The Confluent Terminating Context-Free Substitutive Rewriting System for the lambda-Calculus with Surjective Pairing and Terminal Type

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Abstract

For the lambda-calculus with surjective pairing and terminal type, Curien and Di Cosmo, inspired by Knuth-Bendix completion, introduced a confluent rewriting system of the naive rewriting system. Their system is a confluent (CR) rewriting system stable under contexts. They left the strong normalization (SN) of their rewriting system open. By Girard's reducibility method with restricting reducibility theorem, we prove SN of their rewriting, and SN of the extensions by polymorphism and (terminal types caused by parametric polymorphism). We extend their system by sum types and eta-like reductions, and prove the SN. We compare their system to type-directed expansions.

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1 Introduction

We recall the equational theory $\lambda\beta\eta\pi*$ from [9]. Types are built up from the distinguished type constant $\top$, and type variables by means of the product type $\varphi \times \psi$ and the function type $\varphi \to \psi$. Terms are built up from the distinguished term constant $*^\top$ and term variables $x^\varphi, y^\varphi, \dots, x^\psi, y^\psi, \dots$ by means of λ-abstraction $(\lambda x^\varphi.\, t^\psi)^{\varphi\to\psi}$, term application $(u^{\varphi\to\psi} v^\varphi)^\psi$, pairing $\langle u^\varphi,\, v^\psi\rangle^{\varphi\times\psi}$, left-projection $(\pi_1 t^{\varphi\times\psi})^\varphi$, right-projection $(\pi_2 t^{\varphi\times\psi})^\psi$, where the superscript represents the type. The superscript is often omitted. The set of free variables of a term t is denoted by $\mathrm{FV}(t)$. The equational theory $\lambda\beta\eta\pi*$ consists of the following axioms:

$$
\begin{array}{llll}
(\beta) & (\lambda x.\, u)v = u[x := v]. & & \\
(\pi_1) & \pi_1\langle u,\, v\rangle = u. & (\pi_2) & \pi_2\langle u,\, v\rangle = v. \\
(\eta) & \lambda x.\, tx = t, \quad (x \in \mathrm{FV}(t).) & & \\
(SP) & \langle \pi_1 u,\, \pi_2 u\rangle = u. & & \\
(c) & s^\top = *^\top. & &
\end{array}
$$

By the last equality, the type $\top$ corresponds to the singleton. The singleton does to the terminal object of a cartesian closed category. So $\top$ is called the *terminal type*.

A confluent (CR for short) and weakly normalizable (WN for short) reduction system generating this equational theory $\lambda\beta\eta\pi*$ is important in relation to the coherence problem of cartesian closed categories [28, 29]. By orienting the axioms $(\beta), (\pi_1), (\pi_2), (\eta), (SP)$ left

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to right, we obtain rewriting rule schemata. Let (T) be a rewrite rule schema $s^\top \to *^\top$ ($s^\top \not\equiv *^\top$). Here for terms t and s, we write $t \equiv s$, provided that by renaming bound variables, t becomes identical to s. Let $\to$ be the closure of these rewriting rule schemata by contexts. By abuse of notation, we write $\lambda\beta\eta\pi*$ for a so-obtained rewriting system. The reverse of $\to$ is denoted by $\leftarrow$. $\xrightarrow{*}$ is the reflexive, transitive closure of $\to$. The rewriting system $\lambda\beta\eta\pi*$ is *not* CR, as follows: In each line of the following, x and y are variables, and it is not the case that there is a term t_0 such that $t_1 \xrightarrow{*} t_0 \xleftarrow{*} t_2$:

$$\begin{aligned}
y^{\varphi\to\top} &\leftarrow \lambda x.\,(yx)^\top \to \lambda x.\,*^\top,\\
x &\leftarrow \langle(\pi_1 x)^\top,\,(\pi_2 x)^\top\rangle \to\to \langle *^\top,\,*^\top\rangle,\\
\lambda x^\top.\,y* &\leftarrow \lambda x^\top.\,yx^\top \to y^{\top\to\varphi},\\
\langle \pi_1 x,\,*\rangle &\leftarrow \langle(\pi_1 x)^\varphi,\,(\pi_2 x)^\top\rangle \to x^{\varphi\times\top},\\
\langle *,\,\pi_2 x\rangle &\leftarrow \langle(\pi_1 x)^\top,\,(\pi_2 x)^\varphi\rangle \to x^{\top\times\varphi}.
\end{aligned} \tag{1}$$

If we omit the rewrite rule schemata (T), then the resulting rewrite relation $\to_{\beta\eta\pi\pi_1\pi_2 SP}$ is CR [31]. In the type-free setting, $\to_{\beta SP}$ is not CR [23]. The decidability of the equational theory $\lambda\beta\eta\pi*$ follows from

- Sarkar's algorithm. For an extension of the well-known LF type theory with dependent pair and unit types, Sarkar [32] provided an algorithm that decides type checking and he proved the existence of canonical forms, by using standard techniques introduced by Harper and Pfenning [16].
- A translation that incorporates type-directed expansions by type-indexed functions on terms. The translation reduces the decidability of the equational theory $\lambda\beta\eta\pi*$ to that of the corresponding intensional equational theory, as in [14, 38]. This idea, however, does not yield a decision procedure for the polymorphic equational theory.

1.1 Why do we insist on η-reduction instead of type-directed η-expansion?

For *Unifying Theory of Dependent Types* (UTT for short) (Luo [26]), Goguen [13] defined the *typed operational semantics* (TOS for short). By employing the TOS, he investigated various decidability properties of UTT. Here UTT is Martin-Löf's Logical Framework extended by a general mechanism for inductive types, a predicative universe and an impredicative universe of propositions. Goguen's TOS defines a reduction to normal form for terms which are well-typed in UTT. "Since his approach is based on η-reduction instead of η-expansion, it is not clear whether it scales to a unit type with extensional equality." ([1]). The unit type is exactly the terminal type, and is related to types for enumeration sets and types for proof irrelevance [2].

The TOS is an intermediate induction principle for reducibility proofs. By using Curien-Di Cosmo's idea, we could hopefully formulate the reduction system for UTT+unit type, so that the TOS for UTT+unit type can be defined, where all terms of unit type reduce to the unique inhabitant, and then used to show termination.

We conjecture that for each term t, the minimum length of the normalization sequence from t with respect to $(\lambda\beta\eta\pi*)'$ is smaller than the minimum length of the normalization sequence from t with respect to $\beta\pi_1\pi_2 T$-reduction union the type-directed ηSP-expansions. If so, a type-checker of dependent type theories perhaps run faster by using $(\lambda\beta\eta\pi*)'$ instead of type-directed expansions, since a type-checker tests term equivalence.

1.2 Curien and Di Cosmo's rewriting system based on η-reduction

For the equational theory $\lambda\beta\eta\pi *$, Curien and Di Cosmo, inspired by completion of TRS, introduced a rewriting system $(\lambda\beta\eta\pi *)'$ in [9]. First they inductively defined the types *"isomorphic to"* the terminal type $\top$ and the *canonical terms* of such types.

▶ **Definition 1** ($(\lambda\beta\eta\pi *)'$).

- $\top$ is "isomorphic to" $\top$ and the canonical term of $\top$ is $*^{\top}$.
- Suppose φ is a type and τ is a type "isomorphic to" $\top$. Then the type $\varphi \to \tau$ is "isomorphic to" $\top$ and the canonical term $*^{\varphi\to\tau}$ of $\varphi \to \tau$ is $\lambda x^{\varphi}. *^{\tau}$.
- If each type τ_i is "isomorphic to" $\top$ $(i = 1, 2)$, then the type $\tau_1 \times \tau_2$ is "isomorphic to" $\top$ and the canonical term $*^{\tau_1\times\tau_2}$ of $\tau_1 \times \tau_2$ is $\langle *^{\tau_1}, *^{\tau_2}\rangle$.

The set of types "isomorphic to" $\top$ is denoted by $\mathit{Iso}(\top)$.

Whenever we write $*^{\varphi}$, we tacitly assume $\varphi \in \mathit{Iso}(\top)$.

The rewrite relation $\to$ of the rewriting system $(\lambda\beta\eta\pi *)'$ is defined by the rewrite rule schemata obtained from the first five equational axioms $(\beta), (\pi_1), (\pi_2), (\eta)$, and (SP) of $\lambda\beta\eta\pi *$ by orienting left to right, and the following four rewrite rule schemata:

$$
\begin{array}{lcr}
(g) & u^{\tau} \to *^{\tau}, & (u \text{ is not canonical.}) \\
(\eta_{top}) & \lambda x^{\tau}. t *^{\tau} \to t, & (x \notin \mathrm{FV}(t).) \\
(SP_{top1}) & \langle \pi_1 u, *^{\tau}\rangle \to u, & (u \text{ has type } \varphi \times \tau.) \\
(SP_{top2}) & \langle *^{\tau}, \pi_2 u\rangle \to u, & (u \text{ has type } \tau \times \psi.)
\end{array}
$$

g stands "gentop." In [9], Curien and Di Cosmo proved that the rewriting system $(\lambda\beta\eta\pi *)'$ is CR and WN, by using an ingenuous lemma for abstract reduction system. $(\lambda\beta\eta\pi *)'$ is non-left-linear and has a rewrite rule schema with side conditions. We cannot apply criteria for CR of left-linear (higher-order) term rewriting system based on closed condition of (parallel) critical pairs (e.g., [35, 36]). $\beta\eta\eta_{top}T$-reduction is the *triangulation* [37] of $\beta\eta T$-reduction, and thus CR by [37, Corollary 2.6]. However, $(\lambda\beta\eta\pi *)'$ is not a triangulation of the rewriting system $\lambda\beta\eta\pi *$. As we see (1), g-rule schema rewrites the one-step reduct $u^{\top\times\top}$ of $\langle \pi_1 u, \pi_2 u\rangle$ to the *two-step* reduct of $\langle \pi_1 u, \pi_2 u\rangle$. This does not fit to the definition of the triangulation.

