\mathrm{CSP}(\Gamma)$ is trivial. In the second case, by taking the image of $\Gamma$ under $g_<$ we obtain a structure that is isomorphic to a reduct of the rational order $(\mathbb{Q};<)$ (and homomorphically equivalent to $\Gamma$). Hence $\mathrm{CSP}(\Gamma)$ belongs to the temporal CSPs classified in [8]. This applies for instance to Example 2 from the introduction, where $\mathrm{CSP}(P;<)$ is essentially the same problem as $\mathrm{CSP}(\mathbb{Q},<)$. Case 3 similarly allows a reduction of $\mathrm{CSP}(\Gamma)$ to the CSP of a reduct of $(\mathbb{Q};\neq)$. These CSPs were already studied in [7] and are also part of the classification [8].

By Proposition 5 only the CSPs of the reducts in case 4 are left unclassified. There are exactly five different possible automorphism groups of reducts of $\mathbb{P}$. If we turn the random partial order $\mathbb{P}$ upside-down, the obtained partial order is again isomorphic to $\mathbb{P}$. Hence there exists a bijection $\updownarrow\colon P \to P$ such that for all $x, y \in P$ we have $x < y$ if and only if $\updownarrow(y) < \updownarrow(x)$. By $\circlearrowright\colon P \to P$ we denote the *rotation on a random filter*, a bijective function introduced in [23] and further studied in [24] that rotates the partial order, in a certain way, around a generic upwards closed set.

Let $F$ be a subset of the symmetric group of $P$. For short let $\langle F\rangle$ denote the smallest closed group containing $\mathrm{Aut}(\mathbb{P})$ and the functions in $F$. Let $\overline{F}$ be the topological closure of $F$ in the topological space of all functions in $P^P$. Then the reducts of $\mathbb{P}$ are classified up to automorphisms by the following result:

▶ **Theorem 6** (Theorem 1 in [23]). *Let $\Gamma$ be a reduct of $\mathbb{P}$. Then $\mathrm{Aut}(\Gamma)$ is equal to $\mathrm{Aut}(\mathbb{P})$, $\langle\updownarrow\rangle$, $\langle\circlearrowright\rangle$ or $\langle\updownarrow,\circlearrowright\rangle$ or the full symmetric group of $P$.*

We define the following relations on $P$ that can be seen as generalisation of the betweenness relation, the cyclic ordering and the separation relation on the rational order (cf. [15]):

$$\begin{aligned}
\mathrm{Betw}(x,y,z) &:= (x<y \wedge y<z) \vee (z<y \wedge y<x),\\
\mathrm{Cycl}(x,y,z) &:= (x<y \wedge y<z) \vee (z<x \wedge x<y) \vee (y<z \wedge z<x) \vee\\
&\quad (x<y \wedge x\perp z \wedge y\perp z) \vee (y<z \wedge x\perp y \wedge x\perp z) \vee (z<x \wedge y\perp z \wedge y\perp x),\\
\mathrm{Sep}(x,y,z,t) &:= (\mathrm{Cycl}(x,y,z) \wedge \mathrm{Cycl}(y,z,t) \wedge \mathrm{Cycl}(x,y,t) \wedge \mathrm{Cycl}(x,z,t)) \vee\\
&\quad (\mathrm{Cycl}(z,y,x) \wedge \mathrm{Cycl}(t,z,y) \wedge \mathrm{Cycl}(t,y,x) \wedge \mathrm{Cycl}(t,z,x)).
\end{aligned}$$

These relations allow us to describe the topological closures of the groups in Theorem 6 as endomorphism monoids:

▶ **Lemma 7.**
1. $\mathrm{End}(P;<,\perp) = \overline{\mathrm{Aut}(\mathbb{P})}$
2. $\mathrm{End}(P;\mathrm{Betw},\perp) = \overline{\langle\updownarrow\rangle}$
3. $\mathrm{End}(P;\mathrm{Cycl}) = \overline{\langle\circlearrowright\rangle}$
4. $\mathrm{End}(P;\mathrm{Sep}) = \overline{\langle\updownarrow,\circlearrowright\rangle}$

Note that the topological closure of the symmetric group is the set of injective functions on $P$, therefore it contains the function $g_\perp$. So in order to prove Proposition 5 we need to show that whenever $\mathrm{End}(\Gamma)$ is not equal to one of the monoids in Lemma 7, it contains a constant function, $g_<$ or $g_\perp$. This can be proven, similarly to Theorem 6, with the method of canonical functions.

