Fast(er) Reasoning in Interval Temporal Logic*

Davide Bresolin[1], Emilio Muñoz-Velasco[2], and Guido Sciavicco[3]

1 **Department of Mathematics, University of Padova, Italy**
`davide.bresolin@unipd.it`
2 **Department of Applied Mathematics, University of Málaga, Spain**
`ejmunoz@uma.es`
3 **Department of Mathematics and Computer Science, University of Ferrara, Italy**
`guido.sciavicco@unife.it`

Abstract

Clausal forms of logics are of great relevance in Artificial Intelligence, because they couple a high expressivity with a low complexity of reasoning problems. They have been studied for a wide range of classical, modal and temporal logics to obtain tractable fragments of intractable formalisms. In this paper we show that such restrictions can be exploited to lower the complexity of interval temporal logics as well. In particular, we show that for the Horn fragment of the interval logic $\mathbf{A}\overline{\mathbf{A}}$ (that is, the logic with the modal operators for Allen's relations *meets* and *met by*) without diamonds the complexity lowers from NExpTime-complete to P-complete. We prove also that the tractability of the Horn fragments of interval temporal logics is lost as soon as other interval temporal operators are added to $\mathbf{A}\overline{\mathbf{A}}$, in most of the cases.

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1 Introduction

Sub-propositional logics are particularly interesting from a practical point of view, because they couple a high expressivity with a low complexity of reasoning problems. There are two standard ways to weaken the classical propositional language based on the clausal form of formulas: the *Horn fragment*, that only allows clauses with at most one positive literal [18], and the *Krom fragment*, that only allows clauses with at most two (positive or negative) literals [20]. The *core fragment* combines both restrictions. In the case of *modal* logics two more possible restrictions arise, namely by excluding the use of *universal* or the use of *existential* modal operators, giving rise, respectively, to the *diamond fragment* and the *box fragment* of modal logic. Such restrictions apply, as well, to temporal, spatial, and description logic, and, in each case, combined with classical sub-propositional restriction, they generate a complex scenario encompassing several fragments.

Weakening a logic is motivated by the search of computationally well-behaved fragments. For example, the satisfiability problem for classical propositional Horn logic is P-complete, while the classical propositional Krom logic satisfiability problem (also known as the 2-SAT problem) is NLogSpace-complete [23], and the same holds for the core fragment, where

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the classical NLogSpace-hardness theorem for 2-SAT can be easily strengthened to deal with the case of binary Horn clauses. The satisfiability problem for quantified propositional logic (**QBF**), which is PSpace-complete in its general form, becomes P when formulas are restricted to binary (Krom) clauses [7]. Horn fragments of modal logic **K** and of several of its axiomatic extensions have been considered in [12, 13, 15, 16, 22]; in particular, it is known that **K**, **T**, **K4**, and **S4** are all PSpace-complete even under the Horn restriction, but they become P-complete under the Horn restriction without diamonds [12]. In [4, 14], the authors study different sub-propositional fragments LTL. By excluding the operators Since and Until from the language, and keeping only the Next/Previous-time operators and the box version of Future and Past, it is possible to prove that the Krom and core fragments are NP-hard, while the Horn fragment is still PSpace-complete (the same complexity of the full language). Moreover, the complexity of the Horn, Krom, and core fragments without Next/Previous-time operators range from NLogSpace (core), to P (Horn), to NP-hard (Krom). Where only a *global* (anywhere in time) modality is allowed, their complexity is even lower (from NLogSpace to P). Temporal extensions of the description logic DL-Lite have been studied under similar sub-propositional restrictions, and similar improvements in the complexity have been obtained [5].

In this paper we show that clausal forms can be exploited to lower the complexity of interval temporal logics as well. Halpern and Shoham's Modal Logic of Allen's Relations (**HS**) [17] is a multi-modal logic where each world is interpreted as an *interval* in time, and accessibility relations are built over Allen's relations [2]. So, **HS** is the classical propositional logic extended with six modal operators *later* (denoted by $\langle L\rangle$), *meets* ($\langle A\rangle$), *overlaps* ($\langle O\rangle$), *during* ($\langle D\rangle$), *finishes* ($\langle E\rangle$), and *starts* ($\langle B\rangle$), plus their inverse ones (equality does not play any role in modal logic). The finite satisfiability problem for fragments of **HS** (that is, logics that emerge from a systematic exclusion of some of these modalities) has been studied in [9], resulting in a very informative picture of the various classes of fragments classified by their computational complexity. The Horn, Krom, and core restrictions of **HS** are still undecidable [11], but weaker restrictions have shown positive results. In particular, the Horn fragment of **HS** without diamonds becomes P-complete in two interesting cases [6, 8]: first, when it is interpreted over dense linear orders, and, second, when the semantics of its modalities becomes reflexive. On the other hand, in the classical irreflexive semantics, interpreted on finite/discrete linear orders, the Horn fragment of **HS** without diamonds is still undecidable, and this justifies our interest in finding well-behaved syntactical fragments of it. Without restrictions, the satisfiability for $\mathbf{A}$ and for $\mathbf{A}\overline{\mathbf{A}}$ is NExpTime-complete in the finite case, the case of natural numbers, the integers, and the class of all discrete linear orders [10]. The purpose of this paper is to prove that their Horn fragments without diamonds become P-complete at least in the finite case, which is (usually) emblematic for the behaviour of interval temporal logics in the whole range of classes of discrete linear orders. We shall also prove that tractability of these fragments is lost as soon as other modal operators from the **HS** machinery are added, in most of the cases.

2 Preliminaries

Let $\mathbb{D} = \langle D, <\rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in D$ and $x < y$[1]. There are 12 different relations between two intervals in a linear order,

[1] This definition excludes point intervals $[x, x]$, and conforms to the one adopted by Allen in [1] and by most of the recent literature; it differs from the one in [17], where point intervals are allowed.

HS	Allen's relations	Graphical representation
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$	
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$	
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$	
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$	
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$	
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$	

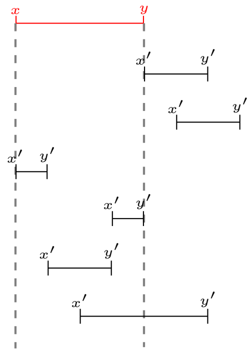

Figure 1 Allen's interval relations and **HS** modalities.

often called *Allen's relations* [1]: the six relations R_A (adjacent to), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Fig. 1, and their inverses $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

Halpern and Shoham's logic **HS** [17] is a multi-modal logic with formulae built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (or proposition letters), the classical propositional connectives, and a modality for each Allen relation:

$$\varphi ::= \bot \mid p \mid \neg\psi \mid \psi \vee \xi \mid \psi \wedge \xi \mid \langle X \rangle \psi \mid \langle \overline{X} \rangle \psi,$$

where $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O\}$. The other propositional connectives and constants (e.g., $\rightarrow$, and $\top$), and the universal modalities $[X]$ and $[\overline{X}]$ can be derived in the standard way. A *syntactical fragment* of **HS** is any fragment obtained by selecting a specific subset $\mathcal{F} = \{X_1, X_2, \ldots, X_n\}$ of relations, and denoted by $\mathrm{X_1X_2 \ldots X_n}$.

The semantics of **HS** is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula φ on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulae, as follows:

- $M, [x, y] \Vdash p$ if $[x, y] \in V(p)$, for $p \in \mathcal{AP}$;
- $M, [x, y] \Vdash \neg\psi$ if $M, [x, y] \nVdash \psi$;
- $M, [x, y] \Vdash \psi \vee \xi$ if $M, [x, y] \Vdash \psi$ or $M, [x, y] \Vdash \xi$;
- $M, [x, y] \Vdash \psi \wedge \xi$ if $M, [x, y] \Vdash \psi$ and $M, [x, y] \Vdash \xi$;
- $M, [x, y] \Vdash \langle X \rangle \psi$ if there exists $[z, t]$ such that $[x, y]R_X[z, t]$ and that $M, [z, t] \Vdash \psi$;
- $M, [x, y] \Vdash \langle \overline{X} \rangle \psi$ if there exists $[z, t]$ such that $[x, y]R_{\overline{X}}[z, t]$ and that $M, [z, t] \Vdash \psi$.

