$\exists\mathbb{R}$-Complete Decision Problems about Symmetric Nash Equilibria in Symmetric Multi-Player Games

Vittorio Bilò[1] and Marios Mavronicolas[2]

1 **Department of Mathematics and Physics, University of Salento, Lecce, Italy**
vittorio.bilo@unisalento.it
2 **Department of Computer Science, University of Cyprus, Nicosia, Cyprus**
mavronic@cs.ucy.ac.cy

Abstract

We study the complexity of decision problems about symmetric Nash equilibria for symmetric multi-player games. These decision problems concern the existence of a symmetric Nash equilibrium with certain natural properties. We show that a handful of such decision problems are $\exists\mathbb{R}$-complete; that is, they are exactly as hard as deciding the *Existential Theory of the Reals.*

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1 Introduction

1.1 The Setting

Algorithmic Game Theory has come to take into account and accomodate computational issues into the classical setting of Game Theory. Thus, one of its major trends seeks to determine tight complexity bounds for algorithmic problems originating from Game Theory. Along this trend, strong emphasis has been put on algorithmic problems about *Nash equilibria* [19, 20], states where no player could improve her payoff by unilateral switching. This is no surprise given that the Nash equilibrium is the most influential equilibrium concept in Game Theory. In this work, we continue the complexity-theoretic study of Nash equilibria. We focus on *decision problems,* asking whether or not a given game has a Nash equilibrium with some natural property; recently, such decision problems about Nash equilibria, whose complexity-theoretic study dates back to the seminal work of Gilboa and Zemel [17], attracted a lot of flourishing interest and attention – see, e.g., [2, 3, 4, 5, 10, 11, 16, 23].[1] The complexity of deciding the existence of *approximate* Nash equilibria with certain properties has been studied in [1, 6, 13, 18].

A key factor affecting the complexity of decision problems about Nash equilibria is the number of players. Due to the rationality of Nash equilibria for bimatrix games, decision problems about Nash equilibria for bimatrix games are placed in $\mathcal{NP}$; on the other hand, there were early found 3-player games whose Nash equilibria were all irrational [19, 20]; under

[1] There is a distinct thread of breakthrough results providing exact characterizations of the complexity of the *search problem* about Nash equilibria for r-player games with $r \geq 2$ – see [9, 12, 14, 21]. The characterizations amount to completeness for the complexity classes $\mathcal{PPAD}$ [21] and $\mathcal{FIXP}$ [14]. By a recent breakthrough result in [22], there is no PTAS for a Nash equilibrium in bimatrix games, asssuming the Exponential Time Hypothesis for $\mathcal{PPAD}$.

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standard complexity-theoretic assumptions, this dashes the hope that decision problems about Nash equilibria for multi-player games could be placed in $\mathcal{NP}$. Such considerations suggest that the *tight* bound for the complexity of decision problems about Nash equilibria for multi-player games must be some complexity class encompassing $\mathcal{NP}$. Recent pioneering work by Schaefer and Štefankovič [23] identified $\exists\mathbb{R}$, a complexity class associated with the *Existential Theory of the Reals* [24], as such a class; it is known that $\exists\mathbb{R} \subseteq \mathcal{PSPACE}$ [8].

The first decision problem about Nash equilibria shown $\exists\mathbb{R}$-complete for multi-player games was the problem asking, given an r-player game with $r \geq 3$ and a rational ϱ, whether there is a Nash equilibrium with no probability exceeding ϱ [23, Corollary 3.5]. Using this, four additional decision problems were subsequently shown, via a chain of problem-specific reductions, $\exists\mathbb{R}$-complete for multi-player games in [16, Section 3]. The present authors presented in [4], via a *single* unifying reduction, a catalog of such $\exists\mathbb{R}$-complete problems for multi-player games, encompassing all the decision problems that had been shown $\mathcal{NP}$-complete for (symmetric) bimatrix games in [2, 11, 17]; the catalog encompassed the four $\exists\mathbb{R}$-complete decision problems for r-player games with $r \geq 3$ from [16].

In this work, we focus on *symmetric* multi-player games; symmetric means that all players are identical and indistinguishable: they have the same strategy sets and there is a common payoff function depending only on the player's chosen strategy and on the number of players choosing each strategy. Symmetric games have been studied extensively; already in 1951 Nash proved that every symmetric game has a *symmetric* Nash equilibrium [20]: one where all players play the same mixed strategy. Very recently decision problems about the existence of a symmetric Nash equilibrium with some additional property for a symmetric game were introduced by Garg *et al.* [16]; they showed that deciding the existence of a symmetric Nash equilibrium where all strategies played with non-zero probability are in a given set (resp., a given set of strategies are played with non-zero probability) is $\exists\mathbb{R}$-complete for symmetric multi-player games with a constant number $r \geq 3$ of players [16, Theorem 23].

1.1.1 Contribution, Techniques and Significance

As our main result, we present a catalog with ten $\exists\mathbb{R}$-complete decision problems about the existence of a symmetric Nash equilibrium with certain properties for symmetric r-player games with constant $r \geq 3$ (Theorem 10). Such decision problems include the ones of deciding the existence of a second Nash equilibrium, or of a Nash equilibrium where the expected payoff to each player is at most (resp., at least) a given number. The properties associated with the decision problems in our catalog come originally from the decision problems about Nash equilibria in bimatrix games, which were shown $\mathcal{NP}$-complete in [2, 11, 17]; the same properties are also found in the decision problems about the existence of a Nash equilibrium with certain properties for multi-player games, which were shown $\exists\mathbb{R}$-complete in [4, 16].

To show the $\exists\mathbb{R}$-completeness results, we present a unifying, polynomial time, many-to-one reduction from the decision problem asking about the existence of a symmetric Nash equilibrium where all strategies played with non-zero probability come from a given set T; this is $\exists\mathbb{R}$-complete [16, Theorem 23]. The many-to-one reduction amounts to a simple *symmetric game reduction*[2] we design and analyze (Section 4), which maps a pair of symmetric r-player games $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, called the *input game* and the *gadget game,* respectively, to a symmetric r-player game G with a larger set of strategies; $\widetilde{\mathsf{G}}$ is accompanied with a set of strategies T.

[2] By *symmetric game reduction* we mean any transformation of some game(s) into a symmetric game that preserves certain properties.

Thus, $\left\langle \widetilde{\mathsf{G}}, \mathsf{T} \right\rangle$ represents an instance of the decision problem above, which the many-to-one reduction reduces from; the symmetric game reduction is tailored to this decision problem.

The symmetric game reduction provides certain correspondences between the symmetric Nash equilibria for G and those for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively. Going backwards, a symmetric Nash equilibrium for G subsumes a symmetric Nash equilibrium for either $\widetilde{\mathsf{G}}$ or $\widehat{\mathsf{G}}$ (Lemma 7). Going forward, a symmetric Nash equilibrium for $\widehat{\mathsf{G}}$ always induces a symmetric Nash equilibrium for G (Lemma 8); but a symmetric Nash equilibrium for $\widetilde{\mathsf{G}}$ induces a symmetric Nash equilibrium for G if and only if all strategies played by it with non-zero probability come from the given set of strategies T (Lemma 9).

