On the Number of Ordinary Lines Determined by Sets in Complex Space*†

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Abstract

Kelly's theorem states that a set of n points affinely spanning $\mathbb{C}^3$ must determine at least one ordinary complex line (a line passing through exactly two of the points). Our main theorem shows that such sets determine at least $3n/2$ ordinary lines, unless the configuration has $n-1$ points in a plane and one point outside the plane (in which case there are at least $n-1$ ordinary lines). In addition, when at most $n/2$ points are contained in any plane, we prove a theorem giving stronger bounds that take advantage of the existence of lines with four and more points (in the spirit of Melchior's and Hirzebruch's inequalities). Furthermore, when the points span four or more dimensions, with at most $n/2$ points contained in any three dimensional affine subspace, we show that there must be a quadratic number of ordinary lines.

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1 Introduction

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of n points in $\mathbb{C}^d$. We denote by $\mathcal{L}(\mathcal{V})$ the set of lines determined by points in $\mathcal{V}$, and by $\mathcal{L}_r(\mathcal{V})$ (resp. $\mathcal{L}_{\geq r}(\mathcal{V})$) the set of lines in $\mathcal{L}(\mathcal{V})$ that contain exactly (resp. at least) r points. Let $t_r(\mathcal{V})$ denote the size of $\mathcal{L}_r(\mathcal{V})$. Throughout the write-up we omit the argument $\mathcal{V}$ when the context makes it clear. We refer to $\mathcal{L}_2$ as the set of *ordinary lines*, and $\mathcal{L}_{\geq 3}$ as the set of *special lines*.

A well known result in combinatorial geometry is the Sylvester-Gallai theorem.

▶ **Theorem 1** (Sylvester-Gallai theorem). *Let $\mathcal{V}$ be a set of n points in $\mathbb{R}^2$ not all on a line. Then there exists an ordinary line determined by points of $\mathcal{V}$.*

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The statement was conjectured by Sylvester in 1893 [18], and the first published proof is by Melchior [14]. Later proofs were given by Gallai in 1944 [8] and others; there are now several different proofs of the theorem. Of particular interest is the following result by Melchior [14].

▶ **Theorem 2** (Melchior's inequality [14]). *Let $\mathcal{V}$ be a set of n points in $\mathbb{R}^2$ that are not collinear. Then*

$$t_2(\mathcal{V}) \geq 3 + \sum_{r \geq 4} (r-3) t_r(\mathcal{V}).$$

Theorem 2 in fact proves something stronger than the Sylvester-Gallai theorem, i.e. there are at least three ordinary lines. A natural question to ask is how many ordinary lines must a set of n points, not all on a line, determine. This led to what is known as the *Dirac-Motzkin conjecture.*

▶ **Conjecture 3** (Dirac-Motzkin conjecture). *For every $n \neq 7, 13$, the number of ordinary lines determined by n noncollinear points in the plane is at least $\lceil \frac{n}{2} \rceil$.*

There were several results on this question (see [15, 13, 5]), before Green and Tao [9] resolved it for large enough point sets.

▶ **Theorem 4** (Green and Tao [9]). *Let $\mathcal{V}$ be a set of n points in $\mathbb{R}^2$, not all on a line. Suppose that $n \geq n_0$ for a sufficiently large absolute constant n_0. Then $t_2(\mathcal{V}) \geq \frac{n}{2}$ for even n and $t_2(\mathcal{V}) \geq \lfloor \frac{3n}{4} \rfloor$ for odd n.*

[9] provides a nice history of the problem, and there are several survey articles on the topic, see for example [3].

The Sylvester-Gallai theorem is not true when the field $\mathbb{R}$ is replaced by $\mathbb{C}$. The well known Hesse configuration, realized by the nine inflection points of a non-degenerate cubic, provides a counter example. A more general example is the following:

▶ **Example 5** (Fermat configuration). For any positive integer $k \geq 3$, let $\mathcal{V}$ be inflection points of the Fermat Curve $X^k + Y^k + Z^k = 0$ in $\mathbb{PC}^2$. Then $\mathcal{V}$ has $n = 3k$ points, in particular

$$\mathcal{V} = \bigcup_{i=1}^{k} \{[1 : \omega^i : 0]\} \cup \{[\omega^i : 0 : 1]\} \cup \{[0 : 1 : \omega^i]\},$$

where ω is a k^{th} root of -1.

It is easy to check that $\mathcal{V}$ determines three lines containing k points each, while every other line contains exactly three points. In particular, $\mathcal{V}$ determines no ordinary lines.[1]

In response to a question of Serre [17], Kelly [12] showed that when the points span more than two dimensions, the point set must determine at least one ordinary line.

▶ **Theorem 6** (Kelly's theorem [12]). *Let $\mathcal{V}$ be a set of n points in $\mathbb{C}^3$ that are not contained in a plane. Then there exists an ordinary line determined by points of $\mathcal{V}$.*

Kelly's proof of Theorem 6 used a deep result of Hirzebruch [11] from algebraic geometry. More specifically, it used the following result, known as Hirzebruch's inequality.

[1] We note that the while Fermat configuration as stated lives in the projective plane, it can be made affine by any projective transformation that moves a line with no points to the line at infinity.

▶ **Theorem 7** (Hirzebruch's inequality [11]). *Let $\mathcal{V}$ be a set of n points in $\mathbb{C}^2$, such that $t_n(\mathcal{V}) = t_{n-1}(\mathcal{V}) = t_{n-2}(\mathcal{V}) = 0$. Then*

$$t_2(\mathcal{V}) + \frac{3}{4}t_3(\mathcal{V}) \geq n + \sum_{r \geq 5}(2r-9)t_r(\mathcal{V}).$$

More elementary proofs of Theorem 6 were given in [7] and [6]. To the best of our knowledge, no lower bound greater than one is known for the number of ordinary lines determined by point sets spanning $\mathbb{C}^3$. Improving on the techniques of [6], we make the first progress in this direction.