1.3 Curien and Di Cosmo's attempted to prove SN of their rewriting system

All variations (e.g., Girard [12], Blanqui (computability closure [6])) of Tait's reducibility method require to show a key statement like *"if $v[x := u]$ is reducible for all reducible u, then $\lambda x. v$ is reducible,"* where we say a term t of type $\varphi \to \psi$ is reducible if for all reducible term u of type φ, tu is reducible. An auxiliary property which is available is that, a term tu is reducible, as soon as s is reducible for all reducts s of tu, for example, in [12]. So the proof of the key statement amounts to the proof that all reducts of $(\lambda x. v)u$ are reducible. Now, if $v \equiv (v' *)$ with $x \notin \mathrm{FV}(v')$, then the rule schema (η_{top}) can rewrite $(\lambda x. v)u$ to $(v'u)$ which is not $v[x := u] \equiv v$, and we do not know if $(v'u)$ *is reducible.*

Curien and Di Cosmo proved SN of g-normal forms without the second-order β-rewrite rule schema (β^2) but with the second-order η-rewrite rule schema (η^2), and SN of all the terms with both (β^2) and (η^2) but without (η_{top}), (SP_{top1}) and (SP_{top2}). But these do not lead to SN of all terms for the full reduction.

1.4 Our SN proof of Curien-Di Cosmo's reduction systems

To carry out Girard's reducibility [12] for $(\lambda\beta\eta\pi*)'$,

1. We prove

   (CR0) "every canonical term is reducible,"

   besides Girard's three properties (CR1) ("the reducibility implies SN."), (CR2) ("the reducibility is closed under $\rightarrow$."), and (CR3).
2. In the reducibility method for $(\lambda\beta\eta\pi*)'$, as a sufficient condition for $\lambda x.\, v$ to be reducible, we consider the conjunction of (a) $v[x := u]$ is reducible for all reducible u and (b) v' is reducible whenever $v \equiv v'*$.
3. We will restrict the reducibility theorem "all terms substituted by reducible terms are reducible" to $*$-*free* terms.

▶ **Definition 2.** Let t be a term of $(\lambda\beta\eta\pi*)'$.

1. t is called $*$-*free*, if the term constant $*^{\top}$ does not occur in t.
2. Let $\tilde{t}$ be a $*$-free term such that all the occurrences of $*^{\top}$ in t is replaced by variables $x^{\top}$.

None of the problematic rewrite rule schemata (η_{top}), (SP_{top1}), and (SP_{top2}) applies for a $*$-free term. So we can prove the restricted reducibility theorem, as usual, by induction on t, but we use

▶ **Lemma 3.** *Let t be a term of $(\lambda\beta\eta\pi*)'$. If $\tilde{t}$ is reducible, so is t.*

Proof. By $\tilde{t} \overset{*}{\rightarrow}_T t$ and (CR2). ◀

From the restricted reducibility theorem, we derive the reducibility of *all* terms, again by Lemma 3. The condition (CR1) establishes SN of $(\lambda\beta\eta\pi*)'$. Thus, we reduced SN of the terms to SN of the same rewriting relation for the $*$-free terms. This kind of trick to restrict terms is also found in the normalization by evaluation for a dependent type theory with enumeration sets and types for proof irrelevance ([2]).

The rest of paper is organized as follows: In Section 2, we prove SN of $(\lambda\beta\eta\pi*)'$. In Section 3, we prove (1) SN of $\left(\lambda^2\beta\eta\pi*\right)'$, the extension by the second-order $\beta\eta$-rewriting and (2) SN of $\left(\lambda^2\beta\eta\pi*\right)''$, the extension by the second-order $\beta\eta$-rewriting where we also consider terminal types caused by parametric polymorphism [17]. In Section 4, we show the worst-case derivational complexity of $(\lambda\beta\eta\pi*)'$ is smaller than that of so-called type-direwected expansions. For type-directed expansions, see [28, 29, 15, 3, 8, 10, 20, 25], to cite a few. In the appendix, we prove SN of $\left(\lambda^{\top,\rightarrow,\times,+}\right)'$, the extension of $(\lambda\beta\eta\pi*)'$ by sum types with weak extensionality.

2 SN proof of $(\lambda\beta\eta\pi*)'$, by restricted reducibility theorem

In our SN proofs, we will use a *well-founded induction on a well-founded relation.* A *well-founded* relation is, by definition, $\mathcal{A} = (A, \succ)$ such that $\emptyset \neq \succ \subseteq A \times A$ and there is no infinite chain $a \succ a' \succ a'' \succ \cdots$. The *well-founded induction* on a well-founded relation $\mathcal{A} = (A, \succ)$ is, by definition,

$$\mathrm{WFI}\,(\mathcal{A}): \quad \forall P \subseteq A \left[\forall x \in A\big(\forall x'\,(x \succ x' \Rightarrow x' \in P) \implies x \in P\big) \implies \forall x \in A(x \in P)\right].$$

We call the subformula $\forall x'\,(x \succ x' \Rightarrow x' \in P)$ the *WF induction hypothesis.* For $n \geq 1$ well-founded relations $\mathcal{A}_i = (A_i, \succ_i)$ $(i = 1, \dots, n)$, we define a binary relation $\mathcal{A}_1 \# \cdots \# \mathcal{A}_n = (A_1 \times \cdots \times A_n,\ \succ_1 \# \cdots \# \succ_n)$ by: $(x_1, \dots, x_n)\ (\succ_1 \# \cdots \# \succ_n)\ (y_1, \dots, y_n)$, if there exists i such that $x_i \succ_i y_i$ but $x_j = y_j$ $(j \neq i)$. Then $\mathcal{A}_1 \# \cdots \# \mathcal{A}_n$ is a well-founded relation.

If the redex of $t \to t'$ is Δ, we write $t \stackrel{\Delta}{\to} t'$. Below, "$\subseteq$" reads "is a subterm occurrence of."

▶ **Definition 4** (Neutral [12]). A term is called *neutral* if it is not of the form $\langle u, v\rangle$ or $\lambda x.\, v$.

By an *atomic type*, we mean the distinguished type constant $\top$ or a type variable.

▶ **Definition 5** (Reducibility [12]).
(a) A term of atomic type is *reducible*, if the term is SN.
($\times$) A term $t^{\varphi\times\psi}$ is *reducible*, if so are $(\pi_1 t)^\varphi$ and $(\pi_2 t)^\psi$.
($\to$) A term $t^{\varphi\to\psi}$ is *reducible*, if for any reducible term u^φ, $(tu)^\psi$ is reducible.

We state and prove four properties (CR0), (CR1), (CR2) and (CR3) of the reducibility, where the last three are the same as those Girard proved [12] for $\beta\pi_1\pi_2$-reduction.

▶ **Lemma 6.** *Let t^φ be a term.*
(CR0) *If t is canonical, then t is reducible.*
(CR1) *If t is reducible, then t is SN.*
(CR2) *if t is reducible and $t \to t'$, then t' is reducible.*
(CR3) *if t is neutral, and t' is reducible whenever $t \to t'$, then t is reducible.*
To prove Lemma 6, we first note the following:

▶ **Lemma 7.** *By (CR0) and (CR3), we have*
(CR4) *If t is a variable, then t is reducible.*

Proof. Let $t \to t'$. Then t' is canonical. By (CR0), t' is reducible. By (CR3), t is too. ◀

Proof of Lemma 6. By induction on φ.
φ is atomic.
(CR0) t is $*^\top$, and SN. So t is reducible. (CR1) is clear. (CR2) As t is SN, so is every reduct t' of t. (CR3) If all reducts are SN, then it is SN.
$\varphi = \varphi_1 \times \varphi_2$.
(CR0) As $*^{\varphi_1\times\varphi_2}$ is a normal form $\langle *^{\varphi_1}, *^{\varphi_2}\rangle$, the reduct of $\pi_i *^{\varphi_1\times\varphi_2}$ is $*^{\varphi_i}$, which is reducible by induction hypothesis (CR0). By induction hypothesis (CR3) for φ_i, $\pi_i *^{\varphi_1\times\varphi_2}$ is reducible. Hence $*^{\varphi_1\times\varphi_2}$ is reducible.
(CR1) Suppose that t is reducible. Then $\pi_i t$ is reducible. By induction hypothesis (CR1) for φ_i, $\pi_i t$ is SN. So t is SN.
(CR2) If $t \to t'$, then $\pi_i t \to \pi_i t'$. As t is reducible by hypothesis, so are $\pi_i t$. By induction hypothesis (CR2) for φ_i, $\pi_i t'$ is reducible, and so t' is reducible.
(CR3) Let $\pi_i t \stackrel{\Delta}{\to} s$. We have two cases.

1. $\Delta \equiv \pi_i t$: Then $\varphi_i \in \mathit{Iso}(\top)$ and $s \equiv *^{\varphi_i}$, because t is neutral. By induction hypothesis (CR0) for φ_i, s is reducible.
2. $\Delta \not\equiv \pi_i t$: Then $s \equiv \pi_i t'$ for some t' such that $t \to t'$. By the hypothesis, t' is reducible. So s is reducible. $\pi_i t$ is neutral, and all the terms s with $\pi_i t \to s$ are reducible. By induction hypothesis (CR3) for φ_i, $\pi_i t$ is reducible. Hence t is reducible.

$\varphi = \varphi_1 \to \varphi_2$.
(CR0) Let u be a reducible term of type φ_1. By induction hypothesis (CR1) for φ_1, u is SN. So we can use WFI $((\{u^{\varphi_1} \mid u^{\varphi_1} \text{ is reducible}\}, \to))$ where $\to$ is the rewrite relation. We will verify that $*^{\varphi_1\to\varphi_2} u$ is reducible. Suppose $*^{\varphi_1\to\varphi_2} u \stackrel{\Delta}{\to} s$. As $*^{\varphi_1\to\varphi_2}$ is in normal form, we have two cases.

1. $\Delta \equiv *^{\varphi_1 \to \varphi_2} u$: Then $s \equiv *^{\varphi_2}$ is reducible by induction hypothesis (CR0) for φ_2.
2. Otherwise, $s \equiv *^{\varphi_1 \to \varphi_2} u'$ with $u \to u'$. Then u' is reducible by induction hypothesis (CR2) for φ_1. So, by the WF induction hypothesis, $s \equiv *^{\varphi_1 \to \varphi_2} u'$ is reducible.

In any case, the neutral term $*^{\varphi_1 \to \varphi_2} u$ rewrites to reducible terms only. By induction hypothesis (CR3) for φ_2, $*^{\varphi_1 \to \varphi_2} u$ is reducible. So $*^{\varphi_1 \to \varphi_2}$ is reducible.

