Strategies with Parallel Causes

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Abstract

We imagine a team Player engaging a team Opponent in a distributed game. Such games and their strategies have been formalised within event structures. However there are limitations in founding strategies on traditional event structures. Sometimes a probabilistic distributed strategy relies on benign races where, intuitively, several members of team Player may race each other to make a common move. Although there exist event structures which support such parallel causes, in which an event is enabled in several compatible ways, they do not support an operation of hiding central to the composition of strategies; nor do they support probability adequately. An extension of traditional event structures is devised which supports parallel causes and hiding, as well as the mix of probability and nondeterminism needed to account for probabilistic distributed strategies. The extension is located within existing models for concurrency and tested in the construction of a bicategory of probabilistic distributed strategies with parallel causes.

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1 Introduction

This article addresses a fundamental, potentially widespread issue of which few are aware. It concerns the accurate modelling of parallel causes in probabilistic distributed strategies; we are thinking for instance of a strategy in which it is advantageous to allow two or more members of the same team to race each other cooperatively, without conflict, to perform some common move. It fixes the absence of a computational model which simultaneously handles parallel causes, probability and an operation of hiding internal events; it provides such a model, locates it via adjunctions within existing models and tests it in the construction of a bicategory of probabilistic distributed strategies supporting parallel causes.

Consider probabilistic distributed games between two teams, Player and Opponent. To set the scene, imagine a simple distributed game in which team Opponent can perform two moves, called 1 and 2, far apart from each other, and that team Player can just make one move, 3. Suppose that for Player to win they must make their move iff Opponent makes one or more of their moves. Informally Player can win by assigning two members of their team, one to watch out for the Opponent move 1 and the other Opponent move 2. When either watcher sees their respective Opponent move they run back and make the Player move 3. Opponent could possibly play both 1 and 2 in which case both watchers would run back and could make their move cooperatively together. Provided the watchers are perfectly reliable this provides a winning probabilistic strategy for Player. No matter how Opponent chooses to play or not play their moves, Player will win; if Opponent is completely inactive the watchers wait forever but then Player does win, eventually.

We can imagine variations in which the watchers are only reliable with certain probabilities, independent or correlated, with a consequent reduction in the probability of Player winning

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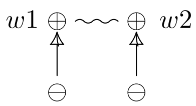
against Opponent strategies. In such a probabilistic strategy Player can only determine probabilities of their moves conditionally on those of Opponent. Because Player has no say in the probabilities of Opponent moves beyond those determined by causal dependencies of the strategy we are led to a *Limited Markov Condition*, of the kind discussed in [6]:

> **(LMC)** In a configuration x in which both a Player move $\oplus$ and an Opponent move $\ominus$ could occur individually, if the Player move and the Opponent move are causally independent, then they are probabilistically independent; in a strategy for Player, $\text{Prob}(\oplus \mid x, \ominus) = \text{Prob}(\oplus \mid x)$.

Note we do not expect that in all strategies for Player that two causally independent Player moves are necessarily probabilistically independent; in fact, because composition of strategies involves hiding internal moves such a property would not generally be preserved by composition.

Let us try to describe the informal winning strategy above in terms of event structures. In 'prime' event structures in which causal dependency is expressed as a partial order, an event is causally dependent on a unique set of events, *viz.* those events below it in the partial order. For this reason within prime event structures we are forced to split the Player move 3 into two events one for each watcher making the move, one $w1$ dependent on Opponent move 1 and the other $w2$ on Opponent move 2. The two moves of the two watchers stand for the same Player move in the game. Because of this they are in conflict (or inconsistent) with each other.[1] We end up with the event structure drawn below:



The polarities + and − signify moves of Player and Opponent, respectively. The arrows represent the (immediate) causal dependencies and the wiggly line conflict. As far as purely nondeterministic behaviour goes, we have expressed the informal strategy well: no matter how Opponent makes or doesn't make their moves any maximal play of Player is assured to win. However consider assigning conditional probabilities to the watcher moves. Suppose the probability of $w1$ conditional on Opponent event 1 is p_1, *i.e.* $\text{Prob}(w1 \mid 1) = \text{Prob}(w1, 1 \mid 1) = p_1$ and that similarly for $w2$ its conditional probability $\text{Prob}(w2 \mid 2) = p_2$. Given that move $w1$ of Player and move 2 of Opponent are causally independent, from (LMC) we expect that $w1$ is probabilistically independent of move 2, *i.e.*whether Opponent chooses to make move 2 or not should have no influence on the watcher of move 1:

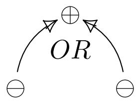
$$\text{Prob}(w1 \mid 1, 2) = \text{Prob}(w1 \mid 1) = p_1\,; \text{ and similarly, } \text{Prob}(w2 \mid 1, 2) = \text{Prob}(w2 \mid 2) = p_2\,.$$

But $w1$ and $w2$ are in conflict, so mutually exclusive, and can each occur individually when 1 and 2 have occurred, ensuring that $p_1 + p_2 \leq 1$ – we haven't insisted on one or the other occurring, the reason why we have not written equality. The best Player can do is assign $p_1 = p_2 = 1/2$. Against a counter-strategy with Opponent playing one of their two moves with probability 1/2 this strategy only wins half the time. We have clearly failed to express the informal winning strategy accurately!

Present notions of "concurrent strategies," the most general of which are presented in [11], are or can be expressed using prime event structures. If we are to be able to express the

[1] Technically, the conflict is forced by the nature of maps of event structures; a map reflects the atomicity of events and cannot send distinct consistent events to a common event.

intuitive strategy which wins with certainty we need to develop distributed probabilistic strategies which allow *parallel causes* in which an event can be enabled in distinct but compatible ways. 'General' event structures are one such model [10]. In the informal strategy described above both Opponent moves would individually enable the Player move, with all events being consistent, illustrated below:



But as we shall see general event structures do not support an appropriate operation of hiding central to the composition of strategies. Nor is it clear how within general event structures one could express the variant of the strategy above in which the two watchers succeed in reporting with different probabilities while respecting LMC – see Section 3.1.

It has been necessary to develop a new model – *event structures with disjunctive causes* (edc's) – which support hiding and probability adequately, and into which both prime and general event structures embed. Conceptually, one is forced to objectify cause in a way that is reminiscent of formal proof being an objectification of theoremhood. Formally, this is achieved by extending prime event structures with an equivalence relation; the equivalence classes are thought of as 'disjunctive events' of which the representatives are 'prime causes.' In this way causes may conflict or not, possess probabilities, and be correlated or independent. The new model provides a foundation on which to build a rich theory of probabilistic distributed strategies with parallel causes. Even without probability, it provides a new bicategory of *deterministic* parallel strategies, including *e.g.* deterministic strategies for "parallel or" and McCarthy's amb [4].

Full proofs can be found in [12]*, Chapter 16, 17. Appendix A summarises the simple instances of concepts we borrow from enriched categories* [2, 3] *and 2-categories* [7]*.*

2 Event structures

We start with event structures. In their simplest form, that of 'prime' event structures, they occupy a central position in models for concurrent computation, both "interleaving" and "causal" [13], and can claim to be the concurrent or causal analogue of trees.

2.1 Prime event structures

A *(prime) event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E. The relation $e' \leq e$ expresses that event e causally depends on the previous occurrence of event e'. That a finite subset of events is consistent conveys that its events can occur together by some stage in the evolution of the process. Together the relations satisfy several axioms:

$[e] =_{\mathrm{def}} \{e' \mid e' \leq e\}$ is finite for all $e \in E$,
$\{e\} \in \mathrm{Con}$ for all $e \in E$,
$Y \subseteq X \in \mathrm{Con}$ implies $Y \in \mathrm{Con}$, and
$X \in \mathrm{Con}$ & $e \leq e' \in X$ implies $X \cup \{e\} \in \mathrm{Con}$.

There is an accompanying notion of state, or history, those events that may occur up to some stage in the behaviour of the process described. A *configuration* is a, possibly infinite, set of events $x \subseteq E$ which is: *consistent,* $X \subseteq x$ and X is finite implies $X \in \mathrm{Con}$; and *down-closed,* $e' \leq e \in x$ implies $e' \in x$.

Two events e, e' are considered to be causally independent, and called *concurrent* if the set $\{e, e'\}$ is in Con and neither event is causally dependent on the other. The relation of *immediate* dependency $e \rightarrowtriangle e'$ means e and e' are distinct with $e \leq e'$ and no event in between. We write $[X]$ for the down-closure of a subset of events X. Write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations.

It will be very useful to relate event structures by maps. A *map* of event structures $f : E \to E'$ is a partial function f from E to E' such that the image of a configuration x is a configuration fx and any event of fx arises as the image of a unique event of x; the map is thus locally injective w.r.t. a configuration x. Maps compose as partial functions. Write $\mathcal{E}$ for the ensuing category.

