The Quest for Strong Inapproximability Results with Perfect Completeness[*][†]

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────────── **Abstract** ──────────

The Unique Games Conjecture (UGC) has pinned down the approximability of all constraint satisfaction problems (CSPs), showing that a natural semidefinite programming relaxation offers the optimal worst-case approximation ratio for any CSP. This elegant picture, however, does not apply for CSP instances that are perfectly satisfiable, due to the imperfect completeness inherent in the UGC. For the important case when the input CSP instance admits a satisfying assignment, it therefore remains wide open to understand how well it can be approximated.

This work is motivated by the pursuit of a better understanding of the inapproximability of perfectly satisfiable instances of CSPs. Our main conceptual contribution is the formulation of a (hypergraph) version of Label Cover which we call "V label cover." Assuming a conjecture concerning the inapproximability of V label cover on perfectly satisfiable instances, we prove the following implications:

- There is an absolute constant c_0 such that for $k \geq 3$, given a satisfiable instance of Boolean k-CSP, it is hard to find an assignment satisfying more than $c_0 k^2/2^k$ fraction of the constraints.
- Given a k-uniform hypergraph, $k \geq 2$, for all $\epsilon > 0$, it is hard to tell if it is q-strongly colorable or has no independent set with an ϵ fraction of vertices, where $q = \lceil k + \sqrt{k} - \frac{1}{2} \rceil$.
- Given a k-uniform hypergraph, $k \geq 3$, for all $\epsilon > 0$, it is hard to tell if it is $(k-1)$-rainbow colorable or has no independent set with an ϵ fraction of vertices.

We further supplement the above results with a proof that an "almost Unique" version of Label Cover can be approximated within a constant factor on satisfiable instances.

1   Introduction

The sustained progress on approximation algorithms and inapproximability results for optimization problems since the early 1990s has been nothing short of extraordinary. This has led to a sharp understanding of the approximability threshold of many fundamental problems, alongside the development of a rich body of techniques on the algorithmic, hardness,

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and mathematical programming aspects of approximate optimization. Yet there also remain many problems which have resisted resolution and for some there are in fact large gaps between the known algorithmic and hardness results. Examples include vertex cover, graph coloring, max-cut, feedback vertex set, undirected multicut, densest subgraph, and so on.

The Unique Games Conjecture of Khot [38] postulates a strong inapproximability result for a particular class of arity two constraint satisfaction problems. This single assumption has a remarkable array of consequences, and implies *tight* inapproximability results for numerous problems including Vertex Cover [41], max-cut and indeed all constraint satisfaction problems (CSPs) [39, 49, 52], maximum acyclic subgraph and all ordering CSPs [20], scheduling problems [4, 3], graph pricing [44], and cut problems like directed multicut [45], to name a few. Furthermore, for CSPs, the UGC implies that a standard semidefinite programming relaxation gives the best approximation ratio [52, 53, 8].

While the UGC has identified a common barrier against progress on a host of approximation problems, there are still several situations it does not apply to. Crucially, imperfect completeness, where Yes instances are only almost satisfiable, is inherent in the UGC, and this feature is inherited by the problems it reduces to. In particular, the UGC does not say *anything* about problems with *perfect completeness*, where Yes instances have a perfect solution obeying all the constraints. Important classes of such problems include satisfiable instances of CSPs (which have a perfect satisfying assignment and the goal is maximize the number of satisfied constraints) and coloring graphs/hypergraphs with approximately optimal number of colors.

Our understanding of approximating satisfiable instances of CSPs still has many gaps. Håstad's tight hardness result for approximating Max 3-SAT on satisfiable instances was much harder to prove than the analogous result for near-satisfiable instances, and was an early sign of the subtleties of ensuring perfect completeness. The approximability of satisfiable CSPs corresponds via a direct translation to the power of probabilistically checkable proof (PCP) systems with perfect completeness – the best soundness error one can achieve with a k query (non-adaptive) PCP is equal to the best inapproximability factor one can prove for a satisfiable arity k CSP. For $k = 3$, the best soundness is $5/8 + \epsilon$ for any $\epsilon > 0$, and this was established only recently via an intricate proof of the approximation resistance of satisfiable NTW (the arity 3 No-Two predicate which stipulates the number of true literals must be either $0, 1$ or 3) [35]. As a basic open question that still remains wide open, we do not know the approximability of satisfiable Max NAE-3-SAT (not-all-equal 3-SAT) under any plausible (or even not so plausible!) conjecture.

The above-mentioned Unique Games hardness results consist of two components: (i) a dictatorship test that gives a way to test if a function is a dictator or is far from a dictator (e.g., has no influential coordinates), using constraints corresponding to the problem at hand (for NAE-3-SAT this would be checking if certain triples of function values are not all equal), and (ii) a reduction from Unique Games via the dictatorship test that establishes inapproximability under the UGC. The second step is standard, and it gives a "free pass" from the world of combinatorics/analysis of Boolean functions to the complexity world. When we require perfect completeness, no such conjectured off-the-shelf compiler from dictatorship tests to hardness is known (and such a passage even appears unlikely). For instance, dictatorship tests with perfect completeness and optimal soundness are known or Max k-CSP [59] and Max NAE-3-SAT [1]. However, in both cases we do not have matching inapproximability results under any plausible conjecture.

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[1] Folklore, and this has connections to robust forms of Arrow's theorem [36] and [50, Sec. 4].

The closest to a UGC surrogate in the literature is the d-to-1 conjecture also made in [38]. The Unique Games problem is an arity 2 CSP whose constraints are bijections; the d-to-1 Label Cover is an arity 2 CSP whose constraints are d-to-1 functions. When $d \geq 2$, deciding satisfiability of a d-to-1 Label Cover instance is NP-complete, unlike Unique Games whose satisfiability is trivial to ascertain. Khot's d-to-1 conjecture states that d-to-1 Label Cover is also hard to approximate within any constant factor, even on satisfiable instances. Note that the UGC and d-to-1 conjecture are incomparable in strength; the UGC has simpler bijective constraints but the d-to-1 conjecture asserts perfect completeness which the UGC cannot.

The d-to-1 conjecture has been used to show some strong inapproximability results with perfect completeness. Such applications are, however, sporadic and also typically do not yield tight results. Some of these results are conditioned specifically on the 2-to-1 conjecture, such as a $\sqrt{2} - \epsilon$ inapproximability for vertex cover (mentioned in [38] and explicit in [40]), Max k-coloring with perfect completeness [27], and coloring 4-colorable graphs [13]. The d-to-1 conjecture, for any fixed d, has been used to show the approximation resistance of NTW [51] and a similar result for larger arity [32],[2] and finding independent sets in 2-colorable 3-uniform hypergraphs [43]. Yet, the implications of the d-to-1 conjecture are limited, and it has become apparent that it is not a versatile starting point for hardness results with perfect completeness.

1.1 Our contributions

Given the above context, our work is motivated by the quest for a better starting point than 2-to-1 Label Cover for inapproximability results with perfect completeness, and which might be able to give striking consequences similar to the UGC.

Aggressive Unique Games variant. One version of Label Cover that is most similar to Unique Games, which we call (L, s)-nearly unique Label Cover, has constraint relations in[3] $[L] \times [L]$ consisting of a matching and s additional edges, for a small s that is a constant independent of L. For this version, it is NP-hard to decide satisfiability, and in fact one can give strong reductions matching the performance of dictatorship tests from it. However, this nearly unique form of Label Cover has a constant factor approximation algorithm with ratio depending only on s. We prove this result in the full version of the paper.

