# The Directed Disjoint Shortest Paths Problem*

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**Abstract**

In the $k$ disjoint shortest paths problem ($k$-DSPP), we are given a graph and its vertex pairs $(s_1, t_1)$, …, $(s_k, t_k)$, and the objective is to find $k$ pairwise disjoint paths $P_1, \dots, P_k$ such that each path $P_i$ is a shortest path from $s_i$ to $t_i$, if they exist. If the length of each edge is equal to zero, then this problem amounts to the disjoint paths problem, which is one of the well-studied problems in algorithmic graph theory and combinatorial optimization. Eilam-Tzoreff [5] focused on the case when the length of each edge is positive, and showed that the undirected version of 2-DSPP can be solved in polynomial time. Polynomial solvability of the directed version was posed as an open problem in [5]. In this paper, we solve this problem affirmatively, that is, we give a first polynomial time algorithm for the directed version of 2-DSPP when the length of each edge is positive. Note that the 2 disjoint paths problem in digraphs is NP-hard, which implies that the directed 2-DSPP is NP-hard if the length of each edge can be zero. We extend our result to the case when the instance has two terminal pairs and the number of paths is a fixed constant greater than two. We also show that the undirected $k$-DSPP and the vertex-disjoint version of the directed $k$-DSPP can be solved in polynomial time if the input graph is planar and $k$ is a fixed constant.

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## 1 Introduction

### 1.1 Disjoint paths problem and disjoint shortest paths problem

The *vertex-disjoint paths problem* is one of the classic and well-studied problems in algorithmic graph theory and combinatorial optimization. In the problem, the input is a graph (or a digraph) $G = (V, E)$ and $k$ pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$, and the objective is to find $k$ pairwise vertex-disjoint paths from $s_i$ to $t_i$, if they exist. If $k$ is part of the input, the vertex-disjoint paths problem is NP-hard [9], and it remains NP-hard even if the input graph is constrained to be planar [12]. The vertex-disjoint paths problem in undirected graphs can be solved in polynomial time when $k = 2$ [17, 19, 22], and Robertson and Seymour's graph minor theory gives an $O(|V|^3)$-time algorithm for the problem when $k$ is a fixed constant [15]. The running time of this algorithm is improved to $O(|V|^2)$ in [10]. The vertex-disjoint paths problem in digraphs is much harder than the undirected version. Indeed, the directed version is NP-hard even when $k = 2$ [6]. The vertex-disjoint paths problem in planar digraphs can

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be solved in polynomial time for fixed $k$ [16], and it is fixed parameter tractable with respect to parameter $k$ [3].

The vertex-disjoint paths problem has many applications, for example in transportation networks, VLSI-design [7, 14], or routing in networks [13, 20]. When we deal with such practical applications, it is natural to generalize the problem to finding *short* or *cheap* vertex-disjoint paths. There are many results on the problem to find disjoint paths minimizing a given objective function such as the total length of the paths or the length of the longest path (see Section 1.2). In this paper, we consider the disjoint shortest paths problem introduced in [5], in which each path has to be a shortest path from $s_i$ to $t_i$. Note that, in contrast to the other problems, the length of each path appears in the constraint of the problem. For an integer $k$, our problem is formally described as follows.

$k$ Disjoint Shortest Paths Problem ($k$-DSPP)

**Input.** A digraph (or a graph) $G = (V, E)$ with a length function $\ell : E \to \mathbb{R}_+$ and $k$ pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ in $G$.

**Find.** Pairwise disjoint (vertex-disjoint or edge-disjoint) paths $P_1, \ldots, P_k$ such that $P_i$ is a shortest path from $s_i$ to $t_i$ for $i = 1, 2, \ldots, k$, if they exist.

Note that $\mathbb{R}_+$ denotes the set of non-negative real numbers. We can consider both directed and undirected variants of this problem, which we call the *directed $k$-DSPP* and the *undirected $k$-DSPP*, respectively. For each problem, we can consider vertex-disjoint and edge-disjoint versions. If the length of each edge is equal to zero, then these problems amount to the directed or the undirected version of the $k$ disjoint paths problem. With this observation, most hardness results on the $k$ disjoint paths problem can be extended to the directed (or undirected) $k$-DSPP. In particular, since the $k$ disjoint paths problem in digraphs is NP-hard even when $k = 2$ [6], almost all variants of the directed $k$-DSPP are hard.

Only few positive results are known for $k$-DSPP. An important positive result is a polynomial time algorithm of Eilam-Tzoreff [5] for the undirected 2-DSPP, in which the length of each edge is positive. It is interesting to note that the algorithm in [5] is completely different from the algorithms for the 2 disjoint paths problem in [17, 19, 22]. This means that properties or tractability of $k$-DSPP will be different from those of the $k$ disjoint paths problem by assuming that the length of each edge is positive. This fact motivates us to study polynomial solvability of the directed $k$-DSPP under this assumption. Indeed, for the case when $k$ is a fixed constant and the length of each edge is positive, polynomial solvability of the directed $k$-DSPP was posed as an open problem in [5].

## 1.2 Related work

There are many results on the problem in which we find $k$ disjoint paths minimizing a given objective function. Such a problem is sometimes called the *shortest disjoint paths problem*. A natural objective function is the total length of the paths. That is, the aim of the problem is to find disjoint paths $P_1, \ldots, P_k$ that minimize $\sum_i \ell(P_i)$ when we are given a length function $\ell : E \to \mathbb{R}_+$, which we call the *min-sum $k$ disjoint paths problem*. Here, $\ell(P_i)$ denotes the length of $P_i$. We note that a solution of the $k$ disjoint shortest paths problem must be an optimal solution of the corresponding min-sum $k$ disjoint paths problem, which shows that if we can solve the min-sum $k$ disjoint paths problem, then we can also solve the $k$ disjoint shortest paths problem. Another objective function is the length of the longest path. That is, the aim of the problem is to find disjoint paths $P_1, \ldots, P_k$ that minimize $\max_i \ell(P_i)$, which we call the *min-max $k$ disjoint paths problem*.

