On Optimal 2- and 3-Planar Graphs*

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Abstract

A graph is k-planar if it can be drawn in the plane such that no edge is crossed more than k times. While for $k = 1$, *optimal* 1-planar graphs, i.e. those with n vertices and exactly $4n - 8$ edges, have been completely characterized, this has not been the case for $k \geq 2$. For $k = 2, 3$ and 4, upper bounds on the edge density have been developed for the case of simple graphs by Pach and Tóth, Pach et al. and Ackerman, which have been used to improve the well-known "Crossing Lemma". Recently, we proved that these bounds also apply to non-simple 2- and 3-planar graphs without homotopic parallel edges and self-loops.

In this paper, we completely characterize optimal 2- and 3-planar graphs, i.e., those that achieve the aforementioned upper bounds. We prove that they have a remarkably simple regular structure, although they might be non-simple. The new characterization allows us to develop notable insights concerning new inclusion relationships with other graph classes.

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1 Introduction

Topological graphs, i.e. graphs with a representation of the edges as Jordan arcs between corresponding vertex points in the plane, form a well-established subject in the field of geometric graph theory. Besides the classical problems on crossing numbers and crossing configurations [3, 19, 25], the well-known "Crossing Lemma" [2, 18] stands out as a prominent result. Researchers on graph drawing have followed a slightly different research direction, based on extensions of planar graphs that allow crossings in some restricted local configurations [7, 12, 15, 14, 17]. The main focus has been on *1-planar graphs*, where each edge can be crossed at most once, with early results dating back to Ringel [23] and Bodendiek et al. [8]. Extensive work on generation [24], characterization [16], recognition [11], coloring [9], page number [5], etc. has led to a very good understanding of properties of 1-planar graphs.

Pach and Tóth [22], Pach et al. [21] and Ackerman [1] bridged the two research directions by considering the more general class of k*-planar graphs*, where each edge is allowed to be

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crossed at most k times. In particular, Pach and Tóth provided significant progress, as they developed techniques for upper bounds on the number of edges of simple k-planar graphs, which subsequently led to upper bounds of $5n - 10$ [22], $5.5n - 11$ [21] and $6n - 12$ [1] for simple 2-, 3- and 4-planar graphs, respectively. An interesting consequence was the improvement of the leading constant in the "Crossing Lemma". Note that for general k, the current best bound on the number of edges is $4.1\sqrt{k}n$ [22].

Recently, we generalized the result and the bound of Pach et al. [21] to non-simple graphs, where non-homotopic parallel edges as well as non-homotopic self-loops are allowed [6]. Note that this non-simplicity extension is quite natural and not new, as for planar graphs, the density bound of $3n - 6$ still holds for such non-simple graphs.

In this paper, we now completely characterize optimal non-simple 2- and 3-planar graphs, i.e. those that achieve the bounds of $5n-10$ and $5.5n-11$ on the number of edges, respectively; refer to Theorems 9 and 17. In particular, we prove that the commonly known 2-planar graphs achieving the upper bound of $5n - 10$ edges, are in fact, the only optimal 2-planar graphs. Such graphs consist of a crossing-free subgraph where all not necessarily simple faces have size 5. At each face there are 5 more edges crossing in its interior. We correspondingly show that the optimal 3-planar graphs have a similar simple and regular structure where each planar face has size 6 and contains 8 additional crossing edges.

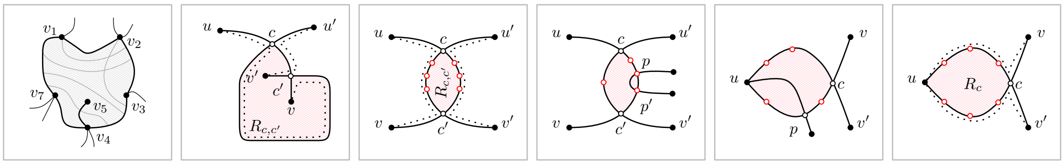
2 Preliminaries

Let G be a (not necessarily simple) *topological graph*, i.e. G is a graph drawn on the plane, so that the vertices of G are distinct points in the plane, its edges are Jordan curves joining the corresponding pairs of points, and:

(i) no edge passes through a vertex different from its endpoints,
(ii) no edge crosses itself and
(iii) no two edges meet tangentially.

Let $\Gamma(G)$ be such a drawing of G. The *crossing graph* $\mathcal{X}(G)$ of G has a vertex for each edge of G and two vertices of $\mathcal{X}(G)$ are connected by an edge if and only if the corresponding edges of G cross in $\Gamma(G)$. A connected component of $\mathcal{X}(G)$ is called *crossing component*. Note that the set of crossing components of $\mathcal{X}(G)$ defines a partition of the edges of G. For an edge e of G we denote by $\mathcal{X}(e)$ the crossing component of $\mathcal{X}(G)$ which contains e.

An edge e in $\Gamma(G)$ is called a *topological edge* (or simply *edge*, if it is clear in the context). Edge e is called *true-planar*, if it is not crossed in $\Gamma(G)$. The set of all true-planar edges of $\Gamma(G)$ forms the so-called *true-planar skeleton* of $\Gamma(G)$, which we denote by $\Pi(G)$. Since G is not necessarily simple, we will assume that $\Gamma(G)$ contains neither *homotopic parallel edges* nor *homotopic self-loops*, that is, both the interior and the exterior regions defined by any self-loop or by any pair of parallel edges contain at least one vertex. For a positive integer s, a cycle of length s is called *true-planar s-cycle* if it consists of true-planar edges of $\Gamma(G)$. If e is a true-planar edge, then $\mathcal{X}(e) = \{e\}$, while for a chord e of a true-planar s-cycle that has no vertices in its interior, it follows that all edges of $\mathcal{X}(e)$ are also chords of this s-cycle. Let $\mathcal{F}_s = \{v_1, v_2, \ldots, v_s\}$ be a facial s-cycle of $\Pi(G)$ with length $s \geq 3$. The order of the vertices (and subsequently the order of the edges) of $\mathcal{F}_s$ is determined by a walk along the boundary of $\mathcal{F}_s$ in clockwise direction. Since $\mathcal{F}_s$ is not necessarily simple, a vertex or an edge may appear more than once; see Fig. 1a. More general, a *region* in $\Gamma(G)$ is defined as a closed walk along non-intersecting segments of Jordan curves that are adjacent either at vertices or at crossing points of $\Gamma(G)$. The *interior* and the *exterior* of a connected region are defined as the topological regions to the right and to the left of the walk.



(a) **(b)** **(c)** **(d)** **(e)** **(f)**

Figure 1 (a) A non-simple face $\{v_1, \ldots, v_7\}$, where v_6 is identified with v_4. Different configurations used in (b–d) Lemma 1, and (e–f) Lemma 2.

If every edge in $\Gamma(G)$ is crossed at most k times, $\Gamma(G)$ is called *k-planar*. A graph is *k-planar* if it has a k-planar drawing. An *optimal k-planar* graph is a k-planar graph with the maximum number of edges. In particular, we consider optimal 2- and 3-planar graphs achieving the best-known upper bounds of $5n - 10$ and $5.5n - 11$ edges. A k-planar drawing $\Gamma(G)$ of an optimal k-planar graph G is called *planar-maximal crossing-minimal* or PMCM-drawing, if and only if $\Gamma(G)$ has the maximum number of true-planar edges among all k-planar drawings of G and, subject to this, $\Gamma(G)$ has the minimum number of crossings.

