# On Dinaturality, Typability and $\beta\eta$-Stable Models

**Paolo Pistone**

**Dipartimento di Matematica e Fisica, Università Roma Tre, Rome, Italy**
paolo.pistone@uniroma3.it

## Abstract

We present two results which relate dinaturality with a syntactic property (typability) and a semantic one (interpretability by $\beta\eta$-stable sets). First, we prove that closed dinatural $\lambda$-terms are simply typable, that is, the converse of the well-known fact that simply typable closed terms are dinatural. The argument exposes a syntactical aspect of dinaturality, as $\lambda$-terms are type-checked by computing their associated dinaturality equation. Second, we prove that a closed $\lambda$-term belonging to all $\beta\eta$-stable interpretations of a simple type must be dinatural, that is, we prove dinaturality by semantical means. To do this, we show that such terms satisfy a suitable version of binary parametricity and we derive dinaturality from it.

By combining the two results we obtain a new proof of the completeness of the $\beta\eta$-stable semantics with respect to simple types. While the completeness of this semantics is well-known in the literature, the technique here developed suggests that dinaturality might be exploited to prove completeness also for other, less manageable, semantics (like saturated families or reducibility candidates) for which a direct argument is not yet known.

**1998 ACM Subject Classification** F.4.1 Mathematical Logic

**Keywords and phrases** Dinaturality, simply-typed lambda-calculus, $\beta\eta$-stable semantics, completeness

**Digital Object Identifier** 10.4230/LIPIcs.FSCD.2017.29

## 1 Introduction

Dinaturality is a property often considered in investigations on polymorphism and can be described in at least three ways. Firstly, as a categorial property, that is, as the fact that a program of type $\sigma \to \tau$ (or proofs of an implication) corresponds to a *dinatural transformation* between the multivariant[1] functors $F_\sigma, F_\tau$ associated to the types $\sigma$ and $\tau$. Secondly, it can be described as a purely syntactic relation between a $\lambda$-term and its type: since the functorial action $F_\sigma[f, B], F_\sigma[A, f]$ of a type $\sigma$ can be expressed by simply typable $\lambda$-terms $H_\sigma, K_\sigma$, the dinaturality of a term $M$ with respect to $\sigma$ can be expressed by an equation relating $M, H_\sigma$ and $K_\sigma$. Thirdly, it can be presented as a special case of Reynolds' relational parametricity ([10]), namely the case in which only functional relations are considered (see [9]).

It is a well-known fact that simply-typed $\lambda$-terms are dinatural in all three senses just mentioned (see [1, 5, 9]). In this paper we establish two facts about dinaturality which put this notion at the center of a chain of arrows from a semantic notion (interpretability by $\beta\eta$-stable sets) and a syntactic notion (simple typability):

**Dinaturality $\Rightarrow$ Typability.** We prove that the converse of the aforementioned fact that simply-typable closed $\lambda$-terms are dinatural holds: if a closed $\beta\eta$-normal $\lambda$-term is dinatural (in the syntactical sense), then it can be simply typed (Theorem 14). It is

[1] that is, functors whose arguments can be divided into a covariant and a contravariant part, corresponding to variables occurring positively and negatively in the type.


2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017).
Editor: Dale Miller; Article No. 29; pp. 29:1–29:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

shown that the dinaturality condition can be exploited to recursively type-check $\lambda$-terms: a typing derivation for a $\lambda$-term is constructed from the computation of the equation expressing its dinaturality.

**Interpretability $\Rightarrow$ Dinaturality.** The semantics of $\beta\eta$-stable sets of $\lambda$-terms was introduced by Krivine as a semantics of second order functional arithmetic ([7]). We prove that an adapted version of binary parametricity (Definition 21) can be expressed in this semantics and we establish that, for all simple type $\sigma$, a closed $\lambda$-term belonging to all interpretations of $\sigma$ must be parametric in $\sigma$ (Theorem 24). Syntactic dinaturality is then shown to follow from parametricity with respect to a particular relation assignment (Theorem 27).

The parametricity and dinaturality of a $\lambda$-term are usually established by induction over a typing derivation of the term; on the contrary, parametricity (and *a fortiori* dinaturality, Theorem 27) is here established for a class of untyped $\lambda$-terms (those which are interpretable in the $\beta\eta$-stable semantics) by induction over simple types (Theorem 24). Moreover, this last result can be thought of as a first step towards the investigation of parametricity and dinaturality in the context of the reducibility semantics of typed $\lambda$-calculi (like the different saturated families considered in [7] or Girard's reducibility candidates [4]). These semantics are commonly used to prove normalization results but their relationship with parametricity and dinaturality, as well as the completeness issue, are still unclear.

By composing the two arrows above we obtain a new argument for the completeness of the $\beta\eta$-stable semantics: for every simple type $\sigma$, if a closed $\lambda$-term belongs to $|\sigma|_{\mathcal{M}}$ for all interpretation $\mathcal{M}$, then it is $\beta\eta$-equivalent to a term having type $\sigma$. This result had already been established by a direct argument (see [6, 8]) and extended to so-called *positive second order types* in Krivine's system $\mathcal{AF}$ (see [3]). However, the arguments developed in these papers cannot be generalized in a straightforward way to the reducibility semantics mentioned above. It might be interesting to see if a semantic proof of parametricity (and, as a consequence, of completeness) can be reproduced in such frames.

The paper is organized as follows: in Section 2 we introduce a tree characterization of simple typability, which will be exploited in the proof of Theorem 14; in Section 3 we recall syntactic dinaturality and in Section 4 we prove Theorem 14. In Section 5 we recall the $\beta\eta$-stable semantics and we define a variant of parametricity based on $\beta\eta$-stable relations; in Section 6 we prove the parametricity Theorem 24 and in Section 7 we apply it to obtain dinaturality (Theorem 27); finally, in Section 8 we briefly discuss some issues related to completeness and other reducibility semantics.

## 2 Tree representation of simple typability

In this section we present a characterization of typability for closed $\beta\eta$-normal $\lambda$-terms based on a confrontation between the tree of the term $\mathcal{T}(M)$ and the tree of its assigned type $\mathcal{T}(\sigma)$.

In the following, by the letters $M, N, P, \dots$ we will indicate elements of the set $\Lambda$ of untyped $\lambda$-terms, subject to usual $\alpha$-equivalence. By the letters $x, y, z, \dots$ we will indicate *$\lambda$-variables*, i.e. the variables that might occur free or bound in untyped $\lambda$-terms. By the expression $M_1M_2M_3 \dots M_n$ we will indicate the term $(\dots((M_1M_2)M_3)\dots M_n)$.

As usual, the relation $M \to_\beta^* N$ indicates the existence of a sequence of $\beta$-reduction steps from $M$ to $N$ and the relation $M \simeq_\beta N$ (resp. $M \simeq_{\beta\eta} N$) indicates that $M$ and $N$ are $\beta$-equivalent (resp. $\beta\eta$-equivalent).

▶ **Definition 1** (The simply typed $\lambda$-calculus $\lambda_\to$ "à la Curry")**.** Given a countable set $\mathcal{V} = \{Z_1, Z_2, Z_3, \dots, \}$ of symbols, called *type variables* (or, simply, variables when no ambiguity occurs), the set $\mathcal{T}$ of simple types is defined inductively as follows:

$$Z_u \in \mathcal{V} \;\Rightarrow\; Z_u \in \mathcal{T} \tag{1}$$
$$\sigma, \tau \in \mathcal{T} \;\Rightarrow\; \sigma \to \tau \in \mathcal{T} \tag{2}$$

A *type declaration* is an expression of the form $x : \sigma$, where $x$ is a term variable and $\sigma$ is a type. A *context* $\Gamma$ is a finite set of type declarations (where no two distinct declarations $x : \sigma, x : \tau$, with $\sigma \neq \tau$ appear). A *judgement* is an expression of the form $\Gamma \vdash M : \sigma$, where $\Gamma$ is a context, $M$ a term and $\sigma$ a type.

The typing derivations of $\lambda_\to$ are generated by the following rules:

$$\boxed{\begin{array}{c} \dfrac{}{\Gamma \cup \{x : \sigma\} \vdash x : \sigma}\ (id) \\[2ex] \dfrac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\ (\to E) \qquad \dfrac{\Gamma \cup \{x : \sigma\} \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}\ (\to I) \end{array}} \tag{3}$$

The *arity* $ar(M)$ of a $\lambda$-term is the positive integer $n$ such that $M$ can be written as $\lambda x_1. \dots .\lambda x_n.M'$ with $M'$ either a variable or an application. The *arity* $ar(\sigma)$ of a simple type $\sigma \in \mathcal{T}_0$ is the positive integer $n$ such that $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$. We indicate by $FV(M)$ (resp. $BV(M)$) the set of free (resp. bound) variables of $M$.

▶ **Definition 2.** A *path* $\pi = p_1 \dots p_k$ is a finite sequence of non zero positive integers. We denote $\epsilon$ the empty path and $\mathbb{N}^*$ the set of all paths. If $\pi$ is the path $p_1 \dots p_k$, then for every $p \in \mathbb{N}$, $\pi * p$ denotes the path $p_1 \dots p_k p$ and $p * \pi$ the path $pp_1 \dots p_n$. The *length* of a path $\ell(\pi)$ is the length of the sequence $\pi$. A partial order $\leq$ over paths is defined by letting $\pi \leq \pi'$ if $\pi = p_1 \dots p_k$ and $\pi' = p_1 \dots p_k p_{k+1} \dots p_{k'}$ for some $k, k' \geq 0$.

A *tree* is a set $T \subseteq \mathbb{N}^*$ containing $\epsilon$ and such that, if $\pi \in T$ and $\pi' \leq \pi$, then $\pi' \in T$. If $T_1, \dots, T_n$ are trees, for some $n \geq 1$, then $(T_1, \dots, T_n)$ is the tree consisting of the empty sequence and all sequences of the form $i * \pi$, for $\pi \in T_i$ and $i \leq n$.

For $\sigma$ and $M$, respectively, a simple type and a closed $\beta\eta$-normal $\lambda$-term, we define the trees $\mathcal{T}(\sigma)$ and $\mathcal{T}(M)$:

▶ **Definition 3** (Tree of a type)**.** Let $\sigma$ be a simple type. We associate with $\sigma$ a tree $\mathcal{T}(\sigma)$ defined by induction as follows:
- if $\sigma = Z_u$, then $\mathcal{T}(\sigma) = \{\epsilon\}$;
- if $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$, then $\mathcal{T}(\sigma) = \big(\mathcal{T}(\sigma_1), \dots, \mathcal{T}(\sigma_n), \mathcal{T}(Z_u)\big)$

Conversely, with every $\pi \in \mathcal{T}(\sigma)$ we can associate a unique subtype $\sigma_\pi$ of $\sigma$:
- $\sigma_\epsilon := \sigma$;
- $\sigma_{\pi * i} := \tau_i$, for $1 \leq i \leq ar(\sigma_\pi) + 1$, where $\sigma_\pi = \tau_1 \to \dots \to \tau_{ar(\sigma_\pi)} \to Z_u$ where we put $\tau_{ar(\sigma_\pi)+1} = Z_u$.