In this paper we are interested in finite models, so that for all intent and purposes, we assume that $D = \{0, 1, \ldots, N - 1\}$ is a finite domain. If a formula φ is satisfiable on a model of length N, we say that φ is *N-satisfiable*. The finite satisfiability problems of **HS** is undecidable, and the computational complexity of its decidable (syntactical) fragments range from P-complete to non-primitive recursive (see [9] and references within).

In order to define sub-propositional fragments of **HS** we start from the *clausal form* of formulas of **HS**, whose building blocks are the *positive literals*:

$$\lambda \quad ::= \quad \top \mid p \mid \langle X \rangle \lambda \mid [X] \lambda \mid \langle \overline{X} \rangle \lambda \mid [\overline{X}] \lambda, \tag{1}$$

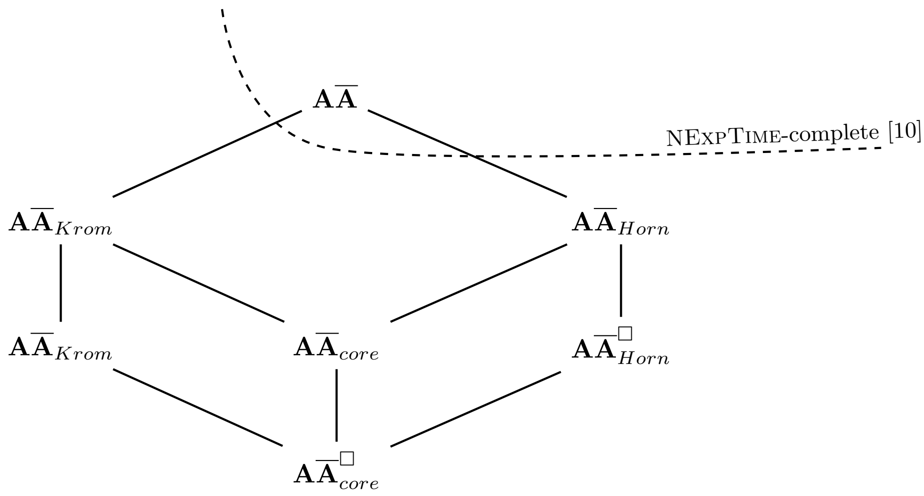

Figure 2 Sub-propositional fragments of $\mathbf{A\overline{A}}$.

and we say that φ is in *clausal form* if it can be generated by the following grammar:

$$\varphi \quad ::= \quad \lambda \mid \psi \wedge \xi \mid [U](\neg\lambda_1 \vee \ldots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \ldots \vee \lambda_{n+m}), \tag{2}$$

where $[U](\neg\lambda_1 \vee \ldots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \ldots \vee \lambda_{n+m})$ is a *clause*, and $[U]$ is the *global operator*. A formula $[U]\psi$ is true on the interval $[x, y]$ if and only if ψ holds on every interval $[z, t]$ of the model. Depending on the language, the global operator can be either definable using the other modalities, or included as a primitive modality. From now on we follow the latter approach and we consider $[U]$ as part of the language. In the following, we use $\varphi_1, \varphi_2, \ldots$ to denote clauses, $\lambda_1, \lambda_2, \ldots$ to denote positive literals, and $\varphi, \psi, \xi, \ldots$ to denote formulas. We write clauses in their implicative form $[U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1} \vee \ldots \vee \lambda_{n+m})$, and use $\perp$ as a shortcut for $\neg\top$. It is known that every modal logic formula can be rewritten in clausal form [22], and this holds for **HS**, as well. Thus, any formula φ can be written as a conjunction $\varphi_1 \wedge \ldots \wedge \varphi_n$ of positive literals and global clauses and, from now on, will be represented as the set $\{\varphi_1, \ldots, \varphi_n\}$ of positive literals and global clauses.

Sub-propositional fragments of **HS** can be defined by constraining the cardinality and the structure of clauses: the fragment in which each clause in (2) is such that $m \leq 1$ is called *Horn* fragment, and denoted by $\mathbf{HS}_{Horn}$, while the fragment in which each clause is such that $n + m \leq 2$ it is called *Krom* fragment, and it is denoted by $\mathbf{HS}_{Krom}$; when both restrictions apply we obtain the *core* fragment, denoted by $\mathbf{HS}_{core}$. We constrain fragments also by allowing only the use of universal modalities and only existential modalities in positive literals, obtaining the *box* and *diamond* fragments of modal logic **HS**. In this way, we define $\mathbf{HS}^{\Box}_{Horn}$ and $\mathbf{HS}^{\Diamond}_{Horn}$ as, respectively, the box and the diamond fragments of $\mathbf{HS}_{Horn}$. By applying the same restrictions to $\mathbf{HS}_{Krom}$ and $\mathbf{HS}_{core}$, we obtain the pair $\mathbf{HS}^{\Box}_{Krom}$ and $\mathbf{HS}^{\Diamond}_{Krom}$ from the former, and the pair $\mathbf{HS}^{\Box}_{core}$ and $\mathbf{HS}^{\Diamond}_{core}$, from the latter. These definitions are borrowed from [13, 12]. In this paper we are interested in studying sub-propositional syntactical fragments of **HS**, and, in particular, the fragment $\mathbf{A\overline{A}}$ under the Horn restriction without diamonds, that is, $\mathbf{A\overline{A}}^{\Box}_{Horn}$; the interesting fragments are shown in Fig. 2, ordered by syntactical containment.

The fragment $\mathbf{A\overline{A}}^{\Box}_{Horn}$ is very similar to its reflexive version (see [6, 8]) but there are significant differences. For instance, in the finite/discrete case the satisfiability problem for irreflexive $\mathbf{HS}^{\Box}_{Horn}$ is still undecidable, while it is P-complete in its reflexive version, which means that, in general, the good computational behaviour of a fragment of **HS** is not necessarily related to that of its reflexive variant. As we have already pointed out, in this paper we limit ourselves to study the complexity of the satisfiability problem in the finite case: this is not only interesting on its own, but, also, it is emblematic for the entire range of (strongly) discrete linearly ordered sets, such as $\mathbb{N}$ or $\mathbb{Z}$.

Finally, observe that the fragment $\mathbf{A\overline{A}}^{\Box}_{Horn}$, despite its weakness, can be still used to describe meaningful situations. First, notice that $[U](\langle A\rangle\lambda \wedge \varphi \rightarrow \lambda')$ can be written as $[U](\lambda \rightarrow [\overline{A}](\varphi \rightarrow \lambda'))$, which is to say that $\mathbf{A\overline{A}}^{\Box}_{Horn}$ allows a (limited) use of diamonds. Then, for example, we can express a medical statement such as *During therapy no patient is allowed to smoke* with a simple combination of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$:

$$\begin{array}{ll} [U](therapy \rightarrow [A](stop \wedge [A]stop)) & \textit{therapy's boundaries} \\ \left.\begin{array}{l} [U](therapy \rightarrow [\overline{A}][A](\langle A\rangle\langle A\rangle(smoking \wedge \langle A\rangle stop) \rightarrow \bot)) \\ [U](therapy \rightarrow [\overline{A}][A](\langle A\rangle(smoking \wedge \langle A\rangle stop) \rightarrow \bot)) \end{array}\right\} & \textit{smoking constraint} \end{array}$$

3 P-Completeness of $\mathbf{A\overline{A}}^{\Box}_{Horn}$

The complexity of the finite satisfiability problem for $\mathbf{A\overline{A}}$ is NExpTime-complete. In this section, we prove that the same problem for $\mathbf{A\overline{A}}^{\Box}_{Horn}$ is P-complete (and, therefore, for $\mathbf{A\overline{A}}^{\Box}_{core}$ it is in P), and hence that, at least in this case, interval logics are showing a similar behaviour to modal logic [12].