Consider two edges (u, v) and (u', v') that cross at least twice in $\Gamma(G)$. Let c and c' be two crossing points of (u, v) and (u', v') that appear consecutively along (u, v) in this order from u to v (i.e., there is no other crossing point of (u, v) and (u', v') between c and c'). W.l.o.g. we assume that c and c' appear in this order along (u', v') from u' to v' as well. In Figs. 1b and 1c we have drawn two possible crossing configurations. First we drew (u, v) as an arc with u above v and the edge-segment of (u', v') between u and c to the right of (u, v). The edge-segment of (u', v') between c and c', starts at c and ends at c' either from the right (Fig. 1b) or from the left (Fig. 1c) of (u, v), yielding two crossing configurations.

▶ **Lemma 1.** *For $k \in \{2, 3\}$, let $\Gamma(G)$ be a PMCM-drawing of an optimal k-planar graph G in which two edges (u, v) and (u', v') cross more than once. Let c and c' be two consecutive crossings of (u, v) and (u', v') along (u, v), and let $R_{c,c'}$ be the region defined by the walk along the edge segment of (u, v) from c to c' and the one of (u', v') from c' to c. Then, $R_{c,c'}$ has at least one vertex in its interior and one in its exterior.*

Proof. Consider first the crossing configuration of Fig. 1b. Since crossings c and c' are consecutive along (u, v) and (u', v') does not cross itself, it follows that vertex u' lies in the exterior of $R_{c,c'}$, while vertex v' in the interior of $R_{c,c'}$. Hence, the lemma holds. Consider now the crossing configuration of Fig. 1c. Since c and c' are consecutive along (u, v), vertices u' and v' are in the exterior of $R_{c,c'}$. Assume now, to the contrary, that $R_{c,c'}$ contains no vertices in its interior. W.l.o.g. we further assume that (u, v) and (u', v') is a *minimal crossing pair* in the sense that, $R_{c,c'}$ cannot contain another region $R_{p,p'}$ defined by any other pair of edges that cross twice; for a counterexample see Fig. 1d. Let $nc(u, v)$ and $nc(u', v')$ be the number of crossings along (u, v) and (u', v') that are between c and c', respectively (red in Fig. 1c). Observe that by the "minimality" criterion of (u, v) and (u', v') we have $nc(u, v) = nc(u', v')$. We redraw edges (u, v) and (u', v') by exchanging their segments between c and c' and eliminate both crossings c and c' without affecting the k-planarity of G; see the dotted edges of Fig. 1c. This contradicts the crossing minimality of $\Gamma(G)$. ◀

▶ **Lemma 2.** *For $k \in \{2, 3\}$, let $\Gamma(G)$ be a PMCM-drawing of an optimal k-planar graph G in which two edges (u, v) and (u, v') incident to a common vertex u cross. Let c be the first crossing of them starting from u and let R_c be the region defined by the walk along the edge* *segment of (u,v) from u to c and the one of (u,v') from c to u. Then, R_c has at least one vertex in its interior and one in its exterior.*

Proof Sketch. The proof is analogous to the one of Lemma 1; see Figs. 1e-1f and [20]. ◀

In our proofs by contradiction we usually deploy a strategy in which starting from an optimal 2- or 3-planar graph G, we modify G and its drawing $\Gamma(G)$ by adding and removing elements (vertices or edges) without affecting its 2- or 3-planarity. Then, the number of edges in the derived graph forces G to have either fewer or more edges than the ones required by optimality (contradicting the optimality or the 3-planarity of G, resp.). To deploy the strategy, we must ensure that we introduce neither homotopic parallel edges and self-loops nor self-crossing edges. We next show how to select and draw the newly inserted elements.

A Jordan curve $[u,v]$ connecting vertex u to v of $\Gamma(G)$ is called *potential edge* if and only if $[u,v]$ does not cross itself and is not a homotopic self-loop in $\Gamma(G)$, that is, either $u \neq v$ or $u = v$ and there is at least one vertex in the interior and the exterior of $[u,v]$. Note that u and v are not necessarily adjacent in G. However, since each topological edge $(u,v) \in E$ of G is represented by a Jordan curve in $\Gamma(G)$, it follows that edge (u,v) is by definition a potential edge of $\Gamma(G)$ among other potential edges that possibly exist. Furthermore, we say that vertices $v_1, v_2, \ldots, v_s$ define a *potential empty cycle* $\mathcal{C}_s$ in $\Gamma(G)$, if there exist potential edges $[v_i, v_{i+1}]$, for $i = 1, \ldots, s-1$ and potential edge $[v_1, v_s]$ of $\Gamma(G)$, which

- **(i)** do not cross each other and
- **(ii)** the walk along the curves between $v_1, v_2, \ldots, v_s, v_1$ defines a region in $\Gamma(G)$ that has no vertices in its interior.

Note that $\mathcal{C}_s$ might be non-simple.

▶ **Lemma 3.** *For $k \in \{2,3\}$, let $\Gamma(G)$ be a PMCM-drawing of a k-planar graph G. Let also $\mathcal{C}_s$ be a potential empty cycle of length s in $\Gamma(G)$ and assume that κ edges of $\Gamma(G)$ are drawn completely in the interior of $\mathcal{C}_s$, while λ edges of $\Gamma(G)$ are crossing[1] the boundary of $\mathcal{C}_s$. Also, assume that if one focuses on $\mathcal{C}_s$ of $\Gamma(G)$, then μ pairwise non-homotopic edges can be drawn as chords completely in the interior of $\mathcal{C}_s$ without deviating k-planarity.*

- **(i)** *If $\mu > \kappa + \lambda$, then G is not optimal.*
- **(ii)** *If G is optimal and $\mu = \kappa + \lambda$, then all boundary edges of $\mathcal{C}_s$ exist[2] in $\Gamma(G)$.*

Proof. (i) If we could replace the $\kappa + \lambda$ edges of $\Gamma(G)$ that are either drawn completely in the interior of $\mathcal{C}_s$ or cross the boundary of $\mathcal{C}_s$ with the μ ones that one can draw exclusively in the interior of $\mathcal{C}_s$, then the lemma would trivially follow. However, to do so we need to ensure that this operation introduces neither homotopic parallel edges nor homotopic self-loops. Since the edges that we introduce are potential edges, it follows that no homotopic self-loops are introduced. We claim that homotopic parallel edges are not introduced either. In fact, if e and e' are two homotopic parallel edges, then both must be drawn completely in the interior of $\mathcal{C}_s$, which implies that e and e' are both newly-introduced edges; a contradiction, since we introduce μ pairwise non-homotopic edges. (ii) In the exchanging scheme that we just described, we drew μ edges as chords exclusively in the interior of $\mathcal{C}_s$. Of course, one can also draw the boundary edges of $\mathcal{C}_s$. Since G is optimal, these edges must exist in $\Gamma(G)$. ◀

[1] Note that the boundary edges of $\mathcal{C}_s$ are not necessarily present in $\Gamma(G)$.

[2] We say that a Jordan curve $[u,v]$ *exists* in $\Gamma(G)$ if and only if $[u,v]$ is homotopic to an edge in $\Gamma(G)$.