▶ **Theorem 8.** *Let $\mathcal{V}$ be a set of $n \geq 24$ points in $\mathbb{C}^3$ not contained in a plane. Then $\mathcal{V}$ determines at least $\frac{3}{2}n$ ordinary lines, unless $n-1$ points are on a plane in which case there are at least $n-1$ ordinary lines.*

Clearly if $n-1$ points are coplanar, it is possible to have only $n-1$ ordinary lines. In particular, let $\mathcal{V}$ consist of the Fermat Configuration, for some $k \geq 3$, on a plane and one point v not on the plane. Then $\mathcal{V}$ has $3k+1$ points, and the only ordinary lines determined by $\mathcal{V}$ are lines that contain v, so there are exactly $3k$ ordinary lines. We are not aware of any examples that achieve the $\frac{3}{2}n$ bound when at most $n-2$ points are contained in any plane. Using a similar argument, for point sets in $\mathbb{R}^3$, Theorems 4 and 8 give us the following easy corollary.

▶ **Corollary 9.** *Let $\mathcal{V}$ be a set of n points in $\mathbb{R}^3$ not contained in a plane. Suppose that $n \geq n_0$ for a sufficiently large absolute constant n_0. Then $\mathcal{V}$ determines at least $\frac{3}{2}n - 1$ ordinary lines.*

When $\mathcal{V}$ is sufficiently non-degenerate, i.e. no plane contains too many points, we are able to give a more refined bound in the spirit of Melchior's and Hirzebruch's inequalities, taking into account the existence of lines with more than three points. In particular, we show the following (the constant $1/2$ in Theorem 10 is arbitrary and can be replaced by any positive constant smaller than 1):

▶ **Theorem 10.** *There exists an absolute constant $c > 0$ and a positive integer n_0 such that the following holds. Let $\mathcal{V}$ be a set of $n \geq n_0$ points in $\mathbb{C}^3$ with at most $\frac{1}{2}n$ points contained in any plane. Then*

$$t_2(\mathcal{V}) \geq \frac{3}{2}n + c\sum_{r \geq 4} r^2 t_r(\mathcal{V}).$$

Suppose that $\mathcal{V}$ consists of $n-k$ points on a plane, and k points not on the plane. There are at least $n-k$ lines through each point not on the plane, at most $k-1$ of which could contain three or more points. So we get that there are at least $k(n-2k)$ ordinary lines determined by $\mathcal{V}$. Then if $k = \epsilon n$, for $0 < \epsilon < 1/2$, we get that $\mathcal{V}$ has $\Omega_\epsilon(n^2)$ ordinary lines, where the hidden constant depends on ϵ. Therefore, the bound in Theorem 10 is only interesting when no plane contains too many points.

We note that having at most a constant fraction of the points on any plane is necessary to obtain a bound as in Theorem 10. Indeed, let $\mathcal{V}$ consist of the Fermat Configuration for some $k \geq 3$ on a plane and $o(k)$ points not on the plane. Then $\mathcal{V}$ has $O(k)$ points and determines $o(k^2)$ ordinary lines. On the other hand, $\sum_{r \geq 4} r^2 t_r(\mathcal{V}) = \Omega(k^2)$.

Finally, when a point set $\mathcal{V}$ spans four or more dimensions in a sufficiently non-degenerate manner, i.e. no three dimensional affine subspace contains too many points, we prove that there must be a quadratic number of ordinary lines.

▶ **Theorem 11.** *There exists an absolute constant $c' > 0$ and a positive integer n_0 such that the following holds. Let $\mathcal{V}$ be a set of $n \geq n_0$ points in $\mathbb{C}^4$ with at most $\frac{1}{2}n$ points contained in any three dimensional affine subspace. Then*

$$t_2(\mathcal{V}) \geq c'n^2.$$

Here, again, the constant $1/2$ is arbitrary and can be replaced by any positive constant less than 1. However, increasing this constant will shrink the constant c' in front of n^2. A quadratic lower bound may also be possible if at most $\frac{1}{2}n$ points are contained in any two dimensional space, but we have no proof or counterexample.

Note that while we state Theorems 8 and 10 over $\mathbb{C}^3$ and Theorem 11 over $\mathbb{C}^4$, the same bounds hold in higher dimensions as well since we may project a point set in $\mathbb{C}^d$ onto a generic lower dimensional subspace, preserving the incidence structures. In addition, while these theorems are proved over $\mathbb{C}$, these results are also new and interesting over $\mathbb{R}$.

Organization. In Section 2 we give a short overview of the new ideas in our proof (which builds upon [6]). In Section 3 we develop the necessary machinery on matrix scaling and Latin squares. In Section 4, we prove some key lemmas that will be used in the proofs of our main results. Section 5 gives the proof of Theorem 8, which is considerably simpler than Theorems 10 and 11. In Section 6, we develop additional machinery needed for the proof of Theorem 10 and describe the basic proof idea. The complete proofs of Theorems 10 and 11 can be found in the full version of the paper.

2 Proof overview

The starting point for the proofs of Theorems 8, 10 and 11 is the method developed in [2, 6] which uses rank bounds for *design matrices* – matrices in which the supports of different columns do not intersect in too many positions. We augment the techniques in these papers in several ways which give us more flexibility in analyzing the number of ordinary lines. We devote this short section to an overview of the general framework (starting with [6]) outlining the places where new ideas come into play.

Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be points in $\mathbb{C}^d$ and denote by V the $n \times (d+1)$ matrix whose i^{th} row is the vector $(v_i, 1) \in \mathbb{C}^{d+1}$, i.e. the vector obtained by appending a 1 to the vector v_i. The dimension of the (affine) space spanned by the point set can be seen to be equal to $\text{rank}(V)-1$. We would now like to argue that too many collinearities in $\mathcal{V}$ (or too few ordinary lines) imply that all (or almost all) points of $\mathcal{V}$ must be contained in a low dimensional affine subspace, i.e. $\text{rank}(V)$ is small. To do this, we construct a matrix A, encoding the dependencies in $\mathcal{V}$, such that $AV = 0$. Then we must have $\text{rank}(V) \leq n - \text{rank}(A)$, and so it suffices to lower bound the rank of A.

We construct the matrix A in the following manner so that each row of A corresponds to a collinear triple in $\mathcal{V}$. For any collinear triple $\{v_i, v_j, v_k\}$, there exist coefficients a_i, a_j, a_k such that $a_iv_i + a_jv_j + a_kv_k = 0$. We can thus form a row of A by taking these coefficients as the nonzero entries in the appropriate columns. By carefully selecting the triples using constructions of Latin squares (see Lemma 22), we can ensure that A is a design matrix.