▶ **Definition 4** (Tree of a closed $\beta\eta$-normal $\lambda$-term)**.** Let $M$ be a closed $\beta\eta$-normal $\lambda$-term. We associate with $M$ a tree $\mathcal{T}(M)$ defined by induction on the number of applications of $M$ as follows:
- if $M = \lambda x_1. \dots .\lambda x_h.x_i$, then $\mathcal{T}(M) = \epsilon$;
- if $M = \lambda x_1. \dots .\lambda x_h.x_i M_1 \dots M_p$, for some $h \geq 0$, $1 \leq i \leq h$ and $p \geq 1$, then $\mathcal{T}(M) = \big(\mathcal{T}(\overline{M_1}), \dots, \mathcal{T}(\overline{M_p})\big)$, where $\overline{M_j} = \lambda x_1. \dots .\lambda x_h.M_j$, for $1 \leq j \leq p$.

Conversely, with every $\pi \in \mathcal{T}(M)$ we can associate a unique subterm $M_\pi$
- $M_\epsilon := M$;
- $M_{\pi * j} := N_j$, for $1 \leq j \leq q$, where $M_\pi = \lambda x_1. \dots .\lambda x_h.yN_1 \dots N_q$.

For $\pi \in \mathcal{T}(M)$, we let $p_M(\pi)$ be the minimum positive integer $p \geq 0$ such that $\pi * (p+1) \notin \mathcal{T}(M)$.

Let $Var(M) \subset \mathcal{T}(M) \times \mathbb{N}$ indicate the set of all pairs $(\pi, i)$ such that $1 \leq i \leq ar(M_\pi)$. Any pair $(\pi, i) \in Var(M)$ uniquely determines a variable appearing in $M$: it is the $i$-th variable abstracted in $M_\pi$.

Given $M$ a closed $\beta\eta$-normal $\lambda$-term and $\sigma$ a simple type, we define two partial maps:

1. a partial map $\mathtt{var} : Var(M) \to \mathcal{T}(\sigma)$ associating variables of $M$ with subtypes of $\sigma$;
2. a partial map $\mathtt{subt} : \mathcal{T}(M) \to \mathcal{T}(\sigma)$ associating subterms $M_\pi$ with subtypes of $\sigma$.

The maps $\mathtt{var}$ and $\mathtt{subt}$ approximate the maps (which are always defined for typed $\lambda$-terms) associating each variable and each subterm of $M$ with its associated subtype of $\sigma$.

Let $h(\pi)$ indicate the pair $(\pi', i)$ associated with the head variable of $M_\pi$. Remark that $\ell(h(\pi)_1) \leq \ell(\pi)$, where $h(\pi)_1$ indicates the first component of the pair $h(\pi)$. The maps $\mathtt{var}$ and $\mathtt{subt}$ are defined as follows:

$$\begin{aligned} \mathtt{var}(\epsilon, i) &= i \\ \mathtt{var}(\pi * j, i) &= \mathtt{var}(h(\pi)) * j * i \end{aligned} \qquad\qquad \begin{aligned} \mathtt{subt}(\epsilon) &= \epsilon \\ \mathtt{subt}(\pi * j) &= \mathtt{var}(h(\pi)) * j \end{aligned} \tag{4}$$

where it is intended that $\mathtt{var}(\epsilon, i)$ is defined only if $i \leq ar(\sigma)$ and $\mathtt{var}(\pi * j, i)$ is defined only if $\mathtt{var}(h(\pi))$ is defined and if $j \leq ar(\sigma_{\mathtt{var}(h(\pi))})$ and $i \leq ar(\sigma_{\mathtt{var}(h(\pi))*j})$. The definition of $\mathtt{var}(\pi * j, i)$ is a correct recursion since $\ell(h(\pi)_1) < \ell(\pi * j)$. Moreover, $\mathtt{subt}(\pi * j)$ is defined only if $\mathtt{var}(h(\pi))$ is defined and $j \leq ar(\sigma_{\mathtt{var}(h(\pi))})$.

The following theorem shows that typability for a closed $\beta\eta$-normal $\lambda$-term $M$ and a simple type $\sigma$ can be expressed as a relation between $\mathcal{T}(M)$ and $\mathcal{T}(\sigma)$:

▶ **Theorem 5.** *Let $M$ be a $\beta\eta$-normal term, $\sigma_1, \dots, \sigma_n, \tau$ simple types and $\Gamma$ the context $\{x_1 : \sigma_1, \dots, x_n : \sigma_n\}$. Then $\Gamma \vdash M : \tau$ is derivable if and only if the following hold (where $N = \lambda x_1. \dots .\lambda x_n.M$, $\sigma = \sigma_1 \to \dots \to \sigma_n \to \tau$):*

1. *$\mathtt{var}, \mathtt{subt}$ are defined for all $\pi \in \mathcal{T}(N)$ and $i \geq 1$ such that $(\pi, i) \in Var(N)$;*
2. *$ar(\sigma_{\mathtt{var}(h(\pi))}) = p_N(\pi) + ar(\sigma_{\mathtt{subt}(\pi)}) - ar(N_\pi)$ for all $\pi \in \mathcal{T}(N)$;*
3. *$\sigma_{\mathtt{var}(h(\pi))*(p_N(\pi)+l)} = \sigma_{\mathtt{subt}(\pi)*(ar(N_\pi)+l)}$, for all $1 \leq l \leq ar(\sigma_{\mathtt{subt}(\pi)}) - ar(N_\pi) + 1$.*

**Proof.** Proof in Appendix A. ◀

From the characterization above we introduce the following notion of *p-fitness*:

▶ **Definition 6.** Let $M$ be a closed $\beta\eta$-normal $\lambda$-term and $\sigma$ a simple type. For $p = 1, 2, 3$ and $\pi \in \mathcal{T}(M)$, we say that $M$ is *p-fit* to $\sigma$ at $\pi$ if $M$ satisfies condition $p$. of Theorem 5 restricted to $\pi$. $M$ is *p-fit* if, for all $\pi \in \mathcal{T}(M)$, $M$ is $p$-fit at $\pi$.

## 3 Syntactic dinaturality

We provide a description of the dinaturality condition for simple types and pure $\lambda$-terms. A similar description can be found in [2]. Let $\mathcal{V}_0, \mathcal{V}_1$ be a partition of $\mathcal{V}$ into two disjoint countable sets of type variables $\mathcal{V}_0 = \{X_0, X_1, X_2, \dots\}$ and $\mathcal{V}_1 = \{Y_0, Y_1, Y_2, \dots\}$. For any type $\sigma$, whose free variables are $Z_{u_1}, \dots, Z_{u_k} \in \mathcal{V}$, let $\sigma^X_Y$ (resp. $\sigma^Y_X$) denote the result of replacing, in $\sigma$, all positive occurrences of $Z_{u_l}$, for $1 \leq l \leq k$, by the variable $X_{u_l}$ (resp. $Y_{u_l}$), and all negative occurrences of $Z_{u_l}$ by the variable $Y_{u_l}$ (resp. $X_{u_l}$). Let then $\sigma_X, \sigma_Y$ denote, respectively, $\sigma^X_X$ and $\sigma^Y_Y$.

Let, for each $u \geq 0$, $f_u$ be a fresh variable (to which we will assign type $X_u \to Y_u$). For each type $\sigma$, we define well-typed terms $H_\sigma, K_\sigma$ coding the functorial action of $\sigma$: if, for any type $\sigma$, we indicate by $\mathbf{F}_\sigma$ its associated functor, then $H_\sigma$ codes both morphisms $\mathbf{F}_\sigma[X, f]$ and $\mathbf{F}_\sigma[Y, f]$, as it can be assigned both types $\sigma_X \to \sigma^Y_X$ and $\sigma^X_Y \to \sigma_Y$; similarly, $K_\sigma$ codes both morphisms $\mathbf{F}_\sigma[f, X]$ and $\mathbf{F}_\sigma[f, Y]$, as it can be assigned both types $\sigma^X_Y \to \sigma_X$ and $\sigma_Y \to \sigma^Y_X$.

The dinaturality condition will be given by the equation below, in which $g$ is a fresh variable (to be declared of type $\sigma^X_Y$)

$$H_\tau(M(K_\sigma g)) \simeq_{\beta\eta} K_\tau(M(H_\sigma g)) \tag{5}$$

Equation 5 can be illustrated by the usual hexagonal diagram:

$$\begin{tikzcd}
& \sigma_X \arrow[r, "M_X"] & \tau_X \arrow[dr, "H_\tau"] & \\
\sigma^X_Y \arrow[ur, "K_\sigma"] \arrow[dr, "H_\sigma"'] & & & \tau^Y_X \\
& \sigma_Y \arrow[r, "M_Y"'] & \tau_Y \arrow[ur, "K_\tau"'] &
\end{tikzcd} \tag{6}$$

We define simultaneously the terms $H_\sigma, K_\sigma$ by induction over $\sigma$:

- if $\sigma = Z_u$, then $H_\sigma := f_u$ and $K_\sigma := \lambda x.x$;
- if $\sigma = \sigma_1 \to \cdots \to \sigma_k \to Z_u$, then

$$H_\sigma := \lambda g.\lambda h_1.\dots.\lambda h_k.f_u\big(g(K_{\sigma_1}h_1)(K_{\sigma_2}h_2)\dots(K_{\sigma_k}h_k)\big) \tag{7}$$

and

$$K_\sigma := \lambda g.\lambda h_1.\dots.\lambda h_k.g(H_{\sigma_1}h_1)(H_{\sigma_2}h_2)\dots(H_{\sigma_k}h_k) \tag{8}$$

By a simple computation, equation 5 can be rewritten as follows:

$$f_u\big(M(K_{\sigma_1}g_1)\dots(K_{\sigma_n}g_n)\big) \ \simeq_{\beta\eta} \ M(H_{\sigma_1}g_1)\dots(H_{\sigma_n}g_n) \tag{9}$$

where $\sigma = \sigma_1 \to \cdots \to \sigma_n \to Z_u$ and $g_1, \dots, g_n$ are fresh variables.

▶ **Proposition 7.** *For all simple type $\sigma$, let $\Gamma$ be the set of all declarations $f_u : X_u \to Y_u$ such that the variable $Z_u$ occurs in $\sigma$. Then $\Gamma \vdash H_\sigma : \sigma_X \to \sigma^Y_X$, $\Gamma \vdash H_\sigma : \sigma^X_Y \to \sigma_Y$, $\Gamma \vdash K_\sigma : \sigma^X_Y \to \sigma_X$ and $\Gamma \vdash K_\sigma : \sigma_Y \to \sigma^Y_X$ are derivable.*

We define the set $DIN_\sigma$ of $\sigma$-dinatural terms:

▶ **Definition 8** (dinatural term)**.** Let $\sigma = \sigma_1 \to \cdots \to \sigma_n \to Z_u$ be a simple type and $M$ be a closed $\lambda$-term. Then $M \in DIN_\sigma$ if

$$f_u\big(M(K_{\sigma_1}g_1)\dots(K_{\sigma_n}g_n)\big) \ \simeq_{\beta\eta} \ M(H_{\sigma_1}g_1)\dots(H_{\sigma_n}g_n) \tag{10}$$

holds.