A map $f : E \to E'$ reflects causal dependency locally, in the sense that if e, e' are events in a configuration x of E for which $f(e') \leq f(e)$ in E', then $e' \leq e$ also in E; the event structure E inherits causal dependencies from the event structure E' via the map f. Consequently, a map preserves concurrency: if two events are concurrent, then their images if defined are also concurrent. In general a map of event structures need not preserve causal dependency.

2.2 General event structures

In contrast, a *general event structure* [9, 10] permits an event to be caused disjunctively in several ways, possibly coexisting in parallel, *i.e.* parallel causes. A general event structure comprises $(E, \mathrm{Con}, \vdash)$ where E is a set of event occurrences, the consistency relation Con is a non-empty collection of finite subsets of E, and the *enabling relation* $\vdash$ is a relation in $\mathrm{Con} \times E$ such that

$$X \subseteq Y \in \mathrm{Con} \implies X \in \mathrm{Con}\,, \text{ and}$$
$$Y \in \mathrm{Con} \;\&\; Y \supseteq X \;\&\; X \vdash e \implies Y \vdash e\,.$$

A *configuration* is a subset of E which is: *consistent,* $X \subseteq_{\mathrm{fin}} x \implies X \in \mathrm{Con}$; and *secured,* $\forall e \in x,\ \exists e_1, \cdots, e_n \in x,\ \ e_n = e\ \ \&\ \forall i \leq n, \{e_1, \cdots, e_{i-1}\} \vdash\ e_i$. Again we write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations.

The notion of an event e being enabled in a configuration has been expressed through the existence of a securing chain $e_1, \cdots, e_n$, with $e_n = e$, within the configuration. The securing chain represents a *complete enabling* of e in the sense that every event in the securing chain is itself enabled by earlier members of the chain. But just as mathematical proofs need not be sequences, so can one imagine more refined ways in which to express complete enablings. Later the idea that complete enablings can be more generally expressed as partial orders of events in which all events are enabled by earlier events in the order – "causal realisations" – will play an important role in unfolding general event structures to structures supporting hiding and parallel causes.

A *map* $f : (E, \mathrm{Con}, \vdash) \to (E', \mathrm{Con}', \vdash')$ of general event structures is a partial function $f : E \rightharpoonup E'$ such that

$$X \in \mathrm{Con} \implies fX \in \mathrm{Con}'\,,$$
$$\forall e_1, e_2 \in X \in \mathrm{Con},\ f(e_1) = f(e_2) \implies e_1 = e_2\,, \text{ and}$$
$$X \vdash e \;\&\; f(e) \text{ is defined } \implies fX \vdash' f(e)\,.$$

Maps compose as partial functions with identity maps being identity functions. Write $\mathcal{G}$ for the category of general event structures.

We can characterise those families of configurations arising from a general event structure. W.r.t. a family of subsets $\mathcal{F}$, a subset X of $\mathcal{F}$ is *compatible* (in $\mathcal{F}$), written $X\uparrow$, if there is $y \in \mathcal{F}$ such that $x \subseteq y$ for all $x \in X$; we write $x \uparrow y$ for $\{x, y\}\uparrow$. Say a subset is *finitely compatible* iff every finite subset is compatible.

A *family of configurations* comprises a family $\mathcal{F}$ of sets such that if $X \subseteq \mathcal{F}$ is finitely compatible in $\mathcal{F}$ then $\bigcup X \in F$; and if $e \in x \in \mathcal{F}$ there is a securing chain $e_1, \cdots, e_n = e$ in x such that $\{e_1, \cdots, e_i\} \in \mathcal{F}$ for all $i \leq n$. The elements of the underlying set $\bigcup \mathcal{F}$ are its *events*. Such a family is *stable* when for any compatible non-empty subset X of $\mathcal{F}$ its intersection $\bigcap X$ is a member of $\mathcal{F}$.

For configurations x, y, we use $x {-\!\!\subset} y$ to mean y covers x, *i.e.* $x \subset y$ with nothing in between, and $x \overset{e}{-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for an event $e \notin x$. We sometimes use $x \overset{e}{-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{-\!\!\subset} y$ for some y.

A map between families of configurations from $\mathcal{A}$ to $\mathcal{B}$ is a partial function $f : \bigcup \mathcal{A} \rightharpoonup \bigcup \mathcal{B}$ between their events such that $fx \in \mathcal{B}$ if $x \in \mathcal{A}$ and any event of fx arises as the image of a unique event of x. Maps compose as partial functions.

The forgetful functor taking a general event structure to its family of configurations has a left adjoint, which constructs a canonical general event structure from a family: given $\mathcal{A}$, a family of configurations with underlying events A, construct a general event structure $(A, \text{Con}, \vdash)$ with $X \in \text{Con}$ iff $X \subseteq_{\text{fin}} y$, for some $y \in \mathcal{A}$; and with $X \vdash a$ iff $a \in A$, $X \in \text{Con}$ and $e \in y \subseteq X \cup \{a\}$, for some $y \in \mathcal{A}$.

The above yields a coreflection of families of configurations in general event structures. It cuts down to an equivalence between families of configurations and *replete* general event structures. A general event structure $(E, \text{Con}, \vdash)$ is *replete* iff

$$\forall e \in E,\ \exists X \in \text{Con},\ X \vdash e\,,$$
$$\forall X \in \text{Con},\ \exists x \in \mathcal{C}(E),\ X \subseteq x \text{ and}$$
$$X \vdash e \implies \exists x \in \mathcal{C}(E),\ e \in x \ \&\ x \subseteq X \cup \{e\}\,.$$

2.3 On relating prime and general event structures

Clearly a prime event structure $(P, \leq, \text{Con})$ can be identified with a (replete) general event structure $(P, \vdash, \text{Con})$ by taking $X \vdash p$ iff $X \in \text{Con}$ & $[p] \subseteq X \cup \{p\}$. Indeed under this identification there is a full and faithful embedding of $\mathcal{E}$ in $\mathcal{G}$. However (contrary to the claim in [10]) there is no adjoint to this embedding. This leaves open the issue of providing a canonical way to describe a general event structure as a prime event structure. This issue has arisen as a central problem in reversible computation [1] and now more recently in the present limitation of concurrent strategies described in the introduction. A corollary of our work will be that the embedding of prime into general event structures does have a *pseudo* right adjoint which unfolds a general event structure to a prime event structure, got at the slight cost of enriching prime event structures with equivalence relations.

3 Problems with general event structures

Why not settle for general event structures as a foundation for distributed strategies? Because although they allow parallel causes, they don't generally support hiding, so composition of strategies; nor do they support probability generally enough.[2]

[2] Should we only be interested in deterministic, non-probabilistic strategies, general event structures do support pullback and hiding required in the composition of strategies [12]. Nondeterministic or probabilistic strategies with parallel causes require an extension such as ese's or edc's, defined shortly.

3.1 Probability and parallel causes

We return to the general-event-structure description of the strategy in the introduction. To turn this into a probabilistic strategy for Player we should assign probabilities to configurations conditional on Opponent moves. The watcher of Opponent move 1 is causally independent of Opponent move 2. Given this we might expect that the probability of the watcher of 1 making the Player move 3 should be probabilistically independent of move 2; after all, both moves 3 and 2 can occur concurrently from configuration $\{1\}$. Applying LMC naively would yield $\text{Prob}(1, 3 \mid 1) = \text{Prob}(1, 2, 3 \mid 1, 2)$. But similarly, $\text{Prob}(2, 3 \mid 2) = \text{Prob}(1, 2, 3 \mid 1, 2)$, which forces $\text{Prob}(1, 3 \mid 1) = \text{Prob}(2, 3 \mid 2)$, *i.e.* that the conditional probabilities of the two watchers succeeding are the same! In blurring the distinct ways in which move 3 can be caused we have obscured causal independence which has led us to identify possibly distinct probabilities.

3.2 Hiding

With one exception, all the operations used in building strategies and, in particular, the bicategory of concurrent strategies [8], extend to general event structures. The one exception, that of hiding, is crucial in ensuring composition of strategies yields a bicategory.

Consider a general event structure with *events* a, b, c, d and e; *enabling* (1) $b, c \vdash e$ and (2) $d \vdash e$, with all events other than e being enabled by the empty set; and *consistency* in which all subsets are consistent unless they contain the events a and b. Any configuration will satisfy the assertion $(a \wedge e) \implies d$ because if e has occurred it has to have been enabled by (1) or (2) and if a has occurred its conflict with b has prevented the enabling (1), so e can only have occurred via enabling (2).

Now imagine the event b is hidden, so allowed to occur invisibly in the background. The configurations after hiding are those obtained by hiding (*i.e.* removing) the invisible event b from the configurations of the original event structure. The assertion $(a \wedge e) \implies d$ will still hold of the configurations after hiding.

There isn't a general event structure with events a, c, d and e, and configurations those which result when we hide (remove) b from the configurations of the original event structure.[3]

Precisely the same problem can arise in the composition (with hiding) of nondeterministic strategies based on general event structures. To obtain a bicategory of strategies with disjunctive causes we need to support hiding. We need to look for structures more general than general event structures. The example above gives a clue: inconsistency should be lifted from an inconsistency between events to an inconsistency between enablings.

