28th International Conference on Concurrency Theory

CONCUR 2017, September 5–8, 2017, Berlin, Germany

Edited by

Roland Meyer

Uwe Nestmann

Editors

Roland Meyer
TU Braunschweig
roland.meyer@tu-braunschweig.de

Uwe Nestmann
TU Berlin
uwe.nestmann@tu-berlin.de

ACM Classification 1998
D. Software, E. Data, F. Theory of Computation

ISBN 978-3-95977-048-4

Published online and open access by
Schloss Dagstuhl – Leibniz-Zentrum für Informatik GmbH, Dagstuhl Publishing, Saarbrücken/Wadern, Germany. Online available at http://www.dagstuhl.de/dagpub/978-3-95977-048-4.

Publication date
August, 2017

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Digital Object Identifier: 10.4230/LIPIcs.CONCUR.2017.0

LIPIcs – Leibniz International Proceedings in Informatics

LIPIcs is a series of high-quality conference proceedings across all fields in informatics. LIPIcs volumes are published according to the principle of Open Access, i.e., they are available online and free of charge.

ISSN 1868-8969

http://www.dagstuhl.de/lipics

Contents

28th International Conference on Concurrency Theory (CONCUR 2017).
Editors: Roland Meyer and Uwe Nestmann
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Preface

This volume contains the proceedings of the 28th Conference on Concurrency Theory, which was held in Berlin, Germany, on September 5–8, 2017. CONCUR 2017 was organized by Technische Universität Berlin and Technische Universität Braunschweig.

CONCUR is a forum for the development and dissemination of leading research in concurrency theory and its applications. The aim is to bring together researchers, developers, and students, exchange and discuss latest theoretical developments and learn about challenging practical problems. CONCUR is the reference annual event for researchers in the field.

The principal topics include basic models of concurrency such as abstract machines, domain-theoretic models, game-theoretic models, process algebras, graph transformation systems, Petri nets, hybrid systems, mobile and collaborative systems, probabilistic systems, real-time systems, biology-inspired systems, and synchronous systems; logics for concurrency such as modal logics, probabilistic and stochastic logics, temporal logics, and resource logics; verification and analysis techniques for concurrent systems such as abstract interpretation, atomicity checking, model checking, race detection, pre-order and equivalence checking, run-time verification, state-space exploration, static analysis, synthesis, testing, theorem proving, type systems, and security analysis; distributed algorithms and data structures: design, analysis, complexity, correctness, fault tolerance, reliability, availability, consistency, self-organization, self-stabilization, protocols. Also the theoretical foundations of more applied topics like architectures, execution environments, and software development for concurrent systems such as geo-replicated systems, communication networks, multiprocessor and multi-core architectures, shared and transactional memory, resource management and awareness, compilers and tools for concurrent programming, programming models such as component-based, object- and service-oriented can be found at CONCUR.

This edition of the conference attracted 86 full paper submissions. We would like to thank the authors for their interest in CONCUR 2017. After careful reviewing and discussions, the Program Committee selected 36 papers for presentation at the conference. Each submission was reviewed by at least three reviewers who wrote detailed evaluations and gave insightful comments. The Conference Chairs warmly thank the members of the Program Committee and the additional reviewers for their excellent work, as well as for the constructive discussions. The full list of reviewers is available as part of these proceedings.

The conference program was greatly enriched by the invited talks by Hongseok Yang (University of Oxford, UK), Azadeh Farzan (University of Toronto, Canada), Madan Musuvathi (Microsoft Research, USA), and Jean-Francois Raskin (Université libre de Bruxelles, Belgium). Moreover, Alastair Donaldson (Imperial College London, UK), Pawel Sobocinski (University of Southampton, UK), and Viktor Vafeiadis (Max Planck Institute for Software Systems, Germany) kindly agreed to contribute tutorials. The invited and the tutorial talks cover a broad range of topics from traditional concurrency theory and domains through games to the analysis of GPU kernels and the semantics of C++ in the presence of concurrency. The abstracts and invited papers are available as part of these proceedings. We thank the speakers for having accepted our invitation.

This year, the conference was jointly organized with the 14th International Conference on Quantitative Evaluation of SysTems (QEST), the 15th International Conference on Formal Modelling and Analysis of Timed Systems (FORMATS), and the 14th European Performance Engineering Workshop (EPEW) in an overarching event QONFEST. QONFEST included five more satellite events, the combined 24rd International Workshop on Expressiveness in

28th International Conference on Concurrency Theory (CONCUR 2017).
Editors: Roland Meyer and Uwe Nestmann
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Concurrency and 14th Workshop on Structural Operational Semantics (EXPRESS/SOS), the 1st Workshop on Recent Advances in Concurrency and Logic (RADICAL), the 7th Young Researchers Workshop on Concurrency Theory (YR-CONCUR), the 6th IFIP WG 1.8 Workshop on Trends in Concurrency Theory (TRENDS), and the 9th International Workshop on Practical Applications of Stochastic Modelling (PASM).

The CONCUR proceedings are available for open access via LIPIcs, and we thank the Marc Herbstritt and Sebastian Schweizer for helping us with the preparation.

Last, but not least, we thank the authors and the participants for making this year's CONCUR a successful event.

Roland Meyer (TU Braunschweig)
Uwe Nestmann (TU Berlin)

Committees

Program Committee

Jade Alglave

Mohamed Faouzi Atig

Paolo Baldan

Johannes Borgström

Luis Caires

Pedro R. D'Argenio

Josée Desharnais

Constantin Enea

Javier Esparza

Wan Fokkink

Rob van Glabbeek

Stefan Göller

Thomas Hildebrandt

Naoki Kobayashi

Antonín Kučera

Jérôme Leroux

Roland Meyer

K Narayan Kumar

Uwe Nestmann

Catuscia Palamidessi

Alexander Rabinovich

Davide Sangiorgi

Pawel Sobocinski

Vasco Thudicum Vasconcelos

Walter Vogler

Tomáš Vojnar

Igor Walukiewicz

Heike Wehrheim

Josef Widder

Thomas Wies

Gianluigi Zavattaro

Lijun Zhang

Co-Chairs

Roland Meyer

Uwe Nestmann

Steering Committee

Jos Baeten

Pedro R. D'Argenio

Javier Esparza

Joost-Pieter Katoen

Kim Guldstrand Larsen

Ugo Montanari

Catuscia Palamidessi

Organizing Committee

Uwe Nestmann, TU Berlin
(QONFEST General Co-Chair)

Katinka Wolter, FU Berlin
(QONFEST General Co-Chair)

Kirstin Peters, TU Berlin
(CONCUR Workshop Chair)

28th International Conference on Concurrency Theory (CONCUR 2017).
Editors: Roland Meyer and Uwe Nestmann
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

External Reviewers

Samy Abbes
Antonis Achilleos
C. Aiswarya
S. Akshay
Robert Atkey
Giorgio Bacci
Luca Bernardinello
Nathalie Bertrand
Michael Blondin
Benedikt Bollig
Tomas Brazdil
Yuliya Butkova
Marco Carbone
Raffaella Carbone
Sofia Cassel
Andrea Cerone
Yu-Fang Chen
Peter Chini
Dmitry Chistikov
Corina Cirstea
Lorenzo Clemente
Emanuele D'Osualdo
Søren Debois
Giorgio Delzanno
Romain Demangeon
Yuxin Deng
Brijesh Dongol
Adrien Durier
Christian Eisentraut
Jerome Feret
Jan Fiedor
Adrian Francalanza
Hongfei Fu
Paul Gastin
Marie-Claude Gaudel
Daniel Gburek
Dan R. Ghica
Daniele Gorla
Nicolas Guenot
Peter Habermehl
Christopher Hampson
Rolf Hennicker
Frédéric Herbreteau
Tom Hirschowitz
Lukas Holik
Radu Iosif
Stefan Jaax
Swen Jacobs
Marie-Christine Jakobs
Petr Jancar
Peter Jipsen
Edon Kelmendi
Henning Kerstan
Sophia Knight
Jan Kretinsky
Siddharth Krishna
Denis Kuperberg
Jürgen König
Marijana Lazic
Ondrej Lengal
Yong Li
Sylvain Lombardy

Robert Lorenz
Michele Loreti
Gerald Luettgen
Konstantinos Mamouras
Radu Mardare
Andrea Marin
Nicolas Markey
Paulo Mateus
Richard Mayr
Claudio Antares Mezzina
Samuel Mimram
Benjamin Monmege
Andrzej Murawski
Sebastian Muskalla
Bernhard Möller
Elisabeth Neumann
Petr Novotný
Jan Obdrzalek
Joachim Parrow
Kirstin Peters
Gustavo Petri
M. Praveen
Gabriele Puppis
Tahiry Rabehaja
Ahmed Rezine
Christina Rickmann
Adam Rogalewicz
Jurriaan Rot
David Šafránek
Prakash Saivasan
Arnaud Sangnier
Ocan Sankur
Zdenek Sawa
Alceste Scalas
Carsten Schuermann
Henning Seidler
Salomon Sickert
Michał Skrzypczak
Ana Sokolova
Marco Solieri
Sadegh Soudjani
Sam Staton
Ilina Stoilkovska
K V Subrahmanyam
Kohei Suenaga
Grégoire Sutre
Géraud Sénizergues
Daniel Thoma
Simone Tini
Takeshi Tsukada
Andrea Turrini
Valeria Vignudelli
Mahesh Viswanathan
Christoph Wagner
Herbert Wiklicky
Sebastian Wolff
Damien Zufferey

List of Authors

Parosh Aziz Abdulla
Uppsala University
Sweden
parosh@it.uu.se

Luca Aceto
Reykjavik University
Iceland
luca@ru.is

Ki Yung Ahn
Nanyang Technological University
Singapore
yaki@ntu.edu.sg

C. Aiswarya
Chennai Mathematical Institute
India
aiswaryanitc@gmail.com

S. Akshay
Indian Institute of Technology Bombay
India
akshayss@cse.iitb.ac.in

Mohamed Faouzi Atig
Uppsala University
Sweden
mohamed_faouzi.atig@it.uu.se

Guy Avni
IST Austria
Austria
guy.avni@ist.ac.at

Nicolas Basset
Université libre de Bruxelles
Belgium
nicolas.basset@ulb.ac.be

Nathalie Bertrand
IRISA
France
nathalie.bertrand@inria.fr

Benedikt Bollig
CNRS, ENS Paris-Saclay
France
bollig@lsv.fr

Filippo Bonchi
CNRS, ENS-Lyon
France
filippo.bonchi@ens-lyon.fr

Ahmed Bouajjani
IRIF, University Paris Diderot
France
abou@irif.fr

Patricia Bouyer
Université Paris-Saclay
France
bouyer@lsv.fr

Romain Brenguier
Oxford University
United Kingdom
romain.brenguier@gmail.com

Franck van Breugel
York University, Toronto
Canada
franck@eecs.yorku.ca

Paul Brunet
ENS de Lyon, France
University College London, UK
paul@brunet-zamansky.fr

Ilaria Castellani
Université Côte d'Azur, INRIA
France
ilaria.castellani@inria.fr

Andrea Cerone
Imperial College London
United Kingdom
a.cerone@imperial.ac.uk

Supratik Chakraborty
Indian Institute of Technology Bombay
India
akshayss@gmail.com

Thomas Chatain
École Normale Supérieure de Cachan
France
thomas.chatain@lsv.ens-cachan.fr

28th International Conference on Concurrency Theory (CONCUR 2017).
Editors: Roland Meyer and Uwe Nestmann
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Krishnendu Chatterjee
IST Austria
Austria
krish.chat@ist.ac.at

Taolue Chen
Birkbeck, University of London
United Kingdom
taolue.chen@gmail.com

Ventsislav Chonev
Max Planck Institute for Software Systems
Germany
vencho@mpi-sws.org

Ankush Das
Carnegie Mellon University
USA
ankushd@cs.cmu.edu

Nicolas David
University of Nantes
France
nicolas.david1@univ-nantes.fr

Normann Decker
Universtität zu Lübeck
Germany
decker@isp.uni-luebeck.de

Miheer Dewaskar
University of North Carolina
USA
miheer.dew@gmail.com

Mariangiola Dezani-Ciancaglini
Università di Torino
Italy
dezani@di.unito.it

Rayna Dimitrova
Max Planck Institute for Software Systems
Germany
rayna@mpi-sws.org

Alastair Donaldson
Imperial College London
United Kingdom
alastair.donaldson@imperial.ac.uk

Ulrich Dorsch
Friedrich-Alexander-Universität
Erlangen-Nürnberg
Germany
ulrich.dorsch@fau.de

Adrien Durier
École Normale Supérieure de Lyon,
Università di Bologna
France
adrien.durier@gmail.com

Constantin Enea
IRIF, University Paris Diderot
France
cenea@irif.fr

Ignacio Fábregas
IMDEA Software Institute, Madrid
Spain
fabregas@ucm.es

Azadeh Farzan
University of Toronto
Canada
azadeh@cs.toronto.edu

Yuan Feng
University of Technology Sydney
Australia
yuan.feng@uts.edu.au

Nathanaël Fijalkow
The Alan Turing Institute of Data Science,
London
United Kingdom
nfijalkow@turing.ac.uk

Wan Fokkink
Vrije Universiteit Amsterdam
The Netherlands
w.j.fokkink@vu.nl

Adrian Francalanza
University of Malta
Malta
adrian.francalanza@um.edu.mt

Yuxi Fu
Shanghai Jiao Tong University
China
fu-yx@cs.sjtu.edu.cn

Álvaro García-Pérez
IMDEA Software Institute, Madrid
Spain
alvaro.garcia.perez@imdea.org

Paul Gastin
ENS Paris-Saclay
France
paul.gastin@lsv.fr

Ivan Gavran
Max Planck Institute for Software Systems
Germany
gavran@mpi-sws.org

Blaise Genest
IRISA
France
bgenest@irisa.fr

Paola Giannini
Università del Piemonte Orientale
Italy
paola.giannini@uniupo.it

Hugo Gimbert
LaBRI
France
hugo.gimbert@labri.fr

Rob van Glabbeek
Data61, CSIRO, Sydney
Australia
rvg@cs.stanford.edu

Alexey Gotsman
IMDEA Software Institute, Madrid
Spain
alexey.gotsman@imdea.org

Julian Gutierrez
University of Oxford
United Kingdom
julian.gutierrez@cs.ox.ac.uk

Peter Habermehl
IRIF, Univ Paris Diderot
France
haberm@irif.fr

Serge Haddad
Université Paris-Saclay
France
haddad@lsv.ens-cachan.fr

Ernst Moritz Hahn
Institute of Software, Chinese Academy of Sciences, Beijing
China
hahn@ios.ac.cn

Paul Harrenstein
University of Oxford
United Kingdom
paul.harrenstein@cs.ox.ac.uk

Thomas A. Henzinger
IST Austria
Austria
tah@ist.ac.at

Daniel Hirschkoff
École Normale Supérieure de Lyon
France
daniel.hirschkoff@ens-lyon.fr

Joshua Holland
University of Southampton
United Kingdom
josh@inv.alid.pw

Ross Horne
Nanyang Technological University
Singapore
rhorne@ntu.edu.sg

Mingzhang Huang
Shanghai Jiao Tong University
China
mingzhanghuang@gmail.com

Anna Ingólfsdóttir
Reykjavik University
Iceland
annai@ru.is

Vishal Jagannath
Indian Institute of Technology Bombay
India
vishal_rjagan@cse.iitb.ac.in

Claude Jard
University of Nantes
France
claude.jard@univ-nantes.fr

Vincent Jugé
Université Paris-Saclay
France
vincent.juge@polytechnique.edu

Tobias Kappé
University College London
United Kingdom
tkappe@cs.ucl.ac.uk

Jeroen Ketema
Embedded Systems Innovation by TNO
Netherlands
jeroen.ketema@tno.nl

Zachary Kincaid
Princeton University
USA
zkincaid@cs.princeton.edu

Shankara Narayanan Krishna
IIT Bombay
India
krishnas@cse.iitb.ac.in

Orna Kupferman
The Hebrew University
Israel
orna@cs.huji.ac.il

Martin Leucker
Universtität zu Lübeck
Germany
leucker@isp.uni-luebeck.de

Didier Lime
École Centrale de Nantes
France
didier.lime@ec-nantes.fr

Bas Luttik
Eindhoven University of Technology
The Netherlands
s.p.luttik@tue.nl

Jean Mairesse
CNRS - Sorbonne Universités
France
jean.mairesse@lip6.fr

Rupak Majumdar
Max Planck Institute for Software Systems
Germany
rupak@mpi-sws.org

Stefan Milius
Friedrich-Alexander-Universität
Erlangen-Nürnberg
Germany
stefan.milius@fau.de

Andrzej S. Murawski
University of Warwick, Coventry
United Kingdom
A.Murawski@warwick.ac.uk

Yolanda Ortega-Mallén
Universidad Complutense de Madrid
Spain
yolanda@ucm.es

Jan Otop
University of Wrocław
Poland
jotop@cs.uni.wroc.pl

Loïc Paulevé
CNRS
France
loic.pauleve@lri.fr

Arno Pauly
Université Libre de Bruxelles, U.L.B.
Belgium
arno.m.pauly@gmail.com

Dusko Pavlovic
University of Hawaii at Manoa
Hawaii
dusko@hawaii.edu

Giuseppe Perelli
University of Oxford
United Kingdom
giuseppe.perelli@cs.ox.ac.uk

Damien Pous
ENS de Lyon
France
damien.pous@ens-lyon.fr

Vinayak Prabhu
Max Planck Institute for Software Systems
Germany
vinayak@mpi-sws.org

Karin Quaas
Universität Leipzig
Germany
quaas@informatik.uni-leipzig.de

Jean-François Raskin
Université Libre de Bruxelles, U.L.B.
Belgium
jraskin@ulb.ac.be

Cristian Riveros
Pontificia Universidad Católica de Chile
Chile
cristian.riveros@uc.cl

Olivier H. Roux
École Centrale de Nantes
France
olivier-h.roux@ec-nantes.fr

Sai Sandeep
Indian Institute of Technology Bombay
India
saisandeep@cse.iitb.ac.in

Davide Sangiorgi
Università di Bologna
Italy
davide.sangiorgi@gmail.com

Arnaud Sangnier
IRIF, Univ Paris Diderot
France
sangnier@irif.fr

Ocan Sankur
CNRS - IRISA Rennes
France
ocan.sankur@irisa.fr

Ilias Sarkar
IIT Bombay
India
ilias@cse.iitb.ac.in

Uli Schlachter
Carl von Ossietzky Universität, Oldenburg
Germany
uli.schlachter@informatik.uni-oldenburg.de

Lutz Schröder
Friedrich-Alexander-Universität
Erlangen-Nürnberg
Germany
lutz.schroeder@fau.de

Alexandra Silva
University College London
United Kingdom
alexandra.silva@ucl.ac.uk

Pawel Sobocinski
University of Southampton
United Kingdom
ps@ecs.soton.ac.uk

Ana Sokolova
University of Salzburg
Austria
ana.sokolova@cs.uni-salzburg.at

Fu Song
ShanghaiTech University
China
songfu@shanghaitech.edu.cn

Tyler Sorensen
Imperial College London
United Kingdom
tyler.sorensen15@imperial.ac.uk

Michèle Soria
UPMC - Sorbonne Universités
France
michele.soria@lip6.fr

Sadegh Soudjani
Max Planck Institute for Software Systems
Germany
sadegh@mpi-sws.org

Georg Struth
The University of Sheffield
United Kingdom
g.struth@sheffield.ac.uk

Qiyi Tang
York University, Toronto
Canada
qiyitang@eecs.yorku.ca

Daniel Thoma
Universtität zu Lübeck
Germany
thoma@isp.uni-luebeck.de

Alwen Tiu
Nanyang Technological University
Singapore
atiu@ntu.edu.sg

Andrea Turrini
Institute of Software, Chinese Academy of Sciences, Beijing
China
turrini@ios.ac.cn

Nikos Tzevelekos
Queen Mary University of London
United Kingdom
nikos.tzevelekos@qmul.ac.uk

Gal Vardi
The Hebrew University
Israel
gal.vardi@mail.huji.ac.il

Chao Wang
IRIF, University Paris Diderot
France
wangch@irif.fr

John Wickerson
Imperial College London
United Kingdom
j.wickerson@imperial.ac.uk

Harro Wimmel
Carl von Ossietzky Universität, Oldenburg
Germany
harro.wimmel@informatik.uni-oldenburg.de

Thorsten Wißmann
Friedrich-Alexander-Universität
Erlangen-Nürnberg
Germany
thorsten.wissmann@fau.de

Michael Wooldridge
University of Oxford
United Kingdom
mjw@cs.ox.ac.uk

James Worrell
University of Oxford
United Kingdom
james.worrell@cs.ox.ac.uk

Zhilin Wu
Chinese Academy of Sciences
China
wuzl@ios.ac.cn

Hongseok Yang
University of Oxford
United Kingdom
hongseok.yang@cs.ox.ac.uk

Qiang Yin
Beihang University
China
yinqiang@buaa.edu.cn

Shenggang Ying
University of Technology Sydney
Australia
shenggang.ying@uts.edu.au

Fabio Zanasi
University College London
United Kingdom
f.zanasi@ucl.ac.uk

Forward Progress on GPU Concurrency

Alastair F. Donaldson[1], Jeroen Ketema[2], Tyler Sorensen[3], and John Wickerson[4]

1 **Department of Computing, Imperial College London, UK**
`alastair.donaldson@imperial.ac.uk`
2 **Embedded Systems Innovation by TNO, Eindhoven, the Netherlands**
`jeroen.ketema@tno.nl`
3 **Department of Computing, Imperial College London, UK**
`tyler.sorensen15@imperial.ac.uk`
4 **Department of Electrical and Electronic Engineering, Imperial College London, UK**
`j.wickerson@imperial.ac.uk`

Abstract

The tutorial at CONCUR will provide a practical overview of work undertaken over the last six years in the Multicore Programming Group at Imperial College London, and with collaborators internationally, related to understanding and reasoning about concurrency in software designed for acceleration on GPUs. In this article we provide an overview of this work, which includes contributions to data race analysis, compiler testing, memory model understanding and formalisation, and most recently efforts to enable portable GPU implementations of algorithms that require forward progress guarantees.

1998 ACM Subject Classification D.1.3 Concurrent Programming

Keywords and phrases GPUs, concurrency, formal verification, memory models, data races

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.1

Category Invited Talk

1 Introduction

Graphics processing units (GPUs) offer a large degree of parallelism at a relatively low cost, and are now routinely applied to the acceleration of a wide variety of computational tasks that go well beyond the domain of graphics (see e.g. [39] for details of many application areas and developments).

It is invariably harder to design a software application that takes advantage of GPU parallelism than it is to write a sequential version of the application that runs only on the CPU. Furthermore, parallel programming for GPUs is in many ways more complicated than parallel programming for multi-core CPUs. This is because achieving high performance requires working in low level languages, such as CUDA [33] and OpenCL [24], which provide close-to-the-metal language features to enable mapping an algorithm to the architectural capabilities of a device. Numerous high level programming models have been proposed to ease the burden of GPU programming, via automatic generation of low level code, but as yet are not widely adopted.

Low level programming of GPUs is made challenging by traditional concurrency bugs such as data races, by issues related to relaxed memory, and by the constrained hardware execution model on which software executes. Furthermore, reliable production compilers for

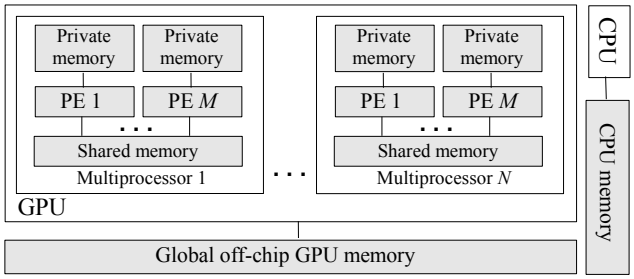


Figure 1 Overview of a typical GPU architecture.

GPUs are hard to maintain due to the challenge of keeping pace with changing architectures and evolving source languages, and because GPU-specific optimisations—e.g. ones that aim to reduce control-flow divergence between threads—are important for performance but error-prone to implement.

After giving a brief overview of the GPU programming model (Section 2), we present an overview of a line of work pursued by the Multicore Programming Group at Imperial College London, with various collaborators, which has contributed to: automated data race analysis (Section 3), compiler testing (Section 4), memory models (Section 5), and forward progress guarantees to support blocking algorithms (Section 6). While we principally focus on the approach taken by our line of work, we briefly discuss related approaches. In each area, we also touch on open problems that will require future research to solve.

2 Overview of the GPU Programming Model

GPUs are programmed using an SPMD (single program, multiple data) model, in which a large number of *processing elements* (PEs) all execute the same program, with each PE operating on a different subset of some shared data. At the hardware level, PEs are typically grouped into *multiprocessors* (see Figure 1). Each PE generally has access to a few hundred bytes of *private memory*; all the PEs in the same multiprocessor have a few kilobytes of *shared memory* available; and a large amount of *global memory* is shared between all multiprocessors.

The two main programming languages for writing SPMD programs are OpenCL [24] and CUDA [33], both of which derive from the C programming language. The languages follow the above sketched hardware hierarchy quite closely by subdividing the execution of a program among a number of *work-items* (or *threads*)—mapping to PEs. These work-items are a grouped into (multi-dimensional) *work-groups* (or *thread-blocks*)—mapping to multiprocessors—which in turn are grouped into (multi-dimensional) *NDRanges* (or *grids*). A programmer specifies the program to be executed by a single work-item—called a *GPU kernel*—and defines the work-group and NDRange sizes that should be used for execution. The OpenCL programming model uses the terms *work-items*, *work-groups*, and *NDRanges*, while CUDA uses *threads*, *thread-blocks*, and *grids*. For the remainder of the article we use the OpenCL terminology, except that we find it more natural to use *thread* rather than *work-item*.

An OpenCL kernel computing a prefix-sum (or scan) [25] is presented in Figure 2. The kernel is intended to be executed by a single 1-dimensional work-group. The keyword **`global`** indicates that the arrays `in` and `out` reside in global memory. Similarly, the keyword **`local`** (not used in the example) indicates that an array resides in shared memory. Any variable declared without the **`global`** or **`local`** keyword is thread-private. To enable each thread to

```
kernel void KoggeStone(global int *in, global int *out) {
    unsigned tid = get_local_id(0);
    out[tid] = in[tid];
    barrier(CLK_GLOBAL_MEM_FENCE);
    for (unsigned offset = 1; offset < tid; offset *= 2) {
        int temp;
        if (tid >= offset)
          { temp = out[tid - offset]; }
        barrier(CLK_GLOBAL_MEM_FENCE);
        if (get_local_id(0) >= offset)
          { out[tid] = temp + out[tid]; }
        barrier(CLK_GLOBAL_MEM_FENCE);
    }
}
```

Figure 2 A Kogge-Stone prefix-sum GPU kernel in OpenCL.

operate on different data, several functions are provided that allow a thread to access its unique id. One of these functions is `get_local_id`, which yields the unique id of a thread within a work-group.

Except when performing the most basic computations, threads typically need to share intermediate results. In OpenCL and CUDA, these intermediate results are traditionally shared with the help of *synchronisation barriers*: each thread writes its intermediate results to local or global memory and then calls the **`barrier`** function. The **`barrier`** function stalls the thread until all other threads *within the same work-group* have also called the function and all outstanding memory operations have been committed. The argument of the **`barrier`** function indicates whether operations on local or global memory need to be committed. In the case of the GPU kernel of Figure 2, only operations on global memory are committed.

As stressed above, barriers can *only* be used for communication within work-groups; they cannot be used for communication between work-groups. The same holds for early implementations of atomic operations on GPUs. This was rectified with OpenCL 2.0, which defines atomic operations that can operate between work-groups. The semantics of OpenCL 2.0 atomics are subtle, and expressing them precisely is a research endeavour we are pursuing, thus we postpone a more detailed discussion of them to Section 5. Finally, it is worth mentioning that the new NVIDIA CUDA 9.0 provides primitives for barrier synchronisation across work-groups [34].

3 Static Data Race Analysis

Data races are an important kind of defect that affect shared memory concurrent programs, and GPU kernels in particular. Informally, a GPU kernel exhibits a *data race* if it is possible for two distinct threads, t_1 and t_2, to issue memory operations accessing a common location m such that: at least one of the operations is non-atomic, at least one of the operations modifies m, and t_1 and t_2 do not synchronise between the operations. The semantics of data races are not well defined in CUDA nor in older versions of OpenCL, and data races are a form of undefined behaviour in OpenCL 2.0 [24] and later. As well as being defects in their own right, data races can lead to unwanted nondeterminism, which can cause other bugs that are hard to reproduce and fix.

To aid programmers in reasoning about data races in GPU kernels we spent several years designing GPUVerify, a static data race analysis tool [11, 10, 4]. GPUVerify caters primarily for the *traditional* style of GPU programming where threads in distinct work-groups do not

communicate during the lifetime of a kernel, and where intra-work-group communication is exclusively via synchronisation barriers.

The traditional setting affords a key property: when a GPU kernel is executed on a given input, if *any* thread schedule can exhibit a data race, then *all* thread schedules exhibit a data race. We briefly sketch the argument for this. Let us call a data race occurring in a given thread schedule a *principal race* if the associated memory accesses a_1 and a_2 are guaranteed to occur, regardless of the instructions executed by other threads. That is, the conditions for the associated threads to issue a_1 and a_2 arise directly from the input on which the GPU kernel is executed, and do not depend on the thread schedule. Clearly, if a thread schedule exhibits a principal race involving accesses a_1 and a_2, then *every* thread schedule must exhibit this principal race. Furthermore, in the traditional GPU programming model, every thread schedule that exhibits a race is guaranteed to exhibit a principal race. This is because a traditional GPU kernel behaves *entirely deterministically* in the absence of a data race [16].[1] Threads in different work-groups have no means of synchronising, and thus can only influence one another by racing. Threads in the same work-group can only synchronise via barriers. Hence, in the absence of data races, the execution of each thread is independent of the actions of all other threads until barrier synchronisation occurs, and the state of each thread upon reaching a barrier is independent of the chosen thread interleaving. Repeating this argument for each following barrier, we see that traditional GPU kernel execution is deterministic. Because a race-free GPU kernel is deterministic, a kernel exhibiting a race is guaranteed to have a fixed set of principal races and, hence, if any schedule exhibits a race then all do.

GPUVerify leverages the above result to reduce the problem of reasoning about data races in highly parallel GPU kernels to reasoning about assertions in a sequential program. This is achieved by translating a parallel kernel into a sequential program that models one particular thread schedule between each pair of barrier synchronisation points in the kernel, tracking the reads and writes of threads, and using assertions to determine the conditions under which data races occur. In practice, GPUVerify uses a schedule in which threads execute in lock-step [18].

Because GPU kernels are often executed by hundreds or even thousands of threads, it would not be practical to model all these threads explicitly. Instead, GPUVerify exploits the fact that data races occur *pairwise*, and only models execution of a kernel by a pair of threads. The identities of these threads are made symbolic in the sequential program that GPUVerify generates, so that reasoning is in fact performed over all possible pairs. To over-approximate the effects of additional threads, the shared state of the GPU kernel is made abstract. By default GPUVerify uses an extremely coarse abstraction, where other threads are assumed to have arbitrary effects on the shared state [10].

The sequential program generated by GPUVerify is encoded in the Boogie intermediate verification language [5], whose supporting tool—also called Boogie—generates verification conditions that are discharged by SMT solvers such as Z3 [19] and CVC4 [6]. To handle loops, Boogie requires invariants to be supplied. For this purpose we have equipped GPUVerify with a tailored engine for loop invariant inference, which is by now relatively mature. The engine works by speculating candidate loop invariants via a set of rules, which we devised through careful study of a large set of GPU kernels [9]. The candidate invariants speculated for a kernel are fed to the Houdini algorithm [20], which computes the largest conjunction of

[1] We assume here that concurrency is the only possible cause of nondeterminism; sources of nondeterminism due to e.g. accessing invalid memory locations are an orthogonal concern.

candidates that form an inductive invariant for the kernel. The computed invariant is then used in an attempt to prove freedom from data races.

Several other methods for testing and verifying properties of GPU kernels have been proposed. These include approaches based on dynamic analysis [35, 48, 28], verification via SMT solving [29, 38, 44], symbolic execution [30, 17], and program logic [12, 26].

Open problems

Despite operating reasonably automatically on a relatively large set of GPU kernels [4, 9], the GPUVerify technique has three main limitations.

First, the loop invariant inference method used by the tool can be inefficient, due to the sheer number of candidate invariants that are speculated. Many of the candidates turn out to be incorrect or non-inductive, so that many iterations of the Houdini algorithm are required to refute them, each iteration requiring one or more expensive calls to an SMT solver. We believe that a significantly more efficient mechanism for computing relevant loop invariants could be built by performing abstract interpretation over carefully-designed abstract domains that capture the memory access patterns of GPU kernels.

Second, the shared state abstraction employed by GPUVerify leads to false alarms for kernels whose race-freedom depends on richer properties of the shared state. To aid in the verification of such kernels, GPUVerify supports *barrier invariants*—properties of the shared state that are proven to be satisfied by threads individually on entry to a barrier synchronisation operation, and which can thus be assumed to hold for all threads on exit from the barrier [15]. However, barrier invariants must currently be specified manually, and have only been investigated practically in one domain so far: reasoning about parallel prefix sums. Relatedly, our recent work on termination analysis for GPU kernels shows that the coarse shared state abstraction used by GPUVerify often suffices for proving kernel termination in a thread-modular manner, but identifies a number of cases where richer shared state abstractions are required [23].

Third, GPUVerify fundamentally exploits the traditional GPU programming model where barriers are the only means for synchronisation. As discussed further in Sections 5 and 6, modern GPU kernels increasingly go beyond the boundaries of this simple computational model, using atomic operations to implement fine-grained concurrent algorithms. In general, reasoning about data race-freedom of GPU kernels that use atomic operations is just as hard as reasoning about data race-freedom for arbitrary concurrent programs. In particular, a race-free GPU kernel need not behave deterministically if it uses atomic operations. This breaks the property that allows GPUVerify to reason about a kernel via translation to a sequential program. Although GPUVerify can handle some limited use cases for atomic operations [7], the tool over-approximates more general uses of atomics, so that it will report false alarm data races for algorithms that, for instance, protect shared data structures using mutexes built via atomic operations. Nevertheless, we conjecture that the contexts in which GPU programmers use atomic operations in practice are likely to be limited and idiomatic, and that there may be scope for targeted analyses that exploit this idiomatic nature.

In related work, we note that GKLEE, another GPU race detection tool [30], has also been extended to reason about atomics in some scenarios [14], and that reasoning about atomic operations in GPU kernels via *resource invariants* in separation logic has recently been proposed [3].

4 Many-Core Compiler Fuzzing

The race analysis provided by GPUVerify (see Section 3) operates on the intermediate representation of the LLVM compiler framework, generated by Clang's OpenCL and CUDA front-ends. More generally, it is common for static race analysis tools to operate at the level of program source code, or at some intermediate representation associated with a particular tool chain. Either way, an analysis that establishes properties of a program at some higher-than-binary level of abstraction relies on correct downstream compilation.

Aware that guarantees provided by GPUVerify are conditional on the reliability of CUDA and OpenCL compilers, we undertook research into testing such compilers. We focused on testing OpenCL compilers using two methods: *random differential testing* [32], where compilers are cross-checked against each other using randomly-generated programs, and *equivalence modulo inputs testing* [27], where a single compiler is tested via a family of programs that, for a given input, ought to yield equivalent results. In both cases, mismatches are indicative of compiler bugs. Both methods are examples of *fuzzing*, where a system is tested against randomly-generated inputs.

We built a tool, CLsmith [31], which extends the Csmith generator for C programs [47] to the domain of OpenCL. It was relatively simple to extend Csmith to generate "embarrassingly parallel" OpenCL kernels in which threads do not communicate at all. More challenging was to devise methods for generating OpenCL kernels in which threads *do* communicate—either using barriers or atomic operations—but in a manner such that the generated kernels are guaranteed to be free from data races and to compute deterministic results.[2]

We armed CLsmith with three modes for generating such kernels: *barrier* mode, where generated kernels are equipped with shared arrays, indexed in a manner that avoids data races; *atomic-sections* mode, where atomic operations are used to build sections of code that can only be executed by a single thread, structured such that the particular thread that executes a section depends on the thread schedule, but such that the side-effects associated with executing the section are independent of the specific thread that executes the section; and *atomic-reductions* mode, where associative, commutative atomic operations (such as `atomic_add` and `atomic_min`) are used to perform reductions on data values computed locally by individual threads, ultimately leading to deterministic results.

Via an experimental campaign applying CLsmith to 21 OpenCL (device, compiler)-configurations, covering a range of GPU, CPU, FPGA, and emulator implementations [31], we discovered more than 50 OpenCL compiler bugs, most affecting commercial implementations. Surprisingly, and disappointingly from an academic perspective, these bugs were almost exclusively *sequential*: basic compilation bugs that could manifest in a single-threaded kernel. We found a large number of defects affecting programs that manipulate data via user-defined structures. While we found a small number of bugs that required barrier synchronisation operations to be present in order to manifest, even these did not appear to be directly concurrency-related: they could affect kernels that, despite featuring barriers, did not depend on inter-thread communication for result computation.

[2] As argued in Section 3, kernels that only use barrier operations for inter-thread communication are guaranteed to be deterministic if they are race-free. However, atomic operations open the possibility for race-free kernels to be nondeterministic.

Open problems

The fact that our testing methods did not identify any compiler bugs that were directly concurrency-related could be because (a) OpenCL compilers do not yet perform sophisticated concurrency-related optimisations, (b) our method did in fact trigger concurrency-related compiler bugs, but in addition triggered so many basic sequential compiler bugs that the effects of the concurrency-related bugs were masked, or (c) the methods we investigated so far for detecting concurrency-related bugs are too simple.

Possibility (a) could be investigated with reference to open source compilers, such as the Intel back-end for the Beignet open source implementation of OpenCL.[3] Running a similar testing campaign against more mature OpenCL compilers in the future might shed light on possibility (b).

As we discuss further in Section 5, OpenCL 2.0 includes a full suite of well-defined atomic operations, together with a memory model specification. With respect to possibility (c), this raises the problem of how to generate random OpenCL kernels that exercise these concurrency-related features in non-trivial yet predictable ways, to test more thoroughly the compilation of concurrency primitives.

5 GPU Memory Models

As mentioned in Section 3, many GPU programmers have been going beyond the traditional barrier synchronous model of computation, writing fine-grained concurrent algorithms that manipulate irregular data structures, e.g. graphs. This raises the question of what sort of *memory models* GPU architectures present, and in particular, what values a thread might observe when reading from a shared memory location.

Empirically, we used *litmus tests* to investigate the memory models of several NVIDIA and AMD GPUs [1], showing (a) that they *do* exhibit relaxed (i.e. non-sequentially consistent) behaviour, and (b) that several programming idioms relied on by high level algorithms assume that such relaxed behaviour *cannot* occur, and thus are not guaranteed to work correctly on these platforms. In follow-up work we substantiated this possibility by designing a testing framework geared towards automatically provoking relaxed memory bugs in GPU-accelerated applications [40].

In large part, the issues that we discovered in these works stem from the lack of a memory model specification in CUDA, and in OpenCL prior to OpenCL 2.0. The OpenCL 2.0 specification [24] provides a memory model specification based on the C11 memory model [22]. However, the memory model is complicated by the fact that OpenCL features a hierarchical organisation of threads and memory spaces, as discussed in Section 2 and illustrated by Figure 1. For example, threads in the same work-group can use fast shared memory to communicate, while threads in different work-groups must communicate via slower global memory. OpenCL also features *shared virtual memory* (not depicted in the overview of Figure 1), through which device threads can communicate with threads running on the host processor. The situation is further complicated by the fact that memory accesses are *scoped*. For example, a write to global memory can have *device* scope, so that its effects are visible to all threads running on a device, or the more restricted *work-group* scope, so that although the write will eventually propagate to all threads, the write can only contribute to reliable synchronisation with threads in the same work-group. These subtleties

[3] `https://www.freedesktop.org/wiki/Software/Beignet/`

are intended to provide optimisation opportunities for OpenCL programmers and language implementers. Our attempt to formalise the memory model [8] led to the identification of some fundamental flaws in the way the model handles the class of *sequentially-consistent* (SC) memory accesses. SC is the default mode for accessing memory, and is intended to be the easiest for programmers to reason about (but not the most efficient), because all SC accesses in a program are guaranteed to be executed in some total order on which all threads agree. Our formalisation exposed a tension between this total order (which is visible to *all* threads), and OpenCL's scoping mechanism (which is intended to restrict visibility to just *some* threads). We were able to construct a small OpenCL program that involves SC accesses on two different devices, and show that even if these accesses are scoped to be visible only within the device that performs them, the compiler is still obliged to enforce an inter-device order between them [8, Example 10]. As such, the existing OpenCL memory model provides guarantees that are too strong to be efficiently implemented by a compiler. In our work, we suggest how the situation could be resolved in a future revision of the OpenCL memory model by enforcing a total order only between SC memory accesses that have the same scope [8, §5].

Although OpenCL's scoping mechanism allows some synchronisation patterns to be optimised, its usefulness is hampered by its requirement that two memory accesses can only synchronise if both use a scope that is wide enough to encompass the other. This symmetry inhibits the popular work-stealing pattern [13], which is often used to implement algorithms with irregular workloads. In order for a work-group to allow its work to be stolen by another work-group, it must scope all of its memory accesses widely enough to encompass all the work-groups that might wish to steal. Using such a wide scope is unnecessary in the common case when no stealing occurs and visibility is only needed within the current work-group. To rectify this shortcoming, Orr et al. have proposed *remote-scope promotion* (RSP) [36]. In their proposal, memory accesses can default to the intra-work-group scope (for good performance in the common case), and when a thread wishes to steal from another work-group, it invokes RSP to temporarily widen the scope of the victim's memory accesses, so that it can perform the necessary synchronisation to steal work.

Using the Isabelle proof assistant and the Herd memory model simulator [2], we formalised Orr et al.'s RSP extension and their proposed mapping to AMD's prototype GPU architecture [45]. These efforts brought to light several corner cases in the design, which we discussed in detail with Orr et al. We also discovered two serious flaws in the proposed mapping, both of which could lead to incorrect results being calculated. We were able to propose fixes in both cases. This work emphasises the value of applying formal methods to the early-stage design of hardware support for concurrent programming language features.

In more recent work, we have gone on to develop techniques for making the bug-finding process applied to Orr et al.'s work more automatic. Our Memalloy tool [46] takes any source language memory model, any target architecture memory model, and a description of a compilation mapping from source language to target architecture, and uses automatic constraint-solving technology to seek programs with behaviours that are forbidden at the source language level but are nonetheless observable on the target architecture when the compilation mapping is applied. For instance, the tool is able to reproduce (and minimise) our two manually-found RSP-related bugs in under two hours. We also used Memalloy to explore the compilation process from OpenCL to NVIDIA GPUs, and thereby discovered that the memory model we had deduced for NVIDIA GPUs through litmus testing (see the beginning of this section) was too weak to support the natural compilation scheme for OpenCL. After strengthening (and re-validating) the NVIDIA model, Memalloy was able to prove (up to a bound) the absence of memory-related bugs in the compilation scheme.

Open problems

There remains work to be done to investigate how the OpenCL memory model is implemented on the other kinds of devices that are OpenCL targets, such as FPGAs and digital signal processors (DSPs). For instance, Intel's OpenCL compiler for its FPGAs supports some of OpenCL's atomic operations, and it may be possible to employ formal methods to help ascertain whether they are implemented correctly and as efficiently as possible.

Also, existing studies of GPU memory models (both at the language level and the architecture level) have focused primarily on the single-device setting. However, OpenCL is designed to allow multiple devices to communicate with each other and with the host processor (via shared virtual memory). There is a need for techniques to help ascertain whether the OpenCL memory model has been implemented correctly and efficiently in this case. Furthermore, when shared virtual memory is used in 'fine-grained system' mode (whereby devices have simultaneous access to the entirety of the host's memory), there is a need to understand how the OpenCL memory model (as implemented on those devices) interacts with the memory model of the host processor. A general theory of how to *compose* memory models may be called for.

6 Blocking Algorithms and Forward Progress

Many parallel programs on traditional multicore systems (e.g. CPUs) have blocking behaviours. That is, one thread in the system will spin, waiting for another thread in the system to reach a certain point in the program. These interactions are typically orchestrated using flag values in a shared memory region. Common examples of concurrent idioms with blocking behaviours are mutexes and synchronisation barriers. In order to execute correctly (e.g. terminate or make progress), blocking behaviours require certain properties from the thread scheduler. Namely, if a thread t_1 is waiting for a thread t_2, the scheduler must ensure that t_2 is allowed to execute, thus freeing t_1 from waiting. If the system, for whatever reason, cannot guarantee that t_2 will eventually execute, then t_1 might wait indefinitely. A scheduler which guarantees the relative execution of threads is said have the *forward progress* property.

Currently, GPU specifications do not provide forward progress properties between threads in different work-groups, effectively disallowing popular existing idioms (and thus, programs) from multicore CPU systems to be ported to GPUs in a reliable manner. However, GPU programmers have discovered that by exploiting quirks in today's GPUs, certain blocking idioms can be made to execute as expected. This can be achieved pragmatically by determining (via trial and error) the number of work-groups for which forward progress appears to be guaranteed, and hard coding this number into the GPU kernel. This can be fiddly, since in practice the number is influenced by the program (e.g. due to register usage), the target GPU (e.g. due to hardware resource limits), and the GPU driver (e.g. due to its policy on how resources are allocated). Moreover, this approach to writing applications suffers from two drawbacks. First, applications that exploit such quirks are *not portable* even across GPUs from the same designer: undesirable behaviour (e.g. indefinite spinning) may occur when executing the program on a different GPU model, applying otherwise semantics-preserving changes to the program, or updating the GPU driver. Second, because the behaviour is not guaranteed by GPU programming language specifications, and applications written in this manner are generally not tested on a GPUs from a wide range of vendors [41], an application tested on multiple GPU models from one vendor may nevertheless fail when executed on a GPU from another vendor. Indeed, we observed that an inter-work-group synchronisation barrier written for one GPU (e.g. the NVIDIA GTX Titan) might deadlock when run on a different GPU (e.g. the Intel HD5500).

To address the above drawbacks, we performed a large empirical study (across eight GPUs spanning four vendors) to search for a forward progress abstraction that was (a) able to account soundly for the behaviour across the GPUs that we tested, and (b) useful enough to allow the implementation of popular blocking idioms [42]. After probing our zoo of GPUs with litmus tests, we determined that all GPUs had the following property: once a work-group begins executing a kernel (i.e. the work-group becomes *occupant* on a hardware resource), it will continue executing until it reaches the end of the kernel. We call this execution model the *occupancy bound* execution model, because the number of work-groups for which relative forward progress is guaranteed is bound by the hardware resources available for executing work-groups; i.e. the hardware resources determine how many work-groups can be occupant.

The occupancy bound execution model naturally allows mutex synchronisation, as a thread that acquires a lock is guaranteed to continue executing, thus eventually releasing the lock.[4] A portable synchronisation barrier across work-groups is, however, much more difficult to achieve. To address this, we developed a *discovery protocol*, which dynamically, at kernel launch, determines a safe, lower-bound estimate of the number of occupant work-groups. Work-groups that are not part of the initial wave of occupant work-groups discover that this is the case, and immediately exit the kernel. As a result, they do not participate in any blocking synchronisation. Kernels must be designed to be agnostic to the number of executing work-groups, to account for the fact that the number of occupant work-groups may vary across GPUs and between driver versions, and due to the fact that the discovery protocol does not guarantee discovering all work-groups that are initially occupant, and may discover a different number of work-groups during different executions of the same kernel. Previous work shows that it is usually possible to design kernels to satisfy this constraint, and programs with this property are said to use the *persistent thread* programming model [21].

The discovery protocol uses a polling mechanism, protected by a mutex, to allow work-groups to mark themselves as executing. Because the poll is only open for a finite amount of time, it is possible that the occupant work-groups may not be able to mark themselves, and contribute to the computation, thus under utilising the GPU. We show through a wide range of experiments that on current GPUs, we are able to define a discovery protocol that has a recall of nearly 100%. Our main insight is that using a *fair* mutex, such as a ticket lock, to guard the polling data structure leads to better recall than is achievable using a more traditional compare-and-swap-based spin-lock.

Open problems

As part of our work, we benchmarked a set of applications that use our discovery protocol and inter-work-group synchronisation barrier against versions of the same applications that use a *multi-kernel* approach to achieve global synchronisation (by explicitly ending and re-launching the kernel). We observed a wide range of performance profiles across the eight GPUs that we tested. For example, on Intel GPUs, our inter-work-group barrier provides a median 28% performance improvement compared with the multi-kernel approach, while on ARM GPUs we observed a median 50% *slowdown* using the inter-work-group barrier. An interesting open problem is to investigate whether a performance model can be developed that can predict when it is beneficial to use an inter-work-group barrier. Recent work has started to develop such a model for NVIDIA GPUs, but is yet to be extended to GPUs of other vendors [37].

[4] Of course, it is possible as usual for deadlock to occur due to mismanagement of multiple mutexes.

In our work so far we have only examined the relative forward progress properties associated with threads in *different* work-groups. Forward progress properties between threads at different levels of the GPU execution hierarchy may not be the same. As with inter-work-group forward progress, GPU specifications do not provide much guidance with respect to intra-work-group forward progress. In this context we think the following are interesting research topics: (a) an empirical investigation of the forward progress properties that GPUs appear to provide at each level of the execution hierarchy, and (b) the identification of applications that might benefit from blocking idioms at additional levels of this hierarchy.

Finally, because GPU specifications *do not* provide forward progress guarantees at present, we believe that working with standards committees in this area is vital. Our understanding from talking with various contacts in the GPU industry is that one reason vendors are reluctant to provide forward progress guarantees is so that they can reserve the right to dynamically change the computational resources assigned to a given GPU kernel, e.g. to make computational resources available to other tasks, or to reduce energy consumption. In our most recent work we have proposed an extension to the GPU programming model called *cooperative kernels*, specifically designed with blocking algorithms in mind, which we hope may allow the forward progress requirements of blocking algorithms to co-exist with the need for dynamic changes in the hardware resources that are assigned to kernels [43].

Acknowledgements. The line of work on which we report was contributed to by many co-authors: Jade Alglave, Ethel Bardsley, Mark Batty, Bradford Beckmann, Adam Betts, Nathan Chong, Peter Collingbourne, Pantazis Deligiannis, Hugues Evrard, Ganesh Gopalakrishnan, Daniel Liew, Daniel Poetzl, Shaz Qadeer, Zvonimir Rakamarić, and Paul Thomson.

References

1 Jade Alglave, Mark Batty, Alastair F. Donaldson, Ganesh Gopalakrishnan, Jeroen Ketema, Daniel Poetzl, Tyler Sorensen, and John Wickerson. GPU concurrency: Weak behaviours and programming assumptions. In *ASPLOS*, pages 577–591, 2015.

2 Jade Alglave, Luc Maranget, and Michael Tautschnig. Herding cats: Modelling, simulation, testing, and data mining for weak memory. *ACM Trans. Program. Lang. Syst.*, 36(2):7:1–7:74, 2014.

3 Afshin Amighi, Saeed Darabi, Stefan Blom, and Marieke Huisman. Specification and verification of atomic operations in GPGPU programs. In *SEFM*, pages 69–83, 2015.

4 Ethel Bardsley, Adam Betts, Nathan Chong, Peter Collingbourne, Pantazis Deligiannis, Alastair F. Donaldson, Jeroen Ketema, Daniel Liew, and Shaz Qadeer. Engineering a static verification tool for GPU kernels. In *CAV*, pages 226–242, 2014.

5 Michael Barnett, Bor-Yuh Evan Chang, Robert DeLine, Bart Jacobs, and K. Rustan M. Leino. Boogie: A modular reusable verifier for object-oriented programs. In *FMCO*, pages 364–387, 2005.

6 Clark Barrett, Christopher L. Conway, Morgan Deters, Liana Hadarean, Dejan Jovanović, Tim King, Andrew Reynolds, and Cesare Tinelli. CVC4. In *CAV*, pages 171–177, 2011.

7 Ethel Barsdley and Alastair F. Donaldson. Warps and atomics: Beyond barrier synchronisation in the verification of GPU kernels. In *NFM*, 2014.

8 Mark Batty, Alastair F. Donaldson, and John Wickerson. Overhauling SC atomics in C11 and OpenCL. In *POPL*, pages 634–648. ACM, 2016.

9 Adam Betts, Nathan Chong, Pantazis Deligiannis, Alastair F. Donaldson, and Jeroen Ketema. Implementing and evaluating candidate-based invariant generation. *IEEE Trans. Software Eng.*, 2017. To appear.

10 Adam Betts, Nathan Chong, Alastair F. Donaldson, Jeroen Ketema, Shaz Qadeer, Paul Thomson, and John Wickerson. The design and implementation of a verification technique for GPU kernels. *ACM Trans. Program. Lang. Syst.*, 37(3):10:1–10:49, 2015.

11 Adam Betts, Nathan Chong, Alastair F. Donaldson, Shaz Qadeer, and Paul Thomson. GPUVerify: A verifier for GPU kernels. In *OOPSLA*, pages 113–132, 2012.

12 Stefan Blom, Marieke Huisman, and Matej Mihelčić. Specification and verification of GPGPU programs. *Science of Computer Programming*, 95(3):376–388, 2014.

13 Robert D. Blumofe and Charles E. Leiserson. Scheduling multithreaded computations by work stealing. *Journal of the ACM*, 46(5):720–748, 1999.

14 Wei-Fan Chiang, Ganesh Gopalakrishnan, Guodong Li, and Zvonimir Rakamarić. Formal analysis of GPU programs with atomics via conflict-directed delay-bounding. In *NFM*, pages 213–228, 2013.

15 Nathan Chong, Alastair F. Donaldson, Paul Kelly, Jeroen Ketema, and Shaz Qadeer. Barrier invariants: A shared state abstraction for the analysis of data-dependent GPU kernels. In *OOPSLA*, pages 605–622, 2013.

16 Nathan Chong, Alastair F. Donaldson, and Jeroen Ketema. A sound and complete abstraction for reasoning about parallel prefix sums. In *POPL*, pages 397–410, 2014.

17 Peter Collingbourne, Cristian Cadar, and Paul H. J. Kelly. Symbolic crosschecking of data-parallel floating-point code. *IEEE Trans. Software Eng.*, 40(7):710–737, 2014.

18 Peter Collingbourne, Alastair F. Donaldson, Jeroen Ketema, and Shaz Qadeer. Interleaving and lock-step semantics for analysis and verification of GPU kernels. In *ESOP*, pages 270–289, 2013.

19 Leonardo Mendonça de Moura and Nikolaj Bjørner. Z3: An efficient SMT solver. In *TACAS*, pages 337–340, 2008.

20 Cormac Flanagan and K. Rustan M. Leino. Houdini, an annotation assistant for ESC/Java. In *FME*, pages 500–517, 2001.

21 Kshitij Gupta, Jeff Stuart, and John D. Owens. A study of persistent threads style GPU programming for GPGPU workloads. In *InPar*, pages 1–14, 2012.

22 ISO/IEC. *Programming languages – C.* International standard 9899:2011, 2011.

23 Jeroen Ketema and Alastair F. Donaldson. Termination analysis for GPU kernels. *Science of Computer Programming*, 2017. To appear.

24 Khronos Group. The OpenCL specification version: 2.0 (rev. 29), July 2015. `https://www.khronos.org/registry/cl/specs/opencl-2.0.pdf`.

25 Peter M. Kogge and Harold S. Stone. A parallel algorithm for the efficient solution of a general class of recurrence equations. *IEEE Trans. Comp.*, C-22(8):786—793, 1973.

26 Kensuke Kojima and Atsushi Igarashi. A Hoare logic for SIMT programs. In *APLAS*, pages 58–73, 2013.

27 Vu Le, Mehrdad Afshari, and Zhendong Su. Compiler validation via equivalence modulo inputs. In *PLDI*, pages 216–226, 2014.

28 Alan Leung, Manish Gupta, Yuvraj Agarwal, Rajesh Gupta, Ranjit Jhala, and Sorin Lerner. Verifying GPU kernels by test amplification. In *PLDI*, pages 383–394, 2012.

29 Guodong Li and Ganesh Gopalakrishnan. Scalable SMT-based verification of GPU kernel functions. In *FSE*, pages 187–196, 2010.

30 Guodong Li, Peng Li, Geoffrey Sawaya, Ganesh Gopalakrishnan, Indradeep Ghosh, and Sreeranga P. Rajan. GKLEE: Concolic verification and test generation for GPUs. In *PPoPP*, pages 215–224, 2012.

31 Christopher Lidbury, Andrei Lascu, Nathan Chong, and Alastair F. Donaldson. Many-core compiler fuzzing. In *PLDI*, pages 65–76, 2015.

32 William M. McKeeman. Differential testing for software. *Digital Technical Journal*, 10(1):100–107, 1998.

33 NVIDIA. CUDA C programming guide, version 5.5, 2013.

34 NVIDIA. CUDA 9 features revealed: Volta, cooperative groups and more, accessed 2017. `https://devblogs.nvidia.com/parallelforall/cuda-9-features-revealed/`.

35 NVIDIA. CUDA-MEMCHECK, accessed 2017. `http://docs.nvidia.com/cuda/cuda-memcheck/`.

36 Marc S. Orr, Shuai Che, Ayse Yilmazer, Bradford M. Beckmann, Mark D. Hill, and David A. Wood. Synchronization using remote-scope promotion. In *ASPLOS*, pages 73–86, 2015.