**Table 1** Results on the $k$ disjoint paths problem and the $k$-DSPP. In the results with $(*)$, we assume that the length of each edge is positive.

| | Conditions | Disjoint Paths | Disjoint Shortest Paths |
|---|---|---|---|
| $k = 2$ | undirected | P [17, 19, 22] | P [5] $(*)$ |
| | directed | NP-hard [6] | NP-hard (Proposition 1) |
| | | | P (Theorem 2) $(*)$ |
| $k$: fixed | undirected | P [14] | OPEN |
| | planar, vertex-disjoint | | P (Corollary 11) |
| | planar, edge-disjoint | | P (Theorem 5) |
| | directed | NP-hard [6] | OPEN $(*)$ / NP-hard |
| | planar, vertex-disjoint | P [16] | P (Theorem 4) |
| | planar, edge-disjoint | OPEN | OPEN |
| | acyclic | P [6] | P (Proposition 10) |
| $k$: general | undirected/directed | NP-hard [9] | NP-hard |

Since the min-sum or min-max $k$ disjoint paths problem is a generalization of the $k$ disjoint paths problem, hardness results on the $k$ disjoint paths problem can be extended to the optimization problem. See [11] for classical results on the min-sum and min-max $k$ disjoint paths problems. We now describe several positive results on the min-sum $k$ disjoint paths problem. Colin de Verdière and Schrijver [4] presented a polynomial time algorithm for the case when the input digraph (or graph) is planar, $s_1, \dots, s_k$ are on the boundary of a common face, and $t_1, \dots, t_k$ are on the boundary of another face. Kobayashi and Sommer [11] gave a polynomial time algorithm for the case when the graph is planar, $k = 2$, and the terminals are on at most two faces. Borradaile et al. [2] gave a polynomial time algorithm for the case when the graph is planar, the terminals are ordered nicely on a common face. Björklund and Husfeldt [1] gave a randomized polynomial time algorithm for the case when $k = 2$ and each edge has a unit length, which is based on interesting algebraic techniques. This result was recently generalized to the case with two terminal pairs by Hirai and Namba [8].

## 1.3 Our results

In this subsection, we describe our results, which are summarized in Table 1.

As mentioned in Section 1.1, it is not difficult see that the directed $k$-DSPP is NP-hard even when $k = 2$ if the length of each edge can be zero.

▶ **Proposition 1.** *Both vertex-disjoint and edge-disjoint versions of the directed $k$-DSPP are NP-hard even when $k = 2$.*

**Proof.** Suppose that the length of each edge is equal to zero. In this case, since any path is a shortest path, the directed $k$-DSPP is equivalent to finding two vertex-disjoint (or edge-disjoint) paths $P_1$ and $P_2$ such that $P_i$ is from $s_i$ to $t_i$. This problem is known to be NP-hard [6], and hence the directed $k$-DSPP is NP-hard even when $k = 2$. ◀

The main result of this paper is to show that the directed $k$-DSPP can be solved in polynomial time when the length of each dicycle (directed cycle) is positive and $k = 2$.

▶ **Theorem 2.** *If the length of each dicycle is positive, both vertex-disjoint and edge-disjoint versions of the directed 2-DSPP can be solved in polynomial time. In particular, the directed 2-DSPP can be solved in polynomial time if each edge has a positive length.*

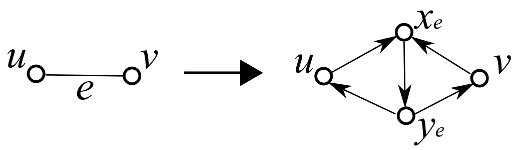


**Figure 1** Reduction to the directed case.

The proof of this theorem is given in Section 3. It is posed as an open problem by Eilam-Tzoreff [5] to determine whether or not the directed $k$-DSPP can be solved in polynomial time when each edge has a positive length and $k$ is a fixed constant. Theorem 2 answers this problem affirmatively for the case of $k = 2$. It is interesting to note that the assumption on the edge length affects the polynomial solvability of the problem as we can see in Proposition 1 and Theorem 2. We also note that a polynomial time algorithm for the undirected version can be derived from Theorem 2, that is, we obtain an alternative elementary proof for the following result.

▶ **Corollary 3** (Eilam-Tzoreff [5]). *If each edge has a positive length, both vertex-disjoint and edge-disjoint versions of the undirected* 2*-DSPP can be solved in polynomial time.*

**Proof.** Suppose we are given an instance of the undirected 2-DSPP in which $\ell(e) > 0$ for every $e \in E$. Replace each edge $e = uv$ with two new vertices $x_e, y_e$ and five new directed edges $ux_e, vx_e, x_ey_e, y_eu, y_ev$ (see Fig. 1). Define a new length function $\ell'$ by $\ell'(ux_e) = \ell'(vx_e) = \ell'(x_ey_e) = \ell'(y_eu) = \ell'(y_ev) = \frac{\ell(uv)}{3}$. Then, each edge has a positive length in the obtained digraph. In this way, we can reduce the undirected 2-DSPP to the directed 2-DSPP, which shows the corollary by Theorem 2. ◀

Theorem 2 can be extended to the case when the input digraph contains two terminal pairs and $k$ is a fixed constant, which is discussed in Section 4.

We also discuss the case when the input (di)graph is restricted to be planar in Section 5. We first show that the vertex-disjoint version of the directed $k$-DSPP can be solved in polynomial time in planar digraphs.

▶ **Theorem 4.** *If $k$ is a fixed constant and the input digraph is planar, the vertex-disjoint version of the directed $k$-DSPP can be solved in polynomial time.*

The proof is given in Section 5. Our proof is based on the reduction technique used in the proof of Theorem 2 and the algorithm for the disjoint paths problem in planar digraphs proposed in [16]. Note that this result implies that we can also solve the undirected version in polynomial time. Since Schrijver's algorithm for the disjoint paths problem [16] works only for the vertex-disjoint case, the proof of Theorem 4 cannot be extended to the edge-disjoint case directly. However, when the graph is undirected, we can show the following theorem, whose proof is given in Section 5.

▶ **Theorem 5.** *If $k$ is a fixed constant and the input graph is planar, the edge-disjoint version of the undirected $k$-DSPP can be solved in polynomial time.*

## 2 Preliminary

For a digraph $G = (V, E)$, a directed edge from $u$ to $v$ is denoted by $uv$. For a directed edge $e$ in $G$, the head and the tail of $e$ are denoted by $\text{head}_G(e)$ and $\text{tail}_G(e)$, respectively, that is,

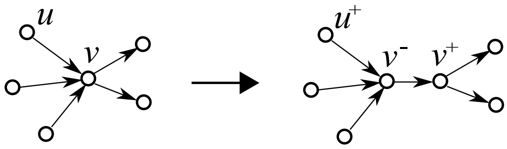


**Figure 2** Reduction to the edge-disjoint version.

$e$ is a directed edge from $\mathrm{tail}_G(e)$ to $\mathrm{head}_G(e)$. A *dipath* (or a *directed path*) is a sequence $(v_0, e_1, v_1, e_2, \ldots, e_p, v_p)$ such that $v_0, v_1, \ldots, v_p \in V$ are distinct vertices and $e_i = v_{i-1}v_i \in E$ for each $i$. If $v_0 = v_p$ in the definition of a dipath, the sequence is called a *dicycle* (or a *directed cycle*). If no confusion may arise, a dicycle, a dipath, and a directed edge are simply called a cycle, a path, and an edge, respectively. For a dipath, a dicycle, or a subgraph $Q$, its vertex set and edge set are denoted by $V(Q)$ and $E(Q)$, respectively. For a length function $\ell : E \to \mathbb{R}_+$ and for an edge set $F \subseteq E$, we denote $\ell(F) = \sum_{e \in F} \ell(e)$. For a dipath or a dicycle $Q$, we identify $Q$ with its edge set, and $\ell(E(Q))$ is simply denoted by $\ell(Q)$.