The proof in [6] now proceeds to prove a general rank lower bound on any such design-matrix. To understand the new ideas in our proof, we need to 'open the box' and see how the rank bound from [6] is actually proved. The proof in [6] relies on *matrix scaling* techniques to gain control of the matrix. We are allowed to multiply each row and each column of A by a nonzero scalar and would like to reduce to the case where the entries of A are 'mostly balanced' (see Theorem 14 and Corollary 15). Once scaled, we can consider $M = A^*A$ (note that $rank(M) = rank(A)$). The design properties of A are then used to show that the diagonal entries of M are *large* and the off-diagonal entries are *small*. Such matrices are referred to as *diagonal dominant* matrices, and it is easy to lower bound their rank using trace inequalities (see Lemma 16).

Our proof introduces two new main ideas into this picture. The first idea has to do with the conditions needed to scale A. It is known (see Corollary 15) that a matrix A has a good scaling if it does not contain a 'too large' zero submatrix. This is referred to as having Property-S (see Definition 13). The proof of [6] uses A to construct a new matrix B, whose rows are the same as those of A but with some rows repeating more than once. Then one shows that B has Property-S and continues to scale B (which has rank at most that of A) instead of A. This loses the control on the exact number of rows in A which is crucial for bounding the number of ordinary lines. We instead perform a more careful case analysis: If A has Property-S then we scale A directly and gain more information about the number of ordinary lines. If A does not have Property-S, then we carefully examine the large zero submatrix that violates Property-S. Such a zero submatrix corresponds to a set of points and a set of lines such that no line passes through any of the points. We argue in Lemma 26 that such a submatrix implies the existence of many ordinary lines.

The second new ingredient in our proof comes into play only in the proof of Theorem 10. Here, our goal is to improve on the rank bound of [6] using the existence of lines with four or more points. Recall that our goal is to give a good upper bound on the off-diagonal entries of $M = A^*A$. Consider the (i, j)'th entry of M, obtained by taking the inner product of columns i and j in A. The i'th column of A contains the coefficients of v_i in a set of collinear triples containing v_i (we might not use all collinear triples). In [6] this inner product is bounded using the Cauchy-Schwartz inequality, and uses the fact that we picked our triple family carefully so that v_i and v_j appear together in a small number of collinear triples. One of the key insights of our proof is to notice that since the entries come from linear dependencies, having more than three points on a line gives rise to cancellations in the inner products (which increase the more points we have on a single line).

3 Preliminaries

3.1 Matrix Scaling and Rank Bounds

One of the main ingredients in our proof is rank bounds for design matrices. These techniques were first used for incidence type problems in [2] and improved upon in [6]. We first set up some notation. For a complex matrix A, let A^* denote the matrix conjugated and transposed. Let A_{ij} denote the entry in the i^{th} row and j^{th} column of A. For two complex vectors $u, v \in \mathbb{C}^d$, we denote their inner product by $\langle u, v\rangle = \sum_{i=1}^{d} u_i \cdot \overline{v_i}$.

Central to the obtaining rank bounds for matrices is the notion of matrix scaling. We now introduce this notion and provide some definitions and lemmas.

▶ **Definition 12** (Matrix Scaling)**.** Let A be an $m \times n$ matrix over some field $\mathbb{F}$. For every $\rho \in \mathbb{F}^m, \gamma \in \mathbb{F}^n$ with all entries nonzero, the matrix A' with $A'_{ij} = A_{ij} \cdot \rho_i \cdot \gamma_j$ is referred to as a scaling of A. Note that two matrices that are scalings of each other have the same rank.

We will be interested in scalings of matrices that control the row and column sums. The following property provides a sufficient condition under which such scalings exist.

▶ **Definition 13** (Property-S)**.** Let A be an $m \times n$ matrix over some field. We say that A satisfies Property-S if for every zero submatrix of size $a \times b$, we have

$$\frac{a}{m} + \frac{b}{n} \leq 1.$$

The following theorem is given in [16].

▶ **Theorem 14** (Matrix Scaling theorem)**.** *Let A be an $m \times n$ real matrix with non-negative entries satisfying Property-S. Then, for every $\epsilon > 0$, there exists a scaling A' of A such that the sum of every row of A' is at most $1 + \epsilon$, and the sum of every column of A' is at least $m/n - \epsilon$. Moreover, the scaling coefficients are all positive real numbers.*

We may assume that the sum of every row of the scaling A' is exactly $1 + \epsilon$. Otherwise, we may scale the rows to make the sum $1 + \epsilon$, and note that the column sums can only increase.

The following Corollary to Theorem 14 appeared in [2].

▶ **Corollary 15** (ℓ_2 scaling)**.** *Let A be an $m \times n$ complex matrix satisfying Property-S. Then, for every $\epsilon > 0$, there exists a scaling A' of A such that for every $i \in [m]$*

$$\sum_{j \in [n]} \left|A'_{ij}\right|^2 \leq 1 + \epsilon,$$

and for every $j \in [n]$

$$\sum_{i \in [m]} \left|A'_{ij}\right|^2 \geq \frac{m}{n} - \epsilon\,.$$

Moreover, the scaling coefficients are all positive real numbers.

Corollary 15 is obtained by applying Theorem 14 to the matrix obtained by squaring the absolute values of the entries of the matrix A. Once again, we may assume that $\sum_{j \in [n]} \left|A'_{ij}\right|^2 = 1 + \epsilon$.

To bound the rank of a matrix A, we will bound the rank of the matrix $M = A'^* A'$, where A' is some scaling of A. Then we have that $\text{rank}(A) = \text{rank}(A') = \text{rank}(M)$. We use Corollary 15, along with rank bounds for diagonal dominant matrices. The following lemma is a variant of a folklore lemma on the rank of diagonal dominant matrices (see [1]) and appeared in this form in [6].

▶ **Lemma 16.** *Let A be an $n \times n$ complex hermitian matrix, such that $|A_{ii}| \geq L$ for all $i \in n$. Then*

$$\textit{rank}(A) \geq \frac{n^2 L^2}{nL^2 + \sum_{i \neq j} |A_{ij}|^2}.$$

The matrix scaling theorem allows us to control the ℓ_2 norms of the columns and rows of A, which in turn allow us to bound the sums of squares of entries of M. To this end, we use the following lemma which appeared in [6].