The *type* of a tuple $\bar{a}$ of elements of $\Gamma$ is the set of all the first-order formulas $\phi(\bar{x})$ such that $\phi(\bar{a})$ holds in $\Gamma$. A function from a structure $\Delta$ to a structure $\Gamma$ is called *canonical* if it maps tuples of the same type in $\Delta$ to tuples of the same type in $\Gamma$.

There is an extension of $\mathbb{P}$ by a linear order relation $\prec$ such that $x < y$ implies $x \prec y$ and $(P;\leq,\prec)$ is a homogeneous structure. We know from [25] that $(P;\leq,\prec)$ is a *Ramsey* *structure*. Without specifying the details, this fact together with results from [13], guarantees the existence of canonical functions with helpful properties for our analysis of endomorphism monoids:

▶ **Lemma 8** (Lemma 14 in [13]). *Let $f : P \to P$ and $c_1, \dots, c_n \in P$. Then there exists a function $g : P \to P$ such that*
1. *$g$ is generated by $f$ and $\mathrm{Aut}(\mathbb{P})$, i.e., $g$ lies in the smallest closed monoid containing $f$ and $\mathrm{Aut}(\mathbb{P})$*
2. *$g(c_i) = f(c_i)$ for $i = 1, \dots n$.*
3. *Regarded as a function from $(\mathbb{P}, \prec, c_1, \dots, c_n)$ to $\mathbb{P}$, $g$ is canonical.*

**Proof sketch of Proposition 5.** Let $\Gamma$ be a reduct of $\mathbb{P}$. Assume for example that $\mathrm{End}(\Gamma)$ is not contained in $\overline{\langle \updownarrow, \circlearrowright \rangle}$. By Lemma 7 there is a tuple $\bar{c} \in P^4$ and a function $f \in \mathrm{End}(\Gamma)$ such that $\bar{c} \in \mathrm{Sep}$ but $f(\bar{c}) \notin \mathrm{Sep}$. By Lemma 8 we can assume that $f$ is canonical, when seen as a function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$. A combinatorial analysis of possible canonical functions shows that then $f$ generates $g_<$, $g_\perp$ or a constant. The details of this analysis of canonical functions are quite technical and left out in this short version of the paper. Following this strategy one can show that, whenever $\mathrm{End}(\Gamma)$ is not equal to one of the monoids in Lemma 7, it has to contain $g_<$, $g_\perp$ or a constant. ◀

## 4 The case where $<$ and $\perp$ are pp-definable

In this section we consider all reducts of $\mathbb{P}$ whose endomorphism monoids are equal to $\overline{\mathrm{Aut}(\mathbb{P})}$, trying to identify NP-hard relations that can appear. The following lemma will be of help in this analysis.

▶ **Lemma 9** (Bodirsky, Kára [8]). *Let $\Gamma$ be a relational structure and let $R \subseteq D^k$ be a union of at most $m$ orbits (i.e. minimal invariant sets) of the component-wise action of $\mathrm{Aut}(\Gamma)$ on $D^k$. If $\Gamma$ has a polymorphism $f$ that violates $R$, then $\Gamma$ also has an at most $m$-ary polymorphism that violates $R$.*

By Lemma 9 the relations $<$ and $\perp$ are primitive positive definable in a reduct $\Gamma$ of $\mathbb{P}$ if and only if $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$.

It is straightforward to see that $\mathrm{CSP}(P; <, \perp)$ and $\mathrm{CSP}(P; \leq, \perp)$ both are in P. They are part of a bigger class of CSPs that can be computed in polynomial time. Let $e_< : P^2 \to P$ be an embedding of the structure $(P; <)^2$ into $(P; <)$; in other words $e_<$ is an injective function such that $e_<(x, y) < e_<(x', y')$ if and only if $x < y$ and $x' < y'$. Similarly let $e_\leq$ be an embedding of $(P; \leq)^2$ into $(P; \leq)$. A more general result in [5] implies that these two operations can be used to characterised so-called *Horn-tractable* structures:

▶ **Lemma 10.** *Let $\Gamma$ be a reduct of $\mathbb{P}$. Suppose that $e_\leq \in \mathrm{Pol}(\Gamma)$. Then $\mathrm{CSP}(\Gamma)$ is in P and every relation in $\Gamma$ is equivalent to a conjunction of Horn formulas in $(P; \leq)$, i.e. formulas of the form:*

$$x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} \to x_{i_{k+1}} \leq x_{j_{k+1}} \text{ or}$$
$$x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} \to \perp.$$

*Suppose that $e_< \in \mathrm{Pol}(\Gamma)$. Then $\mathrm{CSP}(\Gamma)$ is in P and every relation in $\Gamma$ is equivalent to a conjunction of Horn formulas in $(P; <)$, i.e. formulas of the form:*

$$x_{i_1} \lhd_1 x_{j_1} \wedge x_{i_2} \lhd_2 x_{j_2} \wedge \dots \wedge x_{i_k} \lhd_k x_{j_k} \to x_{i_{k+1}} \lhd_{k+1} x_{j_{k+1}} \text{ or}$$
$$x_{i_1} \lhd_1 x_{j_1} \wedge x_{i_2} \lhd_2 x_{j_2} \wedge \dots \wedge x_{i_k} \lhd_k x_{j_k} \to \perp,$$

*where $\lhd_i \in \{<, =\}$ for all $i = 1, \dots, k+1$.*

A polynomial time algorithm for these Horn-tractable structures can be constructed similarly to the algorithm for Horn clauses in Boolean SAT: By resolution, one can substitute the input in polynomial time by a set of positive literals. The satisfiability of these literals can then also be checked in polynomial time, as in Example 2.

We are now going to show that every reduct $\Gamma$ of $\mathbb{P}$ with $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$ that does not have $e_<$ or $e_\leq$ as polymorphism induces an NP-complete CSP. Let us define the relation

$$\mathrm{Low}(x, y, z) := (x < y \wedge x \bot z \wedge y \bot z) \vee (x < z \wedge x \bot y \wedge y \bot z).$$

It is easy to see that every endomorphism of Low is injective and that Low is not preserved by $e_<$ or $e_\leq$. In fact Low can be used to characterise all reducts that do not have $e_<$ or $e_\leq$ as polymorphisms:

▶ **Proposition 11.** *Let $\Gamma$ be a reduct of $\mathbb{P}$ such that $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$. Then either*

- *the relation* Low *is pp-definable in $\Gamma$ or,*
- *one of the functions $e_<$, $e_\leq$ is a polymorphism of $\Gamma$.*

The proof of this result is quite technical and makes up a large part of the long version of this paper. Here we only give a short proof sketch.

**Proof sketch of Proposition 11.** Assume that Low is not pp-definable in $\Gamma$. By Lemma 9 there has to be a binary $f \in \mathrm{Pol}(\Gamma)$ that violates Low. Using the homogeneity of $\mathbb{P}$ we can even further assume that there are elements $a, b, c \in P$ such that $(a, b, c) \in \mathrm{Low}$, but $(f(a, a), f(b, c), f(c, b)) \notin \mathrm{Low}$. We now make again use of canonical functions, however, with the following generalisation of Lemma 8:

▶ **Lemma 12** (Lemma 21 in [13]). *Let $f : P^r \to P$ and $c_1, \ldots, c_n \in P$. Then there exists a function $g : P^r \to P$ such that*

1. *$g$ is generated by $f$ and $\mathrm{Aut}(\mathbb{P})$, i.e. lies in the smallest closed clone containing $f$ and $\mathrm{Aut}(\mathbb{P})$;*
2. *the restriction of $g$ to $\{c_1, \ldots, c_n\}^r$ is equal to $f$;*
3. *regarded as a function from $(\mathbb{P}, \prec, c_1, \ldots, c_n)^r$ to $\mathbb{P}$, $g$ is canonical.*

By Lemma 12 we can assume that $f$ is canonical, when seen as a function from the structure $(P; \leq, \prec, a, b, c)^2$ to $(P; \leq)$. An analysis of all possible such canonical functions shows that $f$ together with $\mathrm{End}(\Gamma)$ has to generate $e_<$ or $e_\leq$. For reasons of space, we omit this combinatorial analysis here; it can be found in Section 7 of the extended paper [20]. ◀

We remark that one can even show that every $f : P^2 \to P$ preserving Low has to be *dominated*, meaning that $x < x'$ implies $f(x, y) < f(x', y')$ and $x \bot x'$ implies $f(x, y) \bot f(x', y')$, or that $f$ satisfies the symmetric conditions for its second coordinate. This fact is proven in the extended version of the paper. It only remains to show that $\mathrm{CSP}(P; \mathrm{Low})$ is NP-complete to prove the complexity dichotomy for the case where $<$ and $\bot$ are pp-definable.