▶ **Example 9.** Let $\sigma$ be the type $(Z_u \to Z_u) \to (Z_u \to Z_u)$. The dinaturality condition associated with a closed $\lambda$-term $M$ and $\sigma$ is the equation

$$f_u\big(M \ \lambda h.g_1(f_u h) \ g_2\big) \ \simeq_{\beta\eta} \ M \ \lambda h.f_u(g_1 h) \ f_u g_2 \tag{11}$$

which corresponds to the diagram below (where $\tau$ is $Z_u \to Z_u$):

$$\begin{tikzcd}
 & X_u \to X_u \arrow[r, "M_X"] & X_u \to X_u \arrow[dr, "H_\tau"] & \\
Y_u \to X_u \arrow[ur, "K_\tau"] \arrow[dr, "H_\tau"'] & & & X_u \to Y_u \\
 & Y_u \to Y_u \arrow[r, "M_Y"'] & Y_u \to Y_u \arrow[ur, "K_\tau"'] &
\end{tikzcd} \tag{12}$$

If $M$ is a closed term of type $\sigma$, the $\beta\eta$-normal form of $M$ is of the form $\lambda y.\lambda x.y^n x$, for some $n \geq 0$. Then equation 11 reduces to the true equation

$$f_u\big(\underbrace{g_1(f_u \dots g_1(f_u}_{n \text{ times}} g_2) \dots)\big) \simeq_{\beta\eta} \underbrace{f_u(g_1 \dots f_u(g_1}_{n \text{ times}}(f_u g_2)) \dots) \tag{13}$$

showing that $M \in DIN_\sigma$.

The theorem below generalizes this example to an arbitrary closed simply-typed $\lambda$-term.

▶ **Theorem 10.** *Let $\sigma$ be a simple type and $M$ be a closed $\lambda$-term. If $\vdash M : \sigma$, then $M \in DIN_\sigma$.*

**Proof.** Several proofs exist in the literature. [5] prove the fact for the categorial notion of dinaturality, and derive the result for syntactic dinaturality by considering a syntactic category. [2] proves this directly for the syntactic notion of dinaturality. [9] proves dinaturality as a consequence of parametricity. ◀

In the next section we will prove the converse of Theorem 10, showing that dinaturality characterizes closed simply-typable $\lambda$-terms.

## 4 Relating dinaturality and typabililty

The result of this section is that, if $M \in DIN_\sigma$ is closed and $\beta\eta$-normal, then $\vdash M : \sigma$ is derivable in $\lambda_\to$. We will rely on the characterization of typability given by Theorem 5 and on two lemmas relating the dinaturality condition with the fitness conditions (Definition 6). In particular, it will be shown, first, that the dinaturality equation implies all fitness conditions relative to the empty path and then that, given fitness at path $\pi$, by reducing the dinaturality equation for $M_\pi$, a dinaturality equation for the subterms $M_{\pi * p}$ is obtained (whence fitness at paths $\pi * p$). Hence, simple typability for a closed $\lambda$-term is checked recursively by reducing the associated dinaturality equation.

The first lemma relates dinaturality with $1, 2$-fitness:

▶ **Lemma 11.** *Let $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$ be a simple type and $M$ a closed $\beta\eta$-normal $\lambda$-term satisfying:*

$$f_u\big(M(K_{\sigma_1} g_1) \dots (K_{\sigma_n} g_n)\big) \simeq_{\beta\eta} M(H_{\sigma_1} g_1) \dots (H_{\sigma_n} g_n) \tag{14}$$

*for some $n \geq 1$ and $\lambda$-variables $g_1, \dots, g_n$. Then*

$$M = \lambda x_1. \dots .\lambda x_h.(x_i) Q_1 \dots Q_p \tag{15}$$

*for certain terms $Q_1, \dots, Q_p$, where $h \leq n$, $1 \leq i \leq h$, and $ar(\sigma_i) = p + (n - h)$.*

**Proof.** Let $k_i = ar(\sigma_i)$ and, for $1 \leq j \leq n$, $P_j = K_{\sigma_j} g_j$ and $P'_j = H_{\sigma_j} g_j$. For a $\lambda$-term $U$, we let $U^*$ (resp. $U^{**}$) indicate $U[P_1/x_1, \ldots, P_n/x_n]$ (resp. $U[P'_1/x_1, \ldots, P'_n/x_n]$). We will show that $h \leq n$. We consider two cases:

- if $h > n$ and $i > n$, then we can write $M = \lambda x_1. \ldots .\lambda x_n.\lambda y_1. \ldots .\lambda y_m.y_j M_1 \ldots M_p$ where $m \geq 1$, and $1 \leq j \leq m$; then

$$f_u\big(MP_1 \ldots P_n\big) \ \to^*_\beta \ f_u\big(\lambda y_1. \ldots .\lambda y_m.y_j M^*_1 \ldots M^*_p\big) \tag{16}$$

cannot be $\beta\eta$-equivalent to

$$MP'_1 \ldots P'_n \ \to^*_\beta \ \lambda y_1. \ldots .\lambda y_m.y_j M^{**}_1 \ldots M^{**}_p \tag{17}$$

- if $n \leq h$ and $i \leq n$, then we can write $M = \lambda x_1. \ldots .\lambda x_n.\lambda y_1. \ldots .\lambda y_m.x_i M_1 \ldots M_p$ where $m \geq 0$, and $1 \leq i \leq n$; then we claim that

$$f_u\big(MP_1 \ldots P_n\big) \ \to^*_\beta \ f_u\big(\lambda y_1. \ldots .\lambda y_m.K_{\sigma_i} g_i M^*_1 \ldots M^*_p\big) \tag{18}$$

can be $\beta\eta$-equivalent to

$$MP'_1 \ldots P'_n \ \to^*_\beta \ \lambda y_1. \ldots .\lambda y_m.H_{\sigma_i} g_i M^{**}_1 \ldots M^{**}_p \tag{19}$$

only if $p = k_i$ and $m = 0$ (that is, $h = n$). We divide the argument in three parts:

1. suppose $p < k_i$; then

$$MP'_1 \ldots P'_n \ \to^*_\beta \ \lambda y_1. \ldots .\lambda y_m.\lambda z_{k_i - p}. \ldots .\lambda z_{k_i}.f_u\big(g_i Q_1 \ldots Q_p \ldots Q_{k_i}\big) \tag{20}$$

for some terms $Q_1, \ldots, Q_{k_i}$, so it cannot be $\beta\eta$-equivalent to $f_u(MP_1 \ldots P_n)$.

2. suppose now $p > k_i$; then

$$MP'_1 \ldots P'_n \ \to^*_\beta \ \lambda y_1. \ldots .\lambda y_m.f_u\big(g_i Q_1 \ldots Q_{k_i}\big) M^{**}_{k_i+1} \ldots M^{**}_p \tag{21}$$

so it is $\beta\eta$-equivalent to $f_u(MP_1 \ldots P_n)$ only if $m = p - k_i$ and, for $1 \leq j \leq m$, $M^{**}_{k_i+j} \simeq_{\beta\eta} y_j$ and $y_j$ is not free in $Q_1, \ldots, Q_p$. We claim that $M^{**}_{k_i+j} \simeq_{\beta\eta} y_j$ only if $M_{k_i+j} \simeq_{\beta\eta} y_j$. It will follow that, if $m = p - k_i > 0$, then $M$ is not in $\beta\eta$-normal form, against the hypothesis. To show this we will use an induction on the number of applications in $M_{k_i+j}$. If $M^{**}_{k_i+j} \simeq_{\beta\eta} y_j$, then $M_{k_i+j}$ is either of the form $\lambda u_1. \ldots .\lambda u_q.x_v N_1 \ldots N_r$, for some $q, r \geq 0$ and $1 \leq v \leq n$, either of the form $\lambda u_1. \ldots .\lambda u_q.y_v N_1 \ldots N_r$, for some $q, r \geq 0$ and $1 \leq v \leq m$ or of the form $\lambda u_1. \ldots .\lambda u_q.u_l N_1 \ldots N_r$, for some $q, r \geq 0$ and $1 \leq l \leq q$; in the first case $M^{**}_{k_i+j}$ reduces to a term of the form $\lambda u_1. \ldots .\lambda u_q.\lambda h_1. \ldots .\lambda h_{q'}.f_u N'_1 \ldots N'_{r'}$, for some $q', r' \geq 0$, so it cannot be $\beta\eta$-equivalent to $y_j$; in the second case $M^{**}_{k_i+j} \simeq_{\beta\eta} \lambda u_1. \ldots .\lambda u_q.y_v N^{**}_1 \ldots N^{**}_r$, so it reduces to $y_v$ only if either $q = r = 0$ and $v = j$, or $q = r$, $v = j$ and $N^{**}_k \simeq_{\beta\eta} u_k$, for $1 \leq k \leq q$; in this latter case we can apply the induction hypothesis on the terms $N_k$: $N^{**}_k \simeq_{\beta\eta} u_k$ only if $N_k \simeq_{\beta\eta} u_k$. In both cases we conclude that $M^{**}_{k_i+j} \simeq_{\beta\eta} y_j$ only if $M_{k_i+j} \simeq_{\beta\eta} y_j$; in the third case $M^{**}_{k_i+j} \simeq_{\beta\eta} \lambda u_1. \ldots .\lambda u_q.u_l N^{**}_1 \ldots N^{**}_r$ cannot be $\beta\eta$-equivalent to $y_l$.

3. Finally, suppose $m \geq 1$; then $MP'_1 \ldots P'_n$ reduces to $\lambda y_1. \ldots .\lambda y_m.f_u\big(g_i Q_1 \ldots Q_p\big)$, so it cannot be $\beta\eta$-equivalent to $f_u(MP_1 \ldots P_n)$.

We have so far shown that $h \leq n$ and moreover that, if $h = n$, then $p = k_i$. Let now $M = \lambda x_1. \ldots .\lambda x_h.x_i M_1 \ldots M_p$, with $h < n$ and $i \leq h$. We prove that $k_i = p + (n - h)$: if $p + (n - h) < k_i$, then $MP'_1 \ldots P'_n$ reduces to $\lambda z_{k_i - d}. \ldots .\lambda z_{k_i}.f_u\big(g_i Q_1 \ldots Q_p \ldots Q_{k_i}\big)$, where $d = p + (n - h)$, for some terms $Q_1, \ldots, Q_{k_i}$, so it cannot be $\beta\eta$-equivalent to $f_u(MP_1 \ldots P_n)$. If $p+(n-h) > k_i$, then $MP'_1 \ldots P'_n$ reduces to $f_u(g_i Q_1 \ldots Q_{k_i}) M^{**}_{p-e} \ldots M^{**}_p$, where $e = k_i - (n - h)$, so it cannot be $\beta\eta$-equivalent to $f_u(MP_1 \ldots P_n)$. ◀

Lemma 11 says that a closed $\beta\eta$-normal dinatural $\lambda$-term is 1, 2-fit at $\epsilon$. Indeed, from $h \leq ar(\sigma)$ it follows that $\mathtt{var}(\epsilon, i)$ is defined for all $(\epsilon, i) \in Var(M)$ (and $\mathtt{subt}(\epsilon)$ is always defined); moreover, we have $ar(\sigma_{\mathtt{var}(h(\epsilon))}) = ar(\sigma_i) = p+(n-h) = p_M(\epsilon)+ar(\sigma_{\mathtt{subt}(\epsilon)})-ar(N_\epsilon)$. The next lemma relates dinaturality and 3-fitness:

▶ **Lemma 12.** *For all simple types $\sigma, \tau$ the equation below*

$$H_\sigma(K_\tau g) \simeq_{\beta\eta} K_\sigma(H_\tau g) \tag{22}$$

*holds if and only if $\sigma = \tau$.*

**Proof.** We argue by induction on $\sigma$: if $\sigma = Z_u$ and $\tau = \tau_1 \to \dots \to \tau_k \to Z_v$, for some $k \geq 0$, then $H_\sigma(K_\tau g) \simeq_{\beta\eta} K_\sigma(H_\tau g)$ becomes

$$\begin{aligned} &H_{Z_u}(K_\tau g) \simeq_\beta f_u(K_\tau g) \simeq_\beta f_u\big(\lambda h_1.\dots.\lambda h_k.g(H_{\tau_1}h_1)\dots(H_{\tau_k}h_k)\big) \simeq_{\beta\eta} \\ &\qquad \lambda h_1.\dots.\lambda h_k.f_v\big(g(K_{\tau_1}k_1)\dots(K_{\tau_k}h_k)\big) \simeq_\beta H_\tau g \simeq_\beta K_{Z_v}(H_\tau g) \end{aligned} \tag{23}$$

and the central equation holds if and only if $k = 0$ and $u = v$, i.e. if and only if $\sigma = \tau = Z_u$.

