# Pricing Social Goods*†

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## Abstract

Social goods are goods that grant value not only to their owners but also to the owners' surroundings, be it their families, friends or office mates. The benefit a non-owner derives from the good is affected by many factors, including the type of the good, its availability, and the social status of the non-owner. Depending on the magnitude of the benefit and on the price of the good, a potential buyer might stay away from purchasing the good, hoping to free ride on others' purchases. A revenue-maximizing seller who sells social goods must take these considerations into account when setting prices for the good. The literature on optimal pricing has advanced considerably over the last decade, but little is known about optimal pricing schemes for selling social goods. In this paper, we conduct a systematic study of revenue-maximizing pricing schemes for social goods: we introduce a Bayesian model for this scenario, and devise nearly-optimal pricing schemes for various types of externalities, both for simultaneous sales and for sequential sales.

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## 1 Introduction

Many goods exhibit a positive externality not only on their owner, but also on other parties. For instance, a coffee machine purchased by an employee benefits all of her office mates, and essentially reduces the probability of another coffee machine to be purchased. Examples of these kinds of goods are abundant: A high-schooler who has many friends with cars that can drive him around might be less tempted to buy a new car. A reputable store might draw large customer traffic and benefit other stores in the shopping mall. Therefore, an aggressive advertising campaign carried out by such a store might reduce the likelihood of another store running a campaign in parallel. In all of these scenarios the externalities depend on the type of good, on the social status of the party with whom the good is shared, and on the set of parties who own the good. In the coffee machine example, the machine is typically used by all the individuals sharing the office space. In the shopping mall, some types of stores (*e.g.*, fast food restaurants) might benefit from any traffic in the shopping mall, whereas more specialized stores may benefit from ad campaigns that draw costumers interested in a similar

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kind of product (*e.g.*, `Staples` may attract costumers similar to those interested in `Office Depot` products). The benefit of a high school student depends on his social status and on the set of friends who own a car.

Because of the abundance of goods that exhibit externalities similar to the ones in the examples above, their study is of great applicability. We term these goods *social goods*. When selling social goods, a seller must take into account the types of buyers in the market and the benefit they derive from other sets of buyers purchasing the good. Our main goal is to study how to sell goods in a way that approximately maximizes the seller's revenue in the presence of externalities.

To study this problem, we consider a setting with a single type of good, of unlimited supply, and a set of $n$ agents; each agent $i \in [n]$ has a non-negative valuation $v_i$ for purchasing the good, drawn independently from a distribution $F_i$. We denote the product distribution by $\mathcal{F} = \times_{i\in[n]} F_i$. Unless stated otherwise, we assume the $F_i$'s are regular.[1]

If an agent does not purchase the good, but the good is purchased by others, then this agent derives only a fraction of her value, depending on the set of agents and the type of externality the good exhibits on the agent. This type of externality is captured in our model by an *externality function* $x_i : 2^{[n]} \to [0, 1]$, where $x_i(S)$ denotes the fraction of $v_i$ an agent $i$ derives when the good is purchased by the set of agents $S$. We assume that $x_i$ is publicly known (as it captures the agent's externalities), monotonically non-decreasing and normalized; i.e., $x_i(\emptyset) = 0$, for every $T \subseteq S$, $x_i(T) \leq x_i(S)$, and $x_i(S) = 1$ whenever $i \in S$. We consider three structures of the function $x_i$, corresponding to three types of externalities of social goods.

**(a)** *Full externalities* (commonly known as "public goods"): in this scenario all agents derive their entire value if the good is purchased by any agent. Therefore, $x_i(S) = 1$ if and only if $S \neq \emptyset$. This model captures goods that are non-excludable, such as a coffee machine in a shared office. A special case of this scenario, where valuations are independently and identically distributed, has been studied in [10].

**(b)** *Status-based externalities*: in this scenario, agent $i$'s "social status" is captured by some *discount factor* $w_i \in [0, 1]$, which corresponds to the fraction of the value an agent $i$ derives from a good when purchased by another party. That is,

$$x_i(S) = \begin{cases} 1 & i \in S, \\ w_i & i \notin S \text{ and } S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This model captures settings that exhibit asymmetry with respect to the benefit different agents derive from goods they do not own (*e.g.*, a fast food restaurant or a popular high-school student in the above examples).

**(c)** *Availability-based externalities*: in this scenario, the availability of a good increases as more agents purchase a good, and therefore, an agent derives a larger fraction of her value as more agents purchase a good. This is captured by the following externality function.

$$x_i(S) = \begin{cases} 1 & i \in S, \\ w(|S|) & i \notin S. \end{cases} \tag{2}$$

Here, $w : \{0, \ldots, n-1\} \to [0, 1]$ is a monotonically non-decreasing function with $w(0) = 0$. Examples of such scenarios include objects that are often shared by neighbors (e.g., snow blowers, lawn mowers), office supplies, etc.

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[1] This means that the virtual valuation function $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is non-decreasing.

Notice that the full externalities scenario is a special case of both the social-status (where $w_i = 1$ for every $i$) and the availability (where $w(k) = 1$ for every $k > 0$) models.

Our focus is on posted-price mechanisms, which exhibit many desired properties: they are simple, distributed, straightforward, and strategyproof. Our goal is to maximize the revenue extracted by the seller. We distinguish between discriminatory and non-discriminatory prices. Naturally, using discriminatory prices can often lead to higher revenue for the seller [18, 15]. Price discrimination is commonly used in the US [12], but user studies reveal that many users believe that this practice is illegal, and consider these acts to be an invasion of privacy [5]. Therefore, offering non-discriminatory prices may be critical for maintaining the seller's reputation. We show scenarios in which setting the same price for all users produces (almost) as much revenue as engaging in price discrimination.