As a tool for the symmetric game reduction, we construct and use the symmetric r-player gadget game $\widehat{\mathsf{G}}[m]$, for any integer $m \geq 3$ (Section 3). $\widehat{\mathsf{G}}[m]$ generalizes the classical *rock-paper-scissors game* to an arbitrary number of players r and an arbitrary number of strategies $m \geq 3$. By construction, $\widehat{\mathsf{G}}[m]$ has the crucial property of zero-sum.

We use the forward and backward correspondences between symmetric Nash equilibria given by the symmetric game reduction to conclude that the constructed symmetric game G has a symmetric Nash equilibrium with the considered property if and only if the input game $\widetilde{\mathsf{G}}$ has a symmetric Nash equilibrium where all strategies played by it with non-zero probability come from the given set T (Lemmas 11, 12, 13 and 14). These yield the $\exists\mathbb{R}$-hardness of deciding the existence of a symmetric Nash equilibrium with the property (Theorem 10). These $\exists\mathbb{R}$-completeness results essentially settle the chapter on the complexity of decision problems about symmetric Nash equilibria for symmetric r-player games, with $r \geq 3$.

1.2 Related Work and Comparison

The symmetric game reduction in Section 4 follows the structure of the *game reduction* in [4, Section 3]. Specifically, what the two reductions have in common is the idea of transforming a pair of a gadget game $\widehat{\mathsf{G}}$ and an input game $\widetilde{\mathsf{G}}$ into a game G. Each of the two reductions is such that the Nash equilibria of the gadget game $\widehat{\mathsf{G}}$ are always preserved while only the Nash equilibria of the input game that have certain properties are preserved. The game reduction in [4, Section 3] is tailored to the decision problem asking, given a 3-player game and a rational ϱ, whether there is a Nash equilibrium with no probability exceeding ϱ; on the other hand, the symmetric game reduction is tailored to the decision problem asking, given a symmetric 3-player game with a strategy set T, whether there is a Nash equilibrium where all strategies played with non-zero probability all come from T.

Our proof techniques (specifically, the symmetric game reduction) apply directly to r-player symmetric games with $r \geq 3$, thanks to the r-player symmetric gadget game $\mathsf{G}[m]$. In contrast, the corresponding techniques in [16, Section 4] deal first with symmetric 3-player games [16, Theorems 20 & 21], obtained by employing the technique of *symmetrization* (cf. [7]) to transform a 3-player game into a symmetric 3-player game; then, they apply separately a rather lengthy and complicated reduction from symmetric 3-player games to symmetric r-player games with $r > 3$ [16, Theorem 23].[3] Thus, our direct reduction yields a handful of $\exists\mathbb{R}$-completeness results while it is much simpler and more elegant and transparent than the long sequence of reductions in [16, Section 4]. Note also that the symmetric game reduction is the *first* symmetric reduction that works entirely within the realm of decision problems

[3] Note that there is no known trivial reduction of symmetric 3-player games to symmetric r-player games with $r > 3$; this is unlike the case of the trivial reduction of 3-player games to r-player games with $r > 3$, which allows establishing complexity results for decision problems about Nash equilibria for r-player games with $r \geq 3$ by focusing on the case $r = 3$ (cf. [4] and [16, Section 3]).

about Nash equilibria: the reduction in [11], though yielding a symmetric game, involves SAT; the reductions in [4, 16] are not symmetric – in fact, [16] resorts to *symmetrization* (cf. [7]) in order to extend ∃ℝ-hardness from arbitrary to symmetric 3-player games.

2 Framework

Our presentation closely follows [4, Section 2].

2.1 Games and Nash Equilibria

A *game* is a triple $\mathsf{G} = \left\langle [r], \{\Sigma_i\}_{i\in[r]}, \{\mathsf{U}_i\}_{i\in[r]} \right\rangle$, where (i) $[r] = \{1,\ldots,r\}$ is a finite set of *players* with $r \geq 2$, and (ii) for each player $i \in [r]$, Σ_i is the set of *strategies* for player i, and U_i is the *payoff function* $\mathsf{U}_i : \times_{k\in[r]}\Sigma_k \to \mathbb{R}$ for player i; G is called an *r-player game.*

For each player $i \in [r]$, set $\Gamma_{-i} := \times_{k\in[r]\setminus\{i\}}\Sigma_k$; set $\Gamma := \times_{k\in[r]}\Sigma_k$. A *profile* is a tuple $\mathbf{s} \in \Gamma$ of r strategies, one per player. The vector $\mathsf{U}(\mathbf{s}) = \langle \mathsf{U}_1(\mathbf{s}), \ldots, \mathsf{U}_r(\mathbf{s})\rangle$ is the *payoff vector* for $\mathbf{s}$. A *partial profile* $\mathbf{s}_{-i}$ is a tuple of $r-1$ strategies, one for each player other than i; so $\mathbf{s}_{-i} \in \Gamma_{-i}$. For a profile $\mathbf{s}$ and a strategy $t \in \Sigma_i$, denote as $\mathbf{s}_{-i} \diamond t$ the profile obtained by substituting the strategy t for s_i in $\mathbf{s}$. Denote as $\underline{u}(\mathsf{G}) = \min_{\mathbf{s}\in\Sigma, i\in[r]} \{\mathsf{U}_i(\mathbf{s})\}$ and $\overline{u}(\mathsf{G}) = \max_{\mathbf{s}\in\Sigma, i\in[r]} \{\mathsf{U}_i(\mathbf{s})\}$ the *minimum payoff* and the *maximum payoff* for G, respectively.

The game G is *symmetric* if $\Sigma_1 = \Sigma_2 = \ldots = \Sigma_r := \Sigma$, and for each permutation π of the set of players $[r]$, for each player $i \in [r]$, $\mathsf{U}_i(s_1,\ldots,s_i,\ldots,s_r) = \mathsf{U}_{\pi(i)}\left(s_{\pi(1)},\ldots,s_{\pi(i)},\ldots, s_{\pi(r)}\right)$; so, the payoff to a player playing a particular strategy is determined by the multiset of strategies played by the other players, and there is no discrimination among the players.

A *mixed strategy* for player $i \in [r]$ is a probability distribution σ_i on her strategy set Σ_i: a function $\sigma_i : \Sigma_i \to [0,1]$ such that $\sum_{s\in\Sigma_i} \sigma_i(s) = 1$. Denote as Δ_i the set of mixed strategies on Σ_i. Denote as $\mathsf{Supp}(\sigma_i)$ the set of strategies $s \in \Sigma_i$ with $\sigma_i(s) > 0$. A *mixed profile* $\boldsymbol{\sigma} = \{\sigma_i\}_{i\in[r]}$ is a tuple of mixed strategies, one per player. So, a profile is the degenerate case of a mixed profile where all probabilities are either 0 or 1. Set $\Delta = \Delta(\mathsf{G}) := \times_{k\in[r]}\Delta_k$; so, $\boldsymbol{\sigma} \in \Delta$. A *partial mixed profile* $\boldsymbol{\sigma}_{-i}$ is a tuple of $r-1$ mixed strategies, one per player other than i. For a mixed profile $\boldsymbol{\sigma}$ and a mixed strategy τ_i of player $i \in [r]$, denote as $\boldsymbol{\sigma}_{-i} \diamond \tau_i$ the mixed profile obtained when substituting τ_i for σ_i in the mixed profile $\boldsymbol{\sigma}$.