For the induction step, let $\sigma = \sigma_1 \to \dots \to \sigma_k \to Z_u$ and $\tau = \tau_1 \to \dots \to \tau_{k'} \to Z_v$. We can suppose w.l.o.g. $k' = k + d$; now $H_\sigma(K_\tau g)$ reduces to

$$\lambda h_1.\dots.\lambda h_k.f_u\Big(\lambda h_{k+1}.\dots.\lambda h_{k'}.g(K_{\sigma_1}(H_{\tau_1}h_1))\dots(K_{\sigma_k}(H_{\tau_k}h_k))(H_{\tau_{k+1}}h_{k+1})\dots(H_{\tau_{k'}}h_{k'})\Big) \tag{24}$$

and $K_\sigma(H_\tau g)$ reduces to

$$\lambda h_1.\dots.\lambda h_k.\lambda h_{k+1}.\dots.\lambda h_{k'}.f_v\Big(g(H_{\sigma_1}(K_{\tau_1}h_1))\dots(H_{\sigma_k}(K_{\tau_k}h_k))(K_{\tau_{k+1}}h_{k+1})\dots(K_{\tau_{k'}}h_{k'})\Big) \tag{25}$$

and the two terms are $\beta\eta$-equivalent if and only if $d = 0$, $u = v$ and, for all $1 \leq i \leq k$, $\sigma_i = \tau_i$ (by induction hypothesis), i.e if and only if $\sigma = \tau$. ◀

Lemma 12 says that a closed $\beta\eta$-normal dinatural $\lambda$-term is 3-fit at $\epsilon$: if $M \in DIN_\sigma$ and $M = \lambda x_1.\dots.\lambda x_h.x_i M_1 \dots M_p$, then, by Lemma 11, $h \leq n$ and $k_i = p + (n-h)$; so, equation 9 reduces to

$$\begin{aligned} &f_u\Big(g_i(H_{\sigma_{i1}}M_1^*)\dots(H_{\sigma_{ip}}M_p^*)(H_{\sigma_{i(p+1)}}(K_{\sigma_{h+1}}g_{h+1}))\dots(H_{\sigma_{i(ar(\sigma_i))}}(K_{\sigma_n}g_n))\Big) \\ \simeq_{\beta\eta}\ &f_u\Big(g_i(K_{\sigma_{i1}}M_1^{**})\dots(K_{\sigma_{ip}}M_p^{**})(K_{\sigma_{i(p+1)}}(H_{\sigma_{h+1}}g_{h+1}))\dots(K_{\sigma_{i(ar(\sigma_i))}}(H_{\sigma_n}g_n))\Big) \end{aligned} \tag{26}$$

where $M_j^*, M_j^{**}$ are as in the proof of Lemma 11. Now, by Lemma 12, we get that, for $1 \leq j \leq n-h$, $\sigma_{h+j} = \sigma_{i(p+j)}$, i.e. $\sigma_{\mathtt{subt}(\epsilon)*(ar(M)+j)} = \sigma_{\mathtt{var}(h(\epsilon))*(p_M(\epsilon)+j)}$.

We will now show that, for $M \in DIN_\sigma$ closed and $\beta\eta$-normal, $p$-fitness can be extended to all paths $\pi \in \mathcal{T}(M)$: by computing the dinaturality equation, and exploiting lemmas 11 and 12, we obtain a dinaturality equation for all subterms $M_\pi$.

▶ **Proposition 13.** *Let $\sigma$ be a simple type and $M$ be a closed $\beta\eta$-normal $\lambda$-term in $DIN_\sigma$. Then, for every $\pi \in \mathcal{T}(M)$, $M_\pi$ is $p$-fit for $\sigma_{\mathtt{subt}(\pi)}$, for $p = 1, 2, 3$.*

**Proof.** We argue by induction on the number of applications in $M$. If $M = \lambda x_1.\dots.\lambda x_h.x_i$, then $\mathcal{T}(M) = \{\epsilon\}$ so the claim holds since $M_\epsilon = M$ is $p$-fit for $p = 1, 2, 3$ by lemmas 11 and 12. Let then $M = \lambda x_1.\dots.\lambda x_h.x_i M_1 \dots M_p$ for some $p \geq 1$ and let $N_j = \lambda x_1.\dots.\lambda x_n.M_j$ and

$\tau_j = \sigma_1 \to \cdots \to \sigma_n \to \sigma_{ij}$. Again, by Lemma 11 and 12, $M_\epsilon$ is $p$-fit to $\sigma_\epsilon$, for $p = 1, 2, 3$. Let then $\pi = j * \pi' \in \mathcal{T}(M)$, with $\pi' \in \mathcal{T}(N_j)$ and $1 \leq j \leq p$. Since, by Lemma 11, $h \leq n$ and $k_i = p + (n - h)$ (where $k_i$ is $ar(\sigma_i)$), equation 9 reduces to

$$f_u\big(K_{\sigma_i} g_i M_1^* \dots M_p^* (K_{\sigma_{p+1}} g_{h+1}) \dots (K_{\sigma_n} g_n)\big) \ \simeq_{\beta\eta} \ H_{\sigma_i} g_i M_1^{**} \dots M_p^{**} (H_{\sigma_{p+1}} g_{h+1}) \dots (H_{\sigma_n} g_n) \tag{27}$$

where $M_j^*, M_j^{**}$ are as in the proof of Lemma 11; this in turn reduces to

$$\begin{aligned} & f_u\Big(g_i\big(H_{\sigma_{i1}} M_1^*\big) \dots \big(H_{\sigma_{ip}} M_p^*\big)\big(H_{\sigma_{i(p+1)}}(K_{\sigma_{p+1}} g_{p+1})\big) \dots \big(H_{\sigma_{i(ar(\sigma_i))}}(K_{\sigma_n} g_n)\big)\Big) \\ \simeq_{\beta\eta} \ & f_u\Big(g_i\big(K_{\sigma_{i1}} M_1^{**}\big) \dots \big(K_{\sigma_{ip}} M_p^{**}\big)\big(K_{\sigma_{i(p+1)}}(H_{\sigma_{p+1}} g_{p+1})\big) \dots \big(K_{\sigma_{i(ar(\sigma_i))}}(H_{\sigma_n} g_n)\big)\Big) \end{aligned} \tag{28}$$

By Lemma 12, $\sigma_{i(p+j)} = \sigma_{p+j}$, for all $1 \leq j \leq n - h$. From equation 28 we obtain then $H_{\sigma_{ij}} M_j^* \simeq_{\beta\eta} K_{\sigma_{ij}} M_j^{**}$. Now we have that

$$\begin{aligned} & f_v\Big(N_j(K_{\sigma_1} g_1) \dots (K_{\sigma_h} g_h)(K_{\sigma_{ij1}} h_1) \dots (K_{\sigma_{ijd}} h_d)\Big) \ \simeq_\beta \ H_{\sigma_{ij}} M_j^* \\ \simeq_{\beta\eta} \ & K_{\sigma_{ij}} M_j^{**} \ \simeq_\beta \ N_j(H_{\sigma_1} g_1) \dots (H_{\sigma_h} g_h)(H_{\sigma_{ij1}} h_1) \dots (H_{\sigma_{ijd}} h_d) \end{aligned} \tag{29}$$

where $d = ar(\sigma_{ij})$, which implies that $N_j \in DIN_{\tau_j}$. We can then apply the induction hypothesis to $N_j$: by letting $h_j : \mathcal{T}(N_j) \to Var(N_j), \texttt{var}_j : Var(N_j) \to \mathcal{T}(\tau_j), \texttt{subt}_j : \mathcal{T}(N_j) \to \mathcal{T}(\tau_j)$ indicate the $h, \texttt{var}, \texttt{subt}$ functions, respectively, defined for $N_j$, we have that, for any $\pi' \in \mathcal{T}(N_j)$, $(N_j)_{\pi'}$ is $p$-fit to $(\tau_j)_{\texttt{subt}_j(\pi')}$, for $p = 1, 2, 3$.

In order to prove that $M_\pi = (N_j)_{\pi'}$ is $p$-fit to $\sigma_{\texttt{subt}(\pi)}$, it is enough to verify that, for all $\lambda \in \mathcal{T}(N_j)$, the following equalities hold:

$$(\tau_j)_{\texttt{var}_j(h_j(\lambda))} = \sigma_{\texttt{var}(h(j*\lambda))} \qquad\qquad (\tau_j)_{\texttt{subt}_j(\lambda)} = \sigma_{\texttt{subt}(j*\lambda)} \tag{30}$$

We postpone this technical verification to Appendix A. ◀

▶ **Theorem 14.** *Let $\sigma$ be a simple type and $M \in DIN_\sigma$ be closed and $\beta\eta$-normal. Then $\vdash M : \sigma$ is derivable.*

**Proof.** By applying Proposition 13 and Theorem 5. ◀

## 5 Parametricity in the $\beta\eta$-stable semantics

In this section we recall the interpretation of $\lambda_\to$ by means of $\beta\eta$-stable sets of $\lambda$-terms (see [7]) and we define a variant of Reynolds' binary parametricity in this frame.

Let $S_{\beta\eta}$ be the set of all sets $s \subseteq \Lambda$ stable by $\beta\eta$-equivalence. By an interpretation we mean any function $\mathcal{M} : \mathcal{V} \to S_{\beta\eta}$. For any $T, U \in S_{\beta\eta}$, we let $T \to U := \{M \in \Lambda | N \in T \ \Rightarrow MN \in U\} \in S_{\beta\eta}$.

▶ **Definition 15.** For any simple type $\sigma$ and assignment $\mathcal{M} : \mathcal{V} \to \mathcal{S}_{\beta\eta}$, we define an interpretation $|\sigma|_\mathcal{M} \in S_{\beta\eta}$:

$$\begin{aligned} |Z_u|_\mathcal{M} &:= \mathcal{M}(Z_u) \\ |\sigma \to \tau|_\mathcal{M} &:= |\sigma|_\mathcal{M} \to |\tau|_\mathcal{M} \end{aligned} \tag{31}$$

For any simple type $\sigma$, we define its *uniform interpretation* $|\sigma| := \bigcap_\mathcal{M} |\sigma|_\mathcal{M}$. A closed $\lambda$-term $M$ will be said *interpretable (for $\sigma$)* when $M \in |\sigma|$.

