Irrational Guards are Sometimes Needed*

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Abstract

In this paper we study the *art gallery problem*, which is one of the fundamental problems in computational geometry. The objective is to place a minimum number of guards inside a simple polygon so that the guards together can see the whole polygon. We say that a guard at position x sees a point y if the line segment xy is contained in the polygon.

Despite an extensive study of the art gallery problem, it remained an open question whether there are polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. We give a positive answer to this question by constructing a *monotone* polygon with integer coordinates that can be guarded by three guards only when we allow to place the guards at points with irrational coordinates. Otherwise, four guards are needed. By extending this example, we show that for every n, there is a polygon which can be guarded by $3n$ guards with irrational coordinates but needs $4n$ guards if the coordinates have to be rational. Subsequently, we show that there are rectilinear polygons given by integer coordinates that require guards with irrational coordinates in any optimal solution.

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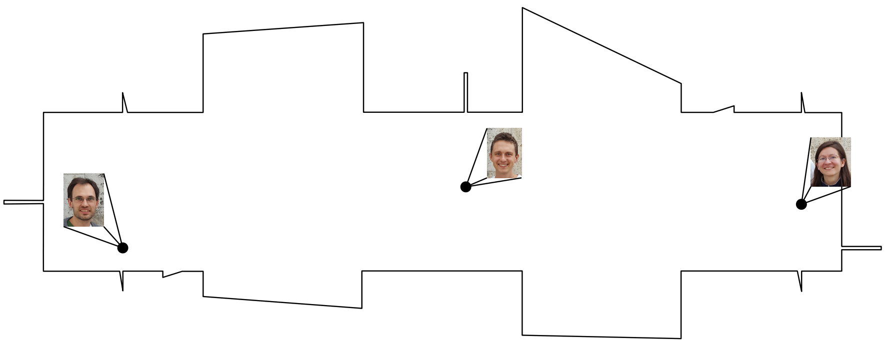

Figure 1 Till, Mikkel, and Anna are meticulously guarding the polygon. They are a little irrational, but pretty optimal.

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1 Introduction

For a polygon $\mathcal{P}$ and points $x, y \in \mathcal{P}$, we say that x *sees* y if the line segment xy is contained in $\mathcal{P}$. A *guard set* S is a set of points in $\mathcal{P}$ such that every point in $\mathcal{P}$ is seen by some point in S. The points in S are called *guards*. The *art gallery problem* is to find a minimum cardinality guard set for a given simple polygon $\mathcal{P}$ on n vertices. Such a guard set is called *optimal*. The polygon $\mathcal{P}$ is considered to be filled, i.e., it consists of a closed, simple polygonal curve in the plane and the bounded region enclosed by this curve.

This classical version of the art gallery problem has been originally formulated in 1973 by Victor Klee (see the book of O'Rourke [20, page 2]). It is often referred to as the *interior-guard art gallery problem* or the *point-guard art gallery problem*, to distinguish it from other versions that have been introduced over the years.

Chvátal proved in 1975 that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary to guard a polygon with n vertices [9]. A simpler proof was later found by Fisk [15]. Since then, the art gallery problem has been extensively studied, both from the combinatorial and the algorithmic perspective. Most of this research, however, is not focused directly on the classical art gallery problem, but on its numerous versions, including different definitions of visibility, restricted classes of polygons, restrictions on the positions of the guards, etc. For more detailed information we refer the reader to the surveys [26, 28, 20, 22].

Despite extensive research on the art gallery problem, no combinatorial algorithm for finding an optimal solution, or even for deciding whether a guard set of a given size k exists, is known. The only exact algorithm is attributed to Micha Sharir (see [12]), who has shown that in $n^{O(k)}$ time one can decide whether a guard set consisting of k guards exists. This result is obtained by using standard tools from real algebraic geometry [2], and it is not known how to find an optimal solution without using this powerful machinery (see [3] for an analysis of the very restricted case of $k = 2$). Some recent lower bounds [5] based on the exponential time hypothesis suggest that there might be no better exact algorithms than the one by Sharir.

To explain the difficulty in constructing exact algorithms, we want to emphasize that it is *not* known whether the decision version of the art gallery problem (i.e., the problem of deciding whether there is a guard set consisting of k guards, where k is a parameter) lies in the complexity class NP. While NP-hardness and APX-hardness of the art gallery problem have been shown for different versions of the problem [18, 25, 27, 6, 13, 21, 17], the question of whether the point-guard art gallery problem is in NP remains open. A simple way to show NP-membership would be to prove that there always exists an optimal set of guards with *rational* coordinates of polynomially bounded description.

Sándor Fekete posed at MIT in 2010 and at Dagstuhl in 2011 an open problem, asking whether there are polygons requiring irrational coordinates in an optimal guard set [14, 1]. The question has been raised again by Günter Rote at EuroCG 2011 [23]. It has also been mentioned by Rezende *et al.* [10]: "it remains an open question whether there are polygons given by rational coordinates that require optimal guard positions with irrational coordinates". A similar question has been raised by Friedrichs *et al.* [16]: "[...] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits".

Our results. We answer the open question of Sándor Fekete by proving the following result. Recall that a polygon $\mathcal{P}$ is called *monotone* if there exists a line l such that the intersection between any line orthogonal to l and $\mathcal{P}$ is either empty or a single line segment.

▶ **Theorem 1.** *There is a simple monotone polygon $\mathcal{P}$ with integer vertex coordinates such that*
1. *$\mathcal{P}$ can be guarded by 3 guards, and*
2. *an optimal guard set of $\mathcal{P}$ with guards at points with rational coordinates has size 4.*

An interesting consequence of Theorem 1 is that there is no optimal guard set of $\mathcal{P}$ among a candidate set of guard positions consisting of intersections between extensions of chords and edges of $\mathcal{P}$. It does not help to expand the candidate set by adding a line through each pair of candidates, thus creating new intersections to be added to the set of candidates, or to repeat this procedure any finite number of iterations, since all candidate points created by such a process must inevitably have rational coordinates. This shows that algorithms based on this procedure, as well as other algorithms for the art gallery problem which consider only rational points as possible guard positions, will in general not find an optimal guard set.

We then extend Theorem 1 by providing a family of polygons for which the ratio between the size of an optimal rational guard set and the size of an optimal set with irrational guards allowed is 4/3.

▶ **Theorem 2.** *There is a family of simple polygons $(\mathcal{P}_n)_{n\in\mathbb{Z}_+}$ with integer vertex coordinates such that*
1. *$\mathcal{P}_n$ can be guarded by $3n$ guards, and*
2. *an optimal guard set of $\mathcal{P}_n$ with guards at points with rational coordinates has size $4n$.*

Moreover, the coordinates of the points defining the polygons $\mathcal{P}_n$ are polynomial in n.