A mixed profile $\boldsymbol{\sigma}$ induces a probability measure $\mathbb{P}_{\boldsymbol{\sigma}}$ on Γ in the natural way; so, for a profile $\mathbf{s} \in \Gamma$, $\mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) = \prod_{k\in[r]} \sigma_k(s_k)$. Say that the profile $\mathbf{s}$ (resp., the partial profile $\mathbf{s}_{-i}$) is *supported* in the mixed profile $\boldsymbol{\sigma}$ (resp., the partial mixed profile $\boldsymbol{\sigma}_{-i}$), and write $\mathbf{s} \sim \boldsymbol{\sigma}$ (resp., $\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}$), if $\mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) > 0$ (resp., $\mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) > 0$. Under the mixed profile $\boldsymbol{\sigma}$, the payoff of each player becomes a random variable. So associated with $\boldsymbol{\sigma}$ is the *expected payoff* for each player $i \in [r]$, denoted as $\mathsf{U}_i(\boldsymbol{\sigma})$, which is the expectation according to $\mathbb{P}_{\boldsymbol{\sigma}}$ of her payoff; so, clearly, $\mathsf{U}_i(\boldsymbol{\sigma}) = \sum_{\mathbf{s}\in\Gamma} \left(\prod_{k\in[r]} \sigma_k(s_k)\right) \cdot \mathsf{U}_i(\mathbf{s})$.

A *pure Nash equilibrium* is a profile $\mathbf{s} \in \Gamma$ such that for each player $i \in [r]$ and for each strategy $t \in \Sigma_i$, $\mathsf{U}_i(\mathbf{s}) \geq \mathsf{U}_i(\mathbf{s}_{-i} \diamond t)$. A *mixed Nash equilibrium,* or *Nash equilibrium* for short, is a mixed profile $\boldsymbol{\sigma}$ such that for each player $i \in [r]$ and for each mixed strategy τ_i, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond \tau_i)$. Denote as $\mathcal{NE}(\mathsf{G})$ the set of Nash equilibria for G. For each game G, $\mathcal{NE}(\mathsf{G}) \neq \emptyset$ [19, 20]. We shall make extensive use of the following characterization of Nash equilibria.

▶ **Lemma 1.** *The mixed profile $\boldsymbol{\sigma}$ is a Nash equilibrium if and only if for each player $i \in [r]$, (1) for each strategy $t \in \mathsf{Supp}(\sigma_i)$, $\mathsf{U}_i(\boldsymbol{\sigma}) = \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond t)$, and (2) for each strategy $t \notin \mathsf{Supp}(\sigma_i)$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond t)$.*

For an arbitrary number δ, denote as $\mathsf{G} + \delta$ the game obtained from G by adding δ to each possible value of the payoff function. We recall a simple fact:

▶ **Lemma 2.** *Consider the pair of r-player games G and $\widehat{\mathsf{G}} := \mathsf{G} + \delta$, for some number δ. Then, $\mathcal{NE}(\mathsf{G}) = \mathcal{NE}(\widehat{\mathsf{G}})$. Moreover, for each pair of a player $i \in [r]$ and a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$, $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \mathsf{U}_i(\boldsymbol{\sigma}) + \delta$.*

A Nash equilibrium $\boldsymbol{\sigma}$ is *fully mixed* if for each player $i \in [r]$, $\mathsf{Supp}(\sigma_i) = \Sigma_i$. A Nash equilibrium is *symmetric* if all mixed strategies are identical. For a symmetric Nash equilibrium $\boldsymbol{\sigma}$, denote as $\mathsf{Supp}(\boldsymbol{\sigma})$ the common support of all mixed strategies. Denote as $\mathcal{SNE}(\mathsf{G})$ the set of symmetric Nash equilibria for G. For each symmetric game G, $\mathcal{SNE}(\mathsf{G}) \neq \emptyset$ [19, 20].

2.2 Decision Problems about Symmetric Nash Equilibria

Here are the formal statements of the decision problems about symmetric Nash equilibria for symmetric games we shall consider; they are given in the style of Garey and Johnson [15], where I. and Q. stand for INSTANCE and QUESTION, respectively.

∃ SECOND SNE

I.:	A symmetric game G.
Q.:	Is there a second symmetric Nash equilibrium?

∃ SNE WITH LARGE PAYOFFS

I.:	A symmetric game G and a number u.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq u$?

∃ SNE WITH SMALL PAYOFFS

I.:	A symmetric game G and a number u.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) \leq u$?

∃ SNE WITH LARGE TOTAL PAYOFF

I.:	A symmetric game G and a number u.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t $\sum_{i \in [r]} \mathsf{U}_i(\boldsymbol{\sigma}) \geq u$?

∃ SNE WITH SMALL TOTAL PAYOFF

I.:	A symmetric game G and a number u.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t $\sum_{i \in [r]} \mathsf{U}_i(\boldsymbol{\sigma}) \leq u$?

∃ SNE WITH LARGE SUPPORTS

I.:	A symmetric game G and an integer $k \geq 1$.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t for each player $i \in [r]$, $\lvert\mathsf{Supp}(\sigma_i)\rvert \geq k$?

∃ SNE WITH SMALL SUPPORTS

I.:	A symmetric game G and an integer $k \geq 1$.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t for each player $i \in [r]$, $\lvert\mathsf{Supp}(\sigma_i)\rvert \leq k$?

∃ SNE WITH RESTRICTING SUPPORTS

I.:	A symmetric game G and a subset of strategies $\mathsf{T} \subseteq \Sigma$.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t $\mathsf{T} \subseteq \mathsf{Supp}(\boldsymbol{\sigma})$?

∃ SNE WITH RESTRICTED SUPPORTS

I.:	A symmetric game G and a subset of strategies $\mathsf{T} \subseteq \Sigma$.
Q.:	Is there a symmetric Nash equilibrium $\boldsymbol{\sigma}$ s/t $\mathsf{Supp}(\boldsymbol{\sigma}) \subseteq \mathsf{T}$?