V label cover. Our main conceptual contribution is the formulation of a (hypergraph) version of Label Cover which we call "V label cover." This is an extension of 2-to-1 Label Cover, where the constraint predicates are 2-to-1 maps from $[2L]$ to $[L]$, whose relation graph can be visualized as L disjoint "V's." In V label cover of arity k, we have "longer V's" where the two branches involve k variables which coincide in single variable.[4] This is best illustrated by Figure 1 in Section 3.

We put forth the *V label cover conjecture*, which asserts a strong inapproximability result for this problem. For completeness, we want an assignment where for every constraint, the k variables involved get values in a single "V-branch." For soundness, we insist that no assignment even *weakly satisfies* more than a tiny fraction of constraints, where a constraint

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[2] These were later improved to NP-hardness in [35] and [63].
[3] We denote $[L] = \{1, \ldots, L\}$.
[4] We should mention that our path to the formulation of V label cover was more circuitous, and has its origins in attempts to define hypergraph versions of the "α Label Cover" problem of [13].

is weakly satisfied if two of its k variables get values in some V-branch. [5] For this to make sense, the "junction" of the V's cannot all be on the same variable (as in 2-to-1 Label Cover), as in that case we will have a Unique Label Cover constraint between the other $(k-1)$ variables, which we can perfectly satisfy. Therefore, in our V label cover constraints, we have V's with junctions at all the k variables involved in the constraint. At a high level, this is similar to the correlation-breaking constraints of Chan [10].

Near-optimal inapproximability for Max k-CSP with perfect completeness. Assuming the V label cover conjecture, we prove a near-tight inapproximability result for approximating satisfiable Max k-CSP over any fixed domain.

▶ **Theorem 1.1.** *Assume the V label cover conjecture. There is an absolute constant c_0 such that for $k \geq 3$, given a satisfiable instance of Boolean k-CSP, it is hard to find an assignment satisfying more than $c_0 k^2/2^k$ fraction of the constraints. For CSP over domain size $q \geq 3$, where q is a prime power, it is hard to satisfy more than $c_0 k^3 q^3/q^k$ of the constraints.*

The approximability of Max k-CSP has been the subject of many papers in the past two decades since the advent of Håstad's optimal inapproximability results [29]; a partial list includes [58, 57, 16, 30, 17, 56, 25, 2, 10, 33] on the hardness side, and [60, 61, 28, 11, 25, 46] on the algorithmic side.

   The best known approximation guarantee for Max k-CSP over domain size q is $\Omega(kq/q^k)$ (for $k \geq \Omega(\log q)$, and $0.62k/2^k$ for the Boolean case [46]. This tight up to constant factors, due to Chan's inapproximability factor of $O(kq/q^k)$ [10]. However, this hardness does not apply for satisfiable instances. For satisfiable instances, the best hardness factor is $2^{O(k^{1/3})}/2^k$ for Boolean Max k-CSP [33], and $q^{O(\sqrt{k})}/q^k$ for Max k-CSP over domain size a prime q [30]. Note that our improved hardness factors (conditioned on the V label cover conjecture) from Theorem 1.1 are the first to get $\mathrm{poly}(k,q)/q^k$ type hardness for satisfiable instances (albeit only for prime powers) and are close to optimal. We note that satisfiable instances can be easier to approximate – Trevisan gave an elegant linear-algebra based factor $(k+1)/2^k$ approximation algorithm for satisfiable Boolean Max k-CSP [61] long before Hast's $\Omega(k/2^k)$ algorithm for the general case [28].

Inapproximability for strong and rainbow colorable hypergraphs. Our other application of the V label cover conjecture is to hypergraph coloring, another fundamental problem where perfect completeness is crucial. We say a hypergraph is c-colorable if there is a coloring of its vertices with c colors so that no hyperedge is monochromatic. Given a 2-colorable k-uniform hypergraph for $k \geq 3$, strong inapproximability results that show the NP-hardness of coloring with any fixed ℓ number of colors are known [21, 14], and recent developments show hardness (for $k \geq 8$) even for $\ell = \exp((\log n)^{\Omega(1)})$ where n is the number of vertices [42, 62, 34]. However, these results do not apply when the hypergraph has some form of balanced coloring that is stronger than just being 2-colorable. Specifically, we consider the notions of strong and rainbow colorability in this work. A hypergraph is q-strongly colorable, $q \geq k$ (resp. q-rainbow colorable, $q \leq k$) if it can be colored with q colors so that in every hyperedge, all vertices get distinct colors (resp. all q colors are represented). We refer the reader to the recent work [23, 6, 5] for further context on these notions. When $k = q$, so that there

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[5] This stronger requirement in soundness is common in hypergraph versions of Label Cover. For general Label Cover the stronger soundness guarantee can be ensured with a minor loss in parameters, but for V label cover we do not know such a reduction.

is a perfectly balanced k-coloring where each hyperedge has exactly one vertex of each of the k colors, one can in polynomial time find a 2-coloring without any monochromatic hyperedge [47]. Here we prove a strong hardness result for coloring hypergraphs (in fact for finding sizable independent sets), when this perfect balance condition is relaxed even slightly (specifically, $q = k - 1$ for rainbow coloring, and $q = k + o(k)$ for strong coloring).

A q-strong coloring of a hypergraph is also a legal q-coloring of the graph obtained by converting each of its hyperedges into a clique. For this reason, our hardness result for strongly colorable hypergraphs also implies hardness results in the more elementary setting of *approximate graph coloring*. There are several "pure" NP-hardness results known for graph coloring (e.g., the best known results in different regimes are [37, 22, 34, 6]), but there is a gigantic gap between these results and the known algorithms. [13] establishes much improved results, assuming variants of both the 2–to–1 conjecture as well as a new variant known as *alpha label cover*. Their main result is that for all $\epsilon > 0$, given a 3–colorable graph G, under these assumptions, it is NP–hard to locate an independent set with $|G|\epsilon$ vertices. In this work, assuming the V label cover–conjecture, we give a substantial generalization of this hardness.

▶ **Theorem 1.2.** *Assume the V label cover conjecture.*[6]
- *Given a k-uniform hypergraph, $k \geq 2$, for all $\epsilon > 0$, it is hard to tell if it is q-strongly colorable or has no independent set with an ϵ fraction of vertices, where $q = \lceil k + \sqrt{k} - \frac{1}{2} \rceil$.*
- *Given a k-uniform hypergraph, $k \geq 3$, for all $\epsilon > 0$, it is hard to tell if it is $(k-1)$-rainbow colorable or has no independent set with an ϵ fraction of vertices.*

The authors of [23] showed that for any $\epsilon > 0$, it is NP-hard to distinguish if a k-uniform hypergraph (k even) is a $k/2$-rainbow colorable or does not have a independent set with ϵ fraction of the vertices. The results of [6] give results for strong coloring, but they only apply when $k = 2$ or when the weak coloring has only two colors. Thus, modulo the V label cover–conjecture, our results improve on those in the literature.

These proofs are included in the full version of the paper.

A path to NP-hardness results? In several cases, the UGC conditioned hardness results were later replaced by NP-hardness results. Examples include some geometric inapproximability results [26], hardness of Unique Coverage [24], inapproximability results for agnostic learning [18], tight hardness results for scheduling [55], Chan's breakthrough showing an asymptotically tight inapproximability result for (near-satisfiable) Max k-CSP [10], etc. We hope that establishing a similar body of conditional results for perfect completeness, based on the V label cover conjecture or related variants, will point to strong inapproximability results and spur unconditional results in this domain.