37 Sreepathi Pai and Keshav Pingali. A compiler for throughput optimization of graph algorithms on GPUs. In *OOPSLA*, pages 1–19, 2016.

38 Phillipe A. Pereira, Higo F. Albuquerque, Hendrio Marques, Isabela Silva, Celso Carvalho, Lucas C. Cordeiro, Vanessa Santos, and Ricardo Ferreira. Verifying CUDA programs using SMT-based context-bounded model checking. In *SAC*, pages 1648–1653, 2016.

39 Hamid Sarbazi-Azad, editor. *Advances in GPU Research and Practice*. Morgan Kaufmann, 2017.

40 Tyler Sorensen and Alastair F. Donaldson. Exposing errors related to weak memory models in GPU applications. In *PLDI*, pages 100–113, 2016.

41 Tyler Sorensen and Alastair F. Donaldson. The hitchhiker's guide to cross-platform OpenCL application development. In *IWOCL*, pages 2:1–2:12, 2016.

42 Tyler Sorensen, Alastair F. Donaldson, Mark Batty, Ganesh Gopalakrishnan, and Zvonimir Rakamaric. Portable inter-workgroup barrier synchronisation for gpus. In *OOPSLA*, pages 39–58, 2016.

43 Tyler Sorensen, Hugues Evrard, and Alastair F. Donaldson. Cooperative kernels: GPU multitasking for blocking algorithms. In *ESEC/FSE*, 2017. To appear.

44 Stavros Tripakis, Christos Stergiou, and Roberto Lublinerman. Checking equivalence of SPMD programs using non-interference. In *HotPar*, pages 1–5, 2010.

45 John Wickerson, Mark Batty, Bradford M. Beckmann, and Alastair F. Donaldson. Remote-scope promotion: clarified, rectified, and verified. In *OOPSLA*, pages 731–747, 2015.

46 John Wickerson, Mark Batty, Tyler Sorensen, and George A. Constantinides. Automatically comparing memory consistency models. In *POPL*, pages 190–204, 2017.

47 Xuejun Yang, Yang Chen, Eric Eide, and John Regehr. Finding and understanding bugs in C compilers. In *PLDI*, pages 283–294, 2011.

48 Mai Zheng, Vignesh T. Ravi, Feng Qin, and Gagan Agrawal. GRace: A low-overhead mechanism for detecting data races in GPU programs. In *PPoPP*, pages 135–146, 2011.

Admissibility in Games with Imperfect Information*

Romain Brenguier[1], Arno Pauly[2], Jean-François Raskin[3], and Ocan Sankur[4]

1 **Oxford University, UK**
2 **Université Libre de Bruxelles, U.L.B., Belgium**
3 **Université Libre de Bruxelles, U.L.B., Belgium**
4 **CNRS – IRISA Rennes, France**

Abstract

In this invited paper, we study the concept of *admissible strategies* for two player win/lose infinite sequential games with *imperfect information*. We show that in stark contrast with the perfect information variant, admissible strategies are *only* guaranteed to exist when players have objectives that are *closed sets*. As a consequence, we also study decision problems related to the existence of admissible strategies for regular games as well as finite duration games.

1998 ACM Subject Classification F.1.1 Models of Computation, B.1.2 [Automatic Synthesis] Formal Models

Keywords and phrases Admissibility, non-zero sum games, reactive synthesis, imperfect information

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.2

Category Invited Talk

1 Introduction

Two-player zero-sum perfect information games played on finite (directed) graphs are the canonical model to formalize the reactive synthesis problem [24, 1]. Unfortunately, this mathematical model is often a too coarse abstraction of reality. First, realistic systems are usually made up of several components, each of them with its own objective. These objectives are not necessarily antagonistic. Hence, the setting of non-zero sum graph games needs to be investigated, see [9] and additional references therein. Second, in systems made of several components, each component has usually a partial view on the entire system. Hence it is natural to study games with imperfect information [25, 13]. In this paper, we investigate the notion of *admissible strategies* for infinite duration non-zero sum games played on graphs in which players have *imperfect information*.

The *objective* W of a player in such a game is a set of infinite paths, those that model behaviors of the system that are regarded as satisfactory by the player. During the course of the game, players move a token from vertices to adjacent vertices. Each player applies one *strategy* which is a function that maps histories of plays that ends in a vertex owned by the player to adjacent vertices. The strategy instructs the player where to move the token

* Work partially supported by the ERC Starting grant 279499 (inVEST) and the ARC project "Non-Zero Sum Game Graphs: Applications to Reactive Synthesis and Beyond" (Fédération Wallonie-Bruxelles). J.-F. Raskin is Professeur Francqui de Recherche.

according to the prefix of play constructed so far. For a player with objective W, a strategy σ is said to be *dominated* by a strategy σ' if σ' does as well as σ with respect to W against all the strategies of the other players and strictly better for some of them. A strategy σ is *admissible* for a player if it is not dominated by any other of his strategies. Clearly, playing a strategy which is not admissible is sub-optimal, and as a consequence a rational player should favor admissible strategies.

While admissibility is a classical concept for games in normal form, i.e. matrix games, it was first studied in this context of infinite duration games by Berwanger in [6]. In that paper, it was shown that admissibility, i.e. the avoidance of dominated strategies, is well-behaved in infinite duration n-player non-zero sum turn-based games with *perfect information* and *Boolean outcomes* (two possible payoffs: win or lose). This framework encompasses games with omega-regular objectives. The main contributions of Berwanger were to show that:

- **(i)** In all n-player game structures, for all objectives, players have admissible strategies. Berwanger even shows the existence of strategies that survive the iterated elimination of dominated strategies.
- **(ii)** Every strategy that is dominated by a strategy is dominated by an admissible strategy.
- **(iii)** For finite game structures, the set of admissible strategies forms a regular set.

Contributions

In this paper, we study the notion of admissible strategies in the more general setting of infinite duration games with *imperfect* information. We obtain results that are in stark contrast with the results obtained by Berwanger:

- **(i)** In a 2-player game with imperfect information, players may have no admissible strategies at all, even for reachability objectives (Example 9). In particular, there are strategies that are dominated but not dominated by any admissible strategy.
- **(ii)** Admissible strategies always exist for the class of closed objectives, i.e. safety winning conditions, and their existence is only guaranteed for this class of objectives (Theorem 10).
- **(iii)** The set of admissible strategies of a player depends on the informedness of the other players: this set is the largest when the other players are perfectly informed (Theorem 17).
- **(iv)** For the special case of finite duration games, we know by point (ii) that admissible strategies always exist, but we show that simple queries about this set are NP-complete (Theorem 28).
- **(v)** For infinite duration 2-player games, we characterize precisely the set of admissible strategies when the second player is fully informed and we show how to decide the existence of admissible strategies (Theorem 20). When the second player is not fully informed but more informed than the first player, then we show how to decide the existence of admissible strategies for the first player by a reduction to the model-checking problem of Strategy Logic with (hierarchical) imperfect information (Theorem 32). In the general case, we show how to decide, given a regular observation-based strategy if this strategy is admissible or not (Theorem 35). For this last case, we left open the decidability status of the non-emptiness problem for the set of admissible strategies.

Related works

The iterated elimination of dominated strategies formalizes a strong notion of rationality [2]. In [18], Faella considers games played on finite graphs and focuses on the vertices from which one designated player cannot force a win. He compares several criteria for establishing what is the preferable behavior of this player from those vertices, eventually settling on the notion of admissible strategy. In [11], starting from the notion of admissible strategy, we have defined a novel rule for the compositional synthesis of reactive systems, applicable to systems made of n components which have each their own objective. We have shown that this synthesis rule leads to solutions which are robust and resilient. In [12], we have shown how to solve the central algorithmic questions related to admissibility for omega-regular objectives. In all those works, players are assumed to have perfect information.

In [10], we have studied the notion of admissible strategies for non-zero quantitative games with perfect information. Similarly to games with imperfect information, in the quantitative case, admissible strategies may not exist. Nevertheless, in all games where the *adversarial* and the *cooperative* values can be realized, i.e. games in which there exist worst-case optimal strategies and cooperative optimal profiles, players are guaranteed to have admissible strategies.

In [4], we have studied the notion of admissibility for concurrent games. Concurrent games [17] are n-player games played on graphs in which players take moves concurrently: at each round, all players choose an action at the same time and without informing the other players. This set of actions determines the next vertex. Because players choose actions without being informed of the concurrent choices of the other players, concurrent games are a special case of imperfect information games. In concurrent games, in contrast with general games with imperfect information studied here, admissible strategies are guaranteed to exist. While in this paper, we limit our study to deterministic strategies, in [4], we have also considered the more general case of randomized strategies and the notion of almost-sure winning.

In [16], Damm and Finkbeiner use the notion of *dominant strategy* to provide a compositional semi-algorithm for the (undecidable) distributed synthesis problem. So while we use the notion of admissible strategy, they use a notion of dominant strategy. The notion of dominant strategy is *strictly stronger*: every dominant strategy is admissible but an admissible strategy is not necessary dominant.

In normal form games it is trivial to decide whether a strategy dominates another one, or is dominated by an arbitrary strategy. Iterated elimination of dominated strategies does yield a relevant computational problem here, though. Depending on the type of dominance used (weak dominance, dominance, strict dominance) and the number of distinct payoffs, these problems were shown to be complete for L, NL, P or NP in [23, 15, 20, 8]. See [23] for an overview.

Structure of the paper

In Section 3, we discuss the existence of admissible strategies in two player win/lose infinite sequential games with imperfect information. In Section 4, we discuss the impact of informedness of the opponent on the set of admissible strategies. In Section 5, we characterize the set of admissible strategies when the opponent is perfectly informed. In Section 6, we consider the special case of games on finite trees. In Section 7, we investigate decision problems related to dominance in regular games of infinite duration.

2 Definitions

Games of imperfect information

Given some finite word w, let $\ell(w)$ denote its last element and $|w|$ its length. For an finite or infinite word w, if $|w| \geq n$, let $w_{\leq n}$ denote the prefix of w of length n, and for i, $0 \leq i < |w|$, let $w(i)$ denote the letter in position i in w. Given an infinite word w, let $\inf(w)$ denote the letters in w that appear infinitely often along w.

▶ **Definition 1.** A two player win/lose infinite sequential game with imperfect information (short: infinite game with imperfect information) between Player 1 (sometimes called the protagonist) and Player 2 (sometimes called the opponent) $G = (d, W, \cong_1, \cong_2)$ is given by the following:

1. a function $d : \{0,1\}^* \to \{1,2\}$ assigning a player to each vertex in the full infinite binary tree.
2. a winning condition $W \subseteq \{0,1\}^\omega$ for the first player.
3. equivalence relations $\cong_i$ on $\{0,1\}^*$ for $i \in \{1,2\}$ satisfying the following properties:
   a. $v \cong_i u \Rightarrow d(v) = d(u)$ (knowledge of who is playing is guaranteed).
   b. If $v \ncong_i u$, then $vb \ncong_i ub'$ for any $b, b' \in \{0,1\}$ (perfect recall).
   c. $v \cong_i u \Rightarrow |v| = |u|$ (ability to count moves).

Player i is *perfectly informed* if $\cong_i$ is the identity noted id_i. We only specify an objective for Player 1 as we will characterize the admissible strategies of this player, and to do this, the objective of Player 2 is irrelevant.

▶ **Definition 2.** Elements of $\{0,1\}^*$ (resp. $\{0,1\}^\omega$) are called *finite plays* or histories (resp. *plays*). A strategy of Player i is a function $\sigma : d^{-1}(i) \to \{0,1\}$. It is *observation-based*, if $v \cong_i u$ implies $\sigma(v) = \sigma(u)$. A play $p \in \{0,1\}^\omega$ is *compatible* with a strategy σ of Player i, if for all $n \in \mathbb{N}$, if $d(p_{\leq n}) = i$, then $\sigma(p_{\leq n}) = p(n+1)$. We write Σ_1 ($\Sigma_1^{\cong_1}$) and Σ_2 ($\Sigma_2^{\cong_2}$) for the set of all (observation-based) strategies for Player 1 and Player 2 respectively.

▶ **Definition 3.** Given strategies $\sigma \in \Sigma_1^{\cong_1}$, $\tau \in \Sigma_2^{\cong_2}$, we let $\mathsf{Out}(\sigma, \tau)$ be the unique $p \in \{0,1\}^\omega$ compatible with both σ and τ. A strategy $\sigma \in \Sigma_1^{\cong_1}$ of Player 1 is an *adversarially winning strategy* in G if $\forall \tau \in \Sigma_2^{\cong_2} : \mathsf{Out}(\sigma, \tau) \in W$, we denote this by $G, \sigma \models W$. A strategy $\sigma \in \Sigma_1^{\cong_1}$ is *cooperatively winning* if $\exists \tau \in \Sigma_2^{\cong_2} : \mathsf{Out}(\sigma, \tau) \in W$.

▶ **Definition 4.** A *regular* game $R(v_0) = (V, E, V_1, V_2, \mathcal{O}^1, \mathcal{O}^2, \Omega)$ is given by

1. a finite directed graph (V, E) with a designated starting vertex $v_0 \in V$ where all vertices have out-degree 1 or 2, moreover, the successors of each vertex $v \in V$ are ordered, and denoted by $s(v,0)$ and $s(v,1)$ (if v has out-degree 1, they are equal).
2. a control partition $V = V_1 \cup V_2$, V_i being the vertices controlled by Player i, v_0 is the starting vertex.
3. two observation partitions $V = \bigcup_{j \in I^i} \mathcal{O}^i_j$ for $i \in \{1,2\}$ which are both refinements of the control partition. We denote by $\mathcal{O}^i_{\mathsf{turn}}$ the set of observations of Player i that contain vertices that belong to Player i.
4. an ω-regular winning condition $\Omega \subseteq V^\omega$ defined by a parity condition $pr : V \to \{1, 2, \ldots, d\}$ and such that $\rho \in \Omega$ if and only if $\min\{k \mid v \in \inf(\rho) \wedge pr(v) = k\}$ is even.

▶ Remark. For convenience, we will give examples of regular games violating the condition that the out-degree is 1 or 2. In this case, we understand implicitly that the following

modifications are employed to satisfy this condition: If 2^d is an upper bound for the out-degree, then any non-sink vertex is replaced by a full binary tree of height d, with the partitions and winning condition being extended accordingly.

▶ **Definition 5.** From a regular game $R(v_0)$, we obtain an infinite sequential game $R^U(v_0) = (d, W, \cong_0, \cong_1)$ by unfolding as follows:

1. Let $\iota : \{0,1\}^* \to V^*$ be defined via $\iota(\langle\rangle) = v_0$ and $\iota(wb) = \iota(w) \cdot s(\ell(\iota(w)), b)$. Let $d(w) = i$ if $\ell(\iota(w)) \in V_i$.
2. Let $\widehat{\iota} : \{0,1\}^\omega \to V^\omega$ be the limit induced by ι. Let $W = \widehat{\iota}^{-1}(\Omega)$.
3. Let $\cong_i$ be the smallest equivalence relation that satisfies the following constraints:
   a. If $w \cong_i w'$ and $d(w) = 3 - i$ and $\exists j\ \ell(\iota(wb)) \in \mathcal{O}^i_j \wedge \ell(\iota(w'b')) \in \mathcal{O}^i_j$, then $wb \cong_i w'b'$.
   b. If $w \cong_i w'$ and $d(w) = i$ and $\exists j\ \ell(\iota(wb)) \in \mathcal{O}^i_j \wedge \ell(\iota(w'b)) \in \mathcal{O}^i_j$, then $wb \cong_i w'b$.

We also consider the special case in which Player 2 is *perfectly informed*. In that case, we do not take into account the partition $\mathcal{O}^2$ and $R^U(v_0) = (d, W, \cong_1, \cong_2)$ is defined as above but $\cong_2 = id_2$ is the identity relation.

To ease presentation, we make a limited use of *computation tree logic (CTL)* to state simple facts on strategies in regular games. We refer the interested reader for instance to [3] for formal definitions. Recall that for subsets $A, B \subseteq V$, $\mathbf{F}A$ (resp. $\mathbf{G}A$) is the set of plays that visit A (resp. that always stay in A). We use the following CTL formulas. $G, \sigma \models \mathbf{AF}A$ means that all plays compatible with σ eventually reach A. Furthermore $G, \sigma \models \mathbf{EF}A$ means that there exists $\tau \in \Sigma^{\cong_{3-i}}_{3-i}$ such that $\iota(\mathsf{Out}(\sigma, \tau))$ eventually reaches A.

For a subset $A \subseteq V$ of vertices, $G(A)$ denotes the game that starts with nondeterministically moving to any vertex of A, while both players receive respective observations of the chosen vertex. Strategy σ is *winning from $v \in V$ (resp. $A \subseteq V$)* if it is winning in $G(\{v\})$ (resp. $G(A)$).

▶ **Definition 6.** We consider a regular game $R(v_0)$. Given some finite sequence or infinite sequence $w \in \{0,1\}^{\leq\omega}$, the *derived observation sequence* for Player i is $\mathcal{O}^i(w) = \mathcal{O}^i_{j_0}\mathcal{O}^i_{j_1}\dots$ where $\iota(w)(k) \in \mathcal{O}^i_{j_k}$ for all $k \leq |w|$. An observation-based strategy for Player i in $R(v_0)$ is equivalent to a function $\sigma : (\mathcal{O}^i)^* \cdot \mathcal{O}^i_{\mathsf{turn}} \to \{0,1\}$. Now a strategy σ of Player i is *regular*, if there exists a finite deterministic automaton $\mathcal{A} = (Q^{\mathcal{A}}, q_0^{\mathcal{A}}, \mathcal{O}^i, \delta^{\mathcal{A}}, F^{\mathcal{A}})$, where $Q^{\mathcal{A}}$ is a finite set of states, $q_0^{\mathcal{A}}$ is the initial state, $\mathcal{O}^i$ is the alphabet, $\delta : Q^{\mathcal{A}} \times \mathcal{O}^i \to Q^{\mathcal{A}}$, and $F^{\mathcal{A}} \subseteq Q^{\mathcal{A}}$ is the set of accepting states, and the observation-based strategy σ is encoded by the language of the automaton as follows: when reading words $\mathcal{O}^i_{j_0}\mathcal{O}^i_{j_1}\dots$, and whenever $d(w) = i$, then the automaton accepts the observation sequence $\mathcal{O}^i(w)$ induced by w iff $\sigma(\mathcal{O}^i(w)) = 0$.

▶ **Definition 7.** Given a regular game $R(v_0)$, the *knowledge* of Player i playing the observation-based strategy σ after a sequence of observations $\rho = o_1 o_2 \dots o_n \in (\mathcal{O}^i)^*$, denoted $K(\sigma, \rho)$, is the set of vertices v such that there exists $w \in \{0,1\}^*$:

1. $\mathcal{O}^i(\iota(w)) = \rho$
2. for all j, $1 \leq j < n$, if $o_j \in \mathcal{O}^i_{\mathsf{turn}}$, then $\sigma(\rho_{\leq j}) = w_j$
3. v is equal to $\ell(\iota(w))$.

Dominance and admissibility

▶ **Definition 8.** For $\sigma, \sigma' \in \Sigma^{\cong_1}_1$, we say that σ' *weakly dominates* σ and write $\sigma' \succeq \sigma$ if $\forall \tau \in \Sigma^{\cong_2}_2$, $\mathsf{Out}(\sigma, \tau) \in W \Rightarrow \mathsf{Out}(\sigma', \tau) \in W$. If in addition $\exists \tau \in \Sigma^{\cong_2}_2$, $\mathsf{Out}(\sigma, \tau) \notin W$, and $\mathsf{Out}(\sigma', \tau) \in W$, then σ' *dominates* σ and write $\sigma' \succ \sigma$. We say that a strategy $\sigma \in \Sigma^{\cong_1}_1$ is *admissible* if there is no $\sigma' \in \Sigma^{\cong_1}_1$ such that $\sigma' \succ \sigma$. Note that if $\sigma \preceq \sigma'$ but $\sigma \not\succ \sigma'$, then $\sigma' \preceq \sigma$, justifying the notation.

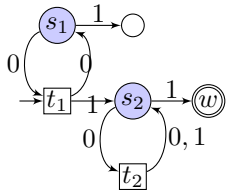


Figure 1 Player 1 can not differentiate between s_1 and s_2, and not between t_1 and t_2. In this example, no strategy is admissible for Player 1.

3 Existence of Admissible Strategies

A starting observation for our investigation is that the existence of admissible strategies in infinite sequential games with imperfect information is not guaranteed, even for regular games and reachability objectives. This is in stark contrast to the case where both players are perfectly informed [6].

▶ **Example 9.** We exhibit a regular game with a reachability objective (to reach the state marked w) lacking observation-based admissible strategies for Player 1. The graph is depicted in Figure 1. Player 1 controls circle vertices, Player 2 controls box vertices. Player 1 wins all plays that eventually enter the vertex w. Player 1 has imperfect information: he can not differentiate between s_1 and s_2, and not between t_1 and t_2. As a consequence, the observation-based strategies available to Player 1 are essentially the strategies σ_n, one for each $n \in \mathbb{N}$, which play the action "0" for n consecutive steps followed by the action "1", and σ_∞, which always plays the action "0". Then it is easy to show that $\sigma_\infty \prec \sigma_0 \prec \sigma_1 \ldots$, hence there is no observation-based admissible strategy for Player 1 in this game.

We can characterize exactly the properties of the winning condition W that ensures the existence of admissible strategies, provided we use colours to define the winning condition. Let C be a finite set of colours, and $c : \{0,1\}^* \to C$ be a colouring function. Let $\widehat{c} : \{0,1\}^\omega \to C^\omega$ be defined via $\widehat{c}(p)(n) = c(p_{\leq n})$. We say that $W \subseteq \{0,1\}^\omega$ is induced by $W_C \subseteq C^\omega$ if there exists a colouring function c such that $W = \widehat{c}^{-1}(W_C)$. Note that functions of the form $\widehat{c}$ are exactly the 1-Lipschitz functions from $\{0,1\}^\omega$ to C^ω.

▶ **Theorem 10.** *The following are equivalent for $W_C \subseteq C^\omega$:*

1. *W_C is closed (i.e. corresponds to a safety condition).*
2. *All imperfect information games with winning set induced by W_C have admissible strategies.*

Our proof of this theorem heavily relies on topological arguments. Note that we can conceive of $\Sigma_i^{\cong_i}$ as a metric space by setting $d(\sigma, \sigma') = 2^{-k}$ for the least $k \in \mathbb{N}$ such that $\exists w \in \{0,1\}^k \cap d^{-1}(i)$ with $\sigma(w) \neq \sigma'(w)$. With this metric, $\Sigma_i^{\cong_i}$ is a compact metric space, hence every sequence of strategies has an accumulation point. Moreover, the map $(\sigma, \tau) \mapsto \mathsf{Out}(\sigma, \tau) : \Sigma_1^{\cong_1} \times \Sigma_2^{\cong_2} \to \{0,1\}^\omega$ is continuous. We will write $B(\sigma, k) := \{\sigma' \in \Sigma_i^{\cong_i} \mid d(\sigma, \sigma') < 2^{-k}\}$ and utilize the fact that there are only countably many distinct sets $B(\sigma, k)$.

▶ **Lemma 11.** *Let W be closed. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma_1^{\cong_1}$ with $\sigma_n \preceq \sigma_{n+1}$ for all n, and let σ be an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$. Then $\forall n \in \mathbb{N} : \sigma_n \preceq \sigma$.*

Proof. Assume that for some $i \in \mathbb{N}$ we have $\sigma_i \not\preceq \sigma$. Then there is some $\tau \in \Sigma_2^{\cong_2}$ such that $\mathsf{Out}(\sigma, \tau) \notin W$ and $\mathsf{Out}(\sigma_n, \tau) \in W$. As W is closed, $\mathsf{Out}(\sigma, \tau) \notin W$ depends only on some finite prefix of σ, i.e. whenever $d(\sigma, \sigma') < 2^{-k}$ for a suitably large k, then also $\mathsf{Out}(\sigma', \tau) \notin W$. As σ is an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$, there is some $j \geq i$ with $d(\sigma, \sigma_j) < 2^{-k}$.

By assumption and transitivity of $\preceq$, we find that $\sigma_i \preceq \sigma_j$. But $\mathsf{Out}(\sigma_i, \tau) \in W$, whereas $\mathsf{Out}(\sigma_j, \tau) \notin W$ – contradiction. ◀

▶ **Corollary 12.** *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of protagonist strategies with $\sigma_n \prec \sigma_{n+1}$ for all n, and let σ be an accumulation point of $(\sigma_n)_{n\in\mathbb{N}}$. Then $\forall n \in \mathbb{N}\ \sigma_n \prec \sigma$.*

▶ **Lemma 13.** *Let $(\sigma_\beta)_{\beta<\alpha}$ be an ordinal-indexed sequence of protagonist strategies such that $\sigma_\beta \prec \sigma_\gamma$ for any $\beta < \gamma < \alpha$. Then α is countable.*

Proof. By assumption, $p_\gamma \not\preceq p_\beta$ for any $\gamma > \beta$. This is witnessed by some $\tau \in \Sigma_2^{\cong_2}$ such that $\mathsf{Out}(\sigma_\gamma, \tau) \in W$ and $\mathsf{Out}(\sigma_\beta, \tau) \notin W$. As W is closed, there is some $k \in \mathbb{N}$ such that $\mathsf{Out}(\sigma_\beta, \tau') \notin W$ for all $\tau' \in B_{\beta,\gamma} := B(\tau, k)$.

As there are only countably many distinct sets of the form $B(\tau, k)$, if α were uncountable, there would have to be $\beta < \gamma < \beta' < \gamma'$ with $B_{\beta,\gamma} = B_{\beta',\gamma'}$. By construction we have that $\exists \tau \in B_{\beta,\gamma} : \mathsf{Out}(\sigma_\gamma, \tau) \in W$ and $\forall \tau \in B_{\beta',\gamma'} : \mathsf{Out}(\sigma'_\beta, \tau) \notin W$. But this contradicts that $\sigma_\gamma \preceq \sigma_{\beta'}$, hence α has to be countable. ◀

▶ **Lemma 14.** *If $W_C \subseteq C^\omega$ is not closed, then there is an imperfect information game with winning set induced by W_C without an admissible strategy.*

Proof. This is a generalization of the Example 9. As W_C is not closed, we can pick a sequence $(p_n)_{n\in\mathbb{N}}$ and path p_∞ in C^ω such that $d(p_n, p_\infty) < 2^{-2n}$, $p_n \in W_C$ and $p_\infty \notin W_C$.

The two players take turns, i.e. $d(w) = 1$ iff $|w|$ is even. It is unknown to Player 1 which moves Player 2 has taken, i.e. $\cong_1$ is the coarsest equivalence relation satisfying the criteria. The colouring ensures that any sequence of the form $0^{2j+1}1w1q$ is mapped to p_{j+l+1} where $|w| = 2l$ and w contains no 1 in an odd position, while any other sequence gets mapped to p_∞. The sequence $(p_n)_{n\in\mathbb{N}}$ and p_∞ were chosen in a way to make this possible.

Intuitively, this means that Player 1 wins iff he plays a 1 for the first time at some turn after Player 2 already has played a 1. As Player 1 cannot observe the moves of Player 2, we find that as in Example 9 the observation based strategies available to Player 1 are essentially σ_n, which plays 0 the first n times it is Player 1's turn and then 1, and σ_∞, which always plays 1. Then again $\sigma_\infty \prec \sigma_0 \prec \sigma_1 \ldots$, hence there is no admissible strategy. ◀

Proof of Theorem 10. If W_C is closed, then any W induced by it is closed, too. We can start with some strategy p_0, and if it is dominated, move to some strategy p_1 it is dominated by, etc., and create a strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$, unless we hit a non-dominated strategy. Then pick some accumulation point p_ω (as the space of strategies is compact). If p_ω is dominated, pick a witness dominating strategy $p_{\omega+1}$, etc. If we would never reach a non-dominated strategy, then this would by Lemma 11 create an increasing Ω_1-sequence (where Ω_1 is the first uncountable ordinal), but this would contradict Lemma 13.

The second implication is the statement of Lemma 14. ◀

Some further remarks

▶ **Observation 15.** *In infinite trees, there are strictly increasing sequences of any countable length.*

Proof. By induction. If T is a tree with a strictly increasing sequence $(p_\beta)_{\beta<\alpha}$ of length α, let opponent choose between playing in T or letting protagonist choose between losing and winning. Starting with protagonist choosing to lose outside of T, and playing p_0 inside T, he can improve inside T for α steps and then decide to win if outside of T, yielding an improvement sequence of length $\alpha + 1$.

If for each $i \in \mathbb{N}$, T_i is a tree with an improvement sequence $(p^i_\beta)_{\beta<\alpha_i}$, then by letting opponent choose in which tree to play (or non at all), we can obtain an improvement sequence of length $\sup_{i\in\mathbb{N}} \alpha_i$. ◀

▶ **Observation 16.** *The construction in Lemma 14 is sufficiently uniform that the preceding argument also shows that for some ω-regular winning condition W all imperfect information games played on finite graphs have admissible strategies iff W is a safety condition.*

4 The Impact of the Informedness of the Opponent

The following theorem characterizes how the information available to the opponent (Player 2) impacts dominance between strategies of Player 1. When the second player is less informed then there are more dominated strategies for Player 1 and so less admissible strategies for Player 1.

We consider two infinite sequential games with imperfect information that only differ in the equivalence relation of Player 2, with one having $\cong_2$ and the other having $\cong'_2$. The set of (observation-based) strategies for Player 1 in both games is the same. We assume that $\cong_2$ is coarser than $\cong'_2$, i.e. $w \cong'_2 w' \Rightarrow w \cong_2 w'$. Then $\Sigma_2^{\cong'_2} \supseteq \Sigma_2^{\cong_2}$. We write $\sigma \prec \sigma'$ ($\sigma \prec' \sigma'$) if σ is dominated by σ' in the game built with $\cong_2$ (with $\cong'_2$).

▶ **Theorem 17.** *$G = (d, W, \cong_1, \cong_2)$ and $G' = (d, W, \cong_1, \cong'_2)$, where $\cong_2$ is coarser than $\cong'_2$, let $\prec$ be the dominance ordering in G and $\prec'$ in G', for all observation-based strategies σ and σ' of Player 1 if $\sigma \prec' \sigma'$, then $\sigma \prec \sigma'$.*

Proof. By the definition of $\preceq$ and $\preceq'$, we see that $\Sigma_2^{\cong'_2} \supseteq \Sigma_2^{\cong_2}$ implies that if $\sigma \preceq' \sigma'$, then $\sigma \preceq \sigma'$. To extend this to $\prec$ and $\prec'$, we need show that if there is some strategy $\tau' \in \Sigma_2^{\cong'_2}$ such that $\mathsf{Out}(\sigma, \tau') \notin W$ but $\mathsf{Out}(\sigma', \tau') \in W$, then there already is some $\tau \in \Sigma_2^{\cong_2}$ with this property.

Consider only word-strategies played by Player 2, i.e. strategies not depending on the actions of the opponent at all. If there is a word-strategy obtaining different outcomes against σ and σ', which by assumption of $\sigma \preceq' \sigma'$ can only mean that the outcome of the word strategy against σ is not in W while the outcome of the word strategy against σ' is in W, we would be done – as all word-strategies are in $\Sigma_2^{\cong_2}$ by the requirement that players can count the number of moves played so far.

If τ is played against σ, it will act like some particular word strategy (denoted by τ^σ), likewise τ acts like the word strategy $\tau^{\sigma'}$ when played against σ'. If $\tau^\sigma = \tau^{\sigma'}$, then this is the witness we are looking for. Otherwise, $\tau^\sigma = h\rho$ and $\tau^{\sigma'} = h\rho'$ for some maximal common prefix h, and after playing h against either σ or σ', Player 2 can distinguish w.r.t. $\cong'_2$ which of the two strategies he is playing against. Thus, the following strategy τ' is also in $\Sigma_2^{\cong'_2}$: Play h. If faced against σ, play ρ', if faced against σ', play ρ. Now τ' behaves like $\tau^{\sigma'}$ if played against σ, and like τ^σ if played against σ'. If all word strategies would yield the same outcomes against σ and σ', then τ' wins against σ' and loses against σ, a contradiction to $\sigma \prec' \sigma'$. Thus, there has to be a word strategy yielding different outcomes, which as explained above, provides the witness for $\sigma \prec \sigma'$. ◀

▶ **Corollary 18.** *If a strategy is not admissible w.r.t. perfect information strategies of the opponent, then it is not admissible w.r.t. observation-based strategies of the opponent.*

To conclude this section, we shall demonstrate that Player 2 having imperfect information can indeed lead to strictly less admissible strategies for Player 1 compared to a perfectly informed Player 2:

▶ **Example 19.** Consider a game where Player 1 moves first and plays **a** or **b**. Then Player 2 responds with **x**, **y** or **z**. Player 1 wins with the combinations **ax**, **bx** and **by**. If Player 2 can observe the move of Player 1, then both strategies of Player 1 are admissible. If Player 2 cannot observe the first move, then only playing **b** is admissible for Player 1.

5 Characterizing Admissibility with Perfectly Informed Opponents

We will provide a characterization of the admissible strategies of Player 1 under the assumption that Player 2 is perfectly informed. Corollary 18 shows that this is in a sense a conservative information, as any strategy found to be not admissible here will not be admissible against arbitrary opponents.

Fix some $\sigma \in \Sigma_1^{\cong_1}$. We say that $M \subseteq \{0,1\}^*$ is a σ-monochromatic set, if M is maximal under the two following constraints:

1. For any $v, u \in M$ we find that $v \cong_1 u$.
2. Any $v \in M$ is compatible with σ.

Given some σ-monochromatic set M, we partition M into $M_w^\sigma \uplus M_c^\sigma \uplus M_\ell^\sigma$ such that:

- $M_w^\sigma = \{v \in M \mid \forall \tau \in \Sigma_2 : v \text{ is compatible with } \tau \Rightarrow \mathsf{Out}(\sigma,\tau) \in W\}$, i.e. the set of histories in M from which the strategy σ is adversarially winning;
- $M_c^\sigma = \{v \in M \wedge v \notin M_w^\sigma \mid \exists \tau \in \Sigma_2 : v \text{ is compatible with } \tau \wedge \mathsf{Out}_\in(\sigma,\tau) W\}$, i.e. the set of histories in M from which the strategy σ is cooperatively winning;
- $M_\ell^\sigma = \{v \in M \mid \forall \tau \in \Sigma_2 : v \text{ is compatible with } \tau \Rightarrow \mathsf{Out}(\sigma,\tau) \notin W\}$, i.e. the set of histories in M from which the strategy σ is losing against all the strategies of Player 2.

Note that in the definition of monochromatic set, we quantify over the *entire* set of strategies of Player 2 as Player 2 is assumed to be perfectly informed.

▶ **Theorem 20.** *An observation-based strategy $\sigma \in \Sigma_1^{\cong_1}$ of Player 1 is* dominated, *when Player 2 is perfectly informed, if and only if there exists a σ-monochromatic set K such that*

1. *there exists another strategy $\sigma' \in \Sigma_1^{\cong_1}$ such that K is also σ'-monochromatic,*
2. *the strategies σ and σ' induce partitions $(K_w^\sigma, K_c^\sigma, K_\ell^\sigma)$ and $(K_w^{\sigma'}, K_c^{\sigma'}, K_\ell^{\sigma'})$ of K such that:*
   - **(a)** $K_w^\sigma \cup K_c^\sigma \subseteq K_w^{\sigma'}$;
   - **(b)** *and either*
     - *(i)* $K_\ell^\sigma \cap (K_w^{\sigma'} \cup K_c^{\sigma'}) \neq \emptyset$,
     - *(ii)* *or* $K_c^\sigma \neq \emptyset$.

Proof. We first establish the right to left direction. We construct an observation-based strategy σ'' that dominates σ when Player 2 is perfectly informed. The strategy σ'' is defined as follows. In all histories that are not extensions of those in K, σ'' plays exactly as σ. For all histories extending some history in K, then σ'' plays as σ'. Let us prove that σ'' has all the required properties to dominate σ:

- First, we note that σ'' is observation-based as σ and σ' are both observation-based, and extensions of K cannot be $\cong_1$-equivalent to non-extensions of K.

- Second, let π be a strategy of Player 2, and assume that $\mathsf{Out}(\sigma, \pi) \in W$. Then we need to show that $\mathsf{Out}(\sigma'', \pi) \in W$. We consider two different cases:
  1. If no prefix of $\mathsf{Out}(\sigma, \pi)$ is in K then we have that $\mathsf{Out}(\sigma, \pi) = \mathsf{Out}(\sigma'', \pi)$ so $\mathsf{Out}(\sigma'', \pi) \in W$.
  2. If a prefix w of $\mathsf{Out}(\sigma, \pi)$ is in K, then once we reach this prefix σ'' is now acting as σ'. As $\mathsf{Out}(\sigma, \pi) \in W$, w is such that $w \in K_w^\sigma \cup K_c^\sigma$, then we can conclude that $w \in K_w^{\sigma'}$, and so $\mathsf{Out}(\sigma'', \pi) \in W$.
- Third, we need to establish the existence of one strategy π of Player 2 such that $\mathsf{Out}(\sigma'', \pi) \in W$ and $\mathsf{Out}(\sigma, \pi) \notin W$. For that we consider a history $w \in K$ such that either: (i) $w \in K_c^\sigma$ (and so $w \notin K_w^\sigma$), or (ii) $w \in K_\ell^\sigma \wedge w \in K_c^{\sigma''} \cup K_w^{\sigma''}$. Such a w always exists by hypothesis. We design π as follows. First, π plays compatible with w. As σ and σ'' disagree at the history w, w.l.o.g. let us consider that σ plays 0 while σ' plays 1. As Player 2 is perfectly informed, he observes this and behaves in a way that:
  - if (i) $w \in K_c^\sigma$ and thus $w \in K_w^{\sigma'}$, then π does not help σ (this is possible as $w \notin K_w^\sigma$), and ensures that the outcome is outside W, while σ'' behaves like σ' and is thus winning from w no matter what π is.
  - if (ii) $w \in K_\ell^\sigma \wedge w \in K_c^{\sigma''} \cup K_w^{\sigma''}$, then π helps Player 1 when he plays σ'' to ensure that the outcome is in W and as $w \in K_\ell^\sigma$, we know that the outcome when σ is played is not in W.

We now establish the left to right direction. Assume that σ is dominated by some other strategy σ''. By definition of dominance, there exists a strategy π of Player 2 such that $\mathsf{Out}(\sigma, \pi) \notin W$ and $\mathsf{Out}(\sigma'', \pi) \in W$. Let w be the longest common prefix of $\mathsf{Out}(\sigma, \pi)$ and $\mathsf{Out}(\sigma'', \pi)$, and let K be the σ-monochromatic set containing w (K is then also σ''-monochromatic). Let us consider the partition of K for σ and σ''. First, we know that $w \notin K_w^\sigma$ and $w \in K_c^{\sigma''} \cup K_w^{\sigma''}$. Also, because σ is dominated by σ'' then we have that:

- for all $v' \in K_w^\sigma : v' \in K_w^{\sigma''}$,
- for all $v' \in K_c^\sigma : v' \in K_w^{\sigma''}$,
- for all $v' \in K_\ell^{\sigma''} : v' \in K_\ell^\sigma$,

Otherwise, it is easy to obtain a contradiction with the fact that σ is dominated by σ''. All this implies the right properties on the respective partitions. ◀

▶ Remark. This characterization of dominance is particularly useful in the case of regular games. There the question whether for some monochromatic K there exists a strategy σ inducing a particular partition $(K_w^\sigma, K_c^\sigma, K_\ell^\sigma)$ depends only on the set of last vertices of the histories of K. This is because winning sets in regular games are defined with parity conditions that lead to prefix independence. In particular, there are only finitely many cases to check. We will exploit this algorithmically in Subsection 7.2.

6 Games on Finite Trees

The simplest non-trivial subclass of infinite sequential games with imperfect information are those with clopen winning sets W. Essentially, this means that whether or not $p \in W$ for some run p only depends on some fixed-length prefix of p. Hence, we can consider these as *finite tree games*. Alternatively, we could conceive of these as regular games where the underlying graph is a tree.

The initial vertex is the root of the tree, and we denote by $\mathsf{Leaves}(G)$ the leaves of the tree, i.e. the finite histories determining membership in W of any run passing through them. W.l.o.g., we assume that all players know when a leaf is reached, and that both players can distinguish winning from losing leaves; in other terms, no $\cong_i$-equivalence class contains both

a leaf and a non-leaf, and no class contains both a winning and a losing leaf. We denote the set of winning leaves by Ω_T.

In this section, for convenience, we assume that each vertex of Player i has k_i successors for a given $k_i \geq 1$, and that these successors are labeled with numbers between 0 and $k_i - 1$. We sometimes call these numbers *actions*, and denote $\mathsf{Act}_i = \{0, 1, \ldots, k_i\}$, and $\mathsf{Act} = (\mathsf{Act}_1, \mathsf{Act}_2)$. For any vertex $v \in V$, and $a \in \mathsf{Act}$, let $\delta(v, a)$ denote the a-th successor of v. Games with $|\mathsf{Act}_i| > 2$ can be equivalently modeled using two actions by adding intermediate states (see the remark after Definition 4).

We denote the finitely many $\cong_i$-equivalence classes that matter (i.e. are not proper extensions of the leaves) by $\mathcal{O}^i = \{\mathcal{O}^i_j \mid j \in I^i\}$. An observation-based strategy then is, up to irrelevant moves, simply a function $\sigma : \mathcal{O}^i \to \mathsf{Act}_i$, which can be stored in linear space.

6.1 Characterization of Domination

We are going to give an algorithm to decide whether a given strategy is dominated in a finite tree game. To do so, we are going to simultaneously simulate σ and another strategy that is to be chosen by Player i, in a product construction. We need the following additional definition:

▶ **Definition 21** (Switching strategies)**.** For any player i, strategies $\tau, \tau' \in \Sigma_i^{\cong_i}$ and an observation $o \in \mathcal{O}_i$, we denote $\tau[o/\tau']$ the strategy that plays τ but upon visiting o switches to τ'. Formally,

$$\tau[o/\tau'](o_1 \cdots o_n) = \begin{cases} \tau'(o_1 \cdots o_n) \text{ if } \exists i, o_i = o \\ \tau(o_1 \cdots o_n) \text{ otherwise} \end{cases}$$

Let us consider a finite tree game $G = (V, E, V_1, V_2, \mathcal{O}^1, \mathcal{O}^2, \Omega_T)$ and let us note $\Sigma_i^{\cong_i}(G)$ the observation-based strategies of Player i in G. For strategy $\sigma \in \Sigma_i^{\cong_i}(G)$, and vertex $v \in V_i$, let $\delta^*(v, \sigma)$ be the vertex obtained by repeatedly applying σ as long as the vertex stays in V_i. Thus, $\delta^*_\sigma(v)$ is either a leaf, or a vertex of V_{3-i}. For $v \notin V_i$, let $\delta^*(v, \sigma) = v$. Given action a and strategy σ, let us define $a \cdot \sigma$ that plays a in the first step, and then switches to σ.

▶ **Definition 22.** Given a finite tree game $G(v_{\mathsf{init}}) = (V, E, V_1, V_2, \mathcal{O}^1, \mathcal{O}^2, \Omega)$ with actions Act, and strategy $\sigma \in \Sigma_i^{\cong_i}(G)$, define $G_\sigma(v^\sigma_{\mathsf{init}}) = (V^\sigma, E^\sigma, V_1^\sigma, V_2^\sigma, \mathcal{O}^1_\sigma, \mathcal{O}^2_\sigma, \Omega^\sigma)$ with actions Act^σ, where

- $V^\sigma = V \times V$, $V_i^\sigma = V \times V_i$, and $V_{3-i}^\sigma = V^\sigma \setminus V_i^\sigma$,
- $v^\sigma_{\mathsf{init}} = (\delta^*(v_{\mathsf{init}}, \sigma), v_{\mathsf{init}})$,
- $\mathsf{Act}_i^\sigma = \mathsf{Act}_i$, and $\mathsf{Act}_{3-i}^\sigma = \mathsf{Act}_{3-i} \times \mathsf{Act}_{3-i}$,
- $\mathcal{O}_i^\sigma = \{V \times \mathcal{O}^i_j \mid \mathcal{O}_j \in \mathcal{O}^i\}$, $\mathcal{O}^\sigma_{3-i} = \{\mathcal{O}^{3-i}_j \times \mathcal{O}^{3-i}_{j'} \mid \mathcal{O}^{3-i}_j, \mathcal{O}^{3-i}_{j'} \in \mathcal{O}^{3-i}\}$.
- For all $(v, v') \in V_i^\sigma$, $a \in \mathsf{Act}_i^\sigma$,

  $$\delta^\sigma((v, v'), a) = (v, \delta(v', a)).$$

  For all $(v, v') \in V_{3-i}^\sigma$ with $\mathcal{O}^{3-i}(v) = \mathcal{O}^{3-i}(v')$, for all $(a, b) \in \mathsf{Act}_{3-i}^\sigma$,

  $$\delta_\sigma((v, v'), (a, b)) = (\delta^*(v, a \cdot \sigma), \delta(v', a)).$$

  For all $(v, v') \in V_{3-i}^\sigma$, if $\mathcal{O}^{3-i}(v) \neq o_{3-i}(v')$, for all $(a, b) \in \mathsf{Act}_{3-i}^\sigma$,

  $$\delta^\sigma((v, v'), (a, b)) = \begin{cases} (\delta^*(v, a \cdot \sigma), \delta(v', b)) & \text{if } v, v' \notin \mathsf{Leaves}(G), \\ (\delta^*(v, a \cdot \sigma), v') & \text{if } v \notin \mathsf{Leaves}(G), v' \in \mathsf{Leaves}(G), \\ (v, \delta(v', b)) & \text{if } v \in \mathsf{Leaves}(G), v' \notin \mathsf{Leaves}(G), \end{cases}$$

Informally, Player i only sees the second component and plays as in G; in fact, the second component of G_σ reproduces precisely G. The first component always moves according to σ from Player-i vertices, and according to Player $3-i$'s actions otherwise. Player $3-i$ plays the same actions in both components as long as her observations match, but can choose different actions otherwise. Observe that we "accelerate" the transitions in the first component in Player-i vertices, so we ensure that the first component is always either a Player $3-i$ vertex, or a leaf.

Note that $\Sigma_i^{\cong_i}(G) = \Sigma_i^{\cong_i}(G_\sigma)$ since the observations and actions for Player i are identical in both games.

Consider game G, player i, and $\sigma, \sigma' \in \Sigma_i^{\cong_i}(G)$. For any strategy $\underline{\tau} \in \Sigma_{3-i}^{\cong_{3-i}}(G)$, let us call $n_{\underline{\tau}}^{\sigma,\sigma'}$ the smallest integer such that the observation of Player $3-i$ at the $n_{\underline{\tau}}^{\sigma,\sigma'}$-th step is different between the plays $\rho = \iota(\mathsf{Out}_G(\sigma,\tau))$ and $\rho' = \iota(\mathsf{Out}_G(\sigma',\tau))$, that is, $n_{\underline{\tau}}^{\sigma,\sigma'} = \min\{n | \mathcal{O}^{3-i}(\rho_n) \neq \mathcal{O}^{3-i}(\rho'_n)\}$ when this minimum is finite, and $n_{\underline{\tau}}^{\sigma,\sigma'} = \infty$ otherwise. Note that the lengths of the plays in the latter case must be equal since observations distinguish leaves from non-leaves. Let $\mathsf{dist}_1^{G,\underline{\tau}}(\sigma,\sigma') = \rho_{n_{\underline{\tau}}^{\sigma,\sigma'}}$ and $\mathsf{dist}_2^{G,\underline{\tau}}(\sigma,\sigma') = \rho'_{n_{\underline{\tau}}^{\sigma,\sigma'}}$ be these vertices where Player $3-i$ distinguishes both plays for the first time.

▶ **Lemma 23.** *Consider any game G, player i, $\sigma_1, \sigma_2 \in \Sigma_i^{\cong_i}(G)$, and $\underline{\tau} \in \Sigma_{3-i}^{\cong_{3-i}}(G_{\sigma_1})$. If we write $(t_1,t_2) = \ell(\iota(\mathsf{Out}_{G_{\sigma_1}}(\sigma_2,\tau)))$, there exists $\tau \in \Sigma_{3-i}^{\cong_{3-i}}(G)$ such that $t_j = \ell(\iota(\mathsf{Out}_G(\sigma_j,\tau)))$ for all $j \in \{1,2\}$.*

Proof. If $n_{\underline{\tau}}^{\sigma_1,\sigma_2} = \infty$, then Player $3-i$ receives the same observation in both components. Let us define τ by $\tau(o) = a$ where $(a,b) = \underline{\tau}((o,o))$. Then choosing $\tau_1 = \tau_2 = \tau$ yields the result.

Assume $n_{\underline{\tau}}^{\sigma_1,\sigma_2} < \infty$, and write $(t_1,t_2) = \ell(\iota(\mathsf{Out}_{G_{\sigma_1}}(\sigma_2,\underline{\tau})))$. We define $\tau(o)$ as the first component of $\underline{\tau}((o,o))$ for all observations $o \in \mathcal{O}_{3-i}$ such that $o \notin \{\mathsf{dist}_j^{G,\underline{\tau}}(\sigma_1,\sigma_2))\}_{j\in\{1,2\}}$ or o is not a descendant of these observations. Let (s_1,s_2) be the $n_{\underline{\tau}}^{\sigma_1,\sigma_2}$-th vertex of $\iota(\mathsf{Out}_{G_{\sigma_1}}(\sigma_2,\underline{\tau}))$. Notice that the run $\iota(\mathsf{Out}_G(\sigma_j,\tau))$ visits s_j, for each $j \in \{1,2\}$. For each j, in $\mathsf{dist}_j^{G,\underline{\tau}}(\sigma_1,\sigma_2)$, there exists strategy τ_j such that $\ell(\iota(\mathsf{Out}_{G_{s_j}}(\sigma_j,\tau_j))) = t_j$. We complete the definition of τ as follows: at each $\mathsf{dist}_j^{G,\underline{\tau}}(\sigma_1,\sigma_2)$, it switches to τ_j. ◀

Let us define $P_\sigma = \{(t,t') \in \mathsf{Leaves}(G) \times \mathsf{Leaves}(G) \mid t \in \Omega_T \Rightarrow t' \in \Omega_T\}$.

▶ **Lemma 24.** *For any $\sigma' \in \Sigma_i^{\cong_i}(G)$, σ is weakly dominated by σ' if, and only if $G_\sigma, \sigma' \models \boldsymbol{AF}P_\sigma$.*

Proof. Assume $G_\sigma, \sigma' \models \mathbf{AF}P_\sigma$. Let $\tau \in \Sigma_{3-i}^{\cong_{3-i}}$ such that $G,\sigma,\tau \models \Omega$; we will show that $G,\sigma',\tau \models \Omega$.

Let $\underline{\tau}$ defined by $\underline{\tau}((o,o')) = (\tau(o),\tau(o'))$ for all $o,o' \in \mathcal{O}_{3-i}$. We have, by assumption, $G_\sigma, \sigma', \underline{\tau} \models \mathbf{F}P_\sigma$. Moreover, the projection of this run to the first component is exactly $\iota(\mathsf{Out}_G(\sigma,\tau))$. The projection to the second component is $\iota(\mathsf{Out}_G(\sigma',\tau))$. By the definition of P_σ, and since $\mathsf{Out}_G(\sigma,\tau)$ is in Ω, $\mathsf{Out}_G(\sigma',\tau)$ is in Ω too.

Conversely, assume that $G_\sigma, \sigma', \underline{\tau} \models \mathbf{G}\neg P_\sigma$. Let $(t,t') = \ell(\mathsf{Out}(\sigma',\underline{\tau})G_\sigma)$, with $t \in \Omega_T$, $t' \notin \Omega_T$. Let $\tau \in \Sigma_{3-i}^{\cong_{3-i}}(G)$ be as given by Lemma 23. We get that $\iota(\mathsf{Out}_G(\sigma,\tau)) = t$ and $\iota(\mathsf{Out}_G(\sigma',\tau)) = t'$, which gives the desired result. ◀

We let $Q_\sigma = \{(t,t') \in \mathsf{Leaves}(G) \times \mathsf{Leaves}(G) \mid t \notin \Omega_T \wedge t' \in \Omega_T\}$.

▶ **Lemma 25.** *For any $\sigma' \in \Sigma_i^{\cong_i}(G)$, σ is dominated by σ' if, and only if $G_\sigma, \sigma' \models \boldsymbol{AF}P_\sigma \wedge \boldsymbol{EF}Q_\sigma$.*

1 \ 2	x	y	z
a a	1	0	0
a b	0	1	1
b a	1	1	0
b b	0	0	1

Figure 2 An admissible strategy of a sub-game may not be a part of an admissible strategy of the whole game.

Proof. Assume $G_\sigma, \sigma' \models \mathbf{AF}P_\sigma \wedge \mathbf{EF}Q_\sigma$. The previous lemma shows that σ' weakly dominates σ. Let $\underline{\tau}$ such that $G_\sigma, \sigma', \underline{\tau}$ ends in $(t, t') \in Q_\sigma$. Let τ given by Lemma 23. We get that $\iota(\mathsf{Out}_G(\sigma, \tau)) = t$ and $\iota(\mathsf{Out}_G(\sigma', \tau)) = t'$, which means that σ is dominated by σ'. ◀

▶ **Lemma 26.** *Given a game G, player i, and subsets $P, Q \subseteq \mathsf{Leaves}(G)$, one can decide in polynomial time if there is a strategy $\sigma \in \Sigma_i^{\cong_i}(G)$ such that $G, \sigma \models \boldsymbol{AF}P \wedge \boldsymbol{EF}Q$.*

Proof. We first show that there exist maximally permissive strategies for objectives of the type $\mathbf{AF}P$. In fact, define

$$\mathsf{Safe} = \{(o, a) \in \mathcal{O}_i \times \mathsf{Act}_i \mid \exists \sigma \in \Sigma_i^{\cong_i}, G, \sigma \models \mathbf{AF}P \wedge \mathbf{EF}o, \sigma(o) = a\}.$$

This is the set of pairs of observation-actions (o, a) such that some Player-i strategy that is winning for $\mathbf{AF}P$ is compatible with o, and chooses a from o. We claim that any $\sigma \in \Sigma_i^{\cong_i}(G)$ such that $\forall o \in \mathcal{O}_i$, $(o, \sigma(o)) \in \mathsf{Safe}$ is winning for objective $\mathbf{AF}P$. This can be proved by induction on the length of plays, and using perfect recall.

Now, the set Safe can be computed bottom-up. Let G' be the game G where we only keep Player-i actions that conform to Safe. There exists a strategy in G winning for $\mathbf{AF}P \wedge \mathbf{EF}Q$ if, and only if some vertex of Q is reachable in G'. ◀

▶ **Theorem 27.** *Given game G on a finite tree, player i and strategy $\sigma \in \Sigma_i^{\cong_i}(G)$, one can decide in polynomial time whether σ is admissible.*

6.2 Hardness

Let us start with a remark showing that one cannot hope to prune some actions of the game locally, to obtain a description of the admissible plays as it is the case in the perfect-information setting, see section on safety games in [12]. In fact, a strategy that only uses actions that appear in (the reachable part of) some admissible strategy may not be admissible itself.

Consider the game in normal form in Figure 2. It is easy to encode it as a game on a tree by ensuring that Player 2 is blind (i.e. all states in V_2 leads to the same observation) and players have perfect recall (in particular, they remember which actions they have played). Player 1 starts choosing a or b, then Player 2 plays (x, y or z), and Player 1 plays again. We have that ab is admissible, so there is an admissible strategy playing a in the first vertex. Furthermore, in the subgame where Player 1 has already played a, playing a is admissible (to see this, compare the first two lines in the figure). However strategy aa is dominated by ba.

Similarly there is an admissible strategy of Player 1 playing b in the first step. Playing b is admissible in the subgame that starts with history b, but the strategy bb is dominated by ab.

We now show that deciding the existence of an admissible strategy choosing a particular action at a given state is NP-complete.

▶ **Theorem 28.** *Given two-player game G on a finite tree, and an action a, deciding whether there is an admissible strategy σ with $\sigma(s_{\mathsf{init}}) = a$ is* NP*-complete.*

Proof. NP-membership holds because strategies of Player 1 can be represented in polynomial size, and having guessed some strategy σ_1 which uses a, whether σ_1 is admissible can be verified in polynomial time by Corollary 27.

For NP-hardness, we encode an instance of the 3-SAT problem of the following form

$$\psi = \exists x_1, \ldots, x_n.\ C_1 \wedge \cdots \wedge C_m,$$

where $C_i = \ell_{i,1} \vee \ell_{i,2} \vee \ell_{i_3}$ and $\ell_{i,j} \in \{x_k, \neg x_k \mid k \in [1,n]\}$, for each $1 \leq i \leq m$, and $1 \leq j \leq 3$.

We define a game G_Ω as follows. From the initial vertex Player 1 either chooses a clause among $C_1, \ldots, C_m$, or goes to a special vertex C_0. Then Player 2 blindly selects a literal ℓ in $\{x_k, \neg x_k \mid k \in [1,n]\}$. At this point, Player 1 only sees the index k such that $\ell \in \{x_k, \neg x_k\}$. Let us write $o^{i,k}$ for this observation, where i is the index of the clause, or 0, and k is the index of the variable chosen by Player 2.

From $o^{0,k}$, Player 1 chooses T or F meaning, intuitively, that x_k is evaluated to true and false respectively. All other observations $o^{i,k}$ with $i > 0$ are leaves. The reachability condition Ω is satisfied when either Player 1 selects C_i with $i > 0$ and ℓ is a literal which does not appear in C_i or Player 1 selected C_0 and the valuation chosen in the last vertex is such that ℓ is evaluated to true.