We consider two natural sale models: (a) a *simultaneous sale*, where the seller simultaneously sets take-it-or-leave-it prices for all agents, after which agents play a simultaneous Bayesian game, and each agent decides whether or not to buy at the price offered to her; and (b) a *sequential sale*, in which the agents arrive sequentially, and each one is offered a take-it-or-leave-it price upon arrival. In this case, the price and the agent's decision may depend on the set of agents that purchased the good before the arrival of the current agent. We distinguish between *adaptive* and *non-adaptive* pricing schemes, which differ in whether the price can depend upon the set of agents who purchased the good prior to the agent's arrival.

In both simultaneous and sequential sales, assuming that agent $i$ is offered a take-it-or-leave-it price $p_i$, and that the good is eventually purchased by a set $S \subseteq [n]$ of agents, the utility of agent $i$ is:

$$u_i(S, p_i) = \begin{cases} v_i - p_i & \text{if } i \in S, \\ v_i \cdot x_i(S) & \text{if } i \notin S. \end{cases} \tag{3}$$

As shown in Section 2, a set of prices induces equilibria of the game (multiple equilibria in the simultaneous model, and a single one in the sequential model). Every equilibrium is characterized by a set of *threshold* strategies for the agents, where an agent buys the good if and only if her value exceeds the threshold.

## 1.1 Our contribution

We provide results for the three aforementioned models. In this section, we provide informal statements of our results. The exact bounds we achieve are summarized in Table 1. Due to space limitation, some of the formal statements and proofs are deferred to the full version.

### (a) Full externalities

**Theorem (informal):** There exist poly-time algorithms for computing pricing schemes for settings with full externalities that give a constant factor approximation to the optimal pricing scheme, for both simultaneous and sequential sales. Moreover, this result can be achieved using non-discriminatory prices, despite asymmetry among buyers.

To derive this result, we first analyze the equilibria in simultaneous and sequential models. We show a surprising equivalence between the revenue attainable in the best equilibrium at simultaneous and sequential sales, albeit induced by different prices. A corollary of this equivalence is that the optimal attainable revenue at a sequential sale does not depend on the order of agents. Furthermore, we observe that in both simultaneous and sequential sales, the revenue attainable is upper bounded by the optimal revenue from selling a single *private* good (i.e., a good that grants value only to their owners)[2].

We proceed as follows. For simultaneous sales, we establish a method for transforming prices for the sale of a single private good *in expectation* into prices for selling public goods, which preserve the revenue up to a constant factor in every equilibrium. Since selling a single good in expectation yields at least as much revenue as selling a single good deterministically, this implies a near-optimal pricing scheme for simultaneous sales of public goods.

For sequential sales, we use the theory of prophet inequalities. Consider prices that induce thresholds that are equal to the prices that emerge from the prophet inequalities. We show that such prices obtain at least half of the revenue obtained from the prophet inequalities prices in the private good model. We use this connection to obtain a pricing scheme that gives 4-approximation to the revenue of the optimal sequential sale of public goods.

Finally, we show how to compute nearly-optimal *non-discriminatory* prices, even for asymmetric agents, in both the simultaneous and sequential models.

### (b) Status-based externalities

**Theorem (informal):** There exist poly-time algorithms for computing pricing schemes for settings with status-based externalities that give a constant factor approximation to the optimal pricing scheme, for both simultaneous and sequential sales.[3]

For sequential sales, we devise a *non-adaptive* pricing scheme, while the benchmark is the optimal adaptive pricing scheme. To obtain this result, we first show that a seller who is restricted to set only two prices per agent can extract as much revenue as one who can present exponentially many prices. We then show that the optimal revenue in this simpler case can be decomposed into two components: a private component (monotonically decreasing in the agents' discount factors) and a public component (monotonically increasing in the discount factors). The private component can be approximated by simulating $n$ private sales, setting thresholds equal to the monopoly prices. The public component can be approximated by similar techniques to the ones introduced for public goods. Therefore, the better of the two mechanisms extracts a constant fraction of the optimal revenue. A similar decomposition technique is established for the case of simultaneous sales. Our result for the sequential case is essentially a reduction: given prices that yield a $c$-approximation for the optimal sequential sale in the full externalities model, one can find prices that $(c+2)$-approximate the optimal sequential sale in the status-based externalities model.

### (c) Availability-based externalities

**Theorem (informal):** There exists a poly-time algorithm for computing a pricing scheme for sequential sales with availability-based externalities, that gives a logarithmic factor approximation (with regard to the number of buyers) to the optimal pricing scheme.

In this case, both the pricing scheme and the benchmark set a pricing function for each agent, which depends on the number of agents who have purchased the good before the arrival of the agent. To obtain this result, we decompose the revenue into $n$ components.

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[2] A similar argument was used in [10] for the special case of simultaneous sales where valuations are identically distributed.

[3] We note that no non-discriminatory prices can achieve a constant approximation in this model. Indeed, the case of private digital goods is a special case of this model, with $w_i = 0$ for every $i$.

