Report from Dagstuhl Seminar 16452

Beyond-Planar Graphs: Algorithmics and Combinatorics

Edited by

Seok-Hee Hong[1], Michael Kaufmann[2], Stephen G. Kobourov[3], and János Pach[4]

1 **The University of Sydney, AU,** `seokhee.hong@sydney.edu.au`
2 **Universität Tübingen, DE,** `mk@informatik.uni-tuebingen.de`
3 **University of Arizona – Tucson, US,** `kobourov@cs.arizona.edu`
4 **EPFL – Lausanne, CH,** `janos.pach@epfl.ch`

Abstract

This report summarizes Dagstuhl Seminar 16452 "Beyond-Planar Graphs: Algorithmics and Combinatorics" and documents the talks and discussions. The seminar brought together 29 researchers in the areas of graph theory, combinatorics, computational geometry, and graph drawing. The common interest was in the exploration of structural properties and the development of algorithms for so-called *beyond-planar graphs*, i.e., non-planar graphs with topological constraints such as specific types of crossings, or with some forbidden crossing patterns. The seminar began with three introductory talks by experts in the different fields. Abstracts of these talks are collected in this report. Next we discussed and grouped together open research problems about beyond planar graphs, such as their combinatorial structures (e.g, thickness, crossing number, coloring), their topology (e.g., string graph representation), their geometric representations (e.g., straight-line drawing, visibility representation, contact representation), and applications (e.g., algorithms for real-world network visualization). Four working groups were formed and a report from each group is included here.

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1 Executive Summary

Seok-Hee Hong
Michael Kaufmann
Stephen G. Kobourov
János Pach

Relational data sets, containing a set of objects and relations between them, are commonly modeled by graphs/networks, with the objects as the vertices and the relations as the edges. A great deal is known about the structure and properties of special types of graphs, in particular *planar graphs*. The class of planar graphs is fundamental for both Graph Theory and Graph Algorithms, and extensively studied. Many structural properties of planar graphs are known and these properties can be used in the development of efficient algorithms for planar graphs, even where the more general problem is NP-hard.

Most real world graphs, however, are *non-planar*. In particular, many scale-free networks, which can be used to model web-graphs, social networks and biological networks, consists of sparse non-planar graphs. To analyze and visualize such real-world networks, we need to solve fundamental mathematical and algorithmic research questions on *sparse non-planar* graphs, which we call *beyond-planar graphs*. The notion of beyond-planar graphs has been established as non-planar graphs with topological constraints such as specific types of crossings or with some forbidden crossing patterns, although it has not been formally defined. Examples of beyond-planar graphs include:

- *k-planar* graphs: graphs which can be embedded with at most k crossings per edge.
- *k-quasi-planar* graphs: graphs which can be embedded without k mutually crossing edges.
- *bar k-visibility* graphs: graphs whose vertices are represented as horizontal segments (bars) and edges as vertical lines connecting bars, intersecting at most k other bars.
- *fan-crossing-free* graphs: graphs which can be embedded without fan-crossings.
- *fan-planar* graphs: graphs which can be embedded with crossings sharing the common vertices.
- *RAC (Right Angle Crossing)* graphs: a graph which has a straight-line drawing with right angle crossings.

The aim of the seminar was to bring together world-renowned researchers in graph algorithms, computational geometry and graph theory, and collaboratively develop a research agenda for the study of beyond-planar graphs. The plan was to work on specific open problems about the structure, topology, and geometry of beyond-planar graphs. One of the outcomes of the workshop might be an annotated bibliography of this new field of study.

On Sunday afternoon, 29 participants met at Dagstuhl for an informal get-together. Fortunately, there were no cancelations and everybody who registered was able to attend. On Monday morning, the workshop officially kicked off. After a round of introductions, where we discovered that eight participants were first-time Dagstuhl attendees, we enjoyed three overview talks about beyond-planar graphs from three different points of view. First, Géza Tóth from the Rényi Institute in Budapest talked about the combinatorics of beyond-planar graphs in connection to graph theory. Next, Giuseppe Liotta from the University of Perugia gave an overview about the connections between graph drawing and beyond-planar graphs and presented a taxonomy of related topics and questions. Finally, Alexander Wolff from the University of Würzburg discussed beyond-planar graphs in the context of geometry and geometric graph representations.

On Monday afternoon, we had lively open problem sessions, where we collected 20 problems covering the most relevant topics. The participants split into four groups based on common interest in subsets of the open problems. The last three days of the seminar were dedicated to working group efforts. Most of the groups kept their focus on the original problems as stated in the open problem session, while one group modified and expanded the problems; see Section 4. We had two progress reports sessions, including one on Friday morning, where group leaders were officially designated and plans for follow-up work were made. Work from one of the groups has been submitted to an international conference, and we expect further research publications to result directly from the seminar.

Arguably the best, and most-appreciated, feature of the seminar was the opportunity to engage in discussion and interactions with experts in various fields with shared passion about graphs, geometry and combinatorics. We received very positive feedback from the participants (e.g., scientific quality: 10.5/11, inspired new ideas: 23/25, inspired joint projects:

21/25) and it is our impression that the participants enjoyed the unique scientific atmosphere at the seminar and benefited from the scientific program. In summary, we regard the seminar as a success, and we are grateful for having had the opportunity to organize it and take this opportunity to thank the scientific, administrative, and technical staff at Schloss Dagstuhl.

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3 Overview of Talks

3.1 Graph Drawing Beyond Planarity

Giuseppe Liotta (University of Perugia, IT)

It is well known that drawings of graphs with many edge crossings are hard to read. On the other hand, edge crossings are simply unavoidable when the graph to be drawn is not planar. As a consequence a number of relaxations of the notion of graph planarity relaxation have been proposed in the literature. They allow edge crossings but forbid specific configurations which would affect the readability of the graph representation. For each such relaxation, different research questions can be asked having both algorithmic and combinatorial nature. Aim of the invited talk "Graph Drawing Beyond Planarity" was to briefly survey this rapidly growing research area by pointing out some of its most investigated questions and some of the most prominent open questions.

Graph Drawing Beyond Planarity

The classical literature on graph drawing showcases elegant algorithms and sophisticated data structures under the assumption that the input relational data set can be displayed as a network where no two edges cross (see, e.g., [5, 10, 11, 14]), i.e. the input is a planar graph. Unfortunately, almost every graph is non-planar in practice and the question on how to simplify the visual analysis of non-planar networks has become a central topic in the graph drawing research agenda.

We recall that planar graphs can be expressed in terms of forbidden subgraphs: A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Then, a fundamental natural step towards understanding non-planar graphs is to consider realizations where some types of crossings are forbidden while some other types are allowed. For example, we recall a sequence of HCI experiments by Huang et al. [7, 8, 9] proving that crossing edges significantly affect human understanding if they form acute angle, while crossing that form angles from about $\frac{\pi}{3}$ to $\frac{\pi}{2}$ guarantee good readability properties. Hence it makes sense to explore complexity issues related to drawings of graphs where such "sharp angle crossings" are forbidden. In addition to these results, Purchase et al. [16, 17, 18]) prove that an edge is difficult to read if it is crossed by many other edges; hence, the current research agenda on graph drawing beyond planarity includes the study of representations where every edge is crossed by at most k other edges, for a given constant k.

Examples of "beyond planar" graphs and problems

A *drawing* of a graph G:

(i) injectively maps each vertex u of G to a geometric body p_u in the plane;
(ii) maps each edge (u, v) of G to a Jordan arc connecting p_u and p_v that does not pass through any other vertex;
(iii) is such that any two edges have at most one point in common.