Given two mixed profiles $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$, denote as $\mathsf{Diff}(\boldsymbol{\sigma}, \widehat{\boldsymbol{\sigma}}) := \{i \in [r] : \sigma_i \neq \widehat{\sigma}_i\}$ the set of players with different mixed strategies in $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$. A Nash equilibrium $\boldsymbol{\sigma}$ is *Strongly Pareto-Optimal* if for each mixed profile $\widehat{\boldsymbol{\sigma}}$ where there is player $i \in [r]$ with $\mathsf{U}_i(\widehat{\boldsymbol{\sigma}}) > \mathsf{U}_i(\boldsymbol{\sigma})$ for some player $i \in [r]$, there is a player $j \in \mathsf{Diff}(\boldsymbol{\sigma}, \widehat{\boldsymbol{\sigma}})$ such that $\mathsf{U}_j(\widehat{\boldsymbol{\sigma}}) \leq \mathsf{U}_j(\boldsymbol{\sigma})$; so, there is no other profile where at least one player is strictly better off and every player using a different strategy is strictly better off. We have two additional decision problems.

∃ ¬ PARETO-OPTIMAL SNE

I.:	A symmetric game G.
Q.:	Is there a symmetric Nash equilibrium which is not Pareto-Optimal?

∃ ¬ STRONGLY PARETO-OPTIMAL SNE

I.:	A symmetric game G.
Q.:	Is there a symmetric Nash equilibrium which is not Strongly Pareto-Optimal?

2.3 The Class ∃ℝ

The *Existential Theory of the Reals,* denoted as ETR, is the set of true sentences of the form $(\exists x_1, \ldots, x_n)(\varphi(x_1, \ldots, x_n))$, where φ is a quantifier-free $(\vee, \wedge, \neg)$-boolean formula over the signature $(0, 1, +, *, <, \leq, =)$, interpreted over the real numbers. ∃ℝ is the complexity class associated with ETR: A decision problem *belongs to* ∃ℝ if there is a polynomial-time, many-to-one reduction from it to ETR, and it is ∃ℝ-*hard* if there is a polynomial-time many-to-one reduction from each problem in ∃ℝ to it; it is ∃ℝ-*complete* if it belongs to ∃ℝ and it is ∃ℝ-hard. Since satisfiability of a propositional boolean formula (SAT) is expressible in ETR, $\mathcal{NP} \subseteq \exists\mathbb{R}$; so, ETR is for ∃ℝ what SAT is for $\mathcal{NP}$, and an ∃ℝ-complete problem is in $\mathcal{NP}$ if and only if ETR is in $\mathcal{NP}$. We shall use a result from [16, Theorem 23]:

▶ **Theorem 3** ([16]). *The problems* ∃ SNE WITH RESTRICTING SUPPORTS *and* ∃ SNE WITH RESTRICTED SUPPORTS *are* ∃ℝ*-complete for symmetric r-player games with constant $r \geq 3$.*

3 The Symmetric Gadget Game $\widehat{\mathsf{G}}[m]$

For any two integers r and m such that $r \geq 3$ and $m \geq 3$, $\widehat{\mathsf{G}}[m]$ is a symmetric r-player game with $\Sigma = [m]$. We assume a cyclic ordering on the strategy set $[m]$ so that strategy m precedes strategy 1 (i.e., "$1 - 1 = m$") and strategy 1 follows strategy m (i.e., "$m + 1 = 1$"). For each strategy $s \in [m]$, say that s ***wins*** against strategy $s - 1$, ***loses*** against $s + 1$ and ***ties*** against any other strategy $s' \notin \{s - 1, s + 1\}$. Say that player i ***wins*** (resp., ***loses*** or ***ties*** against player j if i chooses a strategy which wins (resp., loses or ties) against the strategy chosen by j. The payoff functions of $\widehat{\mathsf{G}}[m]$ are defined as follows. Fix a profile $\mathbf{s}$.

- If $\bigcup_{i \in [r]} \{s_i\} = \{j, j + 1\}$ for some strategy $j \in [m]$, define $\mathsf{W}(\mathbf{s}) := \{i \in [r] : s_i = j + 1\}$ and $\mathsf{L}(\mathbf{s}) := \{i \in [r] : s_i = j\}$. Each player $i \in \mathsf{L}(\mathbf{s})$ gets payoff -1; each player $i \in \mathsf{W}(\mathbf{s})$ gets payoff $\dfrac{|\mathsf{L}(\mathbf{s})|}{|\mathsf{W}(\mathbf{s})|}$.
- Otherwise all players get payoff 0.

Note that, by construction, $\widehat{\mathsf{G}}[m]$ is zero-sum. Also, by the payoff functions, a player gets a positive (resp., negative) payoff in a profile $\mathbf{s}$ only if she is choosing a strategy which wins (resp., loses) in the profile $\mathbf{s}$. We show:

▶ **Lemma 4.** *Fix an odd integer $m \geq 3$. Then, every symmetric Nash equilibrium $\boldsymbol{\sigma}$ for $\widehat{\mathsf{G}}[m]$ is fully mixed and has $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = 0$ for each player $i \in [r]$.*

Case	Condition on the profile $\mathbf{s}$	$\mathsf{U}_i(\mathbf{s})$
(1)	$\mathsf{A}(\mathbf{s}) = [r]$	$\widetilde{\mathsf{U}}_i(\mathbf{s})$
(2)	$\mathsf{C}(\mathbf{s}) = [r]$	$\widehat{\mathsf{U}}_i(\mathbf{s})$
(3)	$s_i \in \mathsf{A}$ and $\mathsf{A}(\mathbf{s}) \neq [r]$	$\phi - 1$
(4)	$s_i = j \in \mathsf{B}$ and $\mathsf{A}^\alpha(\mathbf{s}) = [r] \setminus \{i\}$	$\widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond (j - n))$
(5)	$s_i = j \in \mathsf{B}$, $\mathsf{A}(\mathbf{s}) = [r] \setminus \{i\}$ and $\mathsf{A}^\alpha(\mathbf{s}) \neq [r] \setminus \{i\}$	θ
(6)	$i \in \mathsf{B}(\mathbf{s})$ and $\mathsf{A}(\mathbf{s}) \neq [r] \setminus \{i\}$	ϕ
(7)	$i \in \mathsf{C}(\mathbf{s})$ and $\mathsf{C}(\mathbf{s}) \neq [r]$	$\phi + 1$
(8)	None of the above	$\phi - 1$

Figure 1 The payoff functions for the game G.

Note that in Lemma 4 we are only interested in determining (in order to use later) the cardinality of the support and the utilities in a symmetric Nash equilibrium. Although it may be possible that the symmetric Nash equilibrium is unique, we did not consider this issue since it is beyond our purposes.

4 The Symmetric Game Reduction

The *symmetric game reduction* takes as input a pair of symmetric r-player games, where $r \geq 3$ is a fixed constant:

- The *input game* $\widetilde{\mathsf{G}}$, with $\Sigma(\widetilde{\mathsf{G}}) = [n]$, coming together with a set of strategies $\mathsf{T} \subseteq \Sigma(\widetilde{\mathsf{G}})$ with $|\mathsf{T}| = \alpha$; $\widetilde{\mathsf{G}}$ and T form together an instance of $\exists$ SNE WITH RESTRICTED SUPPORTS. Assume, without loss of generality, that T consists of the first α strategies in $\Sigma(\widetilde{\mathsf{G}})$.
- The *gadget game* $\widehat{\mathsf{G}}$, with $\Sigma(\widehat{\mathsf{G}}) = [m]$.

