Vertex Isoperimetry and Independent Set Stability for Tensor Powers of Cliques*†

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Abstract

The tensor power of the clique on t vertices (denoted by K_t^n) is the graph on vertex set $\{1, \dots, t\}^n$ such that two vertices $x, y \in \{1, \dots, t\}^n$ are connected if and only if $x_i \neq y_i$ for all $i \in \{1, \dots, n\}$. Let the density of a subset S of K_t^n to be $\mu(S) := \frac{|S|}{t^n}$. Also let the vertex boundary of a set S to be the vertices of the graph, including those of S, which are incident to some vertex of S. We investigate two similar problems on such graphs.

First, we study the vertex isoperimetry problem. Given a density $\nu \in [0, 1]$ what is the smallest possible density of the vertex boundary of a subset of K_t^n of density ν? Let $\Phi_t(\nu)$ be the infimum of these minimum densities as $n \to \infty$. We find a recursive relation allows one to compute $\Phi_t(\nu)$ in time polynomial to the number of desired bits of precision.

Second, we study given an independent set $I \subseteq K_t^n$ of density $\mu(I) = \frac{1}{t}(1 - \epsilon)$, how close it is to a maximum-sized independent set J of density $\frac{1}{t}$. We show that this deviation (measured by $\mu(I \setminus J)$) is at most $4\epsilon^{\frac{\log t}{\log t - \log(t-1)}}$ as long as $\epsilon < 1 - \frac{3}{t} + \frac{2}{t^2}$. This substantially improves on results of Alon, Dinur, Friedgut, and Sudakov (2004) and Ghandehari and Hatami (2008) which had an $O(\epsilon)$ upper bound. We also show the exponent $\frac{\log t}{\log t - \log(t-1)}$ is optimal assuming n tending to infinity and ϵ tending to 0. The methods have similarity to recent work by Ellis, Keller, and Lifshitz (2016) in the context of Kneser graphs and other settings.

The author hopes that these results have potential applications in hardness of approximation, particularly in approximate graph coloring and independent set problems.

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1 Introduction

A growing subfield in extremal combinatorics is understanding the structure of combinatorial objects which are close in size to the maximal such objects. In this work, we study such questions in the context of independent sets of tensor power of cliques. We establish this by first understanding the isoperimetric properties of such graphs.

1.1 Vertex isoperimetry

For any undirected graph $G = (V_G, E_G)$ and $S \subseteq V_G$, we define the *vertex boundary* of S to be

$$\partial S := \{x \in V_G \,:\, \text{exists } y \in S \text{ such that } \{x, y\} \in E_G\}.$$

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Furthermore, we define the *density* of S to be

$$\mu(S) := \frac{|S|}{|V_G|}.$$

The relationship between $\mu(S)$ and $\mu(\partial S)$ is typically captured by *vertex isoperimetric inequalities.* Such inequalities are particularly studied when $\mu(S)$ is sufficiently small (typically at most 1/2). These relationships are captured by the *isoperimetric parameter* (or *isoperimetric profile*) of a graph

$$\Phi(G, \nu) = \inf\{\mu(\partial S) \,:\, \mu(S) \geq \nu\}.$$

Proving such inequalities for various graphs is a frequent topic in the literature (e.g., [4, 8]). Typically such works focus on a *linear* or near-linear relationship between $\mu(\partial S)$ and $\mu(S)$, known as the *isoperimetric constant.*

$$h(G) = \inf\left\{\frac{\mu(\partial S)}{\mu(S)} \,\middle|\, S \subset V_G, \mu(S) \in (0, 1/2]\right\}. \tag{1}$$

In this paper, we study graphs for which there is an order-of-magnitude difference between $\mu(S)$ and $\mu(\partial S)$, when $\mu(S)$ is sufficiently small. For example, if $\mu(\partial S) \geq \sqrt{\mu(S)}$ for all S, we would like to say that G expands by a power of 2. Such 'hyper-expansion' can be captured by what we coin as the *isoperimetric exponent.* For all $\epsilon > 0$ consider.

$$\eta(G, \epsilon) = \inf\left\{\frac{\log \mu(S)}{\log \mu(\partial S)} \,\middle|\, S \subset V_G, \mu(S) \in (0, \epsilon)\right\} \tag{2}$$

where log is the natural logarithm. In other words, for every subset S of G of density δ, the boundary of S has density at least $\delta^{1/\eta(G,\epsilon)}$. The larger the parameter $\eta(G)$ is, the more 'expansive' the graph is. It is easy to see that $\eta(G, \epsilon)$ is in general a decreasing function of ϵ. As we often work with large subsets of our graph, we let $\eta(G) := \eta(G, 1)$.