Assume there is a strategy σ_1 choosing C_0 which is not dominated by any strategy not choosing C_0. Now, define the valuation v as follows. For each variable x_k, we set $v(x_k)$ to true if, and only $\sigma_1(o^{0,k}) = T$. Let σ_1' be any strategy of Player 1 choosing some clause C_i with $i > 0$. Because σ_1 is not dominated by σ_1', either: (A) there is a strategy τ^i of Player 2 which makes σ_1 win but not σ_1'; or (B) for all strategies τ of Player 2, σ_1', τ wins implies σ_1, τ wins.

Let us first show that case (B) cannot happen. Recall that σ_1', τ is winning when τ selects a literal outside C_i. If there are more than 2 variables ($n > 2$), which we can assume without loss of generality, then there is one variable x_k and its negation that do not appear in C_i. Consider the strategies τ and τ' that select x_k and $\neg x_k$ respectively. Since they make σ_1' win, they must also make σ_1 win, which means that the valuation defined by σ_1 makes both x_k and $\neg x_k$ true, which is a contradiction.

Thus, only the case (A) is possible. For each $1 \leq i \leq m$, let σ_i' be the strategy that chooses C_i. Let τ^i be a Player-2 strategy such that σ_1, τ^i wins and σ_i', τ^i loses. Let us fix $1 \leq i \leq m$, and let ℓ denote the literal chosen by τ^i. The run of σ_1', τ^i is losing, which means that $\ell \in C_i$. Since σ_1, τ^i wins, ℓ is satisfied by v. Since $\ell \in C_i$, C_i is satisfied by v. Since i was arbitrary, all clauses C_i are satisfied. Hence ψ is satisfiable.

In the other direction, assume the formula is satisfiable. Let v be a valuation satisfying the formula, and σ_v be the strategy that first plays C_0, and then plays according to the valuation v. Let σ_1' be any strategy that first plays C_i for any $i \in [1,n]$. As v satisfies the formula, there is a literal ℓ in C_i such that $v(\ell) = T$. Let τ_ℓ be the strategy of Player 2 that chooses this literal. As $\ell \in C_i$, τ_ℓ makes σ_1' lose, and as $v(\ell) = T$, τ_ℓ makes σ_1 win. So σ_1 is not dominated by σ_1'. ◀ ◀

7 Decision problems for dominance in regular games

In this section, we study decision problems related to admissible strategies for regular infinite sequential games with imperfect information. We have seen in Theorem 17 that a strategy of Player 1 that is dominated when Player 2 is perfectly informed is also dominated when Player 2 is imperfectly informed. The special case of a perfectly informed Player 2 is thus important. In this section, we start by considering this case, exploiting the characterization obtained in Theorem 20. Then we turn to the case when Player 2 is not perfectly informed. Before presenting our results, we need to recall some notions related to infinite trees and automata on infinite trees.

7.1 Automata on infinite trees and tree encodings of strategies

Infinite trees

Following e.g. [21], given a finite set $\mathcal{D}$, a $\mathcal{D}$-*tree* is a prefix closed language $T \subseteq \mathcal{D}^*$. When $\mathcal{D}$ is clear from the context, we call T a tree. Elements of T are called *nodes*, and the empty word ϵ is the *root* of the tree. For every $x \in T$, the nodes $x \cdot d \in T$ for $d \in \mathcal{D}$ are the *children* of x, the children $x \cdot d$ is called the child of direction d of x. A $\mathcal{D}$-tree is a full infinite tree if $T = \mathcal{D}^*$. A *branch* b of T is a set $b \subseteq T$ such that $\epsilon \in b$ and for every $x \in b$ there exists a unique $d \in \mathcal{D}$ such that $x \cdot d \in b$. Given two sets $\mathcal{D}$ and $\mathcal{L}$, a $\mathcal{L}$-labelled $\mathcal{D}$-tree is a pair $\langle T, L \rangle$ where T is a $\mathcal{D}$-tree and $L : T \to \mathcal{L}$ labels each node of T with an element of $\mathcal{L}$.

Tree encoding of observation-based strategies

We encode observation-based strategies $\sigma \in \Sigma_1^{\cong_1}$, i.e. $\sigma : (\mathcal{O}^1)^* \cdot \mathcal{O}^1_{\mathsf{turn}} \to \{0,1\}$, as full infinite $\{0,1,*\}$-labelled $\mathcal{O}^1$-trees with the convention that:

- the root node is always labelled with $*$;
- every node that follows an observation $o \in \mathcal{O}^1_{\mathsf{turn}}$ is labeled with 0 or 1 according to σ: if the node is reached with a sequence of directions $\rho = o_1 o_2 \dots o_n$ that corresponds to observations in $\mathcal{O}^i$ and $o_n \in \mathcal{O}^1_{\mathsf{turn}}$, then the node is labelled by $\sigma(\rho)$;
- every node that follows an observation $o \in \mathcal{O}^1 \setminus \mathcal{O}^1_{\mathsf{turn}}$ is labelled with $*$ as in that case, it is the turn of Player 2 to play and the choice of Player 2 is not visible for Player 1.

Alternating tree automata

We use *alternating tree automata* (AT) to recognize regular sets of infinite trees that encode observation-based strategies. A *AT* $P = (Q, q_0, \Delta, \Psi)$ that operates on $\mathcal{L}$-labelled $\mathcal{D}$-trees is defined by:

- its finite non-empty set of states Q;
- its initial state q_0;
- its transition function $\Delta : Q \times \mathcal{L} \to \mathcal{B}^+(Q \times \mathcal{D})$, where $\mathcal{B}^+(Q \times \mathcal{D})$ is the set of (positive) Boolean formula built from elements in $Q \times \mathcal{D}$ using $\vee$ and $\wedge$. For a set $E \subseteq Q \times \mathcal{D}$ and a formula $\psi \in \mathcal{B}^+(Q \times \mathcal{D})$, we say that E satisfies ψ iff assigning true to elements in E and assigning false to elements in $(Q \times \mathcal{D}) \setminus E$ makes ψ true;
- Ψ is the acceptance condition which is a set of infinite sequences of states in Q that are accepting. Typically, we consider either parity functions, giving alternating parity tree automata (APT), or Muller conditions (that subsume Boolean combinations of parity conditions), giving alternating Muller tree automata (AMT), to define Ψ.

A *run* of $P = (Q, q_0, \Delta, \Psi)$ on a $\mathcal{L}$-labelled $\mathcal{D}$-tree $\langle T, L\rangle$ is $Q \times \mathcal{D}^*$-labelled tree $\langle T_r, r\rangle$ such that:

1. $\epsilon \in T_r$ and $r(\epsilon) = (q_0, \epsilon)$;
2. Let $y \in T_r$ with $r(y) = (q, x)$ and $\Delta(q, L(x)) = \psi$. Then there exists a (possibly empty) set $E = \{(q_0, d_0), (q_1, d_1), \ldots, (q_n, d_n)\} \subseteq Q \times D$ and:
   a. E satisfies ψ;
   b. for all i, $0 \leq i \leq n$, we have $y \cdot i \in T_r$ and $r(y \cdot i) = (q_i, x \cdot d_i)$.

A run is *accepting* iff all its infinite branches b^∞ are such that $b^\infty \in \Psi$, i.e. they satisfy the acceptance condition.

7.2 Player 2 perfectly informed

As we mentioned following Theorem 20, in regular games the validity of the condition in the theorem depends only on the set of last vertices in the histories forming a monochromatic set. In slight abuse of notation, we call a set of vertices arising in this way a monochromatic set, too.

▶ **Lemma 29.** *Let $M = M_w \uplus M_c \uplus M_\ell$ be a monochromatic set. We can construct in* PTIME *an APT that recognizes the set of trees that encode observation-based strategies $\sigma : (\mathcal{O}^1)^* \cdot \mathcal{O}^1_{\mathsf{turn}} \to \{0, 1\}$ of Player 1 and such that $M_w = M^\sigma_w$, $M_c = M^\sigma_c$, and $M_\ell = M^\sigma_\ell$.*

Proof. The main idea of the construction is as follows. The APT has four parts:

1. one that checks that σ ensures a win from all vertices in M_w no matter what are the choices made by Player 2;
2. one that checks that σ can cooperate with Player 2 to win from all vertices in M_c;
3. one that checks that σ can cooperate with Player 2 to lose from all the vertices in M_c;
4. one that checks that all outcomes that are compatible with σ are losing from all the vertices in M_ℓ no matter what are the choices made by Player 2.

Accordingly, its state space is defined as $Q = \{q_0\} \cup V \times \{w, c_w, c_\ell, \ell\}$. The transitions are defined as follows:

- for states in $V \times \{w\}$. If a node n of the tree is reached by a run of the APT that ends up in state (v, w) then we must verify that the strategy encoded in the tree ensures to win no matter what are the choices of Player 2. So, the automaton follows the choices prescribed by the strategy for nodes annotated by $i \in \{0, 1\}$ and branches universally on the choices of Player 2 for nodes annotated by $*$. Accordingly:
  - for $i \in \{0, 1\}$: $\Delta((v, w), i) = ((s(v, i), w), \mathcal{O}^1(s(v, i)))$
  - for $*$: $\Delta((v, w), *) = ((s(v, 0), w), \mathcal{O}^1(s(v, 0))) \wedge ((s(v, 1), w), \mathcal{O}^1(s(v, 1)))$
- for states in $V \times \{c_w\}$. If a node n of the tree is reached by a run of the APT that ends up in state (v, c_w) then the automaton must verify that the strategy σ encoded in the tree ensures that at least one outcome is winning. So, the automaton follows the choices of Player 1 as encoded in the tree and for nodes labelled with $*$ that denote choices of Player 2, the automaton *nondeterministically* chooses one choice for Player 2. Accordingly:
  - for $i \in \{0, 1\}$: $\Delta((v, c_w), i) = ((s(v, i), c_w), \mathcal{O}^1(s(v, i)))$
  - for $*$: $\Delta((v, c_w), *) = ((s(v, 0), c_w), \mathcal{O}^1(s(v, 0))) \vee ((s(v, 1), c_w), \mathcal{O}^1(s(v, 1)))$
- for states in $V \times \{c_\ell\}$. The approach is similar. If a node n of the tree is reached by a run of the APT that ends up in state (v, c_ℓ) then the automaton must verify that the strategy σ encoded in the tree ensures that at least one outcome compatible with σ is losing. So, the automaton follows the choices of Player 1 as encoded in the tree and for nodes labelled with $*$ that denote choices of Player 2, the automaton *nondeterministically*

chooses one choice for Player 2. The difference with the previous case (c_w) is handled by the acceptance condition. Accordingly:
  - for $i \in \{0,1\}$: $\Delta((v,c_\ell),i) = ((s(v,i),c_\ell),\mathcal{O}^1(s(v,i)))$
  - for $*$: $\Delta((v,c_\ell),*) = ((s(v,0),c_\ell),\mathcal{O}^1(s(v,0))) \vee ((s(v,1),c_\ell),\mathcal{O}^1(s(v,1)))$
- for states in $V \times \{\ell\}$. If a node n of the tree is reached by a run of the APT that ends up in state (v,ℓ) then we must verify that the strategy σ encoded in the tree cannot win no matter what Player 2 chooses. So, the automaton follows the choices prescribed by the strategy σ for nodes annotated by $i \in \{0,1\}$ and branches universally on the choices of Player 2 for nodes annotated by $*$. The difference with the first case (w) is handled by the acceptance condition. Accordingly:
  - for $i \in \{0,1\}$: $\Delta((v,\ell),i) = ((s(v,i),\ell),\mathcal{O}^1(s(v,i)))$
  - for $*$: $\Delta((v,\ell),*) = ((s(v,0),\ell),\mathcal{O}^1(s(v,0))) \wedge ((s(v,1),\ell),\mathcal{O}^1(s(v,1)))$

It remains now to define the acceptance condition. First, we remark that a run of the APT directly jump from its initial state q_0 to one of the four parts whose transitions have been described above. When entering one of those parts, the run will stay in that part of the state space for ever. The runs that have have entered the part associated to w and to c_w must simulate paths in the game graph that are in W, those that are in the part associated to c_ℓ and to ℓ must simulate paths that are outside W. W is defined by a parity condition pr. We thus define the party condition pr' of our automaton as follows: for all $v \in V$, $pr'(v,w) = pr'(v,c_w) = pr(v)$ (the parity condition is preserved) and $pr'(v,c_\ell) = pr'(v,\ell) = pr(v)+1$ (the parity condition is inverted). ◀

As the emptiness problem for APT is solvable in ExpTime, we deduce the following corollary:

▶ **Corollary 30.** *Given a partition $M = M_w \uplus M_c \uplus M_\ell$ of a monochromatic set, we can decide in* ExpTime *if there exists an observation-based strategy $\sigma : (\mathcal{O}^1)^* \cdot \mathcal{O}^1_{\mathsf{turn}} \to \{0,1\}$ that induces this partition.*

▶ **Lemma 31.** *We can construct in* ExpTime *an APT P that accepts the $\{0,1,*\}$-labelled $\mathcal{O}^1$-trees that are the tree encodings of the observation-based strategies $\sigma \in \Sigma_1^{\cong_1}$ that are dominated in $R^U(v_0)$ where Player 2 is perfectly informed.*

Proof. Let $R(v_0)$ be a regular game. First, we note that Lemma 29 allows us to compute, for all monochromatic sets K, all the partitions $K_w \uplus K_c \uplus K_\ell$ that are witnessed by observation-based strategies of Player 1. Among those, we can extract all the partitions $K_w \uplus K_c \uplus K_\ell$ that correspond to strategies that are dominated following Theorem 20. There are at most an exponential number of them in the size of the regular game. We note $\mathcal{B}$ all those partitions.

Our APT will first guess a finite observation history ρ and compute the knowledge associated to ρ while following the strategy σ encoded in the tree. Let K be the resulting monochromatic set of vertices. This set is stored in the state of the APT and then the APT nondeterministically chooses a partition $K_w \uplus K_c \uplus K_\ell$ of K in $\mathcal{B}$. Then the APT verifies that the strategy in the tree induces that partition. This is done as in the proof of Lemma 29. ◀

As APT are closed under all Boolean operations, see e.g. [19], the set of admissible observation-based strategies of Player 1 is effectively omega-regular:

▶ **Theorem 32.** *When Player 2 is perfectly informed, the set of admissible observation-based strategies of Player 1 in $R^U(v_0)$ is effectively ω-regular and the emptiness of this set is decidable in* 2ExpTime.

7.3 Player 2 imperfectly informed

The hierarchical case

We start by considering the case in which both players have imperfect information but Player 2 is more informed than Player 1. In this case, the informedness of players is hierarchical in the sense of [7]. For the hierarchical case, we can decide if the set of admissible observation-based strategies of Player 1 is empty or not. We obtain this result by a reduction to the model-checking problem of *Strategy Logic with Imperfect Information* [5], $\mathsf{SL}_{ii}$for short.

We only recall here informally the syntax and semantics of SL_{ii} formulas and refer the interested reader to [5] for formal definitions. We start with the case where the players are perfectly informed. Strategy Logic (SL) extends the linear temporal logic (LTL [3]) and treats strategies x as first-order objects that can be quantified: $\langle\!\langle x \rangle\!\rangle$ reads "there exists a strategy x" while $[\![x]\!]$ reads "for all strategies x". In SL, strategies can be bound to players: $(x, 1)$ reads "Player 1 uses strategy x". As an example, let ϕ be an LTL formula, and consider the following SL formula $\langle\!\langle x \rangle\!\rangle [\![y]\!](x, 1)(y, 2)\phi$. This formula expresses that there exists a strategy x for Player 1 such that for all strategies y of Player 2, when the two players play their respective strategies x and y then the outcome from the initial vertex of the game arena is a path that satisfies ϕ. So, this formula expresses the existence of a *winning strategy* for Player 1 in a two-player zero sum game with objective ϕ for Player 1 and $\neg\phi$ for Player 2. SL can express many interesting game properties such as the existence of Nash equilibria, the existence of dominating strategies, etc., see [14, 22] for more examples.

When strategy logic is interpreted over a game in which players have imperfect information, then the strategy quantifier explicitly limits the quantification to observation-based strategies: $\langle\!\langle x \rangle\!\rangle^{\mathcal{O}}$ reads "there exists an $\mathcal{O}$-observation-based strategy x". However, to obtain the decidability of the model-checking problem for SL_{ii}, quantifiers must respect constraints that ensure "hierarchical instances".

Intuitively, a formula of SL_{ii} is *hierarchical* if, as one goes down the syntactic tree of the formula, the observations annotating strategy quantifications can only become finer. So, for example, if in the syntactic tree, quantification $\langle\!\langle x \rangle\!\rangle^{\mathcal{O}_1}$ is followed by $\langle\!\langle y \rangle\!\rangle^{\mathcal{O}_2}$ then it must be the case that $\mathcal{O}_2$ is finer than $\mathcal{O}_1$ meaning that for all $v_1, v_2 \in V$, if $\mathcal{O}_2$ does not distinguish v_1 and v_2, i.e. $\mathcal{O}_2(v_1) = \mathcal{O}_2(v_2)$, then it is also the case for $\mathcal{O}_1$, i.e. $\mathcal{O}_1(v_1) = \mathcal{O}_1(v_2)$.

Let us now consider a regular game $R(v_0) = (V, E, V_1, V_2, \mathcal{O}^1, \mathcal{O}^2)$ such that $\mathcal{O}_2$ is finer than $\mathcal{O}_2$ that induces the infinite sequential game with imperfect information $R^U(v_0) = (d, W, \cong_1, \cong_2)$. Let ϕ_1 be a LTL specification for the objective of Player 1 then the following SL_{ii} is hierarchical and evaluates to true in v_0, if and only if, Player 1 has admissible $\mathcal{O}_1$-observation-based strategies when Player 2 plays $\mathcal{O}_2$-observation-based strategies:

$$\langle\!\langle x \rangle\!\rangle^{\mathcal{O}_1} [\![x']\!]^{\mathcal{O}_1} \begin{array}{l} \langle\!\langle y \rangle\!\rangle^{\mathcal{O}_2}(x, 1)(y, 2)\phi_1 \wedge (x', 1)(y, 2)\neg\phi_1 \\ \vee [\![y]\!]^{\mathcal{O}_2}(x', 1)(y, 2)\phi_1 \rightarrow (x, 1)(y, 2)\phi_1 \end{array}$$

▶ **Theorem 33.** *Let $R(v_0) = (V, E, V_1, V_2, \mathcal{O}^1, \mathcal{O}^2)$ be a regular game such that $\mathcal{O}_2$ is finer than $\mathcal{O}_1$ and that induces $R^U(v_0) = (d, W, \cong_1, \cong_2)$. The existence of admissible strategies for Player 1 in $R^U(v_0)$ is decidable.*

The general case

This case is more involved and so far we did not succeed to obtain a general characterization of dominated strategies in the form of Theorem 20. Nevertheless, we are able to characterize all the observation-based strategies of Player 1 that dominate a regular observation-based

strategy given as a finite automaton $\mathcal{A}$ (as in Definition 6). Our characterization is effective: we can construct from a regular game R and a finite automaton $\mathcal{A}$, an APT P that accepts the tree encodings of all the observation-based strategies σ that dominate the regular strategy $\sigma_{\mathcal{A}}$ defined $\mathcal{A}$. The construction of the APT is given in the proof of the following lemma:

▶ **Lemma 34.** *Let $R(v_0)$ be a regular game that induces $R^U(v_0) = (d, W, \cong_1, \cong_2)$, and let $\mathcal{A}$ be a finite automaton that encodes a regular observation-based strategy $\sigma_{\mathcal{A}} \in \Sigma_1^{\cong_1}$. We can construct in* PTIME *an AMT P that accepts all the tree encodings of observation-based strategies $\sigma \in \Sigma_1^{\cong_1}$ that dominates $\sigma_{\mathcal{A}}$ when Player 2 plays strategies in $\Sigma_2^{\cong_2}$.*

Proof. Remember that the strategy $\sigma_{\mathcal{A}}$ is dominated by a strategy σ when Player 2 plays observation-based strategies if and only if:

1. $\forall \pi \in \Sigma_2^{\cong_2} : \mathsf{Out}(\sigma_{\mathcal{A}}, \pi) \in W \rightarrow \mathsf{Out}(\sigma, \pi) \in W$
2. $\exists \pi \in \Sigma_2^{\cong_2} : \mathsf{Out}(\sigma_{\mathcal{A}}, \pi) \notin W \wedge \mathsf{Out}(\sigma, \pi) \in W$

Here the strategy $\sigma_{\mathcal{A}}$ is encoded by the automaton $\mathcal{A} = (Q^{\mathcal{A}}, q_0^{\mathcal{A}}, \mathcal{O}^1, \delta^{\mathcal{A}}, F^{\mathcal{A}})$, and the strategy σ is the strategy encoded in the tree. We use a AMT to check those two properties: its state space is structured in two disjunct parts plus an initial state q_0: $Q = \{q_0\} \cup Q^1 \cup Q^2$. States in Q^1 are used to check the first property, and in Q^2 the second property. We now give a detailed description of those sets of states: $Q^i = \{i\} \times V \times V \times \{0,1\} \times Q^{\mathcal{A}}$, $i \in \{1,2\}$. Intuitively, a state $(i, v_1, v_2, \mathsf{dist}, q)$ is reached when:

- if $i = 1$, the ATP is checking the first condition, and if $i = 2$, the AMT is checking the second condition;
- the interaction of the strategy π played by Player 2 against $\sigma_{\mathcal{A}}$ leads to vertex v_1;
- the interaction of the strategy π played by Player 2 against σ leads to vertex v_2;
- dist is true if and only if Player 2 has been able to distinguish, based on its observation $\mathcal{O}^2$, between the history generated by strategy σ against π and the history generated by the strategy $\sigma_{\mathcal{A}}$ against π, and false otherwise;
- q is the state reached by the automaton $\mathcal{A}$, that encodes $\sigma_{\mathcal{A}}$ on the current sequence of $\mathcal{O}^1$-observations.

In the part Q^1 of the AMT, the automaton branches universally on the choices of Player 2 in order to consider all possible strategies π that Player 2 can play and it verifies that, for all of them, if the outcome against $\sigma_{\mathcal{A}}$ is winning then the outcome against σ is also winning. In the part Q^2 of the AMT, the automaton branches existentially on the choices of Player 2 in order to guess a strategy of Player 2 that forces an outcome in the complement of W against $\sigma_{\mathcal{A}}$ and in W against σ. This is realized by the following transition relation and acceptance condition:

- **Transition function.** Let $(i, v_1, v_2, \mathsf{dist}, q^{\mathcal{A}})$, and we distinguish between the universal ($i = 1$) and existential ($i = 2$) cases:
  - in the universal part:
    - if the label of the current node in the tree is $*$ then it is the turn of Player 2 to play. Then either Player 2 has already distinguished the histories that ends up in v_1 and v_2 respectively, i.e. $\mathsf{dist} = 1$ and so he can possibly play differently in the two cases:

$$\Delta((1, v_1, v_2, 1, q^{\mathcal{A}}), *) = \bigwedge_{j_1, j_2 \in \{0,1\}} \left((1, s(v_1, j_1), s(v_2, j_2), 1, \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, j_1)))), \mathcal{O}^1(s(v_2, j_2))\right)$$

or Player 2 has not yet distinguished the histories that ends up in v_1 and v_2 respectively, i.e. $\mathsf{dist} = 0$, and so Player 2 makes the same choices in the two branches:

$$\begin{array}{l}\Delta((1, v_1, v_2, 0, q^{\mathcal{A}}), *)\\ =\\ \bigwedge_{j\in\{0,1\}} \left((1, s(v_1, j), s(v_2, j), \mathsf{dist}'(j), \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, j)))), \mathcal{O}^1(s(v_2, j))\right)\end{array}$$

where $\mathsf{dist}'(j) = (\mathcal{O}^2(s(v_1, j)) \neq \mathcal{O}^2(s(v_2, j)))$, i.e. the new value of dist is set to 1 only if Player 2 received two different observations for $s(v_1, j)$ and $s(v_2, j)$, in that case, he can now distinguish between the two branches.

- If the label of the current node in the tree is $c \in \{0, 1\}$ then it is the turn of Player 1 to play and c is the choice made by σ, while the choice d made by $\sigma_{\mathcal{A}}$ is equal to 0 if $q^{\mathcal{A}} \in F^{\mathcal{A}}$ otherwise it is equal to 1:

$$\begin{array}{l}\Delta((1, v_1, v_2, \mathsf{dist}, q^{\mathcal{A}}), c)\\ =\\ \left((1, s(v_1, d), s(v_2, c), \mathsf{dist}'(j), \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, d)))), \mathcal{O}^1(s(v_2, c))\right)\end{array}$$

where $\mathsf{dist}'(j) = \mathsf{dist} \vee (\mathcal{O}^2(s(v_1, d)) \neq \mathcal{O}^2(s(v_2, c)))$, i.e. the two histories are distinguishable if they were before or if Player 2 gets two different observations in this round.

- in the existential part:
  - If the label of the current node in the tree is $*$ then it is the turn of Player 2 to play. Then either Player 2 has already distinguished the histories that ends up in v_1 and v_2 respectively, i.e. $\mathsf{dist} = 1$:

$$\begin{array}{l}\Delta((1, v_1, v_2, 1, q^{\mathcal{A}}), *)\\ =\\ \bigvee_{j_1, j_2\in\{0,1\}} \left((1, s(v_1, j_1), s(v_2, j_2), 1, \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, j_1)))), \mathcal{O}^1(s(v_2, j_2))\right)\end{array}$$

    or Player 2 has not already distinguished the histories that ends up in v_1 and v_2 respectively, i.e. $\mathsf{dist} = 0$, so Player 2 makes the same choices in the two branches:

$$\begin{array}{l}\Delta((1, v_1, v_2, 0, q^{\mathcal{A}}), *)\\ =\\ \bigvee_{j\in\{0,1\}} \left((1, s(v_1, j), s(v_2, j), \mathsf{dist}'(j), \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, j)))), \mathcal{O}^1(s(v_2, j))\right)\end{array}$$

    where $\mathsf{dist}'(j) = (\mathcal{O}^2(s(v_1, j)) \neq \mathcal{O}^2(s(v_2, j)))$, i.e. the new value of dist becomes equal to 1 only if Player 2 received two different observations for $s(v_1, j)$ and $s(v_2, j)$, if so, he can now distinguish between the two branches. Note that those formulas are the same as the one for the universal part but with the conjunction replaced by a disjunction.
  - If the label of the current node in the tree is $d \in \{0, 1\}$ then it is the turn of Player 1 to play and d is the choice made by σ, while the choice c made by $\sigma_{\mathcal{A}}$ is equal to 0 if $q^{\mathcal{A}} \in F^{\mathcal{A}}$ otherwise it is equal to 1, and then we have that:

$$\begin{array}{l}\Delta((1, v_1, v_2, 1, q^{\mathcal{A}}), c)\\ =\\ \left((1, s(v_1, c), s(v_2, d), \mathsf{dist}'(j), \delta^{\mathcal{A}}(q^{\mathcal{A}}, \mathcal{O}^1(s(v_1, c)))), \mathcal{O}^1(s(v_2, d))\right)\end{array}$$

    where $\mathsf{dist}'(j) = \mathsf{dist} \vee (\mathcal{O}^2(s(v_1, c)) \neq \mathcal{O}^2(s(v_2, d)))$, i.e. the two histories are distinguishable if they were before or if Player 2 gets two different observations now.

- **Acceptance condition.** Again, we distinguish between the universal and existential parts:
  - Let r be a run of the AMT in the universal part and let $\inf(r)$ be the set of states that repeats infinitely often along r . Let P_r be the set of pairs of vertices (v_1, v_2) that appear in states of the automaton in $\inf(r)$, i.e. pairs of vertices that appear infinitely often along the run r. We declare the set of states $\inf(r)$ to be accepting when we verify that: if $\min_{(v_1,v_2)\in P_r} p(v_1)$ is even then $\min_{(v_1,v_2)\in P_r} p(v_2)$ is even, i.e. a run is good if the run simulates an execution in which Player 2 plays a strategy π, and if the outcome of σ_A and π is winning for the parity condition pr, then it is also the case for the outcome of σ and π. This is a valid Muller condition.
  - Let r be a run of the AMT in the existential part and let $\inf(r)$ be the set of states that repeats infinitely often along r . Let P_r be the set of pairs of vertices (v_1, v_2) that are in $\inf(r)$, i.e. the pairs of vertices that appear infinitely often along the run r. We declare the set of state $\inf(r)$ to be accepting when we verify that: $\min_{(v_1,v_2)\in P_r} p(v_1)$ is odd and $\min_{(v_1,v_2)\in P_r} p(v_2)$ is even, i.e. a run is good if the run simulates an execution in which Player 2 plays a strategy π, and the outcome of σ_A and π is losing for the parity condition pr, and the outcome of σ and π is winning for the parity condition pr. This is a valid Muller condition. ◀

As a consequence, we obtain:

▶ **Theorem 35.** *Let $R(v_0)$ be a regular game that induces $R^U(v_0) = (d, W, \cong_1, \cong_2)$, and let $\mathcal{A}$ be a finite automaton that encodes a regular observation-based strategy $\sigma_{\mathcal{A}} \in \Sigma_1^{\cong_1}$. The problem to decide if $\sigma_{\mathcal{A}}$ is dominated when Player 2 plays strategies in $\Sigma_2^{\cong_2}$ is* ExpTime-C.

Proof. We can solve the problem in ExpTime thanks to the AMT construction of polynomial size given in lemma 34 and the fact that emptiness of AMT is solvable in ExpTime. For hardness, it is easy to reduce the problem of deciding the winner in a zero-sum reachability game with imperfect information to our problem, and this problem is complete for ExpTime [13]. ◀

Acknowledgements. We would like to thank Marie Van den Bogaard for carefully reading and commenting on a previous version of this paper.

References

1 Martín Abadi, Leslie Lamport, and Pierre Wolper. Realizable and unrealizable specifications of reactive systems. In *ICALP'89*, volume 372 of *LNCS*. Springer, 1989.

2 Brandenburger Adam, Friedenberg Amanda, H Jerome, et al. Admissibility in games. *Econometrica*, 2008.

3 Christel Baier and Joost-Pieter Katoen. *Principles of model checking*. MIT Press, 2008.

4 Nicolas Basset, Gilles Geeraerts, Jean-François Raskin, and Ocan Sankur. Admissibility in concurrent games. *CoRR*, abs/1702.06439, 2017. URL: `http://arxiv.org/abs/1702.06439`.

5 Raphael Berthon, Bastien Maubert, Aniello Murano, Sasha Rubin, and Moshe Vardi. Strategy logic with imperfect information. In *LICS'14*. IEEE, 2014.

6 Dietmar Berwanger. Admissibility in infinite games. In *STACS 2007, 24th Annual Symposium on Theoretical Aspects of Computer Science, Aachen, Germany, February 22-24, 2007, Proceedings*, volume 4393 of *Lecture Notes in Computer Science*, pages 188–199. Springer, 2007.

7 Dietmar Berwanger, Anup Basil Mathew, and Marie van den Bogaard. Hierarchical information patterns and distributed strategy synthesis. In *Automated Technology for Verification and Analysis - 13th International Symposium, ATVA 2015, Shanghai, China, October 12-15, 2015, Proceedings*, volume 9364 of *Lecture Notes in Computer Science*, pages 378–393. Springer, 2015.

8 Felix Brandt, Markus Brill, Felix Fischer, and Paul Harrenstein. On the complexity of iterated weak dominance in constant-sum games. *Theory of Computing Systems*, 49(1):162–181, 2011. doi:10.1007/s00224-010-9282-7.

9 Romain Brenguier, Lorenzo Clemente, Paul Hunter, Guillermo A. Pérez, Mickael Randour, Jean-François Raskin, Ocan Sankur, and Mathieu Sassolas. Non-zero sum games for reactive synthesis. In *Language and Automata Theory and Applications - 10th International Conference, LATA 2016, Prague, Czech Republic, March 14-18, 2016, Proceedings*, volume 9618 of *Lecture Notes in Computer Science*, pages 3–23. Springer, 2016.

10 Romain Brenguier, Guillermo A. Pérez, Jean-François Raskin, and Ocan Sankur. Admissibility in quantitative graph games. In *36th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2016, December 13-15, 2016, Chennai, India*, volume 65 of *LIPIcs*, pages 42:1–42:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

11 Romain Brenguier, Jean-François Raskin, and Ocan Sankur. Assume-admissible synthesis. *Acta Inf.*, 54(1):41–83, 2017. doi:10.1007/s00236-016-0273-2.

12 Romain Brenguier, Jean-François Raskin, and Mathieu Sassolas. The complexity of admissibility in omega-regular games. In *CSL-LICS '14, 2014*. ACM, 2014. doi:10.1145/2603088.2603143.

13 Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, and Jean-François Raskin. Algorithms for omega-regular games with imperfect information. *Logical Methods in Computer Science*, 3(4), 2007.

14 Krishnendu Chatterjee, Thomas A. Henzinger, and Nir Piterman. Strategy logic. *Inf. Comput.*, 208(6):677–693, 2010.

15 Vincent Conitzer and Tuomas Sandholm. Complexity of (iterated) dominance. In *EC '05: Proceedings of the 6th ACM conference on Electronic commerce*, pages 88–97, New York, NY, USA, 2005. ACM. doi:10.1145/1064009.1064019.

16 Werner Damm and Bernd Finkbeiner. Automatic compositional synthesis of distributed systems. In *FM 2014*, volume 8442 of *LNCS*, pages 179–193. Springer, 2014.

17 Luca de Alfaro, Thomas A. Henzinger, and Orna Kupferman. Concurrent reachability games. In *39th Annual Symposium on Foundations of Computer Science, FOCS '98, November 8-11, 1998, Palo Alto, California, USA*, pages 564–575. IEEE Computer Society, 1998.

18 Marco Faella. Admissible strategies in infinite games over graphs. In *MFCS 2009*, volume 5734 of *Lecture Notes in Computer Science*, pages 307–318. Springer, 2009.

19 Daniel Kirsten. Alternating tree automata and parity games. In *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *Lecture Notes in Computer Science*, pages 153–167. Springer, 2001.

20 Donald Knuth, Christos Papadimitriou, and John Tsitsiklis. A note on strategy elimination in bimatrix games. *Operations Research Letters*, 7(3):103–107, 1988. doi:10.1016/0167-6377(88)90075-2.

21 Orna Kupferman and Moshe Y. Vardi. Synthesis with incomplete informatio. In Howard Barringer, Michael Fisher, Dov Gabbay, and Graham Gough, editors, *Advances in Temporal Logic*, pages 109–127, Dordrecht, 2000. Springer Netherlands. doi:10.1007/978-94-015-9586-5_6.

22 Fabio Mogavero, Aniello Murano, and Moshe Y. Vardi. Reasoning about strategies. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India*, volume 8 of *LIPIcs*, pages 133–144. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010.

23 Arno Pauly. The computational complexity of iterated elimination of dominated strategies. *Theory of Computing Systems*, pages 52–75, 2016. doi:10.1007/s00224-015-9637-1.

24 Amir Pnueli and Roni Rosner. On the synthesis of a reactive module. In *POPL*, pages 179–190, 1989.

25 John H. Reif. The complexity of two-player games of incomplete information. *J. Comput. Syst. Sci.*, 29(2):274–301, 1984. doi:10.1016/0022-0000(84)90034-5.

Probabilistic Programming

Hongseok Yang

University of Oxford, UK
hongseok00@gmail.com

Abstract

Probabilistic programming refers to the idea of using standard programming constructs for specifying probabilistic models from machine learning and statistics, and employing generic inference algorithms for answering various queries on these models, such as posterior inference and estimation of model evidence. Although this idea itself is not new and was, in fact, explored by several programming-language and statistics researchers in the early 2000, it is only in the last few years that probabilistic programming has gained a large amount of attention among researchers in machine learning and programming languages, and that expressive and efficient probabilistic programming systems (such as Anglican, Church, Figaro, Infer.net, PSI, PyMC, Stan, and Venture) started to appear and have been taken up by a nontrivial number of users.

The primary goal of my talk is to introduce probabilistic programming to the CONCUR/QUEST/FORMATS audience. At the end of my talk, I want the audience to understand basic results and techniques in probabilistic programming and to feel that these results and techniques are relevant or at least related to what she or he studies, although they typically come from foreign research areas, such as machine learning and statistics. My talk will contain both technical materials and lessons that I learnt from my machine-learning colleagues in Oxford, who are developing a highly-expressive higher-order probabilistic programming language, called Anglican. It will also include my work on the denotational semantics of higher-order probabilistic programming languages and their inference algorithms, which are jointly pursued with colleagues in Cambridge, Edinburgh, Oxford and Tubingen.

1998 ACM Subject Classification D.3.1 Formal Definitions and Theory, G.3 Probability and Statistics

Keywords and phrases Probabilistic programming, Machine learning, Denotational semantics

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.3

Category Invited Talk

28th International Conference on Concurrency Theory (CONCUR 2017).
Editors: Roland Meyer and Uwe Nestmann; Article No. 3; pp. 3:1–3:1

Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

A New Notion of Compositionality for Concurrent Program Proofs

Azadeh Farzan[1] and Zachary Kincaid[2]

1 **University of Toronto, Ontario, Canada**
azadeh@cs.toronto.edu
2 **Princeton University, NJ, USA**
zkincaid@cs.princeton.edu

Abstract

This paper presents a high level overview of Proof Spaces [11] as an instance of a new approach to compositional verification of concurrent programs and discusses potential future work extending the approach beyond its current scope of applicability.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Concurrency, Proofs, Dynamic Memory, Recursion

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.4

Category Invited Talk

1 Verification of Parameterized Concurrent Programs

Compositional proofs have always been the holy grail of reasoning about concurrent programs. Two seminal contributions, namely Owicki-Gries style [22] and *rely-guarantee* [16], have set the tone for compositional proofs of concurrent programs and inspired decades of research in this area. In both techniques (and the wealth of follow-up work), compositionality strictly means composing correctness arguments over individual threads into a correctness argument for the entire program, with some extra supporting arguments that are preferably kept at a minimum required. In a recent line of work [11, 10, 9, 12], we proposed a new approach to static analysis and verification of concurrent programs which include an unbounded number of concurrent threads with local and global memory, with a different notion of compositionality. This talk will provide an overview of this approach, which takes a different view of compositionality. In this paper, we hope to look forward and through examples discuss early evidence of why we believe our approach to be a promising framework for future research into extensions of the program models (of the already published work [11, 10, 9, 12]) into more expressive models including features like dynamic memory and recursion.

We will do an informal exposition of our approach from [11] using a simple example. The interested reader can refer to [11, 10, 9, 12] for the technical details. The principle behind our methodology is simple. It is difficult to reason about the correctness of a complex concurrent program, however, it is much simpler to reason about the correctness of a single behaviour (i.e. a single run) of the same program. Consider, for example, the complexity of reasoning about unbounded concurrency. Any (terminating) run of such a program always includes a bounded (by the length of the run) number of participating threads. Therefore, the complexity of dealing with unboundedly many threads can be circumvented whilst reasoning about the run. The idea is then to *mine* these simple proofs of program runs for ingredients to construct a correctness proof for the program.

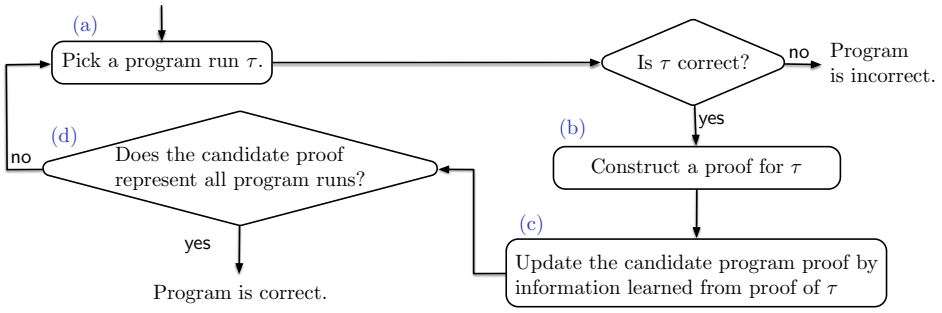


Figure 1 Proof Construction Methodology.

This effectively introduces a different way of decomposing the correctness argument for a concurrent program. The new proof is a non-trivial composition of proofs of correctness constructed for *finitely* many runs of the program. The correctness proof for each run represents a class of behaviours for the program with a common correctness argument. Instead of being forced to decompose the proof argument using the program syntax as a strict guide, one can observe how the program behaves in different *scenarios* and reason that in each case it exhibits the correct behaviour. Our constructed proofs may end up with structures very similar to or wildly different from the original program. This is extra flexibility is useful specially when a compositional proof in the classic sense is either nonexistent or hard to automatically construct.

The diagram in Figure 1 illustrates our methodology. The main goal of this methodology is to introduce a clean separation between

- logical reasoning about data, that is over domains of the program variables, and
- purely combinatorial reasoning about sufficiency of a candidate proof for a given program.

In many settings, distinctly so for concurrent programs, an argument for correctness often involves both types of reasonings. The thesis of this methodology is that this separation could make each argument type simpler, and moreover, each side of the argument becomes amenable to existing technology. In particular, on the logical reasoning side we can make use of the wealth of techniques that have been developed for loop invariant generation, such as abstract interpretation [6, 7], Craig interpolation [15, 18, 2], and constraint-based techniques [5, 14]. On the combinatorial side, we can make use of technology behind finite-state model checking, such as partial order reduction [32, 23, 13] and symbolic state-space exploration [4, 17].

The key steps in this methodology are:

- **Proof generation for a run.** Depending on the richness of the program model, this task can be straightforward with an array of existing solutions, or more tricky where new algorithms need to be proposed for it. In particular:
  - In [9], program runs are simple (straight line sequential) programs with integer typed variables, where constructing a proof for a run is a well-understood task. Since concurrent programs are assumed to have a fixed number of threads, the challenge is the well-known state explosion problem, and therefore the goal is compactness of proofs. The innovation of proof construction step in this case is focused on constructing *parallel* in proofs in form of *inductive data flow graphs*, that would generalize the proof of correctness of a run to many other *similar* runs.
  - In [10], where the program model was extended to include unboundedly many threads, a special type of proof, namely a counting proof, is constructed for a run. A special

```
global t:  int
global s:  int
local m:  int
initially s <= t

do forever {
  m = t++;
  // busy wait until your turn
  while (m > s);
  // enter critical section
  s++;
}
```

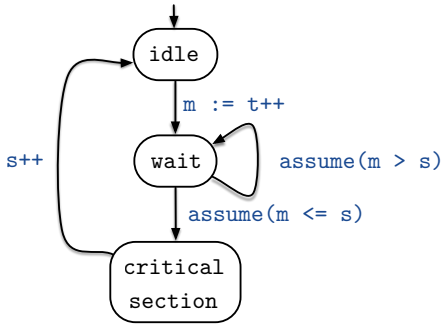


Figure 2 Ticket protocol code and control flow automaton.

technique for proof generation is proposed that automatically constructs a counting argument (i.e. a proof making use of counters) for a run.
- In [11], program runs are syntactically identical to those in [9], however, these program runs are (terminating) runs for a program with unboundedly many threads. A proof generated for a run in this context needs to be generalizable to the proof for a set of runs for such a program, and therefore, a different notion of proof, namely a *proof space* was introduced for these programs, a special proof generalization *symmetry* rule to tackle unbounded concurrency (more on this later).
- In [12], the program model is identical to the previous item, but the runs can potentially be non-terminating, since proving termination (and liveness in general) is the goals. Proofs for such runs, therefore, need to include termination arguments, which were ranking functions in this case.

- **Proof checking.** This is an algorithmic step that needs to (i) check if the candidate proof is indeed a proof of the given property for the given program, and (ii) provide a counter example if it is not, to ensure progress in the overall proof construction scheme (i.e. the choice of the run τ in (a) in Figure 1). This requires effective representations of both the program and the candidate proof so that sufficiency of the proof can be algorithmically checked. We used alternating finite automata in [9], Petri nets in [10], and a new notion of data automata in [11, 12] to represent various different classes of concurrent programs (and their proofs) for this purpose.

Here, we will focus on one instance of this methodology [11] from our past work, and discuss the ideas behind algorithmic solutions to the steps above over a running example.

Consider the program in Figure 2 where the illustrated code is executed by unboundedly many threads.This program implements the *ticket* mutual exclusion protocol. The safety property of interest is that no two threads are ever simultaneously in their critical sections. This program has two global integer typed variables `s` and `t`, while it has unboundedly many integer typed local variables `m`, one per each thread. The challenge of proving this property for this program, beyond the standard challenges of dealing with shared memory and infinite-state programs, is that the local variables of unboundedly many threads effectively require an unbounded sized memory to keep track of their values in the proof.

Figure 2 also illustrates the control flow automaton for each thread in this program. Let us start by looking at a *run* of this program that would lead to two threads being in a

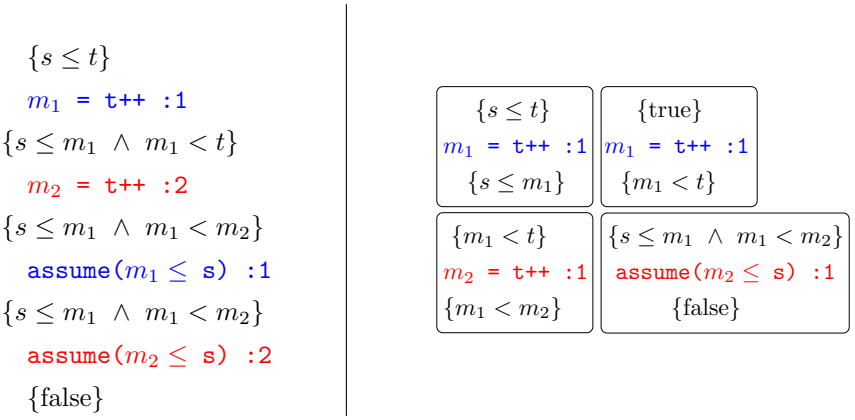


Figure 3 Correctness proof of a run (left) and the Hoare triples extracted from it (right).

critical section at the same time. The run is a string accepted by the parallel composition of unboundedly many of versions of the control flow automaton illustrated in Figure 2. Below, is a run of the program with only two participating threads (blue and red), where each instruction includes the id of the thread executing it in form of $:i$ ($i \in \{1, 2\}$), and the two local variables for the two threads are distinguished by giving them two different subscripts for clarity:

$$m_1 \texttt{ = t++ :1; } m_2 \texttt{ = t++ :2; assume(} m_1 \leq \texttt{ s) :1; assume(} m_2 \leq \texttt{ s) :2;}$$

Note that the run does not correspond to a real execution of the program; in other words, it is *infeasible*. It is easy to argue that such a run is *infeasible* and therefore a spurious counter example to violation of mutual exclusion. One possible proof for this is illustrated in Figure 3. What can we learn from this proof that would help with the construction of a proof of correctness for the entire program?

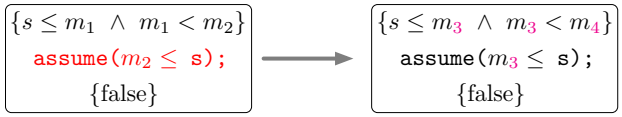
A natural and minimal unit of reasoning that can be extracted from a proof of the correctness of a run is *a set of Hoare triples*. For example, the Hoare triples demonstrate in Figure 3 can be learned from the proof of the run in the same figure. Consider a deductive system in which these triples can be considered as axioms. What should be the inference rules of this deductive system that would produce new judgements based on a *finite* set of base axioms? The answer lies in the definition of a *proof space* [11].

A proof space $\mathcal{H}$ is a set of valid Hoare triples which is:

- Closed under symmetry: Let $\pi : \mathbb{N} \to \mathbb{N}$ be any index permutation.

$$\{\phi\}\ \langle\sigma_1 : \mathtt{i}_1\rangle \cdots \langle\sigma_n : \mathtt{i}_n\rangle\ \{\psi\} \in \mathcal{H} \implies \{\phi[\pi]\}\ \langle\sigma_1 : \pi(\mathtt{i}_1)\rangle \cdots \langle\sigma_n : \pi(\mathtt{i}_n)\rangle\ \{\psi[\pi]\} \in \mathcal{H}$$

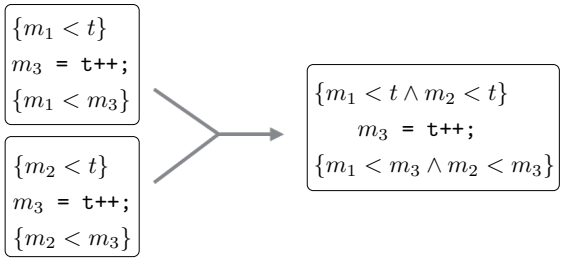
  For example



- Closed under conjunction:

$$\text{If } \{\phi\}\ \tau\ \{\psi\} \in \mathcal{H} \ \wedge\ \{\phi'\}\ \tau\ \{\psi'\} \in \mathcal{H} \implies \{\phi \wedge \phi'\}\ \tau\ \{\psi \wedge \psi'\} \in \mathcal{H}$$

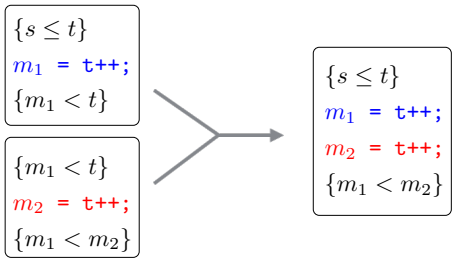
For example



- Closed under sequencing:

$$\text{If } \{\phi_0\}\ \tau_0\ \{\phi_1\} \in \mathcal{H}\ \wedge\ \{\phi_1'\}\ \tau_1\ \{\phi_2\} \in \mathcal{H}\ \wedge\ \phi_1 \Vdash \phi_1'\ \Longrightarrow\ \{\phi_0\}\ \tau_0\tau_1\ \{\phi_2\} \in \mathcal{H}$$

For example



Note how proofs of runs of arbitrary length can be constructed using the sequencing rule.

One can effectively consider symmetry, conjunction, and sequencing as the inference rules of the aforementioned deductive system, and a proof space as a theory of this system. Note that the symmetry rule is the one that facilitates reasoning about unboundedly many threads. Without it, sequencing and conjunction can be used to produce proofs of concurrent programs with a fixed number of threads [9].

The next natural question to ask is whether the closure (under symmetry, conjunction, and sequencing) of the set of Hoare triples in Figure 3 includes the proof of every run of the program in Figure 2. The answer is no. The complete proof corresponds to the *basis* illustrated in Figure 4 which includes one additional Hoare triple. This basis is a finite set of valid Hoare triples that generates the complete proof space for the ticket program through its closure under sequencing, symmetry, and conjunction rules. This additional Hoare triple can be mined out of a second sample run. Therefore, the proof of the program in Figure 2 is decomposed into two correctness arguments of two sample runs (one illustrated in Figure 3 and discussed above, and the other skipped for brevity).

It is important to note that the proof space does not make use of any of the exotic features that are common to logics for reasoning about concurrency, most notably auxiliary variables and universal quantification over threads.

How is the proof in Figure 4 checked? To reduce the problem of checking whether a candidate proof space serves as a correctness proof for a program, we developed the notion of predicate automata (PA), an infinite-state, infinite-alphabet generalization of alternating finite automata. Predicate automata are equipped with a finite vocabulary of predicates parameterized over natural numbers, and their states are propositions over this vocabulary. The transition function of a PA maps each predicate symbol and letter to a positive Boolean formula over its vocabulary. For example, the transition

$$\delta(p(i,j), a : k) = (p(i,j) \wedge i \neq k) \vee (q(i) \wedge q(j) \wedge i = k)$$

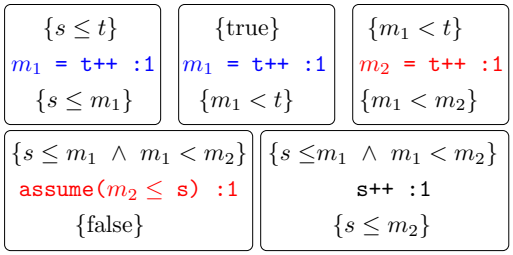


Figure 4 Complete basis for the proof of ticket protocol in Figure 2.

indicates that, if the PA is at state $p(1,2)$ and reads $a:2$, then it transitions to $p(1,2)$; if it then reads $a:1$, then it transitions to both the state $q(1)$ and $q(2)$. A finite basis B of Hoare triples gives rise to a predicate automaton which recognizes the same set of runs as the proof space generated by B.

The *proof checking* problem for proof spaces reduces to the inclusion problem for predicate automata, which in turn reduces to the emptiness problem of predicate automata. Although this problem is undecidable in general, we proposed an algorithm which is a decision procedure for the special case of PAs where each predicate symbol in its vocabulary has arity at most one.

2 Dynamic memory

Separation logic is an extension of Hoare logic for reasoning about memory [21, 26]. It is based on the addition of a new logical connective, the *separating conjunction* $P * Q$, which asserts that the heap can be split into two disjoint parts such that P holds in one and Q in the other. Separating conjunction allows for *local reasoning* in the sense that a specification of a program fragment need only involve the portion of the heap that is relevant to that fragment – the rest (the *frame*) is automatically proved to remain untouched via the Frame rule, stated below:

$$\textsc{Frame}\quad \frac{\{P\}\ S\ \{Q\}}{\{P * F\}\ S\ \{Q * F\}}$$

The capacity of separation logic for reasoning about disjointness of memory makes it appealing for reasoning about concurrency. This power can be illustrated by the appealing inference rule for parallel composition [19]:

$$\textsc{Parallel Composition}\quad \frac{\{P_1\}\ S_1\ \{Q_1\} \qquad \{P_2\}\ S_2\ \{Q_2\}}{\{P_1 * P_2\}\ S_1 \parallel S_2\ \{Q_1 * Q_2\}}$$

This rule gives concurrent separation logic (CSL) an intuitive way to prove properties of concurrent programs where threads act on disjoint memory. Generally, it is too much to expect that threads do not share memory, but a kind of disjointness can be achieved through the notion of *ownership*. The essential idea can be summarized as follows [20]:

- *Ownership Hypothesis.* A code fragment can access only those portions of state that it owns.
- *Separation Property.* At any time, the state can be partitioned into that owned by each process and each grouping of mutual exclusion.

```
type node = { data: int; next: node }
global top :: node
push(item):
  mtop := new node
  mtop.data := item
  do:
    mtop.next := *top
  while(!cas(top, mtop.next, mtop))
```

```
pop():
  do:
    mtop := *top
    if(mtop = null): break
    mnext := mtop.next
  while(!cas(top, mtop, mnext))
```

Figure 5 A nonblocking stack.

The earliest variation of CSL managed ownership using resource invariants, following Owicki and Gries [22]. Subsequently, there has been a great deal of work on increasing the power of CSL, a recent survey of which appears in [3]. Each advance increases the scope of what is possible to prove. However, increasing the expressivity of these logics takes us farther away from the class of proofs that we know how to completely automate. We thus raise a challenge problem: *can we automate proofs for concurrent heap-manipulating programs using the ideas of proof spaces and separation logic?*

A natural starting point in adapting proof spaces to separation logic is to replace the Conjunction rule with the Frame rule. Call a set of valid separation logic triples closed under Sequencing, Symmetry, and Frame a *separation proof space*.

The power of separation proof spaces can be illustrated with the *push* routine of Treiber's non-blocking stack [29], illustrated in Figure 5. The difficulty in verifying memory safety of *push* is that no process owns the global `top` variable. There are various methods that can be used to verify memory safety and even full functional correctness of the lock-free stack [31, 30, 27, 8]. However, such proofs are difficult to automate. Figure 6 pictures a basis for a proof space that proves the memory safety of `push`. We observe the following features:

1. Separating conjunction and the Frame rule allow the proof to scale to an arbitrary number of threads by maintaining disjointness between each thread's local `mtop`.
2. Separation logic triples are ordinary – they do not make use of exotic features that require ingenuity (except perhaps the separating implication $-\!\!*$, the adjoint of $*$).
3. The stack is not owned by any process – the fact that `top` points to the top of the stack is preserved by every statement of the program, either explicitly or by application of the Frame rule.

There are three challenges in making the combination of proof spaces and separation logic practical.

1. **Inference rules.** While it may be possible to get some mileage out of the Frame rule combined with Sequencing and Symmetry, we believe that additional inference rules will be required in order to verify systems of practical interest. For example, existentially quantified variables commonly used in separation logic, and some means for reasoning about them is likely required (in the Treiber stack, for instance, we resorted to using separating implication to avoid proliferation of existential variables).
2. **Proof generation.** While there has been a great deal of work on generating correctness proofs for single runs in Hoare logic, there is relatively little in separation logic [1]. The additional challenge imposed by proof spaces is that we need methods to synthesize proofs that decompose into small, re-usable components.
3. **Proof checking.** The problem of checking that a proof space can prove every run of a program correct is reduced to the emptiness problem of predicate automata. This

$$\{\mathsf{emp}\} \quad \langle \texttt{mtop := new node} : 1\rangle \quad \{\texttt{mtop}_1 \mapsto [\_,\_]\}$$

$$\{\texttt{mtop}_1 \mapsto [\_,\_]\} \quad \langle \texttt{mtop.data := item} : 1\rangle \quad \{\texttt{mtop}_1 \mapsto [\_,\_]\}$$

$$\{\texttt{mtop}_1 \mapsto [\_,\_] * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$
$$\langle \texttt{mtop.next := top} : 1\rangle$$
$$\{\big((\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n)) \mathrel{-\!\!*} \mathsf{list}(\texttt{mtop}_1)\big) * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$

$$\{\texttt{mtop}_1 \mapsto [\_,\_] * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$
$$\langle \texttt{mtop.next := top} : 1\rangle$$
$$\{\texttt{mtop}_1 \mapsto [\_,\_] * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$

$$\{\big((\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n)) \mathrel{-\!\!*} \mathsf{list}(\texttt{mtop}_1)\big) * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$
$$\langle \texttt{assume(cas(\&top, mtop.next, mtop))} : 1\rangle$$
$$\{\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n)\}$$

$$\{\texttt{mtop}_1 \mapsto [\_,\_] * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$
$$\langle \texttt{assume(!cas(\&top, mtop.next, mtop))} : 1\rangle$$
$$\{\texttt{mtop}_1 \mapsto [\_,\_] * (\exists n.\texttt{top} \mapsto [n] * \mathsf{list}(n))\}$$

Figure 6 A basis for a separation proof space for Figure 5.

reduction exploits the duality between disjunction and conjunction to complement the language of a predicate automaton recognizing all runs that are proved correct by a proof space. This duality does not exist between disjunction and separating conjunction. Some new mechanism (e.g., a new class of automaton) is likely needed to encode the proof checking problem for separation proof spaces.