**Table 1** Summary of our results. The columns correspond to sale models, whereas the rows correspond to externality types. The rows are further divided to sales using discriminatory and non-discriminatory prices. All the unreferenced results appear in the full version.

| | | Simultaneous | | | | Sequential | | |
|---|---|---|---|---|---|---|---|---|
| | | disc. | | non-disc. | disc. | | non-disc. |
| Full (Public goods) | i.i.d. | $\geq 4/e$ | | 4 | – | | 4 |
| | non i.i.d. | 5.83 | Thm. 3.5 | $4e$ | 4 | | $4e$ |
| Status-based | | 6.83 | | $\Omega(\log n)$ | 6 | Cor. 4.4 | $\Omega(\log n)$ |
| availability-based | | – | | – | $O(\log n)$ | | – |
| network-based | | $\Omega\left(n^{1-\epsilon}\right)$ | | – | $\Omega\left(n^{1-\epsilon}\right)$ | | – |

Component $k = 1, \ldots, n$ is upper bounded by the optimal revenue obtainable by selling $k$ identical private goods, scaled by $w(k) - w(k-1)$. We then partition the components into buckets, and compute prices based on the sequential posted pricing scheme developed by Chawla et al. [8] for selling private goods.

**General externalities.** Given the near-optimal pricing schemes above, one may be tempted to infer that every social goods scenario is amenable to a near-optimal pricing scheme. We complement our positive results with the following hardness result, refuting this hope. We consider a natural family of social goods proposed by Feldman et al. [10]: network-based externalities. In this model, externalities are represented by a graph, and an agent derives her entire value when a neighboring agent buys a good. We show that there is no poly-time algorithm to compute prices that give a non-trivial approximation to the optimal posted-price mechanism. This negative result holds for both the simultaneous and sequential models. We show that even in very restricted cases (i.e., where agents' valuations are independently and uniformly distributed on $[0, 1]$ in the simultaneous case, and agents' valuations are fixed in the sequential case), it is NP-hard to find prices that approximate the optimal posted-price mechanism to within a factor of $n^{1-\epsilon}$. A $\Theta(n)$ approximation can be trivially achieved by offering the good only to the agent maximizing the monopolist revenue. We note that this negative result rules out other natural externality structures.[4]

**Irregular distributions.** Although our results are stated and proved for regular distributions, some of our results extend to irregular distributions. Namely, we establish near optimal pricing schemes for sequential and simultaneous sales under full externalities and status-based externalities. The results of non-discriminatory prices do not extend to irregular distributions since the anonymous pricing devised in [4] do not perform well for irregular distributions. (there exist irregular distributions that there is no anonymous price that give constant approximation.)

**Organization.** Due to space limitations, some of the results and proofs are deferred to the full version[5]. In the extended abstract, we state two of our main results along with their proof ideas. The provided proofs give the flavor of the techniques that seem to be useful in studying pricing mechanisms for social goods.

[4] Some examples include: (a) for every pair of agents $i, j$, agent $i$ can borrow the good from agent $j$ with some probability $w_{ij}$. Thus, $x_i(S) = 1 - \prod_{j \in S, j \neq i}(1 - w_{ij})$; and (b) for every pair of agents $i, j$, $x_i(S) = \max_{j \in S, j \neq i} w_{ij}$.

[5] The full version appears in `https://arxiv.org/abs/1706.10009`.

The extended abstract is organized as follows. In Section 2 we describe the simultaneous and sequential sale models. In Section 3 we study the case of full externalities: In Section 3.1 we establish useful properties of equilibria, and in Section 3.2 we devise a near-optimal pricing scheme for simultaneous sales. The following are deferred to the full version:
**(a)** a near-optimal pricing scheme for sequential sales,
**(b)** a non-discriminatory pricing scheme, and
**(c)** lower bounds for simultaneous sales.
In Section 4 we present our near-optimal pricing scheme for sequential sales under status-based externalities. The near-optimal pricing scheme for simultaneous sales is deferred to the full version. The case of availability-based externalities is deferred to the full version in its entirety. The same is true for the hardness results for general externalities, as well as a discussion about the irregular case.

## 1.2 Related work

The most famous and well studied instance of social goods is public goods, when all agents derive their full value whenever a good is purchased. The study of public goods was initiated by Samuelson [19], who observed that private provisioning of public goods is not necessarily efficient; see also [17] for an overview.

The closest work to ours is that of Feldman et al. [10]. For their positive results, they consider a special case of our full externalities model — in their model agents arrive simultaneously with valuations that are drawn independently and identically from a known distribution. Our work extends this work in several dimensions. First, we consider more realistic forms of externalities that go beyond public goods. Second, we consider settings where agent valuations are drawn from non-identical distributions. Third, we provide results for settings where agents arrive either sequentially or simultaneously. Finally, some of our results extend to irregular distributions.

A line of work similar in flavor to ours, yet inherently different, is that of revenue maximization in the presence of positive externalities [1, 11, 13, 2, 6]. In this line of work, an agent's value for the good increases as more agents purchase the goods, but only if the agent purchased the good as well. Therefore, an agent is more likely to purchase the good as more agents purchase it. This is in stark contrast to our setting, where agents are less inclined to buy a good as more agents do.

Finally, there is a rich body of literature on the design of posted price mechanisms for the sale of private goods (where agents do not derive value from goods they do not own). See Chapter 4 in [14] for a textbook treatment. A sample of the work can be found in [8, 4, 16, 9, 7]. An overview of some results that are directly referred to in this work is given in the full version.

# 2 Models and preliminaries

**Simultaneous sales model.** We view a simultaneous sale game as the following two-stage game. First, the seller posts a price vector $\mathbf{p} = (p_1, \ldots, p_n)$ to the agents (agent $i$ is offered to purchase an item at price $p_i$). Subsequently, the agents play a simultaneous Bayesian game. In this model, we assume that the probability distribution of every agent is atomless.[6]

[6] Meaning that for every $q$ there exists $p$ for which $F_i(p) = q$.