In this paper, we study the isoperimetric profile of the tensor powers of cliques. For undirected graphs $G = (V_G, E_G), H = (V_H, E_H)$, we define the *tensor product* $G \otimes H$ to be the undirected graph on vertex set $V_1 \times V_2$ such that an edge connects (u_1, v_1) and (u_2, v_2) if and only if $\{u_1, u_2\} \in E_G$, and $\{v_1, v_2\} \in E_H$. Note that up to isomorphism, the tensor product is both commutative and associative. We then denote $\otimes^n G$ to be the tensor product of n copies of G. Since this is the only graph product discussed in this article, we shorten this to G^n. In this article, we focus on the case that $G = K_t$, where K_t is the complete graph on $t \geq 3$ vertices. It turns out for such graphs that for all $\epsilon > \frac{1}{t^n}$, $\eta(G) = \eta(G, \epsilon)$.

In particular, we shall compute the following.

▶ **Theorem 1.1.** *For all $t \geq 3$ and all positive integers n,*

$$\eta(K_t^n) = \eta(K_t) = \frac{\log t}{\log t - \log(t-1)} = t \log t + \Theta(\log t). \tag{3}$$

In addition to this high-level structure, we give a more-fine-tuned analysis of the behavior of $\Phi_t(\eta) := \inf_{n \geq 1} \Phi(K_t^n, \eta)$. (See Theorem 2.8.)

1.2 Independent set stability

Next, we apply these vertex isoperimetric inequalities to understand the structure of near-maximum independent sets of graphs. Such results are known as *stability* results.

Such results are not just of interest within combinatorics, a better understanding of independent set stability of certain graphs, such as K_t^n, have resulted in advances in hardness of approximation, particularly in construct dictatorship tests for approximate graph coloring and independent set problems (e.g., [1, 10, 7]). In fact the investigation which led to the results in this paper was inspired by the pursuit of such results.

A landmark result of this form due to [1] is as follows.

▶ **Theorem 1.2** ([1]). *For all $t \geq 3$ there exist C_t with the following property. For any positive integer n, Let $I \subset K_t^n$ be an independent set such that $\epsilon = 1 - t\mu(I)$, then there exists an independent set $J \subset K_t^n$ of maximum size ($\mu(J) = 1/t$) such that $\mu(I \Delta J) \leq C_t \epsilon$, where $S\Delta T = (S \setminus T) \cup (T \setminus S)$.*

In other words, independent sets of near-maximum size are similar in structure to the maximum independent sets. Note that if J is an independent set of maximum size, then for some $i \in [n]$ and $j \in [t]$, we have that

$$J = [t]^{i-1} \times \{j\} \times [t]^{n-i}.$$

This is a well-known result due to [22] (see [2] for a proof using Fourier analysis).

Ghandehari and Hatami improved this result (Theorem 1 of [21]) to show that if $t \geq 20$ and $\epsilon \leq 10^{-9}$ then C_t can be replaced with $40/t$. Both results were proven using Fourier analysis.

We improve upon this result in two steps. First, by applying Theorem 1.1 we improve Theorem 1.2 in a black-box matter to obtain the following result.

▶ **Theorem 1.3.** *For all $t \geq 3$, there exists $\epsilon_t > 0$ with the following property. For any positive integer n, let $I \subset K_t^n$ be an independent set such that $\epsilon = 1 - t\mu(I) < \epsilon_t$. Then there exists an independent set $J \subset K_t^n$ of maximum size ($\mu(J) = 1/t$) such that*

$$\mu(I \setminus J) \leq 4\epsilon^{\eta(K_t)} = 4\epsilon^{\log t/(\log t - \log(t-1))}. \tag{4}$$

▶ Remark 1.4. Since $\mu(I \setminus J) \leq 4\epsilon^{\eta(K_t)}$,

$$\mu(I\Delta J) = \mu(I \setminus J) + \mu(J \setminus I) = \mu(J) - \mu(I) + 2\mu(I \setminus J) = \frac{\epsilon}{t} + 8\epsilon^{\eta(K_t)},$$

so our result gives the optimal first-order structure for Theorem 1.2 assuming ϵ is sufficiently small. Furthermore, in Appendix B, we give examples of independent sets of K_t^n with arbitrarily small density (assuming $n \to \infty$) for which the exponent $\eta(K_t)$ is optimal.

Next, using a purely combinatorial argument we pin down a precise value for ϵ_t.

▶ **Theorem 1.5.** *In Theorem 1.3, for all $t \geq 3$, one may set $\epsilon_t = 1 - \frac{3}{t} + \frac{2}{t^2}$. In other words, the theorem applies for all independent sets I such that $\mu(I) > \frac{3t-2}{t^3}$.*

The choice of ϵ_t is not arbitrary, it corresponds to the density of the following independent set.

$$I = \{(1, 1, a), (1, a, 1), (a, 1, 1) \,:\, a \in [t]\} \times [t]^{n-3}.$$

Note that $\mu(I) = \frac{3t-2}{t^3}$. This set represents a phase transition in the independent sets from 'dictators' to 'juntas,' as the I constructed above is equally influenced by 3 coordinates (where 'influence' is in the sense of [1]). Such phase transitions have been studied in the literature [10], but this may be the first work to highlight the exact transition point.

Additionally, to the best of the author's knowledge, this is the first known purely combinatorial proof of Theorem 1.2.