It constructs a symmetric r-player game $\mathsf{G} = \mathsf{G}\left(\widetilde{\mathsf{G}}, \widehat{\mathsf{G}}\right)$ having $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ as *subgames*.

4.1 Construction of the Symmetric Game

Set $\theta := \overline{u}(\widetilde{\mathsf{G}}) + 1$ and $\phi := \min\left\{\underline{u}(\widetilde{\mathsf{G}}), \underline{u}(\widehat{\mathsf{G}})\right\} - 1$. We construct the game G as follows:

- $\Sigma(\mathsf{G}) = [p]$, with $p = n + \alpha + m$. We partition $[p]$ into three blocks A, B and C as follows:
  - A is the set of the first n strategies in $\Sigma(\mathsf{G})$, which come from $\Sigma(\widetilde{\mathsf{G}})$.
  - C is the set of the last m strategies in $\Sigma(\mathsf{G})$, which come from $\Sigma(\widehat{\mathsf{G}})$.
  - B is the set of the remaining "middle" α strategies in $\Sigma(\mathsf{G})$, which come from T.

  Denote as A^α the subset of the first α strategies in A.
- Fix a profile $\mathbf{s}$. For any set of strategies $\mathsf{X} \in \{\mathsf{A}, \mathsf{A}^\alpha, \mathsf{B}, \mathsf{C}\}$, set $\mathsf{X}(\mathbf{s}) := \{i \in [r] \mid s_i \in \mathsf{X}\}$; so $\mathsf{X}(\mathbf{s})$ is the set of players choosing strategies from X in the profile $\mathbf{s}$. The payoff functions are given in Figure 1. Since G is symmetric, we only need to specify, for a given profile $\mathbf{s}$, the payoff $\mathsf{U}_i(\mathbf{s})$ for any fixed player $i \in [r]$.

Clearly, G is constructed in time polynomial in the sizes of $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$. Note that by Case (1), $\widetilde{\mathsf{G}}$ is a subgame of G; by Case (2), $\widehat{\mathsf{G}}$ is a subgame of G. So, the blocks A and C correspond to the games $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively.

We shall need some notation. For an n-dimensional vector $\boldsymbol{x} \in \mathbb{R}^n$, denote as $\overrightarrow{\boldsymbol{x}} \in \mathbb{R}^p$ the p-dimensional vector with $\overrightarrow{x}_j = x_j$ for $j \in [n]$ and $\overrightarrow{x}_j = 0$ for $n < j \leq p$. Similarly, for an m-dimensional vector $\boldsymbol{x} \in \mathbb{R}^m$, denote as $\overleftarrow{\boldsymbol{x}} \in \mathbb{R}^p$ the p-dimensional vector with

$\overleftarrow{x}_j = 0$ for $i \in [n+\alpha]$ and $\overleftarrow{x}_j = x_j$ for $n+\alpha < j \leq p$. Thus, for a mixed profile $\boldsymbol{\sigma} \in \Delta(\widetilde{\mathsf{G}})$, $(\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r}) \in \Delta(\mathsf{G})$; for a mixed profile $\boldsymbol{\sigma} \in \Delta(\widehat{\mathsf{G}})$, $(\overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \ldots, \overleftarrow{\sigma_r}) \in \Delta(\mathsf{G})$.

4.2 Correspondences Between Nash Equilibria

We establish correspondences between the Nash equilibria for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, and those for G.

4.2.1 Backward Correspondence: From the Game G to the Subgames

We first prove that a Nash equilibrium for G is induced by a Nash equilibrium for either $\widetilde{\mathsf{G}}$ or $\widehat{\mathsf{G}}$. We start with two claims about a Nash equilibrium for G. We first prove that if some player is playing some strategy outside A, then no other player is playing a strategy in A.

▶ **Lemma 5.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$ for which there is a player $i' \in [r]$ such that $\mathsf{Supp}(\sigma_{i'}) \setminus \mathsf{A} \neq \emptyset$. Then, for every player $i \neq i'$, $\mathsf{Supp}(\sigma_i) \cap \mathsf{A} = \emptyset$.*

Proof. Assume, by way of contradiction, that there is a player $i \neq i'$ with $\mathsf{Supp}(\sigma_i) \cap \mathsf{A} \neq \emptyset$. Fix an arbitrary strategy $k \in \mathsf{Supp}(\sigma_i) \cap \mathsf{A}$. Lemma 1 (Condition (1)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}) \quad = \quad \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond k) \quad = \sum_{\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,.$$

Lemma 1 (Condition (2)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}) \quad \geq \quad \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond (k+n)) \quad = \sum_{\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n)) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,. \tag{1}$$

Consider the choices of the players other than i in the partial profile $\mathbf{s}_{-i}$ supported in $\boldsymbol{\sigma}_{-i}$:

(C.1) *All of them choose a strategy from A^α:* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (1) of the payoff functions, with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k)$; on the other hand, $\mathbf{s}_{-i} \diamond (k+n)$ falls into Case (4), with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n)) = \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k)$. Thus, $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n))$.

(C.2) *All of them choose a strategy from A and at least one chooses a strategy from $\mathsf{A} \setminus \mathsf{A}^\alpha$.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (1) of the payoff functions with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k) < \theta$; on the other hand, $\mathbf{s}_{-i} \diamond (k+n)$ falls into Case (5), with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n)) = \theta$. Thus, $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) < \mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n))$.

(C.3) *At least one player chooses a strategy outside A.* Then, $\mathbf{s}_{-i} \diamond k$ falls into Case (3) of the payoff functions, with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) = \phi - 1$; on the other hand, $\mathbf{s}_{-i} \diamond (k+n)$ falls into Case (6), with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n)) = \phi$. Thus, $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) < \mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n))$.

The assumption that $\mathsf{Supp}(\sigma_{i'}) \setminus \mathsf{A} \neq \emptyset$ implies that there is at least one profile supported in $\boldsymbol{\sigma}_{-i} \diamond (k+n)$ falling into case (C.3) above. Hence, the case analysis implies that

$$\sum_{\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \quad < \quad \sum_{\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond (k+n)) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,,$$

or $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond (k+n))$. A contradiction to (1). ◀

We continue to prove that if some player is playing no strategy from A, then the remaining $r-1$ players are playing only strategies from C.