1.2 Proof overview

We now briefly describe the steps needed to prove Theorem 1.1 and Theorem 1.2.

In each case, we reduce from a V label cover instance to a constraint satisfaction problem (with weighted constraints). In Section 3.3, we detail this reduction. The structure of the reduction has the same standard form as many other inapproximability results. Each vertex of the V label cover instance is replaced by a constellation of variables, known as a long code. Each hyperedge of the V label cover instance is replaced by a probability distribution of

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[6] Technically, we need an "induced" version of the V label cover conjecture for this result.

constraints between the variables in the correspond long codes. This is done carefully as to ensure that perfectly strongly satisfiable V label cover instances map to perfectly satisfiable CSPs.

For each problem type (Max-k-CSP, strong coloring, rainbow coloring), we craft a probability distribution which exploits its underlying structure. The probability distributions need to have a special correlation structure in order to be compatible with the V label cover constraints. We abstract a general notion termed *V label cover–compatibility* (Definition 3.2) which captures the properties common to these distributions. For example, we dictate that each vertex of each long code is sampled uniformly at random. Then, for each application, we outline the additional properties of our probability distributions in order for the reductions to have the proper soundness (Definition 4.1).

For the soundness analysis, given a good approximation to the resulting CSP, we seek to find an approximate weak labeling of the original V label cover instance. To do that, we attempt to decode each long code by finding one (or many) low-degree influential coordinates; these coordinates can be viewed as candidate labels for the associated vertex. We then argue that for a sizable fraction of constraints, two of the decoded labels will belong to the a single V-branch in the constraint. We can then label our V label cover instance by assigning each vertex a label selected at random from among its decoded labels, which in expectation finds a good approximate weak labeling.

In order to guarantee these influential coordinates, we invoke a couple of invariance principles. For Max-k-CSP, we directly invoke a result due to Mossel (Theorem 2.9) on pairwise independent probability distributions. This version guarantees a common influential coordinate between *three* functions that belongs to a common "V." A pigeonhole principle then implies that two of these labels must be in the same branch. For the hypergraph coloring problems, where we do not have pairwise independence of the distributions, we generalize the invariance principles of Mossel [48, 13] to yield a common influential coordinate for two functions that further lie on the same V-branch. This result, is available in the full version of the paper.

1.3   Organization

In Section 2, we outline the necessary background on CSPs and probability spaces. In Section 3, we motivate and detail the V label cover–conjecture. In Section 4, we apply V label cover to the Max-k-CSP problem. In Appendix A, we prove Lemma 4.4.

2   Preliminaries

2.1   Probability distributions

As is now commonplace in hardness of approximation reductions (e.g., [10, 13, 2, 48]), we utilize the following results on correlated probability spaces.

▶ **Definition 2.1** ([31, 19, 54][7]). Let $X \times Y$ be a finite joint probability space with a probability measure μ. The *correlation* between X and Y, denoted $\rho(X, Y)$ is defined to be

$$\rho(X, Y) = \sup_{\substack{f:X \to \mathbb{R}, g:Y \to \mathbb{R} \\ \mathbb{E}[f]=\mathbb{E}[g]=0, \ \ \mathrm{Var}[f]=\mathrm{Var}[g]=1}} \left[\mathop{\mathbb{E}}_{(x,y) \sim \mu} [f(x)g(y)] \right] .$$

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[7] See [1] for a history of this definition.

This is then easily extended to the correlation of $n \geq 3$ spaces.

▶ **Definition 2.2** (Definition 1.9 of [48]). Let $X_1 \times X_2 \times \cdots \times X_n$ be a finite joint probability space. Let $Z_i = X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. Then we define the correlation of $X_1, \ldots, X_n$ to be

$$\rho(X_1, X_2, \ldots, X_n) = \max_{1 \leq i \leq n} \rho(X_i, Z_i).$$

When a probability space can be decomposed into the product of independent subspaces, then the correlation behaves elegantly.

▶ **Lemma 2.3** (Theorem 1 of [64]). *For all $i \in [n]$, let $X_i \times Y_i$ be a probability space with measure μ_i. Assume that $\mu_1, \ldots, \mu_n$ are independent. Then,*

$$\rho(X_1 \times X_2 \times \cdots \times X_n, Y_1 \times Y_2 \times \cdots \times Y_n) = \max_{1 \leq i \leq n} \rho(X_i, Y_i).$$

Often it can be difficult to bound the correlation of a distribution away from 1. The following result is key in reducing these complex correlation problems into rather elementary graph connectivity problems.

▶ **Lemma 2.4** (Lemma 2.9 of [48]). *Let $X \times Y$ be a finite joint probability space with measure μ. Let G be the bipartite graph on $X \cup Y$ such that $(x, y) \in X \times Y$ is an edge iff $\Pr[x, y] > 0$ with respect to μ. Assume that G is connected, and let δ be the minimum nonzero probability in the joint distribution. Then, we have that*

$$\rho(X, Y) \leq 1 - \delta^2/2.$$

2.2 Influences

Recall the influence of a function over a probability space.

▶ **Definition 2.5.** Let $X_1, \ldots, X_n$ be finite independent probability spaces, and let $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ be a function. Let $Y_i = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. The influence is

$$\mathrm{Inf}_i(f) = \mathop{\mathbb{E}}_{x \in Y_i} [\mathrm{Var}_{z \in X_i} f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)].$$

Likewise, we need the notion of low-degree influences. We use the multilinear-polynomial definition used many times previously (e.g., [49, 13, 48]).

▶ **Definition 2.6** (e.g., Definition 3.4, 3.7 of [49]). Let $X_1, \ldots, X_n$ be finite independent probability spaces, and let $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ be a function. For each $i \in [n]$, let q_i be the cardinality of the support of X_i. Let $\alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)} : X_i \to \mathbb{R}$ be an orthonormal basis of functions such that $\alpha_1^{(i)} \equiv 1$. Let $\Sigma = [q_1] \times \cdots [q_n]$. Now, f can be uniquely expressed as

$$f = \sum_{\sigma \in \Sigma} c_\sigma \prod_{i=1}^n \alpha_{\sigma_i}^{(i)}.$$

for $c_\sigma \in \mathbb{R}$, which we call the Fourier coefficients. For $\sigma \in Q$, let $|\sigma| = |\{i \in [n] \mid \sigma_i \neq 1\}|$. The *low-degree influence* for $d \in [n]$ is

$$\mathrm{Inf}_i^{\leq d} f = \sum_{\sigma \in \Sigma, |\sigma| \leq d, \sigma_i \neq 1} c_\sigma^2.$$

The following is a key elementary fact concerning influences.

▶ **Lemma 2.7** (e.g., Proposition 3.8 [49]). *Consider* $f : X_1 \times \cdots \times X_n \to \mathbb{R}$. *For all integers* $d \geq 1$,

$$\sum_{i=1}^{n} \mathrm{Inf}_i^{\leq d} f \leq d \,\mathrm{Var}\, f.$$

In particular, for all $\tau > 0$, $|\{i \in [n] \mid \mathrm{Inf}_i^{\leq d} f \geq \tau\}| \leq \frac{d \,\mathrm{Var}\, f}{\tau}$.