We show that the phenomenon with guards at irrational coordinates occurs already in the much simpler class of rectilinear polygons, i.e., polygons where each edge is parallel to the x-axis or to the y-axis.

▶ **Theorem 3.** *There is a rectilinear polygon $\mathcal{P}_R$ with vertices at integer coordinates satisfying the following properties.*
1. *$\mathcal{P}_R$ can be guarded by 9 guards.*
2. *An optimal guard set of $\mathcal{P}_R$ with guards at points with rational coordinates has size 10.*

The Structure of the Paper. Section 2 contains the description of a monotone polygon $\mathcal{P}$ with vertices at points with rational coordinates that can be guarded by three guards only if the guards are placed at points with irrational coordinates. In Section 3, we describe the intuition behind our construction, and explain how we have found the polygon $\mathcal{P}$. The formal proof of Theorems 1 and 2 is then provided in Section 4. In Section 5, we present the rectilinear polygon $\mathcal{P}_R$ from Theorem 3 requiring guards with irrational coordinates in an optimal guard set. Finally, in Section 6 we suggest some open problems for future research.

2 The Polygon

In Figure 2 we present the polygon $\mathcal{P}$. In Section 4 we will prove that $\mathcal{P}$ can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates.

The polygon $\mathcal{P}$ is constructed as follows. We start with a *basic rectangle* $[0,20]\times[0,4]\subset\mathbb{R}^2$. Then, we append to it six *triangular pockets* (colored with green in the figure), which are triangles defined by the following coordinates:

$$\begin{array}{llll} T_t^\ell: \{(2,4),(2,4.5),(2.1,4)\}, & & T_b^\ell: \{(2,0),(2,-0.5),(1.9,0)\}, \\ T_t^m: \{(16\tfrac{5}{6},4),(17\tfrac{2}{6},4.15),(17\tfrac{2}{6},4)\}, & & T_b^m: \{(3.5,0),(3,-0.15),(3,0)\}, \\ T_t^r: \{(19,4),(19,4.5),(19.1,4)\}, & \text{and} & T_b^r: \{(19,0),(19,-0.5),(18.9,0)\}. \end{array}$$

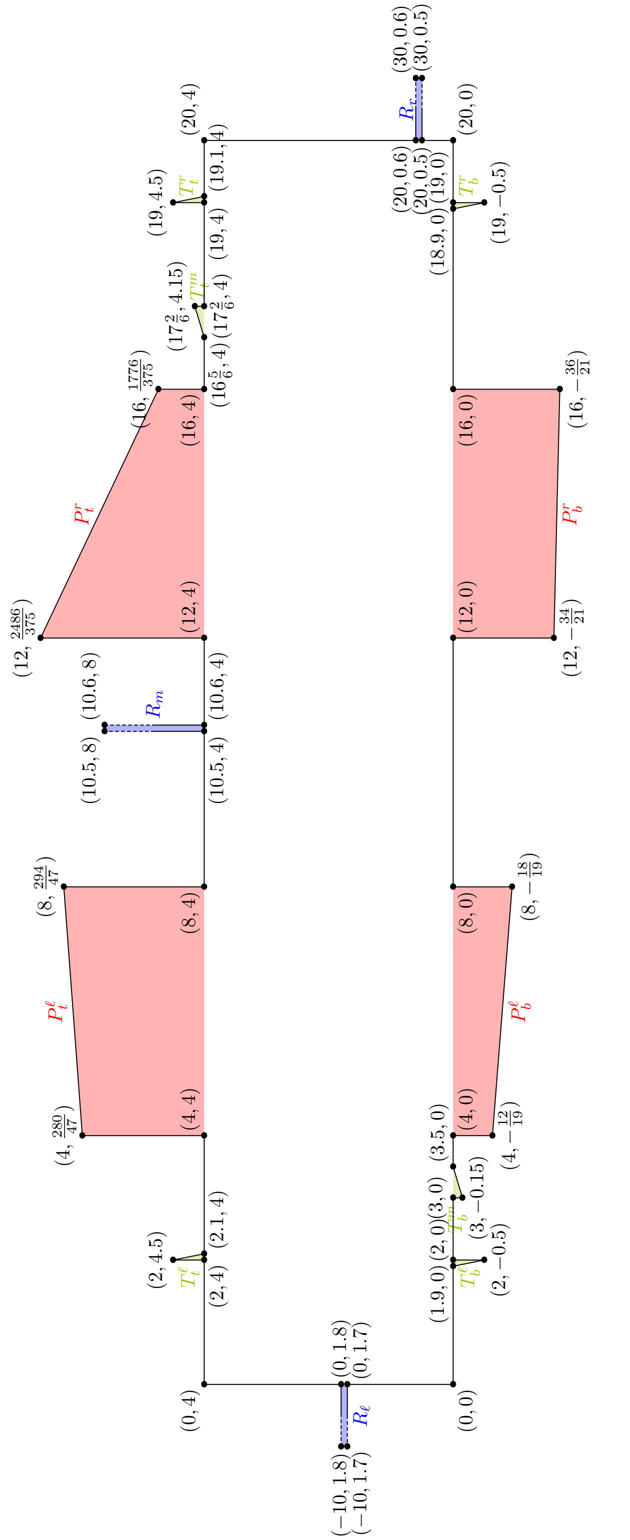
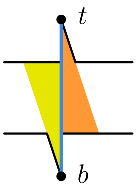
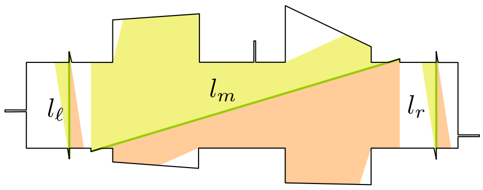


Figure 2 The polygon $\mathcal{P}$. We will show that $\mathcal{P}$ can be guarded by three guards only when we allow the guards to be placed at points with irrational coordinates. For practical reasons, the blue rectangular pockets are drawn shorter than they actually are.



(a) The only way that one guard can see both t and b is when the guard is on the blue line segment.



(b) The only way to guard the polygon with three guards requires one guard on each of the green line segments l_ℓ, l_m, l_r.

Figure 3 Forcing guards to lie on specific line segments.

Next, we append three *rectangular pockets* (colored with blue in the figure, for practical reasons these pockets are drawn in the figure shorter than they actually are), which are rectangles defined in the following way.

R_ℓ: $[-10, 0] \times [1.7, 1.8]$, R_r: $[20, 30] \times [0.5, 0.6]$, and R_m: $[10.5, 10.6] \times [4, 8]$.