3 Recursion

The difficulty of analyzing concurrent programs with recursive procedures is well-known: reachability is undecidable even for simple program models [25]. From the perspective of constructing a proof by considering program runs, the set of runs of a (sequential) program without procedures is regular, and with procedures, it is context-free. However for concurrent programs, adding procedures jumps over the context-freedom, requiring a *multi-stack* automaton to recognize the language of runs.

There are (at least) two possibilities for solving this problem: (1) use an automaton model that is capable of recognizing the languages of multi-threaded recursive program runs; and (2) use summaries to treat complete procedure calls as atomic actions, restoring regularity to the program model. We will discuss both possibilities in turn.

The advantage of using an automaton model that can recognize the interleaved runs of recursive procedures is that the proof system used for proving the correctness of those runs may remain unchanged. In our framework the structure of the proof is divorced from the structure of the program. For example, a proof space may very well cover all the runs of a multi-threaded program with recursive procedures – the added difficulty is only in the proof *checking* problem, where we must *prove* that it does. This problem requires us to devise a class of automata capable of recognizing the language of runs in the program *and* the language of runs proved correct in a proof.

```
int g = 0;
void foo(int r) {
  if (r == 0) {
    foo(r);
  else {
  g++;
  }
  return;
}
```

```
void main() {
  int c = *;
  foo(c);
  assert( g >= 1 );
  return;
}
```

Figure 7 An example recursive concurrent program with unbounded recursion from [24].

In fact, predicate automata are already a suitable model for this task. Procedure calls can be simulated by fork/join parallelism: whenever a thread would make a recursive call, it instead forks a new thread to make the call on its behalf and then immediately joins to retrieve the result. Fork/join parallelism can be modelled with predicate automata using a binary predicate $child(i, j)$ that retains the information that thread j was forked by thread i. However, admitting a binary predicate into the vocabulary of a predicate automaton makes the emptiness problem undecidable. Thus, the research problem is: *can we develop adequate semi-algorithms for emptiness checking for general predicate automata*?

The classical tool for context-sensitive analysis of recursive procedures in sequential programs is to compute *summaries* that abstract the behaviour of each procedure call [28]. Shared memory concurrency breaks the abstraction barrier: the assumption that procedure calls execute atomically will cause us to miss potential bugs. In [24], for the case where concurrent programs have a *fixed number of threads*, a summarization-based technique is proposed where atomicity is argued through Lipton's theory of reduction.

Although naive summarization (that is summarization that is unaware of the environment of concurrently executing threads) is unsound in general, in many cases a software developer's mental model likely assumes *some* atomic specification for each procedure. Consider the example in Figure 7. This example was taken from [24] where the program consisted of parallel composition of two threads each executing the `main` procedure, however the same assertion holds true for unboundedly many threads running `main` in parallel. The main procedure has a local variable `c` which is initialized nondeterministically. Procedure `main` then calls `foo` with `c` as the actual parameter. Procedure `foo` falls into infinite recursion if the parameter r is 0. Otherwise, it increments global variable `g` and returns. After returning from `foo`, the main procedure asserts that `(g >= 1)`. The specification that `foo` increases `g` by 1 or does not return at all is too strong: two threads may concurrently execute `foo` and increment `g` by 2. However, the specification $\{\mathtt{g} \geq 0\}\ \mathtt{foo}(c)\ \{\mathtt{g} \geq 1\}$ is unproblematic (i.e. holds under environment interference) and strong enough to prove that the assertion in `main` holds. Note that this is independent of an argument about the atomicity of the procedure, that is in the style of [24]. This is about atomicity of the specification in a concurrent environment independent of atomicity of the code. The relevant research question is then: *how can our framework discover these atomic specifications for procedures and prove that they hold?*.

References

1 Aws Albarghouthi, Josh Berdine, Byron Cook, and Zachary Kincaid. Spatial interpolants. In *Programming Languages and Systems - 24th European Symposium on Programming, ESOP 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015. Proceedings*, pages 634–660, 2015.

2 Aws Albarghouthi and Kenneth L. McMillan. Beautiful interpolants. In *CAV*, pages 313–329, 2013.

3 Stephen Brookes and Peter W. O'Hearn. Concurrent separation logic. *SIGLOG News*, 3(3):47–65, 2016.

4 Jerry R. Burch, Edmund M. Clarke, Kenneth L. McMillan, David L. Dill, and L. J. Hwang. Symbolic model checking: 10ˆ20 states and beyond. In *Proceedings of the Fifth Annual Symposium on Logic in Computer Science (LICS '90), Philadelphia, Pennsylvania, USA, June 4-7, 1990*, pages 428–439, 1990.

5 Michael Colón, Sriram Sankaranarayanan, and Henny Sipma. Linear invariant generation using non-linear constraint solving. In *CAV*, pages 420–432, 2003.

6 Patrick Cousot and Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *POPL*, pages 238–252, 1977.

7 Patrick Cousot and Nicolas Halbwachs. Automatic discovery of linear restraints among variables of a program. In *POPL*, pages 84–96, 1978.

8 Thomas Dinsdale-Young, Pedro da Rocha Pinto, Kristoffer Just Andersen, and Lars Birkedal. Caper - automatic verification for fine-grained concurrency. In *Programming Languages and Systems - 26th European Symposium on Programming, ESOP 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 420–447, 2017.

9 Azadeh Farzan, Zachary Kincaid, and Andreas Podelski. Inductive data flow graphs. In *The 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '13, Rome, Italy - January 23 - 25, 2013*, pages 129–142, 2013.

10 Azadeh Farzan, Zachary Kincaid, and Andreas Podelski. Proofs that count. In *The 41st Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '14, San Diego, CA, USA, January 20-21, 2014*, pages 151–164, 2014.

11 Azadeh Farzan, Zachary Kincaid, and Andreas Podelski. Proof spaces for unbounded parallelism. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2015, Mumbai, India, January 15-17, 2015*, pages 407–420, 2015.

12 Azadeh Farzan, Zachary Kincaid, and Andreas Podelski. Proving liveness of parameterized programs. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5-8, 2016*, pages 185–196, 2016.

13 Patrice Godefroid. *Partial-Order Methods for the Verification of Concurrent Systems – An Approach to the State-Explosion Problem*. PhD thesis, University of Liege, 1994.

14 Ashutosh Gupta, Rupak Majumdar, and Andrey Rybalchenko. From tests to proofs. In *TACAS*, pages 262–276, 2009.

15 Thomas A. Henzinger, Ranjit Jhala, Rupak Majumdar, and Kenneth L. McMillan. Abstractions from proofs. In *POPL*, pages 232–244, 2004.

16 Cliff B. Jones. Specification and design of (parallel) programs. In *IFIP Congress*, pages 321–332, 1983.

17 Kenneth L. McMillan. *Symbolic model checking*. Kluwer, 1993.

18 Kenneth. L. McMillan. Lazy abstraction with interpolants. In *CAV*, pages 123–136, 2006.

19 Peter W. O'Hearn. Resources, concurrency and local reasoning. In *CONCUR 2004 - Concurrency Theory, 15th International Conference, London, UK, August 31 - September 3, 2004, Proceedings*, pages 49–67, 2004.

20 Peter W. O'Hearn. Resources, concurrency, and local reasoning. *Theor. Comput. Sci.*, 375(1-3):271–307, 2007.

21 Peter W. O'Hearn, John C. Reynolds, and Hongseok Yang. Local reasoning about programs that alter data structures. In *Computer Science Logic, 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL, Paris, France, September 10-13, 2001, Proceedings*, pages 1–19, 2001.

22 Susan Owicki and David Gries. Verifying properties of parallel programs: an axiomatic approach. *CACM*, 19:279–285, May 1976.

23 Doron Peled. All from one, one for all: On model checking using representatives. In *CAV*, pages 409–423, 1993.

24 Shaz Qadeer, Sriram K. Rajamani, and Jakob Rehof. Summarizing procedures in concurrent programs. In *Proceedings of the 31st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2004, Venice, Italy, January 14-16, 2004*, pages 245–255, 2004.

25 G. Ramalingam. Context-sensitive synchronization-sensitive analysis is undecidable. *ACM Trans. Program. Lang. Syst.*, 22(2):416–430, March 2000.

26 John C. Reynolds. Separation logic: A logic for shared mutable data structures. In *17th IEEE Symposium on Logic in Computer Science (LICS 2002), 22-25 July 2002, Copenhagen, Denmark, Proceedings*, pages 55–74, 2002.

27 Ilya Sergey, Aleksandar Nanevski, and Anindya Banerjee. Mechanized verification of fine-grained concurrent programs. In *Proceedings of the 36th ACM SIGPLAN Conference on Programming Language Design and Implementation, Portland, OR, USA, June 15-17, 2015*, pages 77–87, 2015.

28 Micha Sharir and Amir Pnueli. *Two Approaches to Interprocedural Data Flow Analysis*, chapter 7. Prentice-Hall, Inc., 1981.

29 R. Treiber. Systems programming: coping with parallelism. Technical report, Almaden Research Center, 1986.

30 Aaron Turon, Derek Dreyer, and Lars Birkedal. Unifying refinement and Hoare-style reasoning in a logic for higher-order concurrency. In *ACM SIGPLAN International Conference on Functional Programming, ICFP'13, Boston, MA, USA - September 25 - 27, 2013*, pages 377–390, 2013.

31 Viktor Vafeiadis. Rgsep action inference. In *Proceedings of the 11th International Conference on Verification, Model Checking, and Abstract Interpretation*, pages 345–361, 2010.

32 Antti Valmari. Stubborn sets for reduced state space generation. In *Advances in Petri Nets*, pages 491–515, 1991.

Bidirectional Nested Weighted Automata*

Krishnendu Chatterjee[1], Thomas A. Henzinger[2], and Jan Otop[3]

1 **IST Austria, Klosterneuburg, Austria**
krish.chat@ist.ac.at
2 **IST Austria, Klosterneuburg, Austria**
tah@ist.ac.at
3 **University of Wrocław, Wrocław, Poland**
jotop@cs.uni.wroc.pl

Abstract

Nested weighted automata (NWA) present a robust and convenient automata-theoretic formalism for quantitative specifications. Previous works have considered NWA that processed input words only in the forward direction. It is natural to allow the automata to process input words backwards as well, for example, to measure the maximal or average time between a response and the preceding request. We therefore introduce and study bidirectional NWA that can process input words in both directions. First, we show that bidirectional NWA can express interesting quantitative properties that are not expressible by forward-only NWA. Second, for the fundamental decision problems of emptiness and universality, we establish decidability and complexity results for the new framework which match the best-known results for the special case of forward-only NWA. Thus, for NWA, the increased expressiveness of bidirectionality is achieved at no additional computational complexity. This is in stark contrast to the unweighted case, where bidirectional finite automata are no more expressive but exponentially more succinct than their forward-only counterparts.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases weighted automata, nested weighted automata, complexity, bidirectional

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.5

1 Introduction

We study an extension of nested weighted automata (NWA) [13] that can process words in both directions. We show that this new and natural framework can express many interesting quantitative properties that the previous formalism could not. We establish decidability and complexity results of the basic decision problems for the new framework. We start with the motivation for quantitative properties, then describe NWA and our new framework, and finally the contributions.

Weighted automata. Automata-theoretic formalisms provide a natural way to express quantitative properties of systems. Weighted automata extend finite automata where every transition is assigned an integer number called weight. Thus a run of an automaton gives rise to a sequence of weights. A value function aggregates the sequence of weights into a single

* This research was supported in part by the Austrian Science Fund (FWF) under grants S11402-N23,S11407-N23 (RiSE/SHiNE) and Z211-N23 (Wittgenstein Award), ERC Start grant (279307: Graph Games), Vienna Science and Technology Fund (WWTF) through project ICT15-003 and by the National Science Centre (NCN), Poland under grant 2014/15/D/ST6/04543.

value. For non-deterministic weighted automata, the value of a word w is the infimum value of all runs over w. First, weighted automata were studied over finite words with weights from a semiring, and ring multiplication as value function [19], and later extended to infinite words with limit averaging or supremum as value function [12, 11, 10]. While weighted automata over semirings can express several quantitative properties [22], they cannot express long-run average properties that weighted automata with limit averaging can [12]. However, even weighted automata with limit averaging cannot express some basic quantitative properties (see [13]).

Nested weighted automata. A natural extension of weighted automata is to add nesting, which leads to *nested weighted automata (NWA)* [13]. A nested weighted automaton consists of a master automaton and a set of slave automata. The master automaton runs over input infinite words. At every transition the master can invoke a slave automaton that runs over a finite subword of the infinite word, starting at the position where the slave automaton is invoked. Each slave automaton terminates after a finite number of steps and returns a value to the master automaton. Each slave automaton is equipped with a value function for finite words, and the master automaton aggregates the returned values from slave automata using a value function for infinite words.

Advantages of NWA. We discuss the various advantages of NWA.

1. For Boolean finite automata, nested automata are equivalent to the non-nested counterpart, whereas NWA are strictly more expressive than non-nested weighted automata [13, Example 5]. NWA provide a specification framework where many basic quantitative properties can be expressed, which cannot be expressed by weighted automata [13].
2. NWA provide a natural and convenient way to express quantitative properties. Every slave automaton computes a subproperty, which is then combined using the master automaton. Thus NWA allow to decompose properties conveniently, and provide a natural framework to study quantitative run-time verification.
3. Finally, subclasses of NWA are equivalent in expressive power with automata with monitor counters [16], and thus they provide a robust framework to express quantitative properties.

Bidirectional NWA. Previous works considered slave automata that can only process input words in the forward direction (forward-only NWA). However, to specify quantitative properties, it is natural to allow slave automata to run backwards, for example, to measure the maximal or average time between a response and the preceding request. In this work we consider this natural extension of NWA, namely *bidirectional NWA*, where slave automata can process words in the forward as well as the backward direction.

Natural properties. First, we show that many natural properties can be expressed in the bidirectional NWA framework. We present two examples below (details in Section 3).

1. *Average energy level.* Consider a quantitative setting where each weight represents energy gain or consumption, and thus the sum of weights represents the energy level. To express the average energy level property, the master automaton has long-run average as the value function, and at every transition it invokes a slave automaton that walks backward with sum value function for the weights. Thus the average energy level property is naturally expressed by NWA with backward-walking slave automata, while this property is not expressible by NWA with forward-walking slave automata.

2. *Data-consistency property (DCP).* Consider the data-consistency property (DCP) where the input letters correspond to reads, writes, null instructions, and commits. For each read, the distance to the previous commit measures how fresh is the read with respect to the last commit, and this can be measured with a backward-walking slave automaton. For each write, the distance to the next commit measures how fresh is the write with respect to the following commit, and this can be measured with a forward-walking slave automaton. Thus the average freshness, called DCP, is expressed with bidirectional NWA. Moreover, the DCP can neither be expressed by NWA with only forward-walking slave automata nor by NWA with only backward-walking slave automata.

Our contributions. We propose bidirectional NWA as a specification framework for quantitative properties. First, we show that the classes of forward-only NWA and backward-only NWA have incomparable expressiveness, and bidirectional NWA strictly generalize both classes. Second, we establish complexity of the emptiness and universality problems for bidirectional NWA, where we consider the limit-average value function for the master automaton and for the slave automata we consider standard value functions for finite words (such as min, max, and variants of sum). The obtained complexity results coincide with the results for forward-only NWA, and range from NLogSpace-complete, PTime to PSpace-complete to ExpSpace. However the proofs for bidirectional NWA are much more involved than forward-only NWA. Thus bidirectional NWA have all the advantages of NWA but provide a more expressive framework for natural quantitative properties. Moreover, the added expressiveness of bidirectionality is achieved with no increase in the computational complexity of the decision problems (Table 1). We highlight two significant differences as compared to the unweighted case: (1) In the unweighted case bidirectionality does not change expressiveness, whereas we show for NWA it does; and (2) in the unweighted case for deterministic automata bidirectionality leads to exponential succinctness and increase in complexity of the decision problems, whereas for NWA bidirectionality does not change the computational complexity. Thus the combination of nesting and bidirectionality is very interesting in the weighted automata setting, which we study in this work.

Related works. Quantitative automata and logic have been extensively studied in recent years in many different contexts [19, 12, 4, 2]. The book [19] presents an excellent collection of results of weighted automata on finite words. Weighted automata on infinite words have been studied in [12, 11, 20]. Weighted automata over finite words extended with monitor counters have been considered (under the name of cost register automata) in [3, 21]. A version of nested weighted automata over finite words has been studied in [6], and nested weighted automata over infinite words has been studied in [13, 15, 14]. Several quantitative logics have also been studied, such as [5, 7, 1]. However, none of these works consider the rich and expressive formalism of quantitative properties expressible by NWA with slaves that walk both forward and backward, retaining decidability of the basic decision problems.

In the main paper, we present the key ideas and main intuitions of the proofs of selected results, and detailed proofs are presented in the full version [17]. Moreover, to ease of presentation we focus on bidirectional NWA where each slave automaton is either forward walking or backward walking. We can allow slave automata that change direction while running, i.e., allow two-way slave automata and obtain the same complexity results; we discuss nested weighted autmata with two-way slave automata in the full version [17].

2 Definitions

2.1 Words and automata

Words. We consider a finite *alphabet* of letters Σ. A *word* over Σ is a (finite or infinite) sequence of letters from Σ. We denote the i-th letter of a word w by $w[i]$, and for $i < j$ we define $w[i, j]$ as the word $w[i]w[i + 1] \dots w[j]$. The length of a finite word w is denoted by $|w|$; and the length of an infinite word w is $|w| = \infty$. For an infinite word w, word $w[i, \infty]$ is the suffix of w with first $i - 1$ letters removed. For a finite word w of length k, we define the reverse of w, denoted by w^R, as the word $w[k]w[k - 1] \dots w[1]$.

Labeled automata. For a set X, an *X-labeled automaton* $\mathcal{A}$ is a tuple $\langle \Sigma, Q, Q_0, \delta, F, C \rangle$, where (1) Σ is the alphabet, (2) Q is a finite set of states, (3) $Q_0 \subseteq Q$ is the set of initial states, (4) $\delta \subseteq Q \times \Sigma \times Q$ is a transition relation, (5) F is a set of accepting states, and (6) $C : \delta \mapsto X$ is a labeling function. A labeled automaton $\langle \Sigma, Q, \{q_0\}, \delta, F, C \rangle$ is *deterministic* if and only if δ is a function from $Q \times \Sigma$ into Q and Q_0 is a singleton.

Semantics of (labeled) automata. A *run* π of a (labeled) automaton $\mathcal{A}$ on a word w is a sequence of states of $\mathcal{A}$ of length $|w| + 1$ such that $\pi[0]$ belongs to the initial states of $\mathcal{A}$ and for every $0 \leq i \leq |w| - 1$ we have $(\pi[i], w[i + 1], \pi[i + 1])$ is a transition of $\mathcal{A}$. A run π on a finite word w is *accepting* if and only if the last state $\pi[|w|]$ of the run is an accepting state of $\mathcal{A}$. A run π on an infinite word w is *accepting* if and only if some accepting state of $\mathcal{A}$ occurs infinitely often in π. For an automaton $\mathcal{A}$ and a word w, we define $\mathsf{Acc}(w)$ as the set of accepting runs on w. Note that for deterministic automata, every word w has at most one accepting run ($|\mathsf{Acc}(w)| \leq 1$).

Weighted automata and their semantics. A *weighted automaton* is a $\mathbb{Z}$-labeled automaton, where $\mathbb{Z}$ is the set of integers. The labels are called *weights*. We define the semantics of weighted automata in two steps. First, we define the value of a run. Second, we define the value of a word based on the values of its runs. To define values of runs, we will consider *value functions* f that assign real numbers to sequences of integers. Given a non-empty word w, every run π of $\mathcal{A}$ on w defines a sequence of weights of successive transitions of $\mathcal{A}$, i.e., $C(\pi) = (C(\pi[i - 1], w[i], \pi[i]))_{1 \leq i \leq |w|}$; and the value $f(\pi)$ of the run π is defined as $f(C(\pi))$. We denote by $(C(\pi))[i]$ the weight of the i-th transition, i.e., $C(\pi[i - 1], w[i], \pi[i])$. The value of a non-empty word w assigned by the automaton $\mathcal{A}$, denoted by $\mathcal{L}_{\mathcal{A}}(w)$, is the infimum of the set of values of all *accepting* runs; i.e., $\inf_{\pi \in \mathsf{Acc}(w)} f(\pi)$, and we have the usual semantics that the infimum of the empty set is infinite, i.e., the value of a word that has no accepting run is infinite. Every run π on the empty word has length 1 and the sequence $C(\pi)$ is empty, hence we define the value $f(\pi)$ as an external (not a real number) value $\perp$. Thus, the value of the empty word is either $\perp$, if the empty word is accepted by $\mathcal{A}$, or ∞ otherwise. To indicate a particular value function f that defines the semantics, we call a weighted automaton $\mathcal{A}$ with value function f an f-automaton.

Value functions. For finite runs we consider the following classical value functions: for runs of length $n + 1$ we have

- *Max and min:* $\textsc{Max}(\pi) = \max_{i=1}^{n}(C(\pi))[i]$ and $\textsc{Min}(\pi) = \min_{i=1}^{n}(C(\pi))[i]$.
- *Sum and absolute sum:* the sum function $\textsc{Sum}(\pi) = \sum_{i=1}^{n}(C(\pi))[i]$, the absolute sum $\textsc{Sum}^{+}(\pi) = \sum_{i=1}^{n} \mathsf{Abs}((C(\pi))[i])$, where $\mathsf{Abs}(x)$ is the absolute value of x.

- *Variants of bounded sum:* we consider a family of functions called the (variants of) bounded sum value function $\textsc{Sum}^{L,U}$. Each of these functions returns the sum if all the partial sums are in the interval $[L, U]$, otherwise there are many possibilities which lead to multiple variants. For example, we can require that for all prefixes π' of π we have $\textsc{Sum}(\pi') \in [L, U]$. We can impose a similar restriction on all suffixes, all infixes etc. Moreover, if partial sums are not contained in $[L, U]$, a bounded sum can return ∞, the first violated bound, etc.

For infinite runs we consider:

- *Limit average:* $\textsc{LimAvg}(\pi) = \liminf_{k\to\infty} \frac{1}{k} \cdot \sum_{i=1}^{k} (C(\pi))[i]$.

2.2 Nested weighted automata

Nested weighted automata (NWA) have been introduced in [13] and originally allowed slave automata to move only forward. The variant we define here allow two types of slave automata, forward walking and backward walking. The original definition of NWA from [13] is versatile and hence it can be seamlessly extended to the case with bidirectional (forward- and backward-walking) slave automata. We follow the description of [13].

Informal description. A *nested weighted automaton* consists of a labeled automaton over infinite words, called the *master automaton*, a value function f for infinite words, and a set of weighted automata over finite words, called *slave automata*. A nested weighted automaton can be viewed as follows: given a word, we consider the run of the master automaton on the word, but the weight of each transition is determined by dynamically running slave automata; and then the value of a run is obtained using the value function f. That is, the master automaton proceeds on an input word as an usual automaton, except that before taking a transition, it starts a slave automaton corresponding to the label of the current transition. The slave automaton starts at the current position of the master automaton in the input word and works on some finite part of it. There are two types of slave automata: (a) forward walking, which move onward the input word (toward higher positions), and (b) backward walking, which move towards the beginning of the input word. Once a slave automaton finishes, it returns its value to the master automaton, which treats the returned value as the weight of the current transition that is being executed. The slave automaton might immediately accept and return value $\perp$, which corresponds to a *silent transition*, i.e., transition with no weight. If one of slave automata rejects, the nested weighted automaton rejects. We present two examples of properties expressible by NWA. Additional examples are presented in Section 3.

▶ **Example 1** (Average response time and its dual)**.** Consider infinite words over $\{r, g, \#\}$, where r represents requests, g represents grants, and $\#$ represents idle. A basic and interesting property is the average number of letters between a request and the corresponding grant, which represents the long-run *average response time (ART)* of the system. This property cannot be expressed by a non-nested automaton [13]. ART can be expressed by a deterministic nested weighted automaton, which basically implements the definition of ART. This automaton invokes at every request a forward-walking slave automaton with $\textsc{Sum}^{+}$ value function, which counts the number of events until the following grant. On the other events the NWA takes silent transitions. Finally, the master automaton applies $\textsc{LimAvg}$ value function to the values returned by slave automata. Figure 1 presents a run of the NWA computing ART.

We define the average workload property (AW), which measures the average number of pending requests. The average is computed over all positions in a word. Intuitively, if we

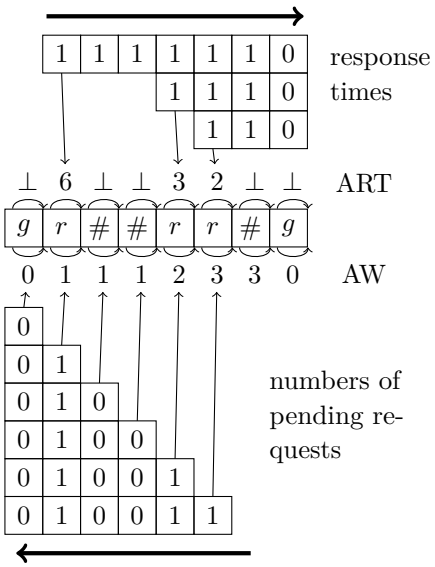


Figure 1 Runs of NWA computing ART (above) and AW (below). Each weight of a transition is dynamically computed as the sum of weights of slave automata. The thick arrows depict directions of slave automata.

pick a position in word w at random, the expected number of pending requests is the average workload of w. Formally, we define the workload at i in w, denoted $wl(w, i)$, as the number of letters r among $w[j, i]$, where j is the last position in $w[1, i]$ where g occurs or 1 if such a position does not exist. The average workload of w is the limit average over all positions i of $wl(w, i)$.

Observe that AW can be expressed by a deterministic (LimAvg; Sum$^+$)-automaton with backward-walking slave automata. Basically, the NWA invokes at every position a slave automaton, which counts the number of r letter from its current position to the first position containing letter g, where it terminates. Since slave automata run backwards, each of them computes the workload at the position of its invocation. Figure 1 presents a run of the NWA computing AW.

Now, we present a formal definition of NWA and their semantics.

Nested weighted automata. A *nested weighted automaton* (NWA) with bidirectional slave automata is a tuple $\langle \mathcal{A}_{mas}; f; \mathfrak{B}_{-m}, \ldots, \mathfrak{B}_0, \ldots, \mathfrak{B}_l \rangle$, with $m, l \in \mathbb{N}$ where (1) $\mathcal{A}_{mas}$, called the *master automaton*, is a $\{-m, \ldots, l\}$-labeled automaton over infinite words (the labels are the indexes of automata $\mathfrak{B}_{-m}, \ldots, \mathfrak{B}_l$), (2) f is a value function on infinite words, called the *master value function*, and (3) $\mathfrak{B}_{-m}, \ldots, \mathfrak{B}_l$ are weighted automata over finite words called *slave automata*. Intuitively, an NWA can be regarded as an f-automaton whose weights are dynamically computed at every step by the corresponding slave automaton. The automata $\mathfrak{B}_{-m}, \ldots, \mathfrak{B}_{-1}$ (resp., $\mathfrak{B}_1, \ldots, \mathfrak{B}_l$) are called *backward walking* (resp., *forward walking*) slave automata. We refer to NWA with both forward and backward walking slave automata as *bidirectional NWA*. The automaton $\mathfrak{B}_0$ immediately accepts and returns no weight; it is used to implement silent transitions, which have no weight. We define an $(f; g)$-*automaton* as an NWA where the master value function is f and all slave automata are g-automata.

Semantics: runs and values. A *run* of $\mathbb{A}$ on an infinite word w is an infinite sequence $(\Pi, \pi_1, \pi_2, \ldots)$ such that (1) Π is a run of $\mathcal{A}_{mas}$ on w; (2) for every $i > 0$ the label $j =$

$C(\Pi[i-1], w[i], \Pi[i])$ pointers at a slave automaton and (a) if $j < 0$, then π_i is a run of the automaton $\mathfrak{B}_j$ on some prefix of the reverse word $(w[1,i])^R$, and (b) if $j \geq 0$, then π_i is a run of the automaton $\mathfrak{B}_j$ on some finite prefix of $w[i,\infty]$. The run $(\Pi, \pi_1, \pi_2, \ldots)$ is *accepting* if all runs $\Pi, \pi_1, \pi_2, \ldots$ are accepting (i.e., Π satisfies its acceptance condition and each $\pi_1, \pi_2, \ldots$ ends in an accepting state) and infinitely many runs of slave automata have length greater than 1 (the master automaton takes infinitely many non-silent transitions). The value of the run $(\Pi, \pi_1, \pi_2, \ldots)$ is defined as $\mathsf{sil}(f)(v(\pi_1)v(\pi_2)\ldots)$, where $v(\pi_i)$ is the value of the run π_i in the corresponding slave automaton, and $\mathsf{sil}(f)$ is the value function that takes its input sequence, removes $\perp$ symbols and applies f to the remaining sequence. The value of a word w assigned by the automaton $\mathbb{A}$, denoted by $\mathcal{L}_{\mathbb{A}}(w)$, is the infimum of the set of values of all *accepting* runs. We require accepting runs to contain infinitely many non-silent transitions as f is a value function over infinite sequences, hence the sequence $v(\pi_1)v(\pi_2)\ldots$ with $\perp$ removed must be infinite.

Deterministic nested weighted automata. An NWA $\mathbb{A}$ is *deterministic* if (1) the master automaton and all slave automata are deterministic, and (2) in all slave automata, accepting states have no outgoing transitions. Intuitively, a slave automaton in an accepting state can choose (non-deteministically) to terminate or continue running; condition (2) removes this source of non-determinism.

Width of NWA. An NWA has *width* k if and only if in every run at every position at most k slave automata are active.

3 Examples

In this section we present several examples of quantitative properties that can be expressed with bidirectional NWA.

▶ **Example 2** (Average energy level)**.** We consider the average energy level property studied in [18, 8]. Consider $W \in \mathbb{N}$ and an alphabet Σ_W consisting of integers from interval $[-W, W]$. These letters correspond to the energy change, i.e., negative values represent energy consumption whereas positive values represent energy gain. For $w \in \Sigma_W$ we define the energy level at i as the sum $w[1] + \ldots + w[i]$. The average energy property (AE) is the limit average of the energy levels at every position. For example, the average energy level of $2(-1)3((-1)1)^\omega$ is 4.

AE can be expressed by a (LimAvg; Sum)-automaton $\mathbb{A}$ with backward-walking slave automata, but it is not expressible by (LimAvg; Sum)-automata with forward-walking slave automata. To express AE, a (LimAvg; Sum)-automaton $\mathbb{A}$ with backward-walking slave automata invokes at every position a slave automaton, which runs backward to the beginning of the word and sums up all the letters. In contrast, (LimAvg; Sum)-automata with forward-walking slave automata can use finite memory of the master automaton, but finite prefixes influence only finitely many values returned by slave slave automata and the limit-average value function neglects finite prefixes. Formally, we can show with a simple pumping argument that for every (LimAvg; Sum)-automaton with forward-walking slave automata, among words $w_i = 1^i 0^\omega$ there exists a pair of words with the same value. In contrast, all these words have different AE (AE of w_i is i).

AE property is often considered in conjunction with bounds on energy values. Typically, energy should not drop below some threshold, in particular, it should not be negative. In addition, the energy storage is limited, which motivates the upper bound on the stored energy,

where the excess energy is released. These two restrictions lead to the *interval constraint* on energy levels, i.e., we require the energy level at every position to belong to a given interval $[L, U]$, which results in a variant of the bounded sum $\textsc{Sum}^{L,U}$.

▶ **Example 3** (Data consistency)**.** Consider a database server, which processes instructions grouped into transactions. There are four instructions: read r, write w, void $\#$ and commit c. The commit instruction applies all writes, finishes the current transaction and starts a new one. The read instructions refer to writes applied before the previous commit.

In the presence of multiple clients connected to the database, there are two options to achieve consistency. One option is to use locks that can limit concurrency. A second approach is *optimistic concurrency* which proceeds without locks, and then rolls back in case there was a collision between transactions. In ordered to limit the number of roll backs, it is preferred that the read instructions occur shortly after commit, while write instructions are followed by the commit instruction as quickly as possible. Formally, we define (a) consistency (or freshness) of a read instruction as the number of steps to the first preceding commit instruction, and (b) consistency of a write instruction as the number of steps to the following commit instruction. The data consistency property (DCP) of w is the limit average of consistency of reads and writes in w.

DCP is expressed by the following deterministic $(\textsc{LimAvg}; \textsc{Sum}^{+})$-automaton $\mathbb{A}$ with bidirectional slave automata. On every read r (resp., w), the NWA $\mathbb{A}$ invokes a slave automaton which walks backward (resp., forward) and counts the number of steps to the first encountered c. On the remaining instructions $c, \#$, the NWA $\mathbb{A}$ invokes a dummy slave automaton which corresponds to a silent transition.

▶ **Example 4.** Consider the framework of Example 3. For every position with read r or write w we define a regret at position i as the minimal distance to the preceding or the following commit c. Intuitively, the regret corresponds to the number of instructions by which we have to prepone or postpone the commit to include the instruction at the current position. We consider the minimal regret property (MR) on words over $\{r, w, c, \#\}$ defined the limit average over positions with r and w of the regret at these positions. MR can be expressed by a non-deterministic $(\textsc{LimAvg}; \textsc{Sum}^{+})$-automaton with bidirectional slave automata, which basically implements the definition of MR (the non-deterministic guess is whether it is the preceding or the following commit). The NWA invokes at every r or w position one of the following two slave automata $\mathfrak{B}_B, \mathfrak{B}_F$. The automaton $\mathfrak{B}_B$ counts the number of steps to the preceding grant, while $\mathfrak{B}_F$ counts the number of steps to the following grant.

4 Decision questions

For NWA with bidirectional slave automata, we consider the quantitative counterparts of the fundamental problems of emptiness and universality. The (quantitative) emptiness and universality problems are defined in the same way for weighted automata and all variants of NWA; in the following definition $\mathcal{A}$ denotes either a weighted automaton or an NWA.

Emptiness and universality. Given an automaton $\mathcal{A}$ and a threshold λ, the *emptiness* (resp. *universality*) problem asks whether there exists a word w with $\mathcal{L}_{\mathcal{A}}(w) \leq \lambda$ (resp., for every word w we have $\mathcal{L}_{\mathcal{A}}(w) \leq \lambda$).

▶ Remark. The emptiness and universality problems have been studied for forward-only NWA in [13].

- For NWA the value functions considered for the master automaton are the infimum (or limit-infimum), the supremum (or limit-supremum), and the limit-average. For all the decidability results for the infimum (limit-infimum) and the supremum (limit-supremum) value functions the techniques are similar to unweighted automata [13], which can be easily adapted to the bidirectional framework. Hence in the sequel we only focus on bidirectional NWA with the limit-average value function for the master automaton.
- Moreover, we study only the emptiness problem for the following reasons. First, for the deterministic case the emptiness and the universality problems are similar and hence we focus on the emptiness problem. Second, in the non-deterministic case the universality problem is already undecidable for LimAvg-automata even with no nesting [9].

4.1 The minimum, maximum and bounded sum value functions

First, we show that for g being Min, Max, or a variant of the bounded sum value function $\text{Sum}^{L,U}$, the emptiness problem for (LimAvg; g)-automata with bidirectional slave automata is decidable in PSpace. To show that, we prove a stronger result, i.e., every (LimAvg; g)-automaton can be effectively transformed to a LimAvg-automaton of exponential size.

Key ideas. Weighted automata with value functions $\text{Min}, \text{Max}, \text{Sum}^{L,U}$ are close to (non-weighted) finite-state automata. In particular, these automata have finite range and for each value λ from the range, the set of words of value λ is regular. Thus, instead of invoking a slave automaton, the master automaton can non-deterministically pick value λ and verify that the value returned by this slave automaton is λ. For backward-walking slave automata the guessing can be avoided as the master automaton can simulate (the reverse of) runs of all backward-walking slave automata until the current position. Thus, we can eliminate slave automata from NWA, i.e., we transform such NWA to weighted automata. Formally, we show that for $g \in \{\text{Min}, \text{Max}, \text{Sum}^{L,U}\}$, every (LimAvg; g)-automaton with bidirectional slave automata can be transformed into an equivalent LimAvg-automaton of exponential size. The emptiness problem for non-deterministic LimAvg-automata is in NLogSpace (assuming weights given in unary) and hence we have the containment part in the following Theorem 5. The hardness part follows from PSpace-hardness of the emptiness problem for (LimAvg; g)-automata with forward-walking slave automata only [13].

▶ **Theorem 5.** *Let $g \in \{\text{Min}, \text{Max}, \text{Sum}^{L,U}\}$. The emptiness problem for non-deterministic* (LimAvg; g)*-automata with bidirectional slave automata is* PSpace*-complete.*

Note. The complexity in Theorem 5 does not depend on encoding of weights in slave automata, i.e., the problem is PSpace-hard even for a fixed set of weights, and it remains in PSpace for weights encoded in binary.

The average energy property from Example 2 with bounds on energy levels can be expressed with (LimAvg; $\text{Sum}^{L,U}$)-automata. The emptiness problem for these automata is decidable by Theorem 5.

▶ Remark (Parametrized complexity). If we assume that the size of slave automata in Theorem 5 is bounded by a constant, the complexity of the emptiness problem drops to NLogSpace-complete. NLogSpace-hardness follows from NLogSpace-hardness of the emptiness problem for LimAvg-automata, which can be considered as a special case of NWA.

The results of this section apply to general bidirectional NWA. In the following section we consider bidirectional NWA with the sum value function, where we consider additional

restrictions of finite width (Section 5) and bounded width (Section 6). We also justify in Remark 5.1 that the finite width restriction is natural.

5 Finite-width case

In this section we study NWA satisfying the *finite width* condition. First, we briefly discuss the finite-width condition and argue that it is a natural restriction. Next, we show that the emptiness problem for (finite-width) (LimAvg; Sum$^+$)-automata with bidirectional slave automata is decidable in ExpSpace. We conclude this section with the expressiveness results; we show that classical NWA with forward-walking slave automata and NWA with backward-walking slave automaton have incomparable expressive power. Hence, (finite-width) (LimAvg; Sum$^+$)-automata with bidirectional slave automata are strictly more expressive than NWA with one-direction slave automata.

5.1 The finite-width condition

Finite width. An NWA $\mathbb{A}$ has finite width if and only if in every accepting run of $\mathbb{A}$ at every position at most finitely many slave automata are active. Classical NWA with forward-walking slave automata only have finite width. Indeed, in any run, at any position i at most i slave automata can be active.

▶ **Example 6.** Consider an NWA over $\{a, b\}$ such that the master automaton accepts a single word ab^ω and all slave automata are backward walking and accept words b^*a. All slave automata terminate at the first position of ab^ω and hence this NWA does not have finite width.

The automata expressing properties from Examples 1, 3 and 4 are finite-width (LimAvg; Sum$^+$)-automata with bidirectional slave automata. Observe that an NWA does not have finite width if and only if it has an accepting run, in which at some position i infinitely many backward-walking slave automata terminate.

▶ Remark (Finite width is natural for positive sum). Let $\mathbb{A}$ be a (LimAvg; Sum$^+$)-automaton with bidirectional slave automata. Except for degenerate cases, runs of $\mathbb{A}$, which do not have finite width, have infinite value. Indeed, consider a run π and a position i_0 at which infinitely many automata are active. Since only finitely many forward-walking slave automata are active at i_0, infinitely many of them are backward-walking and for some position $i < i_0$, infinitely many slave automata S terminate at position i. Then, one of the following holds: either that value of this run is infinite or one of the following two degenerate cases happen: (a) The slave automata from S are invoked with zero density (i.e., if consider the longrun-average of the frequency of invoking slave automata, then it is zero). This situation represents that monitoring with slave automata happens with vanishing frequency which is a degenerate case. (b) The values returned by the slave automata from S are bounded. It follows that these automata take transitions of non-zero weight only in some finite subword $w[i, j]$ of the input word w. This situation represents monitoring of an infinite sequence, in which all events past position j are irrelevant. This is a degenerate case in the infinite-word case.

The finite-width property does not depend on weights and hence we can construct an exponential-size Büchi automaton $\mathcal{A}$, which simulates runs of a given NWA $\mathbb{A}$. Having $\mathcal{A}$, we can check whether it has a run corresponding to an accepting run of $\mathbb{A}$, in which infinitely many backward-walking slave automata terminate at the same position. This check can be done in logarithmic space and hence checking the finite-width property is in PSpace.

A simple reduction from the non-emptiness problem for NWA shows PSPACE-hardness of checking the finite-width property.

▶ **Theorem 7.** *The problem whether a given NWA has finite width is* PSPACE*-complete.*

5.2 The absolute sum value function

We present the main result on NWA of finite width.

▶ **Theorem 8.** *The emptiness problem for finite-width* (LIMAVG; SUM$^+$)*-automata with bidirectional slave automata is* PSPACE*-hard and it is decidable in* EXPSPACE*.*

Key ideas. PSPACE-hardness follows from PSPACE-hardness of the emptiness problem for (LIMAVG; SUM$^+$)-automata with forward-walking slave automata only. Containment in EXPSPACE is shown by reduction to the bounded-width case, which is shown decidable in the following section (Theorem 13). We briefly describe this reduction. Consider a finite-width (LIMAVG; SUM$^+$)-automaton $\mathbb{A}$ with bidirectional slave automata. First, we observe that without loss of generality, we can assume that $\mathbb{A}$ is deterministic. Second, we observe that in every word accepted by $\mathbb{A}$, at almost every position i there exists a *barrier*, which is a word u such that (a) the word $w' = w[1,i]uw[i+1,\infty]$, i.e., w with u inserted at position i, is accepted by $\mathbb{A}$, and the runs on w and w' coincide except for positions in w' corresponding to u, (b) in the run on w', backward-walking slave automata active at the end of u terminate within u, (c) in the run on w', forward-walking slave automata active at the beginning of u terminate within u, and (d) u has exponential length. Basically, active slave automata cannot cross u in w' and in the effect insertion of u bounds the number of active slave automata. Existence of barriers follows from the finite-width property of $\mathbb{A}$.

We insert barriers in w to reduce the number of active slave automata. We show that if at position i in w, exponentially many active slave automata accumulates exponential weight past crossing i (some slave automata walk forward while other backwards), all partial averages (of values returned by slave automata) in w' are bounded by the corresponding partial averages in w. We conclude that for every word w, there exists a word w' such that (i) at every position at most exponentially many slave automata accumulate at least exponential values, and (ii) the value of w' does not exceed the value of w. Thus, to compute the infimum over all runs of $\mathbb{A}$, we can focus on runs in which at every position at most exponentially many slave automata accumulate at least exponential values. Runs of slave automata in which they accumulate bounded (exponential) values can be eliminated as in Theorem 5, i.e., we can construct an exponential size NWA $\mathbb{A}'$, which simulates $\mathbb{A}$, and such that its slave automata run as long as they can accumulate value exponential (in $|\mathbb{A}|$) and otherwise they non-deterministically pick the remaining value and the master automaton verifies that the pick is correct. Therefore, the infimum over all runs of $\mathbb{A}$ coincides with the infimum over all runs of $\mathbb{A}'$ of width exponentially bounded.

▶ Remark (Parametrized complexity). If we assume that the size of slave automata in Theorem 8 is bounded by a constant, the complexity of the emptiness problem drops to NLOGSPACE-complete. NLOGSPACE-hardness follows from NLOGSPACE-hardness of the emptiness problem for LIMAVG-automata, which can be viewed as a special case of NWA.

5.3 Expressive power

DCP defined in Example 3 can be expressed by a deterministic finite-width (LIMAVG; SUM$^+$)-automaton with bidirectional slave automata. We show that both forward-walking and

backward-walking slave automata are required to express DCP. That is, we formally show that DCP cannot be expressed by any (non-deterministic) (LimAvg; Sum$^+$)-automaton with slave automata walking in one direction only.

Classes of NWA. We define $\mathcal{FB}$(LimAvg; Sum$^+$) as the class of all finite-width (LimAvg; Sum$^+$)-automata with bidirectional slave automata. We define $\mathcal{F}$(LimAvg; Sum$^+$) (resp., $\mathcal{B}$(LimAvg; Sum$^+$)) as the subclass of $\mathcal{FB}$(LimAvg; Sum$^+$) consisting of NWA with forward-walking (resp., backward-walking) slave automata only.

We establish that classes $\mathcal{F}$(LimAvg; Sum$^+$) and $\mathcal{B}$(LimAvg; Sum$^+$) have incomparable expressive power and hence they are strictly less expressive than class $\mathcal{FB}$(LimAvg; Sum$^+$).

Key ideas. Consider a word $w = (c\#^N r^{2K} c\#^{2N} r^K)^\omega$ for some big K and much bigger N. An NWA from $\mathcal{B}$(LimAvg; Sum$^+$) computes DCP of w by invoking (non-dummy) slave automata at every r letter and taking silent transitions on letters $\#, c$. We show that an NWA $\mathbb{A}$ from $\mathcal{F}$(LimAvg; Sum$^+$) cannot invoke the right number of slave automata, even if it uses non-determinism. More precisely, we show that $\mathbb{A}$ computing DCP has to invoke at most $O(K)$ non-dummy slave automata on average on subwords $c\#^N r^{2K} c\#^{2N} r^K$. Since N is much bigger than K, we conclude that $\mathbb{A}$ has a cycle over $\#$ letters at which it takes only silent transitions and a cycle over r letters on which it increases the multiplicity of active slave automata. Using these two cycles, we construct a run of value smaller than DCP, which contradicts the assumption that $\mathbb{A}$ computes DCP. Similarly, we can show that an NWA from $\mathcal{B}$(LimAvg; Sum$^+$) cannot compute correctly DCP of words of the form $w = (cw^{2K}\#^N cw^K\#^{2N})^\omega$, while on these words DCP is expressible by an NWA from $\mathcal{F}$(LimAvg; Sum$^+$).

▶ **Lemma 9.** *(1) DCP restricted to alphabet $\{r, \#, c\}$ is expressed by an NWA from $\mathcal{B}$(LimAvg; Sum$^+$), but it is not expressible by NWA from $\mathcal{F}$(LimAvg; Sum$^+$). (2) DCP restricted to alphabet $\{w, \#, c\}$ is expressed by an NWA from $\mathcal{F}$(LimAvg; Sum$^+$), but it is not expressible by NWA from $\mathcal{B}$(LimAvg; Sum$^+$).*

The above lemma implies that DCP over alphabet $\{r, w, \#, c\}$ is not expressible by any NWA from $\mathcal{F}$(LimAvg; Sum$^+$) nor from $\mathcal{B}$(LimAvg; Sum$^+$). In conclusion, we have:

▶ **Theorem 10.** *(1) $\mathcal{F}$(LimAvg; Sum$^+$) and $\mathcal{B}$(LimAvg; Sum$^+$) have incomparable expressive power. (2) $\mathcal{FB}$(LimAvg; Sum$^+$) are strictly more expressive than $\mathcal{F}$(LimAvg; Sum$^+$) and $\mathcal{B}$(LimAvg; Sum$^+$).*

6 Bounded-width case

In this section, we study (LimAvg; Sum)-automata with bidirectional slave automata, which have bounded width. The bounded width restriction has been introduced in [14] to improve the complexity of the emptiness problem and to establish decidability of the emptiness problem for (LimAvg; Sum)-automata. NWA considered in [14] have only forward-walking slave automata, while we extend these results to NWA with bidirectional slave automata. This extension preserves the complexity bounds from [14], i.e., the emptiness problem is in PTime for constant width and PSpace-complete for width given in unary.

The bounded width restriction emerges naturally in examples presented so far. If we bound the number of pending requests, we can express ART and AW (Example 1) by automata of bounded width. If we bound the number of writes and reads between any two

commits, then DCP and MR (Examples 3 and 4) can be expressed by NWA of bounded width. These natural restrictions lead to more efficient decision procedures.

The decision procedure in this section differs from the one from [14]. The key step in the decidability proof from [14] is establishing the following dichotomy: either the infimum over values of all words is $-\infty$ or the infimum is realized by *dense* runs. A run is dense if for the values $v_1, v_2, \ldots$ returned by slave automata invoked at positions $1, 2, \ldots$ we have $\frac{v_i}{i}$ converges to 0, i.e., values returned by slave automata are sublinear in the positions of their invocation. Properties of dense runs allow for further reductions, which lead to a decision procedure. However, we show in the following Example 11 that for NWA with bidirectional slave automata, dense runs may not attain the infimum of all runs.

▶ **Example 11.** Consider a (LimAvg; Sum)-automaton $\mathbb{A}$ with bidirectional slave automata over $\Sigma = \{a, b, c\}$. The NWA $\mathbb{A}$ accepts words $(ab^*c)^\omega$ and it works as follows. On letters a, $\mathbb{A}$ invokes a forward-walking slave automaton $\mathfrak{B}_a$, which returns the number of the following b letters up to c. On letters c, $\mathbb{A}$ invokes a backward-walking slave automaton $\mathfrak{B}_c$, which returns the number of the preceding b letters since a. Finally, on b letters, $\mathbb{A}$ invokes a slave automaton $\mathfrak{B}_b$, which takes a single transition and returns value 0. The NWA $\mathbb{A}$ has width 3. We can show that the value of any dense run, is 2. However, the infimum over values of all words is 1. The partial average of the values returned by slave automata on finite word $u = (ab^*c)^*$ is 2, while the partial average over uab^N is $\frac{2|u|+N}{|u|+N}$. Therefore, the value, which is limit infimum over partial averages, of word $ab^{n_1}c \ldots ab^{n_i}c \ldots$ is 1 if sequence $n_1, n_2, \ldots$ grows rapidly (e.g. doubly-exponentially).

Main ideas. In Example 11, the words attaining the infimum contain long blocks of letter b, at which the NWA $\mathbb{A}$ is (virtually) in the same state, i.e., it loops in this state. On letters b, the sum of all weights collected by all active slave automata is 2, i.e., automata $\mathfrak{B}_a, \mathfrak{B}_c$ collect weight 1 and $\mathfrak{B}_b$ collect 0. However, in computing limit infimum over partial averages, we pick positions just before letter c as they correspond to the local minima, i.e., we compute the partial average over prefixes uab^N, and hence the weights collected by $\mathfrak{B}_c$ do not contribute to this partial average. Then, the sum of all weights collected by slave automata $\mathfrak{B}_a, \mathfrak{B}_b$ over a letter b is 1, which is equal to the least value of the limit infimum of the partial averages. In the following, we extend this idea and present the solution for all bounded-width (LimAvg; Sum)-automata with bidirectional slave automata. We show that the infimum over all words of a given NWA is the least average value over all cycles. In the following, we define appropriate notions of cycles of NWA and their average with exclusion of some slave automata.

The graph of k-configurations. Let $\mathbb{A}$ be a non-deterministic (LimAvg; Sum)-automaton of width k. We define a *k-configuration* of $\mathbb{A}$ as a tuple $(q; q_1, \ldots, q_k)$ where q is a state of the master automaton, and each $q_1, \ldots, q_k$ is either a state of a slave automaton of $\mathbb{A}$ or $\perp$. Given a run of $\mathbb{A}$, we say that $(q; q_1, \ldots, q_k)$ is the k-configuration at position i in the run if q is the state of the master automaton at position i and there are $l \leq k$ active slave automata at position i, whose states are $q_1, \ldots, q_l$ ordered by position of invocation (backward-walking slave automata are invoked past position i). If $l < k$, then $q_{l+1}, \ldots, q_k = \perp$. We say that a k-configuration C_2 is a successor of a k-configuration C_1 if there exists an accepting run of $\mathbb{A}$ and $i > 0$ such that C_1 is the k-configuration at i and C_2 is the k-configuration at $i + 1$. The *graph of k-configurations* of $\mathbb{A}$ is the set of k-configurations of $\mathbb{A}$, which occur infinitely often in some accepting run, with the successor relation.

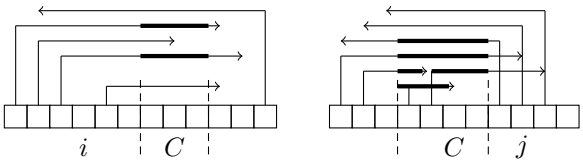


Figure 2 Pictorial explanation of $\textsc{Gain}(\mathcal{C}, FC)$ (on the left) and $\textsc{AvgE}(\mathcal{C}, R)$ (on the right). The gain $\textsc{Gain}(\mathcal{C}, FC)$ on the left is the sum of weights corresponding to thick parts of runs of slave automata invoked before i. The average $\textsc{AvgE}(\mathcal{C}, R)$ corresponds to the average of the thick parts of runs divided by the number of slave automata invoked within C. Slave automata invoked past j are excluded from the average.

Characteristics of cycles. Let $\mathcal{C}$ be a cycle in a graph of k-configurations of $\mathbb{A}$. Let F (resp., B) be the set of forward-walking (resp., backward-walking) slave automata, which are active throughout $\mathcal{C}$, i.e., which are not invoked nor terminated within $\mathcal{C}$. A *focus* Fc (for $\mathcal{C}$) is a downward closed subset of F, i.e., it contains all automata from F invoked before some position. We define a focused gain $\textsc{Gain}(\mathcal{C}, Fc)$ as the sum of weights which automata from Fc accumulate over $\mathcal{C}$. A *restriction* R (for $\mathcal{C}$) is an upward closed subset of B, i.e., it contains all automata from B invoked past certain position. We define an average weight of $\mathcal{C}$ excluding R, denoted by $\textsc{AvgE}(\mathcal{C}, R)$, as the sum of weights of all transitions of slave automata within $\mathcal{C}$, except of transitions of slave automata from R, divided by the number of slave automata invoked within $\mathcal{C}$.

Intuitively, a focused gain refers to the value, which forward-walking slave automata invoked before some position i, accumulate over the part of run corresponding to $\mathcal{C}$ (see Figure 2). If the focused gain $\textsc{Gain}(\mathcal{C}, Fc)$ is negative, then by pumping $\mathcal{C}$ we can arbitrarily decrease the partial average of the values of slave automata invoked before i. In consequence, we can construct a run of the value $-\infty$. Formally, we define condition (*), which implies that there exists a run of value $-\infty$, as follows: (*) there exists a cycle $\mathcal{C}$ in the graph of k-configurations of $\mathbb{A}$ and a focus Fc such that $\textsc{Gain}(\mathcal{C}, Fc) < 0$.

If the focused gain of every cycle is non-negative, we need to examine averages of cycles, while excluding some backward-walking slave automata. The average weight with restriction corresponds to the partial average of values aggregated over $\mathcal{C}$ by all slave automata invoked before position j (which can be past $\mathcal{C}$). Backward-walking slave automata in the restriction correspond to automata invoked past j, and hence their values do not contribute to the partial average (up to i) (see Figure 2). In Example 11, we compute the average of slave automata over letters b, but we exclude the backward-walking slave automaton invoked at the following letter c. Observe that for any cycle $\mathcal{C}$ and any restriction R, having a run containing $\mathcal{C}$ occurring infinitely often, we can repeat each occurrence of cycle $\mathcal{C}$ sufficiently many times so that the partial average of values of slave automata up to position corresponding to j becomes arbitrarily close to the average $\textsc{AvgE}(\mathcal{C}, R)$. The resulting run contains a subsequence of partial averages convergent to $\textsc{AvgE}(\mathcal{C}, R)$ and hence its value does not exceed $\textsc{AvgE}(\mathcal{C}, R)$. We can now state our key technical lemma. This lemma is a direct extension of an intuition behind computing the infimum over values of all words of the NWA $\mathbb{A}$ from Example 11.

▶ **Lemma 12.** *Let $\mathbb{A}$ be a $(\textsc{LimAvg}; \textsc{Sum})$-automaton of bounded width with bidirectional slave automata. (1) If condition (*) holds, then $\mathbb{A}$ has a run of value $-\infty$. (2) If (*) does not hold, then the infimum $\inf_w \mathbb{A}(w)$ equals the infimum $\inf_{\mathcal{C}\in\Lambda,R} \textsc{AvgE}(\mathcal{C}, R)$, where Λ is the set of all cycles $\mathcal{C}$ in the graph of k-configurations of $\mathbb{A}$.*

If the width of $\mathbb{A}$ is constant, then the graph of k-configurations has polynomial size in $|\mathbb{A}|$ and it can be constructed in polynomial time by employing reachability checks on the set

Table 1 The complexity of the emptiness problem for (LimAvg; g)-automata. The columns describe respectively: the value function g, restrictions imposed on the problem, the complexity in the case with bidirectional slave automata, and the complexity in the previously studied [13, 14] case with only forward-walking slave automata. Results presented in this paper are boldfaced.