1.3 Related work

Such stability results for independent sets have also been studied for Kneser graphs. A result similar to that of Theorem 1.2 was proved by [20]. Numerous other works in the literature [9, 11, 18, 19] use Fourier analysis to prove generalized stability results for Kneser graphs or other structures related to intersecting families. Other related works find purely combinatorial characterizations [3, 26, 27]. These results typically have a linear error bound ($\eta = 1$) on the closeness to maximal independent sets.

A result which also finds a "tight" super constant exponent $\eta > 1$ for the independent set stability is proved in some very recent work [14, 13, 16, 29, 28, 15] on Kneser graphs and related structures. (See also [12] and Proposition 4.3 of [17].) The techniques have high-level similarity to the ones adopted here:[1] particularly in their use of compressions to prove a isoperimetric inequality which they then bootstrap to a combinatorial independent set stability result.

1.4 Paper organization

In Section 2 we prove the claimed vertex isoperimetric inequalities. In Section 3, we prove the stability results for near-maximum independent sets in K_t^n. Appendix A proves some algebraic inequalities omitted from the main text. Appendix B shows that the exponent of $\eta(t)$ in Theorems 1.3 and 1.5 is optimal.

2 Vertex isoperimetric Inequalities

In this section, we proceed to prove the isoperimetry results claimed in Section 1.1.

Identify the vertex set of K_t^n with $[t]^n$. Two vertices of $x, y \in [t]^n$ are connected in K_t^n if and only if $x_i \neq y_i$ for all $i \in [n]$. Denote $y_{\neg i} := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. We often write y as $(y_i, y_{\neg i})$ when it is clear from context which coordinate is being inserted.

2.1 Compressions

A useful tool in our study will be the operation of the well-known technique of *compressions* (e.g., [30, 31]). Although compressions are not strictly necessary to prove Theorem 1.1, they are essential in the proof of stronger isoperimetry results as well as Theorem 1.5, so we introduce the machinery now.

For $S \subseteq [t]^n$ be a subset, define the *compression of S in coordinate i* to be

$$c_i(S) = \{x \in [t]^n \,:\, x_i \le |\{y \in S \,:\, y_{\neg i} = x_{\neg i}\}|\} \,. \tag{5}$$

This notion of compression appeared in the work of Bollobás and Leader [5].

Informally, we 'shift' each element of S to be as small as possible in the ith direction. Note that $\mu(c_i(S)) = \mu(S)$ for all $S \subseteq [t]^n$. It is easy to see that c_i is *idempotent*: $c_i(c_i(S)) = c_i(S)$ for all $S \subseteq [t]^n$ and $i \in [n]$.

We say that a set S is *compressed* if $c_i(S) = S$ for all $i \in [n]$. Equivalently, for all $x \in S$ there is no $y \in [t]^n \setminus S$ such that $x_i \le y_i$ for all $i \in [n]$.

▶ Remark 2.1. Note that every time a compression c_i is applied, the quantity

$$\Sigma(S) := \sum_{x \in S} \sum_{j \in [n]} x_j$$

[1] The author became aware of these similar proofs only after writing major portions of the manuscript.

decreases or stays the same (in which case $c_i(S) = S$). Thus, since $\Sigma(S)$ is always positive, there must exist a finite sequence of compressions which can be applied to S to make the set compressed.

Now we show that compressions respect independent sets of K_t^n. This result is not needed until Section 3, but the proof does give intuition for how the compressions work.

▶ **Claim 2.2.** *For all $i \in [n]$ and all $I \subset [t]^n$ independent set of K_t^n, $c_i(I)$ is also an independent set of K_t^n.*

Proof. Assume not, then there exist $x, y \in c_i(I)$ such that $\{x, y\}$ is an edge. In particular, since $x_i \neq y_i$, we must have that $x_i \neq 1$ or $y_i \neq 1$. Assume without loss of generality that $y_i \neq 1$. Then, by definition of $c_i(I)$, there must be $z := (1, y_{\neg i}) \in c_i(I)$. Since $x, y, z \in c_i(I)$, there must be $x', y', z' \in I$ such that

$$\begin{aligned} x_{\neg i} &= x'_{\neg i} \\ y_{\neg i} = z_{\neg i} &= y'_{\neg i} = z'_{\neg i} \\ y'_i &\neq z'_i. \end{aligned}$$

Since $y'_i \neq z'_i$, we must either have that $x'_i \neq y'_i$ or $x'_i \neq z'_i$. In the former case, $\{x', y'\}$ is an edge of K_t^n and in the latter case $\{x', z'\}$ is an edge of K_t^n. This contradicts the fact that I is an independent set. ◀

Next we show that compressions can only decrease the size of the vertex boundary.

▶ **Claim 2.3.** *For all $i \in [n]$ and $S \subseteq [t]^n$, $|\partial c_i(S)| \leq |\partial S|$.*

Proof. Fix $\bar{a} := a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in [t]$. Consider $T = \{(a_1, \ldots, a_{i-1})\} \times [t] \times \{(a_{i+1}, \ldots, a_n\} \subset [t]^n$.