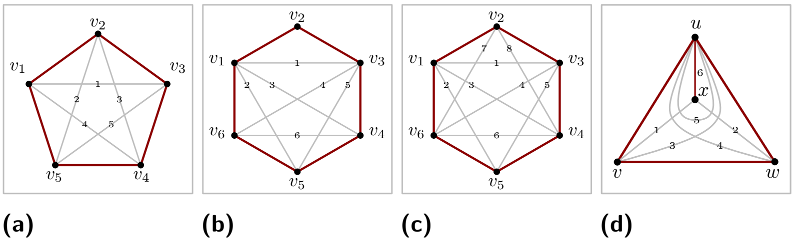


Figure 2 (a–c) A potential empty cycle $\mathcal{C}_s$ with (a) $s = 5$ and five chords with two crossings each, (b) $s = 6$ and six chords with at most two crossings each, and (c) $s = 6$ and eight chords with at most three crossings each. (d) Configuration used in the proof of Property 2.

3 Properties of optimal 2- and 3-planar graphs

In this section, we investigate properties of optimal 2- and 3-planar graphs. We prove that a PMCM-drawing $\Gamma(G)$ of an optimal 2- or 3-planar graph G can contain neither true-planar cycles of a certain length nor a pair of edges that cross twice. We use these properties to show that $\Gamma(G)$ is *quasi-planar*, i.e. it contains no 3 pairwise crossing edges.

Let R be a simple closed region that contains at least one vertex of G in its interior and one in its exterior. Let H_1 (H_2) be the subgraph of G whose vertices and edges are drawn entirely in the interior (exterior) of R. Note that potentially there exist edges that exit and enter R. We refer to H_1 and H_2 as the *compact subgraphs* of $\Gamma(G)$ defined by R. The following lemma bounds the number of edges in any compact subgraph of $\Gamma(G)$.

▶ **Property 1.** *Let $\Gamma(G)$ be a drawing of an optimal 2- or 3-planar graph G and let H be a compact subgraph of $\Gamma(G)$ on n' vertices defined by a closed region R. If $n' \geq 2$, H has at most $5n' - 6$ edges if G is optimal 2-planar, and at most $5.5n' - 6.5$ edges if G is optimal 3-planar. Further, there exists at least one edge of G crossing the boundary of R in $\Gamma(G)$.*

Proof. We prove this property for the class of 3-planar graphs; the proof for the class of 2-planar graphs is analogous. So, let $\Gamma(G)$ be a drawing of an optimal 3-planar graph $G = (V, E)$ with n vertices and m edges. Let H_1 and H_2 be two compact subgraphs of $\Gamma(G)$ defined by a closed region R. For $i = 1, 2$ let n_i and m_i be the number of vertices and edges of H_i. Suppose that $n_1 \geq 2$. In the absence of $\Gamma(H_2)$, drawing $\Gamma(H_1)$ might contain homotopic parallel edges or self-loops. To overcome this problem, we subdivide an edge-segment of the unbounded region of $\Gamma(H_1)$ by adding one vertex.[3] The derived graph, say H_1', has $n_1' = n_1 + 1$ vertices and $m_1' = m_1 + 1$ edges. Since H_1' has no homotopic parallel edges or self-loops and $n_1' \geq 3$, it follows that $m_1' \leq 5.5n_1' - 11$, which gives $m_1 \leq 5.5n_1 - 6.5$.

For the second part, assume to the contrary that no edge crosses the boundary of R. This implies that $m = m_1 + m_2$. We consider first the case where $n_1, n_2 \geq 2$. By the above we have that $m_1 \leq 5.5n_1 - 6.5$and $m_2 \leq 5.5n_2 - 6.5$. Since $n = n_1 + n_2$ and $m = m_1 + m_2$, it follows that $m \leq 5.5n - 13$; a contradiction to the optimality of G. Since a graph consisting only of two non-adjacent vertices cannot be optimal, it remains to consider the case where either $n_1 = 1$ or $n_2 = 1$. W.l.o.g. assume that $n_1 = 1$. Since $n_2 \geq 2$, it follows that $m_2 \leq 5.5n_2 - 6.5$, which implies $m \leq 5.5n - 12$; a contradiction to the optimality of G. ◀

For two compact subgraphs H_1 and H_2 defined by a closed region R, Property 1 implies that the drawings of H_1 and H_2 cannot be "separable". In other words, either there exists

[3] One can view this process as replacing $\Gamma(H_2)$ with a single vertex; thus no homotopic parallel edges exist in $\Gamma(H_1)$. Then we move this vertex towards the edge-segment we want to subdivide until it touches it.

an edge connecting a vertex of H_1 with a vertex of H_2, or there exists a pair of edges, one connecting vertices of H_1 and the other vertices of H_2, that cross in the drawing $\Gamma(G)$.

▶ **Property 2.** *In a PMCM-drawing $\Gamma(G)$ of an optimal 2-planar graph G there is no empty true-planar cycle of length three.*

Proof. Assume to the contrary that there exists an empty true-planar 3-cycle $\mathcal{C}$ in $\Gamma(G)$ on vertices u, v and w. Since G is connected and since all edges of $\mathcal{C}$ are true-planar, there is neither a vertex nor an edge-segment in $\mathcal{C}$, i.e., $\mathcal{C}$ is a chordless facial cycle of $\Pi(G)$. This allows us to add a vertex x in its interior and connect x to vertex u by a true-planar edge. Now vertices u, x, u, w and v define a potential empty cycle of length five, and we can draw five chords in its interior without violating 2-planarity and without introducing homotopic parallel edges or self-loops; refer to Fig. 2d. The derived graph G' has one more vertex than G and six more edges. Hence, if n and m are the number of vertices and edges of G respectively, then G' has $n' = n + 1$ vertices and $m' = m + 6$ edges. Then $m' = 5n' - 9$, which implies that G' has more edges than allowed; a contradiction. ◀

▶ **Property 3.** *The number of vertices of an optimal 3-planar graph G is even.*

Proof. Follows directly from the density bound of $5.5n - 11$ of G. ◀

▶ **Property 4.** *A PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph G has no true-planar cycle of odd length.*

Proof. Let $s \geq 1$ be an odd number and assume to the contrary that there exists a true-planar s-cycle $\mathcal{C}$ in $\Gamma(G)$. Denote by G_1 (G_2, resp.) the subgraph of G induced by the vertices of $\mathcal{C}$ and the vertices of G that are in the interior (exterior, resp.) of $\mathcal{C}$ in $\Gamma(G)$ without the chords of $\mathcal{C}$ that are in the exterior (interior, resp.) of $\mathcal{C}$ in $\Gamma(G)$. For $i = 1, 2$, observe that G_i contains a copy of $\mathcal{C}$. Let n_i and m_i be the number of vertices and edges of G_i that do not belong to $\mathcal{C}$. Based on graph G_i, we construct graph G'_i by employing two copies of G_i that share cycle $\mathcal{C}$. Observe that G'_i is 3-planar, because one copy of G_i can be embedded in the interior of $\mathcal{C}$, while the other one in its exterior. Hence, in this embedding, there exist neither homotopic self-loops nor homotopic parallel edges. Let n'_i and m'_i be the number of vertices and edges of G'_i that do not belong to $\mathcal{C}$. If G has n vertices and m edges, then by construction the following equalities hold:

(i) $n'_i = 2n_i + s$,
(ii) $m'_i = 2m_i + s$,
(iii) $n = n_1 + n_2 + s$, and
(iv) $m = m_1 + m_2 + s$.