Value func. g	Restrictions	Complexity Bidirectional	Complexity Forward case
Min,Max, $\text{Sum}^{L,U}$	None	**PSpace-complete** (Thm 5)	PSpace-complete [13]
Sum^{+}	finite width	**PSpace-hard ExpSpace** (Thm 8)	PSpace-hard ExpSpace [13]
Sum^{+}, Sum	constant width unary weights	**NLogSpace-complete** (Thm 13)	NLogSpace-complete [14]
Sum^{+}, Sum	constant width binary weights	**PTime** (Thm 13)	PTime [14]
Sum^{+}, Sum	width given in unary	**PSpace-complete** (Thm 13)	PSpace-complete [14]

of all k-configurations w.r.t. to relaxation of the successor relation. Therefore, for every focus Fc and every k-configuration c we can check in polynomial time whether there exists a cycle $\mathcal{C}$ such that $\mathcal{C}[1] = c$ and $\text{Gain}(\mathcal{C}, Fc) < 0$. Thus, condition (1) can be check in logarithmic space assuming that weights are given in unary. If weights are given in binary, condition (1) can be checked in polynomial time. Checking condition (2) has the same complexity as condition (1). If the width k is given in unary in input, the graph of k-configurations is exponential in $|\mathbb{A}|$ and conditions (1) and (2) can be checked in polynomial space. Weights in this case are logarithmic in the size of the graph and hence changing representation from binary to unary does not affect the (asymptotic) size of the graph.

▶ **Theorem 13.** *The emptiness problem for* (LimAvg; Sum)*-automaton of width k with bidirectional slave automata is (a)* NLogSpace*-complete for constant k and weights given in unary, (b) in* PTime *for constant k and weights given in binary, and (c)* PSpace*-complete for k given in unary.*

7 Discussion and Conclusion

Discussion. We established decidability of the emptiness problem for classes of bidirectional NWA, which include all NWA presented in the examples. An NWA from Example 2 is covered by Theorem 5, while NWA from Examples 1, 3 and 4 are covered by Theorem 8. The lower bounds in our results follow from the lower bounds of the special case of forward-only NWA. The established complexity (Table 1) coincide with the forward-only case.

Concluding remarks. In this work we present bidirectional NWA as a specification formalism for quantitative properties. There are several interesting directions for future work. The study of bidirectional NWA with other value functions is an interesting direction. The second direction of future work is to consider other formalism (such as a logical framework) which has the same expressive power as bidirectional NWA.

References

1 Shaull Almagor, Udi Boker, and Orna Kupferman. Discounting in LTL. In *TACAS, 2014*, pages 424–439, 2014.

2 Shaull Almagor, Udi Boker, and Orna Kupferman. Formally reasoning about quality. *J. ACM*, 63(3):24:1–24:56, 2016. doi:10.1145/2875421.

3 Rajeev Alur, Loris D'Antoni, Jyotirmoy V. Deshmukh, Mukund Raghothaman, and Yifei Yuan. Regular functions and cost register automata. In *LICS 2013*, pages 13–22, 2013.

4 Mikołaj Bojańczyk and Thomas Colcombet. Bounds in w-regularity. In *21th IEEE Symposium on Logic in Computer Science (LICS 2006), 12-15 August 2006, Seattle, WA, USA, Proceedings*, pages 285–296, 2006. doi:10.1109/LICS.2006.17.

5 Udi Boker, Krishnendu Chatterjee, Thomas A. Henzinger, and Orna Kupferman. Temporal specifications with accumulative values. *ACM TOCL*, 15(4):27:1–27:25, 2014.

6 Benedikt Bollig, Paul Gastin, Benjamin Monmege, and Marc Zeitoun. Pebble weighted automata and transitive closure logics. In *ICALP 2010, Part II*, pages 587–598. Springer, 2010.

7 Patricia Bouyer, Nicolas Markey, and Raj Mohan Matteplackel. Averaging in LTL. In *CONCUR 2014*, pages 266–280, 2014.

8 Patricia Bouyer, Nicolas Markey, Mickael Randour, Kim Guldstrand Larsen, and Simon Laursen. Average-energy games. In *GandALF 2015.*, pages 1–15, 2015. doi:10.4204/EPTCS.193.1.

9 Krishnendu Chatterjee, Laurent Doyen, Herbert Edelsbrunner, Thomas A. Henzinger, and Philippe Rannou. Mean-payoff automaton expressions. In *CONCUR*, pages 269–283, 2010. doi:10.1007/978-3-642-15375-4_19.

10 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Alternating weighted automata. In *FCT'09*, pages 3–13. Springer, 2009.

11 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Expressiveness and closure properties for quantitative languages. *LMCS*, 6(3), 2010.

12 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Quantitative languages. *ACM TOCL*, 11(4):23, 2010.

13 Krishnendu Chatterjee, Thomas A. Henzinger, and Jan Otop. Nested weighted automata. In *LICS 2015*, pages 725–737, 2015.

14 Krishnendu Chatterjee, Thomas A. Henzinger, and Jan Otop. Nested weighted limit-average automata of bounded width. In *MFCS 2016*, pages 24:1–24:14, 2016. doi:10.4230/LIPIcs.MFCS.2016.24.

15 Krishnendu Chatterjee, Thomas A. Henzinger, and Jan Otop. Quantitative automata under probabilistic semantics. In *LICS 2016*, pages 76–85, 2016. doi:10.1145/2933575.2933588.

16 Krishnendu Chatterjee, Thomas A. Henzinger, and Jan Otop. Quantitative monitor automata. In *SAS 2016*, pages 23–38, 2016. doi:10.1007/978-3-662-53413-7_2.

17 Krishnendu Chatterjee, Thomas A. Henzinger, and Jan Otop. Bidirectional nested weighted automata. *CoRR*, abs/1706.08316, 2017. URL: http://arxiv.org/abs/1706.08316.

18 Krishnendu Chatterjee and Vinayak S. Prabhu. Quantitative temporal simulation and refinement distances for timed systems. *IEEE Trans. Automat. Contr.*, 60(9):2291–2306, 2015. doi:10.1109/TAC.2015.2404612.

19 Manfred Droste, Werner Kuich, and Heiko Vogler. *Handbook of Weighted Automata*. Springer, 1st edition, 2009.

20 Manfred Droste and George Rahonis. Weighted automata and weighted logics on infinite words. In *DLT 2006*, pages 49–58, 2006.

21 Filip Mazowiecki and Cristian Riveros. Copyless cost-register automata: Structure, expressiveness, and closure properties. In *STACS, 2016*, pages 53:1–53:13, 2016. doi:10.4230/LIPIcs.STACS.2016.53.

22 Mehryar Mohri. Semiring frameworks and algorithms for shortest-distance problems. *J. Aut. Lang. & Comb.*, 7(3):321–350, 2002.

k-Bounded Petri Net Synthesis from Modal Transition Systems*

Uli Schlachter[1] and Harro Wimmel[2]

1 **Department of Computing Science, Carl von Ossietzky Universität, Oldenburg, Germany**
uli.schlachter@informatik.uni-oldenburg.de

2 **Department of Computing Science, Carl von Ossietzky Universität, Oldenburg, Germany**
harro.wimmel@informatik.uni-oldenburg.de

Abstract

We present a goal-oriented algorithm that can synthesise k-bounded Petri nets ($k \in \mathbb{N}^+$) from hyper modal transition systems (hMTS), an extension of labelled transition systems with optional and required behaviour. The algorithm builds a potential reachability graph of a Petri net from scratch, extending it stepwise with required behaviour from the given MTS and over-approximating the result to a new valid reachability graph. Termination occurs if either the MTS yields no additional requirements or the resulting net of the second step shows a conflict with the behaviour allowed by the MTS, making it non-sythesisable.

1998 ACM Subject Classification D.2.2 Design Tools and Techniques, F.4.1 Mathematical Logic

Keywords and phrases Modal transition system, bounded Petri net, synthesis

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.6

1 Introduction

Petri net synthesis, or more precisely, the problem of finding an unlabelled Petri net implementing a given labelled transition system, goes back to Ehrenfeucht and Rozenberg [13] and was recently comprehensively presented by Badouel, Bernardinello and Darondeau [5]. Petri net synthesis not only yields implementations which are correct by design, but also allows to extract concurrency and distributability information from a sequential specification [6, 8, 21].

Modal transition systems (MTS) are a well-known and useful method for specifying systems [17, 1, 9, 16]. They are an extension of labelled transition systems (LTS) that can distinguish between required behaviour (must edges) and optional behaviour (may edges). Hyper MTS (called disjunctive MTS in [18]) are a further extension, where must edges are defined as hyper edges, requiring at least one out of (possibly) several actions. An LTS implements a (hyper) MTS if it allows the required behaviour and disallows any not specified (optional or required) behaviour, i.e. there may be more than one LTS implementing the same MTS.

Since modal transition systems are extensions of labelled transition systems, it has been suggested to extend Petri net synthesis to cover modal transition systems [12, 5]. The question here is if a given modal transition system can be realised by a Petri net, which means that the reachability graph of a Petri net can implement the modal transition system,

* This work is supported by the German Research Foundation (DFG) projects ARS and ASYST, reference numbers Be 1267/15-1 and Be 1267/16-1.

and if this is answered positively, to produce such a Petri net. Recent results show that this question is undecidable even when asking for pure Petri nets [14] or for bounded Petri nets [20].

We will easily see that for a given $k \in \mathbb{N}^+$ the synthesis of a k-bounded Petri net from some MTS (or some other specification) is decidable, since there is only a finite number of different reachability graphs available for k-bounded Petri nets with a fixed set of transitions. This is, to our knowledge, the first general positive decidability result for Petri net synthesis from modal specifications. Such a brute force approach is not advisable, though, due to the state space explosion with growing k and number of transitions.

Instead, we introduce a goal-oriented algorithm for synthesising k-bounded Petri nets from hyper MTS, where the value k is an input to the algorithm. Our algorithm iteratively applies two operations until a fixed point is reached. The first operation adds missing edges so that its input implements a given modal transition system. The second operation produces minimal Petri net solvable over-approximations of otherwise Petri net unsolvable labelled transition systems. The algorithm will construct all minimal Petri net realisations of a given modal transition system, where minimality is defined with respect to an LTS homomorphism preorder, similar to the simulation preorder. This preorder implies language inclusion, which means that the algorithm also calculates minimal realisations with respect to language inclusion.

The paper is organised as follows: In Section 2 we present the basic terminology regarding LTS and Petri nets, including LTS homomorphisms, the construction of k-bounded regions, and the synthesis of k-bounded Petri nets from such regions. In Section 3 we introduce (hyper) MTS and their relation to LTS. Section 4 contains our main algorithm including some examples. In Section 5 we prove termination and correctness of our algorithm. Section 6 contains some thought on the algorithmic complexity of the algorithm and how to implement it.

2 Petri Net Synthesis

▶ **Definition 1.** A *labelled transition system (LTS)* A is a structure $A = (Q, \Sigma, \to, q_0)$ where Q is a set of *states*, Σ is a set of *actions*, $q_0 \in Q$ is the *initial state* and $\to \subseteq Q \times \Sigma \times Q$ is a set of (action-)labelled *edges*. A tuple $(q, a, q') \in \to$ is also written as $q \xrightarrow{a} q'$ and $q \xrightarrow{a}$ expresses that some q' with $q \xrightarrow{a} q'$ exists. This notation is canonically extended to words $w \in \Sigma^*$ by inductively defining $q \xrightarrow{\varepsilon} q$ for all $q \in Q$ and $q \xrightarrow{wa} q' \iff \exists q'' \colon q \xrightarrow{w} q'' \wedge q'' \xrightarrow{a} q'$. An LTS is *deterministic* if it satisfies $\forall (q, a, q'), (q, a, q'') \in \to \colon q' = q''$. It is called *totally reachable* if $\forall q \in Q \colon \exists w \in \Sigma^* \colon q_0 \xrightarrow{w} q$. It is called *finite* if Q and Σ (and hence also $\to$) are finite. The language $L(A)$ of an LTS A is $L(A) := \{w \in \Sigma^* \mid q_0 \xrightarrow{w} \}$.

Given two LTS $A_i = (Q_i, \Sigma, \to_i, q_{0i})$ with $i \in \{1, 2\}$, an *LTS homomorphism from* A_1 *to* A_2 is a function $f \colon Q_1 \to Q_2$ with $f(q_{01}) = q_{02}$ such that $q \xrightarrow{a} q'$ implies $f(q) \xrightarrow{a} f(q')$ for all $q, q' \in Q_1$ and $a \in \Sigma$. If such a homomorphism f exists we write $A_1 \sqsubseteq A_2$ (via f). $\sqsubseteq$ is reflexive (with the identity homomorphism id) and transitive (as homomorphisms are closed under composition). A bijection f is an *isomorphism* if both, f and f^{-1}, are homomorphisms, in which case we abstract from state names of the isomorphic LTS A_1 and A_2 and identify them, writing $A_1 = A_2$.

▶ Remark. Unless given explicitly, the components of an LTS A will be named canonically as $A = (Q_A, \Sigma_A, \to_A, q_{0,A})$. The same notation will be used for other structures. For example, the set of places of a Petri net N, which will be defined later, is referred to as P_N. For

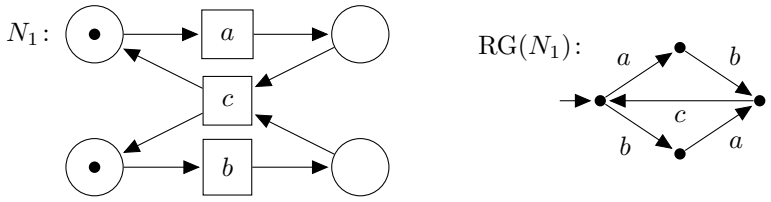


Figure 1 Example of a Petri net and its reachability graph. In the Petri net, places are drawn as circles containing tokens indicating the initial marking. Transitions are drawn as rectangles containing the name of the transition. The reachability graph has states shown as circles where the initial state is indicated by an arrow tip. Edges are drawn as arrows with the edge label next to it.

simplicity, the subscript will be left out if it is clear from the context. This last point will mainly be used for edges, e.g. for $(q, a, q') \in \rightarrow_A$ we might write $q \xrightarrow{a} q'$ instead of $q \xrightarrow{a}_A q'$.

▶ **Lemma 2.** *Let A and B be LTS so that $A \sqsubseteq B$ via f and $A \sqsubseteq B$ via f'. If A is totally reachable and B is deterministic, then $f = f'$.*

Proof. We prove by induction on the length of words $w \in L(A)$ that if $q_{0,A} \xrightarrow{w} q$, then $f(q) = f'(q)$. Since A is totally reachable, we reach all states of A in this way, showing that $f(q) = f'(q)$ for all $q \in Q_A$. The induction basis shows the conclusion for the initial state by applying the definition of $\sqsubseteq$: $f(q_{0,A}) = q_{0,B} = f'(q_{0,A})$.

For the induction step, assume that $q_{0,A} \xrightarrow{w} q$ with $f(q) = f'(q)$ and consider any $q \xrightarrow{a} q'$. Since $A \sqsubseteq B$ via f and f', we have $f(q) \xrightarrow{a} f(q')$ and $f'(q) \xrightarrow{a} f'(q')$. However, because B is deterministic, there cannot be two different states that are reached from $f(q) = f'(q)$ via label a. We conclude $f(q') = f'(q')$. ◀

▶ **Lemma 3.** *Let A and B be totally reachable and deterministic LTS so that $A \sqsubseteq B$ via f_A and $B \sqsubseteq A$ via f_B. Then $A = B$.*

Proof. Consider $A \sqsubseteq A$ (via id) and $A \sqsubseteq B \sqsubseteq A$ via $f_B \circ f_A$. Since Lemma 2 is applicable, we conclude $f_B \circ f_A = \mathrm{id}$, making f_A injective and f_B surjective. With an analogous argument for $f_A \circ f_B = \mathrm{id}$ we see that f_A and f_B are both bijective homomorphisms with $f_A^{-1} = f_B$, i.e. they are isomorphisms. Abstracting from state names for the isomorphic A and B, we may write $A = B$. ◀

▶ **Definition 4.** A *Petri net* is a tuple $N = (P, T, F, M_0)$ where P and T are finite and disjoint sets of *places* and *transitions*, respectively, and $F\colon ((P \times T) \cup (T \times P)) \to \mathbb{N}$ is a *flow relation* specifying *arc weights*. M_0 is the *initial marking* where a *marking* is a mapping $P \to \mathbb{N}$. A transition $t \in T$ is *enabled* in a marking M iff $\forall p \in P\colon F(p,t) \leq M(p)$. An enabled transition can *fire* leading to the marking M' defined by $M'(p) = M(p) - F(p,t) + F(t,p)$. We write $M[t\rangle M'$ and this syntax is inductively extended to label sequences $\sigma \in T^*$. The set of *reachable markings* of a Petri net N is $\mathcal{E}(N) = \{M \mid \exists \sigma \in T^*\colon M_0[\sigma\rangle M\}$. A Petri net is called *k-bounded by $k \in \mathbb{N}$* if $\forall M \in \mathcal{E}(N),\ p \in P\colon M(p) \leq k$. There is a *self-loop* around $p \in P$ and $t \in T$ if $F(p,t) > 0 \wedge F(t,p) > 0$. A Petri net is called *pure* if it does not contain any self-loops. The *reachability graph of N* is the LTS $\mathrm{RG}(N) = (\mathcal{E}(N), T, \rightarrow, M_0)$ with $\rightarrow = \{(M, t, M') \in \mathcal{E}(N) \times T \times \mathcal{E}(N) \mid M[t\rangle M'\}$. An LTS A is called *Petri net solvable* if a Petri net N exists with $A = \mathrm{RG}(N)$. In this situation, N *solves* A.

An example of a Petri net and its reachability graph is shown in Figure 1. The following lemma can easily be verified:

▶ **Lemma 5.** *For a Petri net N, $\mathrm{RG}(N)$ is totally reachable and deterministic. If N is k-bounded (for some $k \in \mathbb{N}^+$), $\mathrm{RG}(N)$ is finite.*

The next lemma shows that there are only finitely many LTS solvable via k-bounded Petri nets. As an immediate consequence, if we can decide whether the reachability graph of a specific k-bounded Petri net "implements" some specification, we can also decide whether any k-bounded Petri net fulfils this, i.e. brute force decisions are possible.

▶ **Lemma 6.** *For any given $k \in \mathbb{N}^+$ and finite set of transitions T, the number of structurally different reachability graphs of k-bounded Petri nets with transitions from T is finite.*

Proof. A weight $F(p,t) > k$ or $F(t,p) > k$ in a k-bounded Petri net means that the transition t can never fire (or the bound would be violated). With weights in $\{0,\dots,k\}$, a place can only be connected in finitely many ways to all the transitions. Exact duplicates of places do not have an effect on the structure of the reachability graph. ◀

A place p of a Petri net has a number of tokens $M(p)$ in each reachable marking M. The next definition formalises this concept on the reachability graph as a so-called region [7, 10, 11].

▶ **Definition 7.** Let $k \in \mathbb{N}^+$. A *k-bounded region of a finite*[1] *LTS A* is a function $r\colon Q_A \to \{0,1,\dots,k\}$ satisfying $\forall a \in \Sigma_A\colon \forall q \xrightarrow{a} q', q'' \xrightarrow{a} q'''\colon r(q') - r(q) = r(q''') - r(q'')$, which means that edges with the same label have the same *gradient*, where the *gradient of $a \in \Sigma_A$* is $\Delta_r(a) = r(q') - r(q)$ for any edge $q \xrightarrow{a} q'$. The value $\mu_r(a) = \min\{r(q) \mid q \xrightarrow{a}\}$ is the minimum value $r(q)$ of a state q with an outgoing edge with label a (generally assuming that every $a \in \Sigma_A$ occurs as a label of some edge).

A region corresponds to a possible place p of a Petri net with initial marking $M_0(p) = r(q_0)$ and flow relation $F(p,t) = \mu_r(t)$, $F(t,p) = \Delta_r(t) + \mu_r(t)$. This correspondence allows to define the Petri net $N(R)$ generated by a set R of regions of an LTS A via $N(R) = (R, \Sigma_A, F, M_0)$ with M_0 and F as above.

This definition adds the maximum possible number of self-loops to the generated place p, because this restricts the behaviour of the Petri net the most. Thus, places with fewer self-loops cannot be generated in this setting, but also will not be needed.

▶ **Lemma 8.** *Let A and B be finite LTS so that $A \sqsubseteq B$ via f. If r is a region of B, then $r' = r \circ f$ is a region of A.*

Proof. Let $q \xrightarrow{a} q'$ and $\widetilde{q} \xrightarrow{a} \widetilde{q}'$ be two edges of A. Then, $f(q) \xrightarrow{a} f(q')$ and $f(\widetilde{q}) \xrightarrow{a} f(\widetilde{q}')$ are edges in B and by the uniqueness of the gradient we get $r'(q') - r'(q) = r(f(q')) - r(f(q)) = r(f(\widetilde{q}')) - r(f(\widetilde{q})) = r'(\widetilde{q}') - r'(\widetilde{q})$. ◀

While r and r' have the same gradient $\Delta_r = \Delta_{r'}$, the values for μ_r and $\mu_{r'}$ may differ, i.e. the places of a Petri net constructed from these regions may have different self-loops.

By definition, there are only a finite number of k-bounded regions. We can use all of them and define a Petri net:

▶ **Definition 9.** For a finite LTS A and a number $k \in \mathbb{N}^+$, let $\mathrm{SN}_k(A)$ be the Petri net generated from all k-bounded regions of A.

[1] We exclude infinite LTS since they lead to unbounded Petri nets, for which the synthesis from modal transition systems is known to be undecidable [14].

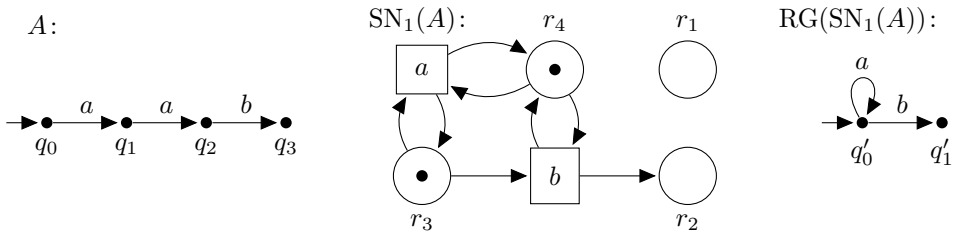


Figure 2 Example for $\mathrm{SN}_k(A)$ and the resulting reachability graph (on the right).

An example for this construction with $k = 1$ is shown in Figure 2. The LTS A has only four regions r_1 to r_4. We identify a region r with the vector $(r(q_0), r(q_1), r(q_2), r(q_3))$:

$$r_1 = (0,0,0,0) \qquad r_2 = (0,0,0,1) \qquad r_3 = (1,1,1,0) \qquad r_4 = (1,1,1,1)$$

We also identify the gradient Δ_r and minimum values μ_r of r with the vector $(\Delta_r(a), \Delta_r(b))$, respectively $(\mu_r(a), \mu_r(b))$:

$$\begin{array}{llll} \Delta_{r_1} = (0,0) & \Delta_{r_2} = (0,1) & \Delta_{r_3} = (0,-1) & \Delta_{r_4} = (0,0) \\ \mu_{r_1} = (0,0) & \mu_{r_2} = (0,0) & \mu_{r_3} = (1,1) & \mu_{r_4} = (1,1) \end{array}$$

The event a always has a gradient of $\Delta_r(a) = 0$, since otherwise at least one of the values $r(q_0)$, $r(q_1)$, or $r(q_2)$ would need to be outside of $\{0,1\}$. For the event b, we can have $\Delta_r(b) \in \{-1,0,1\}$. Altogether, A has four regions which correspond to the places of the Petri net shown in the middle of Figure 2. As can be seen in the reachability graph, the event a forms a loop around the initial state.

▶ **Lemma 10.** *For any $k \in \mathbb{N}^+$ and finite LTS A, the Petri net $\mathrm{SN}_k(A)$ is k-bounded.*

Proof. Assume a reachable marking M where some place p contains more than k tokens, i.e. $M(p) > k$. We use the well-known idea of complement places: With r being the region corresponding to p, define a new region r' via $r'(q) = k - r(q)$. Clearly, r' is also a region of A and corresponds, by definition, to some place p' of $\mathrm{SN}_k(A)$, which is built from all regions of A. Also, for every event $a \in \Sigma_A$ we have $\Delta_r(a) + \Delta_{r'}(a) = 0$ and for every state q we have $r(q) + r'(q) = k$. This also holds for the corresponding places: for any reachable marking M', $M'(p) + M'(p') = k$. Since $M(p) > k$ we find $M(p') < 0$. This contradicts M being a (reachable) marking. ◀

Next, we want to show that $\mathrm{SN}_k(A)$ is the smallest Petri net over-approximation of A. This means that any other Petri net which is an over-approximation of A will also be an over-approximation of the reachability graph of $\mathrm{SN}_k(A)$.

▶ **Theorem 11.** *Given a number $k \in \mathbb{N}^+$ and a finite LTS A, we have that $A \sqsubseteq \mathrm{RG}(\mathrm{SN}_k(A))$ and for all k-bounded Petri nets N with $A \sqsubseteq \mathrm{RG}(N)$, also $\mathrm{RG}(\mathrm{SN}_k(A)) \sqsubseteq \mathrm{RG}(N)$.*

Proof. It is clear that $A \sqsubseteq \mathrm{RG}(\mathrm{SN}_k(A))$, because $\mathrm{SN}_k(A)$ is defined in such a way that it does not prevent edges present in A ($q \xrightarrow{t} \Rightarrow r(q) \geq \mu_r(t) = F(p,t)$).

Then, for any N with $A \sqsubseteq \mathrm{RG}(N)$ via f, each place of N corresponds to a region r on $\mathrm{RG}(N)$. This region can be translated into a region $r \circ f$ of A via Lemma 8, which by

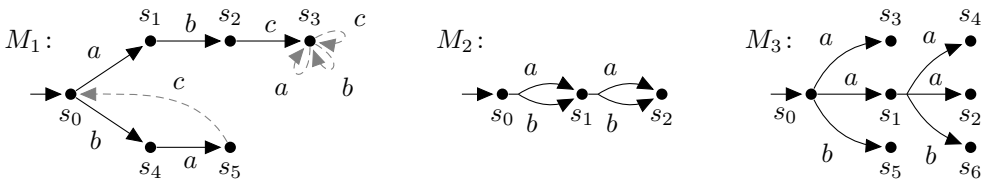


Figure 3 Three different hMTS. Initial states are marked with an arrow tip. Must edges are drawn as solid lines and also indicate may edges as required by the definition. For example, $(s_0, \{(a, s_1), (b, s_1)\}) \in \longrightarrow$ for M_2. Non-must may edges are shown with dashed lines. M_1 is a deterministic MTS, M_2 a deterministic hMTS, M_3 a non-deterministic hMTS.

definition corresponds to a place of $\mathrm{SN}_k(A)$, yielding a mapping of places[2] $\alpha\colon P_N \to P_{\mathrm{SN}_k(A)}$. We can now define a function β that maps markings of $\mathrm{SN}_k(A)$ to markings of N via $\beta(M)(p) = M(\alpha(p))$, i.e. a place p of N is assigned the same number of tokens as its corresponding place $\alpha(p)$ of $\mathrm{SN}_k(A)$ has in M. We can easily verify $M[t\rangle M' \Rightarrow \beta(M)[t\rangle\beta(M')$, i.e. we have $\mathrm{RG}(\mathrm{SN}_k(A)) \sqsubseteq \mathrm{RG}(N)$ via β. ◀

3 Modal Transition Systems

Hyper modal transition systems [18] represent classes of LTS. They allow for multiple initial states and have two different classes of edges. May edges allow an action while must edges require an action. Compared to classical modal transition systems [17], must edges are hyper edges that require at least one out of several possible actions, while in a standard modal transition system no such logical combinations are expressible. A hyper modal transition system is similar to an acceptance specification [19], but its must edges can be considered to be in conjunctive normal form while an acceptance specification uses the disjunctive normal form.

▶ **Definition 12.** A *hyper modal transition system (hMTS)* M is a structure $M = (S, \Sigma, \longrightarrow, \dashrightarrow, S_0)$ where S is a finite set of *states*, Σ is a set of *actions*, $\emptyset \neq S_0 \subseteq S$ is the set of *initial states*, $\dashrightarrow \subseteq S \times \Sigma \times S$ is the set of *may edges* and $\longrightarrow \subseteq S \times (2^{\Sigma \times S} \setminus \emptyset)$ is the set of *must edges* satisfying $\forall (s, D) \in \longrightarrow\colon \forall (a, s') \in D\colon (s, a, s') \in \dashrightarrow$. The syntax $s \overset{w}{\dashrightarrow} s'$ is defined just like $q \overset{w}{\longrightarrow} q'$ was defined for LTS. An hMTS is called *deterministic* iff $S_0 = \{s_0\}$ is a singleton and the LTS $(S, \Sigma, \dashrightarrow, s_0)$ is deterministic. A *modal transition system (MTS)* is an hMTS with $|S_0| = 1$ and $\forall (s, D) \in \longrightarrow\colon |D| = 1$.

As an example, three hMTS are shown in Figure 3. In M_1, the lower path forms a *bac*-loop that can be terminated after an a, the upper path requires *abc*, after which anything is allowed. M_2 contains two hyper edges requiring at least either an a or a b, but not necessarily both. In an LTS meeting the specification of M_3, the three must edges at s_0 demand that both a and b must be available actions, while the hyper edge at s_1 only requires that one of a and b is possible.

The following definition introduces how an LTS implements an hMTS. This implementation relation is similar to a bisimulation, but may and must edges are not treated the same.

[2] A mapped place $\alpha(p)$ has the same initial marking and transitions of $\mathrm{SN}_k(A)$ have the same effect $\Delta_r = \Delta_{r \circ f}$ on it as on p, but self-loops might be added (if $\mu_{r \circ f}(t) > \mu_r(t)$ due to edges not existing in $\mathrm{RG}(\mathrm{SN}_k(A))$ but in $\mathrm{RG}(N)$).

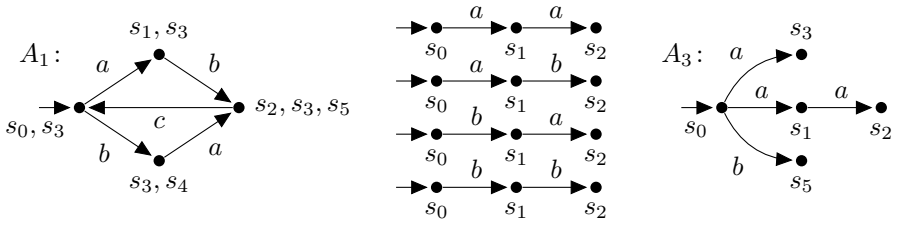


Figure 4 Implementations of the hMTS from Figure 3: A_1 implements M_1, A_3 implements M_3, and the four LTS in the middle are minimal implementations of M_2. A_1 is solved by N_1 from Figure 1. States are named according to the implementation relation R from Definition 13.

A must edge specifies a disjunction of edges, at least one of which must be present, while a may edge defines that some edge is allowed.

▶ **Definition 13.** An LTS A is an *implementation via* $R \subseteq Q_A \times S_M$ of an hMTS M if there is an $s_0 \in S_{0,M}$ with $(q_{0,A}, s_0) \in R$ and for all $(q, s) \in R$:

- $\forall q \xrightarrow{a} q' \colon \exists s \overset{a}{\dashrightarrow} s' \colon (q', s') \in R$,
- $\forall (s, D) \in \longrightarrow \colon \exists (a, s') \in D, q \xrightarrow{a} q' \colon (q', s') \in R$.

If A is an implementation of M via some relation R, we call R an *implementation relation* and write $A \models M$ (via R). If RG(N) is an implementation of M for some Petri net N, we also say that N *realises* M.

Examples for this definition are given in Figure 4. The LTS A_1 is an implementation of M_1 from Figure 3. An implementation relation R can be built as follows. Initially, $(\{s_0, s_3\}, s_0) \in R$. The lower path of M_1 adds first $(\{s_3, s_4\}, s_4)$ and then $(\{s_2, s_3, s_5\}, s_5)$ to R. The must edges of the upper path require $(\{s_1, s_3\}, s_1)$, $(\{s_2, s_3, s_5\}, s_2)$, and $(\{s_0, s_3\}, s_3)$. In the state s_3, all edges are allowed and loop back to s_3, i.e. all states reachable from $\{s_0, s_3\}$ must also be related to s_3: $(\{s_1, s_3\}, s_3), (\{s_3, s_4\}, s_3), (\{s_2, s_3, s_5\}, s_3) \in R$. Thus, the state names of the LTS A_1 show exactly the implementation relation for $A_1 \models M_1$. If we remove in M_1 the may edge labelled with c at s_5, A_1 would not be an implementation any more: $\{s_2, s_3, s_5\}$ must admit a c (forced by the upper path in M_1) and forbid a c (using the lower path) at the same time. We could split up the state $\{s_2, s_3, s_5\}$ into $\{s_2, s_3\}$ and $\{s_5\}$, but then no Petri net could solve the LTS, since the paths ab and ba from the initial state must lead to the same marking[3].

The four LTS in the middle of Figure 4 are all valid implementations of M_2, and they are all minimal according to the LTS homomorphism $\sqsubseteq$, which is a partial order for deterministic and totally reachable LTS according to Lemma 3. There are larger implementations admitting a and b at either s_0 or s_1 (or both).

The LTS A_3 is a minimal implementation of M_3. The states s_1, s_3, and s_5 are forced by the implementation relation; the hyper edge at s_1 allows a choice: at least one of the targets of the hyper edge must exist. Here, we chose s_2. Note that the states s_1 and s_3 cannot be identified in spite of the non-deterministic a-edges with the same source s_0. The implementation relation would contain a pair $(\{s_1, s_3\}, s_3)$ which would in consequence – due to the edge $\{s_1, s_3\} \xrightarrow{a} s_2$ – require s_3 to have at least a may edge in M_3. Since reachability graphs of Petri nets are deterministic, no LTS implementing M_3 can be solved by a Petri

[3] This is due to the well-known Petri net state equation $M[w\rangle M' \Rightarrow M' = M + C \cdot \Psi(w)$ where $C \in \mathbb{Z}^{P \times N}$ with $C(p,t) = F(t,p) - F(p,t)$. This equation only considers the number of times $\Psi(w)(t)$ that event $t \in T$ occurs in the word $w \in T^*$, and not the order of appearances.

net. However, if we omit the edge $(s_0, \{(a, s_3)\}) \in \longrightarrow$ in M_3, Petri net realisations become possible. There are (up to isomorphism) nine different deterministic LTS implementing the modified M_3, five of which can be solved by Petri nets. They differ in whether a, b, or both are possible in s_1, and in the number of states without outgoing edges.

4 Goal-oriented Synthesis

In this section, we will introduce the algorithm that calculates Petri net realisations for hyper modal transition systems. The algorithm works by iteratively adding unimplemented must edges to an LTS and doing minimal Petri net over-approximations. For identifying unimplemented must edges, another kind of relation is needed, which we will define next.

▶ **Definition 14.** Let A be an LTS and M be an hMTS. A relation $R \subseteq Q_A \times S_M$ is called an *expansion relation* if there is a state $s_0 \in S_{0,M}$ with $(q_{0,A}, s_0) \in R$ and for all $(q, s) \in R$ and all $q \xrightarrow{a} q'$, there is an s' with $s \xrightarrow{a} s'$ and $(q', s') \in R$.

Let $R_{\mathrm{E}}(A, M)$ be the set of all expansion relations of A and M. Given LTS A and B, an expansion relation $R \in R_{\mathrm{E}}(A, M)$, and an LTS homomorphism f witnessing $A \sqsubseteq B$, the relation $f(R)$ is defined as $f(R) = \{(f(q), s) \in Q_B \times S_M \mid (q, s) \in R\}$.

Expansion relations are a weaker variant of implementation relations, i.e. any implementation relation is an expansion relation with an additional condition for the must edges.

Our algorithm for realising an hMTS is Algorithm 1. The idea behind the algorithm is to iteratively enlarge a current LTS towards becoming an implementation. The first operation for this is to add edges to the LTS so that currently unimplemented must edges become implemented (Expand). The second operation is the minimal Petri net over-approximation (PNApprox). The algorithm begins in the procedure RealiseMTS, which gets as input the bound $k \in \mathbb{N}^+$ for the Petri nets and the hMTS M to be realised. The first LTS to be considered is the minimal LTS which only has an initial state and no edges. This state is related to each possible initial state of M and the procedure Recurse is called for each such combination.

Recurse identifies unimplemented must edges by negating the corresponding formula in the definition of an implementation relation. If no must edge is missing, a solution is found and returned. Otherwise, the procedure Expand is invoked which will add new states and edges to implement the missing must edges. This procedure returns a set, because, for example, for $m = \{(q, \{(a, s_1), (b, s_2)\})\}$, two LTS are created: In one of them, an edge $q \xrightarrow{a} q_{new}$ is created, while the other one gets the new edge $q \xrightarrow{b} q_{new}$. In the first case, (q_{new}, s_1) is added to the expansion relation R_A while in the second case (q_{new}, s_2) is added. This means the relation R_A is enlarged to keep note on which state of the hMTS the new state should implement. Expand is invoked recursively to obtain all combinations of selections for the missing must edges. In MTS must edges allow no choice, therefore Expand will return one unique result; in hMTS a high number of choices in must edges could lead to an exponential growth, though.

For each result of Expand, the procedure PNApprox is called, yielding a minimal Petri net over-approximation of its argument. As was seen in Figure 2, this can greatly influence the shape of the LTS under consideration. The relation R_A has to be updated to this new LTS. To do so, PNApprox computes a homomorphism witnessing $A \sqsubseteq B$ in line 23. Such a homomorphism exists by Theorem 11. It is unique by Lemma 2, because A is totally reachable by construction and B is deterministic by Lemma 5. This homomorphism f is then used to produce the relation $f(R_A)$ which relates B to M. The set E of all expansion relations

Algorithm 1 Algorithm for finding Petri net realisations for an hMTS.

procedure REALISEMTS(k, M) ▷ $k \in \mathbb{N}^+$ and M is an hMTS
  Let A be the LTS consisting of just an initial state q_0
  return $\bigcup_{s_0 \in S_{0,M}}$ RECURSE($k, A, \{(q_0, s_0)\}, M$)
end procedure
procedure RECURSE(k, A, R_A, M) ▷ R_A is an expansion relation
  $m = \bigcup_{(q,s) \in R_A} m_{q,s}$ where ▷ m collects missing must edges
  $m_{q,s} = \{(q, D) \in Q_A \times (2^{\Sigma \times S_M} \setminus \emptyset) \mid (s, D) \in \longrightarrow_M, \forall (a, s') \in D:$
    $\neg \exists q' \in Q_A : (q \xrightarrow{a} q' \wedge (q', s') \in R_A)\}$
  if $m = \emptyset$ **then return** $\{(A, R_A)\}$ **end if**
  return $\bigcup_{(B,R_B) \in \text{EXPAND}(A,R_A,m)} \bigcup_{(C,R_C) \in \text{PNAPPROX}(k,B,R_B,M)}$ RECURSE(k, C, R_C, M)
end procedure
procedure EXPAND(A, R_A, m)
  if $m = \emptyset$ **then return** $\{(A, R_A)\}$ **end if**
  Select some $(q, D) \in m$, add a new state q_{new} to A and set result $\leftarrow \emptyset$
  for $(a, s') \in D$ **do** ▷ Implement missing must edge
    Let B be a copy of A with an additional edge $q \xrightarrow{a} q_{new}$
    result $\leftarrow$ result $\cup$ EXPAND($B, R_A \cup \{(q_{new}, s')\}, m \setminus \{(q, D)\}$)
  end for
  return result
end procedure
procedure PNAPPROX(k, A, R_A, M)
  $B = \text{RG}(\text{SN}_k(A))$ ▷ Minimal over-approximation
  $f =$ homomorphism witnessing $A \sqsubseteq B$
  $E = \{R_B \in R_E(B, M) \mid f(R_A) \subseteq R_B\}$ ▷ All expansion relations containing $f(R_A)$
  return $\{(B, R_B) \mid R_B \in E \wedge \neg \exists R' \in E : R' \subsetneq R_B\}$ ▷ Select minimal relations
end procedure

containing $f(R_A)$ is computed and the minimal relations are returned. It is possible that this produces several different minimal relations, or none at all. In the latter case, PNAPPROX returns the empty set and this branch of the algorithm fails to find any realisation of the hMTS M.

4.1 Examples

As our first example, we will use M_1 from Figure 3 and realise it with a 1-bounded Petri net.

The call REALISEMTS$(1, M_1)$ begins by creating an LTS A consisting of just an initial state q_0 which gets related to the initial state s_0 of M_1. In RECURSE$(1, A, \{(q_0, s_0)\}, M_1)$, the set $m = \{(q_0, \{(a, s_1)\}), (q_0, \{(b, s_4)\})\}$ is calculated. Therefore, q_0 needs an outgoing edge with label a and another edge with label b. The invocation EXPAND$(A, \{(q_0, s_0)\}, m)$ produces just the pair (A_1, R_1) where A_1 is shown in Figure 5. The same figure also visualises R_1 by labelling a state q in the LTS with s if $(q, s) \in R_1$. The following call to PNAPPROX does not modify this LTS and RECURSE calls itself recursively with (A_1, R_1).

The next call to EXPAND produces the LTS $\widetilde{A_1}$ from Figure 5. This LTS cannot be solved by a Petri net, because if both sequences ab and ba are enabled in a marking, they must lead to the same state. Thus, PNAPPROX produces the LTS A_2. The relation that is indicated by its labelling is $f(\widetilde{R_1})$, i.e. no additional entries need to be added to produce an expansion relation.

This expansion procedure continues for some iterations until the LTS $\widetilde{A_5}$ from Figure 5

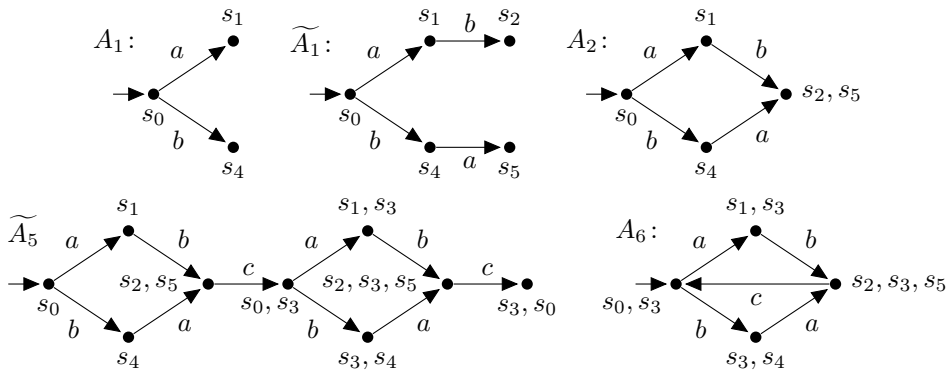


Figure 5 Intermediate LTS from the Petri net synthesis of the deterministic MTS M_1 from Figure 3. Some intermediate results are shown. To visualise a relation R, the states of an LTS are labelled with the states of M_1 that they have to implement.

is created by Expand. This LTS cannot be solved by a 1-bounded Petri net. As we have already seen in Figure 2, when some firing sequence $ww \in \Sigma^*$ is possible in a 1-bounded Petri net, then it has twice the effect of the sequence w on the marking, i.e. it cannot have an effect at all. Thus, it forms a loop in the reachability graph. The same thing happens now with the sequence abc. The minimal over-approximation produces the LTS A_6. The relation returned by PNApprox is just $R_6 = f(\widetilde{R_5})$. At this point all must edges are implemented and we have found a realisation.

As another example, consider RealiseMTS$(2, M_2)$ where M_2 is shown in Figure 3. This produces the four LTS that are displayed in the middle of Figure 4.

If one considers RealiseMTS$(1, M_2)$ instead, i.e. $k = 1$ instead of $k = 2$, then only the LTS corresponding to the words ab and ba are generated. This is because, for example, aa cannot be generated by one-bounded Petri nets without also allowing aaa to occur. This extra behaviour is not allowed by M_2 and so the algorithm will end up with $E = \emptyset$ in line 24 and this aa-branch of the recursion does not find any realisations.

When applying the algorithm to M_3 of Figure 3, we get an LTS with states corresponding to s_0, s_1, s_3, and s_5 in the first expansion phase. The over-approximation will then combine s_1 and s_3 to $\{s_1, s_3\}$ since both a-edges must have the same effect on the marking of the Petri net. In the second expansion phase, one of the edges to s_4, s_2, or s_6 is added to the LTS, yielding three LTS to over-approximate. While the over-approximations exist, no expansion relation can be built for them. For the pair $(\{s_1, s_3\}, s_3)$, the LTS state $\{s_1, s_3\}$ has an edge now (the newly added a or b), but s_3 in M_3 has no may edge at all. Thus, $E = \emptyset$ and PNApprox returns the empty set in all cases. The realisation of M_3 by a Petri net has failed (correctly).

5 Termination and Correctness

In this section, three results will be shown: The algorithm terminates; the algorithm is correct in the sense that it really produces realisations of the hMTS M; and the algorithm is complete in the sense that it always finds a realisation if one exists.

▶ **Lemma 15.** *For an hMTS M and a $k \in \mathbb{N}^+$, a call to* RealiseMTS(k, M) *terminates.*

Proof. If the algorithm does not terminate, it must be due to Recurse because PNApprox contains only finite constructions and Expand has a recursion depth of $|m|$. Since the unions in line 10 of the algorithm are always finite, there must be an infinite recursion of Recurse with arguments $(A_i)_{i\in\mathbb{N}}$. All arguments to Recurse are Petri net solvable by construction. Also, by Lemma 6, there are only finitely many LTS (abstracting state names) over a fixed alphabet which are k-bounded Petri net solvable. Thus, the A_i are all elements of a finite set and form an infinite chain with $A_i \sqsubseteq A_{i+1}$ for $i \in \mathbb{N}$, i.e. some LTS A (up to isomorphism) will occur infinitely often. Let A_j and $A_{j'}$ be any two instances of A with $j < j'$, so $A_j = A_{j'}$. For $j < n < j'$ then $A_j \sqsubseteq A_n \sqsubseteq A_{j'}$ and Lemma 3 can be applied since all A_i are reachability graphs and thus by Lemma 5 totally reachable and deterministic. We conclude $A_j = A_n$ (up to isomorphism), i.e. the infinite chain $(A_i)_{i\in\mathbb{N}}$ becomes stationary.

Consider now some $i \geq j$. Let $\widetilde{A_i}$ be the LTS resulting from applying Expand to A_i, possibly adding one (or more) states q_{new} to A_i. Then, $A_i \sqsubseteq \widetilde{A_i}$ via the identity homomorphism id, where the new state(s) q_{new} do not occur as an image. Now, PNApprox recreates the original LTS, i.e. $\widetilde{A_i} \sqsubseteq A_{i+1} = A_i$ via some homomorphism f. Overall, $A_i \sqsubseteq \widetilde{A_i} \sqsubseteq A_i$ via $f \circ id$, which is uniquely determined according to Lemma 2. Thus, $f \circ id$ must be the identity mapping, and f also must be the identity on all states except the new one(s), q_{new}.

If Expand would add an edge $q \xrightarrow{a} q_{new}$ to A_i, then $f(q) \xrightarrow{a} f(q_{new})$ would also be an edge in $A_i = A_{i+1}$. PNApprox maps $f(q) = q$ and $f(q_{new}) = q' \in Q_{A_i}$. The new element of the expansion relation added by Expand, $(q_{new}, \tilde{s})$ where $\tilde{s}$ is determined by M, is mapped to $(q', \tilde{s}) \in R_{A_{i+1}}$. Note that $(q', \tilde{s}) \notin R_{A_i}$, otherwise Expand would not have added an edge in the first place. Therefore, $|R_{A_i}| < |R_{A_{i+1}}|$. Since the chain $(A_i)_{i\geq j}$ is stationary, the maximal size of an expansion relation is $|Q_{A_j} \times S_M|$, i.e. at some point in the chain, Expand cannot add any further edges. Therefore, the to-be-implemented set m will be empty and Recurse terminates. ◀

The following lemma shows that RealiseMTS only produces realisations of its input:

▶ **Lemma 16.** *For an hMTS M and a $k \in \mathbb{N}^+$, every $(A, R_A) \in$ RealiseMTS(k, M) satisfies $A \models M$ via R_A and A can be solved by a k-bounded Petri net.*

Proof. For (A, R_A) to appear in the result of RealiseMTS, it must satisfy $m = \emptyset$ in line 9 of the algorithm. This means that there are no unimplemented must edges, which shows the first half of $A \models M$ via R_A. The other half, that all edges in A are allowed by may-edges, holds, because every value of R_A in the algorithm is an expansion relation: Initially, A only has an initial state and R_A relates this state to an initial state of M. When Expand adds new edges to an LTS, the relation R_A is updated accordingly. Here, (s, a, s') is a may edge, because every must edge must also be a may edge by definition of an hMTS. Finally, PNApprox explicitly only returns expansion relations.

It remains to be shown that A can be solved by a k-bounded Petri net. This is the case, because every LTS A given to Recurse is either the trivial LTS having no edges or was generated by PNApprox as a minimal Petri net over-approximation. ◀

In the remainder of the section, we will show that RealiseMTS$(k, M) = \emptyset$ can only occur if there are no realisations of M. For this we need the following preorder on LTS enriched with a relation:

▶ **Definition 17.** Let S be an arbitrary set. For two LTS A and B assume relations $R_A \subseteq Q_A \times S$ and $R_B \subseteq Q_B \times S$. We write $(A, R_A) \sqsubseteq (B, R_B)$ if there is an LTS

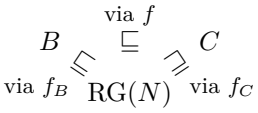


Figure 6 Illustration for part (2) of the proof of Lemma 18.

homomorphism f witnessing $A \sqsubseteq B$ so that $\forall (q,s) \in R_A : (f(q),s) \in R_B$, i.e. $f(R_A) \subseteq R_B$ (cf. Definition 14).

▶ **Lemma 18.** *Let M be an hMTS, $k \in \mathbb{N}^+$, and N be a Petri net with $\mathrm{RG}(N) \models M$ via some relation R_N. There is an $(A, R_A) \in \textsc{RealiseMTS}(k, M)$ so that $(A, R_A) \sqsubseteq (\mathrm{RG}(N), R_N)$.*

Proof. We proof inductively that there is always an (A, R_A) in the current state of the algorithm that satisfies $(A, R_A) \sqsubseteq (\mathrm{RG}(N), R_N)$.

By $\mathrm{RG}(N) \models M$ via R_N, there must be an $s_0 \in S_{0,M}$ with $(M_0, s_0) \in R_N$ (M_0 being the initial marking of N). Since RealiseMTS tries all initial states of M, the same s_0 will be picked at some point, i.e. $\textsc{Recurse}(k, A, R_A, M)$ will be called with A being the LTS having just a single state and the relation $R_A = \{(q_0, s_0)\}$. Here, $(A, R_A) \sqsubseteq (\mathrm{RG}(N), R_N)$ holds.

Next, assume that Recurse is called with arguments A and R_A satisfying $(A, R_A) \sqsubseteq (\mathrm{RG}(N), R_N)$ via some homomorphism f. We have to show two things: (1) One of the pairs (B, R_B) generated by $\textsc{Expand}(A, R_A, m)$ satisfies $(B, R_B) \sqsubseteq (\mathrm{RG}(N), R_N)$ and (2) the same applies to one of the pairs (C, R_C) which are generated by the following call of $\textsc{PNApprox}(k, B, R_B, M)$ in line 10. Since the algorithm always terminates by Lemma 15, eventually $m = \emptyset$ in line 9 will cause a suitable (A, R_A) to be returned. All recursive calls to Recurse pass this on and in the end the pair (A, R_A) is returned by RealiseMTS, completing the proof.

For (1) consider any unimplemented $(q, D) \in m$. By construction of m there must be a $(q, s) \in R_A$ so that $(q, D) \in m_{q,s}$ and $(s, D) \in \longrightarrow_M$. By $(A, R_A) \sqsubseteq (\mathrm{RG}(N), R_N)$ via f, we have $(f(q), s) \in R_N$. By $(s, D) \in \longrightarrow_M$, $(f(q), s) \in R_N$ and $\mathrm{RG}(N) \models M$ via R_N, there must be some $(a, s') \in D$ and $\widetilde{M}$ with $f(q) \xrightarrow{a} \widetilde{M}$ and $(\widetilde{M}, s') \in R_N$. Thus, on our way to B we can pick that LTS generated in the iteration of line 15 of the algorithm which handles the same $(a, s') \in D$. We define a new function f' that is identical to f, except that its domain is extended by q_{new} via $f'(q_{new}) = \widetilde{M}$. Having done this for all unimplemented must edges, the final function f' witnesses $(B, R_B) \sqsubseteq (\mathrm{RG}(N), R_N)$: Most of this property is inherited from f and for any new edge $q \xrightarrow{a} q_{new}$, we constructed f' so that $f'(q) \xrightarrow{a} f'(q_{new})$ is satisfied.

For (2) we assume that we have $(B, R_B) \sqsubseteq (\mathrm{RG}(N), R_N)$ via some function f_B. We have to find some $(C, R_C) \in \textsc{PNApprox}(k, B, R_B, M)$ with $(C, R_C) \sqsubseteq (\mathrm{RG}(N), R_N)$. The LTS relations are illustrated in Figure 6. From the algorithm we know $C = \mathrm{RG}(\mathrm{SN}_k(B))$. From the assumption we get $B \sqsubseteq \mathrm{RG}(N)$ via f_B. Applying Theorem 11 to B, we can deduce $C = \mathrm{RG}(\mathrm{SN}_k(B)) \sqsubseteq \mathrm{RG}(N)$ via some function f_C and $B \sqsubseteq C$ via some function f. We have $B \sqsubseteq C \sqsubseteq \mathrm{RG}(N)$ via $f_C \circ f$ by transitivity of $\sqsubseteq$. Since B is totally reachable by construction and $\mathrm{RG}(N)$ is deterministic by Lemma 5, we can apply Lemma 2 stating that the homomorphism witnessing $B \sqsubseteq \mathrm{RG}(N)$ is unique. We conclude $f_B = f_C \circ f$.

Consider the expansion relation R'_C defined by $R'_C = \{(q, s) \in Q_C \times S_M \mid (f_C(q), s) \in R_N\}$. This relation inherits the expansion property from R_N, i.e. if $(q, s) \in R'_C$ and $(q, a, q') \in \rightarrow_C$, then $(f_C(q), s) \in R_N$ and $(f_C(q), a, f_C(q')) \in \rightarrow_{\mathrm{RG}(N)}$, which means that there is a suitable $s' \in S_M$ with $s \overset{a}{\dashrightarrow} s'$ and $(f_C(q'), s') \in R_N$, so $(q', s') \in R'_C$. By definition we have $(C, R'_C) \sqsubseteq (\mathrm{RG}(N), R_N)$ via f_C. Consider any element $(f(q), s) \in f(R_B)$. By definition of $f(R_B)$, we have $(q, s) \in R_B$. By $(B, R_B) \sqsubseteq (\mathrm{RG}(N), R_N)$ via f_B, we have

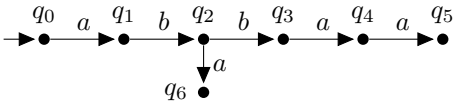


Figure 7 An LTS that can be solved exactly, but only if non-minimal regions are considered.

$(f_B(q), s) \in R_N$. By $f_B = f_C \circ f$, we have $(f_C(f(q)), s) \in R_N$. By definition of R'_C, we have $(f(q), s) \in R'_C$. Thus, $f(R_B) \subseteq R'_C$, which means that R'_C is an element of the set E inside the algorithm, i.e. there is a relation $R_C \in E$ with $f(R_B) \subseteq R_C \subseteq R'_C$ so that $(C, R_C) \in \text{PNAPPROX}(k, B, R_B, M)$. Since $(C, R'_C) \sqsubseteq (\text{RG}(N), R_N)$ via f_C and $R_C \subseteq R'_C$, we also have $(C, R_C) \sqsubseteq (\text{RG}(N), R_N)$ by the definition of $\sqsubseteq$, which needed to be shown. ◀

▶ **Corollary 19.** *For an hMTS M and a $k \in \mathbb{N}^+$, if* $\text{REALISEMTS}(k, M) = \emptyset$ *then there is no k-bounded Petri net N with* $\text{RG}(N) \models M$.

6 Some final thoughts

An aspect of the algorithm that is still open is its complexity. This depends a lot on the complexity of the Petri net over-approximation. The question of whether for a given LTS A any 1-bounded and pure Petri net N exists with $A = \text{RG}(N)$ is NP-complete [4]. The same question can be answered in polynomial time when asking for bounded Petri nets, but without picking a bound $k \in \mathbb{N}^+$ *a priori* [3]. We do not know about any complexity results for the synthesis of k-bounded Petri nets with a given k, nor for the k-bounded over-approximation that is needed in our present setting.