The fragment of $\mathbf{A\overline{A}}^{\Box}_{Horn}$ in which we limit the modal depth to one is *model-extending equivalent* to $\mathbf{A\overline{A}}^{\Box}_{Horn}$, that is, equivalent at the price of adding fresh propositional letters, with a polynomial translation that witnesses such an equivalence. Therefore, for the purposes of this section, we may assume that every literal has modal depth at most one. Given a finitely satisfiable set of clauses φ, among any set of models of fixed length N, we can define a notion of *ordering*: we say that $M_1 \leq_N M_2$ if and only if M_1 and M_2 are based on the same domain (of length N) and, for each $p \in \mathcal{AP}$, $V_1(p) \subseteq V_2(p)$; obviously, $M_1 <_N M_2$ if $M_1 \leq_N M_2$ and $M_1 \neq M_2$.

▶ **Lemma 1.** *Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, and let $\mathcal{M}$ be the (finite) set of models of length N that satisfy φ on the interval $[x, y]$. Then, $(\mathcal{M}, \leq_N)$ has a minimum.*

Proof. Let φ be a satisfiable set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, and let $\mathcal{M}$ be the set of interval models of length N that satisfy φ. Obviously, $\mathcal{M}$ is finite. If $\mathcal{M}$ is a singleton, then the property is trivially true. Suppose, then, that there exist at least two $\leq_N$-incomparable interval models, denoted by M_1 and M_2, in $\mathcal{M}$. Thus, $M_1, [x, y] \Vdash \varphi$ and $M_2, [x, y] \Vdash \varphi$. We define a new model $M' = \langle \mathbb{I}(\mathbb{D}), V'\rangle$ based on the same domain as M_1 and M_2 and such that its valuation V', for each propositional letter p, is $\{[t, t'] \mid [t, t'] \in V_1(p) \cap V_2(p)\}$. We claim that $M', [x, y] \Vdash \varphi$. Suppose, by way of contradiction, that this is not the case. So, there exists a clause φ_i of φ such that $M', [x, y] \not\Vdash \varphi_i$. Several cases arise:

- If $\varphi_i = p$ for some propositional letter p, then we have a contradiction given that $M', [x, y] \not\Vdash \varphi_i$ implies that $M_1, [x, y] \not\Vdash p$ or $M_2, [x, y] \not\Vdash p$;
- If $\varphi_i = [A]p$ (the case for $\varphi_i = [\overline{A}]p$ is symmetrical) for some propositional letter p, then we have a contradiction given that $M', [x, y] \not\Vdash \varphi_i$ implies that for some $z > y$ $M', [y, z] \not\Vdash p$, that is, $M_1, [y, z] \not\Vdash p$ or $M_2, [y, z] \not\Vdash p$, that is, $M_1, [x, y] \not\Vdash [A]p$ or $M_2, [x, y] \not\Vdash [A]p$;

- If $\varphi_i = [U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda)$ or $\varphi_i = [U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \bot)$, then, for some $[t, t']$, we have that $M', [t, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ and $M', [t, t'] \nVdash \lambda$. Consider the generic antecedent λ_j. If it is a propositional letter p, then $M', [t, t'] \Vdash \lambda_j$ implies that both $M_1, [t, t'] \Vdash \lambda_j$ and $M_2, [t, t'] \Vdash \lambda_j$. On the other hand, if $\lambda_j = [A]p$ (the case for $[\overline{A}]p$ is symmetrical) for some propositional letter p, then $M', [t, t'] \Vdash \lambda_j$ implies that, for each $t'' > t'$, $M', [t', t''] \Vdash p$, which, in turn, implies that both $M_1, [t', t''] \Vdash p$ and $M_2, [t', t''] \Vdash p$ for each $t'' > t'$. Either way, if $M', [t, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$, then both $M_1, [t, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ and $M_2, [t, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$. If $\lambda = \bot$, this is already a contradiction. If $\lambda = p$ for some propositional letter p, then both $M_1, [t, t'] \Vdash p$ and $M_2, [t, t'] \Vdash p$, which implies that $M', [t, t'] \Vdash p$, which is a contradiction. If $\lambda = [A]p$ (and, again, the case for $[\overline{A}]p$ is symmetrical) for some propositional letter p, then we have that both $M_1, [t', t''] \Vdash p$ and $M_2, [t', t''] \Vdash p$ for each $t'' > t$, that is, $M', [t, t'] \Vdash [A]p$, which is a contradiction.

This proves that M' is an interval model such that $M', [x, y] \Vdash \varphi$, that is, that $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$ is closed under intersection. Clearly, $M' \leq_N M_1, M_2$ and $M' \in \mathcal{M}$: by iterating this process we prove that $\mathcal{M}$ has a minimum with respect to $\leq_N$. ◀

Notice that the previous result strongly depends on the absence of diamonds in the language.

The above results prove that the finite satisfiability of a set φ of clauses of $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$ can be reduced to the problem of finding an interval model M of length N that satisfies φ on some $[x, y]$ and is the minimum among the models of length N that satisfy φ. Let us call such a model an *N-minimum model* (or, simply, *minimum model*). Now, we want to prove that every set φ of clauses of $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$ is finitely satisfiable if and only if it is satisfiable on a minimum model of cardinality polynomial in $|\varphi|$.

For a set φ of clauses of $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$, let $\mathcal{L}(\varphi)$ be the *literal closure* of φ, that is, the set of all the positive literals that occur as subformulas of φ. Given a model M we can univocally identify the literals that are satisfied on each interval $[z, t]$, and we can define the *label* of $[z, t]$ as $L([z, t]) = \{\lambda \in \mathcal{L} \mid M, [z, t] \Vdash \lambda\}$. Positive literals of the type $[A]p$ (resp., $[\overline{A}]p$) are called *A-temporal* literals (resp., *$\overline{A}$-temporal* literals). It is easy to observe that any two intervals $[z, t]$ and $[z', t]$ that end at the same point should satisfy the same set of A-temporal literals. Thus, it make sense to define the set of *A-requests* of a point t in the model M as the set $A(t)$ of all literals of the type $[A]p$ that occur in labels of intervals that end at t. Symmetrically, one can define the set of *$\overline{A}$-requests* (denoted $\overline{A}(t)$), and, in fact, associate each point t of M with a unique pair $(\overline{A}(t), A(t))$. As we shall see, requests in a minimum model should behave in a regular way: if $M, [x, y] \Vdash \varphi$, we say that M is *standard* if and only if, for each $z \neq x, y$, $A(z) \subseteq A(z + 1)$ and $\overline{A}(z + 1) \subseteq \overline{A}(z)$.

There are two key ingredients to prove that the maximal dimension of a minimum model is polynomial: first, we show that minimum models are necessarily standard, and, then, that the length of minimum standard models can be bounded to a polynomial number in $|\varphi|$.

▶ **Lemma 2.** *Let φ be a set of clauses of $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$, and let M be a N-minimum model that satisfies it. Then, M is standard.*

Proof. Let φ be a set of clauses of $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$, and let $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ be a model that satisfies it, that is, $M, [x, y] \Vdash \varphi$; we proceed by contradiction, and we prove that if M is not standard, then it cannot be minimum. Without loss of generality, let $z > y$ be a point in M such that $A(z) \not\subseteq A(z + 1)$. This means that for some $[A]p$, it is the case that $[A]p \in A(z)$ and $[A]p \notin A(z + 1)$. We define a new model $M' = \langle \mathbb{I}(\mathbb{D}), V' \rangle$ based on the same domain as M and such that its valuation V', for each propositional letter p, is the union of:

1. $\{[t, t'] \mid [t, t'] \in V(p)$ and $t, t' \neq z\}$,

2. $\{[t,z] \mid t < z$ and $[t,z],[t,z+1] \in V(p)$ for each $t < z\}$,
3. $\{[z,z+1] \mid [z,z+1] \in V(p)\}$, and
4. $\{[z,t'] \mid t' > z+1$ and $[z,t'],[z+1,t'] \in V(p)$ for each $t' > z\}$.