Interpretable terms are normalizable:

▶ **Proposition 16.** *If a closed* $\lambda$*-term* $M \in |\sigma|$*, then there exists* $M' \simeq_{\beta\eta} M$ *such that* $M'$ *is in* $\beta\eta$*-normal form.*

**Proof.** Let us call a $\lambda$-term idle if it is $\beta\eta$-equivalent to a $\beta\eta$-normal term which does not begin with a $\lambda$. Let $\mathcal{M}$ be the interpretation such that, for all $Z_u$, $\mathcal{M}(Z_u)$ is the set of all idle terms. Clearly, for all $\sigma$ and variable $x$, $x \in |\sigma|_{\mathcal{M}}$. From $M \in |\sigma|_{\mathcal{M}}$, it follows then that $Mx_1 \dots x_n$ is idle (where $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$), hence there exists $M' \simeq_{\beta\eta} Mx_1 \dots x_n$ $\beta\eta$-normal. We have then $M \simeq_{\beta\eta} \lambda x_1.\dots.\lambda x_n.Mx_1 \dots x_n \simeq_{\beta\eta} \lambda x_1.\dots.\lambda x_n.M'$. Now $\lambda x_1.\dots.\lambda x_n.M'$ is either $\beta\eta$-normal, and in this case we are done, or it is of the form $\lambda x_1.\dots.\lambda x_h.\lambda x_{h+1}.\dots.\lambda x_n.M''x_{h+1} \dots x_n$, for some $1 \leq h < n$, with $x_{h+1}, \dots, x_n$ not free in $M''$. In this case $M \simeq_{\beta\eta} \lambda x_1.\dots.\lambda x_h.M''$ which is $\beta\eta$-normal. ◀

We recall the completeness theorem for the $\beta\eta$-stable semantics:

▶ **Theorem 17** ([6, 3])**.** *Let* $M$ *be a closed* $\lambda$*-term; if* $M \in |\sigma|$*, then there exists* $M' \simeq_{\beta\eta} M$ *such that* $\vdash M' : \sigma$ *is derivable.*

The result in [3] is actually more general, as it holds for *positive types* in Krivine's system $\mathcal{AF}2$, i.e. second order types built from atomic types of the form $X(t_1, \dots, t_n)$, where the $t_i$ are first-order terms, implication, first order quantifiers and second order universal quantifiers (the latter occurring only positively).

In the following we will not use Farkh and Nour's theorem, as we want to prove parametricity and dinaturality by semantic means, that is, without relying on typability.

We introduce the notion of $\beta\eta$-stable relations:

▶ **Definition 18.** Let $s, t \subseteq S_{\beta\eta}$. A *$\beta\eta$-stable relation* is a binary relation $r \subseteq s \times t$ such that for all $M, M' \in s, N, N' \in t$, if $(M, N) \in r$ (what we will note by $M \ r \ N$), $M \simeq_{\beta\eta} M'$ and $N \simeq_{\beta\eta} N'$, then $M' \ r \ N'$.

▶ **Definition 19.** If $r \subseteq s \times t$ and $r' \subseteq s' \times t'$ are $\beta\eta$-stable relations, the $\beta\eta$-stable relation $r \to r' \subseteq (s \to s') \times (t \to t')$ is defined by $P \ (r \to r') \ Q$ if for all $M \in s, N \in t$, if $M \ r \ N$ then $(PM) \ r' \ (QN)$.

In the following lines we reformulate Reynolds' parametricity with respect to $\beta\eta$-stable relations.

▶ **Definition 20** (relation assignment)**.** Let $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ be two interpretations. A *relation assignment* $R$ *over* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ is a map associating, with any simple type $\sigma$, a $\beta\eta$-stable relation $R[\sigma] \subseteq |\sigma|_{\mathcal{M}_1} \times |\sigma|_{\mathcal{M}_2}$ such that, if $\sigma = \sigma_1 \to \sigma_2$, then $R[\sigma] = R[\sigma_1] \to R[\sigma_2]$.

▶ **Definition 21** (parametricity)**.** Let $\sigma$ be a simple type and $M$ be a closed $\lambda$-term. $M$ is *parametric in* $\sigma$ if, for all interpretations $\mathcal{M}_1, \mathcal{M}_2$ and relation assignment $R$ over $\mathcal{M}_1, \mathcal{M}_2$, $M \ R[\sigma] \ M$.

▶ **Example 22.** Let $\sigma$ be $(Z_u \to Z_u) \to (Z_u \to Z_u)$ as in Example 9. Parametricity in $\sigma$ for a closed $\lambda$-term $M$ corresponds to the fact that, for any two $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over $\mathcal{M}_1$ and $\mathcal{M}_2$, and for any $F_0 \in |Z_u|_{\mathcal{M}_1}, G_0 \in |Z_u|_{\mathcal{M}_2}$ and $F_1 \in |Z_u \to Z_u|_{\mathcal{M}_1}, G_1 \in |Z_u \to Z_u|_{\mathcal{M}_2}$, if $F_0 \ R[Z_u] \ G_0$ and $F_1 \ R[Z_u \to Z_u] \ G_1$, then $MF_1F_0 \ R[Z_u] \ MG_1G_0$.

If $M$ is a closed term of type $\sigma$, the $\beta\eta$-normal form of $M$ is of the form $\lambda y.\lambda x.y^n x$, for some $n \geq 0$. Then we can verify that $M$ is parametric in $\sigma$. Let $F_0, G_0, F_1, G_1$ be as before; we have that $MF_1F_0 \simeq_{\beta\eta} F_1^nF_0$ and $MG_1G_0 \simeq_{\beta\eta} G_1^nG_0$. Now we can argue by induction on $n$: if $n = 0$, then $F_1^nF_0 = F_0 \ R[Z_u] \ G_0 = G_1^nG_0$ by hypothesis; if $n = k+1$, then, by induction hypothesis, $F_1^kF_0 \ R[Z_u] \ G_1^kG_0$ and, by hypothesis, $F_1 \ R[Z_u \to Z_u] \ G_1$, that is, for all $U, V$ such that $U \ R[Z_u] \ V$, $F_1U \ R[Z_u] \ G_1V$, so we can conclude $F_1^nF_0 = F_1(F_1^kF_0) \ R[Z_u] \ G_1(G_1^kG_0) = G_1^nG_0$.

The theorem below, known as Reynolds' *abstraction theorem* ([10]), generalizes this example to an arbitrary closed simply-typed $\lambda$-term.

▶ **Theorem 23.** *Let $\sigma$ be a simple type and $M$ be a closed $\lambda$-term. If $\vdash M : \sigma$, then $M$ is parametric in $\sigma$.*

**Proof.** The proof is exactly as in [10]. ◀

In the next section we will prove that closed interpretable $\lambda$-terms are parametric, that is, we will prove parametricity for a term $M$ without relying on a typing of $M$ but rather on the basis of a semantic property of $M$.

## 6 The parametricity theorem

In this section we prove that interpretable closed $\lambda$-terms are parametric (in the sense of Definition 21) by a semantic argument. The proof exploits the idea, appearing in the completeness proofs for the $\beta\eta$-stable semantics in [6, 3], of defining an infinite context $\Gamma^\infty$ made of declarations $(x_i : \tau_i)$, given an enumeration $(x_i)_{i\in\mathbb{N}}$ of $\lambda$-variables and an enumeration $(\tau_i)_{i\in\mathbb{N}}$ of simple types, such that each type receives infinitely many indices. However, the infinite context $\Gamma^\infty$ will not be used, as in those proofs, to define a contextual typability relation, but rather to define contextual notions of interpretation and relation assignment. In particular, a special interpretation $\mathcal{M}_P$ will be defined such that, for any simple type $\sigma$ and $\lambda$-term $P$, if $P \in |\sigma|_{\mathcal{M}_P}$, then for any two interpretations and relation assignment over them, $P$ is contextually related to $P$ relative to $\sigma$.

▶ **Theorem 24** (parametricity)**.** *For any simple type $\sigma$ and closed term $M$, if $M \in |\sigma|$, then $M$ is parametric in $\sigma$.*

**Proof.** Let $(x_i)_{i\in\mathbb{N}}$ be an enumeration of the $\lambda$-variables and $(\tau_i)_{i\in\mathbb{N}}$ an enumeration of simple types such that every type has infinitely many indices. Let $\Gamma^\infty = \{x_i : \tau_i \mid i \in \mathbb{N}\}$ and, for every term $M$, $\Gamma_M$ be the context made by restricting $\Gamma^\infty$ to the free variables occurring in $M$.

We first define contextual interpretations and relations:

- given $\mathcal{M} : \mathcal{V} \to S_{\beta\eta}$, a $\lambda$-term $P$ and a simple type $\tau$, the statement $P \in |\Gamma^\infty \vdash \tau|_{\mathcal{M}}$ holds when, by letting $x_{i_1}, \dots, x_{i_n}$ be the free variables of $P$, for every $Q_1 \in |\tau_{i_1}|_{\mathcal{M}}, \dots, Q_n \in |\tau_{i_n}|_{\mathcal{M}}$, $P[Q_1/x_{i_1}, \dots, Q_n/x_{i_n}] \in |\tau|_{\mathcal{M}}$;
- given $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and $R$ a relation assignment over $\mathcal{M}_1, \mathcal{M}_2$, $\lambda$-terms $P, Q$ and a simple type $\tau$, the statement $P \ R[\Gamma^\infty \vdash \tau] \ Q$ holds when $P \in |\Gamma^\infty \vdash \tau|_{\mathcal{M}_1}$, $Q \in |\Gamma^\infty \vdash \tau|_{\mathcal{M}_2}$ and, by letting $x_{i_1}, \dots, x_{i_n}$ be the free variables of $P$ and $Q$, for every $F_j \in |\tau_{i_j}|_{\mathcal{M}_1}, G_j \in |\tau_{i_j}|_{\mathcal{M}_2}$ such that $F_j \ R[\tau_{i_j}] \ G_j$ for $1 \le j \le n$, we have

$$P[F_1/x_{i_1}, \dots, F_n/x_{i_n}] \ R[\tau] \ Q[G_1/x_{i_1}, \dots, G_n/x_{i_n}] \tag{32}$$

Let now $\mathcal{M}_P : \mathcal{V} \to S_{\beta\eta}$ be the assignment such that, for all $Z_u \in \mathcal{V}$, $\mathcal{M}_P(Z_u)$ is the $\beta\eta$-closure of the set of $\lambda$-terms $P$ such that $P \in |\Gamma^\infty \vdash Z_u|$ and for all $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over $\mathcal{M}_1, \mathcal{M}_2$, $P \ R[\Gamma^\infty \vdash Z_u] \ P$.

We claim that, for any simple type $\sigma$, the following hold:

1. if $P \in |\sigma|_{\mathcal{M}_P}$, then, for any $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over them, $P\ R[\Gamma^\infty \vdash \sigma]\ P$;
2. for every variable $x_i$ such that $\tau_i = \sigma$, $x_i \in |\sigma|_{\mathcal{M}_P}$.