When looking at some algorithms for this problem, we can see that the brute force computation of all regions will certainly be exponential (checking $(k+1)^{|Q|}$ functions) in the size of the LTS, but not all regions might be necessary. Attempts to reduce this problem have been made, e.g. in [10, 11], but this procedure does not guarantee success. It only computes minimal regions, where a region r is *minimal* if there is no other region $r' \neq \mathbf{0}$ with $r' \leq r$ (pointwise). Figure 7 contains an LTS which cannot be solved when just using minimal regions. The following list contains all six minimal regions of this LTS, where a region r is identified with the vector $(r(q_0), r(q_1), \ldots, r(q_6))$ and a gradient Δ_r with the vector $(\Delta_r(a), \Delta_r(b))$:

$$r_1 = (1,0,1,2,1,0,0) \quad \Delta_{r_1} = (-1,1) \quad r_2 = (2,2,1,0,0,0,1) \quad \Delta_{r_2} = (0,-1)$$
$$r_3 = (1,2,1,0,1,2,2) \quad \Delta_{r_3} = (1,-1) \quad r_4 = (0,2,1,0,2,4,3) \quad \Delta_{r_4} = (2,-1)$$
$$r_5 = (0,1,1,1,2,3,2) \quad \Delta_{r_5} = (1,0) \quad r_6 = (0,0,1,2,2,2,1) \quad \Delta_{r_6} = (0,1)$$

It can easily be verified that for all of these regions we have $r(q_6) \geq \mu_r(b) = F(r,b)$, which means that none of these regions prevents transition b in state q_6. However, the region $r_7 = (3,2,2,2,1,0,1)$ does prevent b in q_6 ($r_7(q_6) = 1 < 2 = \mu_{r_7}(b)$). This region is not minimal since $r_1 \leq r_7$, but only with this region can the LTS from Figure 7 be solved.

In the general setting of bounded Petri net synthesis without k given a priori, there are polynomial algorithms based on solving so-called *separation problems* (see e.g. [3, 5]). These algorithms seem to be quite efficient in our experiments, but so far there do not seem to be variants where k is given a priori nor for computing a minimal over-approximation. If further research finds such an algorithm, another optimisation becomes possible: When a must edge is present, the corresponding event does not need to be prevented even if this is

possible. Thus, instead of computing the minimal over-approximation, the Expand-step could be integrated with the synthesis to make it faster.

Coming back to the problem of finding a Petri net realisation of an MTS, the brute force approach would be the construction of all k-bounded Petri nets with a fixed set of transitions. This also requires the computation of all possible regions (places), but afterwards we have to walk through the power set of the places, construct the according reachability graphs, and check if they implement our MTS. As long as the Petri net over-approximation is not used too often, our algorithm will probably be faster.

Still, Expand and PNApprox are independent procedures, so it might be feasible to reduce the number of over-approximations in favour of the Expand step. We can modify our algorithm, doing as many consecutive Expand steps as possible (e.g. until a cycle in the MTS becomes fully implemented) and only then applying one over-approximation.

Of course, with general hMTS, the Expand step can itself create exponentially many LTS, but we might expect that true hyper edges (defining algorithmic points of choice) occur with low frequency in at least human-made hMTS. For MTS (without true hyper edges), every Expand step only computes a single LTS. If an hMTS is deterministic, it can easily be shown that there is at most one expansion relation and so PNApprox would also compute at most a single result. Thus, the algorithm does not branch when given a deterministic MTS.

Overall, we obtain a goal-oriented algorithm and thus a decision procedure for the realisability of hMTS by k-bounded Petri nets with hopefully much lower run times than the brute force approach by enumeration of all candidate nets. We are already close to the barrier of undecidability here, since for bounded Petri nets (without the fixed $k \in \mathbb{N}^+$), the realisability of even MTS is known to be undecidable [20].

Our approach can easily handle restricted Petri net classes such as plain Petri nets (arc weights at most 1) or pure Petri nets (see Definition 4). Instead of constructing $\mathrm{SN}_k(A)$ from all k-bounded regions, simply only regions corresponding to plain, respectively pure, places are considered.

Further research effort can be put into investigating more expressive specifications than hMTS. We would like to have a similar algorithm for finding Petri net realisations of formulas of the modal μ-calculus [15, 2]. The general approach of the algorithm would stay the same, but a suitable replacement for the Expand step is needed.

Acknowledgements. We would like to thank Eike Best and Valentin Spreckels for their help. Also, we are thankful to the anonymous reviewers for their helpful comments.

References

1 Adam Antonik, Michael Huth, Kim G. Larsen, Ulrik Nyman, and Andrzej Wasowski. 20 years of modal and mixed specifications. *Bulletin of the EATCS*, 95:94–129, 2008.

2 André Arnold and Damian Niwiński. *Rudiments of μ-calculus.* North Holland, 2001.

3 Eric Badouel, Luca Bernardinello, and Philippe Darondeau. Polynomial algorithms for the synthesis of bounded nets. In Peter D. Mosses, Mogens Nielsen, and Michael I. Schwartzbach, editors, *TAPSOFT'95*, volume 915 of *LNCS*, pages 364–378. Springer, 1995. doi:10.1007/3-540-59293-8_207.

4 Eric Badouel, Luca Bernardinello, and Philippe Darondeau. The synthesis problem for elementary net systems is NP-complete. *Theoretical Computer Science*, 186(1-2):107–134, 1997. doi:10.1016/S0304-3975(96)00219-8.

5 Eric Badouel, Luca Bernardinello, and Philippe Darondeau. *Petri Net Synthesis*. Texts in Theoretical Computer Science. Springer, 2015. doi:10.1007/978-3-662-47967-4.

6 Eric Badouel, Benoît Caillaud, and Philippe Darondeau. Distributing finite automata through Petri net synthesis. *Formal Aspects of Computing*, 13(6):447–470, 2002. doi:10.1007/s001650200022.

7 Eric Badouel and Philippe Darondeau. Theory of regions. In Wolfgang Reisig and Grzegorz Rozenberg, editors, *Lectures on Petri Nets I: Basic Models, Advances in Petri Nets*, volume 1491 of *LNCS*, pages 529–586. Springer, 1996. doi:10.1007/3-540-65306-6_22.

8 Eike Best and Philippe Darondeau. Petri net distributability. In Edmund M. Clarke, Irina Virbitskaite, and Andrei Voronkov, editors, *PSI 2011*, volume 7162 of *LNCS*, pages 1–18. Springer, 2011. doi:10.1007/978-3-642-29709-0_1.

9 Glenn Bruns. An industrial application of modal process logic. *Science of Computer Programming*, 29(1-2):3–22, 1997. doi:10.1016/S0167-6423(96)00027-5.

10 Josep Carmona, Jordi Cortadella, and Michael Kishinevsky. New region-based algorithms for deriving bounded Petri nets. *IEEE Transactions on Computers*, 59(3):371–384, 2010. doi:10.1109/TC.2009.131.

11 Josep Carmona, Jordi Cortadella, Michael Kishinevsky, Alex Kondratyev, Luciano Lavagno, and Alexandre Yakovlev. A symbolic algorithm for the synthesis of bounded Petri nets. In Kees M. van Hee and Rüdiger Valk, editors, *PETRI NETS 2008*, volume 5062 of *LNCS*, pages 92–111. Springer, 2008. doi:10.1007/978-3-540-68746-7_10.

12 Philippe Darondeau. Distributed implementations of Ramadge-Wonham supervisory control with Petri nets. In Eduardo Camacho, editor, *IEEE CDC-ECC 2005*, pages 2107–2112. IEEE, 2005. doi:10.1109/CDC.2005.1582472.

13 Andrzej Ehrenfeucht and Grzegorz Rozenberg. Partial (set) 2-structures. Part I: basic notions and the representation problem and Part II: state spaces of concurrent systems. *Acta Informatica*, 27(4):315–368, 1990. doi:10.1007/BF00264611.

14 Guillaume Feuillade. *Spécification logique de réseaux de Petri*. PhD thesis, Université de Rennes I, 2005.

15 Dexter Kozen. Results on the propositional mu-calculus. *Theoretical Computer Science*, 27:333–354, 1983. doi:10.1016/0304-3975(82)90125-6.

16 Jan Kretínský and Salomon Sickert. MoTraS: A tool for modal transition systems and their extensions. In Dang Van Hung and Mizuhito Ogawa, editors, *ATVA 2013*, volume 8172 of *LNCS*, pages 487–491. Springer, 2013. doi:10.1007/978-3-319-02444-8_41.

17 Kim Larsen. Modal specifications. In Joseph Sifakis, editor, *AVMFSS*, volume 407 of *LNCS*, pages 232–246. Springer, 1989. doi:10.1007/3-540-52148-8_19.

18 Kim Guldstrand Larsen and Liu Xinxin. Equation solving using modal transition systems. In *LICS 1990*, pages 108–117. IEEE Computer Society, 1990. doi:10.1109/LICS.1990.113738.

19 Jean-Baptiste Raclet. Residual for component specifications. *Electr. Notes Theor. Comput. Sci.*, 215:93–110, 2008. doi:10.1016/j.entcs.2008.06.023.

20 Uli Schlachter. Bounded Petri net synthesis from modal transition systems is undecidable. In Josée Desharnais and Radha Jagadeesan, editors, *CONCUR 2016*, volume 59 of *LIPIcs*, pages 15:1–15:14. Schloss Dagstuhl, 2016. doi:10.4230/LIPIcs.CONCUR.2016.15.

21 Rob J. van Glabbeek, Ursula Goltz, and Jens-Wolfhard Schicke-Uffmann. On characterising distributability. *Logical Methods in Computer Science*, 9(3), 2013. doi:10.2168/LMCS-9(3:17)2013.

A Characterisation of Open Bisimilarity using an Intuitionistic Modal Logic*

Ki Yung Ahn[1], Ross Horne[2], and Alwen Tiu[3]

1 **School of Computer Science and Engineering, Nanyang Technological University, Singapore**
yaki@ntu.edu.sg
2 **School of Computer Science and Engineering, Nanyang Technological University, Singapore**
rhorne@ntu.edu.sg
3 **School of Computer Science and Engineering, Nanyang Technological University, Singapore**
atiu@ntu.edu.sg

Abstract

Open bisimilarity is a strong bisimulation congruence for the π-calculus. In open bisimilarity, free names in processes are treated as variables that may be instantiated; in contrast to late bisimilarity where free names are constants. An established modal logic due to Milner, Parrow, and Walker characterises late bisimilarity, that is, two processes satisfy the same set of formulae if and only if they are bisimilar. We propose an intuitionistic variation of this modal logic and prove that it characterises open bisimilarity. The soundness proof is mechanised in Abella. The completeness proof provides an algorithm for generating distinguishing formulae, useful for explaining and certifying whenever processes are non-bisimilar.

1998 ACM Subject Classification F.4.1 Mathematical Logic

Keywords and phrases bisimulation, modal logic, intuitionistic logic

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.7

1 Introduction

In this work, we consider open bisimilarity [13] which ensures processes equivalence under any context at any point in their execution. Open bisimulation is an appealing choice of equivalence for state-space reduction due to its lazy *call-by-need* approach to inputs, which makes it easier to automate [15]. In such a call-by-need approach, a value received is only observed when it needs to be used. Furthermore, some process calculi have been shown to enjoy sound and complete algebraic characterisations with respect to open bisimilarity.

The fine algebraic properties of open bisimilarity may be desirable for some applications. For many applications, it is desirable to avoid a situation where an equivalence technique proves that two components are equivalent in a sandbox test environment, when, in fact, they are distinguishable when plugged into a larger system. More subtly, processes may change context during execution [10]; for example, virtual machines migrate between devices, and replicas replace components at runtime to keep a system live in the face of unavoidable node

* The authors receive support from MOE Tier 2 grant MOE2014-T2-2-076. The third author receives support from NTU Start Up grant M4081190.020.

failures. For some notions of observational equivalence two processes may be indistinguishable when executed in any context prescribed; however, if the same two processes execute a few steps and then are migrated to another context, then it is possible that, from that point, the processes can exhibit observably distinct behaviours.

Process equivalences for the π-calculus coarser than open bisimilarity are prone to limitations described above. For instance, late bisimilarity [8] is not a congruence, since it is not preserved by input prefixes. Furthermore, even if we take the greatest congruence relation contained in late bisimilarity, called *late congruence*, late congruence is no longer a bisimulation hence is not necessarily preserved during execution. These issues are remedied by open bisimilarity [13].

The problem we address is the nature of a modal logic characterising open bisimilarity, in the tradition pioneered by Hennessy and Milner [6]. A modal logic characterising a bisimulation should have the property that whenever two processes are not bisimilar there should exist a distinguishing formula in the modal logic that holds for one process, but not for the other process. Such distinguishing formulae are useful for explaining why two processes are not bisimilar. Modal logics characterising late bisimilarity and coarser bisimulations were developed early in the literature on the π-calculus, by Milner, Parrow and Walker [9].

A novelty of our modal logic characterising open bisimilarity, which we name $\mathcal{OM}$, is that it is intuitionistic rather than classical. A non-classical feature of $\mathcal{OM}$ is that box and diamond modalities have independent interpretations, except in special cases such as $[\tau]\mathtt{ff}$ which is equivalent to $\neg\langle\tau\rangle\mathtt{tt}$. In general, in $\mathcal{OM}$ it is rarely the case that box can be defined in terms of diamond and negation. This contrasts to a classical modal logic we would expect that $[\pi]\phi$ and $\neg\langle\pi\rangle\neg\phi$ define equivalent formulae, however such de Morgan dualities do not hold for most $\mathcal{OM}$ formulae.

More profoundly, the law of excluded middle does not hold in $\mathcal{OM}$. For example, the process $\overline{a}b \parallel c(x)$ does not satisfy the formula $\langle\tau\rangle\mathtt{tt} \vee \neg\langle\tau\rangle\mathtt{tt}$, that is, $\overline{a}b \parallel c(x) \not\models \langle\tau\rangle\mathtt{tt} \vee \neg\langle\tau\rangle\mathtt{tt}$. The failure of the formula above relies on the fact that we have not yet fixed whether $a = c$ or $a \neq c$, which amounts to the absence of the law of excluded middle for name equality, as observed in related work on logical encodings of open bisimilarity [16]. In open bisimulation, both a and c are variables that may or may not be instantiated with the same value.

As a further example, consider the following two processes.

$$R \triangleq \tau.(\overline{a}b.a(x) + a(x).\overline{a}b + \tau) + \tau.(\overline{a}b.c(x) + c(x).\overline{a}b) \qquad S \triangleq R + \tau.(\overline{a}b \parallel c(x))$$

The above processes are not open bisimilar. Process R satisfies $[\tau]\big(\langle\tau\rangle\mathtt{tt} \vee \neg\langle\tau\rangle\mathtt{tt}\big)$ but process S does not, since there is a τ-transition to process $\overline{a}b \parallel c(x)$ that we just agreed above does not satisfy $\langle\tau\rangle\mathtt{tt} \vee \neg\langle\tau\rangle\mathtt{tt}$. In this example, the absence of the law of excluded middle is necessary for the existence of a formula distinguishing these processes in $\mathcal{OM}$.

The absence of de Morgan dualities discussed above complicates the construction of distinguishing formulae for processes that are not open bisimilar. For example, τ and $[a = c]\tau$ are not open bisimilar, so there should be a formula distinguishing these processes. Such a formula is $\langle\tau\rangle\mathtt{tt}$, for which $\tau \models \langle\tau\rangle\mathtt{tt}$ and $[a = c]\tau \not\models \langle\tau\rangle\mathtt{tt}$. This particular construction has a bias towards τ. In the classical setting of modal logic for late bisimilarity, given such a distinguishing formula, we can dualise it to obtain another distinguishing formula $\neg\langle\tau\rangle\mathtt{tt}$ that has a bias towards $[a = c]\tau$, i.e., $[a = c]\tau \models \neg\langle\tau\rangle\mathtt{tt}$ but $\tau \not\models \neg\langle\tau\rangle\mathtt{tt}$. This dual construction **fails** in the case of our intuitionistic modal logic characterising open bisimilarity. In the intuitionistic setting, we have both $\tau \not\models \neg\langle\tau\rangle\mathtt{tt}$ and $[a = c]\tau \not\models \neg\langle\tau\rangle\mathtt{tt}$. To address this problem, our algorithm (c.f. Section 3) simultaneously constructs two distinguishing formulae, that are not necessarily dual to each other.

$$
\begin{array}{rcll}
\pi & ::= & \tau & \text{(progress)} \\
 & & \overline{x}z & \text{(free out)} \\
 & & \overline{x}(z) & \text{(bound out)} \\
 & & x(z) & \text{(input)} \\
 & & & \\
P & ::= & 0 & \text{(deadlock)} \\
 & & \nu x.P & \text{(nu)} \\
 & & \pi.P & \text{(action)} \\
 & & [x=y]P & \text{(match)} \\
 & & P \parallel P & \text{(par)} \\
 & & P + P & \text{(choice)}
\end{array}
$$

$$\frac{}{\pi.P \xrightarrow{\pi} P} \qquad \frac{P \xrightarrow{\overline{x}z} Q}{\nu z.P \xrightarrow{\overline{x}(z)} Q}\ x \neq z$$

$$\frac{P \xrightarrow{\pi} R}{P + Q \xrightarrow{\pi} R} \qquad \frac{P \xrightarrow{\pi} R}{[x=x]P \xrightarrow{\pi} R}$$

$$\frac{P \xrightarrow{\pi} Q}{\nu x.P \xrightarrow{\pi} \nu x.Q}\ x \notin \mathrm{n}(\pi) \qquad \frac{P \xrightarrow{\pi} Q}{P \parallel R \xrightarrow{\pi} Q \parallel R}\ \begin{array}{l}\text{if } x \in \mathrm{bn}(\pi)\\ \text{then } x \text{ fresh for } R\end{array}$$

$$\frac{P \xrightarrow{\overline{x}(z)} P' \quad Q \xrightarrow{x(z)} Q'}{P \parallel Q \xrightarrow{\tau} \nu z.(P' \parallel Q')} \qquad \frac{P \xrightarrow{\overline{x}y} P' \quad Q \xrightarrow{x(z)} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'\{^{y}/_{z}\}}$$

Figure 1 Syntax and semantics of the π-calculus, plus symmetric rules for choice and parallel composition, where $\mathrm{n}(x(y)) = \mathrm{n}(\overline{x}(y)) = \mathrm{n}(\overline{x}y) = \{x, y\}$, $\mathrm{bn}(x(y)) = \mathrm{bn}(\overline{x}(y)) = \{y\}$ and $\mathrm{n}(\tau) = \mathrm{bn}(\tau) = \mathrm{bn}(\overline{x}y) = \emptyset$; and α-conversion is such that $\nu x.P$, $z(x).P$ and $\bar{z}(x).P$ bind x in P.

The precise semantics is presented in the body of this paper. The techniques are clean and modular, so results extend to open bisimilarity for more expressive process calculi.

Outline

Section 2 introduces the semantics of Open Milner–Parrow–Walker logic ($\mathcal{OM}$) and states the soundness and completeness results. Section 3 presents the proof of the correctness of an algorithm for generating distinguishing formulae, which is used to establish completeness of the logic with respect to open bisimilarity.

2 Open Milner–Parrow–Walker logic ($\mathcal{OM}$)

We recall the syntax and labelled transition semantics for the finite π-calculus (Fig. 1). All features are standard: the deadlocked process that can do nothing, the ν quantifier that binds private names, the output prefix that outputs a name on a channel, the input prefix that binds the name received on a channel, the silent progress action τ, the name match guard, parallel composition and non-deterministic choice. There are four types of action ranged over by π, where a *free output* sends a free name, whereas a *bound output* extrudes a ν-bounded private name. Stylistically, the semantics is the late labelled transition system for the π-calculus, where the name on the input channel is a symbolic place holder for a name that is not chosen until after an input transition.

Histories are used to define both the intuitionistic modal logic and open bisimilarity. Histories represent what is known about free variables due to how they have been communicated previously to the environment. There are two types of event to record in a history: The output of a fresh private name, using action $\overline{a}(x)$, which is denoted x^o; and a (symbolic) input, using action $a(z)$, which is denoted z^i. The only thing that matters about the order of events in the history is the alternation between the bound outputs and symbolic inputs, since an input variable can only be instantiated with private names that were output earlier in the history. E.g., for history $x^o \cdot z^i$, input variable z may be instantiated with private name x; in contrast, for history $z^i \cdot x^o$, input variable z may not be instantiated with private name x. This is reflected by the constraints on substitutions in the following inductive definition.

▶ **Definition 1** (σ respecting h)**.** A substitution σ invariant on names not in $\mathrm{fv}(h)$ is respecting a history h according to the following inductive definition.

$$\frac{}{\sigma \text{ respecting } \epsilon} \qquad \frac{\sigma \text{ respecting } h}{\sigma \text{ respecting } h \cdot x^i} \qquad \frac{x \notin \mathrm{dom}(\sigma) \cup \mathrm{fv}(h\sigma) \qquad \sigma \text{ respecting } h}{\sigma \text{ respecting } h \cdot x^o}$$

Note that the above inductive definition fulfils the role of sets of inequality constraints called *distinctions* in the original work on open bisimilarity [13]. The definition above also captures the alternations between nominal and universal quantifiers in embeddings of open bisimilarity in the intuitionistic logic LINC [16, 3]. Although distinctions are more general than histories, it is shown in [16] that given a history h and its corresponding distinction D, the corresponding definitions of open bisimilarity coincide.

2.1 The semantics of the intuitionistic modal logic $\mathcal{OM}$

The semantics of the modal logic $\mathcal{OM}$ is defined in terms of the late labelled transitions system (Fig. 2) and history respecting substitutions (Definition 1). Intuitively, each judgement must hold for all possible respectful substitutions, which explains the asymmetry between the box and diamond modalities. For the diamond modality $\langle\pi\rangle$, a π transition must be possible regardless of the substitution. It is sufficient to consider the identity substitution because applying a respectful substitution cannot prevent a transition. For the box modality $[\pi]$ there may exist substitutions σ other than the identity substitution enabling a $\pi\sigma$ transition, hence we should consider all respectful substitutions.

▶ **Definition 2** (satisfaction)**.** Process P satisfies formula ϕ with history h, written $P \models^h \phi$, according to the inductive definition in Fig. 2. Satisfaction, written $P \models \phi$, is satisfaction with a history of inputs $x_0^i \cdot \ldots \cdot x_n^i$, where $\mathrm{fv}(P) \subseteq \{x_0, \ldots, x_n\}$.

2.1.1 Why an intuitionistic modal logic?

In the open bisimulation game, every transition step is closed under respectful substitutions. Modal logic $\mathcal{OM}$ reflects in its semantics the substitutions that can be applied to a process. A natural semantics would be a Kripke-like semantics, where *worlds* are process-history pairs and the accessibility relation relates instances of such world. More precisely, consider a relation on worlds as follows: $(P, h) \leq (Q, h')$ iff there exists a substitution σ respecting h such that $P\sigma = Q$ and $h\sigma = h'$. The pair $(\mathcal{P}, \leq)$, where $\mathcal{P}$ is the set of worlds, forms a Kripke frame that is reflexive and transitive. Consequently, we obtain a semantics for an intuitionistic logic, where implication is closed under respectful substitutions as follows.

$$P \models^h \phi_1 \supset \phi_2 \qquad \text{iff} \qquad \forall \sigma \text{ respecting } h, P\sigma \models^{h\sigma} \phi_1\sigma \Longrightarrow P\sigma \models^{h\sigma} \phi_2\sigma$$

Intuitionistic negation $\neg\phi$ can then be defined as $\phi \supset \mathtt{ff}$.

Recall the example from the introduction $\overline{a}b \parallel c(x) \not\models \langle\tau\rangle\mathtt{tt} \vee \neg\langle\tau\rangle\mathtt{tt}$, demonstrating that the law of excluded middle is invalid. Neither $\overline{a}b \parallel c(x) \models \langle\tau\rangle\mathtt{tt}$ nor $\overline{a}b \parallel c(x) \models \neg\langle\tau\rangle\mathtt{tt}$ hold. The former holds only if $\overline{a}b \parallel c(x)$ is guaranteed to make a τ transition; but such a transition is only possible assuming $a = c$, hence $\overline{a}b \parallel c(x) \not\models \langle\tau\rangle\mathtt{tt}$. For the latter, we should consider all substitutions which enable a τ transition; and, since such a substitution $\{^c/_a\}$ exists, $\overline{a}b \parallel c(x) \not\models \neg\langle\tau\rangle\mathtt{tt}$. Notice that the satisfaction would hold by forcing the assumption $a \neq c$. Of course, for open bisimilarity we make no a priori assumption about whether $a = c$ or $a \neq c$, since both are variables that may, or may not, be instantiated with the same value.

$$
\begin{array}{lll}
P \models^h \mathtt{tt} & & \text{always holds.} \\
P \models^h \phi_1 \wedge \phi_2 & \text{iff} & P \models^h \phi_1 \text{ and } P \models^h \phi_2. \\
P \models^h \phi_1 \vee \phi_1 & \text{iff} & P \models^h \phi_1 \text{ or } P \models^h \phi_2. \\
P \models^h \langle x = x \rangle \phi & \text{iff} & P \models^h \phi. \\
P \models^h \langle \alpha \rangle \phi & \text{iff} & \exists Q,\ P \xrightarrow{\alpha} Q \text{ and } Q \models^h \phi. \\
P \models^h \langle \overline{a}(z) \rangle \phi & \text{iff} & \exists Q,\ P \xrightarrow{\overline{a}(z)} Q \text{ and } Q \models^{h \cdot z^o} \phi. \\
P \models^h \langle a(z) \rangle \phi & \text{iff} & \exists Q,\ P \xrightarrow{a(z)} Q \text{ and } Q \models^{h \cdot z^i} \phi. \\
P \models^h [x = y] \phi & \text{iff} & \forall \sigma \text{ respecting } h,\ x\sigma = y\sigma \implies P\sigma \models^{h\sigma} \phi\sigma. \\
P \models^h [\alpha] \phi & \text{iff} & \forall \sigma \text{ respecting } h, \forall Q,\ P\sigma \xrightarrow{\alpha\sigma} Q \implies Q \models^{h\sigma} \phi\sigma. \\
P \models^h [\overline{a}(z)] \phi & \text{iff} & \forall \sigma \text{ respecting } h,\ \forall Q, P\sigma \xrightarrow{\overline{a\sigma}(z)} Q \implies Q \models^{h\sigma \cdot z^o} \phi\sigma. \\
P \models^h [a(z)] \phi & \text{iff} & \forall \sigma \text{ respecting } h, \forall Q, P\sigma \xrightarrow{a\sigma(z)} Q \implies Q \models^{h\sigma \cdot z^i} \phi\sigma.
\end{array}
$$

$$
\begin{array}{lll}
h ::= & \epsilon & \text{(empty)} \\
& h \cdot x^o & \text{(name)} \\
& h \cdot x^i & \text{(variable)}
\end{array}
$$

$$
\begin{array}{lll}
\phi ::= & \mathtt{tt} & \text{(true)} \\
& \mathtt{ff} & \text{(false)} \\
& \phi \wedge \phi & \text{(and)} \\
& \phi \vee \phi & \text{(or)} \\
& \langle x = y \rangle \phi & \text{(dia-match)} \\
& \langle \pi \rangle \phi & \text{(dia-action)} \\
& [x = y] \phi & \text{(box-match)} \\
& [\pi] \phi & \text{(box-action)}
\end{array}
$$

Figure 2 Syntax and semantics of the modal logic $\mathcal{OM}$, where α is τ or $\overline{a}b$; and z is fresh for P, h, and σ.

The intuitionistic implication and negation above are used only to explain the origin of $\mathcal{OM}$ and for contrast with properties expected of classical modal logics. The distinguishing formula algorithm, considered in subsequent sections, does not depend on these connectives.

2.2 Open bisimilarity, soundness and completeness

We recall the definition of open bisimilarity. Open bisimilarity is a greatest fixed point of symmetric relations closed under all respectful substitutions and labelled transitions actions at every step. Notice that a symbolic input or output of a fresh private name updates the history.

▶ **Definition 3** (open bisimilarity)**.** Open bisimilarity with history h is the greatest symmetric relation such that: if $P \sim^h Q$ then, for all substitutions σ respecting h, the following hold, where α is a τ or $\overline{a}b$ action and x is fresh for $P\sigma$, $Q\sigma$ and $h\sigma$:

- $P\sigma \xrightarrow{\alpha\sigma} P' \implies \exists Q', Q\sigma \xrightarrow{\alpha\sigma} Q'$ and $P' \sim^{h\sigma} Q'$.
- $P\sigma \xrightarrow{\overline{a\sigma}(x)} P' \implies \exists Q', Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q'$ and $P' \sim^{h\sigma \cdot x^o} Q'$.
- $P\sigma \xrightarrow{a\sigma(x)} P' \implies \exists Q', Q\sigma \xrightarrow{a\sigma(x)} Q'$ and $P' \sim^{h\sigma \cdot x^i} Q'$.

Open bisimilarity, written $P \sim Q$, is defined to be open bisimilarity with a history $x_0^i \cdot \ldots \cdot x_n^i$ such that $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \subseteq \{x_0, \ldots, x_n\}$.

2.2.1 Soundness and completeness results

The main result of this paper is that, for finite π-calculus processes open bisimilarity ($\sim$) coincides with the relation between processes with no distinguishing formula ($\overset{\models}{\sim}$).

▶ **Definition 4** (logical equivalence). $P \overset{\models}{\sim} Q$ is defined whenever, for all ϕ, $P \models \phi$ iff $Q \models \phi$.

▶ **Theorem 5** (soundness). *For π-calculus processes (including replication), $P \sim Q$ implies $P \overset{\models}{\sim} Q$.*

▶ **Theorem 6** (completeness). *For finite π-calculus processes, $P \overset{\models}{\sim} Q$ implies $P \sim Q$.*

The proof of soundness has been mechanically checked in the proof assistant Abella [2] using the two-level logic approach [4] to reason about the π-calculus semantics specified in λProlog [11]. The proof of soundness proceeds by induction on the structure of the logical formulae in the definition of logical equivalence. The proof of completeness is explained in detail in Section 3. Soundness extends to infinite π-calculus processes with replication, but completeness holds for decidable fragments such as Fig. 1.

Firstly, we provide examples demonstrating the implications of Theorems 5 and 6. Due to soundness, if two processes are bisimilar, we cannot find a distinguishing formula that holds for one process but does not hold for the other process. Due to completeness, if it is impossible to prove that two process are open bisimilar, then we can construct a distinguishing formula that holds for one process but does not hold for the other process. Thus Theorems 5 and 6 guarantee that an $\mathcal{OM}$ formulae can be used to characterise non-bisimilarity.

2.2.2 Example processes distinguishable by postconditions

All modalities are essential for the soundness and completeness of $\mathcal{OM}$. Perhaps the least obvious modality is $\langle x = y \rangle$. When prefixed with a box modality it indicates a postcondition that always holds after an action. To see, this consider the process $[x = y]\tau$. The judgement $[x = y]\tau \models [\tau]\langle x = y\rangle \mathtt{tt}$ holds since for any θ such that $([x = y]\tau)\theta \xrightarrow{\tau} 0$ it must be the case that $x\theta = y\theta$. Hence, by definition of diamond, $0 \models \langle x\theta = y\theta\rangle \mathtt{tt}$ iff $0 \models \mathtt{tt}$. In contrast, $\tau \not\models [\tau]\langle x = y\rangle \mathtt{tt}$ since, taking the identity substitution in the definition of box, $\tau \xrightarrow{\tau} 0$, but $0 \models \langle x = y\rangle \mathtt{tt}$ cannot be proven in general. The formula $[\tau]\langle x = y\rangle \mathtt{tt}$ is therefore a distinguishing formula satisfied by $[x = y]\tau$ but not τ.

The use of $\langle x = y \rangle$ as a postcondition contrasts to the use of $[x = y]$ as a precondition. Consider the same process as above with the formula $[x = y]\langle \tau \rangle \mathtt{tt}$. Observe that, for substitutions θ such that $x\theta = y\theta$, $([x = y]\tau)\theta \xrightarrow{\tau} 0$ and $0 \models \mathtt{tt}$ holds, hence $([x = y]\tau)\theta \models \langle \tau \rangle \mathtt{tt}$; and thereby the judgement $[x = y]\tau \models [x = y]\langle \tau \rangle \mathtt{tt}$ holds. In contrast, $0 \not\models [x = y]\langle \tau \rangle \mathtt{tt}$.

We will return to the above two formulae shortly, as they are critical for the algorithm for generating distinguishing formulae.

2.3 Sketch of algorithm for generating distinguishing formulae

The completeness proof in Section 3 relies on an algorithm for generating distinguishing formulae for non-bisimilar processes. Here, we provide a sketch of the algorithm executed on key examples.

2.3.1 Example requiring intuitionistic assumptions

The algorithm proceeds over the structure of a tree of moves that show two processes are non-bisimilar. In base cases, we have a pair of processes where, under a substitution, one

process can make a transition, but the other process cannot match the transition. We revisit two examples of base cases, discussed previously:

$[x = y]\tau \not\sim 0$ **:** The left process leads by $([x = y]\tau)\{^{y}/_{x}\} \xrightarrow{\tau} 0$, but 0 cannot make a τ transition, under any substitution; hence $[x = y]\tau \models [x = y]\langle\tau\rangle\mathtt{tt}$ and $0 \models [\tau]\mathtt{ff}$ are distinguishing formulae.

$[x = y]\tau \not\sim \tau$ **:** The right process leads by $\tau \xrightarrow{\tau} 0$, but $[x = y]\tau\theta$ can make a τ transition only when $x\theta = y\theta$; hence $[x = y]\tau \models [\tau]\langle x = y\rangle\mathtt{tt}$ and $\tau \models \langle\tau\rangle\mathtt{tt}$ are distinguishing formulae.

In an inductive case, the two processes cannot be distinguished by an immediate transition. However, under some substitutions, one process can make a π transition to a state, say P', that, under the same substitution the other process can only make a corresponding π transition to reach states Q'_i that are non-bisimilar to P'. This allows a distinguishing formula to be inductively constructed from the distinguishing formulae for P' paired with each Q'_i.

For example, consider how the algorithm would find distinguishing formulae for P and Q below.

The first step in the strategy for non-bisimilarity is to show that P can make a τ transition to a state that is not bisimilar to any state reachable by a τ transition from the other process. One possibility is the transition to P' as illustrated above. In reply, Q may attempt a corresponding τ transition either to Q'_1 or Q'_2. Inductively, we require that $P' \nsim Q'_1$ and $P' \nsim Q'_2$. Both are instances of the base case discussed above where we discovered distinguishing formulae for each of them.

This enables us to construct distinguishing formulae for the inductive case. The distinguishing formula satisfied by P is a diamond followed by the conjunction of the left distinguishing formulae that is satisfied by P' in the base cases: $\tau.[x = y]\tau + \tau + \tau.\tau \models \langle\tau\rangle\big([\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}\big)$. The distinguishing formula satisfied by Q is a box followed by the disjunction of the right distinguishing formulae from the base cases: $\tau + \tau.\tau \models [\tau]\big(\langle\tau\rangle\mathtt{tt} \vee [\tau]\mathtt{ff}\big)$.

To confirm that they are indeed distinguishing formulae for P and Q, swap the processes and formulae above to observe that each process fails to satisfy the other formula. To be precise, assume for contradiction that $\tau+\tau.\tau \models \langle\tau\rangle\big([\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}\big)$ holds. By definition of $\langle\tau\rangle$, this holds iff either $0 \models [\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}$ or $\tau \models [\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}$ holds. Now observe that $0 \models [x = y]\langle\tau\rangle\mathtt{tt}$ holds iff we make the additional assumption in the meta framework that x and y are persistently distinct, i.e., for all σ, $x\sigma \neq y\sigma$. In addition, observe that $\tau \models [\tau]\langle x = y\rangle\mathtt{tt}$ holds iff we make the additional assumption in the meta framework that x and y are persistently equal, i.e., for all σ, $x\sigma = y\sigma$. In fact, by these observations we are able to mechanically prove the following in intuitionistic framework Abella: $\tau + \tau.\tau \models \langle\tau\rangle\big([\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}\big)$ iff $\forall x, y.\,(x = y \vee x \neq y)$. Notice that $\forall x, y.\,(x = y \vee x \neq y)$ is an instance of the law of excluded middle; hence, assuming the law of excluded middle, the formula for Q also holds for P; and vice versa. Indeed there would be no distinguishing formulae for these processes; and hence in a classical framework the modal logic would be incomplete. Fortunately, since intuitionistic logics do not assume the law of excluded middle, as long as we evaluate the semantics in an intuitionistic framework, we are able to establish that $Q \not\models \langle\tau\rangle\big([\tau]\langle x = y\rangle\mathtt{tt} \wedge [x = y]\langle\tau\rangle\mathtt{tt}\big)$, as required.

2.3.2 Example involving private names that are distinguishable

The alternation between inputs and outputs in the history affects what counts as a respectful substitution. Intuitively, respectful substitutions ensure that a private name can never be input earlier than it was output. Consider the following processes: $P \triangleq \nu x.\overline{a}x.a(y).\tau \not\sim \nu x.\overline{a}x.a(y).[x=y]\tau \triangleq Q$.

These processes are not open bisimilar because P can make the following three transition steps: $\nu x.\overline{a}x.a(y).\tau \xrightarrow{\overline{a}(x)} a(y).\tau \xrightarrow{a(y)} \tau \xrightarrow{\tau} 0$. However, Q can only match the first two steps. At the third step, a base case of the algorithm for $\tau \not\sim^{a^i x^o y^i} [x=y]\tau$ applies. In this case, any substitution θ respecting $a^i x^o y^i$ where $[x=y]\tau\theta \xrightarrow{\tau} 0$ is such that $y\theta = x$, $x \notin \mathrm{dom}(\theta)$ and $a\theta \neq x$, which is satisfiable. Thus $[x=y]\tau \models^{a^i x^o y^i} [\tau]\langle x=y\rangle \mathtt{tt}$ and $\tau \models^{a^i x^o y^i} \langle\tau\rangle\mathtt{tt}$. By applying inductive cases, we obtain $\nu x.\overline{a}x.a(y).\tau \models \langle\overline{a}(x)\rangle\langle a(y)\rangle\langle\tau\rangle\mathtt{tt}$ and $\nu x.\overline{a}x.a(y).[x=y]\tau \models [\overline{a}(x)][a(y)][\tau]\langle x=y\rangle\mathtt{tt}$.

2.3.3 Example involving private names that are indistinguishable

In contrast to the previous example, consider the following processes where a fresh name is output and compared to a name already known: $\nu x.\overline{a}x \sim \nu x.\overline{a}x.[x=a]\tau$.
These processes are open bisimilar, hence by Theorem 5 there is no distinguishing formula. The existence of a distinguishing formula of the form $\langle\overline{a}(x)\rangle[x=a]\langle\tau\rangle\mathtt{tt}$ is *prevented* by the history. Both $\nu x.\overline{a}x.[x=a]\tau \models \langle\overline{a}(x)\rangle[x=a]\langle\tau\rangle\mathtt{tt}$ and $\nu x.\overline{a}x \models \langle\overline{a}(x)\rangle[x=a]\langle\tau\rangle\mathtt{tt}$ hold. The latter holds since $\nu x.\overline{a}x \models^{a^i} \langle\overline{a}(x)\rangle[x=a]\langle\tau\rangle\mathtt{tt}$ holds if and only if $\nu x.\overline{a}x \xrightarrow{\overline{a}(x)} 0$ and $0 \models^{a^i x^o} [x=a]\langle\tau\rangle\mathtt{tt}$. By definition of $[x=a]$, this holds if only if for all θ respecting $a^i x^o$ and such that $x\theta = a\theta$, $0 \models^{a^i x^o} \langle\tau\rangle\mathtt{tt}$. Clearly 0 cannot make a τ transition, hence $0 \models^{a^i x^o} \langle\tau\rangle\mathtt{tt}$ does not hold. However, fortunately, there is no substitution θ respecting $a^i x^o$ such that $x\theta = a\theta$. By the definition of respecting substitution, θ must satisfy $x \notin \mathrm{dom}(\theta)$ and $x \neq a\theta$, contradicting constraint $x\theta = a\theta$. Thereby $0 \models^{a^i x^o} [x=a]\langle\tau\rangle\mathtt{tt}$ holds vacuously; hence $\nu x.\overline{a}x \models^{a^i} \langle\overline{a}(x)\rangle[x=a]\langle\tau\rangle\mathtt{tt}$ holds as required.

3 Completeness of open bisimilarity with respect to $\mathcal{OM}$

There is a constructive definition of non-bisimilarity. Since bisimilarity is defined in terms of a greatest fixed point of relations satisfying a certain closure property, non-bisimilarity is defined in terms of a least fixed point satisfying the dual property. This leads to the following constructive definition of non-bisimilarity from which a non-bisimilarity algorithm can be extracted. Since non-bisimilarity is defined in terms of a least fixed point, there is a finite winning strategy, consisting of a finite tree of moves such that in each branch eventually a pair of processes and a history is reached such that one process can make a move that the other cannot always match.

▶ **Definition 7** (non-bisimilarity)**.** Firstly, we inductively define the family of relation $\not\sim^h_n$, for $n \in \mathbb{N}$. The base case is when, for some respectful substitution one player can make a move, that cannot be matched by the other player without assuming a stronger substitution. The class of all such pairs of processes form the base case for the construction of the non-bisimilarity relation, say $P \not\sim^h_0 Q$. More precisely, the relation $\not\sim^h_0$ is the least symmetric relation such that for any P and Q, $P \not\sim^h_0 Q$ whenever there exist process P', action π and substitution σ respecting h, such that the following holds.

- $P\sigma \xrightarrow{\pi\sigma} P'$, for $x \in \mathrm{bn}(\pi)$, x is fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, and there is no Q' such that $Q\sigma \xrightarrow{\pi\sigma} Q'$.

Inductively, $\not\sim^h_{n+1}$ is the least symmetric relation extending $\not\sim^h_n$ such that $P \not\sim^h_{n+1} Q$ whenever for some substitution σ respecting h, one of the following holds, where α is τ or $\overline{a}b$:

- $P\sigma \xrightarrow{\alpha\sigma} P'$ and for all Q_i such that $Q\sigma \xrightarrow{\alpha} Q_i$, $P' \not\sim^{h\sigma}_n Q_i$.
- $P\sigma \xrightarrow{\overline{a\sigma}(x)} P'$, and for all Q_i and x fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, such that $Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q_i$, $P' \not\sim^{h\sigma\cdot x^o}_n Q_i$.
- $P\sigma \xrightarrow{a\sigma(x)} P'$, and for all Q_i and x fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, such that $Q\sigma \xrightarrow{a\sigma(x)} Q_i$, $P' \not\sim^{h\sigma\cdot x^i}_n Q_i$.

Thereby, the relation $P \not\sim^h_n Q$ contains all processes that can be distinguished by a strategy with depth at most n, i.e., at most n moves are required to reach a pair of processes in $\not\sim^h_0$, at which point there is an accessible world in which a process can make a move that the other process cannot match.

The relation $\not\sim^h$, pronounced non-bisimilarity with history h, is defined to be the least relation containing $\not\sim^h_n$ for all $n \in \mathbb{N}$, i.e. $\bigcup_{n\in\mathbb{N}} \not\sim^h_n$. Similarly to open bisimulation, $P \not\sim Q$ is defined as $P \not\sim^{x^i_1\cdot\ldots x^i_m} Q$ where $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \subseteq \{x_1, \ldots x_m\}$.

3.1 Preliminaries

We require the following terminology for substitutions, and abbreviations for formulae.

▶ **Definition 8.** Composition of substitutions σ and θ is defined such that $P(\sigma\cdot\theta) = (P\sigma)\theta$, for all processes P. For substitutions σ and θ, $\sigma \leq \theta$ whenever there exists σ' such that $\sigma \cdot \sigma' = \theta$. For a finite substitution $\sigma = \{^{z_1}/_{x_1}\} \cdots \{^{z_n}/_{x_n}\}$ the formula $[\sigma]\phi$ abbreviates the formula $[x_n = z_n] \ldots [x_1 = z_1]\phi$. Similarly, $\langle\sigma\rangle\phi$ abbreviates $\langle x_n = z_n\rangle \ldots \langle x_1 = z_1\rangle\phi$. For finite set of formulae ϕ_i, formula $\bigvee_i \phi_i$ abbreviates $\phi_1 \vee \ldots \vee \phi_n$, where the empty disjunction is $\mathtt{ff}$. Similarly $\bigwedge_i \phi_i$ abbreviates $\phi_1 \wedge \ldots \wedge \phi_n$, where the empty conjunction is $\mathtt{tt}$.

We require the following technical lemmas. The first (image finiteness, as used in [5]) ensures that there are finitely many reachable states in one step, up to renaming. The second extends the definition of the box-match modality to finite substitutions. The third is required in inductive cases involving bound output and input. The fourth is a monotonicity property for satisfaction. The fifth is a monotonicity property for transitions ensuring names bound by label are not changed by a substitution.

▶ **Lemma 9.** *For process P and action π there are finitely many P_i such that $P \xrightarrow{\pi} P_i$.*

▶ **Lemma 10.** *If for all θ respecting h and $\sigma \leq \theta$, it holds that $P\theta \models^{h\theta} \phi\theta$, then $P \models^h [\sigma]\phi$ holds.*

▶ **Lemma 11.** *If $\sigma \cdot \theta$ respects h, then θ respects $h\sigma$.*

▶ **Lemma 12.** *If $P \models^h \phi$ holds then $P\theta \models^{h\theta} \phi\theta$ holds for any θ respecting h.*

▶ **Lemma 13.** *If $P \xrightarrow{\pi} Q$ then $P\theta \xrightarrow{\pi\theta} Q\theta$, for all θ such that if $x \in \mathrm{bn}(\pi)$ and $y\theta = x$ then $x = y$.*

3.2 Algorithm for distinguishing formulae

The constructive definition of non-bisimilarity gives a tree of substitutions and actions forming a strategy showing that two processes are not open bisimilar. The following proposition shows that $\mathcal{OM}$ formulae are sufficient to capture such strategies. For any strategy that distinguishes two processes, we can construct *distinguishing* $\mathcal{OM}$ formulae. A distinguishing

formula holds for one process but not for the other process. Furthermore, there are always at least two distinguishing formulae, one biased to the left and another biased to the right, as in the construction of the proof for the following proposition. As discussed in the introduction, the left bias cannot be simply obtained by negating the right bias and vice versa; both must be constructed simultaneously and may be unrelated by negation.

▶ Proposition 14. If $P \not\sim Q$ then there exists ϕ_L such that $P \models \phi_L$ and $Q \not\models \phi_L$, and also there exists ϕ_R such that $Q \models \phi_R$ and $P \not\models \phi_R$.

Proof. Since $\not\sim^h$ is defined by a least fixed point over a family of relations $\not\sim^h_n$, if $P \not\sim^h Q$, there exists n such that $P \not\sim^h_n Q$, so we can proceed by induction on the depth of a winning strategy.

In the base case, assume that $P \not\sim^h_0 Q$, hence by definition, for substitution σ respecting h, $P\sigma \xrightarrow{\pi\sigma} P'$, for $x \in \mathrm{bn}(\pi)$, x is fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, such that there is no Q' such that $Q\sigma \xrightarrow{\pi\sigma} Q'$, up to symmetry of $\not\sim^h_n$. There exist finitely many pairs of variables x_j and y_j, selected from $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \cup \mathrm{fv}(\pi)$ such that $x_j\sigma \neq y_j\sigma$, and, for any R and substitution θ respecting h, if $Q\theta \xrightarrow{\pi\theta} R$ there exists j such that $x_j\theta = y_j\theta$. To see why, assume for contradiction that there is some θ respecting h such that $Q\theta \xrightarrow{\pi\theta} R$ but there is no x and y in $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \cup \mathrm{fv}(\pi)$ such that $x\sigma \neq y\sigma$ and $x\theta = y\theta$. Stated otherwise, for all x and y in $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \cup \mathrm{fv}(\pi)$ if $x\theta = y\theta$ then $x\sigma = y\sigma$, which is precisely the definition of a function, i.e. substitution, say θ', defined on $\mathrm{fv}(P\theta) \cup \mathrm{fv}(Q\theta) \cup \mathrm{fv}(\pi\theta)$ such that θ' maps $z\theta$ to $z\sigma$. In that case, $\theta \cdot \theta' = \sigma$ on $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \cup \mathrm{fv}(\pi)$; and hence, by Lemma 13, since for $x \in \mathrm{bn}(\pi)$, x is fresh, $Q\theta\theta' \xrightarrow{\pi\theta\theta'} R\theta'$ contradicting the initial assumption for the base case that no transition $Q\sigma \xrightarrow{\pi\sigma} Q'$ exists for any Q'.

In this case, there are two distinguishing formulae $[\sigma]\langle\pi\rangle\mathtt{tt}$ and $[\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$ biased to P and Q respectively. There are four cases to check to confirm that these are distinguishing formulae.

Case $P \models^h [\sigma]\langle\pi\rangle\mathtt{tt}$: Consider all θ respecting h such that $\sigma \leq \theta$. By definition there exists θ' such that $\sigma \cdot \theta' = \theta$, so since $P\sigma \xrightarrow{\pi\sigma} P'$, by Lemma 13, $P\theta \xrightarrow{\pi\theta} P'\theta'$. Thereby, since $P'\theta' \models^{h'} \mathtt{tt}$ holds, $P\theta \models^{h\theta} \langle\pi\theta\rangle\mathtt{tt}$. Hence, by Lemma 10, $P \models^h [\sigma]\langle\pi\rangle\mathtt{tt}$.

Case $Q \not\models^h [\sigma]\langle\pi\rangle\mathtt{tt}$: Assume $Q \models^h [\sigma]\langle\pi\rangle\mathtt{tt}$ for contradiction. Now, since σ respects h and $\sigma \leq \sigma$, by Lemma 10, $Q \models^h [\sigma]\langle\pi\rangle\mathtt{tt}$ holds only if $Q\sigma \models^{h\sigma} \langle\pi\sigma\rangle\mathtt{tt}$ holds; which holds only if there exists Q' such that $Q\sigma \xrightarrow{\pi\sigma} Q'$, contradicting the assumption that no such Q' exists. Thereby $Q \not\models^h [\sigma]\langle\pi\rangle\mathtt{tt}$.

Case $Q \models^h [\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$: Consider substitutions θ respecting h and Q' such that $Q\theta \xrightarrow{\pi\theta} Q'$. It must be the case that there exists j such that $x_j\theta = y_j\theta$, thereby $Q' \models^{h\theta} \langle x_j\theta = y_j\theta\rangle\mathtt{tt}$ holds; hence clearly $Q' \models^{h\theta} \left(\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}\right)\theta$ holds. Hence $Q \models^h [\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$.

Case $P \not\models^h [\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$: Assume for contradiction $P \models^h [\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$. This holds iff for all processes S and substitutions θ respecting h, $P\theta \xrightarrow{\pi\theta} S$ implies $S \models^{h'} \left(\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}\right)\theta$. Since we know that σ respects h and $P\sigma \xrightarrow{\pi\sigma} P'$, we have $P' \models^{h''} \left(\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}\right)\sigma$. This holds only if for some j, $P' \models^{h''} \langle x_j\sigma = y_j\sigma\rangle\mathtt{tt}$; hence, $x_j\sigma = y_j\sigma$ for some j, which contradicts the assumption that $x_j\sigma \neq y_j\sigma$. Thereby $P \not\models^h [\pi]\bigvee_j\langle x_j = y_j\rangle\mathtt{tt}$.

Now consider the inductive cases. Given P, Q, if $P \not\sim^h_{n+1} Q$, up to symmetry of $\not\sim^h_{n+1}$, there are three cases to consider, for some substitution σ respecting h, where α is either τ or $\overline{a}b$:

- $P\sigma \xrightarrow{\alpha\sigma} P'$ and for all Q_i such that $Q\sigma \xrightarrow{\alpha\sigma} Q_i$, $P' \not\sim_n^{h\sigma} Q_i$.
- $P\sigma \xrightarrow{\overline{a\sigma}(x)} P'$, and for all Q_i and x fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, such that $Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q_i$, $P' \not\sim_n^{h\sigma\cdot x^o} Q_i$.
- $P\sigma \xrightarrow{a\sigma(x)} P'$, and for all Q_i and x fresh for $P\sigma$, $Q\sigma$ and $h\sigma$, such that $Q\sigma \xrightarrow{a\sigma(x)} Q_i$, $P' \not\sim_n^{h\sigma\cdot x^i} Q_i$.

We consider the second case above involving bound output only, the other two cases are similar — differing only in the accounting for respectful substitutions according to Def. 1.

For $P\sigma \xrightarrow{\overline{a\sigma}(x)} P'$, by Lemma 9, there exist finitely many Q_i such that $Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q_i$. For each i, since $P' \not\sim_n^{h\sigma\cdot x^o} Q_i$, by the induction hypothesis, there exist ϕ_i^L and ϕ_i^R such that $P' \models^{h\sigma\cdot x^o} \phi_i^L\sigma$ and $Q_i \not\models^{h\sigma\cdot x^o} \phi_i^L\sigma$ and $P' \not\models^{h\sigma\cdot x^o} \phi_i^R\sigma$ and $Q_i \models^{h\sigma\cdot x^o} \phi_i^R\sigma$. Furthermore, assume that σ is minimal with respect to the order over substitutions in the sense that, if $\theta \leq \sigma$ is such that $P\theta \xrightarrow{\overline{a\theta}(x)} P''$, where x is fresh for $P\theta$, $Q\theta$ and $h\theta$, and for all Q'' such that $Q\theta \xrightarrow{\overline{a\theta}(x)} Q''$, $P'' \not\sim_n^{h\theta\cdot x^o} Q''$, then $\theta = \sigma$.

By a similar argument to the base case, there are finitely many pairs of variables x_j and y_j selected from $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \cup \{a\}$ such that $x_j\sigma \neq y_j\sigma$ and, for any substitution θ respecting h, if, for some S, $Q\theta \xrightarrow{\overline{a\theta}(x)} S$ then either: there exists some j such that $x_j\theta = y_j\theta$; or both $\sigma \leq \theta$ and $\theta \leq \sigma$ hold and hence there exist i and θ' such that $\sigma\cdot\theta' = \theta$ and $S_i\theta' = S$. Notice cases where $\theta < \sigma$ are eliminated by minimality of σ.

From the above, distinguishing formulae $[\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$ and $[\overline{a}(x)]\big(\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle\big)$ can be constructed. There are four cases to consider to verify these are indeed distinguishing formulae.

Case $P \models^h [\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$: Consider all θ such that $\sigma \leq \theta$, θ respects h, and without loss of generality x is fresh such that $x \notin \mathrm{dom}(\theta)$ and $x \notin \mathrm{fv}(h\theta)$. By definition, there exists θ' such that $\sigma\cdot\theta' = \theta$. Now since $\sigma\cdot\theta'$ respects h, by Lemma 11, θ' respects $h\sigma$ hence since $x \notin \mathrm{dom}(\theta')$ and $x \notin \mathrm{fv}(h\sigma\theta')$, θ' respects $h\sigma\cdot x^o$. Thereby since θ' respects $h\sigma\cdot x^o$ and also $P' \models^{h\sigma\cdot x^o} \phi_i^L\sigma$ holds, by Lemma 12, it holds that $P'\theta' \models^{h\theta\cdot x^o} \phi_i^L\theta$. The above holds for all i, hence it holds that $P'\theta' \models^{h\theta\cdot x^o} \bigwedge_i \phi_i^L\theta$. Now, since $P\sigma \xrightarrow{\overline{a\sigma}(x)} P'$, by Lemma 13, since x is fresh, $P\theta \xrightarrow{\overline{a\theta}(x)} P'\theta'$ holds; and hence $P\theta \models^{h\theta} \big(\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L\big)\theta$ holds. Thereby, by Lemma 10, $P \models^h [\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$ holds.

Case $Q \not\models^h [\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$: Assume for contradiction that $Q \models^h [\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$. Since σ respects h and $\sigma \leq \sigma$, by Lemma 10, the above assumption holds only if $Q\sigma \models^{h\sigma} \big(\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L\big)\sigma$ holds. Now $Q\sigma \models^{h\sigma} \langle\overline{a\sigma}(x)\rangle\bigwedge_i \phi_i^L\sigma$ holds only if there exists Q' such that $Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q'$ and $Q' \models^{h\sigma\cdot x^o} \bigwedge_i \phi_i^L\sigma$, which holds only if $Q' \models^{h\sigma\cdot x^o} \phi_i^L\sigma$ for all i. Notice that $Q' = Q_k$ for some k, and therefore $Q_k \models^{h\sigma\cdot x^o} \phi_k^L\sigma$; but it was assumed that $Q_k \not\models^{h\sigma\cdot x^o} \phi_k^L\sigma$ leading to a contradiction. Therefore $Q \not\models^h [\sigma]\langle\overline{a}(x)\rangle\bigwedge_i \phi_i^L$.

Case $Q \models^h [\overline{a}(x)]\big(\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle\mathtt{tt}\big)$: Fix Q' and θ respecting h such that $Q\theta \xrightarrow{\overline{a\theta}(x)} Q'$ and without loss of generality assume x is fresh such that $x \notin \mathrm{dom}(\theta)$ and $x \notin \mathrm{fv}(h\theta)$. There are two sub-cases to consider. Firstly consider where for some k, $x_k\theta = y_k\theta$, in which case it holds that $Q' \models^{h\theta\cdot x^o} \langle x_k\theta = y_k\theta\rangle\mathtt{tt}$, and hence $Q' \models^{h\theta\cdot x^o} \big(\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle\mathtt{tt}\big)\theta$, by definition of disjunction. Secondly consider where there exists θ' such that $\sigma\cdot\theta' = \theta$ and for some ℓ, $Q\sigma \xrightarrow{\overline{a\sigma}(x)} Q_\ell$ such that $Q_\ell\theta' = Q'$. Now since $\sigma\cdot\theta'$ respects h, by Lemma 11, θ' respects $h\sigma$, hence by definition of respectful substitutions, since $x \notin \mathrm{dom}(\theta)$ and $x \notin \mathrm{fv}(h\theta)$, θ' respects $h\sigma\cdot x^o$. Thereby, by Lemma 12, since $Q_\ell \models^{h\sigma\cdot x^o} \phi_\ell^R\sigma$ and θ' respects $h\sigma\cdot x^o$, $Q_\ell\theta' \models^{h\theta\cdot x^o} \phi_\ell^R\theta$ holds. Hence $Q' \models^{h\theta\cdot x^o} \big(\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle\mathtt{tt}\big)\theta$, by definition of disjunction. Thus by definition of $[\overline{a}(x)]$, we can conclude that $Q \models^h [\overline{a}(x)]\big(\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle\mathtt{tt}\big)$ holds.