4 Adding disjunctive causes

To cope with disjunctive causes and hiding we must go beyond general event structures. We introduce structures in which we *objectify* cause; a minimal complete enabling is no longer an instance of a relation but a structure that realises that instance (*cf.* a judgement of theorem-hood in contrast to a proof).

[3] One way to see this is to observe that amongst the configurations after hiding we have $\{c\} {-\!\!\subset} \{c, e\}$ and $\{c\} {-\!\!\subset} \{a, c\}$ where both $\{c, e\}$ and $\{a, c\}$ have upper bound $\{a, c, d, e\}$, and yet $\{a, c, e\}$ is not a configuration after hiding as it fails to satisfy the assertion $(a \wedge e) \implies d$. In configurations of a general event structure if $x {-\!\!\subset} y$ and $x {-\!\!\subset} z$ and y and z are compatible, then $y \cup z$ is a configuration.

Fortunately we can do this while staying close to prime event structures. The twist is to regard "disjunctive events" as comprising subsets of events of a prime event structure, the events of which are now to be thought of as representing "prime causes" standing for minimal complete enablings. Technically, we do this by extending prime event structures with an equivalence relation on events.

In detail, an *event structure with equivalence* (an ese) is a structure $(P, \leq, \mathrm{Con}, \equiv)$ where $(P, \leq, \mathrm{Con})$ is a (prime) event structure and $\equiv$ is an equivalence relation on P.

An ese dissociates the two roles of enabling and atomic action conflated in the events of a prime event structures. The intention is that the events p of P, or really their corresponding down-closures $[p]$, describe minimal complete enablings, *prime causes*, while the $\equiv$-equivalence classes of P represent *disjunctive events*: p is a prime cause of the disjunctive event $\{p\}_\equiv$. Notice there may be several prime *disjunctive causes* of the same disjunctive event and that these may be *parallel causes* in the sense that they are consistent with each other and not related in the order $\leq$.

A *configuration* of the ese is a configuration of $(P, \leq, \mathrm{Con})$ and we shall use the notation of earlier on event structures $\mathcal{C}^\infty(P)$ and $\mathcal{C}(P)$ for its configurations, respectively finite configurations. We say a configuration is *unambiguous* if it has no two distinct elements which are $\equiv$-equivalent. We modify the relation of concurrency a little and say $p_1, p_2 \in P$ are *concurrent* and write $p_1 co\, p_2$ iff $[p_1] \cup [p_2]$ is an *unambiguous* configuration of P and neither $p_1 \leq p_2$ nor $p_2 \leq p_1$.

When the equivalence relation $\equiv$ of an ese is the identity we essentially have a prime event structure. This view is reinforced in our choice of maps. A map from ese $(P, \leq_P, \mathrm{Con}_P \equiv_P)$ to $(Q, \leq_Q, \mathrm{Con}_Q, \equiv_Q)$ is a partial function $f : P \rightharpoonup Q$ which *preserves* $\equiv$, *i.e.* if $p_1 \equiv_P p_2$ then either both $f(p_1)$ and $f(p_2)$ are undefined or both defined with $f(p_1) \equiv_Q f(p_2)$), such that for all $x \in \mathcal{C}(P)$ we have (i) the direct image $fx \in \mathcal{C}(Q)$, and (ii) $\forall p_1, p_2 \in x,\ f(p_1) \equiv_Q f(p_2) \implies p_1 \equiv_P p_2$. Maps compose as partial functions with the usual identities. Such maps preserve the concurrency relation. They are only assured to reflect causal dependency locally w.r.t. unambiguous configurations.

We regard two maps $f_1, f_2 : P \to Q$ as equivalent, and write $f_1 \equiv f_2$, iff they are equi-defined and yield equivalent results, *i.e.*if $f_1(p)$ is defined then so is $f_2(p)$ and $f_1(p) \equiv_Q f_2(p)$, and symmetrically. Composition respects $\equiv$: if $f_1, f_2 : P \to Q$ with $f_1 \equiv f_2$ and $g_1, g_2 : Q \to R$ with $g_1 \equiv g_2$, then $g_1 f_1 \equiv g_2 f_2$. Write $\mathcal{E}_\equiv$ for the category of ese's; it is *enriched* in the category of sets with equivalence relations – see [3] and Appendix A.

Ese's support a hiding operation. Let $(P, \leq, \mathrm{Con}_P, \equiv)$ be an ese. Let $V \subseteq P$ be a $\equiv$-closed subset of 'visible' events. Define the *projection* of P on V, to be $P{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V, \equiv_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$ and $v \equiv_V v'$ iff $v \equiv v'$ & $v, v' \in V$.

Hiding is associated with a factorisation of partial maps. Let f be a partial map from $(P, \leq_P, \mathrm{Con}_P, \equiv_P)$ to $(Q, \leq_Q, \mathrm{Con}_Q, \equiv_Q)$. Letting $V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}$, the map f factors into the composition

$$P \xrightarrow{f_0} P{\downarrow}V \xrightarrow{f_1} Q$$

of f_0, a partial map of ese's taking $p \in P$ to itself if $p \in V$ and undefined otherwise, and f_1, a total map of ese's acting like f on V. We call f_1 the *defined part* of the partial map f. Because $\equiv$-equivalent maps share the same domain of definition, $\equiv$-equivalent maps will determine the same projection and $\equiv$-equivalent defined parts. The factorisation is characterised to within isomorphism by the following universal characterisation: for any factorisation $P \xrightarrow{g_0} P_1 \xrightarrow{g_1} Q$ where g_0 is partial and g_1 is total there is a (necessarily total) unique map $h : P{\downarrow}V \to P_1$ such that we obtain the commuting diagram

$$\begin{array}{ccccc} P & \xrightarrow{f_0} & P{\downarrow}V & \xrightarrow{f_1} & Q \\ & \searrow^{g_0} & \downarrow h & \nearrow_{g_1} & \\ & & P_1\,. & & \end{array}$$

The category $\mathcal{E}_\equiv$ of ese's supports hiding in the sense above.

5 Unfolding general event structures to ese's

We next show how replete general event structures embed in ese's as part of a (pseudo) reflection. This fixes the sense in which ese's extend the established model of general event structures in their treatment of parallel causes, while in addition supporting hiding. The relevant (pseudo) adjoint from $\mathcal{G}$ to $\mathcal{E}_\equiv$ is quite subtle and is a form of unfolding of a general event structure into an ese of its prime causes.

The pseudo functor arises as a right adjoint to a more obvious functor from $\mathcal{E}_\equiv$ to $\mathcal{G}$. Given an ese $(P, \leq, \mathrm{Con}, \equiv)$ we can construct a (replete) general event structure $ges(P) =_{\mathrm{def}} (E, \mathrm{Con}_E, \vdash)$ by taking

$E = P_\equiv$, the equivalence classes under $\equiv$,
$X \in \mathrm{Con}_E$ iff $\exists Y \in \mathrm{Con},\ X = Y_\equiv$, and
$X \vdash e$ iff $X \in \mathrm{Con}$ & $e \in E$ & $\exists p \in P,\ e = \{p\}_\equiv$ & $[p]_\equiv \subseteq X \cup \{e\}$.

The construction extends to a functor $ges : \mathcal{E}_\equiv \to \mathcal{G}$ as maps between ese's preserve $\equiv$; the functor takes a map $f : P \to Q$ of ese's to the map $ges(f) : ges(P) \to ges(Q)$ obtained as the partial function induced on equivalence classes. Less obvious is that there is a (pseudo) right adjoint to *ges*. Its construction relies on *extremal causal realisations* which provide us with an appropriate notion of minimal complete enabling of events in a general event structure; these furnish us with the prime causes from which to build the ese unfolding.

5.1 Causal realisations

Let $\mathcal{A}$ be a family of configurations with underlying set A. A *(causal) realisation* of $\mathcal{A}$ comprises a partial order $(E, \leq)$, its *carrier*, such that the set $\{e' \in E \mid e' \leq e\}$ is finite for all events $e \in E$, together with a function $\rho : E \to A$ for which the image $\rho x \in \mathcal{A}$ when x is a down-closed subset of E.

A map between realisations $(E, \leq), \rho$ and $(E', \leq'), \rho'$ is a partial surjective function $f : E \rightharpoonup E'$ which preserves down-closed subsets and satisfies $\rho(e) = \rho'(f(e))$ when $f(e)$ is defined. It is convenient to write such a map as $\rho \succeq^f \rho'$. Occasionally we shall write $\rho \succeq \rho'$, or the converse $\rho' \preceq \rho$, to mean there is a map of realisations from ρ to ρ'. Such a map factors into a "projection" followed by a total map

$$\rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho'\,,$$

where ρ_0 stands for the realisation $(E_0, \leq_0), \rho_0$ where $E_0 = \{r \in R \mid f(r) \text{ is defined}\}$ is the domain of definition of f, $\leq_0$ is the restriction of $\leq$, f_1 is the inverse relation to the inclusion $E_0 \subseteq E$, and f_2 is the total function $f_2 : E_0 \to E'$. We are using $\succeq_1$ and $\succeq_2$ to signify the two kinds of maps. Notice that $\succeq_1$-maps are reverse inclusions. Notice too that $\succeq_2$-maps are exactly the total maps of realisations. Total maps $\rho \succeq_2^f \rho'$ are precisely those functions f from the carrier of ρ to the carrier of ρ' which preserve down-closed subsets and satisfy $\rho = \rho' f$.