We argue for both 1. and 2. by induction on $\sigma$. If $\sigma = Z_u$, then, by definition, any $P \in |\sigma|_{\mathcal{M}_P}$ is such that, for any $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over them, $P\ R[\Gamma^\infty \vdash \sigma]\ P$ holds, so claim 1. holds; moreover, if $\tau_i = Z_u$, then, for any $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over $\mathcal{M}_1, \mathcal{M}_2$, $x_i\ R[\Gamma^\infty \vdash Z_u]\ x_i$: if $F \in \mathcal{M}_1(Z_u), G \in \mathcal{M}_2(Z_u)$ and $F\ R[Z_u]\ G$, then $x_i[F/x_i] = F_i\ R[Z_u]\ G_i = x_i[G/x_i]$. Hence claim 2. holds too.

Let now $\sigma = \sigma_1 \to \sigma_2$. By induction hypothesis, for all $i \geq 0$, if $\tau_i = \sigma_1$ then $x_i \in |\sigma_1|_{\mathcal{M}_P}$; let then $P \in |\sigma|_{\mathcal{M}_P}$ and choose an index $i$, with $\tau_i = \sigma_1$, such that $x_i$ does not occur free in $P$; we have that $Px_i \in |\sigma_2|_{\mathcal{M}_P}$ hence, by induction hypothesis, for any $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over them, $(Px_i)\ R[\Gamma^\infty \vdash \sigma_2]\ (Px_i)$; let then $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and $R$ be a relation assignment over them; by letting $\Gamma_P = \{x_{i_1} : \tau_{i_1}, \dots, x_{i_n} : \tau_{i_n}\}$, suppose $F_j \in |\tau_{i_j}|_{\mathcal{M}_1}, G_j \in |\tau_{i_j}|_{\mathcal{M}_2}$ are such that $F_j\ R[\tau_{i_j}]\ G_j$, for $1 \leq j \leq n$, and moreover suppose $F \in |\sigma_1|_{\mathcal{M}_1}, G \in |\sigma_1|_{\mathcal{M}_2}$ are such that $F\ R[\sigma_1]\ G$; then, since $\Gamma_{Px_i} = \{x_{i_1} : \tau_{i_1}, \dots, x_{i_n} : \tau_{i_n}, x_i : \sigma_1\}$ (remark that $x_i \neq x_{i_1}, \dots, x_{i_n}$ as $x_i$ has been chosen not to occur free in $P$), we have

$$\begin{aligned}
&\big((\lambda x_i.Px_i)[F_1/x_{i_1}, \dots, F_n/x_{i_n}]\big)F \simeq_{\beta\eta} (Py)[F_1/x_{i_1}, \dots, F_n/x_{i_n}][F/y] \\
&\simeq_{\beta\eta} (Px_i)[F_1/x_{i_1}, \dots, F_n/x_{i_n}, F/x_i]\ R[\sigma_2]\ (Px_i)[G_1/x_{i_1}, \dots, G_n/x_{i_n}, G/x_i] \\
&\simeq_{\beta\eta} (Py)[G_1/x_{i_1}, \dots, G_n/x_{i_n}][G/y] \simeq_{\beta\eta} \big((\lambda x_i.Px_i)[G_1/x_{i_1}, \dots, G_n/x_{i_n}]\big)G
\end{aligned} \tag{33}$$

where we can suppose $y$ not occurring free in any of the $F_j$ and $G_j$. We conclude then that $P \simeq_{\beta\eta} \lambda x_i.Px_i\ R[\Gamma^\infty \vdash \sigma]\ \lambda x_i.Px_i \simeq_{\beta\eta} P$, so we proved claim 1.

To prove claim 2., suppose $x_i$ is a variable such that $\tau_i = \sigma$. Let $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_v$ for some $n \geq 1$ and $Q_1, \dots, Q_n$ be terms such that $Q_j \in |\sigma_j|_{\mathcal{M}_P}$, for $1 \leq j \leq n$; by induction hypothesis, for all $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over $\mathcal{M}_1, \mathcal{M}_2$, $Q_j\ R[\Gamma^\infty \vdash \sigma_j]\ Q_j$, for $1 \leq j \leq n$. Let $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$, and $R$ be a relation assignment over $\mathcal{M}_1, \mathcal{M}_2$; moreover let $\{x_i : \tau_i\} \cup \bigcup_i \Gamma_{Q_i}$ be the set $\{x_{i_1} : \tau_{i_1}, \dots, x_{i_r} : \tau_{i_r}\}$ where $i = i_p$ for some fixed $1 \leq p \leq r$; given terms $F_1, G_1, \dots, F_r, G_r$ such that $F_l \in |\tau_{i_l}|_{\mathcal{M}_1}, G_l \in |\tau_{i_l}|_{\mathcal{M}_2}$ and $F_l\ R[\tau_{i_l}]\ G_l$ all hold for $1 \leq l \leq r$, we have that

$$(x_i Q_1 \dots Q_n)\theta_1 \simeq_{\beta\eta}\ F_{i_p}(Q_1\theta_1) \dots (Q_n\theta_1) \tag{34}$$

and

$$(x_i Q_1 \dots Q_n)\theta_2 \simeq_{\beta\eta}\ G_{i_p}(Q_1\theta_2) \dots (Q_n\theta_2) \tag{35}$$

where $\theta_1$ (resp. $\theta_2$) is the substitution $[F_1/x_{i_1}, \dots, F_r/x_{i_r}]$ (resp. $[G_1/x_{i_1}, \dots, G_r/x_{i_r}]$).

Now, from the induction hypothesis (which implies that $Q_j\theta_1\ R[\sigma_j]\ Q_j\theta_2$) and the fact that $F_{i_p}\ R[\sigma]\ G_{i_p}$, it follows that $x_i Q_1 \dots Q_n\ R[\Gamma^\infty \vdash Z_u]\ x_i Q_1 \dots Q_n$. We deduce that $x_i Q_1 \dots Q_n \in |Z_v|_{\mathcal{M}_P}$, that is, $x_i \in |\sigma|_{\mathcal{M}_P}$ and claim 2. is proved.

Finally, let $M$ be closed and interpretable. Then $M \in |\sigma|_{\mathcal{M}_P}$, so, for every $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ and relation assignment $R$ over $\mathcal{M}_1, \mathcal{M}_2$, $M\ R[\Gamma^\infty \vdash \sigma]\ M$, that is $M\ R[\sigma]\ M$, as $M$ is closed. ◀

## 7 From parametricity to dinaturality

In this section we adapt the well-known fact that parametricity implies dinaturality ([9]) to the frame described in the last sections, obtaining a semantic argument that interpretable closed $\lambda$-terms are dinatural.

Let $\mathcal{M}_1, \mathcal{M}_2 : \mathcal{V} \to S_{\beta\eta}$ be given, for any $Z_u$, by $\mathcal{M}_1(Z_u) = \Lambda$ and $\mathcal{M}_2(Z_u) = f_u\Lambda$, where $f_u\Lambda$ is the $\beta\eta$-closure of the set of all $\lambda$-terms of the form $f_uQ$, for some $Q \in \Lambda$. Let moreover the relation assignment $R^f$ over $\mathcal{M}_1, \mathcal{M}_2$ be defined, for any $Z_u$, by $P\ R[Z_u]\ Q$, if $f_uP \simeq_{\beta\eta} Q$. Let $\mathcal{V}_1, \mathcal{V}_2$ be as in Section 3 and $\mathcal{N} : \mathcal{V}_1 \cup \mathcal{V}_2 \to S_{\beta\eta}$ be the interpretation such that, for all $u \geq 0$, $\mathcal{N}(X_u) = \mathcal{M}_1(Z_u)$ and $\mathcal{N}(Y_u) = \mathcal{M}_2(Z_u)$.

▶ **Proposition 25.** *For any type $\sigma$,*

$$\begin{array}{ll} H_\tau \in |\tau_Y^X \to \tau_Y|_{\mathcal{N}} & K_\tau \in |\tau_Y \to \tau_X^Y|_{\mathcal{N}} \\ K_\tau \in |\tau_Y^X \to \tau_X|_{\mathcal{N}} & H_\tau \in |\tau_X \to \tau_X^Y|_{\mathcal{N}} \end{array} \tag{36}$$

**Proof.** We argue by induction on $\tau$: if $\tau = Z_u$, then $H_\tau = f_u \in |X_u \to Y_u|_{\mathcal{N}}$ and $K_\tau = \lambda x.x \in |X_u \to X_u|_{\mathcal{N}}, |Y_u \to Y_u|_{\mathcal{N}}$. If $\tau = \tau_1 \to \dots \to \tau_m \to Z_u$ for some $m \geq 1$, then let $E \in |\tau_Y^X|_{\mathcal{N}}, F \in |\tau_X|_{\mathcal{N}}, G \in |\tau_Y|_{\mathcal{N}}$ and $E_i \in |(\tau_i)_Y^X|_{\mathcal{N}}, F_i \in |(\tau_i)_X|_{\mathcal{N}}, G_i \in |(\tau_i)_Y|_{\mathcal{N}}$, for $1 \leq i \leq m$. Then, by induction hypothesis we have

$$\begin{array}{ll} H_{\tau_i}E_i \in |(\tau_i)_Y|_{\mathcal{N}} & K_{\tau_i}G_i \in |(\tau_i)_X^Y|_{\mathcal{N}} \\ K_{\tau_i}E_i \in |(\tau_i)_X|_{\mathcal{N}} & H_{\tau_i}F_i \in |(\tau_i)_X^Y|_{\mathcal{N}} \end{array} \tag{37}$$

from which we obtain

$$H_\tau E G_1 \dots G_n \simeq_\beta \quad f_u\big(E(K_{\tau_1}G_1)\dots(K_{\tau_n}G_n)\big) \in |Y|_{\mathcal{N}} \tag{38}$$

$$H_\tau F E_1 \dots E_n \simeq_\beta \quad f_u\big(F(K_{\tau_1}E_1)\dots(K_{\tau_n}E_n)\big) \in |Y|_{\mathcal{N}} \tag{39}$$

$$K_\tau E F_1 \dots F_n \simeq_\beta \quad E(H_{\tau_1}F_1)\dots(H_{\tau_n}F_n) \in |X|_{\mathcal{N}} \tag{40}$$

$$K_\tau G E_1 \dots E_n \simeq_\beta \quad G(H_{\tau_1}E_1)\dots(H_{\tau_n}E_n) \in |Y|_{\mathcal{N}} \tag{41}$$

exploiting the fact that $\tau_Y^X = (\tau_1)_X^Y \to \dots \to (\tau_n)_X^Y \to X_u$. ◀

For any $\lambda$-variable $g \neq z$, $H_\tau g \in |\tau_Y|_{\mathcal{N}} = |\tau|_{\mathcal{M}_2}$ and $K_\tau g \in |\tau_X|_{\mathcal{N}} = |\tau|_{\mathcal{M}_1}$. Hence we can ask whether $H_\tau g$ and $K_\tau g$ are related under the relation assignment $R^f$. This is shown by the proposition below:

▶ **Proposition 26.** *For all simple type $\sigma$ and variable $g$,*
**(i)** $(K_\sigma g)\ R^f[\sigma]\ (H_\sigma g)$*;*
**(ii)** *if $P \in |\sigma|_{\mathcal{M}_1}, Q \in |\sigma|_{\mathcal{M}_2}$ and $P\ R^f[\sigma]\ Q$, then $H_\sigma P \simeq_{\beta\eta} K_\sigma Q$.*

**Proof.** We prove both fact simultaneously by induction on $\sigma$. If $\sigma = Z_u$ then $K_\sigma g \simeq_\beta g\ R^f\ f_u g \simeq_\beta H_\sigma g$; moreover, if $P\ R^f[Z_u]\ Q$, then $H_\sigma P \simeq_\beta f_uP \simeq_{\beta\eta} Q \simeq_\beta K_\sigma Q$.