Note that for every vertex $v \in [t]^n$, $\partial\{v\} \cap T$ either has 0 or $t - 1$ elements. Thus, $|T \cap \partial S| \in \{0, t - 1, t\}$. We claim that $|T \cap \partial c_i(S)| \leq |T \cap \partial S|$ for all T.

- If $|T \cap \partial S| = 0$, then there are no edges between S and T and shifting the vertices of S in the ith coordinate cannot change that. Thus, $|T \cap \partial c_i(S)| = 0$.
- If $|T \cap \partial S| = t - 1$, then the set $\partial T \cap S$ must be constant in the ith coordinate. Thus, $c_i(\partial T \cap S) = \partial T \cap c_i(S)$ is constant in the ith coordinate, so $|T \cap \partial c_i(S)| = t - 1$.
- If $|T \cap \partial S| = t$, then trivially $|T \cap \partial c_i(S)| \leq t$.

Thus, summing $|T \cap \partial c_i(S))| \leq |T \cap \partial S|$ across all possible T, we have that $|\partial c_i(S)| \leq |\partial S|$. ◀

▶ Remark 2.4. The proof crucially uses the fact that ∂S can include elements of S. If we instead had defined the vertex boundary to be $\partial S \setminus S$, there is a simple counterexample. Consider $t = 3$ and $n = 2$ and $S = \{(1, 2), (1, 3), (2, 1), (3, 1)\}$. Then it is not hard to check that $|\partial S| = |\partial c_1(S)| = 8$, but $|\partial S \setminus S| = 4 < 5 = |\partial c_1(S) \setminus c_1(S)|$.

2.2 Proof of Theorem 1.1

Define

$$\eta(t) := \frac{\log t}{\log t - \log(t-1)} = t \log t + \Theta(\log t). \tag{6}$$

First, we show that $\eta(K_t^n) \leq \eta(t)$. In fact, we show a whole family of equality cases.

▶ **Claim 2.5.** *For all positive integers n and t such that $t \geq 3$, $\eta(K_t^n) \leq \eta(t)$.*

Proof. For all integers $k \in [n]$, consider $S = \{1\}^k \times [t]^{n-k}$. Then $\partial S = \{2, \ldots, t\}^k \times [t]^{n-k}$. Thus,

$$\eta(K_t^n) \leq \frac{\log \mu(S)}{\log \mu(\partial S)} = \frac{\log t^{-k}}{\log((t-1)^k t^{-k})} = \frac{k \log \frac{1}{t}}{k \log \frac{t-1}{t}} = \eta(t). \qquad ◀$$

The lower-bound is more difficult, we first need the following inequality, proved in Appendix A.

▶ **Claim 2.6.** *Let $t \geq 2$ be a positive integer and let $x \geq y \geq 0$ be real numbers, then*

$$y^{1/\eta(t)} + (t-1)x^{1/\eta(t)} \geq (t-1)(x + (t-1)y)^{1/\eta(t)} \tag{7}$$

▶ **Lemma 2.7.** *For positive integers $n \geq 1$ and $t \geq 3$ and all $S \subseteq [t]^n$, we have that*

$$\mu(\partial(S)) \geq \mu(S)^{1/\eta(t)}. \tag{8}$$

Therefore $\eta(K_t^n) \geq \eta(t)$.

Proof. By Claim 2.3 and Remark 2.1, it suffices to consider the case that S is compressed. We now proceed by induction on n.

For our base case, $n = 1$, we must have that $S = \emptyset$ in which case (8) is trivial, or $S = [k]$ for some positive integer $k \leq t$. If $S = [1]$, then $\partial S = \{2, \ldots, t\}$, in which case we have an equality case of (8) by the proof of Claim 2.5. Otherwise, if $k \geq 2$, then $\partial S = [t]$, so $\mu(\partial S) = 1$, so (8) holds.

For $n \geq 2$, assume by the induction hypothesis that (8) is true for all $S \subseteq \mathbb{Z}_t^m$ where $1 \leq m < n$. For all $i \in [t]$, let

$$S_i := \{x_{\neg n} : x \in S, x_n = i\} \tag{9}$$

$$(\partial S)_i := \{x_{\neg n} : x \in \partial S, x_n = i\}. \tag{10}$$

Since S is compressed for all $1 \leq i \leq j \leq t$, we have that $S_i \supseteq S_j$. Thus, if $i \in \{2, \ldots, t\}$ is nonzero, for any $x \in (\partial S)_i$, there is $y \in S_1$ connected to x by an edge of K_t^{n-1}. Thus, $\partial S_1 \subseteq (\partial S)_i$. Similarly, for any $x \in (\partial S)_1$, there is $y \in S_2$ such that x is disjoint from y. Therefore, $\partial S_2 \subseteq (\partial S)_1$. Putting these together,

$$\begin{aligned}
\mu(\partial S) &= \frac{1}{t} \sum_{i \in [t]} \mu((\partial S)_i) \\
&\geq \frac{1}{t}(\mu(\partial S_2) + (t-1)\mu(\partial S_1)) \\
&\geq \frac{1}{t}\left(\mu(S_2)^{1/\eta(t)} + (t-1)\mu(S_1)^{1/\eta(t)}\right),
\end{aligned}$$