Sometimes, we look at f from the perspective of different marginal distributions. Consider $f : X_1 \times \cdots X_n \to \mathbb{R}$ where the X_i's are independent. Furthermore, assume that each X_i can be written as $X_i = Y_{i,1} \times \cdots Y_{i,\ell_i}$, where these $Y_{i,j}$'s are independent. Then, we let $\mathrm{Inf}_{\overline{X}_i}^{\leq d} f$ denote the low-degree influence of f in the ith coordinate with respect to the X_i's. Likewise, we let $\mathrm{Inf}_{\overline{Y}_{i,j}}^{\leq d} f$ be the influence of the (i,j)th coordinate when viewed from the perspective of $f : Y_{1,1} \times \cdots \times Y_{n,\ell_n} \to \mathbb{R}$.

For each (i,j), let $\beta_1^{(i,j)}, \ldots, \beta_{q_{i,j}}^{(i,j)} : Y_{i,j} \to \mathbb{R}$ be an orthonormal basis of functions such that $\beta_1^{(i,j)} \equiv 1$. Note that $q_i = \prod_{j=1}^{\ell_i} q_{i,j}$. Let $\Sigma' = [q_{1,1}] \times \cdots [q_{n,\ell_n}]$. Then, we have that there exist c_σ's such that $f = \sum_{\sigma \in \Sigma'} c'_\sigma \prod_{i=1}^{n} \alpha_{\sigma_i}^{(i)}$. If $\ell_i \leq D$ for all i, then we have the following result

▶ **Lemma 2.8** (e.g., Claim 2.7 [13]). *If* $\ell_i \leq D$ *for all* $i \in [n]$, *then we have for all* $i, d \in [n]$ *that*

$$\mathrm{Inf}_{\overline{X}_i}^{\leq d} f \leq \sum_{k=1}^{\ell_i} \mathrm{Inf}_{\overline{Y}_{i,k}}^{\leq Dd} f.$$

Thus, there exists $k \in [\ell_i]$ *such that*

$$\frac{1}{D} \mathrm{Inf}_{\overline{X}_i}^{\leq d} f \leq \mathrm{Inf}_{\overline{Y}_{i,k}}^{\leq Dd} f.$$

Proof. The proof is a straightforward adaptation of the proof of Claim 2.7 in [13]. ◀

For our applications, we only need the case $D = 2$.

2.3   Invariance principles

Like [2], we use the following result on pairwise independent probability spaces.

▶ **Theorem 2.9** (Lemmas 6.6, 6.9 [48]). *Fix* $k \geq 3$. *For* $1 \leq i \leq n$, *let* $\Omega_i = X_i^{(1)} \times \cdots \times X_i^{(k)}$ *be finite pairwise independent probability spaces with probability measure* μ_i *such that the probability measures corresponding to* $\mu_1, \ldots, \mu_n$ *are independent. Let* δ *be the minimum positive probability among all the* μ_i. *Let*

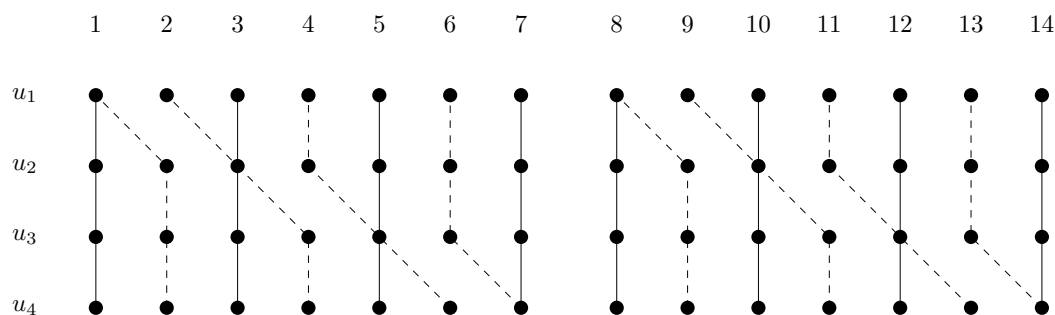$$\rho = \max_{1 \leq i \leq n} \rho(X_i^{(1)}, \ldots, X_i^{(k)})$$

and assume that $\rho < 1$. *For every* $\epsilon > 0$, *there exists* $\tau(\delta, \epsilon, \rho), d(\delta, \epsilon, \rho) > 0$ *such that for any functions* $f_1, \ldots, f_k$ *where* $f_i : X_1^{(i)} \times \cdots \times X_n^{(i)} \to [0,1]$ *if*

$$\forall \ell \in [n], |\{i \mid \mathrm{Inf}_{X_\ell^{(i)}}^{\leq d} f_i > \tau\}| \leq 2$$

then

$$\left| \prod_{i=1}^{k} \mathbb{E}[f_i] - \mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \right| \leq \epsilon.$$

Figure 1 A schematic diagram of the branches for an edge $e = (u_1, u_2, u_3, u_4)$ of V label cover instance Ψ with parameters $k = 4$ and $L = 2$. The ith row represents $\pi_i^{(e)}$ and the jth column represents the input j. The dashed and dotted lines are to indicated the two different branches with the same values with respect to $\pi^{(e)}$. For example, we may deduce from this diagram that $(10, 10, 10, 10)$ and $(9, 10, 11, 11)$ are two branches of e. In particular, we have that $\pi_1^{(e)}(9) = \pi_2^{(e)}(10) = \pi_3^{(e)}(11) = \pi_4^{(e)}(11)$. Note that $\psi_i^{(e)}(j) = \perp$ exactly when the node of the ith row and jth column is at the intersection of two branches. Compare with Figure 1 of [13].

In other words, if the product of the expected values and the expected value of the product significantly differ, then there must exist three functions with a common high low-degree influence coordinate. Note that the number "three" is crucially used in our reduction in Section 4.

3 V label cover

In this section, we propose a variant of hypergraph label cover which seems to plausibly have perfect completeness while also allowing for new hardness reductions. It can be thought of as a generalization of 2-to-1 label cover.

3.1 Definition

Let $k \geq 2$ and $L \geq 1$ be positive integers. An instance of k-uniform V-label cover is a k-uniform hypergraph on vertex set U. The constraints are on k-tuples $E \subseteq U^k$. Each edge $e = (u_1, \ldots, u_k)$ also has projection maps $\pi_1^{(e)}, \ldots, \pi_k^{(e)} : [(2k-1)L] \to [kL]$ with the following special property.

- The maps are surjective, in particular for all $i \in [k]$ and $j \in [kL]$,

$$|(\pi_i^{(e)})^{-1}(j)| = \begin{cases} 1 & i \equiv j \mod k \\ 2 & \text{otherwise} \end{cases}$$

In addition we would like to be able to distinguish between the two labels which map to a common value. To do this, we supplement the projection maps with *distinguishing functions* $\psi_1, \ldots, \psi_k : [(2k-1)L] \to \{0, 1, \perp\}$ such that for all $i \in [k]$, the map $x \mapsto (\pi_i^{(e)}(x), \psi_i(x))$ is injective. Furthermore, if $|(\pi_i^{(e)})^{-1}(\pi_i^{(e)}(x))| = 1$, then we define $\psi_i(x) = \perp$, and otherwise $\psi_i(x) \in \{0, 1\}$. We say that $(t_1, \ldots, t_k) \in [(2k-1)L]^k$ is a *branch* of e if there is $\ell \in [kL]$ and $b \in \{0, 1\}$ such that for all i, $(\pi_i^{(e)}(t_i), \psi_i^{(e)}(t_i))$ equals (ℓ, b) or $(\ell, \perp)$. Note that for each branch, there is exactly one $j \in [k]$ such that $\psi_j^{(e)}(t_j) = \perp$. In fact such such an index satisfies $j \equiv \pi_i^{(e)}(t_i) \mod k$ for all j. We say that i is the *junction* of the branch.