Agents wish to maximize their expected utility. Given a price $p_i$, agent $i$ buys the good if her expected utility from buying, $v_i - p_i$, exceeds the utility from not buying, $v_i \cdot \mathrm{E}_{S \not\ni i}[x_i(S)]$ (where $\mathrm{E}_{S \not\ni i}$ is shorthand for $\mathrm{E}_{S: i \notin S}$). Therefore, an agent buys if and only if $v_i \geq \frac{p_i}{1-\mathrm{E}_{S \not\ni i}[x_i(S)]} =: T_i$. The strategy of every agent $i$ is therefore defined by a threshold $T_i$. Denote by $\mathbf{T} = (T_1, \ldots, T_n)$ a strategy profile, given by a vector of thresholds. A strategy profile $\mathbf{T}$ induces a probability distribution over the set $S$ of agents that purchase the good; denote this distribution by $\mu_{\mathbf{T}}$, and the distribution $\mu_{\mathbf{T}}$ conditioned on $i$ not being in the set of purchasing agents by $\mu_{\mathbf{T}}^{-i}$. A Nash equilibrium is characterized by a threshold vector $\mathbf{T}$ such that:

$$T_i = \frac{p_i}{1 - \mathrm{E}_{S \sim \mu_{\mathbf{T}}^{-i}}[x_i(S)]} \quad \forall i \in [n]. \tag{4}$$

The following theorem establishes the existence of Nash equilibria via a fixed point argument.

▶ **Theorem 2.1.** *In the simultaneous model, for any set of externality functions $\{x_i\}_{i \in [n]}$, for any set of atomless distributions $\mathcal{F}$, and for any price vector $\mathbf{p}$, there exists an equilibrium $\mathbf{T}$.*

One of the challenges in our model stems from the fact that a single price vector may induce multiple equilibria. Consider the simple setting of a single public good and two agents, Alice and Bob, where $F_{\text{Alice}}$ and $F_{\text{Bob}}$ are both uniform on $[0, 1]$, and the seller sets a non discriminatory price of $1/2$. Applying the equilibrium condition in Eq. (4), we get that every tuple $(T_{\text{Alice}}, T_{\text{Bob}}) \in [0, 1]^2$ satisfying $T_{\text{Alice}} \cdot T_{\text{Bob}} = 1/2$ forms an equilibrium strategy.[7] Therefore, in this case, there is a continuum of equilibria. It is not hard to see, however, that a set of thresholds $\mathbf{T}$ can be the consequence of only a single price vector, which can be derived via Eq. (4). This is cast in the following observation:

▶ **Observation 2.2.** *In the simultaneous model, a given price vector can induce multiple equilibria, but any given equilibrium $\mathbf{T}$ can be induced by a single price vector $\mathbf{p}$.*

Let $Eq(\mathcal{F}, \mathbf{p})$ denote the set of equilibria induced by a price vector $\mathbf{p}$, given a product distribution $\mathcal{F}$. For a given price vector $\mathbf{p}$ and an equilibrium $\mathbf{T} \in Eq(\mathcal{F}, \mathbf{p})$, let $\mathcal{R}_{\text{sim}}(\mathcal{F}, \mathbf{p}, \mathbf{T}) = \sum_i p_i \cdot (1 - F_i(T_i))$ denote the seller's expected revenue. Given a price vector $\mathbf{p}$, we define

$$\overline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p}) = \max_{\mathbf{T} \in Eq(\mathcal{F}, \mathbf{p})} \mathcal{R}_{\text{sim}}(\mathcal{F}, \mathbf{p}, \mathbf{T}) \qquad \text{and} \qquad \underline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p}) = \min_{\mathbf{T} \in Eq(\mathcal{F}, \mathbf{p})} \mathcal{R}_{\text{sim}}(\mathcal{F}, \mathbf{p}, \mathbf{T})$$

to be the revenue obtained in the respective best and worst equilibrium induced by $\mathbf{p}$. We refer to these revenues as the *optimistic* and *pessimistic* revenues, respectively.

The strongest approximation results one can hope for are ones that consider the pessimistic revenue obtained by our pricing scheme against an optimistic benchmark. This is exactly the approach we take. In particular, our benchmark is the revenue obtained by the best pricing, assuming the best equilibrium induced by every pricing. We denote the benchmark by $\mathcal{R}^*_{\text{sim}}(\mathcal{F}) = \max_{\mathbf{p}^*} \overline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p}^*)$. The performance of a price vector $\mathbf{p}$ is measured by the worst equilibrium induced by $\mathbf{p}$; i.e., $\underline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p})$. Our goal is to calculate a price vector $\mathbf{p}$ that minimizes the ratio between the former and the latter expressions.

[7] For a comprehensive discussion regarding the equilibrium condition in the public goods model, see Eq.(5) in Section 3.

**Sequential sales model.** In the sequential sales model, $n$ agents arrive one by one according to an order $\sigma : [n] \to [n]$, where agent $i$ is the $\sigma(i)$th agent to arrive. For ease of notation, we assume that agent $i$ is the $i$th agent to arrive, unless explicitly stated otherwise. In sequential sales, the price set by the seller for agent $i$ can depend on the set of agents who have purchased the good prior to agent $i$'s arrival. Thus, it can be viewed as a function $p_i : 2^{[i-1]} \to \mathbb{R}^+$.[8] The subgame perfect equilibrium in this auction is unique and can be found by a (possibly exponential) backward induction. An agent who receives a price buys if and only if her utility from buying exceeds her expected derived value from not buying conditioned on the set of agents that purchased the good prior to her arrival. Of course, this might impose a different threshold for every scenario which might lead to an exponential strategy space for the agents and an exponential time to compute each threshold in the strategy of an agent. As we discuss in the following sections, we devise pricing schemes in which the seller has a simple nearly optimal pricing scheme which leads to a simple strategy space and a poly-time threshold computation.