We shall say a realisation ρ is *extremal* when $\rho \succeq_2^f \rho'$ implies f is an isomorphism, for any realisation ρ'; it is called *prime extremal* when it in addition has a top element, *i.e.* its carrier contains an element which dominates all other elements in the carrier.

In the special case where $\mathcal{A}$ is the family of configurations of a prime event structure, it is easy to show that an extremal realisation ρ forms a bijection with a configuration of the event structure and that the order on the carrier coincides with causal dependency there; the prime extremals correspond to configurations of the form $[e]$ for some event e.

The construction is more interesting when $\mathcal{A}$ is the family of configurations of a general event structure A. In general, there is at most one map between extremal realisations. Hence extremal realisations of $\mathcal{A}$ under $\preceq$ form a preorder. The *order of extremal realisations* has as elements isomorphism classes of extremal realisations ordered according to the existence of a map between representatives of isomorphism classes. In fact:

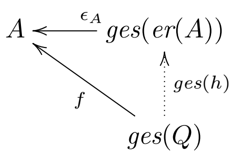
▶ **Theorem 1.** *The order of extremal realisations of $\mathcal{A}$ forms a prime-algebraic domain [5] with complete primes represented by the prime extremal realisations.*

The import of this theorem is that the order of extremal realisations is isomorphic to the configurations of a prime event structure ordered by inclusion. The event structure has events the prime extremal realisations; causal dependency the restriction of the order on extremal realisations; with consistency induced by compatibility there.

5.2 A pseudo adjunction

From Theorem 1, a general event structure A determines a prime event structure with events the prime extremal realisations of $\mathcal{C}^\infty(A)$ [5]. The top element of each prime extremal images to an event of A, providing a map from prime extremals to A. To get the ese-unfolding $er(A)$ of A we further endow the prime event structure with an equivalence, taking two prime extremals as equivalent if their top elements have the same image. Because equivalent prime extremals are sent to the same event of A, we determine a map $\epsilon_A : ges(er(A)) \to A$ of general event structures. It is the component of the counit of the adjunction at A. (See Appendix B for the proof and the detailed construction of er.)

▶ **Theorem 2.** *Let $A \in \mathcal{G}$. For all $f : ges(Q) \to A$ in $\mathcal{G}$, there is a map $h : Q \to er(A)$ in $\mathcal{E}_\equiv$ such that $f = \epsilon_A \circ ges(h)$ i.e. so the diagram below commutes:*

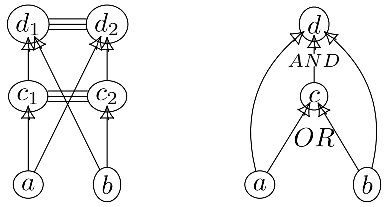


Moreover, if $h' : Q \to er(A)$ is a map in $\mathcal{E}_\equiv$ such that $f = \epsilon_A \circ ges(h')$, then $h' \equiv h$.

The theorem does not quite exhibit a standard adjunction, because the usual cofreeness condition specifying an adjunction is weakened to only having uniqueness up to $\equiv$. However the condition it describes does specify a simple case of *pseudo adjunction* between 2-categories – a set together with an equivalence relation is a very simple example of a category (see Appendix A). As a consequence, whereas with the usual cofreeness condition allows us to extend the right adjoint to arrows, so obtaining a functor, in this case following that same line will only yield a pseudo functor er as right adjoint: thus extended, er will only necessarily preserve composition and identities up to $\equiv$.

The pseudo adjunction from $\mathcal{E}_\equiv$ to $\mathcal{G}$ cuts down to a pseudo reflection (*i.e.* the counit ϵ is a natural isomorphism) when we restrict to the subcategory of $\mathcal{G}$ where all general event structures are replete. Its right adjoint provides a pseudo functor embedding replete general event structures (and so families of configurations) in ese's.

▶ **Example 3.** On the right we show a general event structure and on its left the ese which it unfolds to under *er*:



Although they don't appear in this example, it is possible to have extremal realisations in which an event depends on an event of the family having been enabled in two distinct ways – see Appendix B, Example 10. (Such phenomena will be disallowed in edc's.)

6 EDC'S

Our major motivation in developing and exploring ese's was in order to extend strategies with parallel causes while maintaining the central operation of hiding. What about the other operation key to the composition of strategies, *viz.* pullback?

It is well-known to be hard to construct limits such as pullback within prime event structures, so that we often rely on first carrying out the constructions in stable families, into which there is a coreflection from prime event structures. We might expect an analogous way to construct pullbacks or pseudo pullbacks in $\mathcal{E}_\equiv$.

6.1 Equivalence families

In fact, the pseudo adjunction from $\mathcal{E}_\equiv$ to $\mathcal{G}$ factors through a more basic pseudo adjunction to families of configurations which also bear an equivalence relation on their underlying sets. An *equivalence-family* (ef) is a family of configurations $\mathcal{A}$ with an equivalence relation $\equiv_A$ on its underlying set $\bigcup \mathcal{A}$. We can identify a family of configurations $\mathcal{A}$ with the ef $(\mathcal{A}, =)$, taking the equivalence to be simply equality on the underlying set. A map $f : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$ between ef's is a partial function $f : A \rightharpoonup B$ between their underlying sets which preserves $\equiv$ so that

$$x \in \mathcal{A} \Rightarrow fx \in \mathcal{B} \ \& \ \forall a_1, a_2 \in x,\ f(a_1) \equiv_B f(a_2) \Rightarrow a_1 \equiv_A a_2 \,.$$

Composition is composition of partial functions. We regard two maps $f_1, f_2 : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$ as equivalent, and write $f_1 \equiv f_2$, iff they are equidefined and yield equivalent results. Composition respects $\equiv$. This yields a category of equivalence families $\mathcal{F}am_\equiv$ enriched in the category of sets with equivalence relations.

Clearly we can regard an ese $(P, \leq_P, \mathrm{Con}_P, \equiv_P)$ as an ef $(\mathcal{C}^\infty(P), \equiv_P)$ and a function which is a map of ese's as a map between the associated ef's, and this operation forms a functor. The functor has a pseudo right adjoint built from causal realisations in a very similar manner to *er*. The configurations of a general event structure form an ef with the identity relation as its equivalence. This operation is functorial and has a left adjoint which collapses an ef to a general event structure in a similar way to *ges*; the adjunction is enriched in equivalence relations. In summary, the pseudo adjunction

$$\mathcal{E}_{\equiv} \underset{ges}{\overset{er}{\rightleftarrows}} \mathcal{G} \quad (\top)$$

factors into a pseudo adjunction followed by an adjunction

$$\mathcal{E}_{\equiv} \rightleftarrows \mathcal{F}am_{\equiv} \rightleftarrows \mathcal{G}\,.$$

$\mathcal{F}am_{\equiv}$ has pullbacks and pseudo pullbacks which are easy to construct. For example, let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be total maps of ef's. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\text{def}} \{(a, b) \in A \times B \mid f(a) \equiv_C g(b)\}$ with projections π_1 and π_2 to the left and right components. On D, take $d \equiv_D d'$ iff $\pi_1(d) \equiv_A \pi_1(d')$ and $\pi_2(d) \equiv_B \pi_2(d')$. Define a family of configurations of the *pseudo pullback* to consist of $x \in \mathcal{D}$ iff $x \subseteq D$ such that $\pi_1 x \in \mathcal{A}$ & $\pi_2 x \in \mathcal{B}$, and

$$\forall d \in x \exists d_1, \cdots, d_n \in x,\ d_n = d\ \&$$
$$\forall i \leq n,\ \pi_1\{d_1, \cdots, d_i\} \in \mathcal{A}\ \&\ \pi_2\{d_1, \cdots, d_i\} \in \mathcal{B}\,.$$

The ef $\mathcal{D}$ with maps π_1 and π_2 is the pseudo pullback of f and g. It would coincide with pullback if $\equiv_C$ were the identity.

But unfortunately (pseudo) pullbacks in $\mathcal{F}am_{\equiv}$ don't provide us with (pseudo) pullbacks in $\mathcal{E}_{\equiv}$ because the right adjoint is only a pseudo functor: in general it will only carry pseudo pullbacks to bipullbacks. While $\mathcal{E}_{\equiv}$ does have bipullbacks (in which commutations and uniqueness are only up to the equivalence $\equiv$ on maps) it doesn't always have pseudo pullbacks or pullbacks – Appendix C. Whereas pseudo pullbacks and pullbacks are characterised up to isomorphism, bipullbacks are only characterised up to a weaker equivalence – that induced on objects by the equivalence on maps.[4] While we could develop strategies with parallel causes in the broad context of ese's, defining the composition of strategies via bipullbacks and hiding, doing so would mean that the composition of strategies that ensued was defined only up to equivalence and not isomorphism. Our definition of strategy-composition would be accordingly weaker in that its characterisation could only be up to equivalence.