A drawing of a graph is a *straight-line drawing* if every edge is a straight-line segment, it is a *poly-line drawing* if the edges are polygonal chains and may contain bends. Vertices are typically mapped to points in the plane, but in some cases they can be other geometric objects; for example, in a *rectangle visibility representation* each vertex is represented as

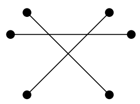
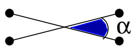
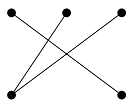
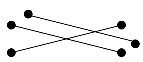
Forbidden configuration	Algorithmic question
 no three mutually crossing edges	Complexity of the recognition problem
 no crossing angle smaller than α	Straight−line drawability for graphs with bounded vertex degree
 no fan crossing	Compute simultaneous emebddings of graph pairs
 no edge with k crossings	Compute compact drawings of planar graphs with lmited number of crossings per edge

Figure 1 A table with some forbidden crossing configurations and related computational questions.

a rectangle and each edge corresponds to a horizontal or vertical line of sight between its end-vertices. Rectangle visibility representations guarantees that edge crossings form $\frac{\pi}{2}$ angles and also that the edges are straight-line segments.

The "beyond planarity" research area could be briefly described as the (potentially uncountable) collection of problems of the type depicted in Figure 1, where the column "Forbidden" describes a forbidden crossing configuration in the drawing of a graph and the column "Question" describes a corresponding computational question of interest in graph drawing. We remark that both the forbidden configurations and the computational questions of Figure 1 are mere examples within a much larger research framework. The interested reader is referred, for example, to recent proceedings of the International Symposium on Graph Drawing [19] for more results and more open problems on the "beyond planarity" topic. (See also http://www.graphdrawing.org/symposia.html.)

For reasons of space we shall make just one concrete example in the next section about how the research on a specific problem has evolved.

A Research Stream Example: Edge Partitions of 1-planar Graphs

A *1-planar drawing* is a drawing where each edge is crossed at most once. A graph is *1-planar* if it admits a 1-planar drawing. A *1-planar embedding* is an embedding that represents an equivalence class of 1-planar drawings. A *1-plane graph* is a graph with a fixed 1-planar embedding. A 1-planar graph G with n vertices has at most $4n - 8$ edges [3, 15], which is a tight bound; namely those 1-planar graphs having n vertices and $4n - 8$ edges are called *optimal 1-planar graphs*.

An *edge partition* of a 1-planar graph G is an edge coloring of G with two colors, say *red* and *blue*, such that both the graph formed by the red edges, called the *red graph*, and the graph formed by the blue edges, called the *blue graph*, are planar. Note that, given a 1-planar embedding of G, an edge partition of G can be constructed by coloring red an edge for each pair of crossing edges, and by coloring blue the remaining edges. Czap and Hudák [4] proved that every optimal 1-planar graph admits an edge partition such that the red graph is a forest. This result has been later extended to all 1-planar graphs by Ackerman [1].

Motivated by visibility representations of 1-planar graphs (see, e.g., [2, 13, 13]), Lenhart et al. [12] and Di Giacomo et al. [6] studied edge partitions such that the red graph has maximum vertex degree that is bounded by a constant independent of the size of the graph. Namely, Lenhart et al. [12] proved that if G is an n-vertex optimal 1-plane graph with a an edge partition with the red graph G_R being a forest, then G_R has n vertices and it is composed of two trees. Based on this finding, they proved that for any constant c, there exists an optimal 1-plane graph such that in any edge partition with the red graph G_R being a forest, the maximum vertex degree of G_R is at least c. On the positive side, if we drop the acyclicity requirement, then every optimal 1-planar graph admits an edge partition such that the red graph has maximum vertex degree at most four, and degree four is sometimes needed [12]. Also, every 3-connected 1-planar graph admits an edge partition such that the red graph has maximum vertex degree at most six, and degree six is sometimes needed, as shown by Di Giacomo et al. [6]. Finally, for every $n > 0$ there exists an $O(n)$-vertex 2-connected 1-planar graph such that in any edge partition the red graph has maximum vertex degree $\Omega(n)$ [6].

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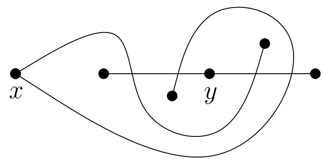

Figure 2 Edge $\{x, y\}$ cannot be added.

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3.2 Saturated Topological Graphs

Géza Tóth (Alfréd Rényi Institute of Mathematics – Budapest, HU)

A *simple topological graph* G is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing. G is called *saturated* if no further edge can be added so that it remains a simple topological graph. Obviously, if G is a *complete* simple topological graph, then it is saturated.

The simple topological graph G_1 on Figure 2, found by Kynčl, [7], has six vertices and if we connect x and y by any curve as an edge, two edges with a common endpoint will cross each other. So the resulting topological graph is not simple anymore. All other egdes can be added, so we obtain a saturated simple topological graph of 6 vertices and 14 edges. From

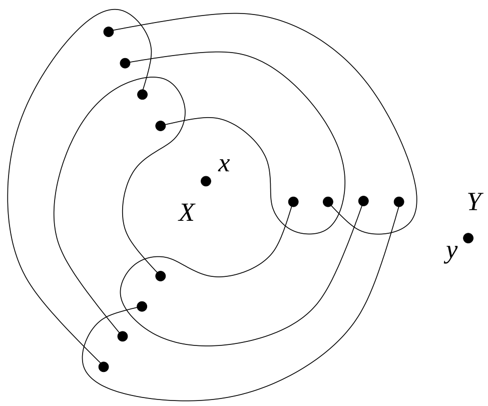

Figure 3 Edge $\{x, y\}$ cannot be added.

this we can construct a saturated simple topological graph of n vertices and $\binom{n}{2} - \lfloor n/6 \rfloor$ edges.

It is a natural question to ask, whether every saturated simple topological graph with n vertices must have $\Omega(n^2)$ edges. It turned out, that there are examples with only a linear number of edges.

▶ **Theorem 1** (Kynčl, Pach, Radoičić, Tóth, [6])**.** *For any $n \geq 4$, let $s(n)$ be the minimum number of edges that a saturated simple topological graph on n vertices can have. Then*

$$1.5n \leq s_1(n) \leq 17.5n.$$

The upper bound construction is an iterated version of the topological graph G_2 on Figure 3. It is a simple topological graph, but if we connect vertex x in region X, and vertex y in Y by a curve, it will cross one of the edges of G_2 at least twice.

For the lower bound, it is proved that in a saturated simple topological graph each vertex has degree at least three. Therefore, the number of edges is at least $1.5n$.

The upper bound has been improved recently by Hajnal, Igamberdiev, Rote, and Schulz [5]. For the lower bound, a natural way to improve it is to show that in a saturated simple topological graph each vertex has degree at least four, or five, or even more. In [5] it is also shown, that we can not expect too much improvement from this simple approach, there could be a vertex of degree four, or many vertices of degree five.

▶ **Theorem 2** (Hajnal, Igamberdiev, Rote, and Schulz [5])**.**
(i) $s(n) \leq 7n$.
(ii) *For every $n \geq 6$ there is a saturated simple topological graph on n vertices with a vertex of degree 4.*
(iii) *For every $m \geq 1$ there is a saturated simple topological graph on $10m$ vertices with m vertices of degree 5.*

▶ **Problem 3.** *Is there a saturated simple topological graph with a vertex of degree three?*

▶ **Problem 4.** *Construct a saturated simple topological graph with many vertices of degree four.*

▶ **Problem 5.** *Improve the bounds for $s(n)$.*

In general, for any positive integer k, a topological graph is called k-*simple* if any two edges have at most k points in common. We also assume that in a k-simple topological graph

Table 1 Upper bounds on the minimum number of edges in saturated k-simple topological graphs.

k	1	2	3	4	5	6	7	8	9	10	≥ 11
upper bound [6]	$17.5n$	$16n$	$14.5n$	$13.5n$	$13n$	$9.5n$	$10n$	$9.5n$	$7n$	$9.5n$	$7n$
upper bound [5]	$7n$	$14.5n$									

no edge crosses itself. A 1-simple topological graph is exactly a simple topological graph. It is not obvious at all how to construct *non-complete* saturated k-simple topological graphs for $k > 1$.

▶ **Theorem 6** (Kynčl, Pach, Radoičić, Tóth, [6]). *For any positive integers k and $n \geq 4$, let $s_k(n)$ be the minimum number of edges that a saturated k-simple topological graph on n vertices can have. Then for $k > 1$ we have*

$$n \leq s_k(n) \leq 16n.$$

For $k = 2$, the upper bound was improved by Hajnal, Igamberdiev, Rote, and Schulz [5].