Last, we append four *quadrilateral pockets* (colored with red in the figure), which are defined by points with the following coordinates:

Top-left pocket P_t^ℓ	$\{(4,4),$	$(4, \frac{280}{47}),$	$(8, \frac{294}{47}),$	$(8,4)\}$
Top-right pocket P_t^r	$\{(12,4),$	$(12, \frac{2486}{375}),$	$(16, \frac{1776}{375}),$	$(16,4)\}$
Bottom-left pocket P_b^ℓ	$\{(4,0),$	$(4, -\frac{12}{19}),$	$(8, -\frac{18}{19}),$	$(8,0)\}$
Bottom-right pocket P_b^r	$\{(12,0),$	$(12, -\frac{34}{21}),$	$(16, -\frac{36}{21}),$	$(16,0)\}.$

The polygon $\mathcal{P}$ is clearly monotone. We will denote by e_t^ℓ, e_t^r, e_b^ℓ, and e_b^r the non-axis-parallel edge within each of the four quadrilateral pockets, respectively.

3 Intuition

In this section, we explain the key ideas behind the construction of the polygon $\mathcal{P}$. Our presentation is informal, but it resembles the work process that lead to the construction of $\mathcal{P}$ more than the formal proof of Theorem 1 in Section 4 does. Here we omit all "scary" computations and focus on conveying the big picture. In the end of this section, we also explain how we actually constructed the polygon $\mathcal{P}$.

Define a *rational point* to be a point with two rational coordinates. An *irrational point* is a point that is not rational. A *rational line* is a line that contains two rational points. An *irrational line* is a line that is not rational.

Forcing a Guard on a Line Segment. Consider the drawing of the polygon $\mathcal{P}$ in Figure 2. We will now explain an idea of how three pairs of triangular pockets, (T_t^ℓ, T_b^ℓ), (T_t^m, T_b^m), and (T_t^r, T_b^r), can enforce three guards on three line segments within $\mathcal{P}$.

Consider the two triangular pockets in Figure 3a. The blue line segment contains one edge of each of these pockets, and the interiors of the pockets are at different sides of the line segment. A guard which sees the point t must be placed within the orange triangular region, and a guard which sees b must be placed within the yellow triangular region. Thus, a single guard can see both t and b only if it is on the blue line segment tb, which is the intersection of the two regions.

Consider now the case that we have k pairs of triangular pockets and no two regions corresponding to different pairs of pockets intersect. In order to guard the polygon with k guards, there must be one guard on the line segment corresponding to each pair. Our

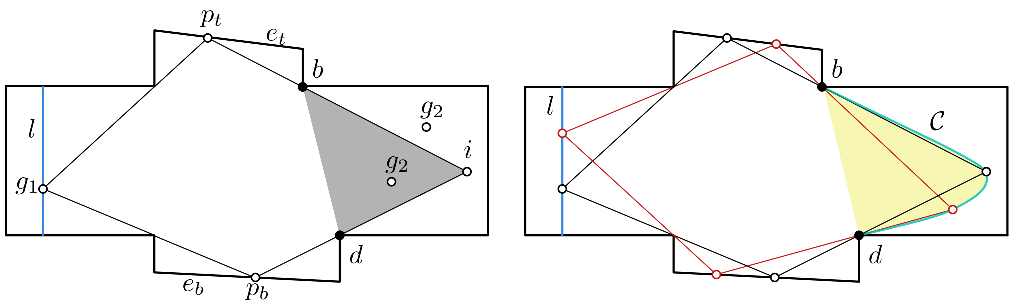


Figure 4 Left: The guard g_2 must be inside the triangular region (or to the left of it) in order to guard the entire part of the polygon that is not seen by g_1. Right: All possible positions of the point i define a simple curve $\mathcal{C}$.

polygon $\mathcal{P}$ has three such pairs of pockets (see Figure 3b), and it can be checked that the corresponding regions do not intersect. Note that in this way we can only enforce a guard to be on a rational line as the line contains vertices of the polygon, which are rational points.

Restricting a Guard to a Region Bounded by a Curve. For the following discussion, see Figure 4 and notation therein. We want to guard the polygon from Figure 4 using two guards, g_1 and g_2. We assume that g_1 is forced to lie on the blue vertical line segment l.

Consider some position of g_1 on l such that g_1 can see at least one point of the top edge e_t of the top quadrilateral pocket and at least one point of the bottom edge e_b of the bottom quadrilateral pocket. Let p_t and p_b denote the leftmost points seen by g_1 on e_t and e_b, respectively. Observe that p_t moves to the right if g_1 moves up and to the left if g_1 moves down. The point p_b behaves in the opposite way when g_1 is moved. Consider some fixed position of g_1 on the blue line segment, and the corresponding positions of p_t and p_b. Let b be the bottom right corner of the top pocket and d the top right corner of the bottom pocket. Let i be the intersection point of the line containing p_t and b with the line containing p_b and d. The points b, d, i define a triangular region Δ. It is clear that if we place the guard g_2 anywhere inside Δ, then g_1 and g_2 will together see the entire polygon. On the other hand, if we place g_2 to the right of Δ, then g_1 and g_2 will not see the entire polygon, as some part of the top or the bottom pocket will not be seen.

Now, let us move the guard g_1 along l. Each position of g_1 yields an intersection point i. We denote the union of all these intersection points by $\mathcal{C}$ (see the right picture in Figure 4). It is easy to see that $\mathcal{C}$ is a simple curve.

Note that g_2 sees a larger part of *both* pockets if it is moved horizontally to the left and a smaller part of *both* pockets if it is moved horizontally to the right. Consider a fixed position of g_2 on or to the right of the segment bd. Let g_2' be the horizontal projection of g_2 on $\mathcal{C}$. Let g_1 be the unique position on l such that g_1 and g_2' see all of the polygon. If g_2 is to the left of $\mathcal{C}$, g_2' sees less of the pockets than g_2, so g_1 and g_2 can together see everything. If g_2 is to the right of $\mathcal{C}$, g_2 sees less of the pockets than g_2' and neither the top nor the bottom pocket are completely guarded by g_1 and g_2. For any higher placement of g_1 even less of the top pocket is guarded and for any lower placement of g_1 even less of the bottom pocket is guarded. Thus, there exists no placement of g_1 such that both pockets are completely guarded by g_1 and g_2. We summarize our reasoning in the following observation.