To better understand the setup, see Figure 1.

The goal of V-label cover is to produce a labeling of the vertices $\sigma : U \to [(2k-1)L]$. We say that a hyperedge $e = (u_1, \ldots, u_k)$ is *strongly satisfied* if $(\sigma(u_1), \ldots, \sigma(u_k))$ is a branch. In other words, for all $i, j \in [k]$, $\pi_i^{(e)}(\sigma(u_i)) = \pi_j^{(e)}(\sigma(u_j))$ *and either* $\psi_i^{(e)}(\sigma(u_i)) = \psi_j^{(e)}(\sigma(u_j)) \neq \perp$ or exactly one of $\psi_i^{(e)}(\sigma(u_i)), \psi_j^{(e)}(\sigma(u_j))$ is $\perp$. Another way to express this is that $(\pi_i^{(e)}(\sigma(u_i)), \psi_i^{(e)}(\sigma(u_i)))$ is uniform except for one i for which $\psi_i^{(e)}(\sigma(u_i)) = \perp$ (the meeting point in the 'V' of the two branches).

We say the hyperedge is *weakly satisfied* if for some distinct $i, j \in [k]$, $\pi_i^{(e)}(\sigma(u_i)) = \pi_j^{(e)}(\sigma(u_j))$ *and $\sigma(u_i)$ and $\sigma(u_j)$ are in the same branch.*

We now formally state our conjectured intractability of approximating V label cover. Below we state an "induced" version where in the soundness guarantee, for every labeling, most of the hyperedges within any subset of vertices of density ϵ fail to be weakly satisfied. The induced version is needed for our reduction to hypergraph coloring (this is similar to the α conjecture of [13] which was also defined in an induced form). For our Max k-CSP result, it suffices to assume the soundness condition that at most ϵ fraction of edges are weakly satisfiable. For simplicity, we only state the stronger induced version below.

▶ **Conjecture 3.1** (V label cover–conjecture, induced version). *For all $k \geq 2$ and $\epsilon > 0$, there exists an $L \geq 1$ such that for any k-uniform V label cover instance Ψ on label set L and vertex set U and hyperedge set E, it is NP-hard to distinguish between*

- *YES: There exists a labeling for which every hyperedge is strongly satisfied.*
- *NO: For every labeling and every subset $U' \subset U$ with $|U'| \geq |U|\epsilon$, less than ϵ fraction of the edges in $(U')^k \cap E$ are weakly satisfied by the labeling.*

3.2   Compatibility

Consider a domain size $q \geq 2$, an arity $k \geq 2$, and a predicate $P \subseteq [q]^k$. In order to understand the "V label cover–hardness" of this predicate P, for each edge $e = (u_1, \ldots, u_k)$ of our V label cover instance we seek to construct probability distributions on $[q]^{k \times (2k-1)L}$ such that the marginal distribution of each branch of e is supported by P. We define the notion of *V label cover–compatibility* in order to capture exactly what we need.

▶ **Definition 3.2.** For a predicate $P \subseteq [q]^k$, consider $\mu_1, \ldots, \mu_k$ supported on P^2. For $i, j \in [k]$, let $X_{i,j} \sim [q]^2$ be the marginal distribution of μ_i on the jth coordinates. That is, for all $(a, b) \in [q]^2$,

$$\Pr_{(x',y') \sim X_{i,j}}[(x', y') = (a, b)] = \Pr_{(x,y) \sim \mu_i}[(x_j, y_j) = (a, b)].$$

We call the distributions $\mu_1, \ldots, \mu_k$ a *V label cover–compatible family* if they satisfy the following properties.

1. For all $i \in [k]$, $X_{i,i}$ is uniform on $\{(a, a) \mid a \in [q]\}$.
2. For all $i, j \in [k]$ with $i \neq j$ and $X_{i,j}$ is uniform on $[q]^2$.
3. For all $i \in [k]$, $\rho(\mu_i) < 1$, which we define to be

$$\rho(\mu_i) := \rho(X_{i,1}, \ldots, X_{i,k}).$$

We say that P is *V label cover–compatible* if a V label cover–compatible family $\mu_1, \ldots, \mu_k$ exists.

The reason we have k different distributions is because the two connected branches can intersect in k different rows (see Figure 1).

Property (3) of Definition 3.2 precludes any algebraic structure in our predicate that would permit a polynomial-time algorithm. For example, the uniform distribution on the predicate $\{x \in \mathbb{Z}_2^n \mid x_1 + \cdots + x_n = 0\}$ has correlation 1 and allows for Gaussian-elimination to solve exactly.

3.3 Reduction from V label cover to P-CSP

Let $P \subseteq [q]^k$ be a predicate for $q, k \geq 2$ which is V label cover–compatible with distributions $\mu_1, \ldots, \mu_k$. In this section, we show how to reduce an arbitrary instance of V label cover into an instance of P-CSP, the constraint satisfaction problem where all clauses are of the form $(x_1, \ldots, x_n) \in P$. Furthermore, we assign weights to the clauses of this CSP, in which the weights are determined by these distributions μ_i. This reduction is the starting point for showing the conditional NP-hardness results in Section 4.

Let $\Psi = (U, E, L, \{\pi_i^{(e)}\}_{e \in E, i \in [k]}, \{\psi_i^{(e)}\}_{e \in E, i \in [k]})$ be our instance of k-uniform V label cover. For each $u \in U$, we construct $q^{(2k-1)L}$ variables $x_s^{(u)}$, where $s \in [q]^{(2k-1)L}$. Now, for every edge $e = (u_1, \ldots, u_k) \in E$ and every $s^{(1)}, \ldots, s^{(k)} \in [q]^{(2k-1)L}$ with the following property:

- For any $t_1, \ldots, t_k \in [(2k-1)L]$ such that $(t_1, \ldots, t_k)$ is a branch of e, we have $(s_{t_1}^{(1)}, \ldots, s_{t_k}^{(k)}) \in P$,

we add the constraint $(x_{s^{(1)}}^{(u_1)}, \ldots, x_{s^{(k)}}^{(u_k)}) \in P$. Looking back at Figure 1, we have that any assignment of values from $[q]$ to the nodes of the schematic such that each branch is an element of P corresponds to some choice $(s^{(1)}, \ldots, s^{(k)})$.

Let Φ be the resulting instance. Although we have described the clauses, we have not yet determined the relative weights of the clauses.

▶ **Claim 3.3.** *If Ψ has a labeling $\sigma : U \to [(2k-1)L]$ which strongly satisfies every hyperedge, then we have that Φ has a perfect satisfying assignment. In other words, this reduction has perfect completeness.*

Proof. For each $u \in U$, and $s \in [q]^{(2k-1)L}$, we let $x_s^{(u)} = s_{\sigma(u)}$. One can verify this assignment satisfies Φ. ◀

Now, fix $e = (u_1, \ldots, u_k) \in E$. For each $\ell \in [kL]$, let $(a_1, \ldots, a_k), (b_1, \ldots, b_k)$ be the two branches of e such that $\pi_i^{(e)}(a_i) = \pi_i^{(e)}(b_i) = \ell$ for all i. Let $j \in [k]$ be the unique index for which $a_j = b_j$, (i.e., j is the junction). Let I be the index set $I := \{(i, a_i) \mid i \in [k]\} \cup \{(i, b_i) \mid i \in [k]\}$; note that $|I| = 2k - 1$. Let $\Omega_\ell^{(e)} \sim [q]^I$ be the probability distribution isomorphic to μ_j such that the marginals $x_1, \ldots, x_k, y_1, \ldots, y_k$ of μ_j correspond to the marginals indexed by $(1, a_1), \ldots, (k, a_k), (1, b_1), \ldots, (k, b_k)$ of $\Omega_\ell^{(e)}$.