If $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$ for some $n \geq 1$, then suppose $P_i \in |\sigma_i|_{\mathcal{M}_1}, Q_i \in |\sigma_i|_{\mathcal{M}_2}$ and $P_i\ R[\sigma_i]\ Q_i$, for all $1 \leq i \leq n$; then

$$P := K_\sigma g P_1 \dots P_n \simeq_\beta g(H_{\sigma_1}P_1)\dots(H_{\sigma_n}P_n) \tag{42}$$

is related to

$$Q := H_\sigma g Q_1 \dots Q_n \simeq_\beta f_u\Big(g(K_{\sigma_1}Q_1)\dots(K_{\sigma_n}Q_n)\Big) \tag{43}$$

Indeed, by induction hypothesis, $H_{\sigma_i}P_i \simeq_{\beta\eta} K_{\sigma_i}Q_i$, hence $g(K_{\sigma_1}P_1)\dots(K_{\sigma_n}P_n) \simeq_{\beta\eta} g(H_{\sigma_1}Q_1)\dots(H_{\sigma_n}Q_n)$ so $f_uP \simeq_{\beta\eta} Q$.

Suppose now $P \in |\sigma|_{\mathcal{M}_1}, Q \in |\sigma|_{\mathcal{M}_2}$ and $P\ R^f[\sigma]\ Q$, then let

$$P' := H_\sigma P g_1 \dots g_n \simeq_\beta f_u\Big(P(K_{\sigma_1}g_1)\dots(K_{\sigma_n}g_n)\Big) \tag{44}$$

and

$$Q' := K_\sigma Q g_1 \dots g_n \simeq_\beta Q(H_{\sigma_1} g_1) \dots (H_{\sigma_n} g_n) \tag{45}$$

By induction hypothesis $(K_{\sigma_i} g_i)\ R^f[\sigma_i]\ (H_{\sigma_i} g_i)$ hence, by hypothesis $P(K_{\sigma_1} g_1) \dots (K_{\sigma_n} g_n)$ is related to $Q'$, that is $P' \simeq_{\beta\eta} Q'$. We conclude that $K_\sigma P \simeq_{\beta\eta} H_\sigma Q$. ◀

We can now apply the parametricity Theorem 24 and obtain the following:

▶ **Theorem 27.** *Let $M$ be a closed $\lambda$-term and $\sigma$ a simple type. If $M \in |\sigma|$ then $M \in DIN_\sigma$.*

**Proof.** Let $M \in |\sigma|$ be a closed term and let $\sigma = \sigma_1 \to \dots \to \sigma_n \to Z_u$, for some $n \geq 0$. By propositions 25 and 26, $H_{\sigma_i} g_i \in |\sigma_i|_{\mathcal{M}_2}$, $K_{\sigma_i} g_i \in |\sigma_i|_{\mathcal{M}_1}$ and $(H_{\sigma_i} g_i)\ R^f[\sigma_i]\ (K_{\sigma_i} g_i)$. By Theorem 24, $M\ R^f[\sigma]\ M$, hence

$$M(K_{\sigma_1} g_1) \dots (K_{\sigma_n} g_n)\ R[Z_u]\ M(H_{\sigma_1} g_1) \dots (H_{\sigma_n} g_n) \tag{46}$$

that is $f_u\big(M(K_{\sigma_1} g_1) \dots (K_{\sigma_n} g_n)\big) \simeq_{\beta\eta} M(H_{\sigma_1} g_1) \dots (H_{\sigma_n} g_n)$, whence $M \in DIN_\sigma$. ◀

## 8 Conclusions

Two results were proved in this paper: first, that dinaturality implies (simple) typability (Theorem 14); second, that interpretability in the $\beta\eta$-stable semantics implies dinaturality (Theorem 27). By putting them together (along with Proposition 16) we obtain the following:

▶ **Theorem 28** (completeness of the $\beta\eta$-stable semantics)**.** *Let $M$ be a closed $\lambda$-term and $\sigma$ a simple type. If $M \in |\sigma|$, there exists $M' \simeq_{\beta\eta} M$ such that $\vdash M' : \sigma$.*

As we said, this completeness result can be proved by a direct argument ([6]) and extended to a restricted class of second order types ([3]). The core idea of the argument is the definition of a particular interpretation $\mathcal{M}$ such that $\mathcal{M}(Z_u)$ contains $\lambda$-terms which are $\beta\eta$-equivalent to terms which can be given type $\sigma$ in an infinite context defined as $\Gamma^\infty$ in the proof of Theorem 24.

This argument cannot be straightforwardly extended to the reducibility semantics commonly used to prove normalization theorems (for instance, Krivine's saturated families or Girard's reducibility candidates, see [4]). Indeed, the sets of $\lambda$-terms there considered have more complex closure properties, making in particular every term of the form $xP_1 \dots P_n$ belong to any considered set. Hence from the fact that a term belongs to the closure of the set of typable $\lambda$-terms one cannot deduce that the term is $\beta\eta$-equivalent to a typable term. In a word, the interpretation constructed over typable terms no longer captures typable terms only. Similarly, the proof of Theorem 24 cannot be straightforwardly extended to the reducibility semantics, as the interpretation $\mathcal{M}_P$ is defined starting from sets of terms which might well have the form $xP_1 \dots P_n$.

However, the results here presented suggest that completeness results might be looked for through a different path, namely that of showing that interpretable $\lambda$-terms satisfy parametricity and/or dinaturality, two semantic properties which allow, as it has been shown, to completely reconstruct the syntactic structure of $\beta\eta$-normal $\lambda$-terms.

## References

1 E. S. Bainbridge, Peter J. Freyd, Andre Scedrov, and Philip J. Scott. Functorial polymorphism. *Theoretical Computer Science*, 70:35–64, 1990.

2 Joachim de Lataillade. Dinatural terms in System F. In *Proceedings of the Twenty-Fourth Annual IEEE Symposium on Logic in Computer Science (LICS 2009)*, pages 267–276, Los Angeles, California, USA, 2009. IEEE Computer Society Press.

3 Samir Farkh and Karim Nour. Résultats de complétude pour des classes de types du système *AF*2. *Informatique Théorique et Applications*, 31(6):513–537, 1998.

4 Jean Gallier. On Girard's "candidats de réductibilité". *Logic and Computer Science*, 1990.

5 Jean-Yves Girard, Andre Scedrov, and Philip J. Scott. Normal forms and cut-free proofs as natural transformations. In Y. Moschovakis, editor, *Logic from Computer Science*, volume 21, pages 217–241. Springer-Verlag, 1992.

6 Roger J. Hindley. The completeness theorem for typing $\lambda$-terms. *Theoretical Computer Science*, 22(1-2):1–17, 1983.

7 Jean-Louis Krivine. *Lambda calculus, types and models.* Ellis Horwood, 1993.

8 R. Labib-Sami. Types avec (ou sans) types auxiliaires. Manuscript, 1986.

9 Gordon Plotkin and Martin Abadi. A logic for parametric polymorphism. In *TLCA'93, International Conference on Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 361–375. Springer Berlin Heidelberg, 1993.

10 John C. Reynolds. Types, abstraction and parametric polymorphism. In R.E.A. Mason, editor, *Information Processing 1983*, pages 513–523. North-Holland, 1983.


# A Postponed proofs

**Proof of Theorem 5.** For one direction first observe that the function $\mathtt{var}$ and $\mathtt{subt}$ are always defined: $\mathtt{var}(x)$ is the path $\pi$ such that $x : \sigma_\pi$ occurs somewhere in the typing derivation of $M$; $\mathtt{subt}(\pi)$ is the path $\pi'$ such that $M_\pi$ has type $\sigma_{\pi'}$. So point 1 is satisfied. For the points 2 and 3 we argue by induction on the number of applications in $M$.

- if $M = \lambda x_{n+1}.\dots.\lambda x_{n+h}.x_j$, for $1 \leq j \leq n+h$, then $\mathcal{T}(N) = \{\epsilon\}$. Then $ar(\sigma_{\mathtt{var}\epsilon}) = ar(\sigma_j) = ar(\sigma) - (n+h) = ar(\sigma_{\mathtt{subt}(\epsilon)}) - ar(N_\epsilon)$, so point 2. holds and, moreover, $\sigma_{jl} = \sigma_{n+h+l}$ (as $\sigma_j = \tau = \sigma_{n+h+1} \to \dots \to \sigma_{n+h+ar(\sigma)} \to Z_u$) for all $1 \leq l \leq ar(\sigma) - (n+h) + 1$, so point 3. holds too.
- if $M = \lambda x_{n+1}.\dots.\lambda x_{n+h}.x_j M_1 \dots M_p$ where $1 \leq j \leq n+h$ and $p \geq 1$, then the typing derivation of $M$ is as follows

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma,\Delta \vdash x_j : \sigma_j \qquad \dfrac{\vdots}{\Gamma,\Delta \vdash M_1 : \sigma_{j1}}}{\Gamma,\Delta \vdash x_j M_1 : \sigma_{j2} \to \dots \to \sigma_{jar(\sigma_j)} \to X}}{\ddots \atop \Gamma,\Delta \vdash x_j M_1 \dots M_{p-1} : \sigma_{j(p)} \to \dots \to \sigma_{jar(\sigma_j)} \to X} \qquad \dfrac{\vdots}{\Gamma,\Delta \vdash M_p : \sigma_{jp}}}{\Gamma,\Delta \vdash x_j M_1 \dots M_p : \sigma_{j(p+1)} \to \dots \to \sigma_{jar(\sigma_j)} \to X}}{\Gamma \vdash M : \tau} \tag{47}
$$

  where $\Delta = \{x_{n+1} : \sigma_{n+1}, \dots, x_{n+h} : \sigma_{n+h}\}$. Hence $\tau = \sigma_{n+1} \to \dots \to \sigma_{n+h} \to \sigma_{j(p+1)} \to \dots \to \sigma_{jar(\sigma_j)} \to Z_u$.
  In order to apply the induction hypothesis to the terms $N_j = \lambda x_1.\dots.\lambda x_h.M_l$ and the types $\tau_j = \sigma_1 \to \dots \to \sigma_n \to \sigma_{ij}$, for $1 \leq l \leq p$, we must consider the maps $\mathtt{var}_j : Var(N_j) \to \mathcal{T}(\tau_j), \mathtt{subt}_j : \mathcal{T}(N_j) \to \mathcal{T}(\tau_j)$ and use equations 30, which are proved in detail below (end of the proof of Proposition 13). We verify now that any $\pi \in \mathcal{T}(N)$ verifies points 2 and 3 (by using equations 30):

1. if $\pi = \epsilon$, then $ar(\sigma_{\texttt{var}(h(\epsilon))}) = ar(\sigma_j) = p + ar(\sigma) - (n+h) = p_N(\epsilon) + ar(\sigma_{\texttt{subt}(\epsilon)}) - ar(N_\epsilon)$. Otherwise $\pi = l * \pi'$, for some $1 \leq l \leq p$ and $\pi' \in \mathcal{T}(N_j)$; then $ar(\sigma_{\texttt{var}(h(\pi))}) = ar(\sigma_{\texttt{var}_j(h(\pi'))}) = p_{N_j}(\pi') + ar(\sigma_{\texttt{subt}_j(\pi')}) - ar((N_j)_{\pi'}) = p_N(\pi) + ar(\sigma_{\texttt{subt}(\pi)}) - ar(N_\pi)$;
2. if $\pi = \epsilon$, then, for $1 \leq d \leq ar(\sigma) - ar(N) + 1$, $\sigma_{j(p+d)} = \sigma_{n+h+d} = \sigma_{ar(N)+d)}$ (since $\tau = \sigma_{n+1} \to \cdots \to \sigma_{n+h} \to \sigma_{j(p+1)} \to \cdots \to \sigma_{jar(\sigma_j)} \to Z_u$). Otherwise, $\pi = l * \pi'$ and, for $1 \leq d \leq ar(\sigma_{\texttt{subt}(\pi)}) - ar(N_\pi) + 1$, $\sigma_{\texttt{var}(h(\pi))*(p_N(\pi)+d)} = \sigma_{\texttt{var}_j(h(\pi'))*(p_{N_j}(\pi')+d)} \stackrel{[i.h.]}{=} \sigma_{\texttt{subt}_l(\pi')*(ar((N_j)_{\pi'})+d)} = \sigma_{\texttt{subt}(\pi)*(ar(N_\pi)+d)}$.