▶ **Lemma 6.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$ for which there is a player i' with $\mathsf{Supp}(\sigma_{i'}) \cap \mathsf{A} = \emptyset$. Then, for each player $i \neq i'$, $\mathsf{Supp}(\sigma_i) \subseteq \mathsf{C}$.*

Proof. Assume, by way of contradiction, that there is a player $i \neq i'$ with $\mathsf{Supp}(\sigma_i) \setminus \mathsf{C} \neq \emptyset$. Fix an arbitrary strategy $k \in \mathsf{Supp}(\sigma_i) \setminus \mathsf{C}$ and an arbitrary strategy $h \in \mathsf{C}$. Since $k \in \mathsf{Supp}(\sigma_i)$, Lemma 1 (Condition (1)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}) \quad = \quad \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond k) \quad = \sum_{\mathbf{s}_{-i} \sim \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,.$$

Lemma 1 (Condition (2)) implies that

$$\mathsf{U}_i(\boldsymbol{\sigma}) \quad \geq \quad \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond h) = \sum_{\mathbf{s}_{-i} \in \boldsymbol{\sigma}_{-i}} \mathsf{U}_i(\mathbf{s}_{-i} \diamond h) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,. \tag{2}$$

Fix now a partial profile $\mathbf{s}_{-i}$ supported in $\boldsymbol{\sigma}_{-i}$. Since $\mathsf{Supp}(\sigma_{i'}) \cap \mathsf{A} = \emptyset$, it follows that $\mathbf{s}_{-i} \diamond k$ falls into either Case (3) or Case (6) of the payoff functions, with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond k) \leq \phi$; on the other hand, $\mathbf{s}_{-i} \diamond h$ falls into either Case (2) or Case (7), with $\mathsf{U}_i(\mathbf{s}_{-i} \diamond h) \geq \phi + 1$. This implies that $\mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond h) > \mathsf{U}_i(\boldsymbol{\sigma})$. A contradiction to (2). ◀

We are now ready to prove:

▶ **Lemma 7.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\mathsf{G})$. Then, there are only two possible cases:*

- $\boldsymbol{\sigma} = (\overrightarrow{\tau_1}, \overrightarrow{\tau_2}, \ldots, \overrightarrow{\tau_r})$ *for some Nash equilibrium* $\boldsymbol{\tau} \in \mathcal{NE}(\widetilde{\mathsf{G}})$.
- $\boldsymbol{\sigma} = (\overleftarrow{\tau_1}, \overleftarrow{\tau_2}, \ldots, \overleftarrow{\tau_r})$ *for some Nash equilibrium* $\boldsymbol{\tau} \in \mathcal{NE}(\widehat{\mathsf{G}})$.

Proof. By Lemmas 5 and 6, either (i) $\boldsymbol{\sigma} = (\overrightarrow{\tau_1}, \overrightarrow{\tau_2}, \ldots, \overrightarrow{\tau_r})$ for some mixed profile $\boldsymbol{\tau} \in \Delta(\widetilde{\mathsf{G}})$, or (ii) $\boldsymbol{\sigma} = (\overleftarrow{\tau_1}, \overleftarrow{\tau_2}, \ldots, \overleftarrow{\tau_r})$ for some mixed profile $\boldsymbol{\tau} \in \Delta(\widehat{\mathsf{G}})$. The conditions that $\boldsymbol{\tau} \in \mathcal{NE}(\widetilde{\mathsf{G}})$ and $\boldsymbol{\tau} \in \mathcal{NE}(\widehat{\mathsf{G}})$ follow from that each of $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ is a subgame of G. ◀

4.2.2 Forward Correspondence: From the Subgames to the Game G

We now characterize the Nash equilibria for the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ that induce corresponding Nash equilibria for G. Specifically, these are all the Nash equilibria for $\widehat{\mathsf{G}}$ (Lemma 8) and every Nash equilibrium for $\widetilde{\mathsf{G}}$ where all players play strategies in the restricted support (Lemma 9), respectively. We first prove:

▶ **Lemma 8.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$. Then, $\langle \overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \ldots, \overleftarrow{\sigma_r} \rangle \in \mathcal{NE}(\mathsf{G})$.*

Proof. By Case (2) of the payoff functions, for each player $i \in [r]$, $\mathsf{U}_i \langle \overleftarrow{\sigma_1}, \overleftarrow{\sigma_2}, \ldots, \overleftarrow{\sigma_r} \rangle = \widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) \geq \phi + 1$. Since $\boldsymbol{\sigma} \in \mathcal{NE}(\widehat{\mathsf{G}})$, no player could improve her payoff by switching to a strategy from C. Switching to a strategy outside C results in a partial mixed profile supporting only profiles from Cases (3) and (6). So the expected payoff of the switching player is at most ϕ. ◀

We continue to prove:

▶ **Lemma 9.** *Fix a Nash equilibrium $\boldsymbol{\sigma} \in \mathcal{NE}(\widetilde{\mathsf{G}})$. Then, $\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle \in \mathcal{NE}(\mathsf{G})$ if and only if for each player $i \in [r]$, $\mathsf{Supp}(\sigma_i) \subseteq [\alpha]$.*

Proof. Assume first that $\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle \in \mathcal{NE}(\mathsf{G})$. Fix an arbitrary player $i \in [r]$ and a strategy $k \in \mathsf{Supp}(\sigma_i)$. Then, by Lemma 1 (Condition (1)),

$$\begin{aligned}
\mathsf{U}_i\left(\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle\right) \quad &= \quad \mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond k) \\
&= \sum_{\mathbf{s}_{-i} \in \Gamma_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond k\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond k\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i})\,.
\end{aligned}$$

Assume, by way of contradiction, that there is a player $i' \in [r]$ for which $\mathsf{Supp}(\sigma_{i'}) \not\subseteq [\alpha]$. Denote as $\widetilde{\Gamma}^{\alpha}_{-i}$ the set of partial profiles in which all players in $[r] \setminus \{i\}$ choose a pure strategy in $[\alpha]$. Note that the assumption implies that there is at least one partial profile $\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i} \setminus \widetilde{\Gamma}^{\alpha}_{-i}$ with $\mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) > 0$. Consider now player i switching to the strategy $k+n$. By Cases (4) and (5) of the payoff functions, we get that

$$\begin{aligned}
\mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond (k+n)) &= \sum_{\mathbf{s}_{-i} \in \Gamma_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond (k+n)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond (k+n)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}^{\alpha}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond (k+n)) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&\quad + \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i} \setminus \widetilde{\Gamma}^{\alpha}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond (k+n)) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}^{\alpha}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i} \setminus \widetilde{\Gamma}^{\alpha}_{-i}} \theta \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&> \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond k) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \mathsf{U}_i\left(\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle\right) .
\end{aligned}$$

a contradiction to the assumption that $\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle \in \mathcal{NE}(\mathsf{G})$.