We claim that $M',[x,y] \Vdash \varphi$. Suppose, by way of contradiction, that this is not the case. So, there exists a clause φ_i of φ such that $M',[x,y] \not\Vdash \varphi_i$. Several cases arise:

- If $\varphi_i = p$ for some propositional letter p, then we have a contradiction given that $M,[x,y] \Vdash \varphi_i$ and M and M' agree on the label of $[x,y]$, as per point (1) of the construction;
- If $\varphi_i = [\overline{A}]p$ for some propositional letter p, then we have a contradiction given that $M,[x,y] \Vdash \varphi_i$ and M and M' agree on the label of every interval of the type $[t,x]$, as per point 1 of the construction, because $t,x \neq z$;
- If $\varphi_i = [A]p$ for some propositional letter p, then, since $M,[x,y] \Vdash \varphi_i$, we know that, in M, for each $t > y$, $M,[y,t] \Vdash p$. By construction (points (1) and (2)), $M',[y,t] \Vdash p$, which implies that $M',[x,y] \Vdash [A]p$, leading to a contradiction;
- If $\varphi_i = [U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda)$ or $\varphi_i = [U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \bot)$, then, for some $[t,t']$, we have that $M',[t,t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ and $M',[t,t'] \not\Vdash \lambda$. There are several sub-cases:
  - Assume, first, that $t,t' < z$, and consider the generic antecedent λ_j. If it is a propositional letter p, then $M',[t,t'] \Vdash \lambda_j$ implies that $M,[t,t'] \Vdash \lambda_j$. On the other hand, if $\lambda_j = [A]p$ for some propositional letter p, then $M',[t,t'] \Vdash \lambda_j$ implies that, for each $t'' > t'$ (including $t'' = z$), $M',[t',t''] \Vdash p$, which in turn, by point (1) and (2) of the construction, implies that $M,[t',t''] \Vdash p$ for each $t'' > t'$. If $\lambda_j = [\overline{A}]p$ for some propositional letter p, then $M',[t,t'] \Vdash \lambda_j$ means that, for each $t'' < t$, $M',[t'',t] \Vdash p$, which implies, by point (1), that $M,[t'',t] \Vdash p$ for each $t'' < t$. Either way, if $M',[t,t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$, then $M,[t,t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$. If $\lambda = \bot$, this is already a contradiction. If $\lambda = p$ for some propositional letter p, then the contradiction is, again, immediate thanks to point (1) of the construction. Now, if $\lambda = [A]p$ for some propositional letter p, then we have that $M',[t',t''] \not\Vdash p$ for some $t'' > t'$; if $t'' \neq z$, then the label of $[t',t'']$ is preserved from M to M', which means that $M,[t',t''] \not\Vdash p$, leading to a contradiction, and, if $t'' = z$, then we have a contradiction with the fact that the label of $[t',z]$ in M' is the intersection of the labels of $[t',z]$ and $[t',z+1]$ in M, due to point (2) of the construction. If, finally, $\lambda = [\overline{A}]p$ for some propositional letter p, then we have that $M',[t'',t'] \not\Vdash p$ for some $t'' < t'$, which is a contradiction again with the fact that the label of each interval of the type $[t'',t']$ ($t' < z$) is preserved from M to M';
  - Suppose, now, that $t' = z$ and $t < t'$. If $\lambda_j = p$ for some propositional letter p, then $M',[t,z] \Vdash p$ implies, by point (2), that $M,[t,z] \Vdash p$ and $M,[t,z+1] \Vdash p$. If $\lambda_j = [A]p$ for some propositional letter p, then $M',[t,z] \Vdash [A]p$ implies that $M',[z,t'] \Vdash p$ for each $t' > z$. But, this implies, by points (3) and (4), that $M,[z,t'] \Vdash p$ and, if $t' > z+1$, also that $M,[z+1,t'] \Vdash p$, which, in turn, implies that $M,[t,z] \Vdash [A]p$ and $M,[t,z+1] \Vdash [A]p$. Finally, if $\lambda_j = [\overline{A}]p$ for some propositional letter p, $M',[t,z] \Vdash [\overline{A}]p$ immediately implies that $M,[t,z] \Vdash [\overline{A}]p$ and $M,[t,z+1] \Vdash [\overline{A}]p$. Therefore, $M',[t,z] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ implies that $M,[t,z] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ and $M,[t,z+1] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$. If $\lambda = \bot$, this is already a contradiction. Otherwise, we have that $M,[t,z] \Vdash \lambda$ and $M,[t,z+1] \Vdash \lambda$. If $\lambda = p$ for some propositional letter p, then $M',[t,z] \not\Vdash p$ is a contradiction, since the label of $[t,z]$ in M' is the intersection of the labels of $[t,z]$ and $[t,z+1]$ in M. If $\lambda = [A]p$ for some propositional letter p, that is, if $M',[z,t''] \not\Vdash p$ for some $t'' > z$, we have a contradiction with the fact that, for each $t'' > z$, the label of $[z,t'']$ in M' is the intersection of the labels of $[z,t'']$ and $[z+1,t'']$ in M. If, finally, $\lambda = [\overline{A}]p$ for some

propositional letter p, then we have that $M', [t'', t] \Vdash \neg p$ for some $t'' < t$, which is a contradiction with the fact that the label of each interval of the type $[t'', t]$ $(t < z)$ is preserved from M to M';
- If $t < z$ and $t' \geq z + 1$, then the contradiction is immediate, as the labels of every interval that may possibly be involved is preserved from M to M';
- Suppose, now, that $t = z$ and $t' \geq z + 1$. If $\lambda_j = p$ for some propositional letter p, then $M', [z, t'] \Vdash p$ implies that $M, [z, t'] \Vdash p$ and (if $t' > z + 1$) $M, [z + 1, t'] \Vdash p$. If $\lambda_j = [A]p$ for some propositional letter p, $M', [z, t'] \Vdash [A]p$ immediately implies that $M, [z, t'] \Vdash [A]p$ and (if $t' > z + 1$) $M, [z + 1, t'] \Vdash [A]p$, since the labels of intervals that start at $t' \geq z + 1$ are preserved from M to M'. Finally, if $\lambda_j = [\overline{A}]p$ for some propositional letter p, $M', [z, t'] \Vdash [\overline{A}]p$ implies that $M', [t'', z] \Vdash p$ for each $t'' < z$; this, in turn, implies that for each $t'' < z$ we have that $M, [t'', z] \Vdash p$ and $M, [t'', z + 1] \Vdash p$, that is, that $M, [z, t'] \Vdash [\overline{A}]p$ and (if $t' > z + 1$) $M, [z + 1, t'] \Vdash [\overline{A}]p$. Therefore, $M', [z, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ implies that $M, [z, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$ and (if $t' > z + 1$) $M, [z + 1, t'] \Vdash \lambda_1 \wedge \ldots \wedge \lambda_n$. If $\lambda = \bot$, this is already a contradiction. Otherwise, we have that $M, [z, t'] \Vdash \lambda$ and (if $t' > z + 1$) $M, [z + 1, t'] \Vdash \lambda$. If $\lambda = p$ for some propositional letter p, then $M', [z, t'] \not\Vdash p$ is a contradiction, since the label of $[z, t']$ in M' is preserved from M if $t' = z + 1$ and it is the intersection of the labels of $[z, t']$ and $[z + 1, t']$ in M otherwise. If $\lambda = [A]p$ for some propositional letter p, that is, if $M', [z, t'] \not\Vdash p$ for some $t' > z$, we have a contradiction with the fact that, for each $t'' > t'$, the label of $[t'', t']$ is preserved from M to M'. If, finally, $\lambda = [\overline{A}]p$ for some propositional letter p, then we have that, in M', $M', [t'', z] \Vdash \neg p$ for some $t'' < z$, that is $M, [t'', z] \Vdash \neg p$ or (if $t > z + 1$) $M, [t'', z + 1] \Vdash \neg p$, which is in contradiction with the fact that both $M, [z, t] \Vdash [\overline{A}]p$ and (if $t > z + 1$) $M, [z + 1, t] \Vdash [\overline{A}]p$;
- If $t \geq z + 1$, then the contradiction is immediate, as the labels of every interval that may possibly be involved are preserved from M to M'.