(CR1) By induction hypothesis (CR4), a variable x^{φ_1} is reducible. So tx is reducible. Hence t is SN.

(CR2) Let u be a reducible term of type φ_1. Then tu is reducible and $tu \to t'u$. By the induction hypothesis (CR2) for φ_2, $t'u$ is reducible. So t' is reducible.

(CR3) Assume t be neutral and suppose all the t' with $t \to t'$ are reducible. Let u be a reducible term of type φ_1. By induction hypothesis (CR1) for φ_1, u is SN. So by $\mathrm{WFI}\left(\left(\{u^{\varphi_1} \mid u^{\varphi_1} \text{ is reducible}\}, \to\right)\right)$, we will verify that tu is reducible.

Suppose $tu \overset{\Delta}{\to} s$. We will show that s is reducible. As t is neutral, we have three cases.

1. $\Delta \equiv tu$: Then, $s \equiv *^{\varphi_2}$ is reducible, by induction hypothesis (CR0) for φ_2.
2. $\Delta \subseteq t$: Then, $s \equiv t'u$ with $t \to t'$. $t'u$ is reducible, because t' is by the assumption,
3. Otherwise, $s \equiv tu'$ with $u \to u'$. Then, u' is reducible by induction hypothesis (CR2) for φ_1. So, by the WF induction hypothesis, tu' is reducible.

In any case, the neutral term tu rewrites to reducible terms only. By induction hypothesis (CR3) for φ_2, tu is reducible. So t is reducible. This completes the proof of Lemma 6. ◀

For pairings and λ-abstractions to be reducible, we consider a sufficient condition stronger than that used in standard reducibility methods (e.g., [12]). In view of the rules (SP_{top1}), (SP_{top2}), and (η_{top}), we newly consider (1(b)), (1(c)) and (2(b)).

▶ **Lemma 8.**

1. *Let u^{φ}, v^{ψ} be any terms. $\langle u^{\varphi}, v^{\psi}\rangle$ is reducible, provided that*
   (a) *u and v are both reducible;*
   (b) *if $u \equiv \pi_1 w$ and $v \equiv *^{\psi}$, then w is reducible; and*
   (c) *if $v \equiv \pi_2 w$ and $u \equiv *^{\varphi}$, then w is reducible.*
2. *Let v^{ψ} be any term. $\lambda x^{\varphi}. v^{\psi}$ is reducible, provided that*
   (a) *$v^{\psi}[x^{\varphi} := u^{\varphi}]$ is reducible for every reducible, possibly non-$*$-free term u^{φ}; and*
   (b) *if $v \equiv w^{\varphi \to \psi} *^{\varphi}$ and $x^{\varphi} \notin \mathrm{FV}(w^{\varphi \to \psi})$, then $w^{\varphi \to \psi}$ is reducible.*

Proof. (1) By the premise and (CR1), u and v are both SN. We can use

$$\mathrm{WFI}\left(\left(\{u^{\varphi} \mid u^{\varphi} \text{ is reducible}\}, \to\right) \# \left(\left\{v^{\psi} \mid v^{\psi} \text{ is reducible}\right\}, \to\right)\right) \tag{2}$$

where $\to$ is the rewrite relation. We will verify that $\pi_1\langle u, v\rangle$ is reducible. Let $\pi_1\langle u, v\rangle \overset{\Delta}{\to} s$. We will prove that s is reducible, by case analysis. We will exhaust the positions of the redexes Δ in $\pi_1\langle u, v\rangle$ from left to right, and the rewrite rule schemata of $\overset{\Delta}{\to}$. We have eight cases.

1. $\Delta \equiv \pi_1\langle u, v\rangle$ is a redex of the rewrite rule (g) and $s \equiv *^{\varphi}$: Then s is reducible by (CR0).
2. $\Delta \equiv \pi_1\langle u, v\rangle$ is a redex of the rewrite rule (π_1) and $s \equiv u$: Then s is reducible by the hypothesis (1(a)).
3. $\Delta \equiv \langle u, v\rangle$ is a redex of (g) and $s \equiv \pi_1(*^{\varphi \times \psi})$: Then $*^{\varphi \times \psi}$ is reducible by (CR0). By the definition of the reducibility for the product type, $s \equiv \pi_1(*^{\varphi \times \psi})$ is reducible.
4. $\Delta \equiv \langle u, v\rangle$ is a redex of (SP) and $s \equiv \pi_1 w$: Then $u \equiv \pi_1 w$ and $v \equiv \pi_2 w$. $s \equiv \pi_1 w$ is reducible by the hypothesis (1(a)).
5. $\Delta \equiv \langle u, v\rangle$ is a redex of (SP_{top1}) and $s \equiv \pi_1 w$: Then $u \equiv \pi_1 w$ and $v \equiv *^{\psi}$. $s \equiv \pi_1 w$ is reducible by the hypothesis (1(b)).

6. $\Delta \equiv \langle u,\, v\rangle$ is a redex of (SP_{top2}) and $s \equiv \pi_1 w$: Then $v \equiv \pi_2 w$ and $u \equiv *^{\varphi}$. $s \equiv \pi_1 w$ is reducible by the hypothesis (1(c)).
7. $\Delta \subseteq u$: Then $s \equiv \pi_1\langle u',\, v\rangle$ with $u \to u'$. u' is reducible by (1(a)) and (CR2). By the WF induction hypothesis, $s \equiv \pi_1\langle u',\, v\rangle$ is reducible.
8. $\Delta \subseteq v$: Then $s \equiv \pi_1\langle u,\, v'\rangle$ with $v \to v'$. v' is reducible by (1(a)) and (CR2). By the WF induction hypothesis, $s \equiv \pi_1\langle u,\, v'\rangle$ is reducible.

In every case, the neutral term $\pi_1\langle u,\, v\rangle$ rewrites to reducible terms only, and by (CR3), $\pi_1\langle u,\, v\rangle$ is reducible. We can similarly prove that $\pi_2\langle u,\, v\rangle$ is reducible. So $\langle u,\, v\rangle$ is reducible.

(2) By (CR4), x^{φ} is reducible. So v^{ψ} is, by the premise. Let u^{φ} be a reducible, possibly non-$*$-free term. By (CR1), both of u, v are SN. By (2), we will verify that $(\lambda x.\, v)u$ is reducible. Assume $(\lambda x.\, v)u \overset{\Delta}{\to} s$. We will exhaust the positions of the redex Δ in $(\lambda x.\, v)u$ from left to right, and the rewrite rule schemata of $\overset{\Delta}{\to}$. Then we have seven cases:

1. $\Delta \equiv (\lambda x.\, v)u$ is a redex of (g) and $s \equiv *^{\psi}$: Then s is reducible by (CR0).
2. $\Delta \equiv (\lambda x.\, v)u$ is a redex of (β) and $s \equiv v[x := u]$: Then s is reducible by hypothesis (2(a)).
3. $\Delta \equiv \lambda x.\, v$ is a redex of (g) and $s \equiv *^{\varphi\to\psi}u$: As $*^{\varphi\to\psi}$ is reducible by (CR0), so is s.
4. $\Delta \equiv \lambda x.\, v$ is a redex of (η) and $s \equiv v[x := u]$: Then, this case is case 2.
5. $\Delta \equiv \lambda x.\, v$ is a redex of (η_{top}) and $s \equiv wu$ with $v \equiv w*^{\varphi}$ and $x \notin \mathrm{FV}(w)$: Then, since w is reducible by hypothesis (2(b)), $s \equiv wu$ is reducible.
6. $\Delta \subseteq v$ and $s \equiv (\lambda x.\, v')u$ with $v \to v'$: Then, by (CR2), v' is reducible. By the WF induction hypothesis, $s \equiv (\lambda x.\, v')u$ is reducible.
7. $\Delta \subseteq u$ and $s \equiv (\lambda x.\, v)u'$ with $u \to u'$: Then, by (CR2), u' is reducible. By the WF induction hypothesis, $s \equiv (\lambda x.\, v)u'$ is reducible.

In every case, the neutral term $(\lambda x.\, v)u$ reduces to reducible terms only. So, by (CR3), $(\lambda x.\, v)u$ is reducible. Hence $\lambda x.\, v$ is reducible. ◀

In the following two theorems, we use Lemma 3. For a term t, a sequence $\vec{x}$ of distinct variables $x_1^{\varphi_1}, \dots, x_n^{\varphi_n}$, and a sequence $\vec{u}$ of terms $u_1^{\varphi_1}, \dots, u_n^{\varphi_n}$, let $t[\vec{x} := \vec{u}]$ be the simultaneous substitution.

▶ **Theorem 9** (Restricted Reducibility). *Assume that*
1. t *is a* $*$-free *term;*
2. *a sequence of distinct variables* $x_1^{\varphi_1}, \dots, x_n^{\varphi_n}$ *contains all free variables of* t*; and*
3. $u_i^{\varphi_i}$ *is reducible and* $*$-free $(i = 1, \dots, n)$.

Then $t[x_1^{\varphi_1}, \dots, x_n^{\varphi_n} := u_1^{\varphi_1}, \dots, u_n^{\varphi_n}]$ *is reducible.*

Proof. We prove that $t\,[\,\vec{x} := \vec{u}\,]$ is reducible, by induction on t. As t is $*$-free, $t \not\equiv *^{\top}$. So, we have five cases.