Assume now that for each player $i \in [r]$, $\mathsf{Supp}(\sigma_i) \subseteq [\alpha]$. Fix an arbitrary player $i \in [r]$. Clearly, by Case (1) of the payoff functions, player i cannot improve her payoff by switching to a strategy from A. Also, by Case (7) of the payoff functions, player i cannot improve her payoff by switching to a strategy from C. So it remains to consider player i switching to the strategy $h+n \in \mathsf{B}$, with $h \in [\alpha]$. Since $h \in [\alpha]$, Lemma 1 (Condition(2)) implies that

$$\mathsf{U}_i\left(\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle\right) \geq \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond h) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) .$$

By Case (4) of the payoff functions, we get that

$$\begin{aligned}
\mathsf{U}_i(\boldsymbol{\sigma}_{-i} \diamond (h+n)) &= \sum_{\mathbf{s}_{-i} \in \Gamma_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond (h+n)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \mathsf{U}_i\left(\mathbf{s}_{-i} \diamond (h+n)\right) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) \\
&= \sum_{\mathbf{s}_{-i} \in \widetilde{\Gamma}_{-i}} \widetilde{\mathsf{U}}_i(\mathbf{s}_{-i} \diamond h) \cdot \mathbb{P}_{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_{-i}) .
\end{aligned}$$

It follows that $\langle \overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, \ldots, \overrightarrow{\sigma_r} \rangle \in \mathcal{NE}(\mathsf{G})$. The proof is now complete. ◀

5 The ∃ℝ-Complete Decision Problems

We now present the ∃ℝ-completeness results. We show:

▶ **Theorem 10.** *Restricted to r-player symmetric games with constant $r \geq 3$, the following decision problems are ∃ℝ-complete:*

Group I	*Group II*
∃ SECOND SNE	∃ SNE WITH LARGE PAYOFFS
∃ SNE WITH SMALL PAYOFFS	∃ SNE WITH LARGE TOTAL PAYOFF
∃ SNE WITH SMALL TOTAL PAYOFF	∃ SNE WITH SMALL SUPPORTS
∃ SNE WITH LARGE SUPPORTS	
∃ SNE WITH RESTRICTING SUPPORTS	
∃ ¬ PARETO-OPTIMAL SNE	
∃ ¬ STRONGLY PARETO-OPTIMAL SNE	

Membership of the decision problems (for r-player games with $r \geq 3$) in $\exists\mathbb{R}$ is established with standard techniques by employing simple ETR formulas to define their properties (cf. [16]).

Proof. By reduction from ∃ SNE WITH RESTRICTED SUPPORT (Theorem 3). Consider an instance $\widetilde{\mathsf{G}}$ of ∃ SNE WITH RESTRICTED SUPPORT, called the *input game*. Assume, without loss of generality, that $\widetilde{\widetilde{\Sigma}} = [n]$. We start with an informal outline of the proof. The reduction will be the composition of (i) the construction of a gadget game, and (ii) the symmetric game reduction from Section 4. For each of *Group I* and *Group II,* we shall employ a suitable game $\widehat{\mathsf{G}}$, called the *gadget game,* which may be constructed from the input game $\widetilde{\mathsf{G}}$. Then, we apply the symmetric game reduction from Section 4 with $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ as the subgames to obtain the game $\mathsf{G} := \mathsf{G}\left\langle \widetilde{\mathsf{G}}, \widehat{\mathsf{G}} \right\rangle$; G is the instance of the decision problem (from the corresponding *Group*) associated with some particular property of symmetric Nash equilibria; to prove that the decision problem is $\exists\mathbb{R}$-hard, we need to establish: The game $\widetilde{\mathsf{G}}$ has a symmetric Nash equilibrium in which all players play strategies from $[\alpha]$ if and only if the game G has a symmetric Nash equilibrium with the property (resp., the set of Nash equilibria for G has the property, as for the decision problem ∃ SECOND SNE). We now present the formal proof. We treat separately each of *Group I* and *Group II.*

Group I: Construct the trivial r-player symmetric gadget game $\widehat{\mathsf{G}}$, where each player $i \in [r]$ has a unique strategy giving her payoff $\overline{u}\left(\widetilde{\mathsf{G}}\right)+1$. Clearly, $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of the inbox game $\widetilde{\mathsf{G}}$. Furthermore, $\widehat{\mathsf{G}}$ has a unique symmetric Nash equilibrium. Apply now the symmetric game reduction from Section 4 to construct the game G from the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$. Since (i) G is constructed in time polynomial in the sizes of $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, and (ii) $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$, it follows that G is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. Lemmas 7, 8 and 9 immediately imply:

▶ **Lemma 11.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has no Nash equilibrium in the restricted support. Then, G has a unique symmetric Nash equilibrium $\boldsymbol{\sigma}$ with the following properties:* (i) *for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) = \overline{u}\left(\widetilde{\mathsf{G}}\right) + 1 = \theta$;* (ii) $\mathsf{Supp}(\boldsymbol{\sigma}) = \{n + \alpha + 1\}$*;* (iii) *$\boldsymbol{\sigma}$ is Pareto-Optimal;* (iv) *$\boldsymbol{\sigma}$ is Strongly Pareto-Optimal.*

On the other hand, Lemma 9 immediately implies:

▶ **Lemma 12.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has a Nash equilibrium in the restricted support. Then, G has a Nash equilibrium $\boldsymbol{\tau}$ with the following properties:* (i) *for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\tau}) \leq \overline{u}(\widetilde{\mathsf{G}}) < \theta$;* (ii) *$\mathsf{Supp}(\boldsymbol{\tau}) \subset [\alpha]$ with $|\mathsf{Supp}(\boldsymbol{\tau})| \geq h$ for some integer $h \geq 2$;* (iii) *$\boldsymbol{\tau}$ is not Pareto-Optimal;* (iv) *$\boldsymbol{\tau}$ is not Strongly Pareto-Optimal.*

Now, given the $\exists\mathbb{R}$-hardness of ∃ SNE WITH RESTRICTED SUPPORTS for games with a constant number of players $r \geq 3$ (Theorem 3), combining corresponding properties from the two families of properties in Lemmas 11 and 12 immediately yields the $\exists\mathbb{R}$-hardness of the following decision problems:

- ∃ SECOND SNE.
- ∃ SNE WITH SMALL PAYOFFS, by taking u with $\overline{u}(\widetilde{\mathsf{G}}) < u \leq \theta$.
- ∃ SNE WITH SMALL TOTAL PAYOFF, by taking u with $r \cdot \overline{u}(\widetilde{\mathsf{G}}) < u \leq r \cdot \theta$.
- ∃ SNE WITH LARGE SUPPORTS, by taking k with $2 \leq k \leq h$.
- ∃ SNE WITH RESTRICTING SUPPORTS, by taking $\mathsf{T} := \{s\}$ for any strategy $s \in [n]$.
- ∃ NON-PARETO-OPTIMAL SNE.
- ∃ NON-STRONGLY PARETO-OPTIMAL SNE.