▶ **Lemma 17.** *Let A be an $m \times n$ matrix over $\mathbb{C}$. Suppose that each row of A has ℓ_2 norm α, the supports of every two columns of A intersect in at most t locations, and the size of the support of every row is q. Let $M = A^* A$. Then*

$$\sum_{i \neq j} |M_{ij}|^2 \leq \left(1 - \frac{1}{q}\right) tm\alpha^4.$$

Lemma 17 is sufficient to prove Theorems 8 and 11. To prove Theorem 10, we need better bounds. A more careful analysis in the proof of Lemma 17 gives us the following lemma. The proof follows the same basic approach and can be found in the full version of the paper. We first need the following definition.

▶ **Definition 18.** Let A be an $m \times n$ matrix over $\mathbb{C}$. Then we define:

$$D(A) := \sum_{i \neq j} \sum_{k<k'} \left| A_{ki}\overline{A_{kj}} - A_{k'i}\overline{A_{k'j}} \right|^2, \text{ and } E(A) := \sum_{k=1}^{m} \sum_{i<j} \left(|A_{ki}|^2 - |A_{kj}|^2 \right)^2.$$

Note that both $D(A)$ and $E(A)$ are non-negative real numbers.

▶ **Lemma 19.** *Let A be an $m \times n$ matrix over $\mathbb{C}$. Suppose that each row of A has ℓ_2 norm α, the supports of every two columns of A intersect in exactly t locations, and the size of the support of every row is q. Let $M = A^*A$. Then*

$$\sum_{i \neq j} |M_{ij}|^2 = \left(1 - \frac{1}{q}\right) tm\alpha^4 - \left(D(A) + \frac{t}{q} E(A)\right).$$

3.2 Latin squares

Latin squares play a central role in our proof. While Latin squares play a role in both [6] and [2], our proof exploits their design properties more strongly.

▶ **Definition 20** (Latin square)**.** An $r \times r$ Latin square is an $r \times r$ matrix L such that $L_{ij} \in [r]$ for all i, j and every number in $[r]$ appears exactly once in each row and exactly once in each column.

If L is a Latin square and $L_{ii} = i$ for all $i \in [r]$, we call it a *diagonal* Latin square.

▶ **Lemma 21.** *For every $r \geq 3$, there exists an $r \times r$ diagonal Latin square. For $r \geq 4$, there exist diagonal Latin squares with the additional property that, for every $i \neq j$, $L_{ij} \neq L_{ji}$.*

Proof. For $r \geq 3$, the existence of $r \times r$ diagonal Latin squares was proved by Hilton [10]. Therefore, we need only show the second part of the theorem. For this we rely on *self-orthogonal Latin squares.*

Two Latin squares L and L' are called *orthogonal* if every ordered pair $(k, l) \in [r]^2$ occurs uniquely as (L_{ij}, L'_{ij}) for some $i, j \in [r]$. A Latin square is called *self-orthogonal* if it is orthogonal to its transpose, denoted by L^T. A theorem of Brayton, Coppersmith, and Hoffman [4] proves the existence of $r \times r$ self-orthogonal Latin squares for $r \in \mathbb{N}$, $r \neq 2, 3, 6$. Let L be a self-orthogonal Latin square. Since $L_{ii} = L^T_{ii}$, the diagonal entries give all pairs of the form (i, i) for every $i \in [r]$, i.e. the diagonal entries must be a permutation of $[r]$. Without loss of generality, we may assume that $L_{ii} = i$ and so L is also a diagonal Latin square. Clearly a self-orthogonal Latin square satisfies the property that $L_{ij} \neq L_{ji}$ if $i \neq j$.

This leaves us only with the case $r = 6$, which requires separate treatment. It is known that 6×6 self-orthogonal Latin squares do not exist. Fortunately, the property we require is weaker and we are able to give an explicit construction of a matrix that is sufficient for our needs. Let L be the following matrix

$$\begin{bmatrix} 1 & 4 & 5 & 3 & 6 & 2 \\ 3 & 2 & 6 & 5 & 1 & 4 \\ 2 & 5 & 3 & 6 & 4 & 1 \\ 6 & 1 & 2 & 4 & 3 & 5 \\ 4 & 6 & 1 & 2 & 5 & 3 \\ 5 & 3 & 4 & 1 & 2 & 6 \end{bmatrix}.$$

It is straightforward to verify that L has the required properties. ◀

The following lemma is a strengthening of a lemma from [2].

▶ **Lemma 22.** *Let $r \geq 3$. Then there exists a set $T \subseteq [r]^3$ of $r^2 - r$ triples, referred to as a* triple system*, that satisfies the following properties:*

1. *Each triple consists of three distinct elements.*
2. *For every pair $i, j \in [r]$, $i \neq j$, there are exactly six triples containing both i and j.*
3. *If $r \geq 4$, for every $i, j \in [r]$, $i \neq j$, there are at least two triples containing i and j such that the remaining elements are distinct.*

Proof. Let L be a Latin square as in Lemma 21. Let T be the set of triples $(i, j, k) \subseteq [r]^3$ with $i \neq j$ and $k = L_{ij}$. Clearly the number of such triples is $r^2 - r$. We verify that the properties mentioned hold.

Recall that we have $L_{ii} = i$ for all $i \in [r]$, and every value appears once in each row and column. So for $i \neq j \in [r]$, it can not happen that $L_{ij} = i$ or $L_{ij} = j$ and we get Property 1, i.e. all elements of a triple must be distinct.

For Property 2, note that a pair i, j appears once as (i, j, L_{ij}) and once as (j, i, L_{ji}). And since every element appears exactly once in every row and column, we have that i must appear once in the j^{th} row, j must appear once in the i^{th} row and the same for the columns. It follows that each of $(*, j, i), (j, *, i), (*, i, j)$ and $(i, *, j)$ appears exactly once, where $*$ is some other element of $[r]$. This gives us that every pair appears in exactly six triples.