Thus, $M', [x, y] \Vdash \varphi$. Now, observe that by hypothesis, in M, $[A]p \in A(z)$ and $[A]p \notin A(z+1)$ for some propositional letter p. Then $M' <_N M$, since, at the very least, p is true in every interval of M starting at z while in M' it is false for some interval starting at z (because in M it is false at some interval starting at $z + 1$). Therefore, M cannot be N-minimum, as we wanted to show. The case in which $z < y$ (recall that $z \neq x$), as well as the case in which M is not standard because of some literal of the type $[\overline{A}]p$, can be treated in a very similar way. ◀

Given a standard model, we say that each $[A]p \in A(z + 1) \setminus A(z)$ (resp., $[\overline{A}]p \in \overline{A}(z) \setminus \overline{A}(z + 1)$), for a generic point z, is a *blocking* temporal literal.

▶ **Lemma 3.** *Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, and let M be the N-minimum standard model that satisfies it. If $N \geq 6 \cdot |\varphi| + 3$, then there exists a minimum standard model M' of length $N' < N$ that satisfies φ.*

Proof. Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, and let $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ be the N-minimum model that satisfies it. Suppose that, for some z such that $z, z + 1 \neq x, y$, it is the case that $(\overline{A}(z), A(z)) = (\overline{A}(z + 1), A(z + 1))$. Consider the model $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ obtained eliminating the point $z + 1$, along with every interval that starts or ends at $z + 1$. Such an elimination implicitly defines a function $(\hat{\cdot})$ from points of M to points of M', which is the identical function for every point smaller than z included, and it is the function "-1" for every other point. We can define V', for each propositional letter p, as the union of:

1. $\{[\hat{t}, \hat{t}'] \mid [t, t'] \in V(p)$ and $t, t' \neq z\}$,

Algorithm 1 Satisfiability Checker for $\mathbf{A\overline{A}}^{\Box}_{Horn}$

```
1: function AĀ□_Horn-Check(φ)
2:     for N ← 2, ..., 6·|φ| + 2 do
3:         let 𝔻 be a linear domain s.t. |𝔻| = N
4:         for [x, y] ∈ 𝕀(𝔻) do
5:             if Saturate(𝔻, x, y, φ) then return True
6:     return False
```

2. $\{[\hat{t}, \hat{z}] \mid t < z$ and $[t, z], [t, z+1] \in V(p)$ for each $t < z\}$, and
3. $\{[\hat{z}, \hat{t}'] \mid t' > z + 1$ and $[z, t'], [z+1, t'] \in V(p)$ for each $t' > z + 1\}$.

Unlike Lemma 2, it is straightforward to see that $M', [\hat{x}, \hat{y}] \Vdash \varphi$, as the sets of requests at z are preserved from M to M' It is clear that $|D'| < |D|$; moreover, M' is a minimum model, as its valuation, for each propositional letter, has been obtained by copying or intersecting different components of the valuation of M (which was minimum to begin with). It remains to be shown that, if $N \geq 6 \cdot |\varphi| + 3$, there must exist some z such that $z, z+1 \neq x, y$ and $(\overline{A}(z), A(z)) = (\overline{A}(z+1), A(z+1))$. Let $[A]p_1, [A]p_2, \ldots$ (resp., $[\overline{A}]p_1, [\overline{A}]p_2, \ldots$) be an arbitrary enumeration of A-temporal (resp., $\overline{A}$-temporal) literals, and let us focus our attention to points greater than y. If $(\overline{A}(y+1), A(y+1)) \neq (\overline{A}(y+2), A(y+2))$, then there must be some blocking temporal literal that witnesses such a difference; without loss of generality, let us assume that $[A]p_1$ is a blocking temporal literal. Since M is standard, $[A]p_1 \in A(y+r)$ cannot be blocking for any $r \geq 2$. By iterating this argument, we may conclude that $A(y+|\varphi|) = A(y+|\varphi|+1)$. Now, if $\overline{A}(y+|\varphi|+1) \neq \overline{A}(y+|\varphi|+2)$, we may assume that $[\overline{A}]p_1$ is blocking for $y+|\varphi|+1$. Since M is standard, we can iterate the same argument as before and conclude that $(\overline{A}(y+2\cdot|\varphi|), A(y+2\cdot|\varphi|)) = (\overline{A}(y+2\cdot|\varphi|+1), A(y+2\cdot|\varphi|+1))$. Therefore, in order to guarantee the existence of at least one pair of successive points with precisely the same temporal requests, we need to account for at least $2 \cdot |\varphi|$ points after y, $2 \cdot |\varphi|$ points between x and y, $2 \cdot |\varphi|$ points before x, plus one, besides x and y themselves. ◀

The above result gives us immediately an NP algorithm for checking the satisfiability of a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$. We show now that the problem is indeed in P by devising a simple deterministic polynomial algorithm. The decision procedure checks all possible minimal models, and for each one of them, all possible starting intervals. For each combination, a saturation procedure iteratively builds a structure $(\mathbb{I}(\mathbb{D}), Hi, Lo)$ that represents a candidate model for the formula, where $\mathbb{D}$ is the underlying domain, Hi labels each interval in $\mathbb{I}(\mathbb{D})$ with the set of formulas 'to be further analyzed' and Lo labels each interval in $\mathbb{I}(\mathbb{D})$ with the set of formulas in $\mathcal{L}(\varphi)$ that holds on it. We first prove that if Saturate terminates with success, then the current structure indeed represents a model for φ on the current starting interval $[x, y]$.

▶ **Lemma 4.** *Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, $\mathbb{D}$ a linear domain of length N, and let $[x, y]$ be an interval in $\mathbb{I}(\mathbb{D})$. If* Saturate*$(\mathbb{D}, x, y, \varphi)$ returns True, then there exists a model M built on $\mathbb{D}$ that satisfies φ on $[x, y]$.*

Proof. Assume that Saturate$(\mathbb{D}, x, y, \varphi)$ returns True, and consider the structure $(\mathbb{I}(\mathbb{D}), Hi, Lo)$ obtained at the end of the procedure. We define a model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ such that, for every $p \in \mathcal{AP}$, $V(p) = \{[z, t] \in \mathbb{I}(\mathbb{D}) \mid p \in Lo([z, t])\}$. We first prove that M respects the following property:

$$M, [z, t] \Vdash \psi \text{ for every } [z, t] \in \mathbb{I}(\mathbb{D}) \text{ and } \psi \in Lo([z, t]) \tag{3}$$

Algorithm 2 Saturation

1: **function** $\textsc{Saturate}(\mathbb{D}, x, y, \varphi)$
2: **for** $[z,t] \in \mathbb{I}(\mathbb{D})$ **do**
3: $Hi([z,t]) \leftarrow \{(\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda) \mid [U](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda) \in \varphi\}$
4: $Lo([z,t]) \leftarrow \{\top\}$
5: $Hi([x,y]) \leftarrow Hi([x,y]) \cup \{\lambda \mid \lambda \in \varphi\}$
6: **while** something changes **do**
7: **let** $[z,t] \in \mathbb{I}(\mathbb{D})$, $\psi \in Hi([z,t])$
8: **if** $\psi = p$ **then**
9: $Hi([z,t]) \leftarrow Hi([z,t]) \setminus \{p\}$
10: $Lo([z,t]) \leftarrow Lo([z,t]) \cup \{p\}$
11: **else if** $\psi = [A]p$ **then**
12: $Hi([z,t]) \leftarrow Hi([z,t]) \setminus \{[A]p\}$
13: $Lo([z,t]) \leftarrow Lo([z,t]) \cup \{[A]p\}$
14: **for** $t' > t$ **do**
15: $Lo([t,t']) \leftarrow Lo([t,t']) \cup \{p\}$
16: **else if** $\psi = [\overline{A}]p$ **then**
17: $Hi([z,t]) \leftarrow Hi([z,t]) \setminus \{[\overline{A}]p\}$
18: $Lo([z,t]) \leftarrow Lo([z,t]) \cup \{[\overline{A}]p\}$
19: **for** $z' < z$ **do**
20: $Lo([z',z]) \leftarrow Lo([z',z]) \cup \{p\}$
21: **else if** $\psi = \lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda$, $\lambda \neq \bot$ **then**
22: **if** $\{\lambda_1, \ldots, \lambda_n\} \subseteq Lo([z,t])$ **then**
23: $Hi([z,t]) \leftarrow (Hi([z,t]) \cup \{\lambda\}) \setminus \{\psi\}$
24: $Lo([z,t]) \leftarrow Lo([z,t]) \cup \{\psi\}$
25: **else if** $\psi = \lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \bot$ **then**
26: **if** $\{\lambda_1, \ldots, \lambda_n\} \subseteq Lo([z,t])$ **then**
27: **return** False
28: **for** $z \in \mathbb{D}$ **do**
29: **for** $[A]p \in \mathcal{L}(\varphi)$ **do**
30: **if** $\forall t > z(p \in Lo([z,t]))$ **then**
31: **for** $z' < z$ **do**
32: $Lo([z',z]) \leftarrow Lo([z',z]) \cup \{[A]p\}$
33: **for** $[\overline{A}]p \in \mathcal{L}(\varphi)$ **do**
34: **if** $\forall z' < z(p \in Lo([z',z]))$ **then**
35: **for** $t > z$ **do**
36: $Lo([z,t]) \leftarrow Lo([z,t]) \cup \{[\overline{A}]p\}$
37: **return** True