6.2 Edc's defined

Fortunately there is a subcategory of $\mathcal{E}_{\equiv}$ which supports pullbacks and pseudo pullbacks, as well as hiding. Define $\mathcal{EDC}$ to be the subcategory of $\mathcal{E}_{\equiv}$ with objects ese's satisfying

$$p_1, p_2 \leq p\ \&\ p_1 \equiv p_2 \implies p_1 = p_2\,.$$

We call such objects *event structures with disjunctive causes* (edc's). In an edc an event can't causally depend on two distinct prime causes of a common disjunctive event, and so rules out realisations such as that mentioned in Example 10. In general, within $\mathcal{E}_{\equiv}$ we lose the local injectivity property that we're used to seeing for maps of event structures; the maps of event structures are injective from configurations, when defined. However for $\mathcal{EDC}$ we recover local injectivity w.r.t. prime configurations, of form $[p]$: if $f : P \to Q$ is a map in $\mathcal{EDC}$, then

$$p_1, p_2 \leq_P p\ \&\ f(p_1) = f(p_2) \implies p_1 = p_2\,.$$

[4] Objects P and Q are equivalent iff there are two maps $f : P \to Q$, $g : Q \to P$ s.t. $gf \equiv \text{id}_P$ and $fg \equiv \text{id}_Q$.

The factorisation property associated with hiding in $\mathcal{E}_\equiv$ is inherited by $\mathcal{EDC}$.

As regards (pseudo) pullbacks, we are fortunate in that the complicated pseudo adjunction between ese's and ef's restricts to a much simpler (pseudo) adjunction, in fact a coreflection, between edc's and *stable* ef's. In an equivalence family $(\mathcal{A}, \equiv_A)$ say a configuration $x \in \mathcal{A}$ is *unambiguous* iff $\forall a_1, a_2 \in x,\ a_1 \equiv_A a_2 \implies a_1 = a_2$. An equivalence family $(\mathcal{A}, \equiv_A)$, with underlying set of events A, is *stable* iff it satisfies

$$\forall x, y, z \in \mathcal{A},\ x, y \subseteq z \ \&\ z \text{ is unambiguous } \Rightarrow\ x \cap y \in \mathcal{A}\,,\ \text{ and}$$
$$\forall a \in A, x \in \mathcal{A},\ a \in x\ \Rightarrow \exists z \in \mathcal{A},\ z \text{ is unambiguous } \& \, a \in z \subseteq x\,.$$

In effect a stable equivalence family contains a stable subfamily of unambiguous configurations out of which all other configurations are obtainable as unions. Local to any unambiguous configuration x there is a partial order on its events $\leq_x$: each $a \in x$ determines a *prime configuration*

$$[a]_x =_{\text{def}} \bigcap \{y \in \mathcal{A} \mid a \in y \subseteq x\}\,,$$

the minimum set of events on which a depends within x; taking $a \leq_x b$ iff $[a]_x \subseteq [b]_x$ defines causal dependency between $a, b \in x$. Write $\mathcal{SFam}_\equiv$ for the subcategory of stable ef's.

(Pseudo) pullbacks in stable ef's are obtained from those in ef's simply by restricting to those configurations which are unions of unambiguous configurations. The configurations of an edc with its equivalence are easily seen to form a stable ef providing a full and faithful embedding of $\mathcal{EDC}$ in $\mathcal{SFam}_\equiv$. The embedding has a right adjoint Pr. It is built out of prime extremals but we can take advantage of the fact that in a stable ef unambiguous prime extremals have the simple form of prime configurations. From a stable ef$(\mathcal{A}, \equiv_A)$ we produce an edc $\text{Pr}(\mathcal{A}, \equiv_A) =_{\text{def}} (P, \text{Con}, \leq, \equiv)$ in which P comprises the prime configurations with

$$[a]_x \equiv [a']_{x'} \text{ iff } a \equiv_A a'\,,$$
$$Z \in \text{Con iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F}\,,\ \text{ and}$$
$$p \leq p' \text{ iff } p, p' \in P \ \&\ p \subseteq p'\,.$$

The adjunction is enriched in the sense that its natural bijection preserves and reflects the equivalence on maps:

$$\mathcal{EDC} \underset{\longrightarrow}{\overset{\text{Pr}}{\underset{\top}{\longleftarrow}}} \mathcal{SFam}_\equiv$$

We can now obtain a (pseudo) pullback in edc's by first forming the (pseudo) pullback of the stable ef's obtained as their configurations and then taking its image under the right adjoint Pr. We now have the constructions we need to support strategies based on edc's.

6.3 Coreflective subcategories of edc's

$\mathcal{EDC}$ is a coreflective subcategory of $\mathcal{E}_\equiv$; the right adjoint simply cuts down to those events satisfying the edc property. In turn $\mathcal{EDC}$ has a coreflective subcategory $\mathcal{E}^0_\equiv$ comprising those edc's which satisfy

$$\{p_1, p_2\} \in \text{Con} \ \&\ p_1 \equiv p_2 \implies p_1 = p_2\,.$$

Consequently its maps are traditional maps of event structures which preserve the equivalence. We derive adjunctions

$$\mathcal{E}^0_\equiv \underset{\longrightarrow}{\underset{\top}{\longleftarrow}} \mathcal{EDC} \underset{\longrightarrow}{\underset{\top}{\longleftarrow}} \mathcal{E}_\equiv \underset{ges}{\underset{\longrightarrow}{\overset{er}{\underset{\top}{\longleftarrow}}}} \mathcal{G}\,.$$

Note the last is only a pseudo adjunction. Consequently we obtain a pseudo adjunction from $\mathcal{E}^0_{\equiv}$, the category of prime event structures with equivalence relations and general event structures – this makes good the promise of Section 2.3. Inspecting the composite of the last two adjunctions, we also obtain the sense in which replete general event structures embed via a reflection in edc's.

There is an obvious 'inclusion' functor from the category of prime event structures $\mathcal{E}$ to the category $\mathcal{EDC}$: it extends an event structure with the identity equivalence. Regarding $\mathcal{EDC}$ as a plain category, so dropping the enrichment by equivalence relations, the 'inclusion' functor $\mathcal{E} \hookrightarrow \mathcal{EDC}$ has a right adjoint, *viz.* the forgetful functor which given an edc $P = (P, \leq, \mathrm{Con}, \equiv)$ produces an event structure $P_0 = (P, \leq, \mathrm{Con}')$ by dropping the equivalence $\equiv$ and modifying the consistency relation to:

$$X \in \mathrm{Con}' \text{ iff } X \subseteq P \ \& \ X \in \mathrm{Con} \ \& \ p_1 \not\equiv p_2, \text{ for all } p_1, p_2 \in X \,.$$

The configurations of P_0 are the unambiguous configurations of P. The adjunction is a coreflection because the inclusion functor is full. Of course it is not the case that the adjunction is enriched: the natural bijection of the adjunction cannot respect the equivalence on maps; it cannot compose with the pseudo adjunction from $\mathcal{EDC}$ to $\mathcal{G}$ to yield a pseudo adjunction from $\mathcal{E}$ to $\mathcal{G}$.

Despite this the adjunction from $\mathcal{E}$ to $\mathcal{EDC}$ has many useful properties. Of importance for us is that the functor forgetting equivalence will preserve all limits and especially pullbacks. It is helpful in relating composition of edc-strategies to the composition of strategies based on prime event structures in [8]. In composing strategies in edc's we shall only be involved with pullbacks of maps $f : A \to C$ and $g : B \to C$ of edc's. (When C is essentially an event structure, *i.e.* an edc in which the equivalence is the identity relation, the construction of such pullbacks coincides with that of pseudo pullbacks.) While this does not entail that composition of strategies is preserved by the forgetful functor – because the forgetful functor does not commute with hiding – it will give us a strong relationship, expressed as a map, between composition of the two kinds of strategies (based on edc's and based on prime event structures) after and before applying the forgetful functor. This has been extremely useful in key proofs of the next section, in importing results about concurrent strategies from [8].

7 Probabilistic strategies based on edc's

The ground is prepared for a general definition of distributed probabilistic strategies, based on edc's. The development follows the same lines as that of probabilistic concurrent strategies [8, 11], to which we refer the reader, and can only be sketched briefly here.

An *edc with polarity* comprises $(P, \leq_P, \mathrm{Con}_P, \equiv, pol)$, an edc $(P, \leq_P, \mathrm{Con}_P, \equiv)$ in which each element $p \in P$ carries a polarity $pol(p)$ which is $+$ or $-$, according as it represents a move of Player or Opponent, and where the equivalence relation $\equiv$ respects polarity. A *map* of edc's with polarity is a map of the underlying edc's which preserves polarity when defined. The adjunctions of the previous section are undisturbed by adding polarity.