Case $P \not\models^h [\overline{a}(x)](\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle \mathtt{tt})$ **:** Assume $P \models^h [\overline{a}(x)](\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle \mathtt{tt})$ for contradition. Since σ respects h and $P\sigma \xrightarrow{\overline{a\sigma}(x)} P'$, the previous assumption can hold only if $P' \models^{h\sigma\cdot x^o} (\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle \mathtt{tt})\sigma$. This holds only if, for some i, $P' \models^{h\sigma\cdot x^o} \phi_i^R\sigma$, or, for some j, $P' \models^{h\sigma\cdot x^o} \langle x_j\sigma = y_j\sigma\rangle \mathtt{tt}$. However, for all i, $P' \not\models^{h\sigma\cdot x^o} \phi_i^R\sigma$; and also, for all j, we have $x_j\sigma \neq y_j\sigma$ and $P' \not\models^{h\sigma\cdot x^o} \langle x_j\sigma = y_j\sigma\rangle \mathtt{tt}$, leading to a contradiction in either case. Thereby $P \not\models^h [\overline{a}(x)](\bigvee_i \phi_i^R \vee \bigvee_j \langle x_j = y_j\rangle \mathtt{tt})$.

By induction we have established that, for any history h, processes P and Q, and any n, if $P \not\sim_n^h Q$ then there exists ϕ_L such that $P \models^h \phi_L$ and $Q \not\models^h \phi_L$, and also there exists ϕ_R such that $Q \models^h \phi_R$ and $P \not\models^h \phi_R$. The result then follows by observing that $\not\sim^h$ is the least relation containing all $\not\sim_n^h$; and, furthermore, $P \not\sim Q$ holds simply when $P \not\sim^{x_1^i\cdot\ldots\cdot x_n^i} Q$ holds, where $\mathrm{fv}(P) \cup \mathrm{fv}(Q) \subseteq \{x_1^i, \ldots, x_n^i\}$. ◀

Since open bisimilarity is decidable for finite π-calculus processes, the constructive non-bisimilarity in Definition 7 coincides with the negation of open bisimilarity.

▶ **Lemma 15.** *For finite processes, $P \not\sim Q$ holds, according to constructive non-bisimilarity in Definition 7, if and only if $P \sim Q$ does not hold.*

Combining Proposition 14 with Lemma 15 yields immediately the completeness of $\mathcal{OM}$ with respect to open bisimilarity. Completeness (Theorem 6) establishes that the set of all pairs of processes that have the same set of distinguishing formulae is an open bisimilarity. The proof can now be stated as follows.

Proof of Theorem 6: Assume that for finite processes P and Q, for all formulae ϕ, $P \models \phi$ iff $Q \models \phi$. Now for contradiction suppose that $P \sim Q$ does not hold. By Lemma 15, $P \not\sim Q$ must hold. Hence by Proposition 14 there exists ϕ_L such that $P \models \phi_L$ but $Q \not\models \phi_L$, but by the assumption above $Q \models \phi_L$, leading to a contradiction. Thereby $P \sim Q$. ◀

Notice that soundness (Theorem 5) and the non-bisimilarity algorithm (Proposition 14) also hold for infinite π-calculus processes (using replication for instance). However, for infinite π-calculus processes, open bisimilarity is undecidable; hence additional insight may be needed to justify whether Lemma 15 holds for infinite processes. Thereby in the infinite case, while it is impossible that $P \sim Q$ and $P \not\sim Q$ holds, it may be the case that neither holds. A possibility is that a more expressive logic is required to completely characterise open bisimilarity for infinite processes.

3.3 Example runs of distinguishing formulae algorithm

We provide further examples of non-bisimilar processes that illustrate subtle aspects of the algorithm. In particular, these examples illustrate the need for disjunctions of postconditions in both the base case and inductive steps.

3.3.1 Multiple postconditions and postconditions in an inductive step

The following example leads to multiple postcondition. Consider the following non-bisimilar processes: $[x = y]\tau + [w = z]\tau \not\sim \tau$. Observe that clearly $\tau \xrightarrow{\tau} 0$ but $([x = y]\tau + [w = z]\tau)\theta \xrightarrow{\tau}$ only if $x\theta = y\theta$ or $w\theta = z\theta$. Thus, $[x = y]\tau + [w = z]\tau \models [\tau](\langle x = y\rangle \mathtt{tt} \vee \langle w = z\rangle \mathtt{tt})$ is a distinguishing formula biased to the left process, while $\tau \models \langle\tau\rangle \mathtt{tt}$ is biased to the right.

Now consider an example where postconditions are required in the inductive case. Firstly observe that $\overline{a}a + \overline{b}b \not\sim \overline{a}a$ are distinguished since $\overline{a}a + \overline{b}b \xrightarrow{\overline{b}b} 0$, but process $\overline{a}a$ can only

make a $\overline{b}b$ transition under a substitution such that $a = b$. Hence we have the distinguishing formulae $\overline{a}a + \overline{b}b \models \langle \overline{b}b \rangle \mathtt{tt}$ and $\overline{a}a \models [\overline{b}b]\langle a = b\rangle \mathtt{tt}$.

For the inductive case, consider $P \triangleq \tau.(\overline{a}a + \overline{b}b) + [x = y]\tau.\overline{a}a \not\sim \tau.(\overline{a}a + \overline{b}b) + \tau.\overline{a}a \triangleq Q$. Let us lead by $Q \xrightarrow{\tau} \overline{a}a$, which can only be matched by $P \xrightarrow{\tau} \overline{a}a + \overline{b}b$. By the above observation, we have distinguishing formulae for $\overline{a}a + \overline{b}b \not\sim \overline{a}a$. Furthermore, $P\theta \xrightarrow{\tau}$ for substitutions θ such that $x\theta = y\theta$.

This leads to the following distinguishing formula for the left side, consisting of a box τ followed by a disjunction of the left distinguishing formula for $\overline{a}a + \overline{b}b \not\sim \overline{a}a$, and the postcondition for any additional τ transitions. $\tau.(\overline{a}a + \overline{b}b) + [x = y]\tau.\overline{a}a \models [\tau](\langle \overline{b}b \rangle \mathtt{tt} \vee \langle x = y\rangle \mathtt{tt})$.

The distinguishing formula for the right process is diamond τ followed by the right distinguishing formula for $\overline{a}a + \overline{b}b \not\sim \overline{a}a$, as follows: $\tau.(\overline{a}a + \overline{b}b) + \tau.\overline{a}a \models \langle\tau\rangle[\overline{b}b]\langle a = b\rangle \mathtt{tt}$.

3.3.2 Formulae generated by substitutions applied to labels

In some cases substitutions applied to labels play a role when generating distinguishing formulae. For a minimal example consider the following non-bisimilar processes: $\overline{a}a \not\sim \overline{a}b$. A distinguishing strategy is where process $\overline{a}b$ makes a $\overline{a}b$ transition, which cannot be matched by $\overline{a}a$. However, $(\overline{a}a)\sigma \xrightarrow{(\overline{a}b)\sigma} 0$ for any substitution such that $a\sigma = b\sigma$, leading to distinguishing formula $[\overline{a}b]\langle a = b\rangle \mathtt{tt}$ biased to $\overline{a}a$. Notice substitution σ is applied to both the process and the label.

For a trickier example consider the following: $\nu b.\overline{a}b.a(x).[x = b]\overline{x}x \not\sim \nu b.\overline{a}b.a(x).\overline{x}x$. After two actions, the problem reduces to base case $[x = b]\overline{x}x \not\sim^{a^i\cdot b^o\cdot x^i} \overline{x}x$, where $\overline{x}x$ can perform a $\overline{x}x$ action, but $[x = b]\overline{x}x$ cannot. However, $([x = b]\overline{x}x)\{{}^{b}/_{x}\} \xrightarrow{\overline{x}x\{{}^{b}/_{x}\}} 0$ does hold, and furthermore $\{{}^{b}/_{x}\}$ respects $a^i \cdot b^o \cdot x^i$. From these observations we can construct a distinguishing formula biased to the left as follows: $\nu b.\overline{a}b.a(x).[x = b]\overline{x}x \models [a(b)][a(x)][\overline{x}x]\langle x = b\rangle \mathtt{tt}$.

3.3.3 Alternative forms for distinguishing formulae

For the two non-bisimilar processes $[x = y]\tau.\tau + \tau \not\sim \tau.\tau + \tau$, we can think of a distinguishing formula biased to the left process: $[x = y]\tau.\tau + \tau \models [\tau][\tau]\langle x = y\rangle \mathtt{tt}$. However, this is different from the left-biased formula generated by the algorithm: $[x = y]\tau.\tau + \tau \models [\tau]([\tau]\mathtt{ff} \vee \langle x = y\rangle \mathtt{tt})$. Thus, there may exist alternative distinguishing formulae other than those generated by the algorithm.

3.3.4 An elaborate example demanding intuitionistic assumptions

For a more elaborate example consider the following.

$$\tau.(\underbrace{\tau + \tau.\tau + \tau.[x = y][w = z]\tau}_{P'}) \triangleq P \quad \not\sim \quad Q \triangleq \tau.(\underbrace{\tau + \tau.\tau + \tau.[x = y]\tau}_{Q'}) + P$$

A non-bisimilarity strategy is as follows: firstly, lead by transition $Q \xrightarrow{\tau} Q'$ on the right, matched by transition $P \xrightarrow{\tau} P'$; secondly, lead by $P' \xrightarrow{\tau} [x = y][w = z]\tau$ on the left, matched in three possible ways by $Q' \xrightarrow{\tau} 0$, $Q' \xrightarrow{\tau} \tau$ and $Q' \xrightarrow{\tau} [x = y]\tau$. To distinguish 0 from $[x = y][w = z]\tau$ observe that $([x = y][w = z]\tau)\{{}^{y}/_{x}\}\{{}^{z}/_{w}\} \xrightarrow{\tau} 0$ but 0 can make no τ transition; hence distinguishing formulae for 0 and $[x = y][w = z]\tau$ are $0 \models [\tau]\mathtt{ff}$ and $[x = y][w = z]\tau \models [x = y][w = z]\langle\tau\rangle\mathtt{tt}$. To distinguish $[x = y]\tau$ from $[x = y][w = z]\tau$, observe that $([x = y]\tau)\{{}^{y}/_{x}\} \xrightarrow{\tau} 0$, but $([x = y][w = z]\tau)\{{}^{y}/_{x}\}$ can only make a τ transition under a substitution such that also $w = z$; hence $[x = y]\tau \models [x = y]\langle\tau\rangle\mathtt{tt}$ and $[x = y][w =$

$z]\tau \models [\tau]\langle w = z\rangle\mathtt{tt}$ are distinguishing formulae. The same formulae also distinguish τ from $[x = y][w = z]\tau$. Thereby the algorithm in the completeness proof generates the following:

$$P \models [\tau]\langle\tau\rangle\big([\tau]\langle w = z\rangle\mathtt{tt} \wedge [x = y][w = z]\langle\tau\rangle\mathtt{tt}\big) \qquad Q \models \langle\tau\rangle[\tau]\big([\tau]\mathtt{ff} \vee [x = y]\langle\tau\rangle\mathtt{tt}\big)$$

The strategy explained above is not unique. An alternative strategy can generate different distinguishing formulae: $P \models [\tau][\tau]\big(\langle\tau\rangle\mathtt{tt} \vee [\tau]\langle w = z\rangle\mathtt{tt}\big)$ and $Q \models \langle\tau\rangle\langle\tau\rangle\big([x = y]\langle\tau\rangle\mathtt{tt} \wedge [\tau]\langle x = y\rangle\mathtt{tt}\big)$. Note if we assume the law of excluded middle, both processes above become equivalent to $\tau.\big(\tau + \tau.\tau\big)$. Fortunately, we do not assume the law of excluded middle.

4 Related work

We consider the relationship between the intuitionistic modal logic for open bisimilarity presented in this work and established classical logics. We also compare this work to existing work claiming to characterise open bisimilarity for the π-calculus.

4.1 Comparison to classical logics for late bisimilarity

The late Milner-Parrow-Walker logic, called $\mathcal{LM}$ [9] for "($\mathcal{L}$) late modality with ($\mathcal{M}$) match" differs from the logic presented in this paper in three significant ways: firstly, free names are a priori assumed to be distinct; secondly, $\mathcal{LM}$ is classical, that is, the law of excluded middle for name equalities is assumed; and thirdly the late input box modality is defined differently as follows — involving an existential quantification over substitutions:

- $P \models^L \big[a(x)\big]^L\phi$ iff for all Q such that $P \xrightarrow{a(x)} Q$ there exists name z such that $Q\{^z/_x\} \models^L \phi\{^z/_x\}$.

To see that logical equivalence for $\mathcal{LM}$ does not define a congruence, consider the processes $[x = y]\overline{x}x$ and 0. These processes satisfy the same set of late formulae (any formula equivalent to $\mathtt{tt}$), since, for $\mathcal{LM}$, x and y are a priori assumed to be distinct names. However, $a(y).[x = y]\overline{x}x$ and $a(y).0$ have distinguishing formulae $a(y).[x = y]\overline{x}x \models^L \big[a(y)\big]^L\langle\overline{x}x\rangle\mathtt{tt}$ biased to the left and its de Morgan complement $a(y).0 \models^L \langle a(y)\rangle\big[\overline{x}x\big]\mathtt{ff}$ biased to the right.

Between open bisimilarity and late bisimilarity there is late congruence, which is the greatest congruence relation contained in late bisimilarity. Late congruence must contain open bisimilarity, since open bisimilarity is contained in late bisimilarity and open bisimilarity is a congruence. Late congruence also has a simpler characterisation: P and Q are late congruent whenever for all substitutions σ, and $P\sigma$ is late bisimilar to $Q\sigma$. The quantification over all substitutions, combined with the law of excluded middle, has the effect that we check late bisimilarity with respect to all combinations of equalities and inequalities between free names.

As for open bisimilarity, $[x = y]\overline{x}x$ and 0 are not late congruent. This is because for substitution $\{^x/_y\}$, $([x = y]\overline{x}x)\{^x/_y\}$ and $0\{^x/_y\}$ are clearly not late bisimilar. This illustrates that late congruence is strictly finer than late bisimilarity. However, open bisimilarity is still strictly finer than late congruence, since $\tau + \tau.\tau + \tau.[x = y]\tau$ and $\tau + \tau.\tau$ are late congruent. Late congruence holds since, for any substitution θ, $(\tau + \tau.\tau)\theta$ and $(\tau + \tau.\tau + \tau.[x = y]\tau)\theta$ are late bisimilar. In contrast, we know these processes are not open bisimilar; and furthermore, have distinguishing formula that rely on the absence of the law of excluded middle.

4.2 Other embeddings into intuitionistic nominal logic

Tiu and Miller [16] studied embeddings of the π-calculus into the logic LINC, as well as late and open bisimilarity and their respectful modal logics. This is the most closely related work since our encodings in Abella were adapted from their work. In their encoding, both late and open bisimilarity are encoded by essentially the same modalities, differing only in the the law of the excluded middle for names and the quantification of free variables. However, no examples of distinguishing formulae for open bisimilarity were provided; and, critically, the proof made flawed assumptions about the existence of a syntactic negation of a formula, which we observe in this work is not permitted.

A problem with the approach of Tiu and Miller is the reuse of the input box modality from $\mathcal{LM}$, which involves an existential quantification over substitutions. In contrast, our input box modality in $\mathcal{OM}$ involves universal quantification over all respectful substitutions. Our choice in $\mathcal{OM}$ is critical for generating distinguishing formulae. For example, the following processes are not open bisimilar: $a(x).\tau + a(x) + a(x).[x = a]\tau \not\sim a(x).\tau + a(x)$.

For the above processes, the algorithm for distinguishing formulae in Proposition 14, correctly generates the following $\mathcal{OM}$ formula biased to the right:
$a(x).\tau + a(x) \models \big[a(x)\big]\big(\langle\tau\rangle\mathtt{tt} \vee \big[\tau\big]\mathtt{ff}\big)$.

However, using only late modalities, as in Tiu and Miller, there is no distinguishing formula for these processes biased to the right: e.g., the formula with a late modality $\big[a(x)\big]^L\big(\langle\tau\rangle\mathtt{tt} \vee \big[\tau\big]\mathtt{ff}\big)$ succeeds for both processes, even when rejecting the law of excluded middle; also the formula $\big[a(x)\big]^L\big(\langle x = a\rangle\big[\tau\big]\mathtt{ff} \vee \langle\tau\rangle\big[x = a\big]\mathtt{ff}\big)$ fails for both processes, despite being distinguishing in classical $\mathcal{LM}$. The choice of modalities we make in $\mathcal{OM}$ make sense, since in open bisimilarity the choice of substitution is deferred as late as possible — possibly several transitions later.

4.3 A generic formalisation using nominal logic

Recently, Parrow et al. [12] provided a general proof of the soundness and completeness of logical equivalence for various modal logics with respect to corresponding bisimulations. The proof is parametric on properties of substitutions, which can be instantiated for a range of bisimulations. Moreover, their proof is mechanised using Nominal Isabelle. The conference version [12], sketches how to instantiate the abstract framework for open bisimilarity in the π-calculus without input prefixes only. However, we understand from communication with the authors that open bisimilarity for the π-calculus with input prefixes will be covered in a forthcoming extended version.

Stylistically, our intuitionistic modal logic is quite different from an instantiation of the abstract framework of Parrow et al. for open bisimilarity. Their framework, is classical and works by syntactically restricting "effect" modalities in formulae, depending on the type of bisimulation. Their effects represent substitutions that reach worlds permitted by the type of bisimulation. In contrast, the modalities of the intuitionistic modal logic $\mathcal{OM}$ in this paper are syntactically closer to long established modalities for the π-calculus [9]; differing instead in their semantic interpretation and in the absence of classical negation. An explanation for the stylistic differences is that for every intuitionistic logic, such as the intuitionistic modal logic in this work, there should be a corresponding classical modal logic based on an underlying Kripke semantics. Such a Kripke semantics would reflect the accessible worlds, as achieved by the syntactically restricted effect modalities in the abstract classical framework instantiated for open bisimilarity.

5 Conclusion

The main result of this paper is a sound and complete logical characterisation of open bisimilarity for the π-calculus. To achieve this result, we introduce modal logic $\mathcal{OM}$, defined in Fig. 2. The soundness of $\mathcal{OM}$ with respect to open bisimilarity, Theorem 5, is mechanically proven in Abella. The details of the completeness, Theorem 6, are provided in Section 3.

There are several novel features of $\mathcal{OM}$ compared to established modal logics for π-calculus, such as $\mathcal{LM}$ characterising late bisimilarity. Firstly, as demonstrated in Examples 2.3.1 and 3.3.4, the absence of the law of excluded middle is essential for the existence of distinguishing formulae in $\mathcal{OM}$ for certain processes that are not open bisimilar (but are late congruent). The absence of the law of excluded middle is an intuitionistic assumption; and, as explained in the introduction, $\mathcal{OM}$ can indeed be considered to be a conservative extension of intuitionistic logic. Furthermore, in contrast to classical modal logics such as $\mathcal{LM}$, diamond and box modalities have independent interpretations, not dual to each other. These properties are expected under criterion set out for intuitionistic modal logics [14]. The absence of de Morgan dualities over modalities complicates the construction of distinguishing formulae.

The completeness proof involves an algorithm, Proposition 14, that constructs distinguishing formulae for non-bisimilar processes. To use this algorithm, firstly attempt to prove that two processes are open bisimilar. If they are non-bisimilar, after a finite number of steps, a distinguishing strategy, according to Def. 7, will be discovered. The strategy can then be used to inductively construct two distinguishing formulae, biased to each process. A key feature of the construction is that there are restricted versions of absolute truth by preconditions ($[\sigma]\langle\pi\rangle\mathtt{tt}$ restricted from $\langle\pi\rangle\mathtt{tt}$) and, dually, there are relaxed versions of absolute falsity by postconditions ($[\pi]\langle\sigma\rangle\mathtt{tt}$ relaxed from $[\pi]\mathtt{ff}$), as demonstrated in Examples 2.2.2 and 3.3.1.

Our logic $\mathcal{OM}$ is suitable for formal and automated reasoning; in particular, it has natural encodings in Abella for mechanised reasoning, used to establish Theorem 5, and Bedwyr [3] for automatic proof search. All bisimulations and satisfactions in examples have been automatically checked in Bedwyr and are available online: `https://github.com/kyagrd/NonBisim2DF`. In addition, our distinguishing formulae generation algorithm is implemented in Haskell [1].

Future work includes justifying whether or not $\mathcal{OM}$ is complete for infinite processes with replication or recursion, as discussed around Lemma 15. A related problem is to extend $\mathcal{OM}$ with fixed points, as in the μ-calculus [7]. Such an extension could lead to intuitionistic model checkers invariant under open bisimulation, where the call-by-need approach to inputs is related to symbolic execution. We are also interested in extensions of $\mathcal{OM}$ for open bisimulation in the spi-calculus [15].

Acknowledgments. We are grateful to Sam Staton for providing an example that helped us in the completeness proof.

References

1 Ki Yung Ahn, Ross Horne, and Alwen Tiu. Generating witness of non-bisimilarity for the pi-calculus. *CoRR*, abs/1705.10908, 2017. URL: `http://arxiv.org/abs/1705.10908`.

2 David Baelde, Kaustuv Chaudhuri, Andrew Gacek, Dale Miller, Gopalan Nadathur, Alwen Tiu, and Yuting Wang. Abella: A system for reasoning about relational specifications. *Journal of Formalized Reasoning*, 7(2):1–89, 2014. `doi:10.6092/issn.1972-5787/4650`.

3 David Baelde, Andrew Gacek, Dale Miller, Gopalan Nadathur, and Alwen Tiu. *The Bedwyr System for Model Checking over Syntactic Expressions*, pages 391–397. Springer Berlin Heidelberg, 2007. doi:10.1007/978-3-540-73595-3_28.

4 Andrew Gacek, Dale Miller, and Gopalan Nadathur. A two-level logic approach to reasoning about computations. *Journal of Automated Reasoning*, 49(2):241–273, 2012. doi:10.1007/s10817-011-9218-1.

5 Maciej Gazda and Wan Fokkink. Modal logic and the approximation induction principle. *Mathematical Structures in Computer Science*, 22(2):175–201, 2012. doi:10.1017/S0960129511000387.

6 Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985. doi:10.1145/2455.2460.

7 Dexter Kozen. Results on the propositional mu-calculus. *Theoretical Computer Science*, 27(3):333–354, 1983. doi:10.1016/0304-3975(82)90125-6.

8 Robin Milner, Joachim Parrow, and David Walker. A calculus of mobile processes, Parts I and II. *Information and Computation*, 100(1):1–77, 1992. doi:10.1016/0890-5401(92)90008-4.

9 Robin Milner, Joachim Parrow, and David Walker. Modal logics for mobile processes. *Theoretical Computer Science*, 114(1):149–171, 1993. doi:10.1016/0304-3975(93)90156-N.

10 Ugo Montanari and Vladimiro Sassone. Dynamic congruence vs. progressing bisimulation for CCS. *Fundamenta informaticae*, 16(2), 1992.

11 Gopalan Nadathur and Dale Miller. An Overview of λProlog. In *Fifth International Logic Programming Conference*. MIT Press, 1988.

12 Joachim Parrow, Johannes Borgström, Lars-Henrik Eriksson, Ramunas Gutkovas, and Tjark Weber. Modal logics for nominal transition systems. In *CONCUR 2015*, volume 42 of *LIPIcs*, pages 198–211, 2015. doi:10.4230/LIPIcs.CONCUR.2015.198.

13 Davide Sangiorgi. A theory of bisimulation for the π-calculus. *Acta Informatica*, 33(1):69–97, 1996. doi:10.1007/s002360050036.

14 Alex K. Simpson. *The proof theory and semantics of intuitionistic modal logic*. PhD thesis, University of Edinburgh, UK, 1994.

15 Alwen Tiu and Jeremy Dawson. Automating open bisimulation checking for the spi calculus. In *Computer Security Foundations Symposium (CSF), 2010 23rd IEEE*, pages 307–321. IEEE, 2010. doi:10.1109/CSF.2010.28.

16 Alwen Tiu and Dale Miller. Proof search specifications of bisimulation and modal logics for the π-calculus. *ACM Transactions on Computational Logic*, 11(2):13:1–13:35, 2010. doi:10.1145/1656242.1656248.

Consistently-Detecting Monitors*

Adrian Francalanza

CS, ICT, University of Malta, Msida, Malta
adrian.francalanza@um.edu.mt

Abstract

We study a contextual definition for deterministic monitoring based on consistent detections. It is defined in terms of the observed behaviour of the monitor when instrumented over arbitrary systems. We give an alternative, coinductive definition based on controllability which does not rely on system quantifications, and show that it is fully-abstract *wrt.* the former definition. We then develop a symbolic counterpart to the controllability definition to facilitate an automated analysis for controllable monitors involving data.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs, F.3.2 Semantics of Programming Languages - Process models, D.2.5 Testing and Debugging

Keywords and phrases Runtime Monitoring, Deterministic Behaviour, Controllability, Compositional Reasoning, Symbolic Analysis

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.8

1 Introduction

Monitors are computational entities that observe the executions of other entities (referred to hereafter as *systems*) with the aim of accruing system information [32, 26], comparing system executions against some behavioural specification [23, 7], or reacting to the observed executions via adaptation or enforcement procedures [13, 36]. They are typically considered to be part of the *Trusted Computing Base (TCB)* and, consequently, their descriptions are expected to be *correct*. A correctness requirement often presumed of monitors is that they should exhibit *deterministic behaviour*. Yet, for most monitoring frameworks, such a requirement is seldom specified in unambiguous terms. In fact, there are a number of viable alternatives that one could consider (*e.g.,* [44, 29, 25, 1]) and it is unclear how to choose one over the other in an objective manner. Moreover, these definitions often fail to account for the instrumentation mechanism used to compose a monitor with the system under scrutiny which may, in turn, affect monitoring behaviour. All of this leads to a poor understanding of what should be expected of a monitor, and may give rise to discrepancies between these expectations and what needs to be guaranteed by the monitor implementer in practice.

Non-determinism is intrinsic to a number of computational models used for expressing monitors and monitored systems. In fact, a substantial body of work on monitors is either cast in terms of inherently non-deterministic formalisms such as Büchi automata [45, 17], or formalisms that admit non-determinism such as process calculi and labelled transition systems [12, 46, 30, 22, 10, 23]. Non-deterministic computation arises naturally in concurrent and distributed programming, used increasingly for runtime monitoring [37, 24, 21, 8, 11]. Furthermore, a growing number of monitoring tools employ automata-based specification languages [16, 5, 41, 18] that offer rudimentary support for ensuring deterministic behaviour:

* Partly supported by UoM and RANNIS projects CPSRP05-04 and TheoFoMon:163406-051.

their respective implementations are either thread-unsafe [41] or admit arbitrary code for transition-triggered actions [14, 16].

This paper sets out to investigate deterministic behaviour for monitors. The study is limited to execution monitors (sequence recognisers) [43, 36], used extensively for Runtime Verification (RV). Our work is developed in terms of an expository formalism (similar to the aforementioned work on transition-based descriptions) expressing monitored systems that can analyse trace events carrying data and admit degrees of non-determinism. We propose a *contextual* definition for deterministic monitor behaviour, founded on the *observational* behaviour that can be discerned when a monitor is instrumented to execute with any arbitrary system under scrutiny. The definition serves two purposes. First, its contextual nature allows us to admit *as many* correct monitors *as possible*, as long as these cannot be *externally* perceived to behave non-deterministically—we contend that the resulting definition is fairly intuitive. Second, it allows us to *justify* design decisions for an alternative definition describing the deterministic behaviour of monitors, based instead on the notion of *controllability* [31]. We show a correspondence between these two definitions. In addition, we demonstrate how the alternative definition (which is arguably less intuitive than its contextual counterpart) is more amenable to automated analyses for assessing the deterministic behaviour exhibited by monitors. In particular, we study how this alternative definition can be reformulated in symbolic terms, to facilitate a tractable handling of infinite-state monitor analysis due to data.

▶ **Example 1.** The monitor description m_1 accepts traces from an authenticator, that challenges (event chl) a supplicant for an *arbitrary* value x followed by the supplicant's authentication (aut) with the (correct) encoding of x, $y = enc(x)$. The authenticator subsequently acknowledges (ack) using the same value y. The guard construct $\mathsf{chl}(x)$ quantifies over *any* value x; guards $\mathsf{aut}\langle v\rangle$ and $\mathsf{ack}\langle v\rangle$ require authentications (*resp.* acknowledgments) for a *specific* value v. $\top$ denotes acceptance.

$$m_1 \triangleq \mathsf{chl}(x).\,\mathsf{let}\ y = enc(x)\ \mathsf{in}\ \big(\mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top\big)$$
$$m_2 \triangleq \mathsf{chl}(x).\mathsf{let}\ y = enc(x)\ \mathsf{in}\ \big(\mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top\ +\ \mathsf{aut}(z).\mathsf{if}\ z\neq y\ \mathsf{then}\ \mathsf{ack}(z').\bot\big)$$
$$m_3 \triangleq \mathsf{chl}(x).\mathsf{let}\ y = enc(x)\ \mathsf{in}\ \big(\mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top + \mathsf{aut}(z).\mathsf{if}\ z\neq y\ \mathsf{then}\ \mathsf{ack}(z').\bot\ \mathsf{else}\ \mathsf{ack}\langle z\rangle.\top\big)$$

It is easy to inadvertently introduce non-determinism. Monitor m_2 extends m_1 (using the choice operator, $+$) with the intention of rejecting (note the verdict $\bot$) acknowledgments following authentications whose value is *not* the encoding of the challenge value x. When $v_2{\neq}enc(v_1)$, the violating trace $t{=}\mathsf{chl}\langle v_1\rangle.\mathsf{aut}\langle v_2\rangle.\mathsf{ack}\langle v_2\rangle\ldots$ is always rejected by m_2. More subtly, however, when $v_2{=}enc(v_1)$ the trace t may cause the monitor to non-deterministically choose either branch, whereby the unintended branch does *not* reach a $\top$ verdict. But non-determinism is not necessarily conducive to inconsistent verdicts and, in cases where verdicts are considered as the only monitor observable behaviour, such non-determinism may be tolerated. Monitor m_3 *deterministically accepts* trace t when $v_2{=}enc(v_1)$, albeit along different execution paths, and *deterministically rejects* it whenever $v_2{\neq}enc(v_1)$. ◀

The main contributions of the paper are:

1. A contextual definition for observationally deterministic monitoring behaviour, Definition 6.
2. An alternative definition based on controllability, Definition 11, that coincides with it, Theorem 13.
3. Symbolic variants of the latter definition enabling tractable automations, Theorems 21 and 25.

Monitors

$$
\begin{array}{rcll}
w, o \in \textsc{Verd} & ::= & \top \quad \text{(accept)} & \mid \bot \quad \text{(reject)} \\
& \mid & \mathbf{0} \quad \text{(inconclusive)} & \\
m, n \in \textsc{Mon} & ::= & w \quad \text{(verdict)} & \mid \mathsf{let}\ x = e\ \mathsf{in}\ m \quad \text{(evaluate)} \\
& \mid & l\langle e \rangle.m \quad \text{(expression guard)} & \mid l(x).m \quad \text{(quantified guard)} \\
& \mid & m + n \quad \text{(choice)} & \mid \mathsf{if}\ b\ \mathsf{then}\ m\ \mathsf{else}\ n \quad \text{(conditional)} \\
& \mid & \mathsf{rec}\, X.m \quad \text{(recursion)} & \mid X \quad \text{(monitor variable)}
\end{array}
$$

$$
\textsc{Ver}\ \frac{}{w \xrightarrow{\eta} w} \qquad \textsc{IfT}\ \frac{[\![b]\!] = \mathsf{true}}{\mathsf{if}\ b\ \mathsf{then}\ m\ \mathsf{else}\ n \xrightarrow{\tau} m} \qquad \textsc{IfF}\ \frac{[\![b]\!] = \mathsf{false}}{\mathsf{if}\ b\ \mathsf{then}\ m\ \mathsf{else}\ n \xrightarrow{\tau} n} \qquad \textsc{Let}\ \frac{[\![e]\!] = v}{\mathsf{let}\ x = e\ \mathsf{in}\ m \xrightarrow{\tau} m[v/x]}
$$

$$
\textsc{GrE}\ \frac{[\![e]\!] = v}{l\langle e \rangle.m \xrightarrow{l\langle v \rangle} m} \qquad \textsc{GrQ}\ \frac{}{l(x).m \xrightarrow{l\langle v \rangle} m[v/x]} \qquad \textsc{Rec}\ \frac{}{\mathsf{rec}\, X.m \xrightarrow{\tau} m[\mathsf{rec}\, X.m/X]} \qquad \textsc{Ch1}\ \frac{m \xrightarrow{\alpha} m'}{m + n \xrightarrow{\alpha} m'}
$$

Instrumentation

$$
\textsc{Mon}\ \frac{s \xrightarrow{\eta} r \quad m \xrightarrow{\eta} n}{s \lhd m \xrightarrow{\eta} r \lhd n} \qquad \textsc{Ter}\ \frac{s \xrightarrow{\eta} r \quad m \overset{\eta}{\not\longrightarrow} \quad m \overset{\tau}{\not\longrightarrow}}{s \lhd m \xrightarrow{\eta} r \lhd \mathbf{0}} \qquad \textsc{AsS}\ \frac{s \xrightarrow{\tau} r}{s \lhd m \xrightarrow{\tau} r \lhd m} \qquad \textsc{AsM}\ \frac{m \xrightarrow{\tau} n}{s \lhd m \xrightarrow{\tau} s \lhd n}
$$

Figure 1 A Model for describing Instrumented Systems.

The paper is structured as follows. Section 2 presents the monitor framework used for this study, allowing us motivate our touchstone definition for deterministic monitor behaviour in Section 3. Section 4 presents a fully-abstract alternative definition that is amenable to compositional reasoning. In Sections 5 and 6 symbolic variants are developed for automation purposes. Section 7 concludes.

2 Systems, Monitors, Instrumentation and Monitored Systems

We perceive *systems* as entities that generate *events* while executing. *Observable* events, $\eta \in \textsc{Evt}$, are those visible to a monitor and have the form $l\langle v \rangle$ where l is an event *label* taken from a set $l, k \in \textsc{Lab}$, and v is an event *payload* taken from some unspecified value domain $v, u \in \textsc{Val}$. Events capture a number of computational notions such as inputs/outputs in message-passing programs [22], or method/function calls and returns [13, 38]. To simplify our technical development, we consider monadic (*i.e.*, single-valued) events but the formalism can be extended to accommodate polyadicity.

Systems may be described as Labelled Transition Systems (LTSs) [2, 34], triples $\langle \textsc{Sys}, \textsc{Act}, \longrightarrow \rangle$ consisting of a set of states, $s, r \in \textsc{Sys}$, a set of actions, $\alpha, \beta \in \textsc{Act} = \textsc{Evt} \cup \{\tau\}$ that include all observable events $\textsc{Evt}$ and a distinguished silent action $\tau \notin \textsc{Evt}$ for *unobservable* events, and a transition relation, $\longrightarrow \subseteq (\textsc{Sys} \times \textsc{Act} \times \textsc{Sys})$. The suggestive notation $s \xrightarrow{\alpha} s'$ denotes $(s, \alpha, s') \in \longrightarrow$, whereas $s \overset{\alpha}{\not\longrightarrow}$ denotes $\neg(\exists s' \cdot\ s \xrightarrow{\alpha} s')$. As usual, we write $s \Longrightarrow s'$ in lieu of $s(\xrightarrow{\tau})^* s'$ and $s \overset{\eta}{\Longrightarrow} s'$ for $s \Longrightarrow \cdot \xrightarrow{\eta} \cdot \Longrightarrow s'$, referring to s' as a *η-derivative* of s; $s \overset{t}{\Longrightarrow}$ stands for $\exists s' \cdot s \overset{t}{\Longrightarrow} s'$. We let *traces*, $t, u \in \textsc{Evt}^*$, range over (finite) sequences of *observable* events and write $s \overset{\eta_1}{\Longrightarrow} \ldots \overset{\eta_n}{\Longrightarrow} s_n$ as $s \overset{t}{\Longrightarrow} s_n$, where $t = \eta_1, \ldots, \eta_n$. The notation $u \ldots$ is occasionally used to denote the existence of some trace t with a prefix u.

We presuppose an expression language $e, d \in \textsc{Exp}$ that ranges over the (event) value domain $\textsc{Val}$ and a denumerable set of *expression variables* $x, y, z \in \textsc{Vars}$; $\vec{e}$ and $\vec{x}$ *resp.*

denote lists of expressions and variables. We also assume a boolean expression language $b, c \in \text{BExp}$ defined over Exp that includes standard constructs for conjunctions, $b \wedge c$, and negations, $\neg b$, but also equality predicates over expressions $e = d$. The meta-functions $\mathbf{fv}(e)$ and $\mathbf{fv}(b)$ return the *free variables* in the *resp.* expressions; expressions are *closed* whenever $\mathbf{fv}(e) = \emptyset$ and *open* otherwise, and similarly for boolean expressions. *Valuations* are total maps from variables to values, $\rho \in (\text{Vars} \to \text{Val})$ whereas *substitutions* are partial maps from variables to expressions $\sigma \in (\text{Vars} \rightharpoonup \text{Exp})$; substitutions are denoted as $[\vec{e}/\vec{x}]$, where $d[\vec{e}/\vec{x}]$ represents the (simultaneous) substitution of all occurrences of $x_i \in \vec{x}$ in d by the corresponding $e_i \in \vec{e}$. We assume an *evaluation function* that takes an expression and a valuation and returns a value, $[\![e\rho]\!] = v$. Similarly, boolean expressions have a semantic function mapping them to the boolean domain via a valuation, $[\![b\rho]\!] \in \{\mathsf{true}, \mathsf{false}\}$, where we presume the expected properties, *e.g.*, $[\![(b \wedge c)\rho]\!] = \mathsf{true}$ iff $[\![b\rho]\!] = \mathsf{true}$ and $[\![c\rho]\!] = \mathsf{true}$. We also assume a classical interpretation of boolean expressions, *i.e.*, $[\![(\neg b)\rho]\!] = \mathsf{true}$ iff $[\![b\rho]\!] = \mathsf{false}$. To alleviate the presentation, we often work up to associativity and commutativity for conjunctions, treating $\langle \text{BExp}, \wedge, \mathsf{true} \rangle$ as an abelian monoid. For closed expressions, we elide the valuation and write $[\![e]\!]$ and $[\![b]\!]$ for $[\![e\rho]\!]$ and $[\![b\rho]\!]$ *resp.* The *satisfiability* judgement for boolean expressions, $\mathbf{sat}(b) \stackrel{\text{def}}{=} \exists \rho \cdot [\![b\rho]\!] = \mathsf{true}$, plays a central role in subsequent development.

Monitors, here defined by the syntax in Figure 1, constitute the focus of our study. They may reach two kinds of verdicts, Verd. *Conclusive verdicts* consist of acceptances, $\top$, and rejections, $\bot$. In addition, a monitor may also reach the *inconclusive verdict*, $\mathbf{0}$, a form of premature termination used when the generated system events of the monitor specification itself does not yield sufficient information so as to reach a definite conclusion. The monitor *expression* guard $l\langle e \rangle.m$ expects events with label l and a payload value matching the evaluation of e, whereas the *quantified* guard $l(x).m$ allows the monitor to *dynamically learn* the payload of an event with label l. Monitors may branch (externally) depending on the events observed, $m+n$, or branch (internally) based on data predicates, $\mathsf{if}\, b\, \mathsf{then}\, m\, \mathsf{else}\, n$. They may also perform internal computation themselves by evaluating expressions, $\mathsf{let}\ x = e\ \mathsf{in}\ m$, or recurse, $\mathsf{rec}\, X.p$, via *term variables* $X, Y, Z \in \text{TVars}$. The constructs $l(x).m$ and $\mathsf{rec}\, X.m$ act as *binders* for x and X *resp.* in m, inducing the usual notions of open/closed (monitor) terms. We work up to alpha-conversion of bound expression/term variables and use the shorthand $\mathsf{if}\, b\, \mathsf{then}\, m$ for $\mathsf{if}\, b\, \mathsf{then}\, m\, \mathsf{else}\, \mathbf{0}$ and $\tau.m$ for $\mathsf{let}\ x = v\ \mathsf{in}\ m$ where $x \notin \mathbf{fv}(m)$.

The semantics of *closed* monitors is also defined in terms of an LTS, via the transition rules in Figure 1: for each $m \in \text{Mon}$ we have a dedicated LTS $\langle M, \text{Act}, \longrightarrow \rangle$ where $M \subseteq \text{Mon}$ are the monitors reachable from m via transitions. The rules model the monitor analysis of observable events. Rule Ver describes how verdicts are *irrevocable*, meaning that a verdict can analyse *any* observable event but always transition to itself. In rule GrE, an expression guard $l\langle e \rangle.m$ only transitions to the continuation m when observing an event matching the label l with the payload equal to $[\![e]\!]$. By contrast, a quantified guard $l(x).m$ transitions by analysing any event with label l, binding x to the event payload v in the continuation, $m[v/x]$; see rule GrQ. The remaining rules are as expected where the term $m[\mathsf{rec}\, X.m/X]$ denotes the term substitution of $\mathsf{rec}\, X.m$ for free occurrences of X in m.

A system s *instrumented* with a monitor m is referred to as a *monitored system* and denoted as $s \triangleleft m$. The semantics of monitored systems is defined by the instrumentation rules in Figure 1. We here adopt the composition relation studied in [22, 23], even though other instrumentation relations could have been used. Note that the chosen composition relation is still quite general: it is parametric *wrt.* the system and monitor abstract LTSs and it is *largely independent* of their specific language specifications, since it only requires the monitor LTS to contain an inconclusive (persistent) verdict state, $\mathbf{0}$. The instrumentation

relation of Figure 1 is *asymmetric*: a monitored system can transition with an observable event *only when* the system can produce that event *i.e.,* monitors are *passive* and cannot instigate transitions. When the system generates an (observable) event that can be analysed by the monitor, the two transition in lockstep according to their respective LTSs (rule Mon). When the monitor *cannot* analyse the event generated [1] and cannot internally transition to a state that enables it to do so (*i.e.,* it is already stable, $m \stackrel{\tau}{\not\longrightarrow}$), the instrumentation does *not* block the monitored system: instead, it allows the system to transition *but* aborts monitoring to the inconclusive verdict (rule Ter). System-monitor synchronisations are limited to observable events, and the specific entities can transition independently *wrt.* their respective internal moves (rules AsS and AsM).

▶ **Example 2.** Monitor m_4 below listens for input and output events $\mathsf{in}\langle v\rangle$ and $\mathsf{out}\langle v\rangle$ where the (integer) payload $v \in \mathbb{N}$ reports the port number over which the communication operation is performed.

$$m_4 \triangleq \mathsf{rec}\, X.\Big(\big(\mathsf{out}\langle 80\rangle.\bot\big) + \big(\mathsf{in}(x).\,\mathsf{if}\ x{=}80\ \ \mathsf{then}\ \ \mathsf{out}\langle 81\rangle.\top\ \ \mathsf{else}\ \ \mathsf{out}\langle x\rangle.X\big)\Big) \tag{1}$$

The monitor rejects system executions starting with an output on port 80 but accepts traces containing an input on port 80 followed by an output on port 81, preceded by an arbitrary number of input-output operations on any matching port other than 80. The execution below shows an *accepted* monitored computation for a system s generating the trace $\mathsf{in}\langle 85\rangle \cdot \mathsf{out}\langle 85\rangle \cdot \mathsf{in}\langle 80\rangle \cdot \mathsf{out}\langle 81\rangle$. In monitor m_4, the binding on $\mathsf{in}(x)$ acts as a *freeze-variable* [19] for the subsequent $\mathsf{out}\langle x\rangle$ guard in the else branch.

$$\begin{aligned} s \lhd m_4 &\xrightarrow{\tau} s \lhd \big(\mathsf{out}\langle 80\rangle.\bot\big) + \big(\mathsf{in}(x).\,\mathsf{if}\ x{=}80\ \mathsf{then}\ \mathsf{out}\langle 81\rangle.\top\ \mathsf{else}\ \mathsf{out}\langle x\rangle.m_4\big) && \textsc{Rec}\\ &\xrightarrow{\mathsf{in}\langle 85\rangle} s' \lhd \mathsf{if}\ 85{=}80\ \ \mathsf{then}\ \mathsf{out}\langle 81\rangle.\top\ \ \mathsf{else}\ \mathsf{out}\langle 85\rangle.m_4 && \textsc{Chr}+\textsc{GrQ}\\ &\xrightarrow{\tau} s' \lhd \mathsf{out}\langle 85\rangle.m_4 \xrightarrow{\mathsf{out}\langle 85\rangle} s'' \lhd m_4 \xRightarrow{\mathsf{in}\langle 80\rangle\cdot\mathsf{out}\langle 81\rangle} s''' \lhd \top && \textsc{If},\ldots \end{aligned}$$

The instrumentation of Figure 1 delays system transitions to allow the monitor to internally transition to a state that can process the event. *E.g.,* if a system r can generate event $\mathsf{out}\langle 80\rangle$, $r \lhd m_4$ postpones this transition (Mon and Ter cannot be applied) until m_4 unfolds.

$$r \lhd m_4 \xrightarrow{\tau} r \lhd \big(\mathsf{out}\langle 80\rangle.\bot\big) + \big(\mathsf{in}(x).\,\mathsf{if}\ x{=}80\ \mathsf{then}\ \ldots\big) \xrightarrow{\mathsf{out}\langle 80\rangle} r' \lhd \bot \qquad \textsc{AsM},\textsc{Mon}$$

Rule Ter is crucial both for allowing monitored computations to proceed when the monitor cannot analyse an event, but also to avoid *unintended detections*. *E.g.,* if system r can generate the trace $\mathsf{out}\langle 90\rangle \cdot \mathsf{in}\langle 80\rangle \cdot \mathsf{out}\langle 81\rangle$, this behaviour should still be permitted when instrumented with the monitor m_4, *but* the behaviour should not be detected according to the description in Equation 1. After the initial unfolding of m_4, Ter allows r to transition with $\mathsf{out}\langle 90\rangle$ but transitions m_4 to the inconclusive state, $\mathbf{0}$, since neither guard $\mathsf{out}\langle 80\rangle$ nor guard $\mathsf{in}(x)$ can process the event.

$$\begin{aligned} r \lhd m_4 &\xrightarrow{\tau} r \lhd \big(\mathsf{out}\langle 80\rangle.\bot\big) + \big(\mathsf{in}(x).\,\mathsf{if}\ x{=}80\ \mathsf{then}\ \ldots\big)\\ &\xrightarrow{\mathsf{out}\langle 90\rangle} r'' \lhd \mathbf{0} \xRightarrow{\mathsf{in}\langle 80\rangle\cdot\mathsf{out}\langle 81\rangle} r''' \lhd \mathbf{0} && \textsc{AsM},\textsc{Ter},\ldots \end{aligned}$$

Had rule Ter been designed otherwise (leaving the monitor state unaltered when transiting with $\mathsf{out}\langle 90\rangle$) the ensuing events $\mathsf{in}\langle 80\rangle \cdot \mathsf{out}\langle 81\rangle$ would lead to the unintended acceptance of the trace. ◀

[1] This may be due to a number of reasons, such as event knowledge gaps or knowledge disagreements [6].

3 Deterministic Monitoring Behaviour

In a monitored system, non-deterministic behaviour can be caused by either the system or the monitor. We focus on identifying *non-determinism attributed to monitors*, teasing it apart from non-determinism caused by system behaviour. This is motivated by the fact that, generally, one has limited control over the behaviour of a system under scrutiny. We target a definition that admits monitor non-determinism that is not externally observable. Concretely, we consider detections (*i.e.,* conclusive verdicts) as the only externally visible aspect of a monitor and base our definition on the notion of *deterministic detections*—in applications such as RV, detections are associated with property satisfactions and violations [23, 7]. This immediately rules out a number of candidate definitions for deterministic monitor behaviour. For instance, a definition that considers a monitor m to be deterministic whenever, for *all* systems s and traces t, $s \triangleleft m \stackrel{t}{\Rightarrow} s' \triangleleft m'$ and $s \triangleleft m \stackrel{t}{\Rightarrow} s' \triangleleft m''$ implies $m' = m''$ is too stringent: it precludes the monitor description below (a slight modification on m_1 of Example 1)

$$m_5 \triangleq \mathsf{chl}(x).\big(\big(\mathsf{let}\ y = enc(x)\ \mathsf{in}\ \mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top\big) + \big(\mathsf{aut}\langle enc(x)\rangle.\mathsf{ack}\langle enc(x)\rangle.\top\big)\big) \tag{2}$$

even though m_5 deterministically accepts traces of the form $\mathsf{chl}\langle v_1\rangle \cdot \mathsf{aut}\langle v_2\rangle \cdot \mathsf{ack}\langle v_2\rangle$ where $v_2 = enc(v_1)$. In fact, after an event $\mathsf{chl}\langle v\rangle$ (for some value v), monitor m_5 can reach two possible internal states, namely $\big(\mathsf{let}\ y = enc(v)\ \mathsf{in}\ \mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top\big) + \big(\mathsf{aut}\langle enc(v)\rangle.\mathsf{ack}\langle enc(v)\rangle.\top\big)$ or $\mathsf{aut}\langle v'\rangle.\mathsf{ack}\langle v'\rangle.\top$ where $v' = enc(v)$. Other candidates (*e.g.,* confluence defined over transitions [25, 40]) are either inadequate or not immediately applicable because they do not account for executions that do *not* lead to detections. *E.g.,* m_6 (below) would *not* be confluent (consider event $\mathsf{in}\langle 81\rangle$), even though it consistently rejects any trace with the prefix $u = \mathsf{in}\langle 80\rangle$ (and consistently does *not* detect all the other traces).

$$m_6 \triangleq \ \big(\mathsf{in}\langle 80\rangle.\bot\big) + \big(\big(\mathsf{in}\langle 81\rangle.\mathsf{out}\langle 81\rangle.\mathbf{0}\big) + \big(\mathsf{in}\langle 81\rangle.\mathsf{out}\langle 81\rangle.\mathsf{in}\langle 82\rangle.\mathbf{0}\big)\big) \tag{3}$$

▶ **Definition 3** (Detected Computations)**.** The transition sequence

$$s \triangleleft m \ \stackrel{t}{\Longrightarrow}\ s_0 \triangleleft m_0 \stackrel{\tau}{\longrightarrow} s_1 \triangleleft m_1 \stackrel{\tau}{\longrightarrow} s_2 \triangleleft m_2 \stackrel{\tau}{\longrightarrow} \dots$$

is called a **t-computation** if it is **maximal** *i.e.,* either it is *infinite* or it is finite and *cannot be extended further* using τ-transitions. The t-computation above is called *accepted* whenever $\exists i \in \mathbb{N} \cdot m_i = \top$ and *rejected* when $\exists i \in \mathbb{N} \cdot m_i = \bot$. A *detected* t-computation is either an accepted or a rejected one. ◀

Detected computations are indexed by their trace to allow us to *partition* computations according to the system behaviour exhibited at runtime, thus accounting for *system non-determinism*. Definition 3 also permits monitors to stabilise and reach verdicts in the trailing τ-sequence following a t-trace.

▶ **Definition 4** (Deterministic Detection and Withholding)**.** Monitor m *deterministically accepts* (*resp. deterministically rejects*) for system s along trace $t \in \text{EVT}^*$, denoted as $\mathsf{da}(m, s, t)$ and $\mathsf{dr}(m, s, t)$ *resp.*, iff *all* t-computation from $s \triangleleft m$ are accepting (*resp.* rejecting). Monitor m *deterministically detects* for s along t, $\mathsf{dd}(m, s, t)$, whenever $\mathsf{da}(m, s, t)$ or $\mathsf{dr}(m, s, t)$. Monitor m *deterministically withholds* for s along trace t, $\mathsf{dw}(m, s, t)$, iff *no* t-computation from $s \triangleleft m$ is accepting or rejecting. ◀

▶ **Example 5.** For arbitrary system s, monitors m_1 of Example 1 and m_5 of Equation 2 deterministically accept traces with the prefix $t = \mathsf{chl}\langle v_1\rangle.\mathsf{aut}\langle v_2\rangle.\mathsf{ack}\langle v_2\rangle$ where $v_2{=}enc(v_1)$ and deterministically withhold on all the other traces. Monitor m_2 deterministically rejects traces with the prefix t above when $v_2{\neq}enc(v_1)$ but does *not* deterministically detect traces with prefix t when $v_2{=}enc(v_1)$. For arbitrary s, monitor m_3 deterministically detects *any* trace with the above prefix t (accepting or rejecting the trace depending on whether $v_2 = enc(v_1)$ or not) and deterministically withholds otherwise. For any system s, monitor m_4 of Example 2 satisfies $\mathsf{dr}(m_4, s, t)$ when the trace t is of the form $t = \mathsf{out}\langle 80\rangle \ldots$, $\mathsf{da}(m_4, s, t)$ when $t = \big(\mathsf{in}\langle v_i\rangle \cdot \mathsf{out}\langle v_i\rangle\big)^i \cdot \mathsf{in}\langle 80\rangle \cdot \mathsf{out}\langle 81\rangle \ldots$ for some $i \in \mathbb{N}$, and $\mathsf{dw}(m_4, s, t)$ otherwise. Similarly, for all systems s, m_6 satisfies $\mathsf{dr}(m, s, t)$ when $t = \mathsf{in}\langle 80\rangle \ldots$ and $\mathsf{dw}(m, s, t)$ otherwise. ◀

For the rest of our study, monitors with deterministic behaviour are defined as *consistently-detecting*.

▶ **Definition 6** (Consistent Detection)**.** Monitor m *consistently detects* for system s, denoted as $\mathsf{cd}(m, s)$ iff for *all* traces t we have $\mathsf{dd}(m, s, t)$ or $\mathsf{dw}(m, s, t)$. A monitor m is *consistently-detecting*, denoted as $\mathsf{cd}(m)$, whenever $\mathsf{cd}(m, s)$ holds for *any* system s. ◀

▶ **Example 7.** Monitors m_1, m_3, m_4, m_5 and m_6 are consistently-detecting, but m_2 is *not*. Definition 6 does *not* require monitors to perform any detections. The monitor $m_7 \triangleq \mathsf{rec}\, X.\big(\mathsf{in}\langle 81\rangle.X\big) + \big(\mathsf{in}\langle 81\rangle.\mathsf{out}\langle 81\rangle.X\big)$ can consistently analyse an infinite number of traces for any s, $\mathsf{cd}(m_7)$, even though it never flags. ◀

A few comments are in order. First, Definition 6 abstracts away from the particular instances of the systems considered, the specifics of the monitor language and instrumentation mechanism used; this makes is applicable to *arbitrary* monitoring setups. Second, $\mathsf{cd}(m, s)$ may be seen as requiring an *ambiguity of* $d{=}1$ from automata theory [29, 4], for the observable behaviour specified in Definition 4. Our setting is however more general, allowing for *infinite* states and alphabets (actions). Moreover, $\mathsf{cd}(m)$ quantifies over *all* possible system compositions. Third, since Definition 6 is defined over *monitored system* behaviour, it allows us to assess the *actual* monitor behaviour at runtime. Particularly, the system quantification in $\mathsf{cd}(m)$ accounts for any (indirect) effects of a system on the execution of a monitor.

▶ **Example 8.** Whereas monitor $m_8 \triangleq \mathsf{in}\langle 81\rangle.\bot$ is (trivially) consistently-detecting in the framework of Figure 1, the monitor $m_9 \triangleq \mathsf{in}(x).\mathsf{if}\ x = 81\ \mathsf{then}\ \bot\ \mathsf{else}\ \mathbf{0}$ is, perhaps surprisingly, *not*. Consider a (diverging) system s with behaviour $s \xrightarrow{\mathsf{in}\langle 81\rangle} s' \xrightarrow{\tau} s'$. Although $s \triangleleft m_9$ can reject the t-computation for $t{=}\mathsf{in}\langle 81\rangle$, another possible t-computation of $s \triangleleft m_9$ is

$$s \triangleleft m_9 \xrightarrow{\mathsf{in}\langle 81\rangle} (s' \triangleleft \mathsf{if}\ 81{=}81\ \mathsf{then}\ \bot\ \mathsf{else}\ \mathbf{0}) \xrightarrow{\tau} (s' \triangleleft \mathsf{if}\ 81{=}81\ \mathsf{then}\ \bot\ \mathsf{else}\ \mathbf{0}) \xrightarrow{\tau} \ldots$$

which *never* reaches a verdict. Therefore, we have $\neg\mathsf{cd}(m_9)$ according to Definition 6. ◀

Fourth, consistently-detecting monitors are *not* compositional, affecting the subsequent machinery.