# 3 Pricing goods with full externalities (public goods)

## 3.1 Equilibrium and revenue equivalence

In this section we focus on the case where all agents derive their entire value from a good if purchased by any agent. We first characterize the equilibrium condition for a simultaneous sale. Given an equilibrium $\mathbf{T} = (T_1, \ldots, T_n)$, the expected value agent $i$ derives from other agents is $\mathrm{E}_{S \sim \mu_{\mathbf{T}}^{-i}} [x_i(S)] = 1 * \Pr[\text{some agent } j \neq i \text{ buys}] = 1 - \Pr[\text{no agent } j \neq i \text{ buys}] = 1 - \prod_{j \neq i} F_j(T_j)$. Plugging this expression into Eq. (4) yields the following equilibrium condition:

$$T_i = \frac{p_i}{\prod_{j \neq i} F_j(T_j)} \quad \text{for all } i. \tag{5}$$

For a given price vector $\mathbf{p}$ and an equilibrium $\mathbf{T} \in Eq(\mathcal{F}, \mathbf{p})$, the expected revenue is

$$\mathcal{R}_{\text{sim}}(\mathcal{F}, \mathbf{p}, \mathbf{T}) = \sum_i p_i \left(1 - F_i(T_i)\right) \overset{(5)}{=} \sum_i T_i \cdot \Big( \prod_{j \neq i} F_j(T_j) \Big) \cdot \left(1 - F_i(T_i)\right). \tag{6}$$

We turn to describe the equilibrium in the sequential sales model. In this case, whenever an agent buys an item, no subsequent agent will ever buy an item. Therefore, we can assume without loss of generality that the seller sets a single price per agent. Let $\mathbf{p} = (p_1, \ldots, p_n)$ denote the vector of offered prices.

We now show how to compute the unique subgame perfect equilibrium of the game. When the last agent (agent $n$) is offered a price, her best strategy is to buy if her value exceeds the price; i.e., $T_n = p_n$. When agent $i = n-1, \ldots, 1$ is offered a price, she faces the following tradeoff: if she buys, her utility is $v_i - p_i$. If she does not buy, her utility is $v_i \Big(1 - \prod_{j > i} \Pr[j \text{ does not buy}]\Big) = v_i \Big(1 - \prod_{j > i} F_j(T_j)\Big)$. Consequently, the unique

[8] Indeed, there are cases where the seller can gain higher revenue by setting such prices (an explicit example for availability-based externalities is given in the full version).

equilibrium $\mathbf{T}$ is given by[9][10]

$$T_i = \frac{p_i}{\prod_{j>i} F_j(T_j)} \quad \forall i \in [n]. \tag{7}$$

Given a product distribution $\mathcal{F}$, a price vector $\mathbf{p}$, and an arrival order $\sigma$, let $\mathbf{T}_{\mathcal{F}}(\sigma, \mathbf{p})$ be the function that returns the unique equilibrium. Since every price vector $\mathbf{p}$ defines a unique strategy vector $\mathbf{T}$, the expected revenue from agent $i$ is also uniquely defined, and can be calculated by

$$\begin{aligned}\prod_{j<i} \Pr[j \text{ does not buy}] \cdot p_i \cdot (1 - F_i(T_i)) &\overset{(7)}{=} \Big(\prod_{j<i} F_j(T_j)\Big) \cdot T_i \cdot \Big(\prod_{j>i} F_j(T_j)\Big) \cdot (1 - F_i(T_i)) \\ &= T_i \cdot \Big(\prod_{j \neq i} F_j(T_j)\Big) \cdot (1 - F_i(T_i)).\end{aligned}$$

Therefore, the expected revenue from all agents can be written as

$$\mathcal{R}_{\text{seq}}(\mathcal{F}, \sigma, \mathbf{p}, \mathbf{T} = \mathbf{T}_{\mathcal{F}}(\sigma, \mathbf{p})) = \sum_i T_i \cdot \Big(\prod_{j \neq i} F_j(T_j)\Big) \cdot (1 - F_i(T_i))\,. \tag{8}$$

Given an arrival order $\sigma$, let $\mathcal{R}^*_{\text{seq}}(\mathcal{F}, \sigma) = \max_{\mathbf{p}} \mathcal{R}_{\text{seq}}(\mathcal{F}, \sigma, \mathbf{p}, \mathbf{T} = \mathbf{T}_{\mathcal{F}}(\sigma, \mathbf{p}))$ denote the highest revenue a seller can obtain. We note that given a threshold vector $\mathbf{T}$ and an arrival order $\sigma$, there is also a unique price vector that produces this threshold vector $\mathbf{T}$, which can be calculated by (7), thus $\mathbf{T}_{\mathcal{F}}(\sigma, \cdot)$ is a bijection. This is cast in the following observation.

▶ **Observation 3.1.** *Fix an arrival order. An equilibrium strategy vector $\mathbf{T}$ is uniquely determined by a price vector $\mathbf{p}$, and a price vector $\mathbf{p}$ is uniquely determined by a strategy vector $\mathbf{T}$.*

Theorem 3.2 establishes revenue equivalence in simultaneous and sequential sales.

▶ **Theorem 3.2.** *For every product distribution $\mathcal{F}$ and for every order of arrival $\sigma$ in the sequential model, we have that $\mathcal{R}^*_{\text{seq}}(\mathcal{F}, \sigma) = \mathcal{R}^*_{\text{sim}}(\mathcal{F})$.*

It immediately follows that the optimal revenue is independent of the arrival order.

▶ **Corollary 3.3.** *For every two arrival orders $\sigma, \sigma'$, $\mathcal{R}^*_{\text{seq}}(\mathcal{F}, \sigma) = \mathcal{R}^*_{\text{seq}}(\mathcal{F}, \sigma')$.*

In the sequel, we use $\mathcal{R}^*_{\text{seq}}(\mathcal{F})$ to denote the optimal revenue in the sequential model.