For $r \geq 4$ and $i \neq j$, since $L_{ij} \neq L_{ji}$, the triples (i, j, L_{ij}) and (j, i, L_{ji}) are sufficient to satisfy Property 3. ◀

4 The dependency matrix

Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be a set of n points in $\mathbb{C}^d$. We will use $\dim(\mathcal{V})$ to denote the dimension of the linear span of $\mathcal{V}$ and by affine-dim$(\mathcal{V})$ the dimension of the affine span of $\mathcal{V}$ (i.e., the minimum r such that points of $\mathcal{V}$ are contained in a shift of a linear subspace of dimension r). We projectivize $\mathbb{C}^d$ and consider the set of vectors $\mathcal{V}' = \{v_1', \dots, v_n'\}$, where $v_i' = (v_i, 1)$ is the vector in $\mathbb{C}^{d+1}$ obtained by appending a 1 to the vector v_i. Let V be the $n \times (d+1)$ matrix whose i^{th} row is the vector v_i'. Now note that

$$\text{affine-dim}(\mathcal{V}) = \dim(\mathcal{V}') - 1 = \text{rank}(V) - 1.$$

We now construct a matrix A, which we refer to as the dependency matrix of $\mathcal{V}$. Note here that the construction we give here is preliminary, but suffices to prove Theorems 8 and 11. A refined construction is given in Section 6, where we select the triples more carefully. The rows of the matrix will consist of linear dependency coefficients, which we define below.

▶ **Definition 23** (Linear dependency coefficients)**.** Let v_1, v_2 and v_3 be three distinct collinear points in $\mathbb{C}^d$, and let $v_i' = (v_i, 1)$, $i \in \{1, 2, 3\}$, be vectors in $\mathbb{C}^{d+1}$. Recall that v_1, v_2, v_3 are collinear if and only if there exist nonzero coefficients $a_1, a_2, a_3 \in \mathbb{C}$ such that

$$a_1 v_1' + a_2 v_2' + a_3 v_3' = 0.$$

We refer to the a_1, a_2 and a_3 as the linear dependency coefficients between v_1, v_2, v_3. Note that the coefficients are determined up to scaling by a complex number. Throughout our proof, the specific choice of coefficients does not matter, so we fix a canonical choice by setting $a_3 = 1$.

▶ **Definition 24** (Dependency Matrix)**.** For every line $l \in \mathcal{L}_{\geq 3}(\mathcal{V})$, let $\mathcal{V}_l$ denote the points lying on l. Then $|\mathcal{V}_l| \geq 3$ and we assign each line a triple system $T_l \subseteq \mathcal{V}_l^3$, the existence of which is guaranteed by Lemma 22. Let A be the $m \times n$ matrix obtained by going over every line $l \in \mathcal{L}_{\geq 3}$ and for each triple $(i, j, k) \in T_l$, adding as a row of A a vector with three nonzero coefficients in positions i, j, k corresponding to the linear dependency coefficients among the points v_i, v_j, v_k. We refer to A as the dependency matrix for $\mathcal{V}$.

Note that we have $AV = 0$. Every row of A has exactly three nonzero entries. By Property 2 of Lemma 22, the supports of any distinct two columns intersect in exactly six entries when the two corresponding points lie on a special line[2], and 0 otherwise. That is, the supports of any two distinct columns intersect in at most six entries.

We say a pair of points v_i, v_j, $i \neq j$, *appears* in the dependency matrix A if there exists a row with nonzero entries in columns i and j. The number of times a pair appears is the number of rows with nonzero entries in both columns i and j.

Every pair of points that lies on a special line appears exactly six times. The only pairs not appearing in the matrix are pairs of points that determine ordinary lines. There are $\binom{n}{2}$ pairs of points, $t_2(\mathcal{V})$ of which determine ordinary lines. So the number of pairs appearing in A is $\binom{n}{2} - t_2$. The total number of times these pairs appear is then $6\left(\binom{n}{2} - t_2\right)$. Every row gives three distinct pairs of points, so it follows that the number of rows of A is

$$m = 6\left(\binom{n}{2} - t_2\right)/3 = n^2 - n - 2t_2(\mathcal{V}). \tag{1}$$

Note that $m > 0$, unless $t_2 = \binom{n}{2}$, i.e. all lines are ordinary.

As mentioned in the proof overview, we will consider two cases: when A satisfies Property-S and when it does not. We now prove lemmas dealing with the two cases. The following lemma deals with the former case.

▶ **Lemma 25.** *Let $\mathcal{V}$ be a set of n points affinely spanning $\mathbb{C}^d$, $d \geq 3$, and let A be the dependency matrix for $\mathcal{V}$. Suppose that A satisfies Property-S. Then*

$$t_2(\mathcal{V}) \geq \frac{(d-3)}{2(d+1)}n^2 + \frac{3}{2}n\,.$$

Proof. Fix $\epsilon > 0$. Since A satisfies Property-S, by Lemma 15 there is a scaling A' such that the ℓ_2 norm of each row is at most $\sqrt{1+\epsilon}$ and the ℓ_2 norm of each column is at least $\sqrt{\frac{m}{n} - \epsilon}$. Let $M := A'^* A'$. Then $M_{ii} \geq \frac{m}{n} - \epsilon$ for all i. Since every row in A has support of size three, and the supports of any two columns intersect in at most six locations, Lemma 17 gives us that $\sum_{i \neq j} |M_{ij}|^2 \leq 4m(1+\epsilon)^2$. By applying Lemma 16 to M we get,

$$\operatorname{rank}(M) \geq \frac{n^2(\frac{m}{n} - \epsilon)^2}{n(\frac{m}{n} - \epsilon)^2 + 4m(1+\epsilon)^2}.$$

Taking ϵ to 0, and combining with (1), we get

$$\begin{aligned}\operatorname{rank}(A) = \operatorname{rank}(A') = \operatorname{rank}(M) &\geq \frac{n^2 \frac{m^2}{n^2}}{n\frac{m^2}{n^2} + 4m} = \frac{mn}{m+4n} \\ &= n - \frac{4n^2}{m+4n} = n - \frac{4n^2}{n^2 - n - 2t_2(\mathcal{V}) + 4n} \\ &= n - \frac{4n^2}{n^2 + 3n - 2t_2(\mathcal{V})}.\end{aligned}$$

[2] Note that while the triple system T_l consists of ordered triples, the supports of the rows of A are unordered.

Recall that affine-dim($\mathcal{V}$) $= d =$ rank$(V) - 1$. Since $AV = 0$, we have that rank$(V) \leq n -$ rank(A). It follows that

$$d + 1 \leq \frac{4n^2}{n^2 + 3n - 2t_2(\mathcal{V})}.$$

Rearranging gives us that

$$t_2(\mathcal{V}) \geq \frac{(d-3)}{2(d+1)}n^2 + \frac{3}{2}n.$$ ◀

We now consider the case when Property-S is not satisfied.