1. $t \equiv x_i$: Then $t\,[\,\vec{x} := \vec{u}\,] \equiv u_i$. Immediate.
2. $t \equiv \pi_i w$ $(i = 1, 2)$: Then by induction hypothesis, $w\,[\,\vec{x} := \vec{u}\,]$ is reducible. So each $\pi_i(w\,[\,\vec{x} := \vec{u}\,])$ is reducible. This term is $\pi_i w\,[\,\vec{x} := \vec{u}\,] \equiv t\,[\,\vec{x} := \vec{u}\,]$.
3. $t \equiv \langle u,\, v\rangle$: Then $t\,[\,\vec{x} := \vec{u}\,] \equiv \langle u\,[\,\vec{x} := \vec{u}\,],\, v\,[\,\vec{x} := \vec{u}\,]\rangle$. By the induction hypotheses, both $u\,[\,\vec{x} := \vec{u}\,]$ and $v\,[\,\vec{x} := \vec{u}\,]$ are $*$-free and reducible. By Lemma 8(1), $t\,[\,\vec{x} := \vec{u}\,]$, that is, $\langle u\,[\,\vec{x} := \vec{u}\,],\, (v\,[\,\vec{x} := \vec{u}\,])\rangle$, is reducible.
4. $t \equiv wv$: Then by induction hypotheses $w\,[\,\vec{x} := \vec{u}\,]$ and $v\,[\,\vec{x} := \vec{u}\,]$ are reducible, and so (by definition) is $w\,[\,\vec{x} := \vec{u}\,]\,(v\,[\,\vec{x} := \vec{u}\,])$; but this term is $t\,[\,\vec{x} := \vec{u}\,]$.
5. $t \equiv \lambda y^{\varphi}.\, w^{\psi}$ with y not free in any $\vec{x}, \vec{u}$: Then $t\,[\,\vec{x} := \vec{u}\,] \equiv \lambda y.\,(w\,[\,\vec{x} := \vec{u}\,])$. Let u^{φ} be a reducible, possibly non-$*$-free term. w and $\tilde{u}$ are $*$-free. By induction hypothesis, a $*$-free term $w[\vec{x}, y := \vec{u}, \tilde{u}] \equiv w\,[\,\vec{x} := \vec{u}\,]\,[y := \tilde{u}]$, is reducible. The last term is $\tilde{v}$ where

$v \equiv w\,[\,\vec{x} := \vec{u}\,]\,[y := u]$, because $w, \vec{u}$ are $*$-free. By Lemma 3, $v \equiv w\,[\,\vec{x} := \vec{u}\,]\,[y := u]$ is reducible. $w\,[\,\vec{x} := \vec{u}\,]$ is $*$-free. So, by Lemma 8(2), $t\,[\,\vec{x} := \vec{u}\,] \equiv \lambda y.\,(w\,[\,\vec{x} := \vec{u}\,])$ is reducible.

Hence we have established the restricted reducibility theorem. ◀

Again we use Lemma 3.

▶ **Theorem 10.** *All terms of* $(\lambda\beta\eta\pi*)'$ *are reducible.*

Proof. Let t be a term. The $*$-free term $\tilde{t}$ is reducible, by (CR4) and by Theorem 9 with $u_i := x_i$, the identity substitution. By Lemma 3, t is reducible. ◀

▶ **Corollary 11.** $(\lambda\beta\eta\pi*)'$ *satisfies SN.*

Proof. By (CR1) and Theorem 10, every term of $(\lambda\beta\eta\pi*)'$ is SN. ◀

▶ Remark. The ordinal number assignment of Howard [18] (Schütte [34], resp.) to typed λ-terms (typed combinators, resp.) proves SN of typed β-reduction (typed combinatory reduction, resp.). Beckmann used cut-elimination procedure [5] of a deduction system to give an optimal upper bound of typed $\beta\eta$-reduction. But these two proofs seem not to generalize for SN of the rewriting system $(\lambda\beta\eta\pi*)'$. In these two proofs, it is not the case that (1) $r*^\tau > r$ and (2) the LHS $\lambda x^\tau.\, t*^\tau$ $(x \notin \mathrm{FV}(t))$ of the rewrite rule schema (η_{top}) is greater than the RHS t.

One may suppose that the higher-order recursive path ordering (HORPO for short) [21] or the General Schema [7], could be extended with surjective pairing and hence be used for proving SN of $(\lambda\beta\eta\pi*)'$. If there is a convenient translation of the rewrite rule schemata $(g), (\eta_{top})$, and (SP_{top}) with type-abstraction to an infinite simply-typed system, such that the translation can also put all the rules of $(\lambda\beta\eta\pi*)'$ in the right kind of format, it is possible that a HORPO-variant (with minimal symbol *) may handle $(\lambda\beta\eta\pi*)'$. However, we need a new HORPO variant, since the conventional ones are troubled with the non-left-linear (SP)-rule `pair(p1(X), p2(X))->X`. There is no type ordering that allows for the extraction of `X` from terms of smaller type in general. The top rule (g): $u^\tau \to *^\tau$ $(\tau \in \mathit{Iso}(\top),\ u \not\equiv *^\tau)$ is also problematic for most HORPO-variants. It could be handled by using a variation of HORPO with minimal symbols, such as the one used in WANDA [24]. Here, WANDA is one of the most powerful automatic termination provers for higher-order rewriting.

3 SN proof of polymorphic extensions by restricted reducibility theorem

In [9], Curien and Di Cosmo introduced the polymorphic extension $\lambda^2\beta\eta\pi*$ of the equational theory $\lambda\beta\eta\pi*$, and the polymorphic extension $\left(\lambda^2\beta\eta\pi*\right)'$ of the rewriting system $(\lambda\beta\eta\pi*)'$. We introduce an extension $(\lambda^2\beta\eta\pi*)''$ of $(\lambda\beta\eta\pi*)'$ by *polymorphism* and *terminal types caused by parametric polymorphism* [17].

▶ **Definition 12** $\left(\left(\lambda^2\beta\eta\pi*\right)'\right)$**.** We will first recall the equational theory $\lambda^2\beta\eta\pi*$ of the polymorphic terms. The types are generated from type variables $X, Y, \dots$ and the distinguished type constant $\top$ by means of the product type $\varphi \times \psi$, the function type $\varphi \to \psi$, and the $\Pi X.\,\varphi$. Terms are built up from the distinguished term constant $*^\top$ and term variables $x^\varphi, y^\varphi, \dots, x^\psi, y^\psi, \dots$ by means of λ-abstraction $(\lambda x^\varphi.\, t^\psi)^{\varphi\to\psi}$, term application $(u^{\varphi\to\psi} v^\varphi)^\psi$, pairing $\langle u^\varphi,\, v^\psi\rangle^{\varphi\times\psi}$, left-projection $(\pi_1 t^{\varphi\times\psi})^\varphi$, right-projection $(\pi_2 t^{\varphi\times\psi})^\psi$,

- *universal abstraction*: if v^φ is a term, then so is $(\Lambda X.\, v^\varphi)^{\Pi X.\,\varphi}$, whenever the variable X is not free in the type of a free variable of v^φ; and
- *universal application*: if $t^{\Pi X.\,\varphi}$ and ψ is a type, then so is $\left(t^{\Pi X.\,\varphi}\psi\right)^{\varphi[X:=\psi]}$.

The superscript representing the type is often omitted. The axioms of the equational theory $\lambda^2\beta\eta\pi*$ are those of $\lambda\beta\eta\pi*$ and the following two:

$$(\beta^2)\ \ (\Lambda X.\,t)\varphi = t[X := \varphi]. \qquad\qquad (\eta^2)\ \ \Lambda X.\,tX = t,\ \ (X \text{ does not occur free in } t).$$

For the definition of $\mathit{Iso}(\top)$ for $\left(\lambda^2\beta\eta\pi*\right)'$, Curien and Di Cosmo added the following clause to the inductive definition of $\mathit{Iso}(\top)$ for $(\lambda\beta\eta\pi*)'$:
$(\top\Pi)$ If τ is "isomorphic to" $\top$, so is $\Pi X.\,\tau$. The canonical term $*^{\Pi X.\,\tau}$ of $\Pi X.\,\tau$ is $\Lambda X.\,*^{\tau}$.

The rewrite rule schemata of the rewriting system $\left(\lambda^2\beta\eta\pi*\right)'$ are those of the rewriting system $(\lambda\beta\eta\pi*)'$ and those obtained from (β^2) and (η^2) by orienting left to right. This completes the definition of $\left(\lambda^2\beta\eta\pi*\right)'$.

Taking the parametricity of the polymorphism [17] into account, we add the following clause to the inductive definition of $\mathit{Iso}(\top)$:
$(\top^{par})$ For every $n \geq 0$, if $\tau_1, \ldots, \tau_n$ are "isomorphic to" $\top$, so is $\Pi X.\,((\tau_1 \to \cdots \to \tau_n \to X) \to X)$. The canonical term $*^{\Pi X.\,((\tau_1 \to \cdots \to \tau_n \to X) \to X)}$ of $\Pi X.\,((\tau_1 \to \cdots \to \tau_n \to X) \to X)$ is $\Lambda X.\lambda x^{\tau_1 \to \cdots \tau_n \to X}.\,x\, *^{\tau_1} \cdots *^{\tau_n}$.
As (g)-rule schema applies for more terms in $\left(\lambda^2\beta\eta\pi*\right)''$ than in $\left(\lambda^2\beta\eta\pi*\right)'$, SN of $\left(\lambda^2\beta\eta\pi*\right)''$ implies SN of $\left(\lambda^2\beta\eta\pi*\right)'$. We will prove SN of $\left(\lambda^2\beta\eta\pi*\right)''$.

In [9], to show SN of the rewriting system $\left(\lambda^2\beta\eta\pi*\right)'$, Curien and Di Cosmo tried to prove that every term of $\left(\lambda^2\beta\eta\pi*\right)'$ in the g-normal form is SN. But they observed that the set of g-normal form is not closed under β^2-reduction; $(\Lambda X.\,\lambda x^{X}.\,\lambda y^{X\to Y}.\,yx)\top$ is in g-normal form, but its reduct $u \equiv \lambda x^{\top}.\,\lambda y^{\top \to Y}.\,yx$ is not, as $u \to_g \lambda x^{\top}.\,\lambda y^{\top\to Y}.\,y*$.

▶ **Definition 13** (Neutral)**.** A term is *neutral* if it is not of the form $\langle u,\, v\rangle$, $\lambda x.\,v$, or $\Lambda X.\,u$.

As in $(\lambda\beta\eta\pi*)'$, we consider (CR0) to define a *reducibility candidate* [12].

▶ **Definition 14.** A *reducibility candidate* (RC for short) of type φ is a set $\mathcal{R}$ of terms of type φ such that:
(CR0) If $\varphi \in \mathit{Iso}(\top)$, then $*^{\varphi} \in \mathcal{R}$.
(CR1) If $t^{\varphi} \in \mathcal{R}$, then t^{φ} is SN.
(CR2) If $t^{\varphi} \in \mathcal{R}$ and $t^{\varphi} \to t'$, then $t' \in \mathcal{R}$.
(CR3) If t^{φ} is neutral, and any reduct of t^{φ} is in $\mathcal{R}$, then $t^{\varphi} \in \mathcal{R}$.