where we applied the inductive hypothesis in the last step. Applying Claim 2.6, using the fact that $0 \le \mu(S_2) \le \mu(S_1)$, we have that

$$\begin{aligned}
\mu(\partial(S)) &\ge \frac{1}{t}\left(\mu(S_2)^{1/\eta(t)} + (t-1)\mu(S_1)^{1/\eta(t)}\right) \\
&\ge \frac{t-1}{t}\left(\mu(S_1) + (t-1)\mu(S_2)\right)^{1/\eta(t)} \\
&\ge \frac{t-1}{t}\left(\sum_{i\in[t]} \mu(S_i)\right)^{1/\eta(t)} \\
&= \left(\frac{1}{t}\sum_{i\in[t]} \mu(S_i)\right)^{1/\eta(t)} \\
&= \mu(S)^{1/\eta(t)},
\end{aligned}$$

as desired. ◀

Claim 2.5 and Lemma 2.7 together imply Theorem 1.1.

2.3 A fine-tuned understanding of the isoperimetric profile

Recall that the (vertex) isoperimetric profile of a graph G is

$$\Phi(G,\nu) := \inf\{\mu(\partial S) \,:\, \mu(S) \ge \nu\}.$$

For $t \ge 3$ fixed, define

$$\Phi_t(\nu) := \inf_{n\ge 1} \Phi(K_t^n, \nu).$$

Note that Φ_t is non-decreasing. To avoid complications with the discrete behavior of $\Phi(K_t^n,\nu)$ when n is small, it is easier to instead work with $\Phi_t(\nu)$. By Theorem 1.1,

$$\Phi_t(\nu) \ge \nu^{1/\eta(t)}. \tag{11}$$

This is tight whenever $\nu = t^{-k}$ for any integer $k \ge 0$, but ceases to be tight when $\log_t(\nu)$ is non-integral (see Figure 1).

The following recursive relationship allows one to compute $\Phi_t(\nu)$ to arbitrary precision.

▶ **Theorem 2.8.** *For all $t \ge 3$,*

$$\Phi_t(\nu) = \begin{cases} \frac{t-1}{t}\Phi_t(t\nu) & \nu < 1/t \\ \frac{t-1}{t} + \frac{1}{t}\Phi_t\left(\frac{t\nu-1}{t-1}\right) & \nu \ge 1/t \end{cases}. \tag{12}$$

Using the simple fact that $\Phi_t(0) = 0$ and $\Phi_t(1) = 1$, the above equation is extremely powerful. For example,

$$\Phi_3\left(\frac{5}{9}\right) = \frac{2}{3} + \frac{1}{3}\Phi_3\left(\frac{1}{3}\right) = \frac{8}{9},$$

which is an exact bound compared to $(\frac{5}{9})^{1/\eta(3)} \approx \frac{7.24}{9}$. This recursion is what allowed the creation of Figure 1.

Theorem 2.8 is proved in the full version. This more refined understanding of Φ_t proves critical in the combinatorial proof of Theorem 1.5.

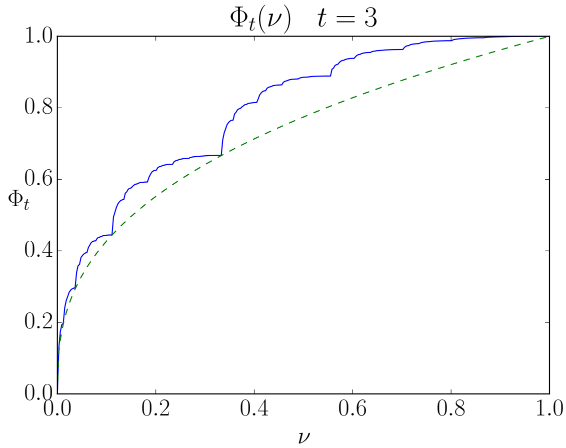

Figure 1 A graph of $\Phi_t(\nu)$ for $t = 3$. The dashed curve $\nu^{1/\eta(t)}$ is for reference.

3 Independent set stability results

In this section, we seek to prove the main independent set stability result, Theorem 1.5. This is done in two stages. First, we prove a simpler version (Theorem 1.3) where we use the weaker vertex isoperimetry inequality to amplify Theorem 1.2 of [1] in a "black-box" manner. Second, we utilize the fine-grained vertex isoperimetry inequality in a fully combinatorial inductive proof to obtain the full Theorem 1.5 without dependence of Theorem 1.2 of [1].

3.1 Black-box result for clique tensor powers

First, we show that if a large independent set I is somewhat close to a maximum-sized independent set J, then it is really close to J. We fix positive integers n and $t \geq 3$.