▶ **Lemma 26.** *Let $\mathcal{V}$ be a set of n points in $\mathbb{C}^d$, and let A be the dependency matrix for $\mathcal{V}$. Suppose that A does not satisfy Property-S. Then, for every integer b^*, $1 < b^* < 2n/3$, one of the following holds:*

1. *There exists a point $v \in \mathcal{V}$ contained in at least $\frac{2}{3}(n+1) - b^*$ ordinary lines;*
2. *$t_2(\mathcal{V}) \geq nb^*/2$.*

Proof. Since A violates Property-S, there exists a zero submatrix supported on rows $U \subseteq [m]$ and columns $W \subseteq [n]$ of the matrix A, where $|U| = a$ and $|W| = b$, such that

$$\frac{a}{m} + \frac{b}{n} > 1.$$

Let $X = [m] \setminus U$ and $Y = [n] \setminus W$ and note that $|X| = m - a$ and $|Y| = n - b$. Let the violating columns correspond to the set $\mathcal{V}_1 = \{v_1, \ldots, v_b\} \subset \mathcal{V}$. We consider two cases: when $b < b^*$, and when $b \geq b^*$.

Case 1: ($b < b^*$). We may assume that U is maximal, so every row in the submatrix $X \times W$ has at least one nonzero entry. Partition the rows of X into three parts: Let X_1, X_2 and X_3 be rows with one, two and three nonzero entries in columns of W respectively. We will get a lower bound on the number of ordinary lines containing exactly one point in $\mathcal{V}_1$ and one point in $\mathcal{V} \setminus \mathcal{V}_1$ by bounding the number of pairs $\{v_i, w\}$, with $v_i \in \mathcal{V}_1$ and $w \in \mathcal{V} \setminus \mathcal{V}_1$, that lie on special lines. Note that there are at most $b(n-b)$ such pairs, and each pair that does not lie on a special line determines an ordinary line.

Each row of X_1 gives two pairs of points $\{v_i, w_1\}$ and $\{v_i, w_2\}$ that lie on a special line, where $v_i \in \mathcal{V}_1$ and $w_1, w_2 \in \mathcal{V} \setminus \mathcal{V}_1$. Each row of X_2 gives two pairs of points $\{v_i, w\}$ and $\{v_j, w\}$, where $v_i, v_j \in \mathcal{V}_1$ and $w \in \mathcal{V} \setminus \mathcal{V}_1$ that lie on special lines. Each row of X_3 has all zero entries in the submatrix supported on $X \times Y$, so does not contribute any pairs. Recall that each pair of points on a special line appears exactly six times in the matrix. This implies that the number of pairs that lie on special lines with at least one point in $\mathcal{V}_1$ and one point in $\mathcal{V} \setminus \mathcal{V}_1$ is $\frac{2|X_1|+2|X_2|}{6} \leq \frac{2|X|}{6}$. Hence, the number of ordinary lines containing exactly one of $v_1, \ldots, v_b$ is then at least $b(n-b) - \frac{|X|}{3}$.

Recall that

$$1 < \frac{a}{m} + \frac{b}{n} = \left(1 - \frac{|X|}{m}\right) + \frac{b}{n}.$$

Substituting $m \leq n^2 - n$, from (1), we get

$$|X| < \frac{bm}{n} \leq b(n-1).$$

This gives that the number of ordinary lines containing exactly one point in $\mathcal{V}_1$ is at least

$$b(n-b) - \frac{|X|}{3} > \frac{2b}{3}n - \frac{3b^2 - b}{3}.$$

We now have that there exists $v \in \mathcal{V}_1$ such that the number of ordinary lines containing v is at least

$$\left\lfloor \frac{2}{3}n - \frac{3b-1}{3} \right\rfloor \geq \left\lfloor \frac{2}{3}n - b^* + \frac{4}{3} \right\rfloor \geq \frac{2}{3}(n+1) - b^*.$$

Case 2: $(b \geq b^*)$. We will determine a lower bound for $t_2(\mathcal{V})$ by counting the number of nonzero pairs of entries $A_{ij}, A_{ij'}$ with $j \neq j'$, that appear in the submatrix $U \times Y$. There are $\binom{n-b}{2}$ pairs of points in $\mathcal{V} \setminus \mathcal{V}_1$, each of which appears at most six times, therefore the number of pairs of such entries is at most $6\binom{n-b}{2}$. Each row of U has three pairs of nonzero entries, i.e. the number of pairs of entries equals $3a$. It follows that

$$3a \leq 6\binom{n-b}{2} \tag{2}$$

Recall equation (1) and that $\frac{a}{m} + \frac{b}{n} > 1$, which gives us

$$a > m\left(1 - \frac{b}{n}\right) = \left(n^2 - n - 2t_2(\mathcal{V})\right)\left(1 - \frac{b}{n}\right). \tag{3}$$

Combining (2) and (3), we get

$$\left(n^2 - n - 2t_2(\mathcal{V})\right)\left(1 - \frac{b}{n}\right) < 2\binom{n-b}{2}.$$

Solving for $t_2(\mathcal{V})$ gives us

$$t_2(\mathcal{V}) > \frac{nb}{2} \geq \frac{nb^*}{2}.$$ ◀

5 Proof of Theorem 8

We note here that the machinery developed so far is sufficient to prove both Theorems 8 and 11. Both proofs are based on similar ideas. We give the proof of Theorem 8 in this section.

The proof relies on Lemmas 25 and 26. Together, these lemmas imply that there must be a point with many ordinary lines containing it, or there are many ordinary lines in total. As mentioned in the proof overview, the theorem is then obtained by using an iterative argument removing a point with many ordinary lines through it, and then applying the same argument to the remaining points. We get the following easy corollary from Lemma 25 and Lemma 26.

► **Corollary 27.** *Let $\mathcal{V}$ be a set of n points in $\mathbb{C}^d$ not contained in a plane. Then one of the following holds:*

1. *There exists a point $v \in \mathcal{V}$ contained in at least $\frac{2}{3}n - \frac{7}{3}$ ordinary lines.*
2. *$t_2(\mathcal{V}) \geq \frac{3}{2}n$.*

Proof. Let A be the dependency matrix for $\mathcal{V}$. If A satisfies Property-S, then we are done by Lemma 25. Otherwise, let $b^* = 3$, and note that Lemma 26 gives us the statement of the corollary when $n \geq 5$. ◀

We are now ready to prove Theorem 8.