The Boolean 3-satisfiability problem is a well-studied NP-complete problem [27]. It can be written as $\mathrm{CSP}(\{0, 1\}; \mathrm{XOR}, 3\mathrm{OR})$ with $\mathrm{XOR} = \{(0, 1), (1, 0)\}$ and $3\mathrm{OR} = \{0, 1\}^3 \backslash \{(0, 0, 0)\}$. In practice we often show the NP-hardness of $\mathrm{CSP}(\Gamma)$ by finding a pp-interpretation of $(\{0, 1\}; \mathrm{XOR}, 3\mathrm{OR})$ or another Boolean NP-complete structure in $\Gamma$. We remark that if such a pp-interpretation exists, *every* finite structure has a pp-interpretation in $\Gamma$.

Also adding constants does not change the complexity of a CSP, by the following lemma:

▶ **Lemma 13** (Corollary 3.6.25 in [4]). *Let $\Gamma$ be an $\omega$-categorical model-complete core and $c_1, c_2, \ldots, c_k$ be constants of $\Gamma$. Then $\mathrm{CSP}(\Gamma)$ is polynomially equivalent to $\mathrm{CSP}(\Gamma, c_1, c_2, \ldots, c_k)$.*

Thus to prove the NP-hardness of a CSP of a $\omega$-categorical model-complete core we often add sufficient constants to the structure and then construct a pp-interpretation of an NP-hard structure in the resulting structure.

▶ **Lemma 14.** *Let $a, b \in P$ be constants such that $a \perp b$. Then there is a primitive positive interpretation of $(\{0,1\}; \mathrm{XOR}, 3\mathrm{OR})$ in $(P; \mathrm{Low}, a, b)$. Hence $\mathrm{CSP}(P; \mathrm{Low})$ is NP-complete.*

**Proof.** Let $D := \{x \in P : \mathrm{Low}(x, a, b)\}$, $D_0 := \{x \in D : x < a\}$ and $D_1 := \{x \in D : x < b\}$. We define $I : D \to \{0,1\}$ by setting $I(x) = 0$ if $x \in D_0$ and $I(x) = 1$ if $x \in D_1$. We claim that $I$ is a pp-interpretation of $(\{0,1\}; \mathrm{XOR}, 3\mathrm{OR})$ in $(P; \mathrm{Low}, a, b)$.

Clearly the domain $D$ and also the sets $D_0$ and $D_1$ are pp-definable in $(P; \mathrm{Low}, a, b)$. Let

$$\mathrm{Abv}(x, y, z) := (y < x \wedge x \perp z \wedge y \perp z) \vee (z < x \wedge x \perp y \wedge z \perp y)$$

be the, in a sense, dual of Low. The relation Abv is not preserved by $e_<$ and $e_\leq$. By Proposition 11 we know that Low is pp-definable in Abv. For symmetry reasons also Abv is pp-definable in Low. The equation $I(x) = I(y)$ holds if and only if $\exists z \, \mathrm{Abv}(b, x, z) \wedge \mathrm{Abv}(b, y, z)$. Hence the preimage of equality under $I$ is pp-definable in $(P; \mathrm{Low})$. Furthermore

$$(I(x), I(y)) \in \mathrm{XOR} \leftrightarrow \exists u, v \in D \ (I(x) = I(u) \wedge I(x) = I(v) \wedge \mathrm{Abv}(a, u, v) \wedge \mathrm{Abv}(b, u, v)).$$