We next draw a connection between selling public goods and selling a single private good. This connection is later used in proving approximation results for mechanisms for the sale of public goods. Let $Myer(\mathcal{F})$ denotes the optimal revenue a seller can obtain by selling a single private good to a set of agents drawn from $\mathcal{F}$ (*i.e.*, the revenue obtained by Myerson's optimal auction). Using similar arguments to ones used in [10], we have the following:

▶ **Lemma 3.4.** *For every product distribution $\mathcal{F}$, $\mathcal{R}^*_{\text{seq}}(\mathcal{F}) \leq Myer(\mathcal{F})$ $\big($and therefore, $\mathcal{R}^*_{\text{sim}}(\mathcal{F}) \leq Myer(\mathcal{F})$ by Theorem 3.2$\big)$.*

[9] Unlike the simultaneous model, an equilibrium exists for non atomless distributions, whenever tie-breaking is done consistently by agents. That is, agents always take the same action when their value is equal to their threshold.

[10] For $n$, we let $\prod_{j>n} F_j(T_j) = 1$.

## 3.2 Near optimal simultaneous sale

In our construction, we use the *ex-ante relaxation* (EAR) [3, 4] for selling a private good. The EAR relaxes the feasibility constraint, so that instead of selling at most one item *ex post*, this constraint holds only in expectation. Since the feasible region increases, the revenue of an optimal mechanism for this case can only be higher than Myerson's optimal mechanism. Combined with Lemma 3.4, it suffices to provide a pricing scheme for our setting that approximates the revenue of the EAR. As it turns out, when agents' values are drawn from regular distributions, the optimal mechanism for the ex-ante setting is a posted price mechanism. These prices can be computed in polynomial time by a convex programming formulation [14].

We use these prices to determine prices for the sale of public goods. To do so, we partition the agents into *valuable* and *non-valuable* agents, based on their contribution to the revenue of the EAR. All the revenue obtained in our pricing scheme comes from the valuable agents. Their prices are set so that if there exists a valuable agent that buys with low probability, the equilibrium condition guarantees that other agents buy with a sufficiently high probability.

▶ **Theorem 3.5.** *For social goods with full externalities and for any regular product distribution $\mathcal{F}$, there exists a poly-time algorithm that computes prices $\mathbf{p}$ for which $\underline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p}) \geq \mathcal{R}^*_{\text{sim}}(\mathcal{F})/5.83$.*

**Proof.** Let $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ be the posted prices that maximize the revenue in the EAR, and let $\mathcal{R} = \sum_i \hat{p}_i(1 - F_i(\hat{p}_i))$ be the optimal revenue of the EAR. As mentioned above, $\mathcal{R} \geq Myer(\mathcal{F})$. Let $c_1, c_2 > 1$ be two parameters, to be determined later. We partition the agents into two groups as follows. Let $B = \{i \in [n] : \hat{p}_i \geq \mathcal{R}/c_1\}$ and $S = [n] \setminus B$. For every agent $i$ we set

$$p_i = \begin{cases} \hat{p}_i/c_2 & i \in B \\ \infty & i \in S \end{cases}.$$

The revenue from the agents in $S$ in the optimal EAR mechanism is bounded by $\sum_{i \in S} \hat{p}_i \cdot \Pr[i \text{ buys}] \leq \frac{\mathcal{R}}{c_1} \sum_{i \in S}(1 - F_i(\hat{p}_i)) \leq \frac{\mathcal{R}}{c_1}$, where the last inequality stems from the fact that the EAR sells at most 1 item in expectation. Therefore, the revenue extracted from agents in $B$ in the EAR is

$$\sum_{i \in B} \hat{p}_i \cdot (1 - F_i(\hat{p}_i)) \geq \mathcal{R} - \frac{\mathcal{R}}{c_1} = (1 - 1/c_1)\,\mathcal{R}. \tag{9}$$

Let $\mathbf{T}$ be an equilibrium induced by the price vector $\mathbf{p} = (p_1, \ldots, p_n)$. We consider two cases:

**Case 1:** $T_i \leq \hat{p}_i$ for every $i \in B$. In this case,

$$\begin{aligned} \mathcal{R}_{\text{sim}}(\mathcal{F}, \mathbf{p}, \mathbf{T}) &= \sum_i p_i \cdot (1 - F_i(T_i)) = \sum_{i \in B} p_i \cdot (1 - F_i(T_i)) \\ &\geq \sum_{i \in B} \frac{\hat{p}_i}{c_2} \cdot (1 - F_i(\hat{p}_i)) \overset{(9)}{\geq} \left(\frac{1 - 1/c_1}{c_2}\right)\mathcal{R}, \end{aligned}$$

where the first inequality follows from case 1 and the monotonicity of $F_i$.

**Case 2:** There exists $i \in B$ such that $T_i > \hat{p}_i$. For such an agent $i$,

$$\frac{\hat{p}_i/c_2}{\prod_{j\neq i} F_j(T_j)} = \frac{p_i}{\prod_{j\neq i} F_j(T_j)} \stackrel{(5)}{=} T_i > \hat{p}_i \quad \Rightarrow \quad \prod_j F_j(T_j) \leq \prod_{j\neq i} F_j(T_j) \leq \frac{1}{c_2}. \tag{10}$$

Let $p_{\min} = \min_i p_i$. The expected revenue in this case is at least

$$p_{\min} \cdot \Pr[\text{at least one agent buys}] \quad \geq \quad \frac{\mathcal{R}}{c_1 c_2}\Big(1 - \prod_j F_j(T_j)\Big) \stackrel{(10)}{\geq} \left(\frac{1 - 1/c_2}{c_1 c_2}\right)\mathcal{R},$$

where the first inequality follows from the fact that all prices are at least $\frac{\mathcal{R}}{c_1 c_2}$.