## 3 Proof of Theorem 2

In this section, we give a proof of Theorem 2, that is, we show that the directed 2-DSPP can be solved in polynomial time if the length of each dicycle is positive. To solve this problem, we will efficiently reduce it to a set of 2 disjoint paths problem in acyclic digraphs. Although the original digraph is not necessarily acyclic, we decompose the digraph into smaller subgraphs and modify each subgraph to an acyclic digraph.

We first note that the vertex-disjoint version of the directed 2-DSPP can be reduced to the edge-disjoint version of the directed 2-DSPP by the following procedure: replace each vertex $v$ with two vertices $v^+$ and $v^-$, replace each edge $uv$ with an edge $u^+v^-$ of the same length, and add an edge $v^-v^+$ of length zero for each $v$ (see Fig. 2). Therefore, it suffices to give a polynomial time algorithm for the edge-disjoint version of the problem.

Suppose we have an instance of the edge-disjoint version of the directed 2-DSPP in which each dicycle is of positive length. For $i = 1, 2$, let $E_i \subseteq E$ be the set of edges that are contained in some shortest path from $s_i$ to $t_i$. By the definition, an $s_i$-$t_i$ path is a shortest $s_i$-$t_i$ path if and only if it consists of edges in $E_i$. Note that we can compute $E_i$ in polynomial time as follows. We first apply a shortest path algorithm (e.g., Dijkstra's algorithm) and obtain the distance $d_i(v)$ from $s_i$ to $v$ for every $v \in V$. Let $E'_i \subseteq E$ be the set of all the edges $uv$ with $d_i(v) - d_i(u) = \ell(uv)$. Then, $\{uv \in E'_i \mid E'_i \text{ contains a } v\text{-}t_i \text{ path}\}$ is the desired set $E_i$. With this observation, the edge-disjoint version of the directed 2-DSPP can be reduced to the following problem: given a digraph $G = (V, E)$, subsets $E_1, E_2 \subseteq E$, and two pairs of vertices $(s_1, t_1)$ and $(s_2, t_2)$ in $G$, find edge-disjoint paths $P_1$ and $P_2$ such that $E(P_i) \subseteq E_i$ and $P_i$ is a path from $s_i$ to $t_i$ for $i = 1, 2$. We now show some properties of $E_i$.

▶ **Lemma 6.** *The edge set $E_i$ forms no dicycle for $i = 1, 2$.*

**Proof.** Assume that $E_i$ forms a dicycle $C$. By the definition of $d_i$ and $E_i$, $d_i(v) - d_i(u) = \ell(uv)$ for each $uv \in E(C)$. This shows that $\ell(C) = \sum_{uv \in E(C)} \ell(uv) = \sum_{uv \in E(C)} (d_i(v) - d_i(u)) = 0$, which contradicts that the length of each dicycle is positive. ◀

For a set $F$ of directed edges, let $\overline{F}$ be the set of directed edges obtained from $F$ by reversing all the edges, that is, $\overline{F} = \{vu \mid uv \in F\}$. Then, we have the following lemma.

▶ **Lemma 7.** *Suppose that $C$ is a dicycle in $E_1 \cup \overline{E}_2$. Then, $E_1 \cap E(C) \subseteq E_2$ and $E_2 \cap \overline{E(C)} \subseteq E_1$.*

**Proof.** Since $C$ is a dicycle in $E_1 \cup \overline{E}_2$, it can be decomposed into subpaths $P_1, \overline{Q_1}, P_2, \overline{Q_2}, \dots, P_r, \overline{Q_r}$ such that $P_i$ is a dipath from $u_i$ to $v_i$ with $E(P_i) \subseteq E_1$ and $Q_i$ is a dipath from $u_{i+1}$ to $v_i$ with $E(Q_i) \subseteq E_2$ for $i = 1, \dots, r$, where we denote $u_{r+1} = u_1$. By the definition of $d_1$ and $E_1$, $d_1(v_i) - d_1(u_i) = \ell(P_i)$ and $d_1(v_i) - d_1(u_{i+1}) \leq \ell(Q_i)$ for $i = 1, \dots, r$. By combining them, we obtain $\sum_{i=1}^r \ell(P_i) \leq \sum_{i=1}^r \ell(Q_i)$. Similarly, by the definition of $d_2$ and $E_2$, $d_2(v_i) - d_2(u_i) \leq \ell(P_i)$ and $d_2(v_i) - d_2(u_{i+1}) = \ell(Q_i)$ for $i = 1, \dots, r$, which shows that $\sum_{i=1}^r \ell(P_i) \geq \sum_{i=1}^r \ell(Q_i)$. Therefore, $\sum_{i=1}^r \ell(P_i) = \sum_{i=1}^r \ell(Q_i)$ and all the above inequalities are tight. That is, $d_1(v_i) - d_1(u_{i+1}) = \ell(Q_i)$ and $d_2(v_i) - d_2(u_i) = \ell(P_i)$ for $i = 1, \dots, r$, which shows that $E(Q_i) \subseteq E'_1$ and $E(P_i) \subseteq E'_2$. Since $E(P_i) \subseteq E_1$ for $i = 1, \dots, r$, there is a $v_i$-$t_1$ path in $E'_1$. This implies that $E'_1$ contains a $v$-$t_1$ path for any $v \in V(Q_i)$, and hence $E(Q_i) \subseteq E_1$. Similarly, since $E(Q_i) \subseteq E_2$ for $i = 1, \dots, r$, there is a $v_i$-$t_2$ path in $E'_2$, which shows that $E(P_i) \subseteq E_2$. ◀

We add four vertices $s'_1, s'_2, t'_1$, and $t'_2$, and four edges $s'_1 s_1$, $s'_2 s_2$, $t_1 t'_1$, and $t_2 t'_2$. We update $E_i \leftarrow E_i \cup \{s'_i s_i, t_i t'_i\}$ for $i = 1, 2$. Then, a path from $s_i$ to $t_i$ is corresponding to a path whose first and last edges are $s'_i s_i$ and $t_i t'_i$, respectively. By using this correspondence, we can rephrase the problem to the following: find edge-disjoint paths $P_1$ and $P_2$ such that $E(P_i) \subseteq E_i$ and $P_i$ is a path whose first and last edges are $s'_i s_i$ and $t_i t'_i$, respectively.

Let $E_0 := E_1 \cap E_2$, $E^*_1 = E_1 \setminus E_0$, and $E^*_2 = E_2 \setminus E_0$. We remove all the edges in $E \setminus (E_1 \cup E_2)$ from $G$, contract all the edges in $E_0$, and reverse all the edges in $E^*_2$. Then, we obtain a digraph $G^* = (V^*, E^*)$. Let $V_0 \subseteq V^*$ be the set of all the vertices in $V^*$ that are newly created by contracting $E_0$. In other words, $V^* \setminus V_0 \subseteq V$ is the set of all original vertices. For $v \in V_0$, let $G_v$ be the subgraph of $G - (E \setminus (E_1 \cup E_2))$ induced by the vertex set corresponding to $v$. For any edge $e$ in $G_v$, by the definition of $G_v$, either $e \in E_0$ or there exist edges $f_1, f_2, \dots, f_r \in E_0$ such that $e, f_1, f_2, \dots, f_r$ form a cycle when we ignore the direction of the edges. In the latter case, these edges induce a dicycle $C$ in $E_1 \cup \overline{E}_2$, which shows that $e \in E_0$ by Lemma 7. Thus, every edge in $G_v$ is in $E_0$, which implies that we can identify $E^*$ with $E^*_1 \cup \overline{E^*_2}$. Furthermore, since every edge in $G_v$ is in $E_0$, $G_v$ is an acyclic digraph by Lemma 6.