Let $R(x, y, z, t) := \exists u \ \mathrm{Abv}(u, y, z) \wedge \mathrm{Abv}(x, u, t)$. Then $R(x, y, z, t)$ implies that $x$ is greater than at least one of the elements $\{y, z, t\}$. One can show the following equivalence:

$$I(x_1, x_2, x_3) \in 3\mathrm{OR} \leftrightarrow \exists y_1, y_2, y_3 \in D \ (R(a, y_1, y_2, y_3) \wedge \bigwedge_{i=1}^{3} I(x_i) = I(y_i)).$$

So $I$ is a pp-interpretation of $(\{0,1\}; \mathrm{XOR}, 3\mathrm{OR})$ in $(P; \mathrm{Low}, a, b)$ and $\mathrm{CSP}(P; \mathrm{Low}, a, b)$ is NP-complete. Adding finitely many constants to a model-complete core does not increase the complexity of the induced constraint satisfaction problem by a result in [3]. So $\mathrm{CSP}(P; \mathrm{Low})$ is NP-complete. ◀

As an immediate consequence we get the following corollary of Proposition 11:

▶ **Corollary 15.** *Let $\Gamma$ be a reduct of $\mathbb{P}$ such that $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$. Then*
- *either the relation Low is pp-definable in $\Gamma$ and $\mathrm{CSP}(\Gamma)$ is NP-complete,*
- *or one of the functions $e_<$, $e_\leq$ is a polymorphism of $\Gamma$ and $\mathrm{CSP}(\Gamma)$ is in P.*

## 5 Hardness of Betw, Cycl and Sep

By Proposition 5 we are left with the reducts $\Gamma$ of $\mathbb{P}$ whose endomorphism monoids are equal to $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ or $\overline{\langle \updownarrow, \circlearrowright \rangle}$. As it turns out, in all these cases the induced CSPs are NP-hard. Interestingly, we can prove this fact for all three cases in the same way: After fixing finitely many constants, which is a feasible reduction by Lemma 13, we can construct a pp-interpretation of $(P; <, \mathrm{Low})$ in the structure. To be more precise the following holds:

▶ **Proposition 16.** *Let $\Gamma$ be a reduct of $\mathbb{P}$ such that $\mathrm{End}(\Gamma)$ is equal to $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ or $\overline{\langle \updownarrow, \circlearrowright \rangle}$. Then there are elements $c_1, \dots, c_n \in P$ and a subset $X \subseteq P$ such that*
- *$(X; \leq)$ is isomorphic to $\mathbb{P}$;*
- *$X$ is pp-definable in $(\Gamma, c_1, \dots, c_n)$;*
- *the restrictions of $<$ and Low to $X$ are pp-definable in $(\Gamma, c_1, \dots, c_n)$.*

*Hence the identity mapping on $X$ is a pp-interpretation of $(P; <, \mathrm{Low})$ in $\Gamma$, and $\mathrm{CSP}(\Gamma)$ is NP-complete.*

**Proof.** We demonstrate the proof only for the case, where $\mathrm{End}(\Gamma) = \overline{\langle \circlearrowright \rangle}$. The other cases are discussed in the extended paper [20]. Note that the relation Cycl is an orbit of $\langle \circlearrowright \rangle$ on $P^3$. So Lemma 9 implies that Cycl is primitive positive definable in $\Gamma$. Without loss of generality let $\Gamma = (P; \mathrm{Cycl})$.

Let $c, d$ be two constants in $P$ such that $c < d$ and let $X := \{x \in P : c < x < d\} = \{x \in P : \mathrm{Cycl}(c, x, d)\}$. By a back-and-forth argument it is easy to show that $\mathbb{P}$ and $(X; \leq)$ are isomorphic. For $x, y \in X$ we have that $x < y$ is equivalent to $\mathrm{Cycl}(c, x, y)$. The incomparability relation $x \bot y$ is also pp-definable in $X$ as follows:

$$\exists a, b, c, d\ (x < a < c \wedge x < b < d \wedge y < c \wedge y < d) \wedge$$
$$(\mathrm{Cycl}(x, a, y) \wedge \mathrm{Cycl}(x, b, y) \wedge \mathrm{Cycl}(y, c, b) \wedge \mathrm{Cycl}(y, d, a) \wedge \mathrm{Cycl}(b, d, c) \wedge \mathrm{Cycl}(a, c, d)).$$

The two maps $e_< : P^2 \to P$ and $e_\leq : P^2 \to P$ do not preserve Cycl. By Proposition 11 we have that Low is pp-definable in $(P; <, \bot, \mathrm{Cycl})$. So the identity on $X$ gives us a pp-interpretation of $(P; \mathrm{Low})$ in $(P; \mathrm{Cycl}, c, d)$, which concludes the proof. ◀

We observed that in the already existing complexity classifications, certain proof parts could have been simplified using the same principle, i.e. fixing finitely many constants in a model-complete core reduct, to obtain the relations of the underlying homogeneous structure.