▶ **Theorem 7** (Hajnal, Igamberdiev, Rote, and Schulz [5]). *$s_2(n) \leq 14.5n$.*

For the best upper bounds see Table 1.

In a graph G, an *isolated triangle* is a triangle (K_3) which is not connected to any other vertices. In the proof of the lower bound $s(n) \geq 1.5n$, an essential step is that we prove that there is no isolated triangle in a saturated simple topological graph. The proof does not work for saturated k-simple topological graphs, for $k > 1$, therefore, in this case we can prove only that every vertex has degree at least 2, which implies $s_k(n) \geq n$.

▶ **Problem 8.** *For $k > 1$, can a saturated k-simple topological graph contain an isolated triangle?*

But unlike in the case of simple topological graphs, even if we knew that a saturated k-simple topological graph can not contain an isolated triangle, we still can not prove that in a saturated k-simple topological graph all vertices have degree at least 3.

▶ **Problem 9.** *Is there a saturated k-simple topological graph for some $k > 1$ with a vertex of degree two?*

Probably the most natural and exciting problem in this topic is the following.

▶ **Problem 10.** *Is it true that every saturated k-simple topological graph is connected?*

The answer might depend on the value of k, and we do not know the answer for any k.

We assumed that in a k-simple topological graph, no edge can cross itself. For any k, a graph drawn in the plane is called a *k-complicated topological graph* if any two edges have at most k points in common, and an edge is allowed to cross itself, at most k times. Somewhat surprisingly, for saturated k-complicated topological graphs we cannot even prove that every vertex has degree at least two! We can only prove that a saturated k-complicated topological graph does not have isolated vertices. Therefore, the best lower bound we have for the minimum number of edges of a saturated k-complicated topological graph is $c_k(n) \geq n/2$.

▶ **Problem 11.** *Is there a saturated k-complicated topological graph with a vertex of degree one, for every $k \geq 1$?*

Now we study a slightly different problem. It is an easy consequence of Euler's Formula, that every planar graph of n vertices has at most $3n-6$ edges. If it has exactly $3n-6$ edges, then it is a triangulation. If it has less edges and it is drawn in the plane without crossings, than we can extend it to a triangulation.

A topological graph is *1-plane*, if each edge is crossed at most once. A graph is *1-planar*, if it has a 1-plane drawing. It is known that the maximum number of edges of a 1-plane or 1-planar graph is $4n-8$. Brandenburg et al. [3] and independently Eades et al. [4] observed a very interesting phenomenon. They noticed that maximal 1-plane or maximal 1-planar graphs can have much fewer edges.

▶ **Theorem 12** (Brandenburg, Eppstein, Gleissner, Goodrich, Hanauer, Reislhuber [3]). *Let $e_1(n)$ (resp. $e'_1(n)$) denote the minimum number of edges of a maximal 1-plane (resp. 1-planar) graph of n vertices. Then we have*

$$2.1n \leq e_1(n) \leq 2.33n, \quad 2.15n \leq e'_1(n) \leq 2.64n.$$

Both lower bounds were recently improved to $2.22n$ [2].

▶ **Problem 13.** *Improve the bounds for $e_1(n)$ and $e'_1(n)$.*

For any n, $e_1(n) \leq e'_1(n)$ since any maximal 1-planar graph has a maximal 1-plane drawing. Now the best known lower bounds are the same.

▶ **Problem 14.** *Is it true that for every n $e_1(n) = e'_1(n)$?*

In general, for every $k \geq 1$, a topological graph is *k-plane*, if each edge is crossed at most k times. A graph is *k-planar*, if it has a k-plane drawing. Let $e_k(n)$ (resp. $e'_k(n)$) denote the minimum number of edges of a maximal k-plane (resp. k-planar) graph of n vertices.

Auer et al. [1] proved that $e_2(n) \leq 1.33n$ and $e'_2(n) \leq 2.63n$. It is not hard to see, that $e_k(n) \leq cn/k$ for some $c > 0$.

▶ **Problem 15.** *Establish some nontrivial bounds for $e_k(n)$ and $e'_k(n)$.*

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3.3 Drawing Graphs: Geometric Aspects Beyond Planarity

Alexander Wolff (Universität Würzburg, DE)

Long before Graph Drawing has been established as scientific field, researchers have been studying ways to draw *planar* graphs. A particularly intriguing question is whether any planar graph can be drawn straight-line, that is, by mapping the vertices to points and the edges to non-crossing line segments between their endpoints. This question was answered in the affirmative, independently by Wagner [25], Fáry [15], and Stein [22]. Koebe [18] showed an even stronger result by proving that any planar graph is a *coin graph*, that is, the vertices can be mapped to pairwise interior-disjoint disks such that two disks touch if and only if the corresponding vertices are adjacent. In a beautiful paper entitled "How to draw a graph", Tutte [23] gave a first constructive, efficient algorithm for drawing any (triconnected) planar graph with straight-line edges (and convex faces). It is easy, however, to give examples (such as the nested-triangles graph) where Tutte's algorithm produces a drawing with an exponential ratio between the lengths of the longest and the shortest edge. This led to the question whether planar graphs can always been drawn straight-line on a grid of polynomial size. Again, this was answered in the affirmative; independently by de Fraysseix et al. [5] and by Schnyder [20].

Few graphs, however, are actually planar. Therefore, researchers in graph drawing and related areas have recently become very interested in studying classes of close-to-planar graphs. For example, Huang et al. [16] did a user study that showed that the crossing angle has a strong influence on the readability of a graph drawing. Based on the results of this user study, Didimo et al. [7] introduced the class of *RAC graphs*, that is, graphs that can be drawn straight-line into the plane such that all crossings are at right angles.

In this abstract, I consider geometric aspects of the recent "beyond planarity" direction in graph drawing. I focus on drawing graphs with low *visual complexity.* By visual complexity I mean the number of geometric primitives needed to represent the graph; for example, slopes, line segments and circular arcs, or lines and, in 3-space, planes (see following sections).

Other obvious geometric aspects "beyond planarity" are ways of dealing with crossings; namely by making crossings nicer (such as insisting on large crossing angles [7] or using edge casing [14]) or by eliminating crossings (such as *confluent drawings* where edges are represented by the existence of locally-monotone curves between their endpoints [6] or *partial edge drawings* where only fractions of each edge are drawn [2]). The layout of graphs can also be subject to other geometric restrictions (such as tracks for the vertices [9]). In this short overview, I will not deal with any of these aspects nor with topological graphs, nor with graph representations beyond dot-link diagrams (such as intersection, contact, or visibility representations, map graphs etc.).

Slope Number

Why are metro maps often drawn with a restricted set of slopes, for example, orthogonal, hexagonal, or, most commonly, octilinear? Arguably, because such drawings have low *visual complexity*, that is, they appear simpler and clearer than drawings where the number of slopes is not restricted; especially if slopes are chosen such that the *angular resolution* – the smallest angle between two edges incident to the same vertex – is large. For example, in an *octilinear* drawing (where edge directions are multiples of 45°), the angular resolution is at least 45°.

Motivated by similar observations, Wade and Chu [24] introduced, for a given graph G, its *slope number*, slope(G), as the minimum number of distinct slopes in a straight-line drawing of G. Among others, they showed that, for $n \geq 3$, slope(K_n) $= n$. Later, Dujmović et al. [10] identified two simple lower bounds, namely slope(G) $\geq \Delta(G)/2$ and slope(G) $\geq \delta(G)$, where $\Delta(G)$ is the maximum degree and $\delta(G)$ is the minumum degree of G. They asked whether there is some universal function f such that, for any graph G, its slope number can be upperbounded by $f(\Delta(G))$, independently of the size of G. Their question was answered to the negative by Pach and Pálvögyi [19] and, independently, by Barát et al. [1]. Pach and Pálvögyi [19] showed that for any sufficiently large integer n and $\Delta \geq 5$, there is an n-vertex graph G of maximum degree Δ whose slope number is larger than $n^{1/2-O(1/\Delta)}$. This bound was later improved to $n^{1-O(1/\Delta)}$ by Dujmović et al. [11]. They also proved positive results for restricted graph classes; namely for interval, co-comparability, and AT-free graphs. Dujmović et al. also showed that, if every edge can have a bend, $\Delta(G)+1$ slopes suffice for any graph G. To name another positive result, Jelínek et al. [17] showed that, for any planar partial 3-tree, $(\Delta(G))^5$ slopes suffice.