▶ **Lemma 3.1.** *Let $I \subset [t]^n$ be an independent set with $\epsilon := 1 - t\mu(I)$. Assume there exists a maximum-sized independent set J such that*

$$\mu(I \setminus J) < \frac{1}{t^3}.$$

Then,

$$\mu(I \setminus J) < 4\epsilon^{\eta(t)}.$$

Proof. Without loss of generality, we may assume that $J = [t]^{n-1} \times [1]$. Pick $J' = [t]^{n-1} \times \{j\}$ such that $j \neq 1$ but otherwise $\mu(I \cap J')$ is maximal. Let $\delta := \mu(I \setminus J)$. Since J and J' are disjoint, we have that

$$\mu(I \cap J') \geq \frac{\mu(I \setminus J)}{t-1} = \frac{\delta}{t-1}.$$

Now, consider $S = \partial(I \cap J')$. Recall the definition of $S_k \subseteq [t]^{n-1}$ from (9). Since $I \cap J' \subseteq J'$ has the property that every element has the same last coordinate, $S_k = S_{k'}$ for all $k, k' \neq j$ and $S_j = \emptyset$. Thus, $\mu(S_k) = \frac{t}{t-1}\mu(S)$ for all $k \neq j$. Therefore,

$$\mu(S \cap J) = \frac{1}{t}\mu((S \cap J)_i) = \frac{1}{t}\mu(S_i) = \frac{1}{t-1}\mu(S).$$

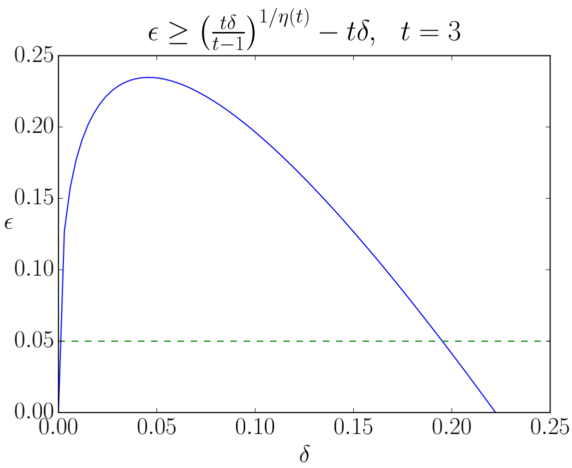

Figure 2 Plot of (15) when $t = 3$. Notice the bifurcation of solutions to (15) for a fixed ϵ (line $\epsilon = 0.05$ is dashed).

Applying Theorem 1.1, we get that

$$\mu(S \cap J) = \frac{1}{t-1}\mu(\partial(I \cap J')) \geq \frac{1}{t-1}\mu(I \cap J')^{1/\eta(t)} \geq \frac{1}{t-1}\left(\frac{\delta}{t-1}\right)^{1/\eta(t)}.$$

Since I is an independent set, ∂I is disjoint from I. Since $S \cap J = \partial(I \cap J') \cap J \subseteq \partial I$, we have that $I \cap J$ and $S \cap J$ are disjoint. Therefore,

$$\mu(I \cap J) \leq \mu(J) - \mu(S \cap J) \leq \frac{1}{t} - \frac{1}{t-1}\left(\frac{\delta}{t-1}\right)^{1/\eta(t)}. \tag{13}$$

But, we also know that

$$\mu(I \cap J) = \mu(I) - \mu(I \setminus J) = \frac{1}{t}(1-\epsilon) - \delta. \tag{14}$$

By (13) and (14)

$$\frac{1}{t}(1-\epsilon) - \delta \leq \frac{1}{t} - \frac{1}{t-1}\left(\frac{\delta}{t-1}\right)^{1/\eta(t)} = \frac{1}{t} - \frac{1}{t}\left(\frac{t\delta}{t-1}\right)^{1/\eta(t)}.$$

Thus,

$$\epsilon \geq \left(\frac{t\delta}{t-1}\right)^{1/\eta(t)} - t\delta \geq \delta^{1/\eta(t)} - t\delta. \tag{15}$$

Consider Figure 2 which has a plot of the RHS of (15) when $t = 3$. If ϵ is sufficiently small, then the inequality holds only when δ is very small (polynomial in ϵ) or very large (about $\frac{1}{t}$). Since is 'moderately' small ($\delta \leq \frac{1}{t^3}$), we must have that δ is very small. Quantitatively, note that

$$\begin{aligned} t\delta &= t\delta^{1/\eta(t)}\delta^{1-1/\eta(t)} \\ &\leq t\delta^{1/\eta(t)}\left(\frac{1}{t^3}\right)^{1-1/\eta(t)} \\ &= t\delta^{1/\eta(t)}\frac{1}{t^3}\left(\frac{t^3}{(t-1)^3}\right) \\ &\leq \frac{t\delta^{1/\eta(t)}}{(t-1)^3}. \end{aligned}$$

So

$$\epsilon \geq \delta^{1/\eta(t)} \left(1 - \frac{t}{(t-1)^3}\right).$$

Therefore,

$$\delta \leq \left(\frac{(t-1)^3}{(t-1)^3 - t}\right)^{\eta(t)} \epsilon^{\eta(t)} \leq 4\epsilon^{\eta(t)},$$

where the last inequality follows from the following claim which is proved in Appendix A.