We now claim that $n'_i \geq 3$. When $s \geq 3$ the claim clearly holds. Otherwise (i.e., $s = 1$), cycle $\mathcal{C}$ is degenerated to a self-loop which must contain at least one vertex in its interior and its exterior. Hence, the claim follows. Property 3 in conjunction with Eq. (i) implies that G'_i is not optimal, that is, $m'_i < 5.5n'_i - 11$. So, by Eq. (ii) it follows that $2m_i + s < 5.5(2n_i + s) - 11$. Summing up over i, we obtain that $2(m_1 + m_2 + s) < 5.5(2n_1 + 2n_2 + 2s) - 22$. Finally, from Eqs. (iii) and (iv) we conclude that $m < 5.5n - 11$; a contradiction to the optimality of G. ◀

▶ **Property 5.** *In a PMCM-drawing $\Gamma(G)$ of an optimal 2-planar graph G there is no pair of edges that cross twice with each other.*

Proof. Assume to the contrary that (u, u') and (v, v') cross twice in $\Gamma(G)$ at points c and c'. By 2-planarity no other edge of $\Gamma(G)$ crosses (u, u') and (v, v'). Let $R_{c,c'}$ be the region

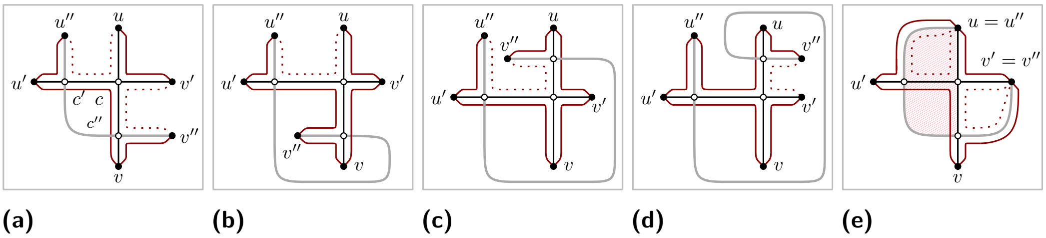


Figure 3 Crossing configurations for three mutually crossing edges. Potential edges are drawn solid red. Jordan curves that can either be potential edges or homotopic self-loops are drawn dotted red.

defined by the walk along the edge segment of (u, u') between c and c' and the edge segment of (v, v') between c' and c. As mentioned in the proof of Lemma 1, there exist two crossing configurations for (u, u') and (v, v'); see Figs. 1b and 1c. In the crossing configuration of Fig. 1b, vertices v and v' are in the interior of $R_{c,c'}$, while vertices u and u' in its exterior. Hence, $u \neq v$ and $u' \neq v'$ hold. We redraw (u, u') and (v, v') by exchanging the middle segments between c and c' and eliminate both crossings c and c' without affecting 2-planarity; see the dotted edges of Fig. 1b. Note that since $u \neq v$ and $u' \neq v'$ the two edges cannot be homotopic self-loops. Also, no homotopic parallel edges are introduced, since this would imply that at least one of the two edges already exists in $\Gamma(G)$ violating 2-planarity. Consider the crossing configuration of Fig. 1c. By Lemma 1, $R_{c,c'}$ has at least one vertex in its interior. By 2-planarity, no edge of G crosses the boundary of $R_{c,c'}$ contradicting Property 1. ◀

▶ **Property 6.** *In a PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph G there is no pair of edges that cross more than once with each other.*

Proof Sketch. As in the proof of Property 5, we show that if (u, v) and (u', v') cross three times, then either $\Gamma(G)$ is not crossing minimal or Property 1 is violated. The rest of the proof needs different arguments, as (u, v) and (u', v') may have one more crossing each; see [20]. ◀

Now assume that $\Gamma(G)$ contains three mutually crossing edges (u, v), (u', v') and (u'', v''). In Figs. 3a–3d we have drawn four the possible crossing configurations depending on the "direction" of the crossing of (u'', v'') along (u, v). Note that the endpoints of the three edges are not necessarily distinct (e.g., in Fig. 3e we illustrate the case where $u = u''$ and $v' = v''$ for the crossing configuration of Fig. 3a). For each crossing configuration, one can draw curves connecting the endpoints of (u, v), (u', v') and (u'', v'') (red colored in Figs. 3a–3d), which define a region that has no vertices in its interior. This region fully surrounds (u, v) and (u', v') and the two segments of (u'', v'') that are incident to vertices u'' and v''.

▶ **Claim 1.** *Each crossing configuration of Figs. 3b-3d induces at least 5 potential edges.*

Proof. All solid-drawn red curves of Figs. 3b–3d are indeed potential edges. ◀

▶ **Claim 2.** *The crossing configuration of Fig. 3a induces at least four potential edges.*

Proof. $[u', u'']$, $[u, v']$, $[u', v]$ and $[v, v'']$ are potential edges. ◀

▶ **Corollary 4.** *The configuration of Fig. 3a induces a potential empty cycle $\mathcal{C}$ of length ≥ 4. Each configuration of Figs. 3b–3d induces a potential empty cycle $\mathcal{C}$ of length ≥ 5.*

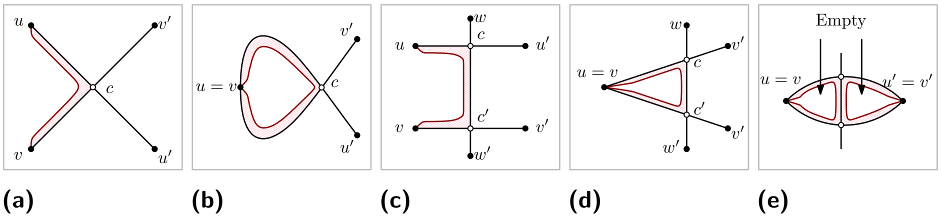


Figure 4 (a–b) vertices u and v form a corner pair; (c–d) vertices u and v form a side pair; (e) at least one of the two potential side-edges exists.

▶ **Claim 3.** *In the case where the crossing configuration of Fig. 3a induces exactly four potential edges, there exists at least one vertex in the interior of region $\mathcal{T}$ defined by the walk along the edge segment of (u, v) between c and c'', the edge segment of (u'', v'') between c'' and c' and the edge segment of (u', v') between c' and c.*

Proof. By Claim 2, $[u, u'']$, and $[v', v'']$ are homotopic self-loops. So, edges (u, v) and (u'', v'') are incident to a common vertex, namely, $u = u''$ and cross. By Lemma 2, $R_{c''}$ (red-shaded in Fig. 3e) has at least one vertex in its interior. Since $R_{c''}$ is the union of the interior of $\mathcal{T}$ and the homotopic self-loop $[u, u'']$, $\mathcal{T}$ contains at least one vertex in its interior. ◀

▶ **Property 7.** *A PMCM-drawing $\Gamma(G)$ of an optimal 2-planar graph G is quasi-planar.*

Proof. Assume to the contrary that (u, v), (u', v') and (u'', v'') mutually cross in $\Gamma(G)$; see Fig. 3. By Corollary 4, there is a potential empty cycle $\mathcal{C}$ of length at least 4. By 2-planarity, there is no other edge crossing (u, v), (u', v') or (u'', v''). So, the only edges that are drawn in the interior of $\mathcal{C}$ are (u, v) and (u', v'), while (u'', v'') is the only crossing the boundary of $\mathcal{C}$.