Segment Number and Arc Number

Another way to keep the visual complexity of a graph drawing low is to use few line segments. This idea is captured by the *segment number* of a graph, that is, the smallest number of line segments that together constitute a straight-line drawing of the given graph. The *arc number* of a graph is defined analogously with respect to circular arcs. For a graph G, we denote its segment number by seg(G) and its arc number by arc(G). So far, both numbers have only been studied for planar graphs. Again, two obvious lower bounds for seg(G) are known [8]; the slope number of G and $\eta(G)/2$, that is, the number of odd-degree vertices of G over 2. Dujmović et al. [8], who introduced the segment number, showed that trees can be drawn such that optimum segment number and slope number are achieved simultaneously. In other words, any tree T admits a drawing with seg(T) $= \eta(T)/2$ and slope(T) $= \Delta(T)/2$. Unfortunately, these drawings need exponential area. Therefore, Schulz [21] suggested to study the arc number of planar graphs. He showed that any tree with m edges can be drawn on a polynomial-size grid ($O(n^{1.81}) \times n$) using at most $3m/4$ arcs. Upper bounds for segment number and arc number (as fractions of the number of edges, ignoring small additive terms) are known for series-parallel graphs (3/4 vs. 1/2), planar 3-trees (2/3 vs. 11/18), and triconnected planar graphs (5/6 vs. 2/3) [8, 21]. The upper bound on the segment number for triconnected planar graphs has been improved for the special cases of triangulations and 4-connected triangulations (from 5/6 to 7/9 and 3/4, respectively) by Durocher and Mondal [12]. Durocher et al. [13] showed that the segment number is NP-hard to compute, even in the special case of arrangement graphs.

Line Cover Number and Plane Cover Number

The *affine cover number* $\rho^d_\ell(G)$ of a graph G is a generalization of planarity. It asks how many ℓ-dimensional planes are needed to cover a straight-line, crossing-free drawing of G in d-dimensional space. Clearly, any graph can be drawn without crossings in 3-space, so only the cases $\ell \in \{1, 2\}$ are interesting. As it turns out, host spaces of dimension greater than three don't help to reduce the affine cover number. Hence, only three combinations of ℓ and d are worth studying for a given graph G: the plane cover number $\rho^3_2(G)$, the line cover number $\rho^3_1(G)$ in 3-space, and, for any planar graph G, the line cover number $\rho^2_1(G)$ in the plane. Chaplick et al. [4] introduced the affine cover number and related it to many known graph parameters. The also study the *weak affine cover number* where only the vertices (but not the edges) of the given graph need to be covered. Concerning computational complexity, Chaplick et al. [3] showed that deciding, for a given graph G and integer k, whether $\rho^2_3(G) \leq k$, $\rho^1_3(G) \leq k$, or $\rho^1_2(G) \leq k$ is (at least) NP-hard. On the positive side, they showed that the two versions of the line cover number are fixed-parameter tractable. For the plane cover number, however, the decision problem is NP-hard even for any *fixed* k; hence, this problem is *not* fixed-parameter tractable.

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4 Working groups

4.1 Working group A: Generalization of the Crossing Lemma

Michael Kaufmann (Universität Tübingen, DE), János Pach (EPFL – Lausanne, CH), Vincenzo Roselli (Roma Tre University, IT), Géza Tóth (Alfréd Rényi Institute of Mathematics – Budapest, HU), Torsten Ueckerdt (KIT – Karlsruher Institut für Technologie, DE), and Pavel Valtr (Charles University – Prague, CZ)

The crossing number is a well-studied quantification on how far a given graph is from being planar. The Crossing Lemma yields a lower bound on the crossing number. The lemma has found many important applications to problems in incidence geometry, to sum-product estimates and other Erdős problems. The lemma was discovered by Ajtai, Chvátal, Newborn, Szemerédi [2], and, independently, by Leighton [1]. Twenty years ago an elegant proof based on randomized amplification was found.

The crossing number $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the smallest number of crossings over all drawings of G in the plane, that is, vertices are represented by points and edges are represented by curves connecting the corresponding points.

▶ **Lemma 1** (Crossing Lemma). *Let $G = (V, E)$ be a simple graph (no loops and multiple edges). If $|V| = n$, $|E| = m$ and $m \geq 4n$, then for the crossing number $\mathrm{cr}(G)$ we have $\mathrm{cr}(G) \geq cm^3/n^2$ for some constant $c > 0$.*

Since there are planar multi-graphs with a fixed number of vertices and an arbitrary number of edges, there is no analog of the Crossing Lemma for general multi-graphs. In the working group we were looking at a special case, where we only consider drawings which are

(i) *simple*, that is, edges are drawn as simple curves such that any two edges have at most one point in common, a crossing or a common endpoint, and
(ii) *non-homotopic*, that is, the drawing has no pair of *homotopic* parallel edges, i.e. , for any pair e_1, e_2 of parallel edges, both of the two regions bounded by $e_1 \cup e_2$ in the drawing contains at least one vertex of G.

Note that with these two assumptions, the maximum number of edges of a planar (multi)graph of n vertices is still $3n - 6$. We call such drawings of (multi-)graphs *simple non-homotopic.*

▶ **Question.** *Does the Crossing Lemma still hold for multigraphs with simple non-homotopic drawings? That is, are there constants* $c', d > 0$ *such that if* $|V| = n$, $|E| = m$ *and* $m \geq dn$, *then any simple non-homotopic drawing of* G *has at least* $c'm^3/n^2$ *crossings?*

It was observed that the restriction to simple drawings is necessary. The following is a description of a non-homotopic, but non-simple drawing where the Crossing Lemma fails: Let G be the vertex-disjoint union of an independent set $\{c_1, \ldots, c_n\}$ and a complete bipartite graph with bipartition classes $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ where each edge has multiplicity n. Hence G has $3n$ vertices and n^3 edges. The vertices of G are placed on three horizontal rows, with $\{a_1, \ldots, a_n\}$ on top, $\{c_1, \ldots, c_n\}$ in the middle, and $\{b_1, \ldots, b_n\}$ below. For each pair (a_i, b_j), $1 \leq i, j \leq n$, the k-th edge between a_i and b_j, $1 \leq k \leq n$, is routed using two straight segments: the first from a_i to a point between c_k and c_{k+1}, and the second from that point to b_j. This way every region defined by parallel edges contains at least one vertex c_k, i.e., no two parallel edges are homotopic. Clearly, each pair of independent edges crosses at most twice and any two adjacent edges cross at most once. Thus the number of crossings in the drawing of G is less than twice its number of edges squared, i.e., $\mathrm{cr} < 2n^6 = O(n^6)$. On the other hand the Crossing Lemma would predict a lower bound of $\Omega(m^3/n^2) = \Omega(n^9/n^2) = \Omega(n^7)$, which is not true for our example.

In this regime of topological non-homotopic drawings we can prove the following partial results.

- The Crossing Lemma still holds for any simple non-homotopic drawing of G in which each edge is an x-monotone curve.
- The Crossing Lemma still holds for any simple non-homotopic drawing of G with the additional property that parallel edges cross exactly the same edges.
- There exist constants $c', d > 0$ such that if $|V| = n$, $|E| = m$ and $m \geq dn$, then any simple non-homotopic drawing of G has at least $c'm^{2.5}/n^{1.5}$ crossings.

There is a lack of constructions for topological non-homotopic drawings with high edge multiplicities. Inspired by Moon's drawing of the complete graph [3] we could find drawings of multi-matchings with $|V| = 2n$, $|E| = 2n(n-1)$ and $\mathrm{cr} \approx n^4$.