Proof of Theorem 8. If $t_2(\mathcal{V}) \geq \frac{3}{2}n$ then we are done. Else, by Corollary 27, we may assume there exists a point v_1 with at least $\frac{1}{3}(2n-7)$ ordinary lines and hence at most $\frac{1}{6}(n+4)$ special lines through it. Let $\mathcal{V}_1 = \mathcal{V} \setminus \{v_1\}$. If $\mathcal{V}_1$ is planar, then there are exactly $n-1$ ordinary lines through v_1. We note here that this is the only case where there exists fewer then $\frac{3}{2}n$ ordinary lines.

Suppose now that $\mathcal{V}_1$ is not planar. Again, by Corollary 27, there are either $\frac{3}{2}(n-1)$ ordinary lines in $\mathcal{V}_1$ or there exists a point $v_2 \in \mathcal{V}_1$ with at least $\frac{2}{3}(n-1) - \frac{7}{3} = \frac{1}{3}(2n-9)$ ordinary lines through it. In the former case, we get $\frac{3}{2}(n-1)$ ordinary lines in $\mathcal{V}_1$, at most $\frac{1}{6}(n+4)$ of which could contain v_1. This gives that the total number of ordinary lines in $\mathcal{V}$ is

$$t_2(\mathcal{V}) \geq \frac{3}{2}(n-1) - \frac{1}{6}(n+4) + \frac{1}{3}(2n-7) = \frac{1}{2}(4n-9).$$

When $n \geq 9$, we get that $t_2(\mathcal{V}) \geq \frac{3}{2}n$.

In the latter case there exists a point $v_2 \in \mathcal{V}_1$ with at least $\frac{1}{3}(2n-9)$ ordinary lines in $\mathcal{V}_1$ through it. Note that at most one of these could contain v_1, so we get at least $\frac{1}{3}(2n-7) + \frac{1}{3}(2n-9) - 1 = \frac{1}{3}(4n-19)$ ordinary lines through one of v_1 or v_2. Note also that the number of special lines through one of v_1 or v_2 is at most $\frac{1}{6}(n+4) + \frac{1}{6}(n+3) = \frac{1}{6}(2n+7)$.

Let $\mathcal{V}_2 = \mathcal{V}_1 \setminus \{v_2\}$. If $\mathcal{V}_2$ is contained in a plane, we get at least $n-3$ ordinary lines from each of v_1 and v_2 giving a total of $2n-6$ ordinary lines in $\mathcal{V}$. It follows that when $n \geq 12$, $t_2(\mathcal{V}) \geq \frac{3}{2}n$.

Otherwise $\mathcal{V}_2$ is not contained in a plane, and again Corollary 27 gives us two cases. If there are $\frac{3}{2}(n-2)$ ordinary lines in $\mathcal{V}_2$, then we get that the total number of ordinary lines is

$$t_2(\mathcal{V}) = \frac{3}{2}(n-2) - \frac{1}{6}(2n+7) + \frac{1}{3}(4n-19) = \frac{1}{2}(5n-21).$$

When $n \geq 11$, we get that $t_2(\mathcal{V}) \geq \frac{3}{2}n$.

Otherwise there exists a point v_3 with at least $\frac{2}{3}(n-2) - \frac{7}{3}$ ordinary lines through it. At most two of these could pass through one of v_1 or v_2, so we get $\frac{2}{3}(n-2) - \frac{7}{3} - 2 = \frac{1}{3}(2n-17)$ ordinary lines through v_3 in $\mathcal{V}$. Summing up the number of lines through one of v_1, v_2 and v_3, we get that

$$t_2(\mathcal{V}) \geq \frac{1}{3}(2n-17) + \frac{1}{3}(4n-19) = 2n-12.$$

When $n \geq 24$, we get that $t_2(\mathcal{V}) \geq \frac{3}{2}n$. ◀

6 Proof Idea of Theorem 10

We first give a more careful construction for the dependency matrix of a point set $\mathcal{V}$. Recall that we defined the dependency matrix in Definition 24 to contain a row for each collinear triple from a triple system constructed on each special line. The goal was to not have too many triples containing the same pair. In this section (Definition 33) we will give a construction of a dependency matrix that will have an additional property (captured in Item 4 of Lemma 31) which is used to obtain cancellation in the diagonal dominant argument.

We denote the argument of a complex number z by $\arg(z)$, and use the convention that for every complex number z, $\arg(z) \in (-\pi, \pi]$.

▶ **Definition 28** (angle between two complex numbers)**.** We define the *angle between two complex numbers a and b* to be the the absolute value of the argument of $a\bar{b}$, denoted by $\left|\arg\left(a\bar{b}\right)\right|$. Note that the angle between a and b equals the angle between b and a.

▶ **Definition 29** (co-factor). Let v_1, v_2 and v_3 be three distinct collinear points in $\mathbb{C}^d$, and let a_1, a_2 and a_3 be the linear dependency coefficients among the three points. Define the *co-factor* of v_3 with respect to (v_1, v_2), denoted by $C_{(1,2)}(3)$, to be $\frac{a_1\overline{a_2}}{|a_1||a_2|}$. Notice that this is well defined with respect to the points, and does not depend on the choice of coefficients.

The following lemma will be used to show that "cancellations" must arise in a line containing four points (as mentioned earlier in the proof overview). We will later use this lemma as a black box to quantify the cancellations in lines with more than four points by applying it to random four tuples inside the line.

▶ **Lemma 30.** *Let v_1, v_2, v_3, v_4 be four collinear points in $\mathbb{C}^d$. Then at least one of the following hold:*

1. *The angle between $C_{(1,2)}(3)$ and $C_{(1,2)}(4)$ is at least $\pi/3$.*
2. *The angle between $C_{(1,3)}(4)$ and $C_{(1,3)}(2)$ is at least $\pi/3$.*
3. *The angle between $C_{(1,4)}(2)$ and $C_{(1,4)}(3)$ is at least $\pi/3$.*

Our final dependency matrix will be composed of blocks, each given by the following lemma. Roughly speaking, we construct a block of rows $A(l)$ for each special line l. The rows in $A(l)$ will be chosen carefully and will correspond to triples that will eventually give non trivial cancellations.