▶ **Lemma 15.** *(CR0) and (CR3) implies*
(CR4) *If t^{φ} is a variable, then t is in $\mathcal{R}$.*

▶ **Definition 16.**
1. Let $\mathcal{SN}^{\psi}$ be the set of SN terms of type ψ.
2. For an RC $\mathcal{R}$ of type φ and an RC $\mathcal{S}$ of type ψ, define

$$\mathcal{R} \times \mathcal{S} = \{t^{\varphi\times\psi} \mid \pi_1 t \in \mathcal{R},\ \pi_2 t \in \mathcal{S}\},\ \text{and}\ \mathcal{R} \to \mathcal{S} = \{t^{\varphi\to\psi} \mid \forall u(u \in \mathcal{R} \implies tu \in \mathcal{S})\}.$$

▶ **Lemma 17.**
1. *For any type ψ, $\mathcal{SN}^{\psi}$ is an RC.*
2. *If $\mathcal{R}, \mathcal{S}$ are RCs of type φ, ψ, then $\mathcal{R} \times \mathcal{S}, \mathcal{R} \to \mathcal{S}$ are RCs of type $\varphi \times \psi, \varphi \to \psi$.*

Proof. (1) (CR0): $*^{\psi} \in \mathcal{SN}^{\psi}$ is SN, if $\psi \in \mathit{Iso}(\top)$. (CR1): By the definition of $\mathcal{SN}^{\psi}$. (CR2): If $t \in \mathcal{SN}^{\psi}$ and $t \to t'$, then $t' \in \mathcal{SN}^{\psi}$. (CR3): Let t be a neutral term of type ψ such that any reduct t' of t is in $\mathcal{SN}^{\psi}$. Then t is in $\mathcal{SN}^{\psi}$. (2) By the proof of Lemma 6 for $\varphi \times \psi$ and $\varphi \to \psi$. ◀

For a type φ, a sequence $\vec{X}$ of distinct type variables $X_1, \ldots, X_m$, and a sequence $\vec{\psi}$ of types $\psi_1, \ldots, \psi_m$, let $\varphi[\vec{X} := \vec{\psi}]$ be the simultaneous substitution.

▶ **Definition 18** (parametric reducibility)**.** Suppose
1. φ is a type;
2. a sequence $\vec{X}$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of φ;
3. $\vec{\psi}$ is a sequence of types $\psi_1, \ldots, \psi_m$; and
4. $\vec{\mathcal{R}}$ is a sequence of RCs $\mathcal{R}_1, \ldots, \mathcal{R}_m$ of corresponding types $\vec{\psi}$.

Define a set $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$ of terms of type $\varphi[\vec{X} := \vec{\psi}]$ as follows:
1. If $\varphi = \top$, $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}] = \mathcal{SN}^\top$;
2. If $\varphi = X_i$, $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}] = \mathcal{R}_i$;
3. If $\varphi \equiv \varphi' \# \varphi''$, $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}] = \mathsf{RED}_{\varphi'}[\vec{X} := \vec{\mathcal{R}}] \# \mathsf{RED}_{\varphi''}[\vec{X} := \vec{\mathcal{R}}]$ ($\# = \to, \times$);
4. If $\varphi \equiv \Pi Y.\, \varphi'$, Y not free in $\vec{\psi}$ and $Y \neq X_i$ ($i = 1, \ldots, m$), then $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$ is the set of terms $t^{\Pi Y.\, \varphi'[\vec{X} := \vec{\psi}]}$ such that for any type ψ and any RC $\mathcal{S}$ of type ψ, $(t\psi)^{\varphi'[\vec{X}, Y := \vec{\psi}, \psi]} \in \mathsf{RED}_{\varphi'}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$.

▶ **Lemma 19.** *Under the conditions of Definition 18,* $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$ *is an RC of type* $\varphi[\vec{X} := \vec{\psi}]$.

Proof. By induction on φ. First consider the case $\varphi \equiv \Pi Y.\, \varphi'$. Let $\mathcal{S}$ be an RC $\mathcal{S}$ of type φ''. By induction hypothesis,

$$\mathcal{T} := \vec{\mathsf{RED}}_{\varphi'}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}], \text{ is an RC.} \tag{3}$$

(CR0) Let $*^{\Pi Y.\, \varphi'[\vec{X} := \vec{\psi}]} \varphi'' \to s$. We will verify $s \in \mathcal{T}$. We have two cases. The first case corresponds to clause ($\top\Pi$) and the second case to clause ($\top^{par}$) in Definition 12.

1. $s \equiv *^{\varphi'[\vec{X}, Y := \vec{\psi}, \varphi'']}$. By (3), $s \in \mathcal{T}$.
2. $s \equiv \lambda y.\, y *^{\tau_1} \cdots *^{\tau_n}$: Then $\varphi' = (\tau_1 \to \cdots \to \tau_n \to Y) \to Y$. We will verify $s \in \mathsf{RED}_{\varphi'}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] = (\mathsf{RED}_{\tau_1}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \cdots \to \mathsf{RED}_{\tau_n}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \mathcal{S}) \to \mathcal{S}$. Take a term

$$u^{\tau_1 \to \cdots \to \tau_n \to \varphi''} \in \mathsf{RED}_{\tau_1}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \cdots \to \mathsf{RED}_{\tau_n}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \mathcal{S}. \tag{4}$$

By induction hypothesis for τ_i,

$$\mathsf{RED}_{\tau_i}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \text{ is an RC.} \tag{5}$$

By Lemma 17 (2), $\mathsf{RED}_{\tau_1}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \cdots \to \mathsf{RED}_{\tau_n}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}] \to \mathcal{S}$ is an RC. By (CR1) of this RC, u is SN. So we can use WFI $\left(\left(\left\{u^{\tau_1 \to \cdots \to \tau_n \to \varphi''} \mid (4) \text{ holds}\right\},\ \to\right)\right)$ where $\to$ is the rewrite relation. We will prove $su \equiv (\lambda y.\, y *^{\tau_1} \cdots *^{\tau_n}) u \in \mathcal{S}$. Let $(su)^{\varphi''} \overset{\Delta}{\to} v$. Then we have four subcases:

a. $\Delta \equiv su$ is a redex of (g): Then v is $*^{\varphi''}$. As $\mathcal{S}$ is an RC, $v \in \mathcal{S}$ by (CR0) of $\mathcal{S}$.

b. $\Delta \equiv su$ is not a redex of (g): Then $v \equiv u *^{\tau_1} \cdots *^{\tau_n}$. By (5) and the (CR0), $*^{\tau_i} \in \mathsf{RED}_{\tau_i}[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$. So $v \in \mathcal{S}$ by (4).

c. $\Delta \subseteq s$: Then $v \equiv \lambda y.\, *^{\tau} *^{\tau_{n-i+1}} \cdots *^{\tau_n}$ for some τ and nonnegative $i \leq n$ with the rewrite rule schema is (g). We see that the normal form of s is canonical. This canonical term is in $\mathcal{T}$ by (3) and (CR0) of $\mathcal{T}$. By repeated applications of induction hypothesis (CR3), $s \in \mathcal{T}$. Hence $v \in \mathcal{S}$.

d. Otherwise, $\Delta \subseteq u$. By the WF induction hypothesis, $v \in \mathcal{S}$.

So, $v \in \mathcal{S}$. By (CR3) for $\mathcal{S}$, $(\lambda y.\, y *^{\tau_1} \cdots *^{\tau_n}) u \in \mathcal{S}$. So, $s \in \mathcal{T}$.

Thus $*^{\Pi Y.\,\varphi'[\vec{X}:=\vec{\psi}]}\varphi'' \in \mathcal{T}$. Hence $*^{\Pi Y.\,\varphi'[\vec{X}:=\vec{\psi}]} \in \mathsf{RED}_{\Pi Y.\,\varphi'}[\vec{X} := \vec{\mathcal{R}}]$.

(CR1) Let $t \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$. Then $t\varphi'' \in \mathcal{T}$ by Definition 18. By (3) and (CR1) of $\mathcal{T}$, $t\varphi''$ is SN. So t is SN.

(CR2) Let $t \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$. Then $t\varphi'' \in \mathcal{T}$ by Definition 18. Assume $t \to t'$. Then $t\varphi'' \to t'\varphi''$. By (3) and (CR2) of $\mathcal{T}$, $t'\varphi'' \in \mathcal{T}$. So $t' \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$.

(CR3) Suppose that t is neutral and that $t' \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$ whenever $t \to t'$. Let $t\varphi'' \overset{\Delta}{\to} s$. As t is neutral, we have two cases:

1. $\Delta \equiv t\varphi''$: Then s is canonical of type $\varphi'[\vec{X} := \vec{\psi}]$, because t is neutral. By (3) and (CR0) of $\mathcal{T}$, $s \in \mathcal{T}$.
2. Otherwise, $s \equiv t'\varphi''$ with $t \overset{\Delta}{\to} t'$. $s \in \mathcal{T}$ by $t' \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$.

By (3) and (CR3) of $\mathcal{T}$, $t\varphi'' \in \mathcal{T}$. So $t \in \mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$.

The cases where φ is other than $\Pi Y.\,\varphi'$ are by induction hypotheses on φ and Lemma 17. ◀

▶ **Lemma 20.** *Suppose that*

1. *φ, ψ are types, Y is a type variable;*
2. *a sequence $\vec{X}$ of distinct type variables $X_1, \dots, X_m$ contains all free type variables of $\varphi[Y := \psi]$;*
3. *$X_i \neq Y$ $(i = 1, \dots, m)$; and*
4. *$\vec{\mathcal{R}}$ is a sequence of RCs $\mathcal{R}_1, \dots, \mathcal{R}_m$.*

Then

$$\mathsf{RED}_{\varphi[Y:=\psi]}[\vec{X} := \vec{\mathcal{R}}] = \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathsf{RED}_\psi[\vec{X} := \vec{\mathcal{R}}]].$$

Proof. By induction on φ. ◀

▶ **Lemma 21** (Universal abstraction). *Suppose that*

1. *φ is a type;*
2. *a sequence $\vec{X}$ of distinct type variables $X_1, \dots, X_m$ contains all free type variables of $\Pi Y.\,\varphi$;*
3. *$X_i \neq Y$ $(i = 1, \dots, m)$, $\vec{\psi}$ is a sequence of types $\psi_1, \dots, \psi_m$;*
4. *$\vec{\mathcal{R}}$ is a sequence of RCs $\mathcal{R}_1, \dots, \mathcal{R}_m$ of types $\vec{\psi}$;*
5. *Y does not occur free in $\vec{\psi}$; and*
6. *$w^{\varphi[\vec{X}:=\vec{\psi}]}$ is a term.*

If for any type ψ and any RC $\mathcal{S}$ of type ψ, $(w[Y := \psi])^{\varphi[\vec{X},Y:=\vec{\psi},\psi]} \in \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$, then $\Lambda Y.\,w \in \mathsf{RED}_{\Pi Y.\,\varphi}[\vec{X} := \vec{\mathcal{R}}]$.