A *game* is represented by an edc with polarity. There are two fundamentally important operations on two-party games. One is that of forming the *dual* game. On a game A this amounts to reversing the polarities of events to produce the dual $A^\perp$. The other operation, a *simple parallel composition* $A \| B$, is achieved on games A and B by simply juxtaposing them, ensuring a finite subset of events is consistent if its overlaps with the two games are individually consistent.

A *pre-strategy* in a game A is a total map $\sigma : S \to A$ of edc's with polarity. A pre-strategy from a game A to a game B is a pre-strategy in the game $A^\perp \| B$. A map $f : \sigma \Rightarrow \sigma'$ of pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ is a map $f : S \to S'$ s.t. $\sigma = \sigma' f$; this determines isomorphism of pre-strategies. The map is *rigid* if it preserves causal dependency.

Two edc pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ compose via *pullback* and *hiding* – with parallel causes, the key features driving our search for edc's. Ignoring polarities, the composite partial map

$$\begin{array}{ccccc} & & A \parallel T & & \\ & \overset{\pi_2}{\nearrow} & & \overset{A\|\tau}{\searrow} & \\ T \circledast S \rangle & & & & A \parallel B \parallel C \rightharpoonup A \parallel C \\ & \underset{\pi_1}{\searrow} & & \underset{\sigma\|C}{\nearrow} & \\ & & S \parallel C & & \end{array}$$

has defined part, yielding the composition $\tau \odot \sigma : T \odot S \to A^\perp \| C$ once we reinstate polarities. (The partial map from $A\|B\|C$ to $A\|C$ acts as the identity but for being undefined on B.)

The copycat strategy comprises $\text{æ}_A : \mathbb{C}_A \to A^\perp \| A$ where $\mathbb{C}_A$ is obtained by adding extra causal dependencies to $A^\perp \| A$ so that any Player move in either component causally depends on its copy, an Opponent move, in the other [8].

In general, copycat may not be an identity w.r.t. composition. However, copycat acts as identity precisely on an edc pre-strategy $\sigma : S \to A$ which is an *edc strategy*, capturing the sense in which Player cannot influence Opponent beyond the constraints of the game:

(i) the image $\sigma_0 : S_0 \to A_0$ of σ (under the right adjoint to the inclusion of event structures in edc's) is a strategy of concurrent games, *i.e.* is *receptive* and *innocent*, as in [8];[5] and

(ii) $s_1 \equiv_S s_2 \iff \sigma(s_1) \equiv_A \sigma(s_2)$, for all $s_1, s_2 \in S$; with

(iii) $x \overset{s}{\longrightarrow\!\!\!\subset} z$ & $x \overset{s'}{\longrightarrow\!\!\!\subset} z'$ & $pol(s) = -$ & $\sigma z \uparrow \sigma z' \implies z \uparrow z'$.

▶ **Theorem 4.** *When $\sigma : S \to A$ is an edc pre-strategy, $\sigma \cong \text{æ}_A \odot \sigma$ iff σ is an edc strategy.*

A *probabilistic edc strategy* in a game A, is an edc strategy $\sigma : S \to A$ together with a configuration-valuation which endows S with probability, while taking account of the fact that in the strategy Player can't be aware of the probabilities assigned by Opponent. We should restrict to race-free games, precisely those for which copycat is deterministic, so that we have probabilistic identity strategies; it follows that S is race-free. A configuration-valuation extends the definition of probabilistic event structure [11] with an axiom (lmc) which implies the Limited Markov Condition, LMC, of the introduction. Precisely, a *configuration-valuation* is a function $v : \mathcal{C}(S) \to [0,1]$ which is: *(normalized)* $v(\emptyset) = 1$; and satisfies:

- *(lmc)* $v(x) = v(y)$ when $x \subseteq^- y$ for finite configurations x, y of S
- *(+ve drop condition)* $d_v[y; x_1, \cdots, x_n] \geq 0$ when $y \subseteq^+ x_1, \cdots, x_n$ for finite configurations.

The 'drop' function, $d_v[y; x_1, \cdots, x_n] =_{\text{def}} v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)$, where the index I ranges over nonempty $I \subseteq \{1, \cdots, n\}$ such that the union $\bigcup_{i \in I} x_i \in \mathcal{C}(S)$. Above we use $x \subseteq^- y$, and $x \subseteq^+ y$, to mean inclusion in which all the intervening events have the indicated polarity.

When there are no Opponent moves in S, a configuration-valuation corresponds to a continuous valuation on the Scott-open sets of $\mathcal{C}^\infty(S)$ and determines a probability distribution on the Borel sets; then $v(x)$ is $\text{Prob}(x)$, the probability that the result includes the events of the finite configuration x [11]. When S has Opponent moves, the reading of a

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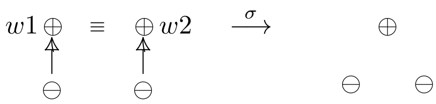
[5] A total map of event structures with polarity $\sigma : S \to A$ is *receptive* if $\sigma x \overset{a}{\longrightarrow\!\!\!\subset}$ & $pol(a) = -$ implies $\exists! s,\ x \overset{s}{\longrightarrow\!\!\!\subset}$ & $\sigma(s) = a$. It is *innocent* if $s \rightarrowtriangle s'$ with $pol(s) = +$ or $pol(s') = -$ implies $\sigma(s) \rightarrowtriangle \sigma(s')$.

configuration-valuation involves conditional probabilities. When $x \subseteq^+ y$ in $\mathcal{C}(S)$, provided $v(x) \neq 0$, the conditional probability of Player making moves $y \setminus x$ given x, is expressed by $\mathrm{Prob}(y \mid x) = v(y)/v(x)$. Because S is race-free, this reading, with (lmc), ensures we obtain LMC directly.

The composition above extends to probabilistic edc strategies. Assuming σ and τ have configuration-valuations v_S and v_T their composition $\tau\odot\sigma$ has configuration-valuation $v(x) =_{\mathrm{def}} v_S([x]_S).v_T([x]_T)$ for x a finite configuration of $T\odot S$; the configuration $[x]_S$ is the S-component in $\mathcal{C}(S)$ of the projection $\pi_1[x]$, and $[x]_T$ the T-component of $\pi_2[x]$. The proof that v is a configuration-valuation relies heavily on properties of "drop" functions.

We obtain a bicategory of probabilistic edc strategies which support parallel causes; its 2-cells are rigid maps of strategies which relate configuration-valuations across 2-cells via a 'push-forward' result – see Appendix D. It has a sub-bicategory of deterministic edc strategies analogous to that of [8]. But now there are deterministic strategies with parallel causes, including the strategy sketched informally in the introduction in which Player makes a move iff Opponent makes one or more of their moves:



Similarly, there are now deterministic strategies for "parallel or" and McCarthy's amb [4].

► **Example 5.** Recall the game of the introduction. In the edc strategy drawn above, individual success of the two watchers $w1$ and $w2$ may be associated with probabilities $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, respectively, and their joint success with $q \in [0, 1]$ provided they form a configuration-valuation v. In other words, $v(x) = p_1$ if x contains $w1$ and not $w2$; $v(x) = p_2$ if x contains $w2$ and not $w1$; and $v(x) = q$ if x contains both $w1$ and $w2$; $v(x) = 1$ otherwise; and $p_1 + p_2 - q \leq 1$, in order to satisfy the $+$-drop condition. To enliven this a little we might imagine the two watchers have a drinking problem and the correlation depends on whether they are sharing from a common bottle: if they had their own bottles we might imagine the drunken unreliability of one independent of that of the other, so $q = p_1.p_2$; as good friends sharing from a common bottle their drunkeness might correlate, so $p_1 = p_2 = q$.

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A Equiv-enriched categories

Here we explain in more detail what we mean when we say "enriched in the category of of sets with equivalence relations" and employ terms such as "enriched adjunction," "pseudo adjunction" and "pseudo pullback." The classic text on enriched categories is [2], but for this paper the articles [3] and [7] provide short, accessible introductions to the notions we use from Equiv-enriched categories and 2-categories, respectively.

Equiv is the category of *equivalence relations*. Its objects are $(A, \equiv_A)$ comprising a set A on which there is an equivalence relation $\equiv_A$. Its maps $f : (A, \equiv_A) \to (B, \equiv_B)$ are total functions $f : A \to B$ which preserve equivalence.

We shall use some basic notions from enriched category theory [2]. We shall be concerned with categories enriched in Equiv, called Equiv-enriched categories, in which the homsets possess the structure of equivalence relations, respected by composition [3]. This is the sense in which we say categories are enriched in (the category of) equivalence relations. We similarly borrow the concept of an Equiv-enriched functor between Equiv-enriched categories which preserve equivalence in acting on homsets. An Equiv-enriched adjunction is a usual adjunction in which the natural bijection preserves and reflects equivalence.

Because an object in Equiv can be regarded as a (very simple) category, we can regard Equiv-enriched categories as (very simple) 2-categories to which notions from 2-categories apply [7].