We proceed by structural induction on ψ. Consider an interval $[z,t] \in \mathbb{I}(\mathbb{D})$ and a formula $\psi \in Lo([z,t])$:

- if $\psi = \top$ then the property trivially holds;
- if $\psi = p$ for some $p \in \mathcal{AP}$, then the property follows from the definition of M;
- if $\psi = [A]p$ (the case for $[\overline{A}]p$ is symmetrical), we first observe that after the initialization of $\textsc{Saturate}$ (lines 2–5), $Lo([z,t])$ contains only $\top$. Hence, at some point during the execution of $\textsc{Saturate}$, one of two things may happen: either lines 11–15 move ψ from

$Hi([z,t])$ to $Lo([z,t])$ and add p to $Lo([t,t'])$ for each $t' > t$, or lines 28–36 put ψ in $Lo([z,t])$ because p is in $Lo([t,t'])$ for each $t' > t$. Either way, by inductive hypothesis, we have that $M,[t,t'] \Vdash p$ for every $t' > t$, and thus that $M,[z,t] \Vdash [A]p$;

- if $\psi = \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$ and $\lambda \neq \bot$, then at some point during the execution of Saturate, lines 21–24 move ψ from $Hi([z,t])$ to $Lo([z,t])$ and add λ to $Hi([z,t])$. Some successive iteration of the **while** loop eventually moves λ from $Hi([z,t])$ to $Lo([z,t])$. By inductive hypothesis, we have that $M,[z,t] \Vdash \lambda$ and thus that $M,[z,t] \Vdash \psi$;
- if $\psi = \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \bot$, then notice that no instruction of Saturate moves ψ from $Hi([z,t])$ to $Lo([z,t])$. Hence, this case cannot occur.

We can now use (3) to conclude. Suppose by contradiction that $M,[x,y] \not\Vdash \varphi$: then there exists a formula $\varphi_i \in \varphi$ such that $M,[x,y] \not\Vdash \psi$. Two cases may arise:

- φ_i is a literal. Then after initialization $\psi \in Hi([x,y])$ (line 5). Since Saturate eventually moves every literal from Hi to Lo, from (3) we can conclude that $M,[x,y] \Vdash \psi$, a contradiction.
- $\varphi_i = [U](\lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda)$ is a global clause. Since $M,[x,y] \not\Vdash \psi$, we can find an interval $[z,t]$ such that $M,[z,t] \not\Vdash \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$, that is, $M,[z,t] \Vdash \lambda_1 \wedge \cdots \wedge \lambda_n$ but $M,[z,t] \not\Vdash \lambda$. We first observe that the initialization of Saturate (lines 2–5) puts $\lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$ in $Hi([z,t])$. Now, let λ_i be some literal in the body of the clause; since $M,[z,t] \Vdash \lambda_i$, the following cases arise:
  - if $\lambda_i = p$ then by definition of M we have that $p \in Lo([z,t])$;
  - if $\lambda_i = [A]p$, from $M,[z,t] \Vdash [A]p$ we have that $M,[t,t'] \Vdash p$ for every $t' > t$. By definition of M we have that $p \in Lo([t,t'])$. Hence, at some iteration of the **while** loop, lines 28–36, put $[A]p$ in $Lo([z,t])$;
  - if $\lambda_i = [\overline{A}]p$, we can prove that $[\overline{A}]p \in Lo([z,t])$ as above.

  In all cases $\lambda_i \in Lo([z,t])$ and thus we have that $\{\lambda_1,\ldots,\lambda_n\} \subseteq Lo([z,t])$. Hence, in the case $\lambda \neq \bot$, lines 21–24 eventually move $\lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$ from $Hi([z,t])$ to $Lo([z,t])$. By (3) we have that $M,[z,t] \Vdash \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$, and a contradiction is found. Finally, whenever $\lambda = \bot$, lines 25–27 imply that Saturate returns False, a contradiction.

Since in all cases we reached a contradiction, we have proved that $M,[x,y] \Vdash \varphi$. ◀

Now we prove that if Saturate fails then φ has no models of a given length with $[x,y]$ as starting interval.

▶ **Lemma 5.** *Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$, $\mathbb{D}$ a linear domain of length N, and let $[x,y]$ be an interval in $\mathbb{I}(\mathbb{D})$. If* Saturate*$(\mathbb{D},x,y,\varphi)$ returns False, then $M,[x,y] \not\Vdash \varphi$ for each model M on $\mathbb{D}$.*

Proof. Suppose by contradiction that Saturate$(\mathbb{D},x,y,\varphi)$ returns False, but there exists a model M on $\mathbb{D}$ that satisfies φ on $[x,y]$. We prove that Saturate respects the following invariant:

$$\text{For every } [z,t] \in \mathbb{I}(\mathbb{D}) \text{ and } \psi \in Hi([z,t]) \cup Lo([z,t]), \text{ we have that } M,[z,t] \Vdash \psi. \tag{4}$$

After initialization (lines 2–5) we have that for $Hi([z,t])$ contains the scope of every global clause in φ, $Lo([z,t])$ contains $\top$ and that $Hi([x,y])$ contains every initial clause in φ. Hence, (4) follows directly from the fact that $M,[x,y] \Vdash \varphi$.

Now, suppose that (4) holds at the k-th iteration of the **while** loop, and consider the $(k+1)$-th iteration. Let $[z,t]$ and ψ be respectively the interval and the formula selected by line 7. Since $\psi \in Hi([z,t])$ we know by inductive hypothesis that $M,[z,t] \Vdash \psi$. The following cases may arise.

- $\psi = p$. Then lines 8–10 move ψ from $Hi([z,t])$ to $Lo([z,t])$, and (4) is still respected.
- $\psi = [A]p$. Then lines 11–15 move ψ from $Hi([z,t])$ to $Lo([z,t])$, and add p to $Lo([t,t'])$ for every $t' > t$. Since, by inductive hypothesis, $M,[z,t] \Vdash [A]p$, we have that $M,[t,t'] \Vdash p$ and thus (4) is respected.
- $\psi = [\overline{A}]p$. As above.
- $\psi = \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \lambda$ with $\lambda \neq \bot$. If $\{\lambda_1,\ldots,\lambda_n\} \not\subseteq Lo([z,t])$ then Hi and Lo do not change and thus (4) is respected. If, otherwise, $\{\lambda_1,\ldots,\lambda_n\} \subseteq Lo([z,t])$ then lines 21–24 move ψ from $Hi([z,t])$ to $Lo([z,t])$ and add λ to $Lo([z,t])$. Since for every λ_i we have that $\lambda_i \in Lo([z,t])$, by inductive hypothesis, $M,[z,t] \Vdash \lambda_i$. This implies that $M,[z,t] \Vdash \lambda$ and hence (4) is respected.
- $\psi = \lambda_1 \wedge \cdots \wedge \lambda_n \rightarrow \bot$. If $\{\lambda_1,\ldots,\lambda_n\} \not\subseteq Lo([z,t])$ then Hi and Lo do not change and thus (4) is respected. If, otherwise, $\{\lambda_1,\ldots,\lambda_n\} \subseteq Lo([z,t])$ then line 27 terminates the execution of SATURATE by returning False. Since for every λ_i we have that $\lambda_i \in Lo([z,t])$, by inductive hypothesis, $M,[z,t] \Vdash \lambda_i$. Since $M,[z,t] \Vdash \psi$, we have a contradiction, and thus line 27 is never executed.