Therefore, we get an approximation factor of $\min\left\{\left(\frac{1-1/c_1}{c_2}\right), \left(\frac{1-1/c_2}{c_1 c_2}\right)\right\}$. Setting $c_1 = \sqrt{2}$ and $c_2 = 1 + \frac{1}{\sqrt{2}}$ optimizes the approximation ratio and gives revenue of at least a $\frac{1}{3+2\sqrt{2}}$ fraction of $\mathcal{R}$. Since $\mathcal{R} \geq Myer(\mathcal{F}) \geq \mathcal{R}^*_{\text{sim}}(\mathcal{F})$ (by Lemma 3.4), we get that $\underline{\mathcal{R}}_{\text{sim}}(\mathcal{F}, \mathbf{p}) \geq \frac{\mathcal{R}^*_{\text{sim}}(\mathcal{F})}{3+2\sqrt{2}} \approx \frac{\mathcal{R}^*_{\text{sim}}(\mathcal{F})}{5.83}$. ◀

▶ Remark. An approximation ratio of 8 is given in [10] for the special case of i.i.d. distributions. The last theorem improves the approximation ratio to 5.83 even for the more general case of non-identical distributions. Moreover, in the full version we give a non-discriminatory pricing that gives 4 approximation for the case of identical distributions. We also show that no pricing scheme can give better approximation than $4/e$, even for identical distributions.

## 4 Near optimal sequential sale under status-based externalities

Recall that in this setting, every agent is associated with a discount factor $w_i \in [0, 1]$, Let $\mathbf{w} = (w_1, \ldots, w_n)$. We devise a non-adaptive pricing scheme (*i.e.*, where an agent's price does not depend on the previous purchases) that approximates the revenue of the optimal adaptive pricing scheme.[11] In our scheme, every agent is assigned with a single price.

Let $\mathbf{p}^0$ and $\mathbf{p}^{>0}$ be the price vectors posted by the seller who uses two price vectors, where $p_i^0$ (resp., $p_i^{>0}$) is the price offered to agent $i$ when no agent (resp., at least one agent) has purchased a good prior to $i$'s arrival. Let $\mathbf{p} = (\mathbf{p}^0, \mathbf{p}^{>0})$. In the full version, we show that we it is without loss of generality to restrict attention to two price vectors.

In contrast to the full externalities settings, agent $i$ may have two different thresholds in the equilibrium — one for the case where no agent bought a good before she arrives, denoted by $T_i^0$, and one for the case where at least one agent buys the good, denoted by $T_i^{>0}$. For every agent $i$, if some agent bought the good before she arrived, she faces the following trade-off — if she buys the good, her utility is $v_i - p_i^{>0}$; otherwise, her utility is $w_i \cdot v_i$. Therefore, the threshold satisfies the following equation:

$$T_i^{>0} - p_i^{>0} = w_i \cdot T_i^{>0} \Rightarrow p_i^{>0} = (1 - w_i) \cdot T_i^{>0}. \tag{11}$$

If no agent bought the good before before agent $i$ arrived, then[12]

$$T_i^0 - p_i^0 = w_i \cdot T_i^0 \cdot \Pr[\text{Agent } j > i \text{ buys a good}] = w_i \cdot T_i^0 \cdot \Big(1 - \prod_{j>i} F_j(T_j^0)\Big)$$
$$\Rightarrow p_i^0 = (1 - w_i) \cdot T_i^0 + w_i \cdot T_i^0 \cdot \prod_{j>i} F_j(T_j^0). \tag{12}$$

[11] Which sets a price for the current agent depending on the set of agents that purchased the good prior her arrival.
[12] For the case of $i = n$, the RHS product is naturally defined to be 1, and therefore $T_n^0 = p_n^0$.

For every agent $i$ and pricing $\mathbf{p} = (\mathbf{p}^0, \mathbf{p}^{>0})$, let $q_i^0 = q_i^0(\mathbf{p})$ (resp., $q_i^{>0} = q_i^{>0}(\mathbf{p})$) denote the probability that no agent (resp., at least one agent) has bought a good before agent $i$ arrived. The revenue can now be written as

$$\begin{aligned}
\mathcal{R}(\mathbf{p}) &= \sum_i \left( q_i^0 \cdot p_i^0 \cdot (1 - F_i(T_i^0)) + q_i^{>0} \cdot p_i^{>0} \cdot (1 - F_i(T_i^{>0})) \right) \\
&\overset{(12)}{=} \sum_i q_i^0 \cdot \left( (1 - w_i) \cdot T_i^0 + w_i \cdot T_i^0 \cdot \prod_{j>i} F_j(T_j^0) \right) \cdot (1 - F_i(T_i^0)) \\
&\quad + \sum_i q_i^{>0} \cdot p_i^{>0} \cdot (1 - F_i(T_i^{>0})) \\
&\overset{(11)}{=} \sum_i q_i^0 \cdot (1 - w_i) \cdot T_i^0 \cdot (1 - F_i(T_i^0)) + \sum_i q_i^0 \cdot w_i \cdot T_i^0 \cdot \left( \prod_{j>i} F_j(T_j^0) \right) \cdot (1 - F_i(T_i^0)) \\
&\quad + \sum_i q_i^{>0} \cdot T_i^{>0} \cdot (1 - w_i) \cdot (1 - F_i(T_i^{>0})).
\end{aligned}$$