▶ **Lemma 31.** *Let l be a line in $\mathbb{C}^d$ and $\mathcal{V}_l = \{v_1, \ldots v_r\}$ be points on l with $r \geq 3$. Let V_l be the $r \times (d+1)$ matrix whose i^{th} row is the vector $(v_i, 1)$. Then there exists an $(r^2 - r) \times r$ matrix $A = A(l)$, which we refer to as the* dependency matrix *of l, such that the following hold:*

1. *$AV_l = 0$;*
2. *Every row of A has support of size three;*
3. *The support of every two columns of A intersects in exactly six locations;*
4. *If $r \geq 4$ then for at least $1/3$ of choices of $k \in [r^2 - r]$, there exists $k' \in [r^2 - r]$ such that following holds: For $k \in [r^2 - r]$, let R_k denote the rth row of A. Suppose $supp(R_k) = \{i, j, s\}$. Then $supp(R_{k'}) = \{i, j, t\}$ (for some $t \neq s$) and the angle between the co-factors $C_{(i,j)}(s)$ and $C_{(i,j)}(t)$ is at least $\pi/3$.*

Proof. Recall that Lemma 22 gives us a family of triples T_r on the set $[r]^3$. For every bijective map $\sigma : \mathcal{V}_l \to [r]$, construct a matrix A_σ in the following manner: Let T_l be the triple system on $\mathcal{V}_l^3$ induced by composing σ and T_r. For each triple $(v_i, v_j, v_k) \in T_l$, add a row with three non-zero entries in positions i, j, k corresponding to the linear dependency coefficients between v_i, v_j and v_k.

Note that for every σ, A_σ has $r^2 - r$ rows and r columns. Since the rows correspond to linear dependency coefficients, clearly we have $A_\sigma V_l = 0$ satisfying Property 1. Properties 2 and 3 follow from properties of the T_l from Lemma 22.

We will use a probabilistic argument to show that there exists a matrix A that has Property 4. Let Σ be the collection of all bijective maps from $[r]$ to the points $\mathcal{V}_l$, and let $\sigma \in \Sigma$ be a uniformly random element. Consider A_σ. Since every pair of points occurs in at least two distinct triples, for every row R_k of A_σ, there exists a row $R_{k'}$ such that the supports of R_k and $R_{k'}$ intersect in two entries. Suppose that R_k and $R_{k'}$ have supports contained in $\{i, j, s, t\}$. Suppose that σ maps $\{v_i, v_j, v_s, v_t\}$ to $\{1, 2, 3, 4\}$ and that $(1, 2, 3)$ and $(1, 2, 4)$ are triples in T_r. Without loss of generality, assume v_i maps to 1. Then by Lemma 30, the angle between at least one of the pairs $\{C_{(i,j)}(s), C_{(i,j)}(t)\}$, $\{C_{(i,s)}(j), C_{(i,s)}(t)\}$, $\{C_{(i,t)}(j), C_{(i,t)}(s)\}$ must be at least $\pi/3$. That is, given that v_i maps to 1, we have that the probability that R_k

satisfies Property 4 is at least 1/3. Then it is easy to see that

$$\Pr(R_k \text{ satisfies Property 4}) \geq 1/3.$$

Define the random variable X to be the number of rows satisfying Property 4, and note that we have

$$\mathbb{E}[X] \geq (r^2 - r)\frac{1}{3}.$$

It follows that there exists a matrix A in which at least 1/3 of the rows satisfy Property 4. ◀

Based on this new construction, we can give improved bound on the sum of the off-diagonal entries. The proof involves somewhat tedious calculations and can be found in the full version.

▶ **Lemma 32.** *There exists an absolute constant $c_0 > 0$ such that the following holds. Let l be a line in $\mathbb{C}^d$ and $\mathcal{V}_l = \{v_1, \dots v_r\}$ be points on l with $r \geq 4$. Let $A(l)$ be the dependency matrix for l, defined in Lemma 31, and A' a scaling of A such that the ℓ_2 norm of every row is α. Let $M = A'^* A'$.*

$$\sum_{i \neq j} |M_{ij}|^2 \leq 4(r^2 - r)\alpha^4 - c_0(r^2 - r)\alpha^2.$$

We are now ready to define the dependency matrix that we will use in the proof of Theorem 10.

▶ **Definition 33** (Dependency Matrix, second construction)**.** Let $\mathcal{V} = \{v_1, \dots v_n\}$ be a set of n points in $\mathbb{C}^d$ and let V be the $n \times (d+1)$ matrix whose i^{th} row is the vector $(v_i, 1)$. For each matrix $A(l)$, where $l \in \mathcal{L}_{\geq 3}(\mathcal{V})$, add $n - r$ column vectors of all zeroes, with length $r^2 - r$, in the column locations corresponding to points not in l, giving an $(r^2 - r) \times n$ matrix. Let A be the matrix obtained by taking the union of rows of these matrices for every $l \in \mathcal{L}_{\geq 3}(\mathcal{V})$. We refer to A as the *dependency matrix* of $\mathcal{V}$.

Note that this construction is a special case of the one given in Definition 24 and so satisfies all the properties mentioned there. In particular, we have $AV = 0$ and the number of rows in A equals $n^2 - n - 2t_2(\mathcal{V})$.

Proof Idea. We now briefly describe the proof idea for Theorem 10, which follows the proof of Theorem 8 closely. Given the dependency matrix A, if A satisfies Property-S, we are able to use matrix scaling along with the improved bound from Lemma 32. If the matrix does not satisfy Property-S, we use Lemma 26. This gives us the following corollary.

▶ **Corollary 34.** *There exists a constant $c_1 > 0$ and a positive integer n_0 such that the following holds. Let $\mathcal{V}$ be a set of $n \geq n_0$ points in $\mathbb{C}^d$ not contained in a plane. Then one of the following must hold:*

1. *There exists a point $v \in \mathcal{V}$ contained in at least $\frac{n}{2}$ ordinary lines.*
2. $t_2(\mathcal{V}) \geq \frac{3}{2}n + c_1 \sum_{r \geq 4}(r^2 - r)t_r(\mathcal{V})$.

To complete the proof, we again use a pruning argument. We use Corollary 34 to find a point with a large number of ordinary lines, "prune" this point, and then repeat this on the smaller set of points. We stop when either we can not find such a point, in which case Corollary 34 guarantees a large number of ordinary lines, or when we have accumulated enough ordinary lines. The assumption that no plane contains more than $n/2$ points guarantees that we are able to continue pruning until we find sufficiently many ordinary lines.

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