▶ **Observation 4.** *Consider a fixed position of g_2 on or to the right of the segment bd. There exists a position of g_1 on l such that the entire polygon is seen by g_1 and g_2 if and only if g_2 lies on or to the left of the curve $\mathcal{C}$.*

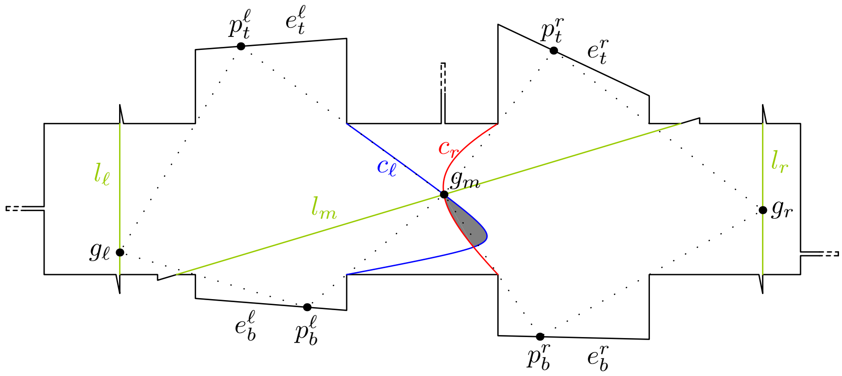


Figure 5 The polygon $\mathcal{P}$.

Restricting a Guard to a Single (Irrational) Point. For this paragraph, let us consider the polygon $\mathcal{P}$ introduced in Section 2, and consider a guard set for $\mathcal{P}$ consisting of three guards. The polygon $\mathcal{P}$ is drawn in Figure 5 with additional labels and information. The three guards g_ℓ, g_m, g_r are forced by the triangular pockets to lie on the three green line segments l_ℓ, l_m, l_r, respectively. Additionally, the three rectangular pockets R_ℓ, R_m, R_r force the guards to lie within one of two or three short intervals within each line segment. (These properties of our construction will be discussed in more detail in Section 4.) With these restrictions, we will show that for the three guards to see the whole polygon, it must hold that the guards g_ℓ and g_m can together see the left pockets P_t^ℓ and P_b^ℓ and the guards g_m and g_r can together see the right pockets P_t^r and P_b^r.

The curve c_ℓ bounds from the right the feasible region for the guard g_m such that g_ℓ and g_m can together see the left pockets P_t^ℓ and P_b^ℓ. Similarly, the curve c_r bounds from the left the feasible region for the guard g_m such that g_r and g_m can together see the right pockets P_t^r and P_b^r. Thus, the only way that g_ℓ, g_m, and g_r can see the whole polygon is when g_m is within the grey region between c_ℓ and c_r. Our idea is to define the line segment l_m so that it contains an intersection point of c_ℓ and c_r while not entering the interior of the grey region. A simple computation with sage [11] outputs equations defining the two curves:

$$c_\ell : 138x^2 - 568xy - 1071y^2 - 3018x + 8828y + 15312 = 0,$$

$$c_r : 138x^2 - 156xy - 356y^2 - 1791x + 3296y + 1620 = 0.$$

One can easily verify that the point $p = (3.5+5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12)$ lies on both curves and also on the line $l_m = \{\, (x, y) : y = 0.3x - 1.05 \,\}$. Therefore, p is a feasible (and at the same time irrational) position for the guard g_m. Moreover, by plotting c_ℓ, c_r, and l_m in $\mathcal{P}$ as in Figure 5, we get an indication that as we traverse l_m from left to right, at the point p we exit the area where g_m and g_l can guard together the two left pockets and at the same time we enter the area where g_m and g_r can guard together the two right pockets. Thus, the only feasible position for the guard g_m is the irrational point p. A formal proof will be given in Section 4.

Searching for the Polygon. The simplicity of the ideas behind our construction does not reflect the difficulty of finding the exact coordinates for the polygon $\mathcal{P}$. The reader might for instance presume that most other choices of horizontal pockets would work if the line segment l_m is changed accordingly. However, this is not the case.

It is easy to construct the pockets so that the corresponding curves c_ℓ and c_r intersect at some point p. We expect p to be an irrational point in general since the curves c_ℓ and c_r are defined by two second degree polynomials, as indicated above. In our construction, we need to force g_m to be on a line segment l_m containing p, but we can only force g_m to be on a rational line. Hence, we require the existence of a rational line that contains p.

As any two rational lines intersect in a rational point, there can be at most one rational line containing the irrational point p. Moreover, there exists a rational line containing p if and only if $p = (r_1 + r_2\alpha, r_3 + r_4\alpha)$ for some $r_1, r_2, r_3, r_4 \in \mathbb{Q}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number. The equation of the rational line containing p is then $y = \frac{r_4}{r_2} \cdot x + (r_3 - r_1 \cdot \frac{r_4}{r_2})$. We say that this line *supports* p. Therefore, we should not hope that the intersection point of the curves c_ℓ and c_r defined by arbitrarily chosen pockets will have a supporting line. Our main idea to overcome this problem has been to reverse-engineer the polygon, after having chosen the positions of the guards. We chose three irrational guards, all with supporting rational lines, and then defined the pockets so that g_m automatically became the intersection point between the curves c_ℓ and c_r associated with the pockets.

We chose all three guards to have coordinates of the form $(r_1 + r_2\sqrt{2}, r_3 + r_4\sqrt{2})$ for $r_1, r_2, r_3, r_4 \in \mathbb{Q}$. Assume, for the ease of presentation, that we already know that we can end up with a polygon described as follows. (In our initial attempts, our polygons were much less regular.) The polygon should consist of the rectangle $R = [0, 20] \times [0, 4]$ with some pockets added. We would like the pockets to extrude vertically from the horizontal edges of R such that the pockets meet R along the segments $(4, 0)(8, 0)$, $(12, 0)(16, 0)$, $(4, 4)(8, 4)$, and $(12, 4)(16, 4)$, respectively.

We now explain the technique for constructing the bottom pocket to the left which should extrude from R vertically downwards from the corners $(4, 0)$ and $(8, 0)$. We have to define the edge e_b^ℓ, which is the bottom edge in the pocket. We want p_b^ℓ to be a point on e_b^ℓ such that g_ℓ can only see the part of e_b^ℓ from p_b^ℓ and to the right, whereas g_m can only see the part of e_b^ℓ from p_b^ℓ and to the left. Therefore, we define p_b^ℓ to be the intersection point between the line containing g_ℓ and $(4, 0)$ and the line containing g_m and $(8, 0)$. It follows that p_b^ℓ is of the form $(r_1 + r_2\sqrt{2}, r_3 + r_4\sqrt{2})$ for some $r_1, r_2, r_3, r_4 \in \mathbb{Q}$. Hence, there is a unique rational line l supporting p_b^ℓ, and e_b^ℓ must be a segment on l. We therefore need that both of the points $(4, 0)$ and $(8, 0)$ are above l, since otherwise we do not get a meaningful polygon. However, this is not the case for arbitrary choices of the guards g_ℓ and g_m. The other pockets add similar restrictions to the positions of the guards.