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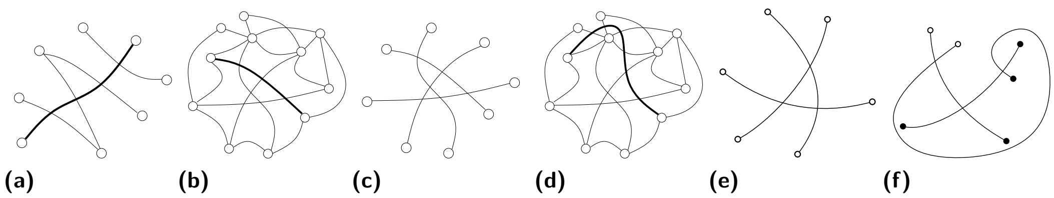

Figure 4 (a) A crossing configuration that is forbidden in a 3-planar topological graph (the thick edge is crossed more than three times). (b) A 3-planar topological graph. (c) A crossing configuration that is forbidden in a 4-quasi planar topological graph. (d) A 4-quasi planar topological graph obtained from the one of Figure (c) by suitably rerouting the thick edge. (e) An untangled 3-crossing; all vertices belong to the same face of the arrangement (the outer face). (f) A tangled 3-crossing; the circled vertices and the solid vertices belong to distinct faces of the arrangement.

4.2 Working group B1: On the Relationship between k-Planar and k-Quasi Planar Graphs

Patrizio Angelini (Universität Tübingen, DE), Michael Bekos (Universität Tübingen, DE), Franz J. Brandenburg (Universität Passau, DE), Giordano Da Lozzo (University of California – Irvine, US), Giuseppe Di Battista (Roma Tre University, IT), Walter Didimo (University of Perugia, IT), Giuseppe Liotta (University of Perugia, IT), Fabrizio Montecchiani (University of Perugia, IT), and Ignaz Rutter (TU Eindhoven, NL)

Introduction and Preliminaries

Drawings of graphs are used in a variety of application domains. The aim of a graph visualization is to clearly convey the structure of the data and their relationships, in order to support users in their analysis tasks. In this respect, there is a general consensus that graph layouts with many edge crossings are hard to read, as also witnessed by several user studies on the subject (see e.g. [9]). An emerging research area, informally recognized as *beyond planarity*, concentrates on different models of graph planarity relaxations, which allow edge crossings but forbid specific configurations that would affect the readability too much.

Two of the most popular families in this context are the *k-planar* and the *k-quasi planar* graphs, which are usually defined in terms of *topological graphs*, i.e., graphs with a geometric representation in the plane with vertices as points and edges as Jordan arcs connecting their endpoints. Namely, a topological graph is *k-planar* ($k \geq 1$) if each edge is crossed at most k times, while it is *k-quasi planar* ($k \geq 2$) if it contains no k pairwise crossing edges. The 2-quasi planar graphs coincide with the planar graphs; also, 3-quasi planar graphs are simply called *quasi planar*. If G and G' are two isomorphic graphs, we write $G \simeq G'$. A graph G' is k-planar (k-quasi planar) if there exists a k-planar (k-quasi planar) topological graph G such that $G \simeq G'$. Figure 4a shows a crossing configuration that is forbidden in a 3-planar graph. Figure 4b depicts a 3-planar topological graph that is not 2-planar (the thick edge is crossed three times). Figure 4c shows a crossing configuration that is forbidden in a 4-quasi planar graph. Figure 4d depicts a 4-quasi planar topological graph that is not 3-quasi planar.

The k-planarity and k-quasi planarity hierarchies have been widely explored in graph theory, graph drawing, and computational geometry, mostly in terms of edge density. Pach and Tóth [7] proved that an n-vertex k-planar *simple* topological graph has at most $1.408\sqrt{k}n$

edges. For $k \leq 4$, they established a finer bound of $(k+3)(n-2)$, which is tight for $k \leq 2$. For $k = 3$, the best known upper bound is $5.5n - 11$, which is tight up to a constant [4, 5].

Concerning k-quasi planar graphs, a 20-year-old conjecture by Pach, Shahrokhi, and Szegedy [6] asserts that, for every fixed k, the maximum number of edges in a k-quasi-planar graph with n vertices is $O(n)$. However, so far, linear bounds have been proven only for $k \leq 4$; see the works of Agarwal et al. [3], Pach et al. [5], Ackerman and Tardos [2], and Ackerman [1]. For $k \geq 5$, several authors proved super-linear upper bounds; the most recent results are due to Suk and Walczak [8], who proved that any k-quasi planar simple topological graph on n vertices has at most $c_k n \log n$ edges, where c_k is a number depending only on k.

During this Dagstuhl Seminar, we studied inclusion relationships between the hierarchies of k-planar graphs and of k-quasi planar graphs. Note that some results on this relationship can be immediately derived from the definition of the two classes and from the previous results on their maximum edge density. For example, 3-quasi planar graphs can be denser than 3-planar graphs, and thus there are infinitely many 3-quasi planar graphs that are not 3-planar. On the other hand, for any $k \geq 1$, every k-planar graph is $(k+2)$-quasi planar, as in a set of $k+2$ mutually crossing edges every edge is crossed at least $k+1$ times. In this work we ask whether every k-planar simple graph is $(k+1)$-quasi planar, and prove that this is true for any $k \geq 3$. Note that for $k = 1$ the answer is trivially negative, since 2-quasi planar graphs are planar. We thus leave the question open for $k = 2$.

Basic Definitions

Two edges *cross* if they share one interior point and alternate around it. Two edges *intersect* if they either cross or share a common endpoint. A graph is *almost simple* if any two edges cross at most once, and *simple* if any two edges intersect at most once. A graph divides the plane into connected regions, called *faces*. The unbounded region is the *outer face*.

Given a subgraph X of a graph G, the *arrangement of* X, denoted by $\mathcal{A}_X$, is the arrangement of the curves corresponding to the edges of X. We denote the vertices and edges of X by $V(X)$ and $E(X)$. A *node* of $\mathcal{A}_X$ is either a vertex or a crossing point of X. A *segment* of $\mathcal{A}_X$ is a part of an edge of X that connects two nodes, i.e., a maximal uncrossed part of an edge of X. A *fan* is a set of edges that share a common endpoint. A set of k vertex-disjoint mutually crossing edges in a topological graph G is called a *k-crossing*. A k-crossing X is *untangled* if in the arrangement $\mathcal{A}_X$ of X all nodes corresponding to vertices in $V(X)$ are incident to a common face. Otherwise, it is *tangled*. For example, the 3-crossing in Figure 4e is untangled, whereas the one in Figure 4f is tangled. We observe the following.

▶ **Observation 1.** *Let $G = (V, E)$ be a k-planar simple topological graph and let X be a $(k+1)$-crossing in G. An edge in $E(X)$ cannot be crossed by any other edge in $E \setminus E(X)$. In particular, for any two distinct $(k+1)$-crossings X and Y in G, $E(X) \cap E(Y) = \emptyset$ holds.*

Edge Rerouting Operations and Proof Strategy

The strategy of our proof works as follows. Let G be any k-planar simple topological graph.

First, we show that it is possible to assume that every $(k+1)$-crossing in G is untangled. For this, we provide a technique to locally redraw the edges of any $(k+1)$-crossing without creating any other crossing, which may be of independent interest; see Figure 5c and 5d.

Second, we define an *edge rerouting* operation to redraw a single edge $e = \{u, v\} \in E(X)$ of an untangled $(k+1)$-crossing X of a k-planar simple topological graph G in order to remove this $(k+1)$-crossing while not introducing any new one. This operation is illustrated in Figure 5a and 5b and formally defined as follows. Consider a vertex $w \in V(X) \setminus \{u, v\}$.

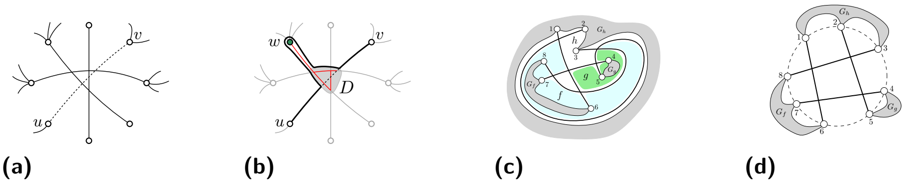

Figure 5 The rerouting operation for dissolving untangled k-crossings. (a) An untangled k-crossing X. (b) The rerouting of the dashed edge (u,v) around the marked vertex w. The arrangement $\mathcal{A}'_X$ is thin red, the removed nodes and segments are gray. Note that the dashed curve is part of $\mathcal{A}'_X$. Illustration of the untangling procedure. (c) A 3-planar simple topological graph with a 4-crossing X. (d) The topological graph resulting from the procedure that untangles X.