For the converse direction, we argue again by induction on the number of applications in $M$:

- if $M = \lambda x_{n+1}.\dots.\lambda x_{n+h}.x_j$, for $1 \leq j \leq n+h$, then from (1),(2) and (3), with $\pi = \epsilon$, it follows that $n+h \leq ar(\sigma)$, $ar(\sigma_j) = ar(\sigma) - (n+h)$ and, for $1 \leq l \leq ar(\sigma) - (n+h) + 1 = ar(\tau) - h + 1 = ar(\tau') + 1$, where $\tau = \sigma_{n+1} \to \cdots \to \sigma_{n+h} \to \tau'$, $\sigma_{jl} = \sigma_{n+h+l}$. We deduce that $\sigma_j = \tau'$ and $\Gamma, \Delta \vdash x_j : \tau'$ is derivable (where $\Delta = \{x_{n+1} : \sigma_{n+1}, \dots, x_{n+h} : \sigma_{n+h}\}$), hence $\Gamma \vdash M : \tau$ is derivable;
- if $M = \lambda x_{n+1}.\dots.\lambda x_{n+h}.x_j M_1 \dots M_p$, with $p \geq 1$, then we can apply the induction hypothesis to the terms $N_1, \dots, N_p$ (defined as above), yielding $\Gamma, \Delta \vdash M_l : \sigma_{jl}$, with $\Delta = \{x_{n+1} : \sigma_{n+1}, \dots, x_{n+h} : \sigma_{n+h}\}$, for all $1 \leq l \leq p$. From (1), (2) and (3), with $\pi = \epsilon$, it follows that $\tau = \sigma_{n+1} \to \cdots \to \sigma_{n+h} \to \tau'$, with $n+h \leq ar(\sigma)$, $ar(\sigma_j) = p + ar(\sigma) - (n+h)$ and, for $1 \leq l \leq ar(\sigma) - (n+h) + 1 = ar(\tau) - h + 1 = ar(\tau') + 1$, $\sigma_{j(p+l)} = \tau'_l$, hence $\sigma_j = \sigma_{j1} \to \cdots \to \sigma_{jp} \to \tau'$. So we can conclude, from $\Gamma, \Delta \vdash M_l : \sigma_{jl}$, for $1 \leq l \leq p$, that $\Gamma, \Delta \vdash x_j M_1 \dots M_p : \tau'$ and finally that $\Gamma \vdash M : \tau$. ◀

**End of the proof of Proposition 13.** We will prove that, for all $\lambda \in \mathcal{T}(N_j)$, the equations 30 hold. For $\lambda \in \mathcal{T}(N_j)$, three cases arise for the variable $x$ corresponding to $h_j(\lambda)$:

- $x$ is one of the $x_1, \dots, x_h$, i.e. $h_j(\lambda) = (\epsilon, l)$, for some $1 \leq l \leq h$;
- $x$ is bound in $M_j$ "at depth 0", i.e. $h_j(\lambda) = (\epsilon, h+l)$, for some $1 \leq l \leq ar(M_j)$;
- $x$ it is bound in $M_j$ "at depth $> 0$", i.e. $h_j(\lambda) = (\lambda' * k, l)$, for some $\lambda'$ and $l \geq 1$.

We claim that:

1. in the first case, $\texttt{var}(h(j * \lambda)) = \texttt{var}_j(h_j(\lambda))$;
2. in the second case, $\texttt{var}(h(j * \lambda)) = i * j * (\texttt{var}_j(h_j(\lambda)) - h)$;
3. in the third case, two subcases must be considered:
   a. either $\texttt{var}_j(h_j(\lambda)) = d * \lambda''$ with $d \leq h$, and $\texttt{var}(h(j * \lambda)) = \texttt{var}_j(h_j(\lambda))$;
   b. or $\texttt{var}_j(h_j(\lambda)) = (h+d) * \lambda''$ with $d \leq ar(M_j)$, and $\texttt{var}(h(j * \lambda)) = i * j * d * \lambda''$.

We prove now our claims:

1. $\texttt{var}(h(j * \lambda)) = \texttt{var}(\epsilon, l) = l = \texttt{var}_j(\epsilon, l) = \texttt{var}_j(h_j(\epsilon))$;
2. $\texttt{var}(h(j * \lambda)) = \texttt{var}(j, l) = \texttt{var}(h(\epsilon)) * j * l = i * j * l = i * j * (l + h - h) = i * j * (\texttt{var}_j(\epsilon, l + h) - h) = i * j * (\texttt{var}_j(h_j(\lambda)) - h)$;
3. we consider the two subcases separately:
   a. if $\texttt{var}_j(h_j(\lambda)) = d * \lambda''$ with $d \leq h$, then we argue by induction on $\ell(\lambda)$: we have $\texttt{var}(h(j * \lambda' * k)) = \texttt{var}(j * \lambda' * k, l) = \texttt{var}(h(j * \lambda')) * k * l \stackrel{*}{=} \texttt{var}_j(h_j(\lambda')) * k * l = \texttt{var}_j(\lambda' * k, l) = \texttt{var}_j(h_j(\lambda))$ where in the starred passage, if $\lambda' = \epsilon$, we apply point 1. as $var_j(h_j(\lambda')) = var_j(\epsilon, d) = d$ (since $\epsilon \notin Im(var_j)$ and $\ell(h_j(\epsilon)_1) \leq 0$), while if $\lambda' \neq \epsilon$, we apply the induction hypothesis to $\lambda'$ as $\ell(\lambda') < \ell(\lambda)$ (since $\ell(h_j(\lambda)_1) \leq \ell(\lambda)$) and $h_j(\lambda') = d * o$ for some path $o$;
   b. if $\texttt{var}_j(h_j(\lambda)) = (h+d) * \lambda''$ with $d \leq h$, then again we argue by induction on $\ell(\lambda)$: we have $\texttt{var}(h(j * \lambda)) = \texttt{var}(j * \lambda' * k, l) = \texttt{var}(h(j * \lambda')) * k * l$. If $\lambda' = \epsilon$ then, by point 2. we have $\texttt{var}(h(j * \lambda')) = i * j * (\texttt{var}_j(h_j(\lambda')) - h)$, hence $\texttt{var}(h(j * \lambda')) * k * l =$

$i * j * (\mathtt{var}_j(h_j(\lambda')) - h) * k * l = i * j * d * k * l = i * j * d * \lambda''$, where $k * l = \lambda''$ follows from $\mathtt{var}_j(h_j(\lambda)) = \mathtt{var}_j(\lambda' * k, l) = \mathtt{var}_j(h_j(\lambda') * k * l) = (h + d) * k * l = (h + d) * \lambda''$. If $\lambda' \neq \epsilon$ then we can apply the induction hypothesis to $\lambda'$ as $\ell(\lambda') < \ell(\lambda)$ and $\mathtt{var}_j(h_j(\lambda')) = (h + d) * o$, for some path $o$; so we get that $\mathtt{var}(h(j * \lambda')) = i * j * d * o$ and then we can compute $\mathtt{var}(h(j * \lambda)) = \mathtt{var}(j * \lambda' * k, l) = \mathtt{var}(h(j * \lambda')) * k * l = i * j * d * o * k * l = i * j * d * \lambda''$, where $o * k * l = \lambda''$ follows from $\mathtt{var}_j(h_j(\lambda)) = \mathtt{var}_j(\lambda' * k, l) = \mathtt{var}_j(h_j(\lambda') * k * l) = (h + d) * o * k * l = (h + d) * \lambda''$.

We can now verify that $(\tau_j)_{\mathtt{var}_j(h_j(\pi'))} = \sigma_{\mathtt{var}(h(\pi))}$, for $\pi = j * \pi'$. We must consider the tree cases above:

1. if $h_j(\pi') = (\epsilon, l)$ with $l \leq h$, then $(\tau_j)_{\mathtt{var}_j(h_j(\pi))} = (\tau_j)_{\mathtt{var}_j(\epsilon,l)} = (\tau_j)_l = \sigma_l$ and $\sigma_{\mathtt{var}(h(j*\pi''))} = \sigma_{\mathtt{var}_j(h_j(\pi'))} = \sigma_{\mathtt{var}_j(\epsilon,l)} = \sigma_l$;
2. if $h_j(\pi') = (\epsilon, h + l)$, then $(\tau_j)_{\mathtt{var}_j(h_j(\pi'))} = (\tau_j)_{(h+l)} = \sigma_{i*j*l} = \sigma_{i*j*(\mathtt{var}_j(h_j(\pi'))-h)} = \sigma_{\mathtt{var}(h(j*\pi'))} = \sigma_{\mathtt{var}(h(\pi))}$;
3. if $h_j(\pi') = (\lambda' * r, l)$, then we consider the two subcases:
   a. if $\mathtt{var}_j(h_j(\pi')) = d*\lambda$ with $d \leq h$, then $(\tau_j)_{\mathtt{var}_j(h_j(\pi'))} = (\tau_j)_{d*\lambda} = \sigma_{d*\lambda} = \sigma_{\mathtt{var}_j(h_j(\pi'))} = \sigma_{\mathtt{var}(h(j*\pi'))} = \sigma_{\mathtt{var}(h(\pi))}$;
   b. if $\mathtt{var}_j(h_j(\pi')) = (h + d) * \lambda$ with $d \leq ar(M_j)$, then $(\tau_j)_{\mathtt{var}_j(h_j(\pi'))} = (\tau_j)_{(h+d)*\lambda'} = \sigma_{i*j*d*\lambda'} = \sigma_{\mathtt{var}(h(j*\pi'))} = \sigma_{\mathtt{var}(h(\pi))}$.

Finally, for $\pi' = \pi'' * k$, we have that $(\tau_j)_{\mathtt{subt}_j(\pi')} = (\tau_j)_{\mathtt{var}_j(h_j(\pi''))*k} = \sigma_{\mathtt{var}(h(j*\pi''))*k} = \sigma_{\mathtt{subt}(\pi)}$. ◀