In the construction we had to take care of other issues as well. In particular, the line l_m which supports the guard g_m cannot enter the grey region between the two curves c_ℓ and c_r, as otherwise the position of g_m would not be unique, and the guard could be moved to a rational point. Also, the three lines l_ℓ, l_m, l_r supporting the three guards g_ℓ, g_m, g_r cannot intersect within the polygon.

4 Proof of Theorems 1 and 2

Basic observations. Recall the construction of the polygon $\mathcal{P}$ as defined in Section 2, and consider a guard set of $\mathcal{P}$ of cardinality at most 3. Let l_ℓ, l_m, l_r, respectively, be the restrictions of the following lines to $\mathcal{P}$:

$$x = 2, \quad y = 0.3x - 1.05, \quad \text{and} \quad x = 19.$$

As argued in Section 3, the triangular pockets enforce a guard onto each of these lines.

▶ **Lemma 5.** *Consider any guard set S for $\mathcal{P}$ consisting of at most* 3 *guards. Then* (i) $|S| = 3$, *and* (ii) *there is one guard on each of the lines l_ℓ, l_m, l_r.*

Now, consider the intervals $i_1 = [0.5, 0.6]$ and $i_2 = [1.7, 1.8]$. Similarly as for the case of triangular pockets, we can show that the rectangular pockets R_ℓ, R_m, R_r enforce a guard with an x-coordinate in $[10.5, 10.6]$, and the two remaining guards with y-coordinates in i_1 and i_2, respectively.

▶ **Lemma 6.** *Consider any guard set for $\mathcal{P}$ consisting of* 3 *guards. Then one of the guards has an x-coordinate in* $[10.5, 10.6]$. *For the remaining two guards, one has a y-coordinate in i_1 and the other has one in i_2.*

Proof. From Lemma 5, there must be one guard g_ℓ on l_ℓ, one guard g_m on l_m, and the last guard g_r on l_r. Recall that the rectangular pockets are as follows R_ℓ: $[-10, 0] \times [1.7, 1.8]$, R_r: $[20, 30] \times [0.5, 0.6]$, and R_m: $[10.5, 10.6] \times [4, 8]$. It is straightforward to check that none of the guards g_ℓ, g_r can see the two top vertices of the pocket R_m. Therefore, the middle guard g_m has to see both of these vertices, so it must have an x-coordinate in $[10.5, 10.6]$.

Then, as $g_m \in l_m$, the y-coordinate of g_m is in $[2.1, 2.13]$. Therefore, g_m cannot see any of the left vertices of R_ℓ or any of the right vertices of R_r. These four vertices must be seen by the guards g_ℓ and g_r.

As some guard must see the bottom-left corner of the pocket R_ℓ, it must be placed at a height of at least 1.7. Then, this guard cannot see any of the right vertices of R_r. Therefore, the last guard must see both right vertices of R_r, and its height must be within $i_1 = [0.5, 0.6]$. Then, this guard cannot see any left vertex of the pocket R_ℓ, and the second guard must see both left vertices of the pocket, so its height must be within $i_2 = [1.7, 1.8]$. ◀

Dependencies between guard positions. Let $\{g_\ell, g_m, g_r\}$ be a guard set of $\mathcal{P}$ with $g_\ell \in l_\ell, g_m \in l_m$, and $g_r \in l_r$. We will now analyze dependencies between the positions of the guards that are caused by the quadrilateral pockets of $\mathcal{P}$. Recall that the non-axis-parallel edges of these pockets are denoted by e^ℓ_t, e^r_t, e^ℓ_b, and e^r_b.

We will first prove two technical lemmas.

▶ **Lemma 7.** *Let $h \in [0, 4]$ be the height of the guard g_ℓ. If $h > \frac{135}{47} \approx 2.87$ then g_ℓ cannot see any point on e^ℓ_t, and otherwise it can see a part of e^ℓ_t starting from the x-coordinate $\frac{908-188h}{181-47h}$ and to the right of it. If $h < \frac{9}{19} \approx 0.47$ then g_ℓ cannot see any point on e^ℓ_b, and otherwise it can see a part of e^ℓ_b starting from the x-coordinate $\frac{76h+12}{19h-3}$ and to the right of it.*

Proof. Consider the guard g_ℓ and the top-left pocket. The left-most point on e^ℓ_t that g_ℓ can see is at the intersection of the following two lines: the line containing g_ℓ and the bottom-left corner of the pocket (i.e., the point $(4, 4)$), and the line containing e^ℓ_t. If $g_\ell = (2, h)$, then the equation of the first line is $y = \frac{4-h}{2}x + (2h - 4)$. The second contains points $(4, \frac{280}{47})$ and $(8, \frac{294}{47})$, and its equation is $y = \frac{7}{94}x + \frac{266}{47}$. The x-coordinate of the intersection is $\frac{908-188h}{181-47h}$. It reaches a value of 8 (i.e., the point coincides with the right endpoint of e^ℓ_t) when $h = \frac{135}{47}$.

Now, consider the guard g_ℓ and the bottom-left pocket. The leftmost point on e^ℓ_b that g_ℓ can see is at the intersection of the following two lines: the line containing g_ℓ and the top-left corner of the pocket (i.e., the point $(4, 0)$), and the line containing e^ℓ_b. The first of these lines has equation $y = -\frac{h}{2}x + 2h$. The second line contains points $(4, -\frac{12}{19}), (8, -\frac{18}{19})$, and its equation is $y = -\frac{3}{38}x - \frac{6}{19}$. The x-coordinate of the intersection is $\frac{76h+12}{19h-3}$, which reaches 8 when $h = \frac{9}{19}$. ◀

▶ **Lemma 8.** *Let $h \in [0,4]$ be the height of the guard g_r. If $h > \frac{507}{250} = 2.028$ then g_ℓ cannot see any point on e_t^r, and otherwise it can see a part of e_t^r starting from the x-coordinate $\frac{4000h-9768}{250h-645}$ and to the left of it. If $h < \frac{17}{14} \approx 1.21$ then g_ℓ cannot see any point on e_b^r, and otherwise it can see a part of e_b^r starting from the x-coordinate $\frac{224h-56}{14h+1}$ and to the left of it.*

Proof. Consider the guard g_r and the top-right pocket. The right-most point on e_t^r that g_r can see is at the intersection of the following two lines: the line containing g_r and the bottom-right corner of the pocket (i.e., the point $(16,4)$), and the line containing e_t^r. If $g_r = (19,h)$, then the equation of the first line is $y = \frac{h-4}{3}x + \frac{76-16h}{3}$. The second contains points $(12, \frac{2486}{375})$ and $(16, \frac{1776}{375})$, and its equation is $y = -\frac{71}{150}x + \frac{4616}{375}$. The x-coordinate of the intersection is $\frac{4000h-9768}{250h-645}$. It reaches a value of 12 (i.e., the point coincides with the left endpoint of e_t^r) when $h = \frac{507}{250} = 2.028$.