Denote by $\mathcal{A}'_X$ the arrangement obtained from the original arrangement $\mathcal{A}_X$ of X by removing all nodes corresponding to vertices in $V(X) \setminus \{u,v,w\}$, together with their incident segments, and by removing edge (u,v). The operation of *rerouting* $e = \{u,v\}$ *around* w consists of redrawing e sufficiently close to the boundary of the outer face of $\mathcal{A}'_X$, choosing the routing that passes close to w, in such a way that e does not cross any edge in $E \setminus E(X)$ except for the fan incident to w. More precisely, let D be a topological disk that encloses all crossing points of X and such each that edge in $E(X)$ crosses the boundary of D exactly twice. Then, the rerouted edge keeps unchanged the parts of e that go from u to the boundary of D and from v to the boundary of D. We call the unchanged parts of a rerouted edge its *tips* and the part that routes around w its *hook*.

We can prove the topological graph $G' \simeq G$ obtained by applying this operation has fewer $(k+1)$-crossings than G and is almost-simple. However, G' may be not simple and may be not k-planar any longer. While simplicity can be later obtained by suitably redrawing some edges in G', as we will discuss later, the fact that G' is not k-planar does not allow us to use an algorithm in which the $(k+1)$-crossings are removed one-by-one by repeatedly applying this operation.

We thus define a *global rerouting* operation, which picks an edge from each $(k+1)$-crossing in G and applies the rerouting operation *simultaneously* for all of these edges. Note that the global rerouting is well-defined since the $(k+1)$-crossings are pairwise edge-disjoint by Observation 1. As discussed above, we proved that a single edge rerouting operation yields an almost-simple graph with fewer $(k+1)$-crossings. We now discuss the analogous properties for a global rerouting. In terms of $(k+1)$-crossings, all those that were present in the original graph have been removed by the single operations composing the global rerouting. Further, we can prove that no new $(k+1)$-crossing can be created by any global rerouting, and hence that the resulting topological graph $G' \simeq G$ is $(k+1)$-quasi planar. We remark that this is the only argument that fails when $k = 2$, and is the reason why we can state the result only for $k \geq 3$. In order to ensure that G' is also almost-simple, on the other hand, we cannot perform any global rerouting, but we have to be careful in the choice of the edges to reroute and of the vertices around which these edges are rerouted. We discuss the conditions for G' to be almost simple in the following lemma.

► **Lemma 2.** *Graph G' is an almost-simple topological graph if and only if the following conditions hold.*

C.1 *No two edges are rerouted around the same vertex; see Figure 6a.*

C.2 *There is no pair of edges e, d such that e is rerouted around an endpoint of d and d is rerouted around an endpoint of e; see Figure 6b.*

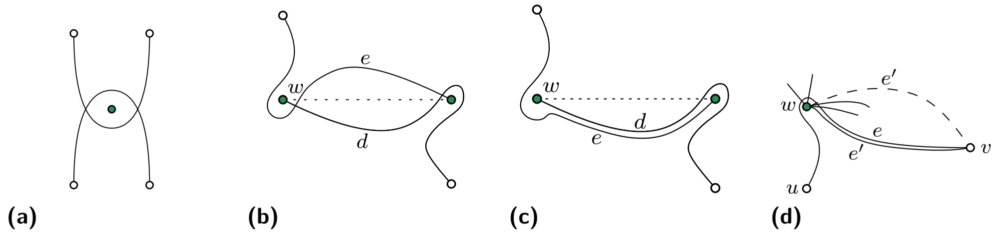

Figure 6 (a–b) Topological graphs that are not almost simple, arising from a global rerouting. (c) Avoiding the non-simplicity in (b) by redrawing one the two rerouted in edges. The vertices used for rerouting are filled green. Dotted parts of the drawing may consist of several vertices and edges. (d) Illustration for the redrawing technique to obtain simplicity.

In order to satisfy Condition C.2, we model the problem of choosing the vertices around which the edges have to be rerouted as a matching problem on a suitably defined bipartite graph, and prove that such a matching always exists by using the Hall's theorem.

Hence, after applying the global rerouting, if two edges cross more than once, then this is due to Condition C.2. In this case, we prove that it is possible to redraw one of these two edges, namely its portion between the two crossing points, without creating new $(k+1)$-crossings and without crossing any other edge more than once; see Figure 6b and 6c.

Once all the pairs of edges that cross more than once have been resolved, hence obtaining an almost-simple topological graph G' that is still $(k+1)$-quasi planar, it only remains to make G' simple, by resolving the possible pairs of adjacent edges that cross with each other. To do so, we again employ suitable redrawing techniques that do not break $(k+1)$-quasi planarity and do not introduce undesired crossings; see Figure 6d for an example.

As a final result of this work, we hence obtain the following theorem.

▶ **Theorem 3.** *For any $k \geq 3$, every k-planar graph is $(k+1)$-quasi planar.*

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4.3 Working group B2: Beyond-Planarity of Graphs with Bounded Degree

Muhammad Jawaherul Alam (University of California – Irvine, US), Kathrin Hanauer (Universität Passau, DE), Seok-Hee Hong (The University of Sydney, AU), Stephen G. Kobourov (University of Arizona – Tucson, US), Quan Nguyen (The University of Sydney), Sergey Pupyrev (Facebook – Menlo Park, US), and Md. Saidur Rahman (Bangladesh University of Eng.& Technology, BD)

We consider several problems related to beyond-planar graphs, that is, non-planar graphs with topological constraints such as specific types of crossings, or with some forbidden crossing patterns. In this context, we study graphs of bounded maximum degree.

Let $G = (V, E)$ be a graph with maximum vertex degree Δ. Furthermore, G is ...

cubic if every vertex of G has degree exactly three.

quasi-planar if G has an embedding so that there are no three pairwise crossing edges.

outer-quasi-planar if G is quasi-planar with the additional restriction that all vertices are incident to the outer face.

2-layer quasi-planar if G is outer-quasi-planar with the vertices being predefined to reside on exactly one of two layers.

1-sided 2-layer quasi-planar if G is 2-layer quasi-planar and the ordering on one layer is fixed.

right-angle crossing (RAC) if G has a straight-line drawing in the plane so that all crossings have a 90 degree angle.

fan-crossing-free if G has an embedding so that no edge crosses two or more other edges that have a common end vertex.

Known Related Results

If G has geometric thickness of at most two, it can be embedded as two stacked planar graphs, which cannot produce a pairwise crossing of more than two edges.

▶ **Proposition 1.** *Every graph with a geometric thickness of at most two is quasi-planar.*

▶ **Proposition 2** ([2]). *Leveled-planarity testing is $\mathcal{NP}$-hard.*

▶ **Proposition 3** ([1]). *Every hamiltonian cubic graph is RAC.*

Discussed Questions

▶ **Question 4.** *What is the computational complexity of recognizing quasi-planar graphs with maximum degree Δ?*

As the (geometric) thickness of graphs with $\Delta \leq 4$ is known to be two, all such graphs are quasi-planar by Proposition 1. An interesting case is $\Delta = 5$. A possibly useful observation is that the graphs of maximum degree five have linear arboricity three, that is, they can be decomposed into three sets of edge-disjoint linear forests (every forest is a set of paths).

▶ **Question 5.** *What is the complexity of recognizing outer-quasi-planar or 2-layer quasi-planar graphs with maximum degree Δ?*

As leveled-planarity testing is $\mathcal{NP}$-hard (see Proposition 2) and every leveled-planar graph is outer-quasi-planar, also testing 2-layer quasi-planarity might be $\mathcal{NP}$-hard. However, the other direction of the reduction is unclear.

The general outer-quasi-planar case remains open.