To conclude, we have to show that (4) is respected also after the execution of lines 28–36. Let $z \in \mathbb{D}$ and $p \in \mathcal{AP}$ be such that $p \in Lo([z,t])$ for every $t > z$. Then, lines 30–32 are eventually executed on z and p, adding $[A]p$ to $Lo([z',z])$ for every $z' < z$. By inductive hypothesis we have that $M,[z,t] \Vdash p$ for every $t > z$ and thus that $M,[z',z] \Vdash [A]p$ for every $z' < z$. This shows that (4) is respected. By the same argument we can also show that for every $z \in \mathbb{D}$ and $p \in \mathcal{AP}$, if $p \in Lo([z',z])$ for every $z' < z$ then the structure Lo is updated in a way that respects the invariant.

Now, observe that we have proved that line 27 cannot be executed, in contradiction with the hypothesis that SATURATE returned False. Hence, M cannot satisfy φ on $[x,y]$. ◀

Lemma 4 and 5 show that SATURATE is correct and complete, and allow us to prove that Algorithm 1 is a deterministic polynomial time solution to the satisfiability problem for $\mathbf{A\overline{A}}^{\Box}_{Horn}$.

▶ **Theorem 6.** *Let φ be a set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$. Then $\mathbf{A\overline{A}}^{\Box}_{Horn}$-CHECK($\varphi$) returns True if and only if φ is satisfiable. Moreover, the time complexity of the procedure is polynomial.*

Proof. Let φ be a satisfiable set of clauses of $\mathbf{A\overline{A}}^{\Box}_{Horn}$. By Lemma 3, we know that there exists an N-minimum standard model M that satisfies φ on some interval $[x,y]$, with $N \leq 6 \cdot |\varphi| + 2$. Since $\mathbf{A\overline{A}}^{\Box}_{Horn}$-CHECK($\varphi$) tries to satisfy φ for all model lengths from 2 to $6 \cdot |\varphi| + 2$ and over all intervals $[z,t]$, we have that SATURATE is eventually called on the starting interval $[x,y]$ of a linear domain $\mathbb{D}$ such that $|\mathbb{D}| = N$, returning True (by Lemma 5). Conversely, suppose that $\mathbf{A\overline{A}}^{\Box}_{Horn}$-CHECK($\varphi$) returns True. This means that a call to SATURATE($\mathbb{D},x,y,\varphi$) returns True for some linear domain $\mathbb{D}$ and initial interval $[x,y] \in \mathbb{I}(\mathbb{D})$. Then, by Lemma 4, we know that φ is satisfiable.

To prove that the time complexity is polynomial, we start from the complexity of a single call to SATURATE. Every iteration of the **while** loop may move the formula ψ from Hi to Lo and add some new formulas to Lo. Since the number of subformulas of φ is linear in $|\varphi|$ and the number of intervals is quadratic in $|\mathbb{D}|$, after a polynomial number of iterations no new formulas are moved nor added to Lo, and SATURATE terminates. Since the number of intervals in a domain of length N is $\frac{N(N-1)}{2}$, we have that the number of calls to SATURATE in the nested loops of $\mathbf{A\overline{A}}^{\Box}_{Horn}$-CHECK($\varphi$) is given by $\sum_{N=2}^{6\cdot|\varphi|+2} \frac{N(N-1)}{2}$, that is a polynomial quantity. Hence, the overall time complexity of $\mathbf{A\overline{A}}^{\Box}_{Horn}$-CHECK is polynomial. ◀

The above theorem shows that the satisfiability problem for $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$is in P. P-hardness follows form the P-completeness of propositional Horn satisfiability.

▶ **Corollary 7.** *The satisfiability problem for* $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$*is* P*-complete.*

4 Intractability

In this section we prove that $\mathbf{A}\overline{\mathbf{A}}^{\Box}_{Horn}$ cannot be extended with modalities from the **HS** machinery preserving its computational good behaviour, at least in most cases. We prove that the finite satisfiability $\mathbf{AB}^{\Box}_{Horn}$ becomes hard for PSPACE; then, we extend the result for the fragments obtained by any combination of $[A]$ or $[\overline{A}]$ with any of $[B], [E], [O]$ and their inverses.

To prove that the finite satisfiability problem for $\mathbf{AB}^{\Box}_{Horn}$ is PSPACE-hard, we encode the halting problem of a *Turing machine with polynomial tape* $\mathcal{T} = (Q, \Gamma, \delta, q_0, q_f)$ in it, where Q is the set of states, q_0 is the initial state, q_f is the final accepting state, δ is the transition function, and Γ is the alphabet that contains $0, 1, \sqcup$ (the latter represents the "blank"). Given $n = |\mathcal{T}|$, we assume that $\mathcal{T}$ can use at most $p(n)$ different tape cells, for some polynomial function $p(\cdot)$; under such conditions, the halting problem is PSPACE-hard. Let $\mathcal{T} = (Q, \Gamma, \delta, q_0, q_f)$ be a Turing machine, and let us define (by abusing of the notation) the following set of propositional variables:

$$\mathfrak{L} = \{c_j \mid c \in \Gamma, 0 \leq j \leq p(n)\} \cup \{h_j \mid 0 \leq j \leq p(n)\} \cup Q.$$

The literal $[B]\bot$ uniquely identifies *unit* intervals, that is, intervals of length 1. We use units to encode configurations of $\mathcal{T}$; this is possible since we only have polynomially many different pairs symbol/position, and we can denote each one of them with a different proposition from $\mathfrak{L}$. Moreover, the position of the reading head can be easily encoded in a similar way; for example, h_j denotes that the head is reading the j-th tape symbol. Consider the following formulas to set the initial configurations and the symbols for tape elements, reading head, and current state:

$$\begin{array}{lr}
\phi_1 = q_0 \wedge (\bigwedge_{j=0,\ldots,p(n)} \sqcup_j) \wedge h_0 & \textit{initial configuration} \\
\phi_2 = [U](u \rightarrow [B]\bot) \wedge [U]([B]\bot \rightarrow u) & \textit{labeling units} \\
\left.\begin{array}{l}
\phi_3 = \bigwedge_{l \in \mathfrak{L}} [U](l \rightarrow u) \\
\phi_4 = \bigwedge_{q \neq q', c_j \neq c'_j} [U](q \rightarrow \neg q') \wedge [U](c_j \rightarrow \neg c'_j) \\
\phi_5 = \bigwedge_{j \neq k} [U](h_j \rightarrow \neg h_k)
\end{array}\right\} & \textit{tape/state/head propositions}
\end{array}$$

As far as transitions are concerned, we separate between the actual transitions and the head movement. The transitions are taken care of as follows:

$$\begin{array}{lr}
\left.\begin{array}{l}
\phi_6 = \bigwedge_{c,c',j}^{\delta(q,c)=(q',c',L/R)} [U](((q \wedge c_j \wedge h_j) \rightarrow [A]P_{q'}) \wedge ((P_{q'} \wedge u) \rightarrow q')) \\
\phi_7 = \bigwedge_{c,c',j}^{\delta(q,c)=(q',c',L/R)} [U](((q \wedge c_j \wedge h_j) \rightarrow [A]P_{c'_j}) \wedge ((P_{c'_j} \wedge u) \rightarrow c'_j))
\end{array}\right\} & \textit{under head} \\
\phi_8 = \bigwedge_{c,j,k \neq j}^{\delta(q,c)=(q',c',L/R)} [U](((c_k \wedge h_j) \rightarrow [A]P_{c_k}) \wedge ((P_{c_k} \wedge u) \rightarrow c_k)) & \textit{far from head}
\end{array}$$