We can also see that, by Lemma 7, $G^*$ is an acyclic digraph. In what follows, roughly, we find two disjoint paths in $G^*$ such that one is from $s'_1$ to $t'_1$ and the other is from $t'_2$ to $s'_2$. Our algorithm is based on the algorithm for finding disjoint paths in digraphs proposed in [6].

We define a new digraph $\mathcal{G}$ whose vertex set is $W = E^*_1 \times \overline{E^*_2}$ as follows. For $(e_1, e_2), (e'_1, e'_2) \in W$, $\mathcal{G}$ has a directed edge from $(e_1, e_2)$ to $(e'_1, e'_2)$ if one of the following holds.

- $e'_1 = e_1$, $\mathrm{head}_{G^*}(e_2) = \mathrm{tail}_{G^*}(e'_2) =: v$, and there is no path in $G^*$ from $\mathrm{head}_{G^*}(e_1)$ to $v$. Furthermore, if $v \in V_0$, then $G_v$ contains a path from $\mathrm{tail}_G(e'_2)$ to $\mathrm{head}_G(e_2)$.
- $e'_2 = e_2$, $\mathrm{head}_{G^*}(e_1) = \mathrm{tail}_{G^*}(e'_1) =: v$, and there is no path in $G^*$ from $\mathrm{head}_{G^*}(e_2)$ to $v$. Furthermore, if $v \in V_0$, then $G_v$ contains a path from $\mathrm{head}_G(e_1)$ to $\mathrm{tail}_G(e'_1)$.
- $\mathrm{head}_{G^*}(e_1) = \mathrm{head}_{G^*}(e_2) = \mathrm{tail}_{G^*}(e'_1) = \mathrm{tail}_{G^*}(e'_2) =: v$. Furthermore, if $v \in V_0$, then $G_v$ contains two edge-disjoint paths such that one is from $\mathrm{head}_G(e_1)$ to $\mathrm{tail}_G(e'_1)$ and the other is from $\mathrm{tail}_G(e'_2)$ to $\mathrm{head}_G(e_2)$.

To construct $\mathcal{G}$, it suffices to solve the two disjoint paths problem in each acyclic digraph $G_v$, which can be done in polynomial time by [6]. We now show that we can solve the edge-disjoint version of the directed 2-DSPP by finding a path in $\mathcal{G}$ from $(s'_1 s_1, t'_2 t_2)$ to $(t_1 t'_1, s_2 s'_2)$.

▶ **Lemma 8.** *There is a path in $\mathcal{G}$ from $(s'_1 s_1, t'_2 t_2)$ to $(t_1 t'_1, s_2 s'_2)$ if and only if $G$ has two edge-disjoint paths $P_1$ and $P_2$ such that $P_i$ is from $s_i$ to $t_i$ and $E(P_i) \subseteq E_i$ for $i = 1, 2$.*

**Proof.** *Sufficiency* ("if" part). Suppose that $G$ has two edge-disjoint paths $P_1$ and $P_2$ such that $P_i$ is from $s_i$ to $t_i$ and $E(P_i) \subseteq E_i$ for $i = 1, 2$. $E(P_1) \setminus E_0$ forms a path $P_1^*$ from $s_1$ to $t_1$ in $G^*$, and $\overline{E(P_2) \setminus E_0}$ forms a path $P_2^*$ from $t_2$ to $s_2$ in $G^*$. Suppose that $P_1^*$ traverses edges $e_1^1, e_1^2, \dots, e_1^p$ in this order, and let $e_1^0 := s'_1 s_1$ and $e_1^{p+1} := t_1 t'_1$. Similarly, suppose that $P_2^*$ traverses edges $e_2^1, e_2^2, \dots, e_2^q$ in this order, and let $e_2^0 := t'_2 t_2$ and $e_2^{q+1} := s_2 s'_2$. It is obvious that $e_1^i \in E_1^*$ for $i = 0, 1, \dots, p+1$ and $e_2^j \in \overline{E_2^*}$ for $j = 0, 1, \dots, q+1$. Since $G^*$ is acyclic, for any $i = 0, 1, \dots, p+1$ and for any $j = 0, 1, \dots, q+1$, at least one of the following holds.

**(1)** There is no dipath in $G^*$ from $\mathrm{head}_{G^*}(e_1^i)$ to $\mathrm{head}_{G^*}(e_2^j)$.

**(2)** There is no dipath in $G^*$ from $\mathrm{head}_{G^*}(e_2^j)$ to $\mathrm{head}_{G^*}(e_1^i)$.

**(3)** $\mathrm{head}_{G^*}(e_1^i) = \mathrm{head}_{G^*}(e_2^j)$.

For each case, we obtain the following by the definition of the edge set of $\mathcal{G}$.

- If (1) holds and $j \neq q+1$, then $\mathcal{G}$ has an edge from $(e_1^i, e_2^j)$ to $(e_1^i, e_2^{j+1})$. Note that if $v := \mathrm{head}_{G^*}(e_2^j) \in V_0$, then $E(P_2) \cap E(G_v)$ forms a path in $G_v$ from $\mathrm{tail}_G(e_2^{j+1})$ to $\mathrm{head}_G(e_2^j)$.
- If (2) holds and $i \neq p+1$, then $\mathcal{G}$ has an edge from $(e_1^i, e_2^j)$ to $(e_1^{i+1}, e_2^j)$. Note that if $v := \mathrm{head}_{G^*}(e_1^i) \in V_0$, then $E(P_1) \cap E(G_v)$ forms a path in $G_v$ from $\mathrm{head}_G(e_1^i)$ to $\mathrm{tail}_G(e_1^{i+1})$.
- If (3) holds, then $\mathcal{G}$ has an edge from $(e_1^i, e_2^j)$ to $(e_1^{i+1}, e_2^{j+1})$. Note that if $v := \mathrm{head}_{G^*}(e_1^i) = \mathrm{head}_{G^*}(e_2^j) \in V_0$, then $E(P_1) \cap E(G_v)$ and $E(P_2) \cap E(G_v)$ form two edge-disjoint paths in $G_v$ such that one is from $\mathrm{head}_G(e_1^i)$ to $\mathrm{tail}_G(e_1^{i+1})$ and the other is from $\mathrm{tail}_G(e_2^{j+1})$ to $\mathrm{head}_G(e_2^j)$.