So it is natural to ask, if Proposition 16 works in general: Let $\Delta$ be a homogeneous structure and $\Gamma$ be a reduct of $\Delta$ that is a model-complete core. Can we then extend $\Delta$ by finitely many constants $c_1, \ldots, c_n$ such that the identity mapping on a definable subset of $(\Gamma, c_1, \ldots, c_n)$ pp-interprets $\Delta$? The answer to this question is negative: A non-trivial counterexample is given by the random ordered graph $\Delta = (G; E, <)$ and the random tournament $\Gamma = (G; T)$ defined by $T(x, y) := x \neq y \wedge (x < y \leftrightarrow E(x, y))$.

## 6 Classification

We sum up the results of the last two sections:

▶ **Theorem 17.** *Let $\Gamma$ be a reduct of $\mathbb{P}$ in a finite relational language and a model-complete core. Then exactly one of the following two cases holds:*

- *One of the relations* Low, Betw, Cycl, Sep *is pp-definable in $\Gamma$. An extension of $\Gamma$ by finitely many constants pp-interprets all finite structures and* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*
- $\mathrm{Pol}(\Gamma)$ *contains $e_<$ or $e_\leq$ and* $\mathrm{CSP}(\Gamma)$ *is in P.*

**Proof.** If Low is pp-definable in $\Gamma$, the statement holds by Lemma 14. Next, assume that one of the relations Betw, Cycl or Sep is pp-definable in $\Gamma$. By Proposition 16 we have a pp-interpretation of $(P; \mathrm{Low})$ in $\Gamma$. By the transitivity of pp-interpretations and Lemma 14 we can obtain a pp-interpretation of $(\{0,1\}; \mathrm{XOR}, \mathrm{3OR})$ in $\Gamma$, extended by finitely many constants. Hence all finite structure are pp-interpretable in an extension of $\Gamma$ by finitely many constants. At last, assume that Low, Betw, Cycl, Sep are not pp-definable in $\Gamma$. By Proposition 5 we have $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$ and therefore, by Corollary 15, $e_<$ or $e_\leq$ is a polymorphism of $\Gamma$ and $\mathrm{CSP}(\Gamma)$ is in P. ◀

Theorem 17 allows us to prove our main result.

**Proof of Theorem 4.** By Proposition 5 we know that the model-complete core of $\Gamma$ is equal to a one-element set, a reduct of $(\mathbb{Q}, <)$ or to $\Gamma$ itself. In the first two cases the dichotomy in Theorem 4 holds by the analysis in [8], see also Theorem 10.1.1. in [4]. If $\Gamma$ itself is a model-complete core, the dichotomy holds by Theorem 17 and the observation that there are endomorphisms $\alpha, \alpha' \in \mathrm{End}(\mathbb{P})$ such that $e_<(x,y) = \alpha(e_<(y,x))$ respectively $e_\leq(x,y) = \alpha'(e_\leq(y,x))$. ◀

We identified the problems Poset-SAT($\Phi$) with CSPs of reducts $\Gamma$ of $\mathbb{P}$, so also the complexity dichotomy for Poset-SAT($\Phi$) in Theorem 1 is true.

We remark that also the "meta-problem" of deciding if Poset-SAT($\Phi$) is NP-complete for a given finite set of formulas $\Phi$, is decidable. The main result in [13] implies that determining whether a quantifier-free formula in $\mathbb{P}$ is pp-definable in a given reduct $\Gamma$ of $\mathbb{P}$, is a decidable problem. Hence it is also decidable to tell if $\Gamma$ is a model-complete core. If $\Gamma$ is a model-complete core, the list of NP-hard relations in Theorem 17 allows us also to decide whether CSP($\Gamma$) is NP-complete. A similar list exists for the case where the model-complete core of $\Gamma$ is a reduct of the rational order in [8].

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