First, consider the case where $\mathcal{C}$ is of length ≥ 5. Since we can draw at least five chords completely in the interior of $\mathcal{C}$ as in Fig. 2a or 2b without violating its 2-planarity, it follows by Lemma 3.(i) (for $\kappa + \lambda = 3$ and $\mu \geq 5$) that G is not optimal; a contradiction. Finally, consider the case where $\mathcal{C}$ is of length four. In this case, we have the crossing configuration of Fig. 3a. By Claim 3 there is at least one vertex in the interior of region $\mathcal{T}$. More in general, let $G_{\mathcal{T}}$ be the compact subgraph of G that is completely drawn in the interior of region $\mathcal{T}$. Since edges (u, v), (u', v') and (u'', v'') have already two crossings, it follows that no edge of G crosses the boundary of $\mathcal{T}$ contradicting Property 1. ◀

▶ **Property 8.** *A PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph G is quasi-planar.*

Proof Sketch. This property is the analogue of Property 7 and its proof is given in [20]. ◀

Two not necessarily distinct vertices u and v of G form a *corner pair* if and only if an edge (u, u') crosses an edge (v, v') for some u' and v' in $\Gamma(G)$; see Fig. 4a. Let c be the crossing point of (u, u') and (v, v'). Then, a Jordan curve $[u, v]$ joining vertices u and v induces a region $R_{u,v}$ that is defined by the walk along the edge-segment of (u, u') from u to c, the edge segment of (v, v') from c to v and the curve $[u, v]$ from v to u. We call $[u, v]$ *corner edge* w.r.t. (u, u') and (v, v') if and only if $R_{u,v}$ has no vertices of $\Gamma(G)$ in its interior.

▶ **Property 9.** *In a PMCM-drawing $\Gamma(G)$ of an optimal k-planar graph G any corner edge $[u, v]$ is a potential edge.*

Proof. By the definition of potential edges, the property holds when $u \neq v$. Otherwise, $[u, v]$ is a self-loop; see Fig. 4b. If the property does not hold, then $[u, v]$ is a self-loop with no vertices either in its interior or in its exterior contradicting Lemma 2. ◀

We say that u and v form a *side pair* if and only if there exist edges (u, u') and (v, v') for some u' and v' such that they both cross a third edge (w, w') in $\Gamma(G)$ and $(u, u') \neq (v, v')$; see Figs. 4c–4d. Let c and c' be the crossing points of (u, u') and (v, v') with (w, w'). Assume w.l.o.g. that c and c' appear in this order along (w, w') from w to w'. Also assume that the edge-segment of (u, u') between u and c is on the same side of (w, w') as the edge-segment of (v, v') between v and c'; see Fig. 4c. Then, any Jordan curve $[u, v]$ joining u and v induces a region $R_{u,v}$ that is defined by the walk along the edge-segment of (u, u') from u to c, the edge segment of (w, w') from c to c', the edge segment of (v, v') from c' to v and the curve $[u, v]$ from v to u. We call $[u, v]$ *side-edge* w.r.t. (u, u') and (v, v') if and only if $R_{u,v}$ has no vertices of $\Gamma(G)$ in its interior. Since by Properties 7 and 8 edges (u, u') and (v, v') cannot cross with each other (as they both cross (w, w')), it follows that region $R_{u,v}$ is well-defined. Symmetrically we define region $R_{u',v'}$ and side-edge $[u', v']$ w.r.t. (u, u') and (v, v').

▶ **Property 10.** *In a PMCM-drawing $\Gamma(G)$ of an optimal k-planar graph G with $k \in \{2, 3\}$ at least one of the side-edges $[u, v]$, $[u', v']$ is a potential edge.*

Proof. Note that since edges (u, u'), (v, v') and (w, w') do not mutually cross, curves $[u, v]$ and $[u', v']$ cannot cross themselves. Assume to the contrary that neither $[u, v]$ nor $[u', v']$ are potential edges. This implies that $u = v$, $u' = v'$ and both $[u, v]$ and $[u', v']$ are self-loops that have no vertices in their interiors or their exteriors. Fig. 4e illustrates the case where both $[u, v]$ and $[u', v']$ are self-loops with no vertices in their interiors; the other cases are similar. It is not hard to see that (u, u') and (v, v') are homotopic side-edges; a contradiction. ◀

Edges (u, u') and (v, v') are called *side-apart* if and only if both side-edges $[u, v]$ and $[u', v']$ are potential edges.

4 Characterization of optimal 2-planar graphs

In this section we examine some more structural properties of optimal 2-planar graphs in order to derive their characterization (see Theorem 9).

▶ **Lemma 5.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 2-planar graph G. Any edge that is crossed twice in $\Gamma(G)$ is a chord of a true-planar 5-cycle in $\Gamma(G)$.*

Proof. Let (u, v) be an edge that is crossed twice in $\Gamma(G)$ by (u', v') and (u'', v'') at points c and c'. By Property 5 edges (u', v') and (u'', v'') are not identical. We assume w.l.o.g. that c and c' appear in this order along (u, v) from vertex u to vertex v. We also assume that the edge-segment of (u', v') between u' and c is on the same side of edge (u, v) as the edge-segment of (u'', v'') between u'' and c'; refer to Fig. 5a. By Property 9 corner edges $[u, u']$, $[u, v']$, $[v, u'']$ and $[v, v'']$ are potential edges. By Property 10 at least one of side-edges $[u', u'']$ and $[v', v'']$ is a potential edge. Assume w.l.o.g. that $[v', v'']$ is a potential edge.

If $[u', u'']$ is a potential edge, vertices u, v', v'', v, u'' and u' define a potential empty cycle $\mathcal{C}$ on six vertices (shaded in gray in Fig. 5b). Edges (u, v), (u', v') and (u'', v'') are drawn in the interior of $\mathcal{C}$, and there exist at most two other edges that cross (u', v') or (u'', v''). In total there exist at most five edges that have an edge-segment within $\mathcal{C}$. However, in the interior of $\mathcal{C}$ one can draw six chords as in Fig. 2b without deviating 2-planarity. By Lemma 3.(i) for $\kappa + \lambda \leq 5$ and $\mu = 6$, it follows that G is not optimal; a contradiction.

If $[u', u'']$ is not a potential edge, $[u', u'']$ is a homotopic self-loop and vertices u, v', v'', v and u' define a potential empty cycle $\mathcal{C}$ on five vertices (gray-shaded in Fig. 5c). In the interior of $\mathcal{C}$ one can draw five chords as in Fig. 2a without deviating 2-planarity. By Lemma 3.(ii), for $\kappa + \lambda \leq 5$ and $\mu = 5$, all boundary edges of $\mathcal{C}$ exist in $\Gamma(G)$ and $\kappa + \lambda = 5$.

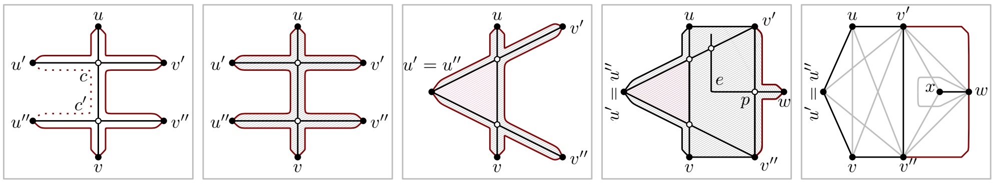


Figure 5 Different configurations used in Lemma 5.