By observing that (1) holds if $i = p+1$ and (2) holds if $j = q+1$, we can see that $\mathcal{G}$ has an edge from $(e_1^i, e_2^j)$ to $(e_1^i, e_2^{j+1})$, $(e_1^{i+1}, e_2^j)$, or $(e_1^{i+1}, e_2^{j+1})$ unless $(i, j) = (p+1, q+1)$. We begin with $(i, j) = (0, 0)$ and find an edge leaving $(e_1^i, e_2^j)$ in $\mathcal{G}$ as above, repeatedly. Then, we obtain a path in $\mathcal{G}$ from $(e_1^0, e_2^0) = (s'_1 s_1, t'_2 t_2)$ to $(e_1^{p+1}, e_2^{q+1}) = (t_1 t'_1, s_2 s'_2)$, which shows the sufficiency.

*Necessity* ("only if" part). Suppose that there is a path in $\mathcal{G}$ from $(f_1^0, f_2^0) := (s'_1 s_1, t'_2 t_2)$ to $(f_1^r, f_2^r) := (t_1 t'_1, s_2 s'_2)$ that traverses vertices $(f_1^0, f_2^0)$, $(f_1^1, f_2^1)$, …, $(f_1^r, f_2^r)$ of $\mathcal{G}$ in this order. In this proof, we regard a path in $G$ as a sequence of edges, and the concatenation of two paths $P$ and $Q$ is denoted by $P \cdot Q$. We define two paths $P_1$ and $P_2$ as follows.

1. Set $P_1 = P_2 = \emptyset$.
2. For $i = 0, 1, 2, \dots, r$, we update $P_i$ as follows.
   - Suppose that $f_1^{i+1} = f_1^i$, $\mathrm{head}_{G^*}(f_2^i) = \mathrm{tail}_{G^*}(f_2^{i+1}) =: v$, and there is no dipath in $G^*$ from $\mathrm{head}_{G^*}(f_1^i)$ to $v$. In this case, let $Q$ be the path in $G_v$ from $\mathrm{tail}_G(f_2^{i+1})$ to $\mathrm{head}_G(f_2^i)$ if $v \in V_0$ and let $Q = \emptyset$ if $v \notin V_0$. Then, update $P_2$ as $P_2 \leftarrow f_2^{i+1} \cdot Q \cdot P_2$.
   - Suppose that $f_2^{i+1} = f_2^i$, $\mathrm{head}_{G^*}(f_1^i) = \mathrm{tail}_{G^*}(f_1^{i+1}) =: v$, and there is no dipath in $G^*$ from $\mathrm{head}_{G^*}(f_2^i)$ to $v$. In this case, let $Q$ be the path in $G_v$ from $\mathrm{head}_G(f_1^i)$ to $\mathrm{tail}_G(f_1^{i+1})$ if $v \in V_0$ and let $Q = \emptyset$ if $v \notin V_0$. Then, update $P_1$ as $P_1 \leftarrow P_1 \cdot Q \cdot f_1^{i+1}$.
   - Suppose that $\mathrm{head}_{G^*}(f_1^i) = \mathrm{head}_{G^*}(f_2^i) = \mathrm{tail}_{G^*}(f_1^{i+1}) = \mathrm{tail}_{G^*}(f_2^{i+1}) =: v$. In this case, if $v \in V_0$, then $G_v$ contains two edge-disjoint paths $Q_1$ and $Q_2$ such that $Q_1$ is from $\mathrm{head}_G(f_1^i)$ to $\mathrm{tail}_G(f_1^{i+1})$ and $Q_2$ is from $\mathrm{tail}_G(f_2^{i+1})$ to $\mathrm{head}_G(f_2^i)$. Let $Q_1 = Q_2 = \emptyset$ if $v \notin V_0$. Then, update $P_1$ and $P_2$ as $P_1 \leftarrow P_1 \cdot Q_1 \cdot f_1^{i+1}$ and $P_2 \leftarrow f_2^{i+1} \cdot Q_2 \cdot P_2$.

Then, $P_1$ and $P_2$ are edge-disjoint paths in $G$ such that $P_i$ is from $s_i$ to $t_i$ and $E(P_i) \subseteq E_i$ for $i = 1, 2$, which shows the necessity. ◀

Since $\mathcal{G}$ contains at most $|E|^2$ vertices, we can detect a path in $\mathcal{G}$ in polynomial time. Thus, Lemma 8 shows that the directed 2-DSPP can be solved in polynomial time.

We note that the most time consuming part of our algorithm is to construct $\mathcal{G}$. We have already seen that, for each pair of vertices in $\mathcal{G}$, the existence of an edge between them can be checked by solving the two disjoint paths problem in an acyclic digraph. Thus, in a naive implementation of our algorithm, we solve the two disjoint paths problem in an acyclic digraph $O(|E|^4)$ times. If we adopt the algorithm of [18] for the two disjoint paths problem, which runs in $O(|V||E|)$ time, the total running time of our algorithm is $O(|V||E|^5)$. Note that a faster algorithm for the two disjoint paths problem is proposed in [21]. Although the above estimation of the running time is very rough, we do not discuss its improvement in this paper, since we focus on the polynomial solvability of the problem.

## 4 Disjoint Shortest Paths with Two Terminal Pairs

In this section, we extend Theorem 2 to the case when the digraph has two terminal pairs. More precisely, for fixed integers $k_1$ and $k_2$, we consider the following problem and give a polynomial time algorithm for it.

Directed Disjoint Shortest Paths Problem with Two Terminal Pairs.

**Input.** A digraph $G = (V, E)$ with a length function $l : E \to \mathbb{R}_+$, two pairs of vertices $(s_1, t_1)$ and $(s_2, t_2)$ in $G$.

**Find.** Internally-vertex-disjoint (or edge-disjoint) paths $P^1_1, \dots, P^1_{k_1}, P^2_1, \dots, P^2_{k_2}$ such that $P^i_j$ is a shortest path from $s_i$ to $t_i$ for $i = 1, 2$ and $j = 1, 2, \dots, k_i$.

Our result is formally stated as follows.

▶ **Theorem 9.** *Let $k_1$ and $k_2$ be fixed integers. If the length of each dicycle is positive, both internally-vertex-disjoint and edge-disjoint versions of the directed disjoint shortest paths problem with two terminal pairs can be solved in polynomial time.*

**Proof.** In the same way as the proof of Theorem 2, it suffices to give an algorithm for the edge-disjoint version. For $i = 1, 2$, let $E_i \subseteq E$ be the set of all the edges that are contained in some shortest path from $s_i$ to $t_i$, which satisfy Lemmas 6 and 7. Then, an $s_i$-$t_i$ path is a shortest $s_i$-$t_i$ path if and only if it consists of edges in $E_i$.