Proof. $\mathcal{SN}^Y$ is an RC, by Lemma 17 (1). By assumption, $w \in \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{SN}^Y]$. By (CR1) of this RC, w is SN. By Definition 18 (4), we have only to verify:

$$(\Lambda Y.\,w)\psi \in \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}], \text{for every type } \psi \text{ and RC } \mathcal{S} \text{ of type } \psi. \tag{6}$$

The proof is by WFI $\left(\left(\left\{w^{\varphi[\vec{X}:=\vec{\psi}]} \mid w \in \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{SN}^Y]\right\}, \to\right)\right)$ where $\to$ is the rewrite relation. Let $(\Lambda Y.\,w)\psi \overset{\Delta}{\to} s$. We have five cases. We verify $s \in \mathcal{T} := \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$.

1. $\Delta \equiv (\Lambda Y.\,w)\psi$ is a redex of (g): Then $s \equiv *^{\varphi[\vec{X},Y:=\vec{\psi},\psi]}$. By (CR0) of $\mathcal{T}$.
2. $\Delta \equiv (\Lambda Y.\,w)\psi$ is a redex of (β^2): Then $s \equiv w[Y := \psi]$. By assumption.
3. $\Delta \equiv (\Lambda Y.\,w)$ is a redex of (g): Then $s \equiv *^{\Pi Y.\,\varphi[\vec{X}:=\vec{\psi}]}\psi$. By $*^{\Pi Y.\,\varphi[\vec{X}:=\vec{\psi}]} \in \mathsf{RED}_{\Pi Y.\,\varphi}[\vec{X} := \vec{\mathcal{R}}]$, $s \in \mathcal{T}$ by Definition 18.
4. $\Delta \equiv (\Lambda Y.\,w)$ is a redex of (η^2): Then this case coincides with the second case.

5. Otherwise, for some w', $s \equiv (\Lambda Y.\, w')\psi$ and $w \to w'$. By the WF induction hypothesis. Thus $s \in \mathcal{T}$. So (6) follows from (CR3) of $\mathcal{T}$. ◀

▶ **Lemma 22** (Universal application). *Suppose that*
1. *φ, ψ are types, Y is a type variable;*
2. *a sequence $\vec{X}$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of $\varphi[Y := \psi]$;*
3. *$X_i \neq Y$ $(i = 1, \ldots, m)$, $\vec{\psi}$ is a sequence of types $\psi_1, \ldots, \psi_m$; and*
4. *$\vec{\mathcal{R}}$ is a sequence of RCs $\mathcal{R}_1, \ldots, \mathcal{R}_m$ of types $\vec{\psi}$.*

Then[1]

$$w \in \mathsf{RED}_{\Pi Y.\,\varphi}[\vec{X} := \vec{\mathcal{R}}] \implies w\left(\psi[\vec{X} := \vec{\psi}]\right) \in \mathsf{RED}_{\varphi[Y:=\psi]}[\vec{X} := \vec{\mathcal{R}}].$$

Proof. By Lemma 19, $\mathsf{RED}_{\psi}[\vec{X} := \vec{\mathcal{R}}]$ is an RC of type $\psi[\vec{X} := \vec{\psi}]$. By the premise and Definition 18 (4), $w\left(\psi[\vec{X} := \vec{\psi}]\right) \in \mathsf{RED}_{\varphi}[\vec{X}, Y := \vec{\mathcal{R}}, \mathsf{RED}_{\psi}[\vec{X} := \vec{\mathcal{R}}]\,]$. So Lemma 20 implies the conclusion. ◀

By the condition $(\top^{par})$, a canonical term $*^{\tau}$ $(\tau \in \mathit{Iso}(\top))$ does not necessarily contain the term constant $*^{\top}$. We generalize the definition of $*$-free and $\tilde{t}$ (Definition 2.)

▶ **Definition 23.**
1. We say a term of $\left(\lambda^2\beta\eta\pi*\right)''$ is *$*$-free* , if it has no canonical subterm of a type of $\mathit{Iso}(\top)$.
2. For any term t of $\left(\lambda^2\beta\eta\pi*\right)''$, let $\tilde{t}$ be a $*$-free term obtained from t by replacing all occurrences of $*^{\tau}$ with variables x^{τ} where τ is any type of $\mathit{Iso}(\top)$.

Similarly as Lemma 3, we can prove:

▶ **Lemma 24.** *Let φ be a type, $\mathcal{R}$ be an RC of φ, and t be a term of $\left(\lambda^2\beta\eta\pi*\right)''$ of φ. If $\tilde{t} \in \mathcal{R}$, $t \in \mathcal{R}$.*

▶ **Lemma 25.** *In $\left(\lambda^2\beta\eta\pi*\right)''$, for every $*$-free term t,*
1. *$t[X_1, \ldots, X_m := \psi_1, \ldots, \psi_m]$ is $*$-free for all distinct type variables $X_1, \ldots, X_m$ and for all types $\psi_1, \ldots, \psi_m$; and*
2. *$t[x_1^{\varphi_1}, \ldots, x_n^{\varphi_n} := u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}]$ is $*$-free for all distinct variables $x_1^{\varphi_1}, \ldots, x_n^{\varphi_n}$ and for all $*$-free terms $u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}$.*

Proof. By induction on t. Let Θ be $[X_1, \ldots, X_m := \psi_1, \ldots, \psi_m]$ and θ be $[x_1^{\varphi_1}, \ldots, x_n^{\varphi_n} := u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}]$. The proof proceeds by cases according to the form of t. By Definition 12, t is not a term constant, because otherwise t is $*^{\top}$.
1. t is a variable: Then (1) holds because no canonical term has a free variable. (2) is clear.
2. t is an abstraction, or an application: Obvious.
3. $t \equiv \Lambda Y.\, w$ such that $X_i \not\equiv Y$ and Y does not occur free in any ψ_i: By induction hypothesis, $w\Theta$ and $w\theta$ are $*$-free. (1) We have only to verify that $t\Theta \equiv \Lambda Y.\, w\Theta$ is not a canonical term:

$$\Lambda Y.\lambda x^{\tau_1 \to \cdots \tau_l \to Y}.\, x *^{\tau_1} \cdots *^{\tau_l} \text{ where } \tau_k \in \mathit{Iso}(\top)\ (k = 1, \ldots, l). \tag{7}$$

Let τ_i be an instance of some type σ by the substitution Θ such that $\sigma \not\equiv \tau$. Then $\sigma \notin \mathit{Iso}(\top)$ because any type of $\mathit{Iso}(\top)$ has no free type variable. So, $*^{\tau_i}$ is an instance

[1] [12, Lemma 14.2.3] corresponding to this lemma has a typo: "tV" should be "$t(V[\underline{U}/\underline{X}])$."

of a non-canonical subterm of t by the substitution Θ. This contradicts against the induction hypothesis. Hence, $t\Theta$ is $\equiv t$, which is $*$-free by the premise.

(2) Assume $t\theta \equiv \Lambda Y.\, w\theta$ is (7). By the premise, no u_j contains $*^{\tau_k}$. We have two subcases.

a. $l = 0$: Then $t\theta \equiv t$ or $t \equiv \Lambda Y.\, x_j^{Y \to Y}$. In the former case, $t\theta$ is $*$-free. The latter case is impossible because of the proviso of *universal abstraction* in Definition 12.

b. $l > 0$: Then, for each k, there is a $*$-free subterm w_k of t such that $w_k\theta \equiv *^{\tau_k}$. This is impossible by induction hypothesis for w_k.

Hence $t\theta$ is $*$-free.

4. $t \equiv w\psi$: Then, by induction hypothesis, $w\Theta$ and $w\theta$ are $*$-free. Hence, none of $t\Theta \equiv w\Theta(\psi\Theta)$ and $t\theta \equiv (w\theta)\psi$ is canonical. ◀

In the following two theorems, we use Lemma 24.

▶ **Theorem 26** (Restricted Reducibility). *Suppose*

1. *t^φ is a $*$-*free *term;*
2. *a sequence of distinct variables $x_1^{\varphi_1}, \ldots, x_n^{\varphi_n}$ contains all free variables of t^φ;*
3. *a sequence $\vec{X}$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of types $\varphi, \varphi_1, \ldots, \varphi_n$;*
4. *$\vec{\mathcal{R}}$ is a sequence of RCs $\mathcal{R}_1, \ldots, \mathcal{R}_m$ of types $\vec{\psi} \equiv \psi_1, \ldots, \psi_m$; and*
5. *$u_i^{\varphi_i[\vec{X}:=\vec{\psi}]}$ is in $\mathsf{RED}_{\varphi_i}[\vec{X} := \vec{\mathcal{R}}]$ and is $*$-*free *($i = 1, \ldots, n$).*
6. *$t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]$ is the term obtained from $t[\vec{X} := \vec{\psi}]$ by simultaneously substitution of $u_1^{\varphi_1[\vec{X}:=\vec{\psi}]}, \ldots, u_n^{\varphi_n[\vec{X}:=\vec{\psi}]}$ into $x_1^{\varphi_1[\vec{X}:=\vec{\psi}]}, \ldots, x_n^{\varphi_n[\vec{X}:=\vec{\psi}]}$.*

Then $t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]$ is in $\mathsf{RED}_\varphi[\vec{X} := \vec{\mathcal{R}}]$.

Proof. By induction on t. The proof proceeds by cases according to the form of t.