A *pseudo functor* between Equiv-enriched categories is like a functor but the usual laws only need hold up to equivalence. A *pseudo adjunction* (or biadjunction) between 2-categories permits a weakening of the usual natural isomorphism between homsets, now also categories, to a natural equivalence of categories. In the special case of a pseudo adjunction between Equiv-enriched categories the equivalence of homset categories amounts to a pair of $\equiv$-preserving functions whose compositions are $\equiv$-equivalent to the identity function. With traditional adjunctions by specifying the action of one adjoint solely on objects we determine it as a functor; with pseudo adjunctions we can only determine it as a pseudo functor – in general a pseudo adjunction relates two pseudo functors. Pseudo adjunctions compose in the expected way. An Equiv-enriched adjunction is a special case of a 2-adjunction between 2-categories and a very special case of pseudo adjunction. In this article there are many cases in which we compose an Equiv-enriched adjunction with a pseudo adjunction to obtain a new pseudo adjunction.

Similarly we can specialise the notions pseudo pullbacks and bipullbacks from 2-categories to Equiv-enriched categories. Let $f : A \to C$ and $g : B \to C$ be two maps in an Equiv-enriched category. A *pseudo pullback* of f and g is an object D and maps $p : D \to A$ and $q : D \to B$ such that $f \circ p \equiv g \circ q$ which satisfy the further property that for any D' and maps $p' : D' \to A$ and $q' : D' \to B$ such that $f \circ p' \equiv g \circ q'$, there is a unique map $h : D' \to D$ such that $p' = p \circ h$ and $q' = q \circ h$. There is an obvious weakening of pseudo pullbacks to the situation in which the uniqueness is replaced by uniqueness up to $\equiv$ and the equalities by $\equiv$ – these are simple special cases of bilimits called *bipullbacks*.

Right adjoints in a 2-adjunction preserve pseudo pullbacks whereas right adjoints in a pseudo adjunction are only assured to preserve bipullbacks.

B The proof of Theorem 2

Here we fill in some details of the proof of Theorem 2 providing a pseudo right adjoint to the functor $ges : \mathcal{E}_\equiv \to \mathcal{G}$: the functor ges quotients an ese down to a general event structure; its right adjoint er constructs an ese out of the prime extremal realisations of a general event structure. Note the adjunction is not an equivalence: whereas it does cut down to a reflection from $\mathcal{G}$ to $\mathcal{E}_\equiv$, where the counit is an isomorphism, the unit is not an isomorphism.

The right adjoint $er : \mathcal{G} \to \mathcal{E}_\equiv$ is defined on objects as follows. Let A be a general event structure. Define $er(A) = (P, \mathrm{Con}_P, \leq_P, \equiv_P)$ where

- P consists of a choice from within each isomorphism class of the prime extremals p of $\mathcal{C}^\infty(A)$ – we write $top_A(p)$ for the image of the top element in A;
- Causal dependency $\leq_P$ is $\preceq$ on P;
- $X \in \mathrm{Con}_P$ iff $X \subseteq_{\mathrm{fin}} P$ and $top_A[X] \in \mathcal{C}^\infty(A)$ – the set $[X]$ is the $\leq_P$-downwards closure of X;
- $p_1 \equiv_P p_2$ iff $p_1, p_2 \in P$ and $top_A(p_1) = top_A(p_2)$.

▶ **Proposition 6.** *The configurations of P, ordered by inclusion, are order-isomorphic to the order of extremal realisations of $\mathcal{C}^\infty(A)$: an extremal realisation ρ corresponds, up to isomorphism, to the configuration $\{p \in P \mid p \preceq \rho\}$ of P; conversely, a configuration x of P corresponds to an extremal realisation $top_A : x \to A$ with carrier $(x, \preceq)$, the restriction of the order of P to x.*

Proof. See the proof of Proposition 16.17 of the ECSYM Notes [12]. ◀

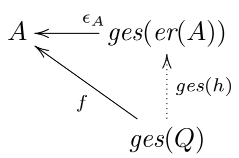
Theorem 1 of the main text, asserting the prime algebraicity of the order of extremal realisations, follows as an immediate corollary of the above proposition.

In defining the right adjoint we rely on the fact that any realisation of a family of configurations can be coarsened to an extremal realisation.

▶ **Lemma 7.** *For any realisation ρ there is an extremal realisation ρ' with $\rho \succeq_2^f \rho'$.*

Proof. See the proof of Lemma 16.4 of the ECSYM Notes [12]. ◀

▶ **Theorem 2** (restated)**.** *Let $A \in \mathcal{G}$. For all $f : ges(Q) \to A$ in $\mathcal{G}$, there is a map $h : Q \to er(A)$ in $\mathcal{E}_\equiv$ such that $f = \epsilon_A \circ ges(h)$ i.e. so the diagram below commutes:*



Moreover, if $h' : Q \to er(A)$ is a map in $\mathcal{E}_\equiv$ such that $f = \epsilon_A \circ ges(h')$, then $h' \equiv h$.

Proof. The component of the counit of the adjunction at A is given by the function ϵ_A taking $\{p\}_\equiv$ to $top_A(p)$; it determines a map $\epsilon_A : ges(er(A)) \to A$ of general event structures.

Let $Q = (Q, \mathrm{Con}_Q, \leq_Q, \equiv_Q)$ be an ese and $f : ges(Q) \to A$ a map in $\mathcal{G}$. We shall define a map $h : Q \to er(A)$ s.t. $f = \epsilon_A \circ ges(h)$. Notice that $\epsilon_A \circ ges(h)(\{q\}_{\equiv_Q}) = top_A(h(q))$, so the requirement that $f = \epsilon_A \circ ges(h)$ amounts to

$$f(\{q\}_{\equiv_Q}) = top_A(h(q))\,, \quad \text{for all } q \in Q.$$

We define the map $h : Q \to er(A)$ by induction on the depth of Q. The depth of an event in an event structure is the length of a longest $\leq$-chain up to it – so an initial event has depth 1. We take the depth of an event structure to be the maximum depth of its events. (Because of our reliance on Lemma 7, the proof of which uses the axiom of choice, we using the axiom of choice implicitly.)

Assume inductively that $h^{(n)}$ defines a map from $Q^{(n)}$ to $er(A)$ where $Q^{(n)}$ is the restriction of Q to depth below or equal to n such that $f^{(n)}$ the restriction of f to $Q^{(n)}$ satisfies $f^{(n)} = \epsilon_A \circ ges(h^{(n)})$. (In particular, $Q^{(0)}$ is the empty ese and $h^{(0)}$ the empty function.) Then, by Proposition 6, any configuration x of $Q^{(n)}$ determines an extremal realisation $\rho_x : h^{(n)}x \to A$ with carrier $(h^{(n)}x, \preceq)$.

Suppose $q \in Q$ has depth $n+1$. If $f(q)$ is undefined take $h^{(n+1)}(q)$ to be undefined. Otherwise, note there is an extremal realisation $\rho_{[q)}$ with carrier $(h[q), \preceq)$. Extend $\rho_{[q)}$ to a realisation $\rho^{\top}_{[q)}$ with carrier that of $\rho_{[q)}$ with a new top element $\top$ adjoined, and make $\rho^{\top}_{[q)}$ extend the function $\rho_{[q)}$ by taking $\top$ to $f(q)$. By Lemma 7, there is an extremal realisation ρ such that $\rho^{\top}_{[q)} \succeq_2 \rho$. Because $\rho_{[q)}$ is extremal $\rho_{[q)} \preceq_1 \rho$, so ρ only extends the order of $\rho_{[q)}$ with extra dependencies of $\top$. (For notational simplicity we identify the carrier of ρ with the set $h[q) \cup \{\top\}$.) Project ρ to the extremal with top $\top$. Define this to be the value of $h^{(n+1)}(q)$. In this way, we extend $h^{(n)}$ to a partial function $h^{(n+1)} : Q^{(n+1)} \to er(A)$ such that $f^{(n+1)} = \epsilon_A \circ ges(h^{(n+1)})$. In showing that $h^{(n+1)}$ is a map we rely on f being a map.

Defining $h = \bigcup_{n \in \omega} h^{(n)}$ we obtain a map $h : Q \to er(A)$ such that $f = \epsilon_A \circ ges(h)$.