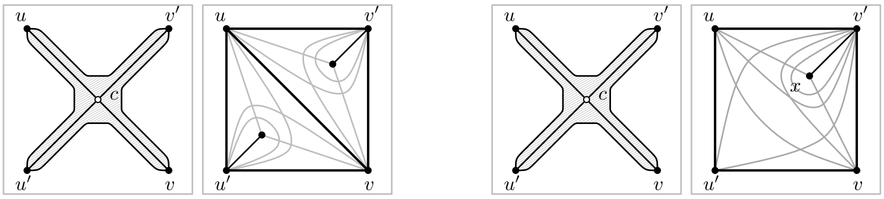


Figure 6 Different configurations used in: (a–b) Lemma 6, and (c–d) Lemma 14.

So, there exist two edges (other than (u, v)), say e and e', that cross (u', v') and (u'', v'') respectively.

If $\mathcal{C}$ is a true-planar 5-cycle in $\Gamma(G)$ the lemma holds. If not, e or e' crosses a boundary edge of $\mathcal{C}$. Suppose w.l.o.g. that e crosses (v', v'') of $\mathcal{C}$ at point p and let w and w' be the endpoints of e. Observe that e already has two crossings in $\Gamma(G)$. By 2-planarity, either the edge-segment of (w, w') between w and p or the one between w' and p is drawn completely in the exterior of $\mathcal{C}$. Suppose w.l.o.g. that this edge-segment is the former one. Then vertices v', w and v'' define a potential empty cycle $\mathcal{C}'$ on three vertices; see Fig. 5d. We proceed as follows: We remove edges (u, v), (u', v'), (u'', v''), e and e' and replace them with five chords drawn in the interior of $\mathcal{C}$ (as in Fig. 5e). The derived graph G' has the same number of edges as G. However, $\mathcal{C}'$ becomes a true-planar 3-cycle in G', contradicting Property 2. ◀

By Lemma 5, any edge of G that is crossed twice in $\Gamma(G)$ is a chord of a true-planar 5-cycle. The following lemma states that there are no edges with only one crossing in $\Gamma(G)$.

▶ **Lemma 6.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 2-planar graph G. Then, every edge of $\Gamma(G)$ is either true-planar or has exactly two crossings.*

Proof. As shown in Lemma 5, for an edge e that is crossed twice in $\Gamma(G)$, both edges that cross e also have two crossings in $\Gamma(G)$. So, the crossing component $\mathcal{X}(e)$ consists exclusively of edges with two crossings. This implies that if (u, v) and (u', v') cross in $\Gamma(G)$ and (u, v) has only one crossing, then the same holds for (u', v'). Vertices u, v', v and u' define a potential empty cycle $\mathcal{C}$ on four vertices (gray-shaded in Fig. 6a). Since (u, v) and (u', v') have only one crossing each, the boundary of $\mathcal{C}$ exists in $\Gamma(G)$ and are true-planar edges. We proceed by removing (u', v'). $\mathcal{C}$ is split into two true-planar 3-cycles; see Fig. 6b. In both of them, we plug the 2-planar pattern of Fig. 2d. In total, we removed one edge and added two vertices and 12 edges, without creating any homotopic parallel edges or self-loops. So, if G has n vertices and m edges, the derived graph G' is 2-planar and has $n' = n + 2$ vertices and $m' = m + 11 = 5n' - 9$, i.e., G' has more edges than allowed; a contradiction. ◀

▶ **Lemma 7.** *The true-planar skeleton $\Pi(G)$ of a PMCM-drawing $\Gamma(G)$ of an optimal 2-planar graph is connected.*

Proof. Assume to the contrary that $\Pi(G)$ is not connected. Let H be a connected component of $\Pi(G)$. By Property 1 either there exists an edge (u, v) with $u \in H$ and $v \in G \setminus H$, or two crossing edges $e_1 \in H$ and $e_2 \in G \setminus H$. In the first case, (u, v) is not true-planar. By Lemma 5, there exists a true-planar 5-cycle with (u, v) as chord; a contradiction. In the second case, e_1 and e_2 belong to the same crossing component and by Lemma 5, there exists a true-planar 5-cycle with e_1 and e_2 as chords; a contradiction. ◀

▶ **Lemma 8.** *The true-planar skeleton $\Pi(G)$ of a PMCM-drawing $\Gamma(G)$ of an optimal 2-planar graph G contains only faces of length 5, each containing 5 crossing edges.*

Proof. Since $\Pi(G)$ is connected (by Lemma 7), all its faces are connected. By Lemmas 5 and 6, all crossing edges are chords of true-planar 5-cycles. We claim that $\Pi(G)$ has no chordless faces. First, $\Pi(G)$ has no chordless face of size ≥ 4, as otherwise one could add in its interior a chord, contradicting the optimality of G. By Property 2, $\Pi(G)$ contains no faces of length 3. Faces of length 1 or 2 correspond to homotopic self-loops and parallel edges. ◀

We are now ready to state the main theorem of this section.

▶ **Theorem 9.** *A graph G is optimal 2-planar if and only if G admits a drawing $\Gamma(G)$ without homotopic parallel edges and self-loops, such that the true-planar skeleton $\Pi(G)$ of $\Gamma(G)$ spans all vertices of G, it contains only faces of length 5 (that are not necessarily simple), and each face of $\Pi(G)$ has 5 crossing edges in its interior in $\Gamma(G)$.*

Proof. For the forward direction, consider an optimal 2-planar graph G. By Lemma 8, the true-planar skeleton $\Pi(G)$ of its 2-planar PMCM-drawing $\Gamma(G)$ contains only faces of length 5 each containing 5 crossing edges in its interior. Since the endpoints of two crossing edges are within a true-planar 5-cycle (by Lemmas 5 and 6) and since $\Pi(G)$ is connected (by Lemma 7), $\Pi(G)$ spans all vertices of G. So, the proof of this direction is complete.

For the reverse direction, denote by n, m and f the number of vertices, edges and faces of $\Pi(G)$. Since $\Pi(G)$ spans all vertices of G, it suffices to prove that G has exactly $5n-10$ edges. The fact that $\Pi(G)$ contains only faces of length 5 implies that $5f = 2m$. By Euler's formula for planar graphs, we have $m = 5(n-2)/3$ and $f = 2(n-2)/3$. Since each face of $\Pi(G)$ contains exactly 5 crossing edges, the total number of edges of G equals $m+5f = 5n-10$. ◀

5 Characterization of optimal 3-planar graphs

In this section we explore structural properties of optimal 3-planar graphs. Following similar arguments as in Section 4 we derive their characterizations (see Theorem 17).

▶ **Lemma 10.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 3-planar graph G, and suppose that there exists a potential empty cycle $\mathcal{C}$ of 6 vertices in $\Gamma(G)$, such that the potential boundary edges of $\mathcal{C}$ exist in $\Gamma(G)$. Let $E_{\mathcal{C}}$ be the set of edge-segments within $\mathcal{C}$. If the following conditions C.1 and C.2 hold, then $\mathcal{C}$ is an empty true-planar 6-cycle in $\Gamma(G)$ and all edges with edge-segments in $E_{\mathcal{C}}$ are drawn as chords in its interior.*

(C.1) $|E_{\mathcal{C}}| \leq 8$, *and,*

(C.2) *every edge-segment of $E_{\mathcal{C}}$ has at least one crossing in the interior of $\mathcal{C}$.*

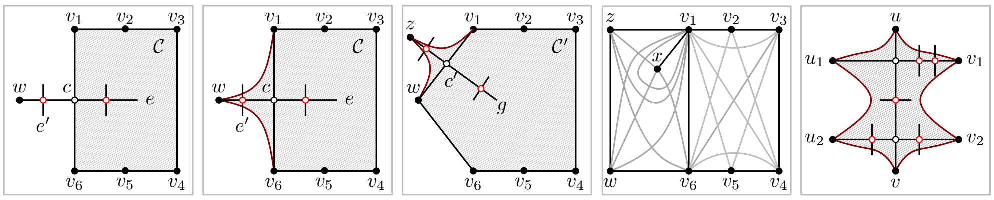


Figure 7 Different configurations used in (a–d) Lemma 10, (e) Lemma 11.