▶ **Claim 3.2.** *For all $t \geq 3$,*

$$\left(\frac{(t-1)^3}{(t-1)^3 - t}\right)^{\eta(t)} \leq 4.$$ ◀

We now use this lemma to 'amplify' Theorem 1.2 to prove Theorem 1.3.

Proof of Theorem 1.3. Set $\epsilon_t := \frac{1}{C_t t^3} > 0$. Consider any independent set I of of K_t^n such that $\epsilon := 1 - t\mu(I) < \epsilon_t$. Pick any maximum-sized J guaranteed by Theorem 1.2 such that

$$\delta := \mu(I \setminus J) \leq \mu(I \Delta J) \leq C_t \epsilon < \frac{1}{t^3}. \tag{16}$$

By Lemma 3.1, we have that

$$\delta \leq 4\epsilon^{\eta(t)},$$

as desired. ◀

3.2 Improved stability result for clique tensor powers

In this section we improve ϵ_t in Theorem 1.3 to an explicit expression. In fact, we may show that

$$\epsilon_t = 1 - \frac{3}{t} + \frac{2}{t^2}$$

which corresponds to independent sets I for which $\mu(I) > \frac{3t-2}{t^3}$.

Proofs of claims and lemmas in this section are reserved for the full version.

First, we try to show that if an independent set I is large enough, then I is either very close to or very far from a maximum-sized independent set. To do this, we show that if I is 'moderately far' from a maximum-sized independent set, then this moderate-sized portion which is not in the maximum-sized independent set has such a large vertex boundary that it precludes a large portion of the maximum-sized independent set from being part of I, forcing the density of I to be at or below our threshold of $\frac{3t-2}{t^3}$.

We need a notation for the maximum sized independent sets. For all $i \in [t]$ and $j \in [n]$ let

$$J_{i,j} = [t]^{j-1} \times \{i\} \times [t]^{n-j}. \tag{17}$$

We say that I is *sorted* if there exists that for all $i_1, i_2 \in [t]$ and $j \in [n]$ we have that $i_1 \leq i_2$ implies that

$$\mu(I \cap J_{i_1,j}) \leq \mu(I \cap J_{i_2,j}).$$

Note that unlike compressions, we may assume without loss of generality that I is sorted since permuting the labels so that an independent set is sorted does not change its intersection sizes with the maximum independent sets.

▶ **Claim 3.3.** *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$ (or $1-t\mu(I) < \epsilon_t$), then for all $j \in [n]$,*

$$\mu(I \setminus J_{1,j}) < \frac{t-1}{t^4} \text{ or } \mu(I \setminus J_{1,j}) > \frac{t-1}{t^3}. \tag{18}$$

From Theorem 2.8, we can attain a bound that is even better.

▶ **Claim 3.4.** *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for all $j \in [n]$,*

$$\mu(I \setminus J_{1,j}) < \frac{t-1}{t^4} \text{ or } \mu(I \setminus J_{1,j}) > \frac{(2t-1)(t-1)}{t^4}. \tag{19}$$

The next key step is to show Theorem 1.5 essentially holds for *compressed* independent sets I.

▶ **Lemma 3.5.** *Let $I \subset [t]^n$ be a compressed independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for some $j \in [n]$,*

$$\mu(I \setminus J_{1,j}) < \frac{t-1}{t^4}. \tag{20}$$

Note that by Lemma 3.1, we immediately have that Theorem 1.5 holds for compressed independent sets.

Now we extend this result to sorted independent sets; and thus all independent sets.

▶ **Lemma 3.6.** *Let $I \subset [t]^n$ be a sorted independent set such that $\mu(I) > \frac{3t-2}{t^3}$, then for some $j \in [n]$,*

$$\mu(I \setminus J_{1,j}) < \frac{t-1}{t^4}. \tag{21}$$

Proof of Theorem 1.5. Let $I \subset [t]^n$ be an independent set with $\mu(I) > \frac{3t-2}{t^3}$. Assume without loss of generality that I is sorted. By Lemma 3.6, we know that there is $j \in [n]$ such that

$$\mu(I \setminus J_{1,j}) \le \frac{t-1}{t^4} < \frac{1}{t^3}.$$

Thus, by Lemma 3.1, we have that

$$\mu(I \setminus J_{1,j}) \le 4\epsilon^{\eta(t)},$$

as desired. ◀

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A Proofs of algebraic inequalities

Proof of Claim 2.6. Let $\alpha(t) = 1/\eta(t)$. For $c \ge 0$, let $f_c(z) = (z+c)^{\alpha(t)} - z^{\alpha(t)}$. Notice that if $z > 0$, then $f_c'(z) = (\alpha(t))((z+c)^{\alpha(t)-1} - z^{\alpha(t)-1}) \le 0$. Thus, we have that $(t-1)f_c(y) \ge (t-1)f_c(x)$ for all $c \ge 0$. Consider $c = (t-1)y$; we then have that

$$\begin{aligned}&(t-1)f_c(y) = (t-1)((ty)^{\alpha(t)} - y^{\alpha(t)}) = (t-1)(t^{\alpha(t)} - 1)y^{\alpha(t)} = y^{\alpha(t)} \ge \\ &(t-1)f_c(x) = (t-1)((x+(t-1)y)^{\alpha(t)} - x^{\alpha(t)}).\end{aligned}$$