The reading head movement is managed as follows:

$$\begin{array}{lr}
\left.\begin{array}{l}
\phi_9 = \bigwedge_{j \neq p(n)}^{\delta(q,c)=(q',c',R)} [U]((h_j \rightarrow [A]P_{h_{j+1}}) \wedge ((P_{h_{j+1}} \wedge u) \rightarrow h_{j+1})) \\
\phi_{10} = \bigwedge^{\delta(q,c)=(q',c',R)} [U]((h_{p(n)} \rightarrow [A]P_{h_{p(n)}}) \wedge ((P_{h_{p(n)}} \wedge u) \rightarrow h_{p(n)}))
\end{array}\right\} & \textit{right} \\
\left.\begin{array}{l}
\phi_{11} = \bigwedge_{j \neq 0}^{\delta(q,c)=(q',c',L)} [U]((h_j \rightarrow [A]P_{h_{j-1}}) \wedge ((P_{h_{j-1}} \wedge u) \rightarrow h_{j-1})) \\
\phi_{12} = \bigwedge^{\delta(q,c)=(q',c',L)} [U]((h_0 \rightarrow [A]P_{h_0}) \wedge ((P_{h_0} \wedge u) \rightarrow h_0))
\end{array}\right\} & \textit{left}
\end{array}$$

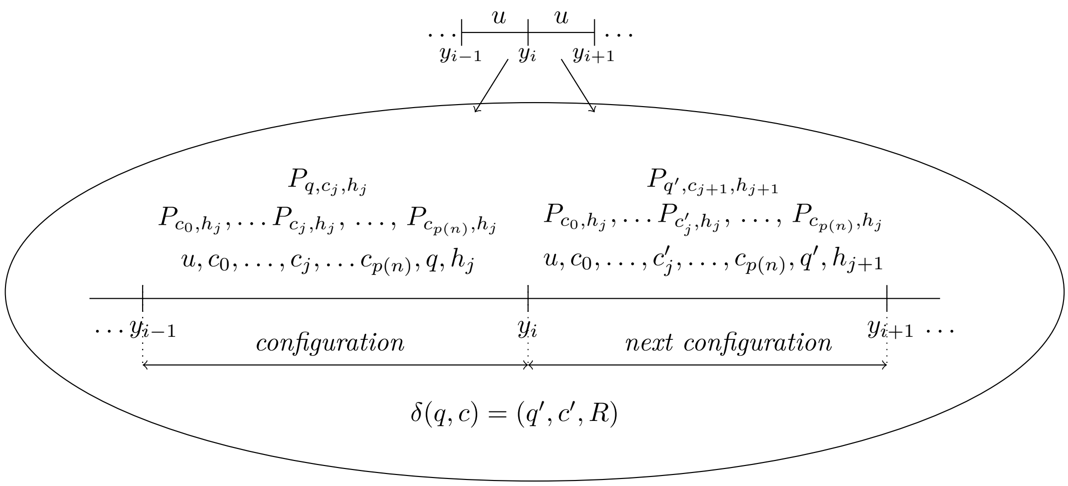

Figure 3 Configurations' structure.

Finally, we make sure that the model is long enough:

$$\phi_{13} = \bigwedge_{q \neq q_f} [U](q \to \neg[A]q)$$

Let $\mathcal{T}$ be a deterministic Turing Machine of size $n = |\mathcal{T}|$, and whose tape is limited by some polynomial function $p(n)$. Then, it is possible to show that $\mathcal{T}$ converges on empty input if and only if the formula:

$$Halts = \phi_1 \wedge \ldots \wedge \phi_{13}$$

is finitely satisfiable. Moreover, since our construction is essentially based on the assumption that the underlying linear order is discrete, it can be easily adapted to obtain an analogous formula for the case of natural numbers, the integers, and the class of all discrete linear ordered sets. An intuitive explanation on the structure of the model obtained in our construction is shown in Fig. 3.

The above construction can be immediately adapted to obtain the same result for any fragment of $\mathbf{HS}^{\Box}_{Horn}$ that includes any combination of $[A]$ or $[\overline{A}]$ with any of $[B], [E]$. For the fragments that include $[O]$ or $[\overline{O}]$, the following considerations are necessary. Consider, for example, the fragment $\mathbf{AO}^{\Box}_{Horn}$. While on infinite discrete linear orders the formula $[O]\bot$ behaves as expected, on finite orders of length N it characterizes unit intervals (as expected) *and* intervals of the type $[x, y_{N-1}]$, where y_{N-1} is the last point of the domain. This does not affect the construction: given a configuration holding on $[x, x+1]$, we may encode the information of the next one in the interval $[x+1, x+2]$ (as desired) as well as in the interval $[x+1, y_{N-1}]$, which in turn, having no A-successors, does not erroneously transmit its information to any other configuration.

▶ **Theorem 8.** *The satisfiability problem for any fragment of $\mathbf{HS}^{\Box}_{Horn}$ that features any combination of $[A]$ or $[\overline{A}]$ with any of $[B], [E], [O]$ and their inverses, interpreted in any class of (finite) discrete linearly ordered sets, is* PSPACE*-hard.*

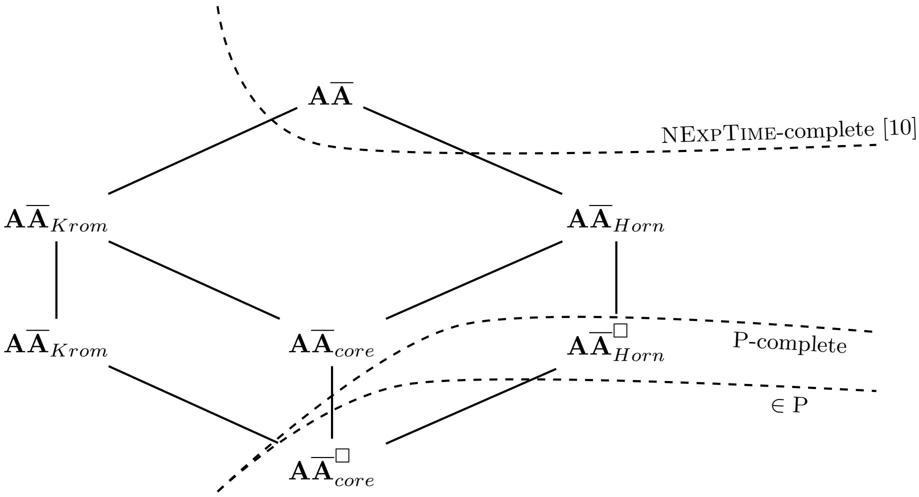

Figure 4 Sub-propositional fragments of $\mathbf{A\overline{A}}$.

5 Conclusions

In this paper we studied the well-behaved fragments of **HS** known as $\mathbf{A\overline{A}}$ under the Horn restriction without diamonds ($\mathbf{A\overline{A}}^{\Box}_{Horn}$); we proved that its finite satisfiability problem is P-complete (in sharp constrast with the same problem for $\mathbf{A\overline{A}}$ without restrictions, which is NExpTime-complete), and that almost every extension of $\mathbf{A\overline{A}}^{\Box}_{Horn}$ obtained by adding modal operators to it is already intractable. The problems that remain open include the status of $\mathbf{A\overline{A}}_{Horn}$, as well as extending our results to the discrete infinite case (e.g., $\mathbb{N}$ or $\mathbb{Z}$).

There are very natural motivations for this work; let us hint to some of them. First, Horn propositional logic is the basis for logic programming, which, in turn, is at the core of many Artificial Intelligence applications; although some attempts to extend logical (programming) languages to include time has been done (see, e.g. [3]), further research in this direction is necessary, expecially in the case of interval-based time. Second, in a different context, consider a generic data classification problem, the most common techniques to solve it being decision trees (see, e.g., [21]) and rule extraction algorithms (see, e.g., [19]): from a purely formal point of view, the result of the application of such common algorithms is a set of propositional logic rules, and, although the temporal data classification problem is not nearly as common as its classical counterpart, its development may undoubtedly benefit from studying the syntax, the semantics, and the properties of (interval) temporal rules. Third, it is well known that temporal databases use intervals to represent time [24], and that **HS** is their the natural logical counterpart, which raises the need of tractable fragments of it. Finally, the recent effort of extending description logics with interval temporal capabilities with the introduction of reflexive **HS** [6] raises the natural question of whether similar extensions may be possible with good-behaved fragments of **HS**.

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