Now, consider the guard g_r and the bottom-right pocket. The rightmost point on e_b^r that g_r can see is at the intersection of the following two lines: the line containing g_r and the top-right corner of the pocket (i.e., the point $(16,0)$), and the line containing e_b^r. The first of these lines has equation $y = \frac{h}{3}x - \frac{16h}{3}$. The second line contains points $(12, -\frac{34}{21}), (16, -\frac{36}{21})$, and its equation is $y = -\frac{1}{42}x - \frac{4}{3}$. The x-coordinate of the intersection is $\frac{224h-56}{14h+1}$, which reaches 12 when $h = \frac{17}{14} \approx 1.21$. ◀

We will now further restrict possible positions of the guards.

▶ **Lemma 9.** *The y-coordinate of the guard g_ℓ is in the interval $i_1 = [0.5, 0.6]$, and the y-coordinate of the guard g_r is in the interval $i_2 = [1.7, 1.8]$.*

Proof. As the guards g_ℓ and g_r lie on line segments l_ℓ and l_r, their x-coordinates are 2 and 19, respectively. From Lemma 6, the x-coordinate of g_m is in the interval $[10.5, 10.6]$. Also, one of the guards g_ℓ, g_r has a y-coordinate in i_1, and the other one in i_2.

Suppose that the y-coordinate of g_r is in i_1, i.e., it is at most 0.6. Let $v = (12, -\frac{34}{21})$ be the left endpoint of the edge e_b^r. We will show that none of the guards can see v. Clearly, as the x-coordinates of g_ℓ and g_m are smaller than 12, neither of them can see v. From Lemma 8, g_r cannot see v. Therefore, the y-coordinate of g_ℓ must be in i_1, and the y-coordinate of g_r in i_2. ◀

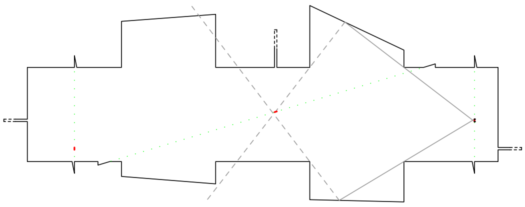
▶ **Lemma 10.** *The guards g_ℓ and g_m must together see all of e_t^ℓ and e_b^ℓ, and the guards g_m and g_r must together see all of e_t^r and e_b^r.*

Proof. By the construction of $\mathcal{P}$, it holds that if a guard sees a point on one of the edges e_t^ℓ, e_t^r, e_b^ℓ, and e_b^r, then the guard sees an interval of the edge containing an endpoint of the edge. It now follows that if three guards together see one of these edges, then two do as well. In order to prove the lemma, it thus suffices to prove that

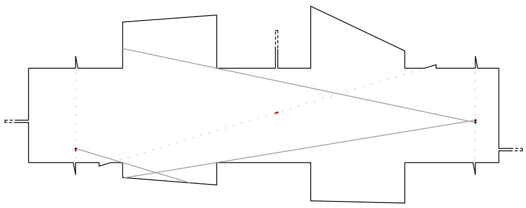
- g_ℓ and g_r cannot together see any of the edges e_t^ℓ, e_b^ℓ, e_t^r, and e_b^r,
- g_ℓ and g_m cannot together see any of the right edges e_t^r and e_b^r, and
- g_m and g_r cannot together see any of the left edges e_t^ℓ and e_b^ℓ.

We now prove that g_ℓ and g_r cannot together see any of the right edges e_t^r and e_b^r (see Figure 6a). Since $h \in i_2$, Lemma 8 gives that g_r cannot see e_t^r to the right of the point $(\frac{742}{55}, \frac{1629}{275})$, and e_b^r to the right of the point $(\frac{1736}{131}, -\frac{216}{131})$. It is now easy to verify that no point on l_ℓ can see any of these two points. Hence, g_ℓ and g_r cannot together see any of the edges e_t^r and e_b^r.

We now prove that g_ℓ and g_r cannot together see e_t^ℓ (see Figure 6b). Since the y-coordinate of g_r is in i_2, it follows that g_r does not see any point on e_t^ℓ. Since the x-coordinate of g_ℓ is less than 4, neither g_ℓ nor g_r can see the left endpoint of e_t^ℓ.



(a) Guards g_ℓ and g_r cannot together see any of the right pockets.



(b) Guards g_ℓ and g_r cannot together see any of the left pockets.

Figure 6 Showing that guards g_ℓ and g_r cannot see together a whole pocket. Possible positions for the guards are pictured in red.

To show that g_ℓ and g_r cannot together see the edge e_b^ℓ, we argue as follows (see Figure 6b). The guard g_ℓ is placed at a height of at most 0.6, and g_r at a height of at most 1.8. It follows from Lemma 7 and from elementary computations that neither of the guards can see the interval of e_b^ℓ with x-coordinates between $\frac{2076}{507} < 4.1$ and $\frac{48}{7} > 6.8$.

As the x-coordinate of both g_ℓ and g_m is smaller than 12, none of these guards can see the left endpoint of the edges e_t^r, e_b^r. Therefore, g_ℓ and g_m cannot together see any of the edges e_t^r, e_b^r. Similarly, as the x-coordinates of g_m and g_r are greater than 8, g_m and g_r cannot together see e_t^ℓ or e_b^ℓ. This completes our proof. ◀

Computing the unique solution. We can now show that there is only one guard set for $\mathcal{P}$ consisting of three guards. Let us start by computing the right-most possible position of g_m such that g_ℓ and g_m can see together both left pockets.

▶ **Lemma 11.** *The maximum x-coordinate of g_m such that g_ℓ and g_m can together see e_t^ℓ and e_b^ℓ is $x = 3.5 + 5\sqrt{2}$. The corresponding position of g_ℓ is $(2, 2 - \sqrt{2})$.*

Proof. Consider the guard g_ℓ at position $(2, h)$. From Lemma 9, we know that $h \in [0.5, 0.6]$. If g_m and g_ℓ together see e_t^ℓ, we know from Lemma 7 that g_m has to be on or below the line containing the vertices $(8, 4)$ and $(\frac{908-188h}{181-47h}, \frac{7}{94} \cdot \frac{908-188h}{181-47h} + \frac{266}{47})$, i.e., the line with equation $y = \frac{92-23h}{-135+47h}x + \frac{-1276+372h}{-135+47h}$. As g_m is at the line $y = 0.3x - 1.05$, its x-coordinate satisfies $0.3x - 1.05 \leq \frac{92-23h}{-135+47h}x + \frac{-1276+372h}{-135+47h}$, i.e., $x \leq \frac{28355-8427h}{2650-742h}$.