By removing factors smaller than 1 ($q_i^0, q_i^{>0}, w_i$) in the last expression, we get

$$\begin{aligned}
\mathcal{R}(\mathbf{p}) &\leq \sum_i (1 - w_i) \cdot T_i^0 \cdot (1 - F_i(T_i^0)) + \sum_i \left( \prod_{j<i} F_j(T_j^0) \right) \cdot T_i^0 \cdot \left( \prod_{j>i} F_j(T_j^0) \right) \cdot (1 - F_i(T_i^0)) \\
&\quad + \sum_i T_i^{>0} \cdot (1 - w_i) \cdot (1 - F_i(T_i^{>0}))
\end{aligned}$$

$$\begin{aligned}
&= \sum_i (1 - w_i) \cdot T_i^0 \cdot (1 - F_i(T_i^0)) + \sum_i (1 - w_i) \cdot T_i^{>0} \cdot (1 - F_i(T_i^{>0})) \\
&\quad + \sum_i T_i^0 \cdot \left( \prod_{j \neq i} F_j(T_j^0) \right) \cdot (1 - F_i(T_i^0)). \qquad (13)
\end{aligned}$$

Given a thresholds vector $\mathbf{T} = (T_1, T_2, \ldots, T_n)$, we define $\mathcal{R}_1(\mathbf{T}, \mathbf{w}) = \sum_i (1 - w_i) \cdot T_i \cdot (1 - F_i(T_i))$ and $\mathcal{R}_2(\mathbf{T}) = \sum_i T_i \cdot \left( \prod_{j \neq i} F_j(T_j) \right) \cdot (1 - F_i(T_i))$. It follows from Eq. (13) that

$$\max_{\mathbf{p}} \mathcal{R}(\mathbf{p}) \leq 2 \max_{\mathbf{T}} \mathcal{R}_1(\mathbf{T}, \mathbf{w}) + \max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T}). \qquad (14)$$

That is, the RHS sum in Eq. (14) is an upper bound on the optimal revenue that can be obtained. $\mathcal{R}_1(\mathbf{T}, \mathbf{w})$ can be viewed as the private component of the revenue, which becomes more significant as $w_i$'s get smaller, while $\mathcal{R}_2(\mathbf{T})$ can be viewed as the public component, which becomes more significant as $w_i$'s grow. Notice that $\max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T})$ is exactly $\mathcal{R}^*_{\text{seq}}(\mathcal{F})$, where $\mathcal{R}^*_{\text{seq}}(\mathcal{F})$ is the optimal posted prices revenue in a sequential sale in the full externalities model, as defined in Section 3.

The following lemmas (4.1 and 4.2) show that it is possible to find prices that approximate $\max_{\mathbf{T}} \mathcal{R}_1(\mathbf{T}, \mathbf{w})$ and prices that approximate $\max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T})$. In fact, they show a stronger result, namely that for each of the terms in the sum, there exists a single price vector $\mathbf{p} = \mathbf{p}^0 = \mathbf{p}^{>0}$ that approximates it.

▶ **Lemma 4.1.** *There exists a poly-time algorithm for computing prices* $\mathbf{p}$ *such that* $\mathcal{R}(\mathbf{p}) \geq \max_{\mathbf{T}} \mathcal{R}_1(\mathbf{T}, \mathbf{w})$.

The following allows us to reduce the problem of finding "good" prices in the status-based externalities model to finding "good" prices in the full externalities model.

▶ **Lemma 4.2.** *Given prices* $\mathbf{p}'$, *there exist poly-time computable prices* $\mathbf{p}$ *such that* $\mathcal{R}(\mathbf{p}) \geq \mathcal{R}_{\text{seq}}(\mathcal{F}, \mathbf{p}')$.

We now present the main result of this section:

▶ **Theorem 4.3.** *Given a c-approximation pricing for sequential sales in the full externalities model, there exists a poly-time computable pricing that guarantees a* $(c+2)$*-approximation for the optimal sequential sales in the model of status-based externalities.*

**Proof.** Since $\max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T}) = \mathcal{R}^*_{\text{seq}}(\mathcal{F})$, if one can find prices that $c$-approximate the optimal prices in the full externalities model, by Lemma 4.2, one can compute prices that $c$-approximate $\max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T})$ in the status-based externalities model.

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the sets of prices for which $\mathcal{R}(\mathbf{p}_1) \geq \max_{\mathbf{T}} \mathcal{R}_1(\mathbf{T}, \mathbf{w})$ and $c \cdot \mathcal{R}(\mathbf{p}_2) \geq \max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T})$, respectively. These prices can be computed in poly time by Lemmas 4.1 and 4.2. We have that

$$\begin{aligned}\max_{\mathbf{p}} \mathcal{R}(\mathbf{p}) &\overset{(14)}{\leq} 2 \max_{\mathbf{T}} \mathcal{R}_1(\mathbf{T}, \mathbf{w}) + \max_{\mathbf{T}} \mathcal{R}_2(\mathbf{T}) \\ &\leq 2 \cdot \mathcal{R}(\mathbf{p}_1) + c \cdot \mathcal{R}(\mathbf{p}_2) \\ &\leq (c+2) \cdot \max\{\mathcal{R}(\mathbf{p}_1), \mathcal{R}(\mathbf{p}_2)\}. \end{aligned}$$ ◀

The following corollary follows from Theorem 4.3 and by the existence of a 4-approximation pricing for sequential sales in the full externalities model, as shown in the full version.

▶ **Corollary 4.4.** *For goods that exhibit status-based externalities, there exists a poly-time algorithm for computing prices that give a* 6*-approximation to the optimal pricing scheme.*

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