Let

$$\nu^{(e)} := \prod_{\ell \in [kL]} \Omega_\ell^{(e)},$$

where the product is over independent distributions. Note that the support of $\nu^{(e)}$ can be identified with $[q]^{[k] \times [(2k-1)L]}$ since each $(i, a_i) \in [k] \times [(2k-1)L]$ is accounted for in some branch. We let $Y_j^{(e,i)}$ be the marginal distribution of coordinate $(i, j) \in [k] \times [(2k-1)L]$ of $\nu^{(e)}$. For any $i \in [k]$ and $\ell \in [kL]$, we let $X_{i,\ell}^{(e)}$ be the marginal distribution on the indices

$\{(i, t) \mid \pi_i^{(e)}(t) = \ell\}$. In particular, if i is a junction, the meeting point of the branches, then $Y_t^{(e,i)} = X_{i,\ell}^{(e)}$. Otherwise, $X_{i,\ell}^{(e)}$ is the product of two Y's:

$$X_{i,\ell}^{(e)} = \prod_{t \in (\pi_i^{(e)})^{-1}(\ell)} Y_t^{(e,i)}.$$

This distribution $\nu^{(e)}$ specifies the probability distribution of the clauses corresponding to a particular edge of the label cover instance. These probabilities are the relative weights of the clauses in the instance.

4   Perfect-completeness approximation resistance and Max-k-CSP$_q$

A natural question to ask concerning V label cover is if it reduces to natural families of predicates which are hard to approximate, even when guaranteed perfect completeness. In the case of imperfect completeness, Austrin and Mossel [2] showed assuming the Unique Games Conjecture that if a predicate $P \subseteq [q]^k$, for some finite domain size q, supports a balanced pairwise independent distribution, then P is *approximation resistant*. That is, for all $\epsilon > 0$, it is NP-hard to distinguish between $1 - \epsilon$-satisfiable and $\frac{|P|}{q^k} + \epsilon$-satisfiable P-CSPs. Only a few years later, in a breakthrough by Chan [9], unconditional approximation resistance was shown for any P which supports a balanced pairwise independent subgroup. We hope that establishing a similar conditional results for perfect completeness will spur unconditional results in this domain.

In order to reduce from V label cover, we need a more stringent criteria than merely supporting a balanced pairwise independent distribution. We call these more structured distributions *pairwise-independent V label cover-compatible*.

▶ **Definition 4.1.** Let $q \geq 2, k \geq 3$ be parameters. Let $P \subseteq [q]^k$ be a predicate. We say that P is *pairwise–independent V label cover–compatible* if there exists a V label cover–compatible family $\mu_1, \ldots, \mu_k$ supported on P^2 (with marginals $X_{i,j}$, $i, j \in [k]$) with the additional property that:
4. For all $i \in [k]$ and $j \neq j' \in [k]$, we have that $X_{i,j}$ and $X_{i,j'}$ are pairwise independent.

To motivate the definition, one way to view property (4), when combined with properties (1) and (2) of Definition 3.2, is that P does not just support a pairwise independent distribution, but that the distribution can preserve pairwise independence even when conditioning on the value of a coordinate.[8] Assuming the V label cover-conjecture, this property suffices to establish perfect-completeness approximation resistance if we allow what are known as *folded* predicates.[9] Assume that $[q]$ has a $+$ operator (e.g., addition modulo q). We specify that we may use folded versions of our predicate P to be the predicates

$$a \in [q]^k, \; P^{(a)} := \{(x_1 + a_1, \ldots, x_k + a_k) \mid (x_1, \ldots, x_k) \in P\}.$$

Each $P^{(a)}$ has the same cardinality, so incorporating these extra predicates can only increase the severity of the hardness of approximation. Thus, more precisely we say that the *family* of predicates $\{P^{(a)} \mid a \in [q]^k\}$ is perfect-completeness approximation resistant. That is, for

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[8] The definition permits a slightly broader class of P (i.e., the distribution can change depending on which coordinate is conditioned on), but our applications will construct P of the type specified in the motivation.

[9] This is a standard assumption in the CSP literature, e.g., [2].

every $\epsilon > 0$, it is NP-hard to distinguish whether a CSP with predicates from $\{P^{(a)} \mid a \in [q]^k\}$ is perfectly satisfiable or is $\frac{|P|}{q^k} + \epsilon$ satisfiable.

▶ **Theorem 4.2.** *Let $P \subseteq [q]^k$ be a predicate which supports a pairwise-independent V label cover–compatible distribution. Then, assuming the V label cover–conjecture, we have that the collection of predicates $\{P^{(a)} \mid a \in [q]^k\}$ is perfect-completeness approximation resistant.*

Proof. The high-level structure of our proof is analogous to that of Austrin and Mossel [2]. The proof proceeds in a couple of stages. First, we describe the reduction from a V label cover instance to an instance of P-CSP, and note that such a reduction preserves perfect completeness. Second, we analyze the soundness of our reduction using Theorem 2.9 to show that if our P-CSP can be well-approximated, then our original V label cover instance also admits an approximation.

Reduction. Let $\Psi = (U, E, L, \{\pi_i^{(e)}\}_{e \in E, i \in [k]}, \{\psi_i^{(e)}\}_{e \in E, i \in [k]})$ be our instance of k-uniform V label cover. Let Φ be the instance of P-CSP guaranteed by the construction in Section 3.3. Let $\nu^{(e)} \in [q]^{[k] \times [(2k-1)L]}$ be the weighting distributions on the clauses corresponding to the hyperedges. Let $\Omega_\ell^{(e)}, X_{i,j}^{(e)}, Y_j^{(e,i)}$ be the marginal distributions described in Section 3.3. By Claim 3.3, our reduction has perfect completeness.

We now modify the CSP Φ into a new CSP Φ' which incorporates folding. For each constraint $(x_{s^{(1)}}^{(u_1)}, \ldots, x_{s^{(k)}}^{(u_k)}) \in P$ and for each $i \in [k]$, let $(s^{(i)})' = s^{(i)} - s_1^{(i)}$ (i.e., subtract $s_1^{(i)}$ from every coordinate). Then, we specify that

$$(x_{(s^{(1)})'}^{(u_1)}, \ldots, x_{(s^{(k)})'}^{(u_k)}) \in P^{(s_1^{(1)}, \ldots, s_1^{(k)})}.$$

One may check that this modification preserves perfect completeness.

Soundness. We view an assignment to Φ' as a collection of functions $\mathcal{F} = \{f_u : [q]^{(2k-1)L} \to [q] \mid u \in U\}$, where $f_u(s)$ is the assigned value for x_s^u. Because of our modification to the CSP, we only specify constraints for $f_u(s)$ when $s_1 = q$. Thus, we may assume that each f_u is *folded*. That is, $f_u(s) + a \equiv f_u(s + (a, \ldots, a)) \mod q$ for all $a \in [q]$. One may check that the f_u's satisfy a clause in Φ' if and only if they satisfy the corresponding clause in Φ. Thus, it is equivalent to focus on the f_u's satisfaction of Φ.