We add $2(k_1 + k_2)$ vertices $s'_{1,1}, \dots, s'_{1,k_1}, s'_{2,1}, \dots, s'_{2,k_2}, t'_{1,1}, \dots, t'_{1,k_1}, t'_{2,1}, \dots, t'_{2,k_2}$, and $2(k_1 + k_2)$ edges $s'_{1,j}s_1$ and $t_1, t'_{1,j}$ for $j = 1, \dots, k_1$, and $s'_{2,j}s_2$ and $t_2, t'_{2,j}$ for $j = 1, \dots, k_2$. We update $E_i \leftarrow E_i \cup \{s'_{i,j}s_i, t_it'_{i,j} \mid j = 1, \dots, k_i\}$ for $i = 1, 2$. Then, we can rephrase the problem to the following: find edge-disjoint paths $P^1_1, \dots, P^1_{k_1}, P^2_1, \dots, P^2_{k_2}$ such that $E(P^i_j) \subseteq E_i$ and $P^i_j$ is a path whose first and last edges are $s'_{i,j}s_i$ and $t_it'_{i,j}$ for each $i$ and $j$.

Define $E_0$, $E^*_1$, $E^*_2$, $G^*$, $V_0$, and $G_v$ for $v \in V_0$ in the same way as the proof of Theorem 2. Let $S_0 := \{(i, j) \mid i = 1, 2,\ j = 1, \dots, k_i\}$. We define a digraph $\mathcal{G}$ whose vertex set is $W = (E^*_1)^{k_1} \times (E^*_2)^{k_2}$ as follows. For $(e^1_1, \dots, e^1_{k_1}, e^2_1, \dots, e^2_{k_2}) \in W$ and $(f^1_1, \dots, f^1_{k_1}, f^2_1, \dots, f^2_{k_2}) \in W$, $\mathcal{G}$ has an edge from $(e^1_1, \dots, e^1_{k_1}, e^2_1, \dots, e^2_{k_2})$ to $(f^1_1, \dots, f^1_{k_1}, f^2_1, \dots, f^2_{k_2})$ if there exists a non-empty set $S \subseteq S_0$ and a vertex $v \in V^*$ such that

> $\mathrm{head}_{G^*}(e^i_j) = \mathrm{tail}_{G^*}(f^i_j) = v$ for $(i, j) \in S$, and $e^i_j = f^i_j$ and there is no path in $G^*$ from $\mathrm{head}_{G^*}(e^i_j)$ to $v$ for $(i, j) \in S_0 \setminus S$. Furthermore, if $v \in V_0$, then $G_v$ contains $|S|$ edges disjoint paths such that each path is from $\mathrm{head}_G(e^1_j)$ to $\mathrm{tail}_G(f^1_j)$ with $(1, j) \in S$ or from $\mathrm{tail}_G(f^2_j)$ to $\mathrm{head}_G(e^2_j)$ with $(2, j) \in S$.

Note that this is a generalization of the construction in the proof of Theorem 2. To construct $\mathcal{G}$, it suffices to solve the disjoint paths problem with at most $k$ terminal pairs in each acyclic digraph $G_v$, which can be done in polynomial time by [6].

In the same way as Lemma 8, there is a path in $\mathcal{G}$ from $(s'_{1,1}s_1, \dots, s'_{1,k_1}s_1, t'_{2,1}t_2, \dots, t'_{2,k_2}t_2)$ to $(t_1t'_{1,1}, \dots, t_1t'_{1,k_1}, s_2s'_{2,1}, \dots, s_2s'_{2,k_2})$ if and only if $G$ has $k_1 + k_2$ edge-disjoint paths $P^1_1, \dots, P^1_{k_1}, P^2_1, \dots, P^2_{k_2}$ such that $E(P^i_j) \subseteq E_i$ and $P^i_j$ is a path whose first and last edges are $s'_{i,j}s_i$ and $t_it'_{i,j}$ for each $i$ and $j$. Since $\mathcal{G}$ has a polynomial size in $|V|$, we can detect a path in $\mathcal{G}$ from $(s'_{1,1}s_1, \dots, s'_{1,k_1}s_1, t'_{2,1}t_2, \dots, t'_{2,k_2}t_2)$ to $(t_1t'_{1,1}, \dots, t_1t'_{1,k_1}, s_2s'_{2,1}, \dots, s_2s'_{2,k_2})$ in polynomial time. Hence, we can solve the directed disjoint shortest paths problem with two terminal pairs in polynomial time. ◀

In order to construct $\mathcal{G}$, we solve the $k$ disjoint paths problem in an acyclic digraph $|E|^{O(k)}$ times. Since the $k$ disjoint paths problem in an acyclic digraph can be solved in $|E|^{O(k)}$ time [6], the total running time of our algorithm is $|E|^{O(k)}$, which is also denoted by $|V|^{O(k)}$.

We note that, by using the same argument as the proofs of Theorems 2 and 9, we can show that the directed $k$-DSPP in acyclic digraphs can be solved in polynomial time if $k$ is a fixed constant.

▶ **Proposition 10.** *If $k$ is a fixed constant and the input graph is acyclic, both vertex-disjoint and edge-disjoint versions of the directed $k$-DSPP can be solved in polynomial time.*

**Proof.** It suffices to consider the edge-disjoint version. For $i = 1, \dots, k$, let $E_i \subseteq E$ be the set of all the edges that are contained in some shortest path from $s_i$ to $t_i$. Then, an $s_i$-$t_i$ path is a shortest $s_i$-$t_i$ path if and only if it consists of edges in $E_i$. We add $2k$ vertices $s'_1, \dots, s'_k, t'_1, \dots, t'_k$, and $2k$ edges $s'_1s_1, \dots, s'_ks_k, t_1t'_1, \dots, t_kt'_k$, and update $E_i \leftarrow E_i \cup \{s'_is_i, t_it'_i\}$ for $i = 1, \dots, k$. The obtained acyclic digraph is also denoted by $G$. Then, the directed $k$-DSPP is equivalent to finding $k$ edge-disjoint paths $P_1, \dots, P_k$ such that $E(P_i) \subseteq E_i$ and $P_i$ is a path whose first and last edges are $s'_is_i$ and $t_it'_i$, respectively.

We define a digraph $\mathcal{G}$ whose vertex set is $W = E_1 \times \dots \times E_k$ as follows. For $(e_1, \dots, e_k) \in W$ and $(f_1, \dots, f_k) \in W$, $\mathcal{G}$ has an edge from $(e_1, \dots, e_k)$ to $(f_1, \dots, f_k)$ if there exists an index $i$ such that $e_j = f_j$ for $j \in \{1, \dots, k\} \setminus \{i\}$, $\mathrm{head}_G(e_i) = \mathrm{tail}_G(f_i) =: v$, and there is no path in $G$ from $\mathrm{head}_G(e_j)$ to $v$ for $j \in \{1, \dots, k\} \setminus \{i\}$.

In the same way as Lemma 8, there is a path in $\mathcal{G}$ from $(s'_1s_1, \dots, s'_ks_k)$ to $(t'_1t_1, \dots, t'_kt_k)$ if and only if $G$ has $k$ edge-disjoint paths $P_1, \dots, P_k$ such that $E(P_i) \subseteq E_i$ and $P_i$ is a path whose first and last edges are $s'_is_i$ and $t_it'_i$ for each $i$. Since $\mathcal{G}$ has $|V|^{O(k)}$ vertices, a path in $\mathcal{G}$ from $(s'_1s_1, \dots, s'_ks_k)$ to $(t'_1t_1, \dots, t'_kt_k)$ can be detected in $|V|^{O(k)}$ time. ◀

## 5 Planar Cases

In this section, we discuss the case when the input (di)graph is planar. We first give a proof of Theorem 4, that is, we show that the vertex-disjoint version of the directed $k$-DSPP can be solved in polynomial time if $k$ is a fixed constant and the input digraph is planar.