1. t is a pairing or a λ-abstraction: We can prove this case, similarly as in the proof of Theorem 9, but we use Lemma 25 to verify the $*$-free condition of Lemma 8.
2. $t \equiv (\Lambda Y.\, w)^{\Pi Y.\, \varphi}$ where $X_i \neq Y$ and Y does not occur free in any $\varphi_i[\vec{X} := \vec{\psi}]$: Then by the induction hypothesis, for any type ψ and any RC $\mathcal{S}$ of ψ, $w[\vec{X}, Y := \vec{\psi}, \psi][\vec{x} := \vec{u}]$ is in $\mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$. Since Y occurs in no $\vec{u}$, we have $w[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}][Y := \psi] \in \mathsf{RED}_\varphi[\vec{X}, Y := \vec{\mathcal{R}}, \mathcal{S}]$. By Lemma 21, $(\Lambda Y.\, w)[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]$ is in $\mathsf{RED}_{\Pi Y.\, \varphi}[\vec{X} := \vec{\mathcal{R}}]$.
3. $t \equiv w^{\Pi Y.\, \varphi}\psi$: Then by the induction hypothesis, $w[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]$ is in $\mathsf{RED}_{\Pi Y.\, \varphi}[\vec{X} := \vec{\mathcal{R}}]$. By Lemma 22, $w[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]\left(\psi[\vec{X} := \vec{\psi}]\right) \in \mathsf{RED}_{\varphi[Y:=\psi]}[\vec{X} := \vec{\mathcal{R}}]$. This term is just $(w\psi)[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}]$.

The other cases are handled similarly as in the proof of Theorem 9, except that we use Lemma 24 instead of Lemma 3. ◀

▶ **Theorem 27.** *If a sequence of distinct type variables $X_1, \ldots, X_m$ contains the free type variables of a type φ, then any term t^φ is in $\mathsf{RED}_\varphi[X_1, \ldots, X_m := \mathcal{SN}^{X_1}, \ldots, \mathcal{SN}^{X_m}]$.*

Proof. Let t be a term. The $*$-free term $\tilde{t}$ is reducible, by (CR4) and by Theorem 26 with $u_i^{\varphi_i} := x_i^{\varphi_i}$, $\psi_j := X_j$ and $\mathcal{R}_j := \mathcal{SN}^{X_j}$. By Lemma 24, t is reducible. ◀

▶ **Corollary 28.** *(1) $(\lambda^2\beta\eta\pi*)''$ is SN. (2) $\left(\lambda^2\beta\eta\pi*\right)'$ satisfies SN.*

Proof. (1) By (CR1) and Theorem 27. (2) If $u \to v$ in $\left(\lambda^2\beta\eta\pi*\right)'$, then $u \to v$ in $\left(\lambda^2\beta\eta\pi*\right)''$. So $\left(\lambda^2\beta\eta\pi*\right)'$ satisfies SN by (1). ◀

4 Derivational complexity of $(\lambda\beta\eta\pi*)'$: comparison to type-directed expansions

For the typed λ-calculus, let a binary relation $\to_{\overline{\eta}}$ ($\to_{\overline{SP}}$) replace a neutral subterm occurrence in a non-elimination context with the η (SP)-expansion [29]. Neither $\to_{\overline{\eta}}$ nor $\to_{\overline{SP}}$ is stable under contexts. We call the relation $\to := \to_{\beta\pi_1\pi_2T\overline{\eta}\overline{SP}}$ *Mints' reduction*, as Mints introduced it in [28, 29]. Mints' reduction generates the equational theory $\lambda\beta\eta\pi*$, and is SN+CR ([3, 20] to cite a few). In [3], the author presented a divide-and-conquer lemma to infer SN+CR property of a reduction system from that property of its subsystems. From this lemma, SN+CR of the $\to$ follows. SN implies $\to_{\overline{\eta}} = \leftarrow_\eta \setminus \leftarrow_\beta$ and $\to_{\overline{SP}} = \leftarrow_{SP} \setminus \leftarrow_{\pi_1} \setminus \leftarrow_{\pi_2}$. The divide-and-conquer lemma suggests a large upper bound of the length of a $\to$-reduction sequence. Čubrić proved CR of $\to$ by development argument of residuals [8] and WN. Although $\to_{\overline{\eta}\overline{SP}}$ is not stable under contexts, Khasidashvili and van Oostrom [22] pointed out that the finite development-like argument based on $\leftarrow_{\eta SP}$ proves CR of $\to$.

In [10], Di Cosmo and Kesner proved CR+SN of a reduction system $\to_\beta \cup \to_{\overline{\eta}} \cup \to_{\pi_1} \cup \to_{\pi_2} \cup \to_{\overline{SP}} \cup \to_T$ union the β-like reductions of sum types. By showing how substitution and the reduction interact with the context-sensitive rules, they proved the WCR. They simulated expansions without expansions, to reduce SN of the reduction to SN for the underlying calculus without expansions, provable by the standard reducibility method.

The rewriting system $(\lambda\beta\eta\pi*)'$ of Curien and Di Cosmo is stable under contexts (i.e., $t \to t' \implies \cdots t \cdots \to \cdots t' \cdots$.) Mints' reduction decides the equational theory $\lambda\beta\eta\pi*$. Mints' reduction is not stable under contexts.

Mints' reduction fits with semantic treatments such as *normalization by evaluation* (e.g., [4]. See [1] in the context of type-checking of dependent type theories). However, because of the complication of Mints' reduction, in his book [30] on selected papers of proof theory, Mints replaced his reduction with the $\beta\eta$-reduction modulo equivalence relation on terms. His purpose is to give a simple proof of difficult theorems of category theory with typed λ-calculus and proof theory by using the correspondence objects = types = propositions and arrows = terms = proofs. Mac Lane is interested in his ambition [27].

In the worst case analysis, normalizing rewriting sequences of Curien and Di Cosmo's rewriting have smaller number of $\beta\pi_1\pi_2T$-reduction steps than those of Mints' reduction.

The derivational complexity of Mints' reduction is higher than that of Curien-Di Cosmo's rewriting in the simply-typed regime. By the *derivational complexity* of a term t, we mean the maximum number of β-reduction steps in a reduction sequence from t. We count only β-reduction steps, because the β-rule is a common rule of Mints' reduction and the $\beta\eta$-calculus, which is Curien-Di Cosmo's rewriting in the simply-typed regime. The optimal bound for the length of a $\beta\eta$-reduction sequence is given in [5] and is also the optimal bound for the length of a β-reduction sequence.

▶ **Theorem 29.** *For every simply-typed λ-term, the derivational complexity of t of Mints' reduction $\to_{\beta\overline{\eta}}$ is greater than or equal to that of the $\beta\eta$-reduction.*

Proof. A $\beta\eta$-reduction sequence S is an alternating sequence of $\xrightarrow{+}_\beta$ and $(\to_\eta \setminus \to_\beta)^+$. Here $(\cdots)^+$ stands for the transitive closure. By [3], (1) $\to_{\overline{\eta}} = \leftarrow_\eta \setminus \leftarrow_\beta$; (2) $\to_{\overline{\eta}}$ is CR; and (3) If $t \to_\beta s$ then the $\overline{\eta}$-normal form of t goes to that of s in positive number of β-reduction steps. Hence, the $\overline{\eta}$-normal forms of terms in S forms an $\overline{\eta}$-normalization sequence followed by a β-reduction sequence such that the number of β-steps is not less than that of S. ◀

Consider the minimum length ℓ_τ of the normalization sequences of variable x^τ with $\tau \in \mathit{Iso}(\top)$. $\ell_\tau = 1$, in the rewriting system $(\lambda\beta\eta\pi*)'$, although ℓ_τ is arbitrary large as

$\tau \in \mathit{Iso}(\top)$ gets complex in the Mints' reduction. Indeed, $x^\top \to *^\top$,

$$x^{\top\to\top} \to \lambda z^\top.\,(xz)^\top \to \lambda z^\top.\,*^\top,$$
$$x^{\top\times\top} \to \langle(\pi_1 x)^\top,\,(\pi_2 x)^\top\rangle \to \langle(\pi_1 x)^\top,\,*^\top\rangle \to \langle *^\top,\,*^\top\rangle.$$

We may be able to introduce SN+CR extensional λ-calculus with surjective pairing, terminal type and empty type, based on type isomorphism.

Lemma 3 and Lemma 24 are described in topological jargon. For any type φ, the reducibility predicate is an open condition, in the topological space of terms where a set of terms is open if and only if the set is closed under reduction. For the set U of $*$-free terms, the maximum open superset of U is the set V of all terms. But U is not dense in V.

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A SN proof for extension by weakly extensional sum types, by restricted reducibility theorem

Recent study of type-directed η-expansion pay attention to sum types (with permutative conversion) and empty types. See [4, 10, 19, 25, 33], to cite a few. For the first step, we introduce Curien-Di Cosmo-style rewrite rule schemata for weak extensionality for sum types, and then prove the SN.

▶ **Definition 30** ($(\lambda^{\top,\to,\times,+})'$)**.** To the equational theory $\lambda\beta\eta\pi*$, we add *sum types* $\varphi_1+\varphi_2$, *the case distinctions* $\mathrm{d}(t^{\varphi_1+\varphi_2},\ t_1^{\varphi_1\to\psi},\ t_2^{\varphi_2\to\psi})^{\psi}$, *injections* $\left(\mathrm{in}^i_{\varphi_1,\varphi_2}(w^{\varphi_i})\right)^{\varphi_1+\varphi_2}$ $(i=1,2)$, and the following two equational axioms:

$$\mathrm{d}(\mathrm{in}^i_{\varphi_1,\varphi_2}(w),\ t_1,\ t_2) = t_i w, \qquad (i=1,2).$$
$$\mathrm{d}(t,\ \lambda x_1.\,\mathrm{in}^1_{\varphi_1,\varphi_2}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\varphi_1,\varphi_2}(x_2)) = t.$$

We write $\lambda^{\top,\to,\times,+}$ for the resulting equational theory. We orient the last two axioms:

$$(+.\beta_i) \qquad \mathrm{d}(\mathrm{in}^i_{\varphi_1,\varphi_2}(w),\ t_1,\ t_2) \to t_i w, \qquad (i=1,2).$$
$$(+.\eta) \qquad \mathrm{d}(t,\ \lambda x_1.\,\mathrm{in}^1_{\varphi_1,\varphi_2}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\varphi_1,\varphi_2}(x_2)) \to t.$$