For $a \in [q]$, we let

$$f_u^{(a)}(x) = \begin{cases} 1 & f_u(x) = a \\ 0 & \text{otherwise} \end{cases}$$

We define the influences and low-degree influences (Definitions 2.5 and 2.6) of the $f_u^{(a)}$'s to be with respect to the uniform distribution.

Let $\Phi(\mathcal{F})$ be the fraction of constraints of Φ satisfied by $\mathcal{F}$, using the weights specified by the $\nu^{(e)}$ distributions. We seek to show for any $\epsilon > 0$ if there exists a $\mathcal{F}$ such that $\Phi(\mathcal{F}) > \frac{|P|}{q^k} + \epsilon$, then there exists $\delta > 0$ and $\sigma : U \to [(2k-1)L]$ such that σ weakly satisfies δ fraction of the constraints of Ψ.

It is evident from the construction, that a group of constraints are associated with each $e \in E$. Let $e(\mathcal{F})$ be the fraction of constraints corresponding to ϕ satisfied by $\mathcal{F}$ (that is the measure with respect to $\nu^{(e)}$ of the clauses satisfied by $\mathcal{F}$). We have that

$$\Phi(\mathcal{F}) = \frac{1}{|E|} \sum_{e \in E} e(\mathcal{F}).$$

Thus, if $\Phi(\mathcal{F}) > \frac{|P|}{q^k} + \epsilon$, there exists a subset $E' \subseteq E$ such that $|E'| > (\epsilon/2)|E|$ and $e(\mathcal{F}) \geq \frac{|P|}{q^k} + \epsilon/2$ for all $e \in E'$; as otherwise, $\Phi(\mathcal{F}) \leq \epsilon/2 \cdot 1 + (1 - \epsilon/2) \cdot \left(\frac{|P|}{q^k} + \epsilon/2\right) < \frac{|P|}{q^k} + \epsilon$.

Fix, $e = (u_1, \ldots, u_k) \in E'$. Note that

$$e(\mathcal{F}) = \mathop{\mathbb{E}}_{(s_1, \ldots, s_k) \sim \nu^{(e)}} [(f_{(u_1)}(s_1), \ldots, f_{(u_k)}(s_k)) \in P]$$

$$= \sum_{r \in P} \mathop{\mathbb{E}}_{(s_1, \ldots, s_k) \sim \nu^{(e)}} [f_{u_1}^{(r_1)}(s_1) \cdots f_{u_k}^{r_k}(s_k)].$$

Thus, for some $r \in P$, we have that

$$\mathop{\mathbb{E}}_{(s_1, \ldots, s_k) \sim \nu^{(e)}} [f_{u_1}^{(r_1)}(s_1) \cdots f_{u_k}^{r_k}(s_k)] > \frac{1}{q^k} + \frac{\epsilon}{2|P|}.$$

Let $\epsilon' = \epsilon/(2|P|) > 0$. Also, for all $i \in [k]$, let $\Pi_i^{(e)} = \prod_{\ell=1}^{kL} X_{i,\ell}^{(e)}$. Since each $\Pi_i^{(e)}$ is uniform and f_{u_i} is folded, we have that

$$\mathop{\mathbb{E}}_{s_i \sim \Pi_i^{(e)}} [f_{u_i}^{(r_i)}(s_i)] = \frac{1}{q}.$$

In particular, this implies that

$$\left| \prod_{i=1}^{\ell} \mathbb{E}[f_{u_i}^{(r_i)}(s_i)] - \mathbb{E}\left[\prod_{i=1}^{\ell} f_{u_i}^{(r_i)}(s_i)\right] \right| > \epsilon'.$$

Note that $\nu^{(e)} = \Omega_1^{(e)} \times \cdots \times \Omega_{kL}^{(e)}$ meets the requirements of Theorem 2.9. Thus, there exists $\tau, d > 0$, which are functions of only ϵ' and parameters of $|P|$, such that

$$\exists \ell \in [kL], |\{i : \mathrm{Inf}_{X_{i,\ell}^{(e)}}^{\leq d} f_{u_i}^{(r_i)} > \tau\}| \geq 3.$$

Let $i_1, i_2, i_3 \in [k]$ be three of these coordinates and let $\ell \in [kL]$ be the guaranteed value of ℓ. Observe that we can also write $\Pi_{i_a}^{(e)}$ as

$$\Pi_{i_a}^{(e)} = \prod_{t \in [(2k-1)L]} Y_t^{(e, i_a)}.$$

Note that each $X_{i_a, \ell}^{(e)}$ can be written as the product distribution of at most 2 $Y_t^{(e, i_a)}$'s, where $\pi_{i_a}^{(e)}(t) = \ell$. By invoking Lemma 2.8 with $D = 2$, we have that there exists t_1, t_2, t_3 such that $\pi_{i_a}^{(e)}(t_a) = \ell$ for all $a \in \{1, 2, 3\}$ and

$$\mathrm{Inf}_{Y_{t_a}^{(e, i_a)}}^{\leq 2d} f_{u_{i_a}}^{(r_i)} = \mathrm{Inf}_{t_a}^{\leq 2d} > \frac{\tau}{2},$$

where the equality is due to the fact that the $Y_{t_a}^{(e, i_a)}$ distributions are uniform distributions on $[q]$.

Note that since each 'component' of (e) has two branches, by the Pigeonhole principle, some two of $\{t_1, t_2, t_3\}$ are in the same branch. Thus, any assignment σ for which $\sigma(u_{i_a}) = t_a$ for all $a \in \{1, 2, 3\}$ weakly satisfies e.

For each $u \in U$. Let $S_u \subseteq [(2k-1)L]$ be the set of labels j for which $\mathrm{Inf}_{j}^{\leq 2d} f_u^{(a)} > \tau/2$ for some $a \in [q]$. Since $\mathrm{Var} f_u^{(a)} \leq \max(f_u^{(a)})^2 = 1$, we have by Lemma 2.7 that $|S_u| \leq 4dq/\tau$, which is independent of L. Construct a random labeling $\sigma : U \to [(2k-1)L]$ by sampling

each $\sigma(u)$ from S_u independently and uniformly at random (if S_u is empty, let $\sigma(u) = 1$). For each $e \in E'$, we established that there exists $i, i' \in [k]$ and $\ell \in S_{u_i}$ and $\ell' \in S_{u_{i'}}$ such that setting $\sigma(u_i) = \ell$ and $\sigma(u_{i'}) = \ell'$ weakly satisfies e. Thus, in expectation at least

$$\frac{|E'|}{|E|} \cdot \frac{1}{(\max |S_u|)^2} = \frac{\tau^2 \epsilon}{16 d^2 q^2} > 0$$

of the edges are weakly satisfied. Note that this expression is independent of L and the size of Ψ, as desired. ◄

We use this theorem to obtain hardness of approximation results for Max-k-CSP$_q$ when $q \geq 2$ is a prime power. We first need the following lemma which follows from standard constructions in BCH codes.