**Proof of Theorem 4.** For $i = 1, \dots, k$, let $E_i \subseteq E$ be the set of all the edges that are contained in some shortest path from $s_i$ to $t_i$. Since an $s_i$-$t_i$ path is a shortest $s_i$-$t_i$ path if and only if it consists of edges in $E_i$, the directed $k$-DSPP in a planar digraph can be reduced to the following problem: given a planar digraph $G = (V, E)$, subsets $E_1, \dots, E_k \subseteq E$, and $k$ pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$ in $G$, find vertex-disjoint paths $P_1, \dots, P_k$ such that $E(P_i) \subseteq E_i$ and $P_i$ is a path from $s_i$ to $t_i$ for $i = 1, \dots, k$. It is shown in [16] that this problem can be solved in $|V|^{O(k)}$ time if $G$ is planar. Therefore, for fixed $k$, the directed $k$-DSPP can be solved in polynomial time if the input digraph is planar. ◀

By replacing each edge with two parallel edges in opposite directions, we can reduce the undirected version to the directed version. Hence, Theorem 4 implies the following as a corollary.

▶ **Corollary 11.** *If the input graph is planar, the vertex-disjoint version of the undirected $k$-DSPP can be solved in $|V|^{O(k)}$ time.*

We note that Schrijver's algorithm for finding disjoint paths $P_1, \ldots, P_k$ with $E(P_i) \subseteq E_i$ [16] works only for the vertex-disjoint case, and no polynomial time algorithm is known for the edge-disjoint version of this problem. However, when the graph is undirected, the edge-disjoint version of $k$-DSPP can be solved in polynomial time (Theorem 5). To prove Theorem 5, we first give a polynomial time algorithm for the case when the obtained paths do not cross each other. Here, we say that two edge-disjoint paths $P$ and $Q$ in a planar graph *cross* at a vertex $v$ if $P$ contains two edges $e_1$ and $e_2$ and $Q$ contains two edges $f_1$ and $f_2$ such that $e_1, f_1, e_2$, and $f_2$ are incident to $v$ clockwise in this order. The problem is formally described as follows.

Undirected $k$ Edge-disjoint Non-crossing Shortest Paths Problem

**Input.** A planar graph $G = (V, E)$ with a length function $\ell : E \to \mathbb{R}_+$ and $k$ pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ in $G$.

**Find.** Pairwise edge-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ is a shortest path from $s_i$ to $t_i$ for $i = 1, 2, \ldots, k$ and they do not cross each other, if they exist.

▶ **Proposition 12.** *The undirected $k$ edge-disjoint non-crossing shortest paths problem can be solved in $|V|^{O(k)}$ time.*

**Proof.** We first reduce the problem to the case when each terminal is of degree one. Suppose we are given an instance of the the undirected $k$ edge-disjoint non-crossing shortest paths problem. For $i = 1, \ldots, k$, we guess the first and last edges of $P_i$, say $s_i u_i$ and $v_i t_i$. Then, replace edge $s_i u_i$ with a new vertex $u'_i$ and a new edge $u'_i u_i$ of length $\ell(s_i u_i)$, and define a new terminal $s'_i = u'_i$. Similarly, replace edge $v_i t_i$ with a new vertex $v'_i$ and a new edge $v_i v'_i$ of length $\ell(v_i t_i)$, and define a new terminal $t'_i = v'_i$. In the obtained graph, we consider the undirected $k$ edge-disjoint non-crossing shortest paths problem with terminal pairs $(s'_1, t'_1), \ldots, (s'_k, t'_k)$. Note that each terminal is of degree one in the obtained instance. Since the number of choices of $s_i u_i$ and $v_i t_i$ is at most $|V|^{O(k)}$, in order to solve the original instance, it suffices to solve $|V|^{O(k)}$ instances in which each terminal is of degree one.

In what follows, we give an algorithm for the case when each terminal is of degree one by using a reduction to the vertex-disjoint version of the undirected $k$-DSPP. Suppose that we are given an instance $G = (V, E)$, $\ell : E \to \mathbb{R}_+$, and $(s_1, t_1), \ldots, (s_k, t_k)$ of the undirected $k$ edge-disjoint non-crossing shortest paths problem in which each terminal is of degree one. For a vertex $v \in V$ of degree at least four, let $e_1, \ldots, e_r$ be the edges that are incident to $v$ clockwise in this order. We replace $v$ with $r$ vertices $w_1, \ldots, w_r$ so that each edge $e_i$ is incident to $w_i$, and add $r$ edges $w_1 w_2, w_2 w_3, \ldots, w_{r-1} w_r, w_r w_1$ of length zero (see Fig. 3). Note that this transformation keeps the planarity of the graph.

By applying this transformation to every vertex $v \in V$ of degree at least four, we obtain a new planar graph $G' = (V', E')$ whose maximum degree is at most three. We can easily see that the undirected $k$ edge-disjoint non-crossing shortest paths problem in $G$ is equivalent to that in $G'$. Since the maximum degree of $G'$ is at most three and the degree of each terminal

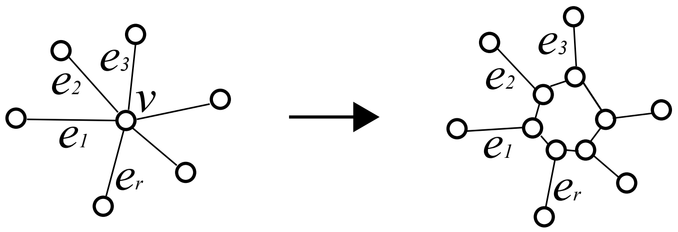


**Figure 3** Reduction to the vertex-disjoint version.

is one, edge-disjoint paths in $G'$ have to be vertex-disjoint, and hence it suffices to solve the vertex-disjoint version of the undirected $k$-DSPP in $G'$. This can be done in $|V|^{O(k)}$ time by Corollary 11, which completes the proof. ◀

We are now ready to prove Theorem 5.

**Proof of Theorem 5.** Suppose we are given an instance of the edge-disjoint version of the undirected $k$-DSPP in a planar graph. We begin with the following claim.