If g_m and g_ℓ together see e_b^ℓ, then g_m has to be on or above the line containing the vertices $(8, 0)$ and $(\frac{76h+12}{19h-3}, -\frac{3}{38} \cdot \frac{76h+12}{19h-3} - \frac{6}{19})$, i.e., the line with equation $y = \frac{3h}{19h-9}x - \frac{24h}{19h-9}$. Hence, the x-coordinate of g_ℓ must satisfy $0.3x - 1.05 \geq \frac{3h}{19h-9}x - \frac{24h}{19h-9}$, i.e., $x(1-h) \leq \frac{81h+189}{54}$. Therefore, since $h < 1$, we must have $x \leq \frac{81h+189}{54-54h}$.

We now know that $x \leq \min\{\frac{28355-8427h}{2650-742h}, \frac{81h+189}{54-54h}\}$. The first of the two values decreases with h, and the second one increases with h. Therefore the maximum is obtained when $\frac{28355-8427h}{2650-742h} = \frac{81h+189}{54-54h}$, i.e., for $h = 2 - \sqrt{2}$. The value of x is then $3.5 + 5\sqrt{2}$. The corresponding position of the guard g_ℓ is $(2, h) = (2, 2 - \sqrt{2})$. ◀

Similarly, we can compute the left-most possible position of g_m such that g_m and g_r can see together both right pockets.

▶ **Lemma 12.** *The minimum x-coordinate of g_m such that g_r and g_m can see both e_t^r and e_b^r is $x = 3.5 + 5\sqrt{2}$. The corresponding position of g_r is $(19, 1 + \frac{\sqrt{2}}{2})$.*

Proof. Consider the guard g_r at position $(19, h)$. From Lemma 9, we know that $h \in [1.7, 1.8]$. If g_m and g_r together see e_t^r, we know from Lemma 8 that g_m has to be on or below the

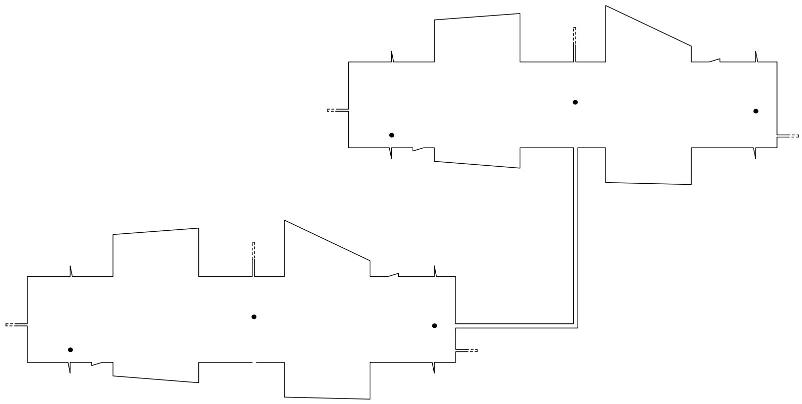


Figure 7 A sketch of a polygon that can be guarded by 6 guards when irrational coordinates are allowed, but needs 8 guards when only rational coordinates are allowed.

line containing the vertices $(12, 4)$ and $(\frac{4000h-9768}{250h-645}, -\frac{71}{150}\frac{4000h-9768}{250h-645} + \frac{4616}{375})$, i.e., the line with equation $y = \frac{46h-184}{250h-507}x + \frac{448h+180}{250h-507}$. As g_m is at the line $y = 0.3x - 1.05$, its x coordinate satisfies: $0.3x - 1.05 \leq \frac{46h-184}{250h-507}x + \frac{448h+180}{250h-507}$, i.e., $x \geq \frac{490h-243}{20h+22}$.

If g_m and g_r together see e^r_b, then g_m has to be on or above the line containing the vertices $(12, 0)$ and $(\frac{224h-56}{14h+1}, -\frac{1}{42}\frac{224h-56}{14h+1} - \frac{4}{3})$, i.e., the line with equation $y = \frac{6h}{17-14h}x - \frac{72h}{17-14h}$. Hence, the x-coordinate of g_r must satisfy $0.3x - 1.05 \geq \frac{6h}{17-14h}x - \frac{72h}{17-14h}$, i.e., $x \geq \frac{34h-7}{4h-2}$.

We have to minimize the value of $\max\{\frac{490h-243}{20h+22}, \frac{34h-7}{4h-2}\}$. When the value of h increases, the first of these two values increases, and the second one decreases. The minimum value is therefore obtained when $\frac{490h-243}{20h+22} = \frac{34h-7}{4h-2}$, i.e., for $h = 1 + \frac{\sqrt{2}}{2}$. The value of x is then $3.5 + 5\sqrt{2}$. ◀

We are now ready to prove our main theorems.

Proof of Theorem 1. Let $\mathcal{P}$ be the polygon constructed as in Section 2, and let S be a guard set for $\mathcal{P}$ consisting of at most 3 guards. From Lemma 5 we have $|S| = 3$, and there is one guard at each of the lines l_ℓ, l_m, l_r. Denote these guards by g_ℓ, g_m, g_r, respectively. From Lemma 10 we know that if g_ℓ, g_m, and g_r together see all of $\mathcal{P}$, then g_ℓ and g_m must see all of e^ℓ_t and e^ℓ_b, and g_m and g_r must see all of e^r_t and e^r_b. It then follows from Lemmas 11 and 12 that g_m must have coordinates $(3.5+5\sqrt{2}, 1.5\sqrt{2}) \approx (10.57, 2.12)$, $g_\ell = (2, 2-\sqrt{2}) \approx (2, 0.59)$, and $g_r = (19, 1 + \frac{\sqrt{2}}{2}) \approx (19, 1.71)$. Thus, indeed, the guards g_ℓ, g_m, and g_r see the entire polygon $\mathcal{P}$ and are the only three guards doing so.

By scaling $\mathcal{P}$ up by the least common multiple of the denominators in the coordinates of the corners of $\mathcal{P}$, we obtain a polygon with integer coordinates. This does not affect the number of guards required to see all of $\mathcal{P}$.