▶ **Lemma 4.3.** *Let $q \geq 2$ be a prime power, and let $\ell \geq 1$ be odd. There exists $S \subset \mathbb{F}_q^\ell$ with $|S| = q^{(\ell-1)/2}$ such that S is 3-wise linearly independent over $\mathbb{F}_q$. That is, each three-element subset of S is linearly independent.*

▶ **Remark.** Because of the recent breakthrough that subsets of $\mathbb{Z}_q^n$ which do not have an arithmetic progress of length three have size at most q^{cn} for some $c < 1$, [12, 15], it is impossible to improve that factor of $1/2$ in the exponent to 1 when $q \geq 3$. In particular, Lemma 4.4 can at best be improved to $O_q(k^{2+\gamma})$ for some $\gamma > 0$ (where the O_q notation hides the dependence of q).

▶ **Lemma 4.4.** *For all $q \geq 2$ a prime power and $k \geq 2$, there exists $P \subseteq [q]^k$ which is pairwise–independent V label cover-compatible with $|P| = 2k^3 q^3$.*

The proof is given in Appendix A.
Using the same proof techniques, we have the following corollary.

▶ **Corollary 4.5.** *For $q = 2$ and all $k \geq 2$, there exists $P \subseteq [2]^k$ which is pairwise–independent V label cover–compatible and $|P| = O(k^2)$.*

Proof. Repeat the proof of Lemma 4.4, but note that $S = \{x \in \mathbb{F}_2^\ell : \sum_{i=1}^\ell x_i = 1\}$ is a 3-wise-independent subset of size $2^{\ell-1}$. ◄

Now we may obtain Theorem 1.1.

Proof of Theorem 1.1. The case $q = 2$ follows immediately from Corollary 4.5 and Theorem 4.2. Similarly, if $q \geq 3$ is a prime power, then the result follows from Lemma 4.4 and Theorem 4.2. ◄

This is the first conditional NP-hardness reduction which obtains a soundness of $\frac{\text{poly}(q,k)}{q^k}$ for even one fixed q. Previously, a long code test due to Tamaki and Yoshida [59] obtained $\frac{O(k)}{2^k}$ for when $q = 2$. The currently best known unconditional result for Max-k-CSP$_2$ is $\frac{2^{O(k^{1/3})}}{2^k}$ due to Huang [33]. For $q \geq 3$, the best known result is [30] [46].

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A   Proof of Lemma 4.4

Proof. We use a modification of the constructions of [2] and [59]. Let $\ell \geq 3$ be the least odd integer such that $q^{(\ell-1)/2} \geq k$. Thus, $q^\ell \leq k^2 q^3$. View $\mathbb{F}_q^\ell$ as a vector space over $\mathbb{F}_q$. By Lemma 4.3 there exists $S \subset \mathbb{F}_q^\ell$ with $|S| \geq q^{(\ell-1)/2} \geq k$ such that S is 3-wise linearly independent (i.e., every 3-element subset is linearly independent). Let $v^{(1)}, \ldots, v^{(k)} \in S$ be k distinct elements from this set. Define $\langle \cdot, \cdot \rangle$ to be the canonical bilinear form on $\mathbb{F}_q^\ell$. That is, $\langle x, y \rangle = \sum_{i=1}^\ell x_i y_i$.

We give an initial attempt to construct our predicate. Let[10]

$$P_0 = \{(\langle v^{(1)}, X \rangle, \ldots, \langle v^{(k)}, X \rangle) : X \in \mathbb{F}_q^\ell\}.$$

We have that $|P_0| \leq q^\ell \leq k^2 q^3$. We show that P_0 satisfies properties (1), (2), and (4). Note that the definition of P_0 defined a natural probability distribution μ. It is clear that μ has

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[10] Note that we identify $[q]$ with $\mathbb{F}_q$ in some canonical way.

uniform marginal distributions (since each $v^{(i)}$ is nonzero and X is uniform). Furthermore, the marginal distributions are 3-wise independent (and thus 3-wise uniform) since the $v^{(i)}$'s are 3-wise linearly independent. (We omit the proof, a similar result for pairwise independence is Lemma 4.2 of [2].)

Now, fix $i \in [k]$, define μ_i to be

$$\mu_i := \{x, y \sim \mu \text{ independent} : x_i = y_i\}.$$

Let $X_{i,j}$ with $j \in [k]$ be the marginals of μ_i. We seek to show μ_i satisfies properties (1), (2), and (4). Property (1) follows immediately from the uniform marginals of μ. Now, fix $j \neq i$, since (x_i, x_j) and $(x_i = y_i, y_j)$ are uniform distributions and x_j and y_j are conditionally independent given x_i, we have that

$$\Pr[x_i \wedge x_j \wedge y_j] = \Pr[x_j \wedge y_j | x_i]\Pr[x_i] = \Pr[x_j | x_i]\Pr[y_j | x_i]\Pr[x_i] = \Pr[x_j]\Pr[y_j]\Pr[x_i].$$

Therefore, (x_i, x_j, y_j) is uniform on $\mathbb{F}_q^{\ell}$. Thus, property (2) and the case $j' = i$ of property (4) follow.

To finish establishing property (4), consider $j \neq j' \in [k] \setminus \{i\}$. We seek to show that $(x_j, x_{j'}, y_j, y_{j'})$ is uniform for which it suffices to show that $(x_i, x_j, x_{j'}, y_j, y_{j'})$ is uniform. Like before,

$$\begin{aligned}
\Pr[x_i &\wedge x_j \wedge x_{j'} \wedge y_j \wedge y_{j'}] \\
&= \Pr[x_j \wedge x_{j'} | x_i]\Pr[y_j \wedge y_{j'} | x_i]\Pr[x_i] \\
&= \Pr[x_j]\Pr[x_{j'}]\Pr[y_j]\Pr[y_{j'}]\Pr[x_i] \text{ (3-wise independence of } \mu).
\end{aligned}$$

Thus, the μ_i's satisfy properties (1), (2), and (4). Sadly, due to the nice algebraic structure of P_0, we have that $\rho(\mu_i) = 1$ for all i. To rectify this, we create a 'noisy' version of P_0. For $x \in \mathbb{F}_q^k$, let $|x|$ be the number of nonzero coordinates of x. Then, we define P to be

$$P := \{x \in \mathbb{F}_q^k \mid \exists y \in P_0, |x - y| \leq 1\}.$$

Note that $|P| \leq (k+1)|P_0| \leq 2k^3q^3$. Now, modify the μ_i's to get μ_i''s by the following procedure.
1. Sample $(x, y) \in \mu_i$.
2. Sample $j \in [k]$ and $a, b \in \mathbb{F}_q$ uniformly.
3. If $i = j$, set $x_i = y_j = a$. Otherwise, set $x_i = a$ and $y_j = b$.

Clearly the support of μ_i' is P^2. Also μ_i' preserves properties (1), (2), and (4) of being V label cover-compatible since re-randomizing coordinates can only assist in maintaining pairwise independent distributions.

It remains to show that μ_i' satisfies property (3). The proof of this is similar to that of Lemma 4.6 of [59]. Let

$$Z_{i,j} := \prod_{j=1, j \neq i}^k X_{i,j}.$$

It suffices to show that $\rho(X_{i,j}, Z_{i,j}) < 1$. To do that, it suffices to show by Lemma 2.4 that the bipartite graph whose edges are the support of $X_{i,j} \times Z_{i,j}$ is connected. For any $(\alpha, \beta) \in X_{i,j} \times Z_{i,j}$, since with nonzero probability the jth coordinate is rerandomized, we have that $(\alpha', \beta) \in X_{i,j} \times Z_{i,j}$ for all α' in the support of $X_{i,j}$. From this connectivity immediately follows.

Therefore, P has the desired properties.   ◀