Proof Sketch. We start with the following observation: If e is an edge of G, then due to 3-planarity at most one edge-segment of e belongs to $E_{\mathcal{C}}$; if $E_{\mathcal{C}}$ contains at least two edge-segments of e, then we claim that e has at least four crossings. By C.2 each of the two edge-segments of e contributes one crossing to e. Since $\mathcal{C}$ is empty and contains two edge-segments of e, edge e exists and enters $\mathcal{C}$. Hence, e has two more crossings, summing up to a total of at least four.

Let $v_1, \ldots, v_6$ be the vertices of $\mathcal{C}$. If all edges with edge-segments in $E_{\mathcal{C}}$ completely lie in $\mathcal{C}$, then $\mathcal{C}$ is a true-planar 6-cycle and the lemma trivially holds. Otherwise, there is at least one edge e with an edge-segment in $E_{\mathcal{C}}$, that crosses a boundary edge of $\mathcal{C}$. W.l.o.g. we can assume that e crosses (v_1, v_6) of $\mathcal{C}$ at point c (refer to Fig. 7a). If w and w' are the two endpoints of e, then by the observation we made at the beginning of the proof it follows that either the edge-segment of (w, w') between w and c or the one between c and w' is drawn completely in the exterior of $\mathcal{C}$ (as otherwise e would have at least two edge-segments in $E_{\mathcal{C}}$). W.l.o.g. assume that this is the edge-segment between w and c. Then, corner edges $[v_1, w]$ and $[w, v_6]$ are potential edges (by Property 9).

Recall that e has one crossing in the interior of $\mathcal{C}$ (follows from C.2) and one more crossing with edge (v_1, v_6). By 3-planarity, e may have at most one more crossing, say with edge e'. Note that e' may or may not have an edge-segment in $E_{\mathcal{C}}$. Vertices w, v_1, $\ldots$, v_6 define a potential empty cycle $\mathcal{C}'$ on 7 vertices; see Fig. 7b. The set $E_{\mathcal{C}'}$ of edge-segments within $\mathcal{C}'$ contains all edge-segments of $E_{\mathcal{C}}$ (i.e., $E_{\mathcal{C}} \subseteq E_{\mathcal{C}'}$) plus at most two additional edge-segments: the one defined by edge (v_1, v_6), and possibly an edge-segment of e'. Hence $|E_{\mathcal{C}'}| \leq 10$. We now make some observations in the form of claims, which we formally prove in [20].

▶ **Claim 4.** *A PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph G is quasi-planar.*

▶ **Claim 5.** *At least one edge with an edge-segment in $E_{\mathcal{C}'}$ crosses one edge of $\mathcal{C}'$.*

By Claim 5, there is an edge g that crosses a boundary edge, say $[w, v_1]$, of $\mathcal{C}'$ at point c'; refer to Fig. 7c.

▶ **Claim 6.** *All boundary edges of $\mathcal{C}'$ exist in $\Gamma(G)$; g has one crossing in the interior of $\mathcal{C}'$.*

We follow an analogous approach to the one we used for expanding $\mathcal{C}$ (that has 6 vertices) to $\mathcal{C}'$ (that has 7 vertices). We can find an endpoint of g, say z, such that w, z, v_1, v_2, $\ldots$, v_6 define a potential empty cycle $\mathcal{C}''$ on 8 vertices. Furthermore, the set $E_{\mathcal{C}''}$ of edge-segments within $\mathcal{C}''$ has at most 12 elements (at most two more than $E_{\mathcal{C}'}$). We proceed by removing all edges with an edge-segment in $E_{\mathcal{C}''}$ and split $\mathcal{C}''$ into two true-planar cycles of length 6 and 4, by adding true-planar chord (v_1, v_6); see Fig. 7d. In the interior of the 6-cycle, we add 8 crossing edges as in Fig. 2c. In the interior of the 4-cycle, we add a vertex x with a true planar edge (v_1, x). Vertices v_1, x, v_1, v_6, w and z define a new potential empty cycle on 6 vertices, allowing us to add 8 more crossing edges. In total, we removed at most 12 edges, added a vertex and 18 edges. If n and m are the number of vertices and edges of G, then the derived graph G' has $n' = n + 1$ vertices and $m' \geq m + 6$ edges. The last equation gives $m' \geq 5.5n' - 10.5$, i.e. G' has more edges than allowed; a contradiction. ◀

Let (u, v) be an edge of G that is crossed by two edges (u_1, v_1) and (u_2, v_2) in $\Gamma(G)$ at points c_1 and c_2. By Property 6 edges (u_1, v_1) and (u_2, v_2) are not identical. We assume w.l.o.g. that c_1 and c_2 appear in this order along (u, v) from u to v. We also assume that the edge-segment of (u_1, v_1) between u_1 and c is on the same side of edge (u, v) as the edge-segment of (u_2, v_2) between u_2 and c_2; refer to Fig. 7e. Vertices u_1, u_2 and v_1, v_2 define two side pairs. By Property 10, at least one of side-edges $[u_1, u_2]$ and $[v_1, v_2]$ is a potential edge of $\Gamma(G)$. Recall that if both side-edges $[u_1, u_2]$ and $[v_1, v_2]$ are potential edges of $\Gamma(G)$, then edges (u_1, v_1) and (u_2, v_2) are called side-apart.

▶ **Lemma 11.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 3-planar graph G. If (u, v) is crossed by side-apart edges (u_1, v_1) and (u_2, v_2) in $\Gamma(G)$, then (u, v) is a chord of an empty true-planar 6-cycle.*

Proof. Refer to Fig. 7e. Since (u_1, v_1) and (u_2, v_2) are side-apart, side-edges $[u_1, u_2]$ and $[v_1, v_2]$ are potential edges. By Property 9, corner edges $[u, u_1]$, $[u, v_1]$, $[u, u_2]$ and $[v, v_2]$ are potential edges. Hence, vertices u, v_1, v_2, v, u_2 and u_1 define a potential empty cycle $\mathcal{C}$ on six vertices (gray-shaded in Fig. 7e). Edges (u, v), (u_1, v_1) and (u_2, v_2) are drawn completely in the interior of $\mathcal{C}$ and there exist at most five other edges either drawn in the interior of $\mathcal{C}$ or crossing its boundary: at most one that crosses (u, v), and at most four others that cross (u_1, v_1) and (u_2, v_2). Since we can draw eight chords in the interior of $\mathcal{C}$ as in Fig. 2c, by Lemma 3.(ii), for $\kappa + \lambda \leq 8$ and $\mu = 8$, all boundary edges of $\mathcal{C}$ exist in $\Gamma(G)$. Furthermore $\kappa + \lambda = 8$ must hold. Note that the set $E_{\mathcal{C}}$ of edge-segments within $\mathcal{C}$ contains only edge-segments of these $\kappa + \lambda$ edges. Also, these 8 edges have exactly one edge-segment within $\mathcal{C}$ that is crossed in the interior of $\mathcal{C}$. Hence, C.1 and C.2 of Lemma 10 are satisfied and there exists an empty true-planar 6-cycle that has (u, v) as chord. ◀