▶ **Claim 13.** *If there exists a solution of the edge-disjoint version of the undirected $k$-DSPP in a planar graph, then there exists a solution $P_1, \ldots, P_k$ such that $P_i$ and $P_j$ cross at most once for every pair $i, j \in \{1, \ldots, k\}$.*

**Proof.** Let $P_1, \ldots, P_k$ be a solution of the edge-disjoint version of the undirected $k$-DSPP that minimizes the total number of crossings of the paths. We show that this solution satisfies the condition in the claim. Assume to the contrary that $P_i$ and $P_j$ cross at two distinct vertices $u$ and $v$. Then, there exists a subpath $Q_i$ of $P_i$ and a subpath $Q_j$ of $P_j$ such that both $Q_i$ and $Q_j$ are paths from $u$ to $v$. Since $P_i$ is a shortest path from $s_i$ to $t_i$ and $P_j$ is a shortest path from $s_j$ to $t_j$, we have $\ell(Q_i) = \ell(Q_j)$. This shows that, we can obtain another solution of the edge-disjoint version of the undirected $k$-DSPP by replacing $P_i$ and $P_j$ with two paths $P'_i$ and $P'_j$ such that $E(P'_i) = (E(P_i) \setminus E(Q_i)) \cup E(Q_j)$ and $E(P'_j) = (E(P_j) \setminus E(Q_j)) \cup E(Q_i)$. We can see that the number of crossings of $P'_i$ and $P'_j$ is strictly smaller than that of $P_i$ and $P_j$. We can also see that, for any $h \in \{1, \ldots, k\} \setminus \{i, j\}$, the number of crossings of $P_h$ and $\{P'_i, P'_j\}$ is at most that of $P_h$ and $\{P_i, P_j\}$. Therefore, the total number of crossings of the obtained solution is smaller than the original solution, which is a contradiction. ◀

Let $P_1, \ldots, P_k$ be a solution of the edge-disjoint version of the undirected $k$-DSPP satisfying the condition in the above claim. For $i = 1, \ldots, k$, by the above claim, there exist at most $k-1$ vertices $u^i_1, u^i_2, \ldots, u^i_{r_i}$ such that $P_i$ crosses another path at some $u^i_j$ and $s_i =: u^i_0, u^i_1, u^i_2, \ldots, u^i_{r_i}, u^i_{r_i+1} := t_i$ appear in this order along $P_i$. Then, $P_i$ can be divided into $r_i + 1 \leq k$ subpaths $Q^i_1, \ldots Q^i_{r_i+1}$, where $Q^i_j$ is a shortest path from $u^i_{j-1}$ to $u^i_j$. By the definition of $Q^i_j$, we can see that $Q^i_j$ $(i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1)$ are edge-disjoint paths and they do not cross each other.

With this observation, we can solve the edge-disjoint version of the undirected $k$-DSPP as follows.

**Step 1.** For $i = 1, \ldots, k$, guess an integer $r_i \leq k-1$ and vertices $u^i_1, u^i_2, \ldots, u^i_{r_i}$.

**Step 2.** Find pairwise edge-disjoint paths $Q^i_j$ $(i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1)$ such that they do not cross each other and $Q^i_j$ is a shortest path from $u^i_{j-1}$ to $u^i_j$, where $u^i_0 = s_i$ and $u^i_{r_i+1} = t_i$.

**Step 3.** For each $i$, define $P_i$ as the concatenation of $Q^i_1, \ldots, Q^i_{r_i+1}$. Check whether or not $P_1, \ldots, P_k$ form a solution of the original instance.

In Step 1, the number of choices of $r_i$ and $u^i_1, u^i_2, \ldots, u^i_{r_i}$ is at most $|V|^{O(k^2)}$. In Step 2, we can find desired edge-disjoint paths $Q^i_j$ ($i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1$) if they exist in $|V|^{O(k^2)}$ time by Proposition 12. Note that the number of terminals is at most $O(k^2)$. In Step 3, we can easily check whether or not $P_1, \ldots, P_k$ are a solution of the original problem in polynomial time. Therefore, the edge-disjoint version of the undirected $k$-DSPP can be solved in $|V|^{O(k^2)}$ time if the input graph is planar. ◀

## References


1 Andreas Björklund and Thore Husfeldt. Shortest two disjoint paths in polynomial time. In *ICALP*, pages 211–222, 2014.

2 Glencora Borradaile, Amir Nayyeri, and Farzad Zafarani. Towards single face shortest vertex-disjoint paths in undirected planar graphs. In *ESA*, pages 227–238, 2015.

3 Marek Cygan, Dániel Marx, Marcin Pilipczuk, and Michał Pilipczuk. The planar directed $k$-vertex-disjoint paths problem is fixed-parameter tractable. In *FOCS*, pages 197–206, 2013.

4 Éric Colin de Verdière and Alexander Schrijver. Shortest vertex-disjoint two-face paths in planar graphs. In *STACS*, pages 181–192, 2008.

5 Tali Eilam-Tzoreff. The disjoint shortest paths problem. *Discrete Applied Mathematics*, 85(2):113–138, 1998.

6 Steven Fortune, John E. Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10:111–121, 1980.

7 András Frank. *Paths, Flows, and VLSI-Layout*, chapter Packing paths, cuts and circuits – a survey, pages 49–100. Springer-Verlag, 1990.

8 Hiroshi Hirai and Hiroyuki Namba. Shortest $(A + B)$-path packing via hafnian. arXiv:1603.08073, 2016.

9 Richard M. Karp. On the computational complexity of combinatorial problems. *Networks*, 5:45–68, 1975.

10 Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce Reed. The disjoint paths problem in quadratic time. *Journal of Combinatorial Theory, Series B*, 102(2):424–435, 2012.

11 Yusuke Kobayashi and Christian Sommer. On shortest disjoint paths in planar graphs. *Discrete Optimization*, 7(4):234–245, 2010.

12 James F. Lynch. The equivalence of theorem proving and the interconnection problem. *SIGDA Newsletter*, 5(3):31–36, 1975.

13 Richard G. Ogier, Vladislav Rutenburg, and Nachum Shacham. Distributed algorithms for computing shortest pairs of disjoint paths. *IEEE Transactions on Information Theory*, 39(2):443–455, 1993.

14 Neil Robertson and Paul D. Seymour. *Paths, Flows, and VLSI-Layout*, chapter An outline of a disjoint paths algorithm, pages 267–292. Springer-Verlag, 1990.

15 Neil Robertson and Paul D. Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63(1):65–110, 1995. doi:10.1006/jctb.1995.1006.

16 Alexander Schrijver. Finding $k$ disjoint paths in a directed planar graph. *SIAM Journal on Computing*, 23(4):780–788, 1994.

17 Paul D. Seymour. Disjoint paths in graphs. *Discrete Mathematics*, 29:293–309, 1980.

18 Y. Shiloach and Y. Perl. Finding two disjoint paths between two pairs of vertices in a graph. *J. ACM*, 25(1):1–9, 1978.

19 Yossi Shiloach. A polynomial solution to the undirected two paths problem. *Journal of the ACM*, 27(3):445–456, 1980. doi:10.1145/322203.322207.

**20** Anand Srinivas and Eytan Modiano. Finding minimum energy disjoint paths in wireless ad-hoc networks. *Wireless Networks*, 11(4):401–417, 2005. `doi:10.1007/s11276-005-1765-0`.

**21** Torsten Tholey. Finding disjoint paths on directed acyclic graphs. In *WG*, pages 319–330, 2005.

**22** Carsten Thomassen. 2-linked graphs. *European Journal of Combinatorics*, 1:371–378, 1980.