Suppose $h' : Q \to er(A)$ is a map sush that $f = \epsilon_A \circ ges(h')$. Then, for any $q \in Q$,

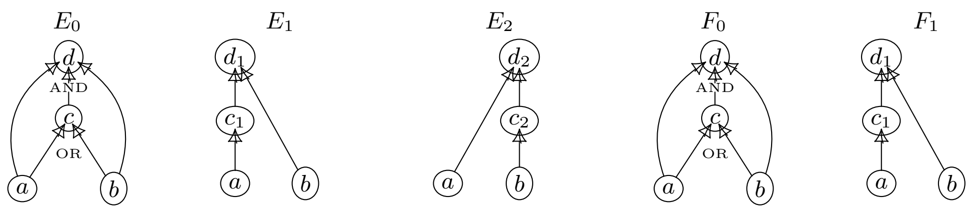
$$top_A(h'(q)) = \epsilon_A \circ ges(h')(\{q\}_{\equiv_Q}) = f(\{q\}_{\equiv_Q}) = \epsilon_A \circ ges(h)(\{q\}_{\equiv_Q}) = top_A(h(q))\,,$$

so $h'(q) \equiv h(q)$ in $er(A)$. Thus $h' \equiv h$. ◀

A configuration $x \in \mathcal{F}$, of a family of configurations $\mathcal{F}$, is *irreducible* iff there is a necessarily unique $e \in x$ such that $\forall y \in \mathcal{F},\ e \in y \subseteq x$ implies $y = x$. Irreducibles coincide with complete join irreducibles w.r.t. the order of inclusion. It is tempting to think of irreducibles as representing minimal complete enablings. But, as sets, irreducibles both (1) lack sufficient structure: in the formulation we are led to of minimal complete enabling as prime extremal realisations, several prime realisations can have the same irreducible as their underlying set; and (2) are not general enough: there are prime realisations whose underlying set is not an irreducible.

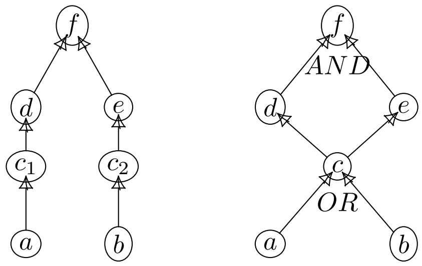
We provide examples illustrating the nature of extremal realisations. In the examples it is convenient to describe families of configurations by general event structures, taking advantage of the economic representation they provide.

▶ **Example 8.** This and the following example shows that prime extremal realisations do not correspond to irreducible configurations. Here, we show a general event structure E_0 with irreducible configuration $\{a, b, c, d\}$ and two prime extremals E_1 and E_2 with tops d_1 and d_2 which both have the same irreducible configuration $\{a, b, c, d\}$ as their image. The lettering indicates the functions associated with the realisations, *e.g.* events d_1 and d_2 in the partial orders map to d in the general event structure.



▶ **Example 9.** On the other hand there are prime extremal realisations of which the image is not an irreducible configuration. We consider the general event structure F_0. The prime extremal F_1 describes a situation where d is enabled by b and c is enabled by a. It has image the configuration $\{a, b, c, d\}$ which is not irreducible, being the union of the two configurations $\{a\}$ and $\{b, c, d\}$.

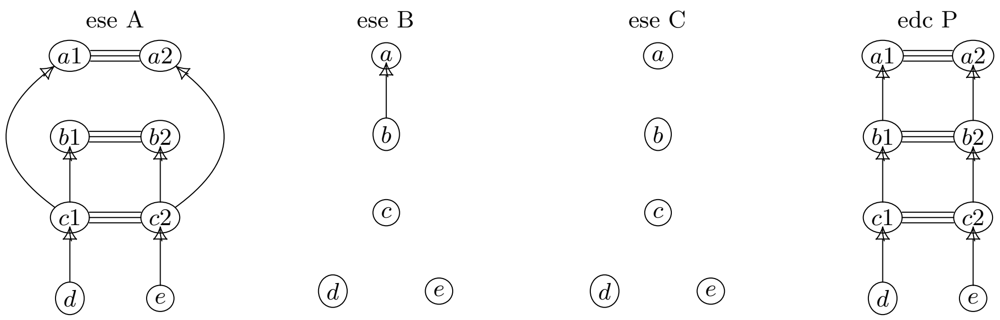
▶ **Example 10.** It is possible to have extremal realisations in which an event depends on an event of the family having been enabled in two distinct ways, as in the following prime extremal realisation, on the left.



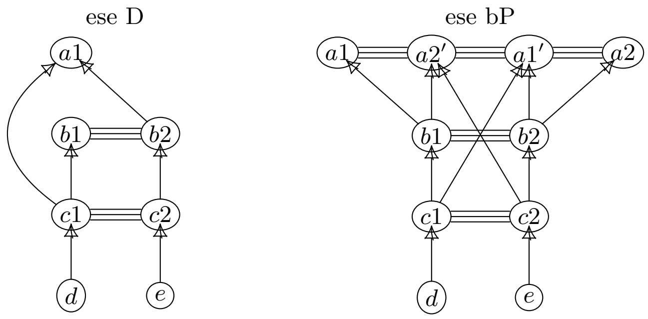
The extremal describes the event f being enabled by d and e where they are in turn enabled by different ways of enabling c. Such phenomena are disallowed in edc's.

C On (pseudo) pullbacks of ese's

We show that the enriched category of ese's $\mathcal{E}_\equiv$ does not always have pullbacks and pseudo pullbacks of maps $f : A \to C$ and $g : B \to C$, the reason why we use the subcategory $\mathcal{EDC}$, which does, as a foundation on which to develop strategies with parallel causes. It suffices to exhibit the lack of pullbacks when C is an (ese of an) event structure as then pullbacks and pseudo pullbacks coincide. Take A, B, C as below, with the obvious maps $f : A \to C$ and $g : B \to C$ (given by the lettering). In fact, A and B are edc's.

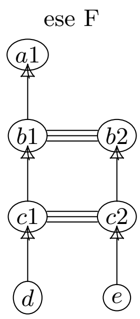


The pullback in edc's $\mathcal{EDC}$ does exist and is given by P with the obvious projection maps. However this is not a pullback in $\mathcal{E}_\equiv$. Consider the ese D with the obvious total maps to A and B; they form a commuting square with f and g. This cannot factor through P: event $b2$ has to be mapped to $b2$ in P, but then $a1$ cannot be mapped to $a1$ (it wouldn't yield a map) nor to $a2$ (it would violate commutation required of a pullback).



There is a bipullback bP got by applying the pseudo functor er to the pullback in ef's. But this is not a pullback because in the ese E the required mediating map is not unique in that $a1$ can go to either $a1$ or $a1'$. In fact, there is no pullback of f and g. To show this we use the additional ese F.

Suppose Q with projection maps to A and B were a pullback of f and g in $\mathcal{E}_\equiv$. Consider the three ese's D, E and F with their obvious maps to A and B; in each case they form a commuting square with f and g. There are three unique maps $h_D : D \to Q$, $h_E : E \to Q$, and $h_F : F \to Q$ such that the corresponding pullback diagrams commute. We remark that there are also obvious maps $k_D : E \to D$ and $k_F : E \to F$ (given by the lettering) which commute with the maps to the components A and B. By uniqueness, we have $h_D \circ k_D = h_E = h_F \circ k_F$, so we have $h_D(a1) = h_F(a1)$. From the definition of the maps, the event $h_D(a1) = h_F(a1)$ has at most one $\leq$-predecessor in Q which is sent to b in C (as D only has one). Because of the projection to B, it has at least one (as B has one). So the event $h_D(a1) = h_F(a1)$ has exactly one predecessor which is sent to b. From the definition of maps, this event is $h_D(b2)$ which equals $h_F(b1)$. But $h_D(b2)$ cannot equal $h_F(b1)$ as they go to two different events of A – a contradiction.



Hence there can be no pullback of f and g in $\mathcal{E}_\equiv$. (By adding intermediary events, we would encounter essentially the same example in the composition, before hiding, of strategies if they were to be developed within the broader category of ese's.)

D The bicategory of probabilistic edc strategies

We obtain a bicategory of probabilistic edc strategies in which objects are race-free games. Maps are probabilistic edc strategies, composition that of strategies and identities are given by copycat strategies, which for race-free games are deterministic, so permit configuration-valuations which are constantly 1.

The 2-cells of the bicategory require consideration. Whereas we can always "push forward" a probability measure from the domain to the codomain of a measurable function this is not true generally for configuration-valuations involving Opponent moves. However:

▶ **Theorem 11.** *Let $f : \sigma \Rightarrow \sigma'$ be a rigid 2-cell between edc strategies $\sigma : S \to A$ and $\sigma' : S' \to A$. Let v be a configuration-valuation on S. Defining, for $y \in \mathcal{C}(S')$,*

$$(fv)(y) =_{\text{def}} \sup_X \sum_{\emptyset \neq Z \subseteq X \ \&\ Z\uparrow} (-1)^{|Z|+1} v(\bigcup Z)$$

as X ranges over finite subsets of $\{x \in \mathcal{C}(S) \mid y = fx\}$, yields a configuration-valuation fv of S' – the push-forward *of v.*

A 2-cell from σ, v to σ', v' is a rigid 2-cell $f : \sigma \Rightarrow \sigma'$ of edc strategies for which the push-forward fv is pointwise less than or equal to v', *i.e.*

$$(fv)(x') \leq v'(x'),$$

for all configurations $x' \in \mathcal{C}(S')$. Vertical composition of 2-cells is their usual composition. Horizontal composition is given by composition $\odot$, which extends to a functor on 2-cells via the universality of pullback and the factorisation property of hiding.