▶ **Lemma 12.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 3-planar graph G. If e is crossed by two side-apart edges in $\Gamma(G)$, $\mathcal{X}(e)$ consists of chords of an empty true-planar 6-cycle.*

Proof. The lemma follows by the observation that since e is a chord of an empty true-planar 6-cycle (by Lemma 11), all edges of $\mathcal{X}(e)$ are also chords of this 6-cycle. ◀

▶ **Lemma 13.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 3-planar graph G. Any edge that is crossed three times in $\Gamma(G)$ is a chord of an empty true-planar 6-cycle in $\Gamma(G)$.*

Proof Sketch. We argue that the preconditions C.1 and C.2 of Lemma 10 are fulfilled, which implies the presence of the empty true-planar 6-cycle in $\Gamma(G)$. For more details refer to [20]. ◀

We next consider edges of G that have two or fewer crossings in $\Gamma(G)$.

▶ **Lemma 14.** *Let $\Gamma(G)$ be a PMCM-drawing of an optimal 3-planar graph G and let $\mathcal{X}$ be a crossing component of $\Gamma(G)$. Then, there is at least one edge in $\mathcal{X}$ that has three crossings.*

Proof. Assume to the contrary that there exists a crossing component $\mathcal{X}$ where all edges have at most two crossings. Assume first that $\mathcal{X}$ does not contain an edge with two crossings. Then, $|\mathcal{X}| = 2$. W.l.o.g. assume that $\mathcal{X} = \{e, e'\}$. The four endpoints of e and e' define a potential empty cycle $\mathcal{C}$ on 4 vertices; see Fig. 6c. Since e and e' have only one crossing each, the boundary of $\mathcal{C}$ exist in $\Gamma(G)$ and is true-planar. Note that there are no other edges passing through the interior of $\mathcal{C}$. We proceed by removing e and e' and replace them with the 3-planar pattern of Fig. 6d, i.e., we add a vertex x in the interior of $\mathcal{C}$ and true-planar edge (v', x). Vertices u, v', x, v', v and u' define a potential empty cycle on six vertices, and we can add 8 edges in its interior as in Fig. 2c. If G has n vertices and m edges, the derived graph G' has $n' = n + 1$ vertices and $m' = m - 2 + 8$ edges. Then, G' is 3-planar and has $m' = 5.5n' - 10.5$ edges, i.e., G' has more edges than allowed by 3-planarity; a contradiction.

Assume now that there exists an edge $(u, v) \in \mathcal{X}$ which has two crossings with edges (u_1, v_1) and (u_2, v_2). By Lemma 11, (u_1, v_1) and (u_2, v_2) are not side-apart. Since all edges in $\mathcal{X}$ have at most two crossings, adopting the proof of Lemma 5 we can prove that the endpoints of (u, v), (u', v') and (u'', v'') define a potential empty cycle $\mathcal{C}$ on five vertices, with at most five edges passing through its interior. We proceed by redrawing these five edges as chords of $\mathcal{C}$ (as in Fig. 2a) and all its boundary edges are true-planar in the new drawing. The derived graph is optimal, since it has at least as many edges as G. Observe, however, that $\mathcal{C}$ becomes a true-planar 5-cycle in the new drawing; a contradiction to Property 3. ◀

The proofs of Lemmas 15 and 16 are similar to the ones of Lemmas 7 and 8; see also [20].

▶ **Lemma 15.** *The true planar skeleton $\Pi(G)$ of a PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph is connected.*

▶ **Lemma 16.** *The true-planar skeleton $\Pi(G)$ of a PMCM-drawing $\Gamma(G)$ of an optimal 3-planar graph G contains only faces of length 6, each containing 8 crossing edges in $\Gamma(G)$.*

We say that a chord of a $2s$-cycle is a *middle chord* if the two paths along the cycle connecting its endpoints both have length s. We now state the main theorem of this section.

▶ **Theorem 17.** *A graph G is optimal 3-planar if and only if G admits a drawing $\Gamma(G)$ without homotopic parallel edges and self-loops, such that the true-planar skeleton $\Pi(G)$ of $\Gamma(G)$ spans all vertices of G, it contains only faces of length 6 (that are not necessarily simple), and each face of $\Pi(G)$ has 8 crossing edges in its interior in $\Gamma(G)$ such that one of the middle chords is missing.*

Proof. For the forward direction, consider an optimal 3-planar graph G. By Lemma 16, the true-planar skeleton $\Pi(G)$ of its 3-planar PMCM-drawing $\Gamma(G)$ contains only faces of length 6, each of which has 8 edges in its interior in $\Gamma(G)$. By Property 8, one of the three middle chords of each face of $\Pi(G)$ cannot be present. Since the endpoints of two crossing edges are within a true-planar 6-cycle (by Lemmas 13 and 14) and since $\Pi(G)$ is connected (by Lemma 15), $\Pi(G)$ spans all vertices of G, which completes the proof of this direction.

For the reverse direction, denote by n, m and f the number of vertices, edges and faces of $\Pi(G)$. Since $\Pi(G)$ spans all vertices of G, it suffices to prove that G has exactly $5.5n - 11$ edges. The fact that $\Pi(G)$ contains only faces of length 6 implies that $6f = 2m$. By Euler's formula, we have $m = 3(n-2)/2$ and $f = (n-2)/2$. Since each face of $\Pi(G)$ contains exactly 8 crossing edges, the total number of edge of G equals to $m + 8f = 5.5n - 11$. ◀

6 Further Insights and Open Problems

The following corollaries are consequences of our new characterizations; for details see [20]. The definitions of bar 1-visibility and fan-planarity can be found in [13, 10] and [17, 7].

▶ **Corollary 18.** *Simple 3-planar graphs have at most $5.5n - 11.5$ edges.*

▶ **Corollary 19.** *Simple optimal 2-planar graphs admit bar 1-visibility representations.*

▶ **Corollary 20.** *Simple optimal* 2*-planar graphs are optimal fan-planar.*

Our characterizations naturally lead to many open questions; we only name a few.

- What is the complexity of the recognition problem for optimal 2- and 3-planar graphs?
- What is the exact upper bound on the number of edges of simple optimal 3-planar graphs? We conjecture that they do not have more than $5.5n - 15$ edges.
- Theorems 9 and 17 imply that optimal 2- and 3-planar graphs have a fully triangulated planar subgraph. Can this property be proved for optimal 4-planar or more in general for optimal k-planar graphs? Proving this property would be useful to derive better density bounds for $k \geq 4$.
- By Properties 7 and 8, optimal 2- and 3-planar graphs are quasi-planar. Angelini et al. [4] proved that every simple k-planar graph is $(k+1)$-quasi planar for $k \geq 3$ (i.e., it can be drawn with no $k+1$ pairwise crossing edges). Our results about optimal 2-planar and even more about optimal 3-planar graphs give indications that the result by Angelini et al. [4] may hold also for $k = 2$.
- We found a RAC drawing (i.e., a drawing in which all crossing edges form right angles) with at most one bend per edge for the optimal 2-planar graph having the dodecahedron as its true-planar structure. Is this generalizable to all simple optimal 2-planar graphs?

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