▶ **Question 6.** *For what values of Δ are the graphs of bounded maximum degree RAC? What is the complexity of recognizing RAC graphs of bounded maximum degree?*

Since K_6 does not admit a RAC drawing and graphs with $\Delta = 2$ are cycles, the above question is interesting for $\Delta = 3$ and $\Delta = 4$.

Bekos et al. show how to construct a RAC drawing for a hamiltonian cubic graph (Proposition 3), but the general case is open.

Note that every RAC drawing is quasi-planar as well as fan-crossing-free (graphs which can be embedded without fan-crossings). Hence, in order to answer Question 6, we first need to resolve the following one:

▶ **Question 7.** *Is it true that every graph of maximum degree $\Delta = 3$ ($\Delta = 4$) admits a (non-geometric, that is, with curves) both quasi-planar and fan-crossing-free drawing?*

Question 7 seems to be straightforward for $\Delta = 3$ (at least for 3-connected cubic graphs that admit a decomposition into three matchings) but less so for $\Delta = 4$.

Another set of questions is related to the density of non-planar graphs:

▶ **Question 8.** *What is the minimum density of maximal quasi-planar/outer-quasi-planar/RAC graphs?*

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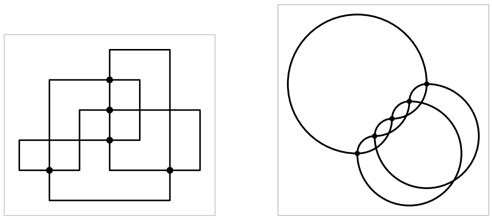
4.4 Working group C: Smooth Crossings

Evmorfia Argyriou (yWorks GmbH Tübingen, DE), Sabine Cornelsen (Universität Konstanz, DE), Martin Nöllenburg (TU Wien, AT), Yoshio Okamoto (The University of Electro-Communications – Tokyo, JP), Chrysanthi Raftopoulou (National Technical University of Athens, GR), and Alexander Wolff (Universität Würzburg, DE)

Orthogonal drawings date back to the 80's, with Valiant's [8], Leiserson's [6] and Leighton's [7] work on VLSI layouts and floor-planning applications and have been extensively studied over the years. The quality of an orthogonal drawing can be judged based on several aestetics criteria such as the required area, the total edge length, the total number of bends or maximum number of bends per edge. Towards the direction of "smoothening" an orthogonal drawing and improving its readability, Bekos et al. [4] introduced *smooth orthogonal drawings* that combine the clarity of orthogonal layouts and the artistic style of Lombardi drawings [5] by "replacing" the bends of the edges with smooth circular arcs.



(a) OC_3-layout **(b)** SC_1-layout

Figure 7 Different 2-planar drawings of K_5.

To the best of our knowledge, smooth orthogonal drawings have only been considered for planar graphs. We are interested in orthogonal and smooth orthogonal layouts of non-planar graphs such as 1-planar graphs that admit a drawing with at most one crossing per edge. Our goal is to study how typical aesthetics criteria for planar (smooth) orthogonal drawings, e.g., edge complexity, extend to non-planar drawings. We consider drawings, where vertices are mapped to points in $\mathbb{R}^2$ and edges are mapped to curves of the following two types.

Orthogonal Layout: Each edge is drawn as a sequence of vertical and horizontal line segments. Two segments meet in a bend.

Smooth Orthogonal Layout [4]: Each edge is drawn as a sequence of vertical and horizontal line segments as well as circular arcs: quarter circles, semicircles, and three-quarter circles. Where segments meet, they must have a common tangent.

The curve complexity of a drawing is the maximum number of segments used for an edge. An *OC_k-layout* is an orthogonal layout with curve complexity k, i.e., an orthogonal layout with at most $k-1$ bends per edge. A *SC_k-layout* is a smooth orthogonal layout with curve complexity k. We concentrate on graphs with a fixed embedding, i.e., a fixed rotation system and a fixed outer face.

As already mentioned, smooth orthogonal drawings have only been considered for planar graphs. Moreover, the vertex degree is usually restricted to four since every vertex has four available *ports* (North, South, East, West), where the edges enter and leave a vertex with horizontal or vertical tangents. In addition, the usual model insists that no two segments incident to the same vertex can use the same port. Note that using the same port necessarily causes overlaps only in the case of straight-line segments.

Biedl and Kant [3] presented a linear-time and -space algorithm that draws any connected graph of maximum degree four orthogonally on a grid of size $n \times n$ with at most $2n+2$ bends, where each edge is bent at most twice. Note that their approach introduces crossings to the produced drawing. For the case that the given graph is planar, they describe how to obtain a planar orthogonal drawing with at most two bends per edge, except possibly for one edge on the outer face.

The smallest non-planar graph is K_5, which is 1-planar. Following the general algorithm of Biedl and Kant, we get an orthogonal drawing of K_5 with edge-complexity three (two bends per edge) as in Figure 7a. The resulting drawing is 2-planar. Bekos et al. [2] have shown that any biconnected graph of maximum degree four admits a (non-planar) SC_1-layout. For a (2-planar) SC_1-drawing of K_5, see Figure 7b.

We focus on 1-planar (and 1-plane) graphs. In the following, we examine the curve complexity of 1-plane drawings in the orthogonal and smooth orthogonal drawing style.

(a) OC_4-layout of K_5 **(b)** OC_4-layout of a 9-vertex graph

Figure 8 Biconnected 1-plane graphs that do not admit an OC_3-layout.

Orthogonal 1-Plane Drawings

In this section, we examine the case of orthogonal 1-plane drawings and we present one negative and one positive result. Namely, we show using a counterexample that not every biconnected graph of maximum degree four with a fixed embedding admits an OC_3-layout, whereas we prove that every biconnected 1-planar graph of maximum degree four admits an OC_5-layout.

▶ **Theorem 1.** *Not every biconnected graph of maximum degree four with a fixed embedding admits an OC_3-layout.*

Proof. The complete graph on five vertices has the above property (Figure 8a). For another example refer to Figure 8b: Vertices a, b and c create a triangle T and all vertices have their two remaining ports in the interior of T. Then, T has at least seven bends, and therefore at least one edge of T has at least three bends and edge-complexity four. ◀

▶ **Corollary 2.** *There is a biconnected 1-planar graph of maximum degree four with n vertices and a given embedding that has $\mathcal{O}(n)$ edges with at least three bends in any OC_4-layout respecting the embedding.*

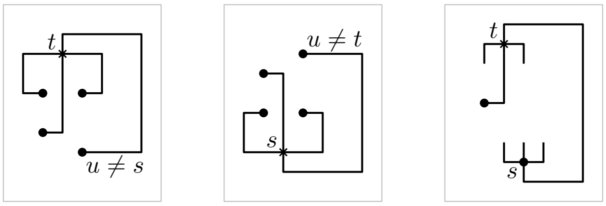
Proof. We use t copies of the graph of Figure 8b in a column by connecting the gray vertices. The graph has $n = 9t$ vertices and $\mathcal{O}(n)$ edges. ◀

▶ **Theorem 3.** *Every biconnected 1-planar graph of maximum degree four admits an OC_5-layout.*

Proof. Our algorithm is a slight modification of the algorithm of Biedl and Kant [3]. First we planarize the given 1-planar embedding by introducing dummy vertices at crossings.

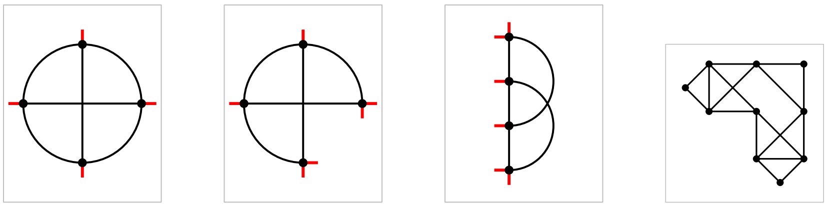
By the algorithm of Biedl and Kant, all edges have at most two bends, except possibly for one edge on the outer face that can have three bends. We only have to make sure that we do not introduce more than four bends in any edge adjacent to the outer face. All other crossing edges have at most four bends and therefore edge-complexity five.