Group II: Fix an odd integer $m > \alpha$ with size polynomial in the size of n. Construct the symmetric r-player gadget game $\widehat{\mathsf{G}} := \mathsf{G}[m] + \underline{u}(\widetilde{\mathsf{G}}) - 1$, where $\mathsf{G}[m]$ is the gadget game from Section 3. Clearly, the game $\widehat{\mathsf{G}}$ is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. (By Lemmas 2 and 4, each symmetric Nash equilibrium $\boldsymbol{\sigma}$ for $\widehat{\mathsf{G}}$ is fully mixed and has $\widehat{\mathsf{U}}_i(\boldsymbol{\sigma}) = \underline{u}(\widetilde{\mathsf{G}}) - 1$ for each $i \in [r]$.) Apply now the symmetric game reduction from Section 4 to construct the game G from the subgames $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$. Note that by construction, G is constructed in time polynomial in the size of $\widetilde{\mathsf{G}}$. Lemmas 7, 8 and 9 immediately imply:

▶ **Lemma 13.** *Assume that the input game $\widetilde{\mathsf{G}}$ has no Nash equilibrium in the restricted support. Then, each symmetric Nash equilibrium $\boldsymbol{\sigma}$ for G has the following properties:* (i) *for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\sigma}) = \underline{u}\left(\widetilde{\mathsf{G}}\right) - 1$;* (ii) *$|\mathsf{Supp}(\boldsymbol{\sigma})| = [m]$.*

On the other hand, Lemma 9 immediately implies:

▶ **Lemma 14.** *Assume that the inbox game $\widetilde{\mathsf{G}}$ has a Nash equilibrium in the restricted support. Then, G has a symmetric Nash equilibrium $\boldsymbol{\tau}$ with the following properties: (i) for each player $i \in [r]$, $\mathsf{U}_i(\boldsymbol{\tau}) \geq \underline{u}(\widetilde{\mathsf{G}})$; (ii) $|\mathsf{Supp}(\boldsymbol{\tau})| \leq \alpha$.*

Now, given the ∃ℝ-hardness of ∃ SNE WITH RESTRICTED SUPPORTS with a constant number of players $r \geq 3$ (Theorem 3), combining corresponding properties from the two families of properties in Lemmas 13 and 14 immediately yields the ∃ℝ-hardness of the following decision problems:

- ∃ SNE WITH LARGE PAYOFFS, by taking u with $\underline{u}(\widetilde{\mathsf{G}}) - 1 < u \leq \underline{u}(\widetilde{\mathsf{G}})$.
- ∃ SNE WITH LARGE TOTAL PAYOFF, by taking u with $r \cdot \left(\underline{u}(\widetilde{\mathsf{G}}) - 1\right) < u \leq r \cdot \underline{u}(\widetilde{\mathsf{G}})$.
- ∃ SNE WITH LARGE SUPPORTS, by taking k with $\alpha \leq k < m$. ◀

We recall that ∃ SNE WITH RESTRICTING SUPPORTS was shown ∃ℝ-complete for symmetric r-player games, $r \geq 3$ also in [16, Theorem 23] (Theorem 3 in the present paper).

6 Conclusion

We presented a handful of ∃ℝ-complete decision problems about symmetric Nash equilibria for symmetric r-player games, where $r \geq 3$ is a fixed constant, which completely settles their precise complexity. Approximate versions of these decision problems are yet to be considered. There so remain decision problems about symmetric Nash equilibria for symmetric *win-lose* r-player games, with $r \geq 3$. It remains very challenging to identify special classes of symmetric multi-player games where such decision problems about symmetric Nash equilibria (or their approximations) become polynomial time solvable.

References

1 P. Austrin, M. Braverman, and E. Chlamtac. Inapproximability of $\mathcal{NP}$-complete variants of Nash equilibrium. *Theory of Computing*, 9:117–142, 2013.

2 V. Bilò and M. Mavronicolas. The complexity of decision problems about Nash equilibria in win-lose games. In *Proceedings of the 5th International Symposium on Algorithmic Game Theory*, volume 7615 of *LNCS*, pages 37–48, 2012.

3 V. Bilò and M. Mavronicolas. Complexity of rational and irrational Nash equilibria. *Theory of Computing Systems*, 54(3):491–527, 2014.

4 V. Bilò and M. Mavronicolas. A catalog of $\exists\mathbb{R}$-complete decision problems about Nash equilibria in multi-player games. In *Proceedings of the 33rd Symposium on Theoretical Aspects of Computer Science*, volume 47, pages 17:1–17:13, 2016.

5 V. Bonifaci, U. Di Orio, and L. Laura. The complexity of uniform Nash equilibria and related subgraph problems. *Theoretical Computer Science*, 401(1–3):144–152, 2008.

6 M. Braverman, Y. Kun-Ko, and O. Weinstein. Approximating the best Nash equilibrium in $O(n^{o(\lg n)})$ time breaks the exponential time hypothesis. In *Proceedings of the 26th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 970–982, 2015.

7 G. W. Brown and J. von Neumann. Solutions of games by differential equations. *Contributions to the Theory of Games, Annals of Mathematics Studies*, (24):73–79, 1950.

8 J. Canny. Some algebraic and geometric computations in $\mathcal{PSPACE}$. In *Proceedings of the 20th Annual ACM Symposium on Theory of Computing*, pages 460–467, 1988.

9 X. Chen, X. Deng, and S.-H. Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM*, 56(3), 2009.

10 B. Codenotti and D. Štefankovič. On the computational complexity of Nash equilibria for $(0,1)$ bimatrix games. *Information Processing Letters*, 94(3):145–150, 2005.

11 V. Conitzer and T. Sandholm. New complexity results about Nash equilibria. *Games and Economic Behavior*, 63(2):621–641, 2008.

12 C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.

13 A. Deligkas, J. Fearnley, and R. Savani. Inapproximability results of approximate Nash equilibria. In *Proceedings of the 12th International Conference on Web and Internet Economics*, volume 10123 of *LNCS*, pages 29–43, 2012.

14 K. Etessami and M. Yannakakis. On the complexity of Nash equilibria and other fixed points. *SIAM Journal on Computing*, 39(6):2531–2597, 2010.

15 M. J. Garey and D. J. Johnson. *Computers and Intractability – A Guide to the Theory of $\mathcal{NP}$-Completeness*. W. H. Freeman, 1979.

16 J. Garg, R. Mehta, V. V. Vazirani, and S. Yazdanbod. $\mathcal{ETR}$-completeness for decision versions of multi-player (symmetric) Nash equilibria. In *Proceedings of the 42nd International Colloquium on Automata, Languages and Programming*, volume 9134 of *LNCS*, pages 554–566, 2015.

17 I. Gilboa and E. Zemel. Nash and correlated equilibria: Some complexity considerations. *Games and Economic Behavior*, 1(1):80–93, 1989.

18 E. Hazan and R. Kraugthgamer. How hard is it to approximate the best Nash equilibrium? *SIAM Journal on Computing*, 40(1):79–91, 2011.

19 J. F. Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences of the United States of America*, 36:48–49, 1950.

20 J. F. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

21 C. H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and System Sciences*, 48(3):498–532, 1994.

22 A. Rubinstein. Settling the complexity of computing approximate two-player Nash equilibria. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*, pages 258–265, 2016.

23 M. Schaefer and D. Štefankovič. Fixed points, Nash equilibria and the existential theory of the reals. *Theory of Computing Systems*, First online: 04 November 2015.

24 A. Tarski. A decision method for elementary algebra and geometry. *RAND Corporation*, 1948.