In order to guard $\mathcal{P}$ using four guards with rational coordinates, we choose two rational guards $g'_{m,1}$ and $g'_{m,2}$ on l_m a little bit to the left and to the right of g_m, respectively. The guard $g'_{m,1}$ sees a little more of both of the edges e^ℓ_t and e^ℓ_b than does g_m, whereas $g'_{m,2}$ sees a little more of e^r_t and e^r_b. Therefore, we can choose a rational guard g'_ℓ on l_ℓ close to g_ℓ such that g'_ℓ and $g'_{m,1}$ together see e^ℓ_t and e^ℓ_b, and a rational guard g'_r on l_r with analogous properties. Thus, $g'_\ell, g'_{m,1}, g'_{m,2}, g'_r$ guard $\mathcal{P}$. ◀

Proof of Theorem 2. We will now construct a polygon $\mathcal{P}_n$ that can be guarded by $3n$ guards placed at points with irrational coordinates, but such that when we restrict guard positions to points with rational coordinates, the minimum number of guards becomes $4n$. We start by making n copies of the polygon $\mathcal{P}$ described above, which we denote by $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n)}$. We connect the copies into one polygon $\mathcal{P}_n$ as follows. Each consecutive pair $\mathcal{P}^{(i)}, \mathcal{P}^{(i+1)}$ is connected by a thin corridor consisting of a horizontal piece $H^{(i)}$ visible by the rightmost guard in $\mathcal{P}^{(i)}$, and a vertical piece $V^{(i)}$ visible to the middle guard in $\mathcal{P}^{(i+1)}$ (see Figure 7 for the case $n = 2$). We can then guard $\mathcal{P}_n$ using $3n$ guards, by placing three guards within each polygon $\mathcal{P}^{(i)}$ in the same way as for $\mathcal{P}$, i.e., at irrational points.

Now, assume that $\mathcal{P}_n$ can be guarded by at most $4n-1$ guards. We will show that at least one guard must be irrational. For formal reasons, we define $H^{(0)} = V^{(0)} = H^{(n)} = V^{(n)} = \emptyset$. The horizontal and vertical corridors $H^{(i)}$ and $V^{(i)}$, for $i \in \{0, \ldots, n\}$, intersect at a rectangular area $B^{(i)} = H^{(i)} \cap V^{(i)}$ which we call a *bend*. For $i \in \{1, \ldots, n-1\}$, the bend $B^{(i)}$ is non-empty and visible from both polygons $\mathcal{P}^{(i)}$ and $\mathcal{P}^{(i+1)}$. Define the *extension* of $\mathcal{P}^{(i)}$, denoted by $E(\mathcal{P}^{(i)})$, to be the union of $\mathcal{P}^{(i)}$ and the adjacent corridors excluding the bends, i.e., $E(\mathcal{P}^{(i)}) = \mathcal{P}^{(i)} \cup (V^{(i-1)} \setminus B^{(i-1)}) \cup (H^{(i)} \setminus B^{(i)})$. Since the extensions are pairwise disjoint, there is an extension $E(\mathcal{P}^{(i)})$ containing at most three guards. If there are no guards in any of the bends $B^{(i-1)}, B^{(i)}$ it follows from Theorem 1 that three guards must be placed inside $\mathcal{P}^{(i)}$ at irrational coordinates, so assume that there is a guard in one or both of the bends. If the adjacent corridors $V^{(i-1)}$ and $H^{(i)}$ are long enough and thin enough, a guard in the bends $B^{(i-1)}$ and $B^{(i)}$ cannot see any of the convex corners of $\mathcal{P}^{(i)}$ in the rectangular pockets, any point in a triangular pocket, or any point in a quadrilateral pocket. Hence, all the features of $\mathcal{P}^{(i)}$ that enforce the irrationality of the guards are unseen by the guards in the bends and it follows that there must be irrational guards in $\mathcal{P}^{(i)}$. Therefore, at least $4n$ guards are needed if we require them to be rational. Similarly as in the proof of Theorem 1, we can show that $4n$ rational guards are enough to guard $\mathcal{P}_n$. ◀

5 Rectilinear Polygon

Figure 8 depicts a rectilinear polygon $\mathcal{P}_R$ with corners at rational coordinates that can be guarded by 9 guards, but requires 10 guards if we restrict the guards to points with rational coordinates. The construction of $\mathcal{P}_R$ starts with the polygon $\mathcal{P}$ from Theorem 1. We extend the non-rectilinear parts by "equivalent" rectilinear parts, colored gray in the figure. The rectilinear pockets are constructed in such a way that each of them requires at least one guard in the interior. Additionally, if the interior of each pocket contains only one guard, then these guards must be placed at specific positions, making the area not seen by these six additional guards exactly the polygon $\mathcal{P}$ described in Section 2 (the white area in Figure 8). Thus, the remaining 3 guards must be placed at three irrational points by Theorem 1.

6 Future Work

One of the most prominent open questions related to the art gallery problem is whether the problem is in NP. Recently, some researchers popularized an interesting complexity class, called $\exists\mathbb{R}$, being somewhere between NP and PSPACE [8, 24, 7, 19]. Many geometric problems for which membership in NP is uncertain have been shown to be complete for the complexity class $\exists\mathbb{R}$. Famous examples are: order type realizability, pseudoline stretchability, recognition of segment intersection graphs, recognition of unit disk intersection graphs, recognition of point visibility graphs, minimizing rectilinear crossing number, linkage realizability. This suggests that there might indeed be no polynomial sized witness for any of these problems as this would imply NP $= \exists\mathbb{R}$. It is an interesting open problem whether the art gallery problem is $\exists\mathbb{R}$-complete or not.

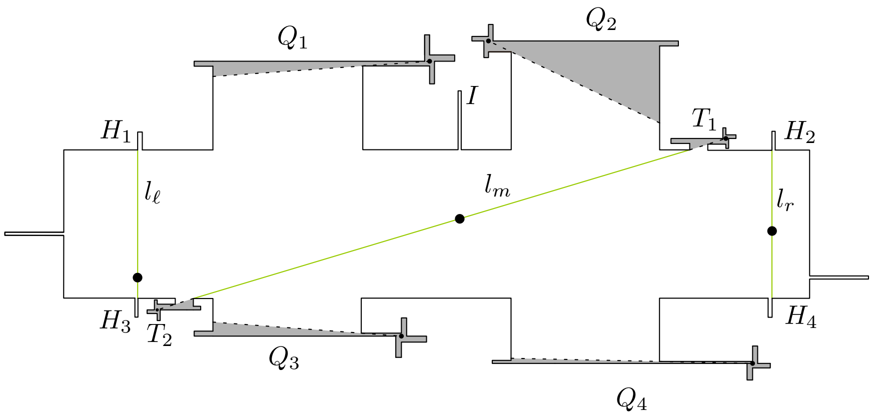


Figure 8 The rectilinear polygon $\mathcal{P}_R$ can be guarded with 9 guards only when we allow placing guards at irrational points.

The irrational coordinates of the guards in our examples are all of degree 2, i.e., they are roots in second-degree polynomials with integer coefficients. We would like to know if polygons exist where irrational numbers of higher degree are needed in the coordinates of an optimal solution.

We show that there exist polygons for which $|OPT_{\mathbb{Q}}| \geq \frac{4}{3}|OPT|$. It follows from the work by Bonnet and Miltzow [4] that it always holds that $|OPT_{\mathbb{Q}}| \leq 9|OPT|$. It is interesting to see if any of these bounds can be improved.

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