Rearranging, we obtain (7). ◀

Proof of Claim 3.2. First, verify the cases $t = 3$ and $t = 4$ using a calculator. Notice that $\eta(t) = \frac{\log t}{\log t - \log(t-1)} \le t \log t$ so

$$\left(\frac{(t-1)^3}{(t-1)^3 - t}\right)^{\eta(t)} \le e^{\frac{t\eta(t)}{(t-1)^3-t}} \le e^{\frac{t^2 \log t}{(t-1)^3-t}}.$$

Also use a calculator to verify that $h(t) := \frac{t^2 \log t}{(t-1)^3-t}$ is less than 1 for $t = 5$. Now observe that when going from t to $t+1$, the numerator increases by

$$\begin{aligned}(t+1)^2 \log(t+1) - t^2 \log t &= (2t+1)\log(t+1) + t^2 \log(1 + \frac{1}{t}) \\ &\le (2t+1)\log(t+1) + t \le (2t+1)t + t \\ &= 2t^2 + 2t.\end{aligned}$$

and the denominator increases by

$$t^3 - (t+1) - (t-1)^3 + t = 3t^2 - 3t$$

Since $2t^2 + 2t \le 3t^2 - 3t$ for all $t \ge 5$ and $h(5) \le 1$, we have by a simple inductive proof that $h(t) \le 1$ for all $t \ge 5$. Thus, for all $t \ge 5$,

$$\left(\frac{(t-1)^3}{(t-1)^3 - t}\right)^{\eta(t)} \le e^1 < 4,$$

as desired. ◀

B Optimality of exponent in Theorem 1.3

In this appendix, we show in (4) of Theorem 1.3 that the exponent $\eta(t) = \frac{\log t}{\log t - \log(t-1)}$ is optimal and that the constant factor of 4 is nearly optimal. In other words, the stability result is optimal up to a constant factor.

▶ **Lemma 2.1.** *For all $t \ge 3$, there exists an infinite sequence of independent sets $\{I_n\}_{n\ge3}$ such that $I_n \subset [t]^n$, $\epsilon_n = 1 - t\mu(I_n) > 0$ tends to 0 as $n \to \infty$, and for any n and any maximum-sized independent set J_n of K_t^n,*

$$\mu(I_n \setminus J_n) > \frac{t-1}{t}\epsilon^{\eta(t)}.$$

Proof. For $n \ge 3$, consider $J_n = [1] \times [t]^{n-1}$ and

$$I_n := (([t] \times [1]^{n-1}) \cup J_n) \setminus ([1] \times \{2, \dots, t-1\}^n) \tag{22}$$

See Figure 3 for a visualization. It has been noted to the author that this construction is similar in structure to the constructions in the Hilton-Milner theorem [24].

One may check that I_n is an independent set of K_t^n and J_n is a maximum-sized independent set which minimizes $\mu(I_n \setminus J_n)$. Furthermore,

$$\mu(I_n) = \frac{t-1}{t^n} + \frac{1}{t} - \frac{(t-1)^{n-1}}{t^n}.$$

Thus,

$$\epsilon_n = \frac{(t-1)^{n-1} - (t-1)}{t^{n-1}} \tag{23}$$

$$\delta_n := \mu(I_n \setminus J_n) = \frac{t-1}{t^n}. \tag{24}$$

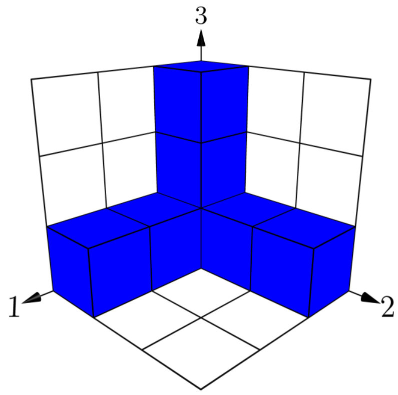

Figure 3 Schematic of I_3 when $t = 3$.

Notice that since $t^{1/\eta(t)} = \frac{t-1}{t}$.

$$\begin{aligned}
\delta_n^{1/\eta(t)} &= \frac{(t-1)^{1/\eta(t)}}{t^{n/\eta(t)}} \\
&= \left(\frac{t-1}{t}\right)^{1/\eta(t)} \left(\frac{t-1}{t}\right)^{n-1} \\
&= \left(\frac{t-1}{t}\right)^{1/\eta(t)} (\epsilon_n + t\delta_n) \\
&> \left(\frac{t-1}{t}\right)^{1/\eta(t)} \epsilon_n.
\end{aligned}$$

Therefore, raising both sides to the $\eta(t)$ power,

$$\delta_n > \frac{t-1}{t}\epsilon_n^{\eta(t)},$$

as desired. ◀