The algorithm of Bield and Kant computes the drawing incrementally based on an st-numbering of the vertices of the graph i.e., each edge must have at least one predecessor (except s) and at least one successor (except t). Also, since our graph is biconnected, for any $s, t \in V$, there exists an st-numbering such that s is the sink vertex and t is the source vertex. We claim that we can choose s and t so that no edge has four bends. We consider the following three cases.



(a) **(b)** **(c)**

Figure 9 Different cases for the outer face.



(a) **(b)** **(c)** **(d)**

Figure 10 (a) SC_1-layouts for K_4 and (b)–(c) for $K_4 - e$ with restricted ports. (d) A biconnected outer-1 plane graph that does not have an SC_1-layout.

The outer face has at least two crossings. Let s and t be the dummy vertices at the crossings. Then the crossing edge that enters t from above (s from below) had at most three bends before entering and at most one after entering; see Figures 9a-9b respectively.

The outer face has one crossing. Let t be the dummy vertex at the crossing. For t we can argue as above. Let s be any vertex on the outer face. If s has degree less than four, we don't use the bottom port of s, and there is no problem. Otherwise s has degree four; there is at least one neighbor $s' \neq t$ on the outer face. We route the edge (s, s') through the bottom port of s. It has no crossing, hence it gets at most three bends.

The outer face has no crossings. No problem (see Figure 9c).

◀

Smooth Orthogonal 1-Plane Drawings

We focus on outerplane 1-planar graphs (in short: outer-1 plane graphs), and start with the following observation. The complete graph on four vertices with free ports towards its outer face has a unique SC_1-layout, shown in Figure 10a. Removing one edge, and restricting all ports towards its outer face, there exist two SC_1-layouts, shown in Figures 10b and 10c.

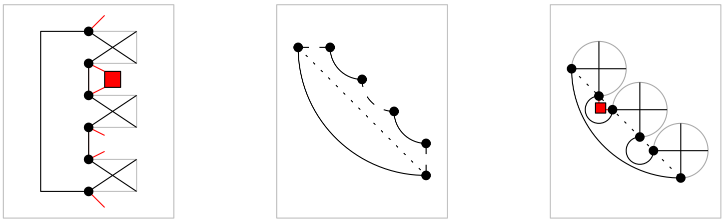
▶ **Theorem 4.** *Not every biconnected outer-1 plane graph has an SC_1-layout.*

Proof. Take the graph in Fig. 10d. It has two subgraphs isomorphic to $K_4 - e$ (with restricted ports) that share a vertex. It is not possible to combine any of the two possible SC_1-layouts for one copy of $K_4 - e$ with any SC_1-layout for the other copy. ◀

▶ **Theorem 5.** *Every biconnected outer-1 plane graph has an SC_4-layout.*

Proof. Planarize and apply the algorithm of Bekos et al. [1] that produces an SC_2-layout. ◀

▶ **Theorem 6.** *Every biconnected outer-1 plane graph where the endpoints of any two crossing edges induce a K_4 has an SC_1-layout.*



(a) **(b)** **(c)**

Figure 11 Example.

Proof. In this case all copies of K_4 are vertex-disjoint. We remove all pairs of crossing edges, producing a biconnected outerplane graph. This turns each copy of K_4 into a face of length four. We use the algorithm of Bekos et al. [1]. Whenever we want to add a "special" face of length four, we use the SC_1-layout of Figure 10a. We have to check and prove that invariants are preserved, that the algorithm can start, and that the diagonal stripes are well-defined. The area can be exponential. ◀

Future Work

Can this be extended to all outer-1 plane graphs where crossings are vertex-disjoint?

- In Figure 11a we have a face defined by vertices and crossing points, and in Figure 11c its SC_1-layout. The free ports, force components attached to the face with two edges, to be drawn before the diagonal.

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5 List of Participants

We invited 40 researchers grouped in three main areas of research, representing 18 different countries; 29 of them attended the seminar. Annotation: 2🏭 (industry), 6♀ (female), 11★ (young researchers).

A) Graph Drawing

1.	**Alam, Md. Jawaherul**	Univ. of California – Irvine	U.S.A.	★
2.	**Angelini, Patrizio**	Univ. of Tübingen	Germany	★
3.	**Bekos, Michael**	Univ. of Tübingen	Germany	★
4.	**Brandenburg, Franz**	Univ. of Passau	Germany	
5.	**Cornelsen, Sabine**	Univ. of Konstanz	Germany	♀
6.	**Da Lozzo, Giordano**	Univ. of California – Irvine	U.S.A.	★
7.	**Di Battista, Giuseppe**	Univ. of Rome III	Italy	
8.	**Didimo, Walter**	Univ. of Perugia	Italy	
9.	**Hanauer, Kathrin**	Univ. of Passau	Germany	♀★
10.	**Hong, Seok-Hee**	Univ. of Sydney	Australia	♀
11.	**Kobourov, Stephen**	Univ. of Arizona	U.S.A.	
12.	**Liotta, Giuseppe**	Univ. of Perugia	Italy	
13.	**Montecchiani, Fabrizio**	Univ. of Perugia	Italy	
14.	**Pupyrev, Sergey**	Facebook	USA	★🏭
15.	**Roselli, Vincenzo**	Univ. of Rome III	Italy	★

B) Computational and Combinatorial Geometry

16.	**Argyriou, Evmorfia**	yWorks	Germany	♀★🏭
17.	**Felsner, Stefan**	TU Berlin	Germany	
18.	**Kaufmann, Michael**	Univ. of Tübingen	Germany	
19.	**Nöllenburg, Martin**	TU Vienna	Austria	★
20.	**Quan Nguyen**	Univ. of Sydney	Australia	★
21.	**Okamoto, Yoshio**	Tokyo Tech	Japan	
22.	**Pach, János**	EPFL	Switzerland	
23.	**Rutter, Ignaz**	TU Eindhoven	Netherlands	★
24.	**Wolff, Alexander**	Würzburg Univ.	Germany	

C) Graph Theory

25.	**Raftopoulou, Chrysanthi**	NTU of Athens	Greece	♀★
26.	**Rahman, Md. Saidur**	Bangladesh Univ.	Bangladesh	
27.	**Tóth, Géza**	Rényi Institute Budapest	Hungary	
28.	**Ueckerdt, Torsten**	KIT	Germany	★
29.	**Valtr, Pavel**	Charles Univ.	Czech Republic	

Participants

- Muhammad Jawaherul Alam
University of California – Irvine, US
- Patrizio Angelini
Universität Tübingen, DE
- Evmorfia Argyriou
yWorks GmbH Tübingen, DE
- Michael Bekos
Universität Tübingen, DE
- Franz J. Brandenburg
Universität Passau, DE
- Sabine Cornelsen
Universität Konstanz, DE
- Giordano Da Lozzo
University of California – Irvine, US
- Giuseppe Di Battista
Roma Tre University, IT
- Walter Didimo
University of Perugia, IT
- Stefan Felsner
TU Berlin, DE
- Kathrin Hanauer
Universität Passau, DE
- Seok-Hee Hong
The University of Sydney, AU
- Michael Kaufmann
Universität Tübingen, DE
- Stephen G. Kobourov
University of Arizona – Tucson, US
- Giuseppe Liotta
University of Perugia, IT
- Fabrizio Montecchiani
University of Perugia, IT
- Quan Nguyen
The University of Sydney, AU
- Martin Nöllenburg
TU Wien, AT
- Yoshio Okamoto
The University of Electro-Communications – Tokyo, JP
- Janos Pach
EPFL – Lausanne, CH
- Sergey Pupyrev
Facebook – Menlo Park, US
- Chrysanthi Raftopoulou
National Technical University of Athens, GR
- Md. Saidur Rahman
Bangladesh University of Eng. & Technology, BD
- Vincenzo Roselli
Roma Tre University, IT
- Ignaz Rutter
TU Eindhoven, NL
- Géza Tóth
Alfréd Rényi Institute of Mathematics – Budapest, HU
- Torsten Ueckerdt
KIT – Karlsruher Institut für Technologie, DE
- Pavel Valtr
Charles University – Prague, CZ
- Alexander Wolff
Universität Würzburg, DE