The rewrite rule schemata $(+.\eta)$ and (g) yields obstructions [2] to CR:

$$\mathrm{d}(y,\ \lambda x_1.\,\mathrm{in}^1_{\varphi_1,\tau}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\varphi_1,\tau}(*^{\tau})) \leftarrow_g \mathrm{d}(y,\ \lambda x_1.\,\mathrm{in}^1_{\varphi_1,\tau}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\varphi_1,\tau}(x_2)) \to_{+.\eta} y.$$
$$\mathrm{d}(y,\ \lambda x_1.\,\mathrm{in}^1_{\tau,\varphi_2}(*^{\tau}),\ \lambda x_2.\,\mathrm{in}^2_{\tau,\varphi_2}(x_2)) \leftarrow_g \mathrm{d}(y,\ \lambda x_1.\,\mathrm{in}^1_{\tau,\varphi_2}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\tau,\varphi_2}(x_2)) \to_{+.\eta} y.$$

Here $\tau \in \mathit{Iso}(\top)$. The set $\mathit{Iso}(\top)$ of "isomorphic to" $\top$ in $\lambda^{\top,\to,\times,+}$ is constructed exactly as that of $\lambda\beta\eta\pi*$ (Definition 1). Below, τ ranges over $\mathit{Iso}(\top)$. The following rule schemata rewrite the leftmost terms of the two obstructions of CR, to the corresponding rightmost terms y.

$$(+.\eta_{top1}) \qquad \mathrm{d}(t,\ \lambda x_1.\,\mathrm{in}^1_{\varphi_1,\tau}(x_1),\ \lambda x_2.\,\mathrm{in}^2_{\varphi_1,\tau}(*^{\tau})) \to t,$$
$$(+.\eta_{top2}) \qquad \mathrm{d}(t,\ \lambda x_1.\,\mathrm{in}^1_{\tau,\varphi_2}(*^{\tau}),\ \lambda x_2.\,\mathrm{in}^2_{\tau,\varphi_2}(x_2)) \to t.$$

Let $(\lambda^{\top,\to,\times,+})'$ be the rewriting system consisting of (g), (β), (η), (π_1), (π_2), (SP), (η_{top}), (SP_{top1}), (SP_{top2}), $(+.\beta_1)$, $(+.\beta_2)$, $(+.\eta)$, $(+.\eta_{top1})$, and $(+.\eta_{top2})$.

[2] We avoid saying "critical pairs," since the notion of critical pairs is not so clear in non-left-linear higher-order rewriting systems.

▶ **Definition 31** (Reducibility). The *reducibility* for terms of $\left(\lambda^{\top,\rightarrow,\times,+}\right)'$ is defined inductively as in Definition 5 but also with the following clause:

(+) A term $t^{\varphi_1+\varphi_2}$ is *reducible*, if t is neutral and t' is reducible whenever $t \to t'$; or $t \equiv \mathrm{in}^i_{\varphi_1,\varphi_2}(u^{\varphi_i})$ for some $i = 1, 2$ and some reducible u^{φ_i}.

▶ **Definition 32** (Neutral). A term is *neutral* if it is not of the form $\langle u, v\rangle$, $\lambda x. v$, $\mathrm{in}^i_{\varphi,\psi}(w)$.

▶ **Lemma 33.** *(CR0), (CR1), (CR2) and (CR3) hold for* $\left(\lambda^{\top,\rightarrow,\times,+}\right)'$.

Proof. By induction on the type φ of a given term t. Consider case where $\varphi = \varphi_1 + \varphi_2$.

(CR0) There is no canonical term of type $\varphi_1 + \varphi_2$.

(CR1) The set of reducible terms is the least set satisfying the three clauses $(a), (\times), (\rightarrow)$ of Definition 5 and the clause $(+)$ of Definition 31. If we replace 'reducible' with 'SN,' the four clauses hold. So, the set of reducible terms is a subset of the set of SN terms.

(CR2) Suppose t is reducible. If t is neutral, then t' is reducible by Definition 31. Otherwise $t \equiv \mathrm{in}^i_{\varphi_1,\varphi_2}(u^{\varphi_i})$, $t' \equiv \mathrm{in}^i_{\varphi_1,\varphi_2}(u'^{\varphi_i})$, $u^{\varphi_i} \to u'^{\varphi_i}$ for some i and some terms $u_i^{\varphi_i}, u_i'^{\varphi_i}$. By induction hypothesis (CR2) for φ_i, u'_i is reducible. Hence t' is reducible by Definition 31.

(CR3) Immediate from Definition 31.

The other cases are checked as Lemma 6 was proved. ◀

We define $*$-*free* terms, exactly as in Definition 2. Then Lemma 3 holds for $\left(\lambda^{\top,\rightarrow,\times,+}\right)'$.

▶ **Lemma 34** (Case). *If t, t_1, and t_2 are reducible. so is* $\mathrm{d}(t,\ t_1,\ t_2)$ *is reducible.*

Proof. By induction on the type ψ of $\mathrm{d}(t^{\varphi_1+\varphi_2},\ t_1^{\varphi_1\to\psi},\ t_2^{\varphi_1\to\psi})$.

ψ is atomic. Let $\mathrm{d}(t,\ t_1,\ t_2) \overset{\Delta}{\to} s$. By (CR1), t, t_1, t_2 are SN. We will verify that s is reducible, by WF induction on

$$\left(\left\{t^{\varphi_1+\varphi_2} \mid t^{\varphi_1+\varphi_2} \text{ is reducible}\right\},\ \to\right) \# \left(\left\{t_1^{\varphi_1\to\psi} \mid t_1^{\varphi_1\to\psi} \text{ is reducible}\right\},\ \to\right)$$
$$\# \left(\left\{t_2^{\varphi_1\to\psi} \mid t_2^{\varphi_2\to\psi} \text{ is reducible}\right\},\ \to\right) \tag{8}$$

where $\to$ is the rewrite relation. If $\Delta \not\equiv \mathrm{d}(t,\ t_1,\ t_2)$, s is reducible by the WF induction hypothesis. Otherwise $\Delta \equiv \mathrm{d}(t,\ t_1,\ t_2)$. We have three cases, as Δ is $*$-free.

1. $s \equiv *$: Then s is reducible by (CR0).
2. $s \equiv t$ by $(+.\eta)$, $(+.\eta_{top1})$ or $(+.\eta_{top2})$: s is reducible by the premise.
3. Otherwise, $s \equiv t_i w$ and $t \equiv \mathrm{in}^i_{\varphi_1,\varphi_2}(w)$. w is reducible by Definition 31. By the premise, s is reducible.

$\psi = \psi_1 \times \psi_2$ ($\psi_1 \to \psi_2$, resp.) We verify the reducibility of $u :\equiv \pi_i(\mathrm{d}(t,\ t_1,\ t_2))$ ($u :\equiv \mathrm{d}(t,\ t_1,\ t_2)r$ for all reducible r^{ψ_1}, resp.). As t is neutral, we use (CR3) for ψ_i (ψ_2, resp.). Since $t,\ t_1,\ t_2$ ($t,\ t_1,\ t_2,\ r$, resp.) are all reducible by the premise, they are all SN by (CR1). When $\psi = \psi_1 \times \psi_2$, we proceed by WF induction on the well-founded relation (8). When $\psi = \psi_1 \to \psi_2$, we proceed by WFI $\left((8) \# \left(\left\{r^{\psi_1} \mid r^{\psi_1} \text{ is reducible}\right\},\ \to\right)\right)$. Let $u \overset{\Delta}{\to} s$. We have three cases.

1. $\Delta \equiv u$: Then $s \equiv *^{\psi_i}$ ($s \equiv *^{\psi_2}, resp.$), which is reducible by (CR0).
2. $\Delta \equiv \mathrm{d}(t,\ t_1,\ t_2)$, we have two subcases, as Δ is $*$-free.
   a. $s \equiv \pi_i t$ (tr, resp.) by $(+.\eta)$, $(+.\eta_{top1})$ or $(+.\eta_{top2})$: Then s is reducible as t is by the hypothesis.
   b. $s \equiv \pi_i(t_j w)$ ($t_j w r$, resp.) and $t \equiv \mathrm{in}^j_{\varphi_1,\varphi_2}(w)$: Then w is reducible by Definition 31, and so s is reducible by the hypothesis.

3. Otherwise, $\Delta \subseteq t$, $\Delta \subseteq t_1$, $\Delta \subseteq t_2$ or $\Delta \subseteq r$. So, s is reducible by the WF induction hypothesis.

Therefore $\pi_i(\mathrm{d}(t,\ t_1,\ t_2))$ $(\mathrm{d}(t,\ t_1,\ t_2)r$, resp.) is reducible.

$\psi = \varphi_1 + \varphi_2$: Similarly as the case where ψ is atomic. ◀

The following two theorems are proved by exactly the same argument for $(\lambda\beta\eta\pi *)'$.

▶ **Theorem 35** (Restricted Reducibility). *The statement of Theorem 9 holds for* $\left(\lambda^{\top,\to,\times,+}\right)'$.

Proof. The proof is exactly the same as that of Theorem 9, but the case for $\mathrm{in}^i_{\varphi_1,\varphi_2}(t)$ is handled by Definition 31, and the case for $\mathrm{d}(t,\ t_1,\ t_2)$ is by Lemma 34. ◀

▶ **Theorem 36.** *All terms of* $\left(\lambda^{\top,\to,\times,+}\right)'$ *are reducible.*

▶ **Corollary 37.** $\left(\lambda^{\top,\to,\times,+}\right)'$ *satisfies SN.*

It is worth checking whether our approach can ease technicality of the following work on strong sums. Consider an equational theory $\mathcal{N}$ where (1) the type is generated from the unique type constant p, the unit type by means of $+$ ("*sum types*"), $\times$, and $\to$; and (2) the equational axioms are $(\beta), (\eta), (\pi_1), (\pi_2), (SP), (c)$, the usual equational axioms for *case-distinction for sum types* and the *general permutative conversion for sum types*. In [11], Dougherty and Subrahmanyam employed $\to_{\overline{\eta SP}}$ to prove "An equation is provable in $\mathcal{N}$, if and only if it is true in *the* set-theoretic model with the unique atomic type p interpreted as an infinite set." In [25], Lindley provided an SN reduction system based on $\to_{\overline{\eta SP}}$ and proved the "CR modulo" in order to decide the equational theory $\mathcal{N}$. A decision procedure for the extensional typed λ-calculus with function types, product types, strong sum types, the terminal types and empty types is given by Scherer [33] based on focusing, and by Balat-Di Cosmo-Fiore [4] based on normalization by evaluation with turning non-standard permutative conversions into an equivalence relation.