▶ **Example 9.** Although m_8 (from Example 8) and monitor $m_{10} = \mathsf{in}\langle 81\rangle.\top$ are both consistently-detecting according to Definition 6, their composition, *i.e.*, $m_8 + m_{10}$, is clearly *not*. ◀

4 Controllability

In spite of its generality and intuitive nature, Definition 6 it is hard to automate directly as a correctness analysis. One major obstacle is the inherent universal quantification over systems and traces defining $\mathsf{cd}(m)$. In this section, we set out to give an alternative definition for describing consistently-detecting monitors that does not suffer from these shortcomings. It is based on the notion of *controllability* [20, 31] which, in discrete event settings, roughly refers to the ability to steer a (passive) entity to designated terminal states via a series of admissible controls. In our case, the monitors will constitute the passive entities to be steered, whereas the monitored systems assume the controller's role: the admissible controls are effectively the observable events in a monitoring setup that cause the monitor to transition, whereas the terminal states of interest are the conclusive verdicts. The proposed definition thus *inverts* the focus from how a system is monitored to how a monitor can be driven.

Before giving the actual definition, we first need to lift the technical machinery of Figure 1 to *sets* of monitors, $M, N \subseteq \textsc{Mon}$: this allows us to express the status whereby a monitor that can be in a *number of potential states* after being driven by a sequence of steering controls, which facilitates the analysis of non-compositional properties such as ours (see Example 9).

▶ **Definition 10.** A monitor-set M *potentially reaches* a verdict w, $\mathbf{pr}(M, w)$, when $\exists m \in M \cdot m \Longrightarrow w$, and *potentially analyses* an event η, $\mathbf{pa}(M, \eta)$, when $\exists m \in M \cdot m \overset{\eta}{\Longrightarrow}$. Function $\mathbf{aft}(M, \eta)$ is defined as:

$$\mathbf{aft}(M,\eta) \overset{\text{def}}{=} \textstyle\bigcup_{m\in M} \mathbf{aft}(m,\eta)$$
$$\mathbf{aft}(m,\eta) \overset{\text{def}}{=} \{\, n \mid m \Longrightarrow \cdot \xrightarrow{\eta} n \,\} \cup \{\, \mathbf{0} \mid \exists n \cdot m \Longrightarrow n \not\xrightarrow{\tau} \text{ and } n \not\xrightarrow{\eta} \,\}$$

◀

Intuitively, $\mathbf{aft}(M, \eta)$ computes the set of reachable states from every $m \in M$ when it is asked by the instrumentation of Figure 1 to analyse an event η. The two conditions defining $\mathbf{aft}(m, \eta)$ correspond to the monitored system transitions dictated by the respective rules Mon and Ter in Figure 1.

▶ **Definition 11** (Controllability)**.** A relation $\mathcal{R} \subseteq \mathcal{P}(\textsc{Mon})$ is *controllable* iff for all $M \in \mathcal{R}$:

1. $\mathbf{pr}(M, w)$ and $w \in \{\top, \bot\}$ implies $M = \{w\}$;
2. $\mathbf{pa}(M, \eta)$ implies $\mathbf{aft}(M, \eta) \in \mathcal{R}$.

Controllability, denoted as the relation $\mathcal{C}$, is the *largest* controllable relation. A monitor m (*resp.* monitor-set M) is said to be controllable iff $\{m\} \in \mathcal{C}$ (*resp.* $M \in \mathcal{C}$). ◀

Controllability is *coinductive*: to show that a monitor m is controllable, *i.e.*, $\{m\} \in \mathcal{C}$, it suffices to provide a witness controllable relation $\mathcal{R}$ such that $\{m\} \in \mathcal{R}$. Condition (i) in Definition 11 requires that if some $m \in M$ can reach a conclusive verdict, then *every* $m' \in M$ must be able to do so *immediately*, without requiring any preceding τ-moves (hence $M = \{w\}$); this rules out the possibility of inconsistent detections and, at the same time, prohibits diverging systems from interfering with the reaching of such verdicts (see Example 8). Condition (ii) in Definition 11 intuitively requires that this condition is satisfied for *any* event η observed, by *all* the states that any $m \in M$ may transition to when analysing η.

▶ **Example 12.** We can show that m_6 (Equation 3) is controllable via the controllable relation $\mathcal{R}_1$ below:

$$\mathcal{R}_1 = \left\{ \begin{array}{l} \{m_6\}, \{\top\}, \{\mathbf{0}, \mathsf{in}\langle 82\rangle.\mathbf{0}\}, \{\mathbf{0}\}, \\ \{\mathsf{out}\langle 81\rangle.\mathbf{0}, \mathsf{out}\langle 81\rangle.\mathsf{in}\langle 82\rangle.\mathbf{0}\} \end{array} \right\}$$

$$\mathcal{R}_2 = \left\{ \begin{array}{l} \{m_7\}, \{m_7, \mathsf{out}\langle 81\rangle.m_7\}, \{\mathbf{0}, m_7\}, \\ \{\mathbf{0}, m_7, \mathsf{out}\langle 81\rangle.m_7\}, \{\mathbf{0}\} \end{array} \right\}$$

Note that $\{m_6\} \in \mathcal{R}_1$. We can also *finitely* determine that the recursive monitor m_7 (Example 7) is controllable via the relation $\mathcal{R}_2$. The reader may want to check that $\mathcal{R}_2$ is controllable. For instance,
$\mathbf{aft}(\{m_7, \mathsf{out}\langle 81\rangle.m_7\}, \mathsf{in}\langle 81\rangle) = \{\mathbf{0}, m_7, \mathsf{out}\langle 81\rangle.m_7\}$, $\mathbf{aft}(\{m_7, \mathsf{out}\langle 81\rangle.m_7\},$
$\mathsf{out}\langle 81\rangle) = \{\mathbf{0}, m_7\}$ and, importantly, $\mathbf{aft}(\{\mathbf{0}, m_7, \mathsf{out}\langle 81\rangle.m_7\}, \mathsf{in}\langle 81\rangle) = \{\mathbf{0}, m_7, \mathsf{out}\langle 81\rangle.m_7\}$ itself. ◀

Controllability coincides with Definition 6: we can use Definition 11 as a *sound* and *complete* proof technique to determine whether a monitor m satisfies $\mathsf{cd}(m)$, side-stepping universal quantifications over systems.

▶ **Theorem 13** (Consistent Detection Full Abstraction). *$\mathsf{cd}(m)$ iff $\{m\} \in \mathcal{C}$* ◀

▶ **Example 14.** As a result of Theorem 13, we can show that m_6 and m_7 are *consistently-detecting* via the controllable relations $\mathcal{R}_1$ and $\mathcal{R}_2$ of Example 12. We can also *indirectly* show that $\neg\mathsf{cd}(m_8 + m_{10})$ from Example 9, by arguing that there *cannot* be a controllable relation $\mathcal{R}$ with $\{m_8 + m_{10}\} \in \mathcal{R}$. For suppose that such an $\mathcal{R}$ exists. By Definition 11(ii) the monitor-set $\mathbf{aft}(\{m_8 + m_{10}\}, \mathsf{in}\langle 81\rangle) = \{\top, \bot\}$ must also be in $\mathcal{R}$; this, in turn, would necessarily mean that $\mathcal{R}$ is *not* controllable since $\{\top, \bot\}$ violates Definition 11(i). ◀

5 Symbolic Controllability

Controllability, Definition 11, is still not adequate for a fully automated analysis of consistently-detecting monitors. Particularly, whenever the *resp.* event value domain is infinite, quantified guards induce an infinite number of transitions, *e.g.*, $l(x).m$ generates a transition with the label $l\langle v\rangle$ for every $v \in$ Val (see rule GrQ). As a result, condition Definition 11(ii) may require the monitor analysis to consider a potentially infinite number of monitor-set states whenever monitor descriptions use quantified guards.

To this end, we define a *symbolic* semantics over *both* open[2] and closed monitor terms using *symbolic* events, $\theta \in$ SEvt. These are similar to the events of Section 2 except that they carry *variables* instead of values as payloads, $l\langle x\rangle$. Symbolic transitions, $m \xrightarrow[b]{\mu} n$ are defined by the rules in Figure 2, where $\mu \in \text{SEvt} \cup \{\tau\}$ ranges over both symbolic and τ events, and the boolean expression b records the condition under which the LTS action may take place. For instance, the term $\mathsf{rec}\, X.m$ may unfold in all circumstances (*i.e.*, $b = \mathsf{true}$ in rule sRec) whereas the term $\mathsf{if}\, b\, \mathsf{then}\, m\, \mathsf{else}\, n$ can either τ-transition to m when b holds, or to n when the converse, $\neg b$, holds (rules sIfT and sIfF). The other key rules in Figure 2 are sGrE and sGrQ: the former transitions with a symbolic event $l\langle x\rangle$ under the condition $x{=}e$, whereas the latter transitions with a similar symbolic event under *any* circumstance. Figure 2 also defines rules for *weak symbolic transitions*, $m \overset{\theta}{\underset{b}{\Longrightarrow}} n$, and *reductions*, $m \underset{b}{\Longrightarrow} n$,

[2] Open *wrt.* expression variables $x, y, \ldots \in$ Vars *not* term variables $X, Y \ldots \in$ TVars.

where both relations *aggregate* boolean constraints via conjunctions. Note that weak symbolic transitions describe transition sequences where τ-transitions *must precede* the (final) symbolic event transition. The predicate $m \overset{\mu}{\not\longrightarrow}$ denotes $\nexists\, b, n \cdot m \overset{\mu}{\underset{b}{\longrightarrow}} n$ whereas $m \overset{\theta}{\underset{b}{\Longrightarrow\!\!\!\!\!\rightarrow}}$ stands for $\exists n \cdot m \overset{\theta}{\underset{b}{\Longrightarrow\!\!\!\!\!\rightarrow}} n$.

A *constrained monitor-set* $\langle b, M\rangle$ is a tuple where every $m \in M$ may be open, and b is a condition constraining free variables in M. Every $\langle b, M\rangle$ abstractly represents a (potentially infinite) set of *closed* monitor-sets for every valuation ρ satisfying b,

$$\{\, m\rho \mid [\![b\rho]\!] = \mathsf{true} \text{ and } m \in M \,\} \tag{4}$$

In this sense, the monitor-sets in Section 4 are special cases of constrained monitor-sets where $b = \mathsf{true}$ and M is closed. Note that whenever $\neg\mathbf{sat}(b)$, the constrained monitor-set $\langle b, M\rangle$ denotes the *empty* set of monitor-sets, $\emptyset$, which is trivially controllable by Definition 11. We lift functions such as that for free variables $\mathbf{fv}(-)$ to constrained monitor-sets in the obvious manner, *e.g.*, $\mathbf{fv}(\langle b, M\rangle) \overset{\mathbf{def}}{=} \mathbf{fv}(b) \cup \mathbf{fv}(M)$.

▶ **Example 15.** The constrained monitor-set $\langle x \geq 3, \{\mathsf{if}\ x = 2\ \mathsf{then}\ \top\ \mathsf{else}\ \bot\}\rangle$ abstractly describes all monitor-sets $\{\mathsf{if}\ x = 2\ \mathsf{then}\ \top\ \mathsf{else}\ \bot\}\,\rho$ where $\rho(x) \geq 3$. For any such ρ, *no* monitor of the form $(\mathsf{if}\ x = 2\ \mathsf{then}\ \top\ \mathsf{else}\ \bot)\rho$ can transition to a $\top$ verdict according to the concrete semantics of Figure 1. Symbolically, this may be expressed as $\neg\mathbf{sat}(x \geq 3 \wedge x = 2)$. ◀

We abstractly model controllability, Definition 11, in terms of constrained monitor-sets, the symbolic semantics of Figure 2 and the satisfiability judgement $\mathbf{sat}(b)$ defined earlier in Section 2.

▶ **Definition 16.** A constrained monitor-set $\langle b, M\rangle$ *potentially reaches* a verdict w, denoted as $\mathbf{spr}\big(\langle b, M\rangle, w\big)$, whenever $\exists m \in M, c \cdot m \underset{c}{\Longrightarrow} w$ and $\mathbf{sat}(b \wedge c)$. Moreover, $\langle b, M\rangle$ *potentially analyses* a symbolic event θ along c, denoted as $\mathbf{spa}\big(\langle b, M\rangle, \theta, c\big)$, whenever $\exists m \in M \cdot m \overset{\theta}{\underset{c}{\Longrightarrow\!\!\!\!\!\rightarrow}}$ and $\mathbf{sat}(b \wedge c)$. ◀

Defining the symbolic counterpart to $\mathbf{aft}(M, \eta)$ of Definition 10 is less straightforward. Intuitively, from all the valuations ρ satisfying (and represented by) b in $\langle b, M\rangle$, only a *subset* of them may satisfy the condition c in a *potentially-analyses* judgement $\mathbf{spa}(\langle b, M\rangle, \theta, c)$ from Definition 16. A correct modelling of Definition 11 therefore requires us to take this fact into account.

▶ **Example 17.** Consider the constrained monitor-set $\langle \mathsf{true}, M\rangle$ where $M = \{m_{10}, m_{11}\}$ and

$$m_{10} \triangleq \mathsf{if}\ x = 2\ \mathsf{then}\ k\langle 1\rangle.\top\ \mathsf{else}\ k\langle 1\rangle.\bot \qquad m_{11} \triangleq \mathsf{if}\ x \leq 1 \vee x \geq 3\ \mathsf{then}\ k\langle 1\rangle.\bot\ \mathsf{else}\ k\langle 1\rangle.\top$$

It turns out that for any ρ satisfying true, $M\rho$ is controllable. Therefore, for the judgement $\mathbf{spa}(\langle \mathsf{true}, M\rangle, k\langle y\rangle, (x{=}2 \wedge y{=}1))$ of Definition 16, which holds since $m_{10} \overset{k\langle y\rangle}{\underset{x=2\wedge y=1}{\Longrightarrow\!\!\!\!\!\rightarrow}}$ and $\mathbf{sat}(\mathsf{true} \wedge x{=}2 \wedge y{=}1)$, the reachable states to be considered by a corresponding symbolic analysis (modelling *Definition* 11(*ii*)) should *not* include the residual state $\bot$, even though it may be reached after the event $k\langle y\rangle$ with $y = 1$ (see rule sGrE). The reason for this is that the conditions required to symbolically reach this state, *i.e.*, $(\neg(x{=}2) \wedge y{=}1)$ or $((x \leq 1 \vee x \geq 3) \wedge y{=}1)$, *cannot* be satisfied by *any* ρ that also satisfies the $\mathbf{spa}(-)$ condition $(x{=}2 \wedge y{=}1)$. Symbolically, this may be expressed as $\neg\mathbf{sat}\big((x{=}2 \wedge y{=}1) \wedge \big(\neg(x{=}2) \wedge y{=}1\big)\big)$ and $\neg\mathbf{sat}\big((x{=}2 \wedge y{=}1) \wedge \big((x \leq 1 \vee x \geq 3) \wedge y{=}1\big)\big)$. ◀

Symbolic Transitions

$$\textsc{sVer}\ \frac{}{w \xrightarrow[\mathsf{true}]{\eta} w} \qquad \textsc{sIfT}\ \frac{}{\mathsf{if}\ b\ \mathsf{then}\ m\ \mathsf{else}\ n \xrightarrow[b]{\tau} m} \qquad \textsc{sIfF}\ \frac{}{\mathsf{if}\ b\ \mathsf{then}\ m\ \mathsf{else}\ n \xrightarrow[\neg b]{\tau} n}$$

$$\textsc{sRec}\ \frac{}{\mathsf{rec}\,X.m \xrightarrow[\mathsf{true}]{\tau} m[\mathsf{rec}\,X.m/X]} \qquad \textsc{sCh1}\ \frac{m \xrightarrow[b]{\mu} m'}{m + n \xrightarrow[b]{\mu} m'} \qquad \textsc{sCh2}\ \frac{n \xrightarrow[b]{\mu} n'}{m + n \xrightarrow[b]{\mu} n'} \qquad \textsc{sGrE}\ \frac{}{l\langle e\rangle.m \xrightarrow[e=x]{l\langle x\rangle} m}$$

$$\textsc{sGrQ}\ \frac{}{l(y).m \xrightarrow[\mathsf{true}]{l\langle x\rangle} m[x/y]} \qquad \textsc{sLet}\ \frac{}{\mathsf{let}\ x = e\ \mathsf{in}\ m \xrightarrow[\mathsf{true}]{\tau} m[e/x]}$$

Weak Symbolic Transitions and Reductions

$$\textsc{sWTr1}\ \frac{m \xrightarrow[b]{\theta} m'}{m \overset{\theta}{\underset{b}{\Longrightarrow}} m'} \qquad \textsc{sWTr2}\ \frac{m \xrightarrow[b]{\tau} m' \qquad m' \overset{\theta}{\underset{c}{\Longrightarrow}} m''}{m \overset{\theta}{\underset{b\wedge c}{\Longrightarrow}} m''} \qquad \textsc{sWRd1}\ \frac{}{m \underset{\mathsf{true}}{\Longrightarrow} m} \qquad \textsc{sWTr2}\ \frac{m \xrightarrow[b]{\tau} m' \qquad m' \underset{c}{\Longrightarrow} m''}{m \underset{b\wedge c}{\Longrightarrow} m''}$$

Figure 2 A Symbolic Semantics for Monitors.

The complications elicited in Example 17 are even more intricate. For instance, for a particular judgement $\mathbf{spa}(\langle b, M\rangle, \theta, c)$, one could have some $m_1, m_2 \in M$ whereby $m_i \overset{\theta}{\underset{c_i}{\Longrightarrow}} m_i'$ and $\mathbf{sat}(b\wedge c\wedge c_i)$ for $i \in \{1,2\}$, but at the same time having c_1 and c_2 being incompatible with one another, *i.e.*, $\neg\mathbf{sat}(c_1\wedge c_2)$. In such cases, the respective residual states m_1' and m_2' should be analysed separately.

▶ **Definition 18.** The *relevant conditions* for a monitor-set M *wrt.* a symbolic event θ are:

$$\mathbf{rc}(M,\theta) \overset{\mathbf{def}}{=} \{\, c \mid \exists m \in M \cdot \big(m \overset{\theta}{\underset{c}{\Longrightarrow}} \ \text{or}\ \exists n \cdot (m \underset{c}{\Longrightarrow} n \ \text{and}\ n \not\xrightarrow{\tau} \ \text{and}\ n \not\xrightarrow{\theta})\big) \,\}$$

The *satisfiability combinations* for a condition-set $\{c_1, \ldots, c_n\}$ *wrt.* a condition b are:

$$\mathbf{sc}(b, \{c_1, \ldots, c_n\}) \overset{\mathbf{def}}{=} \{\, \{b, c_1', \ldots, c_n'\} \mid \forall i \in 1..n \cdot (c_i' = c_i \text{ or } c_i' = \neg c_i) \,\}$$

The *reachable constrained monitor-sets* from $\langle b, M\rangle$ *after* θ with condition c are:

$$\mathbf{saft}(\langle b, M\rangle, \theta, c) \overset{\mathbf{def}}{=} \{\, \langle \wedge B, \mathbf{saft}(M, B, \theta)\rangle \mid B \in \mathbf{sc}(b\wedge c, \mathbf{rc}(M,\theta)) \text{ and } \mathbf{sat}(\wedge B) \,\}$$

$$\mathbf{saft}(M, B, \theta) \overset{\mathbf{def}}{=} \left\{ n \;\middle|\; \begin{array}{l} \exists m \in M, c \cdot\ \mathbf{sat}((\wedge B) \wedge c) \text{ and} \\ \big(m \overset{\theta}{\underset{c}{\Longrightarrow}} n \text{ or } (\exists n' \cdot m \underset{c}{\Longrightarrow} n' \not\xrightarrow{\tau} \text{ and } n' \not\xrightarrow{\theta} \text{ and } n = \mathbf{0})\big) \end{array} \right\}$$

◀

In Definition 18, the *relevant conditions* for M *wrt.* θ, denoted as $\mathbf{rc}(M,\theta)$, are all the symbolic conditions that need to be considered to assess the reachable states from M for the symbolic event θ — they are the symbolic counterpart to the transition sequences defining $\mathbf{aft}(m,\eta)$ in Definition 10. The *satisfiability combinations* of a condition-set B *wrt.* a condition b, denoted as $\mathbf{sc}(b,B)$, capture the maximal condition subsets in B that any valuation ρ satisfying condition b also satisfies. Every condition set B' returned by $\mathbf{sc}(b,B)$

contains b itself and one condition c' for *every* boolean condition $c \in B$ (either c itself or its negation); these combination sets *partition* all the valuations ρ satisfying b. *Symbolic reachability* for $\langle b, M\rangle$ *after* θ with condition c, $\mathbf{saft}(\langle b, M\rangle, \theta, c)$ in Definition 18, is defined *wrt.* all the satisfiability combinations B of $\mathbf{rc}(M, \theta)$ for the (fixed) condition $b \wedge c$. Although $\mathbf{sc}(b \wedge c, \mathbf{rc}(M, \theta))$ partitions all the ρ satisfying $b \wedge c$, some of these partitions are *empty*. Accordingly, $\mathbf{saft}(\langle b, M\rangle, \theta, c)$ only considers the non-empty partitions via the satisfiability condition $\mathbf{sat}(\wedge B)$, where $\wedge B$ returns the syntactic conjunction formula $c_1 \wedge \ldots \wedge c_n$ for a boolean set $B = \{c_1, \ldots, c_n\}$.

It is worth remarking that the symbolic LTS of Figure 2, is image-finite [42], and thus finitely branching when considering the τ-transition graph of a term m. By König's Infinity Lemma [33] the set of constraints $\{\, c \mid m \overset{\theta}{\underset{c}{\Rightarrow\!\!\!|}} \text{ or } \exists n \cdot (m \underset{c}{\Longrightarrow} n \text{ and } n \overset{\tau}{\not\longrightarrow} \text{ and } n \overset{\theta}{\not\longrightarrow}) \,\}$ must be finite and, as a result, $\mathbf{rc}(M, \theta)$ is finite too for a *finite* monitor-set M. This ensures that $\mathbf{saft}(\langle b, M\rangle, \theta, c)$ is well-defined.

▶ **Definition 19** (Symbolic Controllability)**.** The relation $\mathcal{S} \subseteq \big(\textsc{BExp} \times \mathcal{P}(\textsc{Mon})\big)$ is called a *symbolically-controllable* relation iff for all constrained monitor-sets $\langle b, M\rangle \in \mathcal{S}$:

1. $\mathbf{spr}(\langle b, M\rangle, w)$ and $w \in \{\top, \bot\}$ implies $M = \{w\}$;
2. $\mathbf{spa}(\langle b, M\rangle, l\langle x\rangle, c)$ where $\mathbf{frsh}(\mathbf{fv}(\langle b, M\rangle) = x$ implies $\mathbf{saft}(\langle b, M\rangle, l\langle x\rangle, c) \subseteq \mathcal{S}$.

Symbolic Controllability, denoted as $\mathcal{C}_{\text{sym}}$, is the largest symbolically-controllable relation. A (closed) monitor m is symbolically-controllable iff $\langle \mathsf{true}, \{m\}\rangle \in \mathcal{C}_{\text{sym}}$. ◀

The clause Definition 19(ii) assumes a function $\mathbf{frsh}(V)$ that (deterministically) returns the *next* fresh variable x that is *not* in the variable set V. When compared to Definition 11(ii), this allows us to just consider *one* (symbolic) event, $l\langle x\rangle$, for a *finite* set of constraints, as opposed to a potentially infinite set of events, *i.e.,* $l\langle v\rangle$ for every $v \in \textsc{Val}$.

▶ **Example 20.** Recall m_{10} and m_{11} from Example 17. The monitor $m_{12} \triangleq l(x).m_{10} + l(x).m_{11}$ can be shown to be symbolically-controllable via the relation $\mathcal{S}_1$ defined below, where $b_1 = (x = 2 \wedge y = 1)$, $b_2 = (\neg(x = 2) \wedge y = 1)$, $b_3 = ((x \leq 1 \vee x \geq 3) \wedge y = 1)$ and $b_4 = (\neg(x \leq 1 \vee x \geq 3) \wedge y = 1)$; these are obtained from the relevant conditions $\mathbf{rc}(\{m_{10}, m_{11}\}, k\langle y\rangle) = \{b_1, b_2, b_3, b_4\}$.

$$\mathcal{S}_1 = \left\{ \begin{array}{ll} \langle \mathsf{true}, \{m_{12}\}\rangle, & \langle \mathsf{true}, \{m_{10}, m_{11}\}\rangle, \\ \langle (\mathsf{true} \wedge b_1) \wedge b_1 \neg b_2 \wedge \neg b_3 \wedge b_4, \{\top\}\rangle, & \langle (\mathsf{true} \wedge b_4) \wedge b_1 \wedge \neg b_2 \wedge \neg b_3 \wedge b_4, \{\top\}\rangle, \\ \langle (\mathsf{true} \wedge b_2) \wedge \neg b_1 \wedge b_2 \wedge b_3 \wedge \neg b_4, \{\bot\}\rangle, & \langle (\mathsf{true} \wedge b_3) \wedge \neg b_1 \wedge b_2 \wedge b_3 \wedge \neg b_4, \{\bot\}\rangle \end{array} \right\}$$

For illustrative purposes, we do not simplify the constraints in the constrained monitor-sets of $\mathcal{S}$ to show how these are derived. *E.g.,* $\langle (\mathsf{true} \wedge b_1) \wedge b_1 \neg b_2 \wedge \neg b_3 \wedge b_4, \{\top\}\rangle$ is obtained as a result of $\mathbf{saft}(\langle \mathsf{true}, \{m_{10}, m_{11}\}\rangle, l\langle x\rangle, \mathsf{true} \wedge b_1)$. In fact, the combination $\{(\mathsf{true} \wedge b_1), b_1, \neg b_2, \neg b_3, b_4\}$ is the only satisfiable condition-set and all the others are filtered out by $\mathbf{saft}(\langle \mathsf{true}, \{m_{10}, m_{11}\}\rangle, l\langle x\rangle, \mathsf{true} \wedge b_1)$. ◀

Symbolic Controllability, Definition 19, is sound and complete *wrt.* Controllability, Definition 11.

▶ **Theorem 21** (Controllability Full Abstraction)**.** $\{m\} \in \mathcal{C}$ *iff* $\langle \mathit{true}, \{m\}\rangle \in \mathcal{C}_{sym}$ ◀

▶ **Example 22.** Recall $m_3 \triangleq \mathsf{chl}(x).m_3'$ from Example 1, recast in terms of m_3' defined below as:

$$m_3' \triangleq \mathsf{let}\ y = enc(x)\ \mathsf{in}\ \big(\mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top + \mathsf{aut}(z).\mathsf{if}\ z \neq y\ \mathsf{then}\ \mathsf{ack}(z').\bot\ \mathsf{else}\ \mathsf{ack}\langle z\rangle.\top\big)$$

Example 7 stated that m_3 is consistently-detecting. This fact is hard to determine using Definition 6, whereas analyses using Definition 11 are complicated by quantifications over

the values of events. By Theorems 13 and 21, we can show that m_3 is consistently-detecting via the *symbolic* controllability relation:

$$\mathcal{S}_2 = \left\{ \begin{array}{l} \langle \mathsf{true}, \{m_3\}\rangle, \quad \langle \mathsf{true}, m_3'\rangle \ , \\ \langle z = enc(x), \{\mathsf{ack}\langle enc(x)\rangle.\top, \mathsf{if}\ z \neq enc(x)\ \mathsf{then}\ \mathsf{ack}(z').\bot\ \mathsf{else}\ \mathsf{ack}\langle z\rangle.\top\}\rangle \ , \\ \langle (z = enc(x)) \wedge (w = z) \wedge (w = enc(x)), \{\top\}\rangle, \\ \langle \neg(z = enc(x)), \{\mathsf{if}\ z \neq enc(x)\ \mathsf{then}\ \mathsf{ack}(z').\bot\ \mathsf{else}\ \mathsf{ack}\langle z\rangle.\top\}\rangle \ , \quad \langle \neg(z = enc(x)), \{\bot\}\rangle \end{array} \right\}$$

In $\mathcal{S}_2$ and the ensuing discussion, we alleviate our presentation by simplifying the boolean conditions used, *e.g.*, we simply write $(z = enc(x))$ in lieu of $\big(\mathsf{true} \wedge (z = enc(x))\big)$. We highlight a few points.

First, consider the second constrained monitor-set in $\mathcal{S}_2$, namely $\langle \mathsf{true}, m_3'\rangle$. Since the semantics of Figure 2 allows expression guards and quantified guards to transition with the *same* symbolic event (albeit with different conditions) we are able to consider the *resp.* continuations in unison for the event $\mathsf{aut}\langle z\rangle$. Concretely, according to Definition 19(ii), for $\mathbf{spa}(\langle \mathsf{true}, m_3'\rangle, \mathsf{aut}\langle z\rangle, (z{=}enc(x)))$ generated by the *expression* guard weak transition of m_3', we need to ensure that the resulting monitor-set $\langle z = enc(x), \mathbf{saft}(\{m_3'\}, \{(z = enc(x)), \mathsf{true}\}, \mathsf{aut}\langle z\rangle)\rangle$ (which evaluates to the third constrained monitor-set in $\mathcal{S}_2$) is also in the symbolic relation. At the same time, for $\mathbf{spa}(\langle \mathsf{true}, m_3'\rangle, \mathsf{aut}\langle z\rangle, \mathsf{true})$ generated by the *quantified* guard weak transition of m_3', we need to ensure that two monitor-sets are in $\mathcal{S}_2$, namely $\langle z = enc(x), \mathbf{saft}(\{m_3'\}, \{(z = enc(x)), \mathsf{true}\}, \mathsf{aut}\langle z\rangle)\rangle$ (as before) but also the constrained monitor-set $\langle \neg(z = enc(x)), \mathbf{saft}(\{m_3'\}, \{\neg(z = enc(x)), \mathsf{true}\}, \mathsf{aut}\langle z\rangle)\rangle$ (which evaluates to the fifth constrained monitor-set in $\mathcal{S}_2$).

The second point we highlight about $\mathcal{S}_2$ concerns its third constrained monitor-set. In particular, the left branch of the conditional term in this set, namely $\mathsf{ack}(z').\bot$ in the term $\mathsf{if}\ z \neq enc(x)\ \mathsf{then}\ \mathsf{ack}(z').\bot\ \mathsf{else}\ \mathsf{ack}\langle z\rangle.\top$, is not considered by our analysis since its condition, $z{\neq}enc(x)$, is *incompatible* with the constraining condition of the monitor-set, *i.e.*, $\neg\mathbf{sat}((z = enc(x)) \wedge (z \neq enc(x)))$.

Third, we also note how the condition aggregation mechanism for the consecutive symbolic events $\mathsf{aut}\langle z\rangle$ and $\mathsf{ack}\langle w\rangle$ — transferring us from the second constrained monitor-set, $\langle \mathsf{true}, m_3'\rangle$, to the fourth, $\langle (z = enc(x)) \wedge (w = z) \wedge (w = enc(x)), \{\top\}\rangle$, via the third constrained monitor-set in $\mathcal{S}_2$ — enables us to symbolically relate the *expression* guards in m_3, which impose a condition such as $(z = enc(x))$ *upon* transition, with the *quantified* guard that imposes the same condition *after* the transition (by means of a conditional branch in its continuation). We leave it up to the interested reader to check that the remaining monitor-sets in $\mathcal{S}_2$ satisfy the conditions required by Definition 19. ◀

6 On Automating Symbolic Controllability

Despite its merits, a direct implementation of the symbolic controllability from Definition 19 still would *not* perform well for certain recursive monitor descriptions, as shown in the following example.

▶ **Example 23.** Recall monitor m_4 from Example 2. To show that it is controllable, we need to exhibit a symbolic relation that includes $\langle \mathsf{true}, \{m_4\}\rangle$. For some fresh variable x where $\mathbf{frsh}(\mathbf{fv}(\langle \mathsf{true}, \{m_4\}\rangle) = x$, since the judgement $\mathbf{spa}(\langle \mathsf{true}, \{m_4\}\rangle, \mathsf{in}\langle x\rangle, \mathsf{true})$ holds, this relation needs to include the ensuing monitor-set $\langle \mathsf{true}, \{m_4'\}\rangle$ as well, where $m_4' \triangleq \mathsf{if}\ x{=}\ 80\ \mathsf{then}\ \mathsf{out}\langle 81\rangle.\top\ \mathsf{else}\ \mathsf{out}\langle x\rangle.m_4$. In turn, since $\mathbf{spa}\big(\langle \mathsf{true}, \{m_4'\}\rangle, \mathsf{out}\langle y\rangle, (\neg(x = 80) \wedge y = x)\big)$ (where $\mathbf{frsh}(\mathbf{fv}(\langle b, \{m_4'\}\rangle) = y$), the symbolic relation must also contain $\langle \neg(x = 80) \wedge y = x, \{m_4\}\rangle$. We thus reach the original monitor set $\{m_4\}$ but with a stronger condition, namely

$\neg(x=\mathsf{80})\wedge y=x$. By extension of this reasoning, it is not hard to see that the symbolic relation required by Definition 19 needs to be infinitely large. ◀

The problem exhibited by Example 23 is that the condition aggregating mechanism of Definition 19 does not specify any means for consolidating the boolean condition b constraining a monitor set M in $\langle b, M\rangle$, *i.e.,* a form of garbage collection of redundant conditions. For instance, in the constrained monitor-set $\langle\neg(x=\mathsf{80})\wedge y=x, \{m_4\}\rangle$ of Example 23, the condition $(\neg(x=\mathsf{80})\wedge y=x)$ plays *no effective role* in constraining the free variables in $\{m_4\}$, of which there are none. We therefore optimise Definition 19 in a sound (and complete) manner by taking into consideration boolean sub-conditions that can be isolated and discarded. This leads to an improved automated analysis for consistently-detecting monitors.

▶ **Definition 24** (Optimised Symbolic Controllability)**.** The consolidation of a boolean expression b *wrt.* a variable set V, denoted as $\mathbf{cns}(b, V)$, is defined as:

$$\mathbf{cns}(b, V) \stackrel{\text{def}}{=} b_1 \text{ whenever } \mathbf{prt}(b, V) = \langle b_1, b_2\rangle \text{ for some } b_2$$

where the boolean expression partitioning operation $\mathbf{prt}(b, V)$ is defined as:

$$\mathbf{prt}(b, V) \stackrel{\text{def}}{=} \begin{cases} \langle b_1, b_2\rangle & \text{if } \mathbf{sat}(b) \text{ and } b = b_1\wedge b_2 \text{ and } \big(\mathbf{fv}(b_1)\subseteq V\big) \text{ and } \big(V\cap\mathbf{fv}(b_2)=\emptyset\big) \\ \langle b, \mathsf{true}\rangle & \text{otherwise} \end{cases}$$

Let the *optimised symbolic reachability* from $\langle b, M\rangle$ for θ and c, $\mathbf{osaft}(\langle b, M\rangle, \theta, c)$, be defined as:

$$\mathbf{osaft}(\langle b, M\rangle, \theta, c) \stackrel{\text{def}}{=} \left\{ \langle \mathbf{cns}(\wedge B, V), \mathbf{saft}(M, B, \theta)\rangle \;\middle|\; \begin{array}{c} B \in \mathbf{sc}(b\wedge c, \mathbf{rc}(M, \theta)) \text{ and } \mathbf{sat}(\wedge B) \\ \text{and } V = \mathbf{fv}(\mathbf{saft}(M, B, \theta)) \end{array} \right\}$$

A relation $\mathcal{S} \subseteq \big(\text{BExp} \times \mathcal{P}(\text{Mon})\big)$ is called *optimised symbolically-controllable* iff for all $\langle b, M\rangle\in\mathcal{S}$:

1. $\mathbf{spr}(\langle b,M\rangle, w)$ and $w \in \{\top, \bot\}$ implies $M = \{w\}$;
2. $\mathbf{spa}(\langle b,M\rangle, l\langle x\rangle, c)$ where $\mathbf{frsh}(\mathbf{fv}(\langle b, M\rangle))=x$ implies $\mathbf{osaft}(\langle b, M\rangle, l\langle x\rangle, c) \subseteq \mathcal{S}$.

The *largest* optimised symbolically-controllable relation is denoted by $\mathcal{C}^{\text{opt}}_{\text{sym}}$. A (closed) monitor m is said to be optimised symbolically-controllable iff $\langle \mathsf{true}, \{m\}\rangle \in \mathcal{C}^{\text{opt}}_{\text{sym}}$. ◀

We highlight the salient points from Definition 24. First, boolean consolidation in a constrained monitor-set, $\langle \mathbf{cns}(b), M\rangle$, should *not* change the set of concrete monitor sets represented by $\langle b, M\rangle$ and, for this reason, we cannot consolidate *unsatisfiable* boolean conditions. For instance, even if $x\notin \mathbf{fv}(M)$, it is still unsound to optimise $\langle \mathsf{true}\wedge(x\neq x), M\rangle$ to $\langle \mathsf{true}, M\rangle$ based on the fact that $\neg\mathbf{sat}(x\neq x)$. Concretely, from Equation 4 of Section 5 we know that $\langle \mathsf{true}\wedge x\neq x, M\rangle$ denotes the empty set of monitor-sets, $\emptyset$, whereas $\langle \mathsf{true}, M\rangle$ represents the set $\{\, m\rho \mid [\![\mathsf{true}\rho]\!] = \mathsf{true} \text{ and } m \in M\,\}$. Second, consolidation should ideally filter out as much redundant constraints as possible, *e.g.,* in $\langle b_1\wedge b_2, M\rangle$ we should remove b_2 whenever $\mathbf{fv}(b_2)\cap\mathbf{fv}(M) = \emptyset$. In Definition 24 we require the strongest possible condition for the residual condition b_1 in $\langle b_1\wedge b_2, M\rangle$, *i.e.,* $\mathbf{fv}(b_1)\subseteq\mathbf{fv}(M)$, which indirectly implies that the *resp.* condition variables are partitioned $\mathbf{fv}(b_1)\cap\mathbf{fv}(b_1) = \emptyset$. This partitioning is crucial for a sound consolidation, *e.g.,* in $\langle(\neg(x=\mathsf{80})\wedge y=x), M\rangle$, it is unsound to just remove the subcondition $\neg(x=\mathsf{80})$ when $x\notin\mathbf{fv}(M)$ and $y\in\mathbf{fv}(M)$. Although $\mathbf{prt}(b, V)$ can be refined further (while still observing core requirements for soundness such as variable condition partitioning), in Definition 24 we opted for a less elaborate condition that suffices our exposition. Third, we highlight the fact that the conditions specifying $\mathbf{cns}(b)$ in Definition 24 yield a unique consolidated condition up to semantic equivalence meaning that, in an implementation of the framework, this can be defined as a function.

▶ **Theorem 25** (Optimised Controllability). $\langle \mathit{true}, \{m\}\rangle \in \mathcal{C}^{opt}_{sym}$ *iff* $\langle \mathit{true}, \{m\}\rangle \in \mathcal{C}_{sym}$ ◀

▶ **Example 26.** As a result of Theorems 13, 21 and 25, we can show that m_4 of Example 2 is consistently-detecting by exhibiting the optimised symbolic relation $\mathcal{S}_3$ below (m'_4 is the monitor defined earlier in Example 23). For expository purposes, we show how the consolidated boolean expressions are calculated. In particular, the first constrained monitor-set in $\mathcal{S}_3$ denotes both the starting pair $\langle \mathsf{true}, \{m_4\}\rangle$, but also the pair $\langle \mathbf{cns}(x = 80 \wedge y = 81, \mathbf{fv}(\{m_4\})), \{m_4\}\rangle$.

$$\mathcal{S}_3 = \left\{ \langle \underbrace{\mathsf{true}}_{\mathbf{cns}(\neg(x=80)\wedge y=x,\emptyset)}, \{m_4\}\rangle,\ \langle \underbrace{\mathsf{true}}_{\mathbf{cns}(x=80,\emptyset)}, \{\bot\}\rangle,\ \langle \mathsf{true}, \{m'_4\}\rangle,\ \langle \underbrace{\mathsf{true}}_{\mathbf{cns}(x=80\wedge y=81,\emptyset)}, \{\top\}\rangle \right\}$$

The interested reader may check that the conditions of Definition 24 are satisfied by $\mathcal{S}_3$. ◀

Using standard techniques [42, 3], an algorithm constructing symbolically-controllable relations from Definition 24 can be extracted more easily. Moreover, the completeness aspect in Theorems 13, 21, and 25 should enable such an automation to infer a *counter-example* (system and trace) from failed attempts, thereby explaining why a monitor is *not* consistently-detecting.

▶ **Example 27.** Recall monitor m_2 from Example 1. Assuming the shorthand abbreviations $m'_2 \triangleq \mathsf{aut}\langle y\rangle.\mathsf{ack}\langle y\rangle.\top + \mathsf{aut}(z).\mathsf{if}\ z \neq y\ \mathsf{then}\ \mathsf{ack}(z').\bot$ and $M = \{\mathsf{if}\ z \neq enc(x)\ \mathsf{then}\ \mathsf{ack}(z').\bot, \mathsf{ack}\langle enc(x)\rangle.\top\}$, compiling a relation satisfying Definition 24 fails because it needs to include:

$\langle \mathsf{true}, \{\mathsf{let}\ y = enc(x)\ \mathsf{in}\ m'_2\}\rangle$	since	$\mathbf{spa}(\langle \mathsf{true}, \{m_2\}\rangle, \mathsf{chl}\langle x\rangle, \mathsf{true})$
$\langle z = enc(x), M\rangle$	since	$\mathbf{spa}(\langle \mathsf{true}, \{\mathsf{let}\ y = enc(x)\ \mathsf{in}\ m'_2\}\rangle, \mathsf{aut}\langle z\rangle, z = enc(x))$
$\langle \mathsf{true}, \{\top, \mathbf{0}\}\rangle$	since	$\mathbf{spa}(\langle z = enc(x), M\rangle, \mathsf{ack}\langle w\rangle, (w = enc(x)))$

The final pair $\langle \mathsf{true}, \{\top, \mathbf{0}\}\rangle$ violates Definition 24(i) and, via the symbolic events on right-hand column conatining the $\mathbf{spa}(-)$ assertions that lead to it, one can construct the counter-example inducing the inconsistent detection, *i.e.*, a system s producing the trace $\mathsf{chl}\langle v\rangle \cdot \mathsf{aut}\langle u\rangle \cdot \mathsf{ack}\langle u\rangle$ where $u{=}enc(v)$. ◀

7 Conclusion

Monitors should provide guarantees that they will operate correctly when instrumented with *any* system [35]. At the same time, for this requirement to be scalable, the corresponding correctness analysis that determines it must also be *compositional*: monitors should be verified separately, independent of the systems they may be instrumented with. The fact that monitors tend to be considerably smaller (in size and complexity) than the systems they observe further justifies this point. This paper provides two definitions that formalise deterministic monitoring behaviour, Definition 6 (consistently-detecting monitors) and Definition 11 (controllable monitors), that address these seemingly conflicting concerns; it also shows that the two definitions coincide, Theorem 13. In addition, the paper also studies alternative definitions for controllability, Definitions 19 and 24, that enable the implementation of sound and complete symbolic analyses, Theorems 21 and 25.

Our methods provide a systematic way for factoring out auxiliary reasoning on data from the analysis relating to the branching structure of the monitors; the former kind of reasoning can be determined by calling on an independent satisfiability solver. In fact, for the specific case of our expository monitor language in Figure 1, one can show that our methods yield *finite* symbolic transition graphs, making the latter reasoning decidable modulo the expression and boolean language used. The results obtained in this work should also be general enough

to be applicable to other monitoring systems. For instance, Definitions 6, 11, 19, and 24 are independent of the kind of systems monitored, the syntax of the monitor language, and the event value domains and expression languages used. Instead, they are defined in terms of generic characteristics such as their LTS semantics. As a result, extending the monitor language with constructs such as parallel composition would not affect the existing framework. The instrumentation relation one adopts for composing monitors with systems necessarily affects the compositional properties and the correspondence between the respective definitions. However, these changes do not impinge on the general structure of our definitions and should be local to the detection condition in the *resp.* controllability definitions, namely Definition 11(i), 19(i) and 24(i), and the reachability-set definitions $\mathbf{aft}(-), \mathbf{saft}(-)$ and $\mathbf{osaft}(-)$ of Definitions 10, 18, and 24.

Future work. We will further investigate the implementability aspects of our analysis, possibly as an extension to existing model-checking tools. This may raise further issues and adjustments to our definitions *e.g.,* it may be more efficient to batch the satisfiability checks in Definition 24. We plan to apply this to existing transition-based monitor specifications such as [5, 46] and validate its feasibility as an automated specification assistant.

Related Work. The need for determinising monitor syntheses from logical specifications is frequently discussed in the literature [7, 23]. In [1], the authors employ a trace-based definition of deterministic monitors that takes into consideration verdicts (similar to our definition for consistent detections of Definition 6 but without considering universal quantification over system instrumentations) and establish complexity bounds for determinising monitors *wrt.* this definition. Set-simulations, which are related to our monitor-sets, have been used as a proof technique for testing preorders in [15] but do not consider symbolic analyses. Acceptor ambiguity [29] is closely related to our notion of consistent detection with respect to the three outcomes of acceptance, rejection and withholding, as specified in Definition 6. Crucially, however, our definition universally quantifies over all possible system compositions. Subsequently, the main endeavour of our work was that of developing sound and complete compositional techniques to alleviate the analysis for consistent detection; we are unaware of any compositional or coinductive techniques used for determining acceptor ambiguity. Symbolic LTSs were studied extensively for value-passing CCS in [27, 28], but their use in controllability for reasoning about consistent monitor detections is, to our knowledge, novel. The particular setting where it is used, namely the instrumentation composition relation and the use of monitor-sets, also require new technical machinery, such as that of Definition 18. Our definition of controllability, Definition 11, is related to viability (usability) for clients in contract compliance [39] and must-testing [9]. Particularly, in the case of compliance, viability is defined coinductively and is satisfied whenever there *exists* a server that can engage with the client so as to lead it to success whenever interaction terminates. Apart from the universal quantifications over systems (viability existentially quantifies over servers), our work differs from [39, 9] *wrt.* the treatment of verdicts considered, the composition relation used (*i.e.,* instrumentation), and the development of a symbolic analysis for handling of action/event data.

Acknowledgements. The author acknowledges the Dagstuhl seminar 17051 and would like to thank Luca Aceto, Antonis Achilleos, Duncan Attard, Giovanni Bernardi, Ian Cassar, Anna Ingólfsdóttir, Giles Reger and anonymous reviewers for their help and suggestions.

References

1 Luca Aceto, Antonis Achilleos, Adrian Francalanza, Anna Ingólfsdóttir, and Sævar Örn Kjartansson. On the Complexity of Determinizing Monitors. In *CIAA*, pages 1–13, 2017. doi:10.1007/978-3-319-60134-2_1.

2 Luca Aceto, Anna Ingólfsdóttir, Kim Guldstrand Larsen, and Jiri Srba. *Reactive Systems: Modelling, Specification and Verification*. Cambridge Univ. Press, 2007.

3 Luca Aceto, Anna Ingolfsdottir, and Jiri Srba. *Advanced Topics in Bisimulation and Coinduction*, chapter The Algorithmics of Bisimilarity. Cambridge Univ. Press, 2011.

4 Cyril Allauzen, Mehryar Mohri, and Ashish Rastogi. General algorithms for testing the ambiguity of finite automata. In *DTL*, pages 108–120, 2008. doi:10.1007/978-3-540-85780-8_8.

5 Howard Barringer, Yliès Falcone, Klaus Havelund, Giles Reger, and David E. Rydeheard. Quantified Event Automata: Towards Expressive and Efficient Runtime Monitors. In *FM*, pages 68–84, 2012. doi:10.1007/978-3-642-32759-9_9.

6 David Basin, Felix Klaedtke, Srdjan Marinovic, and Eugen Zălinescu. Monitoring compliance policies over incomplete and disagreeing logs. In *RV*, pages 151–167, 2013. doi:10.1007/978-3-642-35632-2_17.

7 Andreas Bauer, Martin Leucker, and Christian Schallhart. Runtime verification for LTL and TLTL. *TOSEM*, 20(4):14, 2011.

8 Shay Berkovich, Borzoo Bonakdarpour, and Sebastian Fischmeister. Runtime verification with minimal intrusion through parallelism. *Formal Methods in System Design*, 46(3):317–348, 2015. doi:10.1007/s10703-015-0226-3.

9 Giovanni Bernardi and Adrian Francalanza. Full-abstraction for must testing preorders. In *COORDINATION*, pages 237–255, 2017. doi:10.1007/978-3-319-59746-1_13.

10 Laura Bocchi, Tzu-Chun Chen, Romain Demangeon, Kohei Honda, and Nobuko Yoshida. Monitoring networks through multiparty session types. *TCS*, 669:33–58, 2017. doi:10.1016/j.tcs.2017.02.009.

11 Borzoo Bonakdarpour, Pierre Fraigniaud, Sergio Rajsbaum, David A. Rosenblueth, and Corentin Travers. Decentralized asynchronous crash-resilient runtime verification. In *CONCUR*, pages 16:1–16:15, 2016. doi:10.4230/LIPIcs.CONCUR.2016.16.

12 Ian Cassar and Adrian Francalanza. Runtime Adaptation for Actor Systems. In *RV*, volume 9333, pages 38–54. Springer, 2015.

13 Ian Cassar and Adrian Francalanza. On Implementing a Monitor-Oriented Programming Framework for Actor Systems. In *iFM*, pages 176–192, 2016.

14 Feng Chen and Grigore Roşu. MOP: An Efficient and Generic Runtime Verification Framework. In *OOPSLA*, pages 569–588, 2007. doi:10.1145/1297027.1297069.

15 Rance Cleaveland and Matthew Hennessy. Testing equivalence as a bisimulation equivalence. *FACS*, 5(1):1–20, 1993.

16 Christian Colombo, Adrian Francalanza, Ruth Mizzi, and Gordon J. Pace. polylarva: Runtime verification with configurable resource-aware monitoring boundaries. In *SEFM*, pages 218–232, 2012.

17 Marcelo d'Amorim and Grigore Roşu. Efficient monitoring of ω-languages. In *CAV*, pages 364 – 378, 2005.

18 Søren Debois, Thomas Hildebrandt, and Tijs Slaats. Safety, liveness and run-time refinement for modular process-aware systems with dynamic sub processes. In *FM*, pages 143–160, 2015. doi:10.1007/978-3-319-19249-9_10.

19 Normann Decker and Daniel Thoma. On freeze LTL with ordered attributes. In *FOSSACS*, pages 269–284, 2016. doi:10.1007/978-3-662-49630-5_16.

20 John Dorsey. *Continuous and Discrete Control Systems: Modeling, Identification, Design, and Implementation*. McGraw-Hill, 2001.

21 Pierre Fraigniaud, Sergio Rajsbaum, and Corentin Travers. On the number of opinions needed for fault-tolerant run-time monitoring in distributed systems. In *RV*, pages 92–107, 2014. doi:10.1007/978-3-319-11164-3_9.

22 Adrian Francalanza. A Theory of Monitors. In *FoSSaCS*, pages 145–161, 2016.

23 Adrian Francalanza, Luca Aceto, and Anna Ingolfsdottir. Monitorability for the Hennessy–Milner logic with recursion. *FMSD*, pages 1–30, 2017. doi:10.1007/s10703-017-0273-z.

24 Adrian Francalanza and Aldrin Seychell. Synthesising Correct concurrent Runtime Monitors. *FMSD*, 46(3):226–261, 2015. doi:10.1007/s10703-014-0217-9.

25 Jan Friso Groote and M.P.A. Sellink. Confluence for process verification. *TCS*, 170(1):47 – 81, 1996. doi:10.1016/S0304-3975(96)80702-X.

26 Yuqin He, Xiangping Chen, and Ge Lin. Composition of monitoring components for on-demand construction of runtime model based on model synthesis. In *Internetware*, pages 1–5, 2013. doi:10.1145/2532443.2532472.

27 Matthew Hennessy and Anna Ingolfsdottir. A Theory of Communicating Processes with Value Passing. *Information and Computation*, 107(2):202 – 236, 1993. doi:10.1006/inco.1993.1067.

28 Matthew Hennessy and Huimin Lin. Symbolic bisimulations. *TCS*, 138(2):353 – 389, 1995. doi:10.1016/0304-3975(94)00172-F.

29 Oscar H. Ibarra and Bala Ravikumar. On Sparseness, Ambiguity and other decision problems for Acceptors and Transducers. In *STACS*, pages 171–179, 1986. doi:10.1007/3-540-16078-7_74.

30 Limin Jia, Hannah Gommerstadt, and Frank Pfenning. Monitors and blame assignment for higher-order session types. In *POPL*, pages 582–594, 2016. doi:10.1145/2837614.2837662.

31 Jerzy Klamka. *Control System, Robotics and Automation*, volume 7, chapter System Characteristics: Stability, Controllability, Observability. EOLLS, 2009.

32 John Klein and Ian Gorton. Runtime Performance Challenges in Bigdata Systems. In *WOSP*, pages 17–22, 2015. doi:10.1145/2693561.2693563.

33 D. König. Über eine Schlussweise aus dem Endlichen ins Unendliche. *Acta Litt. ac. sci. Szeged*, 3, 1927.

34 Dexter Kozen. Results on the propositional μ-calculus. *TCS*, 27:333–354, 1983.

35 Jonathan Laurent, Alwyn Goodloe, and Lee Pike. Assuring the Guardians. In *RV*, pages 87–101, 2015.

36 Jay Ligatti, Lujo Bauer, and David Walker. Edit automata: enforcement mechanisms for run-time security policies. *Int. J. Inf. Secur.*, 4(1-2):2–16, 2005. doi:10.1007/s10207-004-0046-8.

37 Qingzhou Luo and Grigore Roşu. EnforceMOP: A Runtime Property Enforcement System for Multithreaded Programs. In *ISSTA*, New York, NY, USA, 2013. ACM. doi:10.1145/2483760.2483766.

38 Patrick O'Neil Meredith, Dongyun Jin, Dennis Griffith, Feng Chen, and Grigore Roşu. An overview of the MOP runtime verification framework. *STTT*, 14(3):249–289, 2012. doi:10.1007/s10009-011-0198-6.

39 Luca Padovani. Contract-based discovery of web services modulo simple orchestrators. *TCS*, 411(37), 2010.

40 Anna Philippou and David Walker. On confluence in the π-calculus. In *ICALP*, pages 314–324, 1997. doi:10.1007/3-540-63165-8_188.

41 Giles Reger, Helena Cuenca Cruz, and David E. Rydeheard. MarQ: Monitoring at Runtime with QEA. In *TACAS*, pages 596–610, 2015. doi:10.1007/978-3-662-46681-0_55.

42 Davide Sangiorgi. *An introduction to Bisimulation and Coinduction*. Cambridge University Press, 2012.

43 Fred B. Schneider. Enforceable security policies. *ACM Trans. Inf. Syst. Secur.*, 3(1):30–50, 2000. doi:10.1145/353323.353382.

44 Michael Sipser. *Introduction to the Theory of Computation*. Cengage Learning, 3rd edition, 2012.

45 Moshe Y. Vardi and Pierre Wolper. Reasoning about infinite computations. *Inf.& Comp.*, 115(1):1–37, 1994. doi:10.1006/inco.1994.1092.

46 Yoriyuki Yamagata, Cyrille Artho, Masami Hagiya, Jun Inoue, Lei Ma, Yoshinori Tanabe, and Mitsuharu Yamamoto. Runtime monitoring for concurrent systems. In *RV*, pages 386–403, 2016. doi:10.1007/978-3-319-46982-9_24.

Flow Logic*†

Orna Kupferman[1] and Gal Vardi[2]

1 **School of Computer Science and Engineering, The Hebrew University, Israel**
2 **School of Computer Science and Engineering, The Hebrew University, Israel**

Abstract

A flow network is a directed graph in which each edge has a capacity, bounding the amount of flow that can travel through it. Flow networks have attracted a lot of research in computer science. Indeed, many questions in numerous application areas can be reduced to questions about flow networks. This includes direct applications, namely a search for a maximal flow in networks, as well as less direct applications, like maximal matching or optimal scheduling. Many of these applications would benefit from a framework in which one can formally reason about properties of flow networks that go beyond their maximal flow.

We introduce *Flow Logics*: modal logics that treat flow functions as explicit first-order objects and enable the specification of rich properties of flow networks. The syntax of our logic BFL* (Branching Flow Logic) is similar to the syntax of the temporal logic CTL*, except that atomic assertions may be *flow propositions*, like $> \gamma$ or $\geq \gamma$, for $\gamma \in \mathbb{N}$, which refer to the value of the flow in a vertex, and that first-order quantification can be applied both to paths and to flow functions. For example, the BFL* formula $\mathcal{E}((\geq 100) \wedge AG(low \rightarrow (\leq 20))$ states that there is a legal flow function in which the flow is above 100 and in all paths, the amount of flow that travels through vertices with low security is at most 20.

We present an exhaustive study of the theoretical and practical aspects of BFL*, as well as extensions and fragments of it. Our extensions include flow quantifications that range over non-integral flow functions or over maximal flow functions, path quantification that ranges over paths along which non-zero flow travels, past operators, and first-order quantification of flow values. We focus on the *model-checking* problem and show that it is PSPACE-complete, as it is for CTL*. Handling of flow quantifiers, however, increases the complexity in terms of the network to P^{NP}, even for the LFL and BFL fragments, which are the flow-counterparts of LTL and CTL. We are still able to point to a useful fragment of BFL* for which the model-checking problem can be solved in polynomial time.

1998 ACM Subject Classification F.4.1 Temporal Logic

Keywords and phrases Flow Network, Temporal Logic

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.9

1 Introduction

A *flow network* is a directed graph in which each edge has a capacity, bounding the amount of flow that can travel through it. The amount of flow that enters a vertex equals the amount of flow that leaves it, unless the vertex is a *source*, which has only outgoing flow, or a *target*, which has only incoming flow. The fundamental *maximum-flow problem* gets as input a flow

* This research has received funding from the European Research Council under the EU's 7-th Framework Programme (FP7/2007-2013) / ERC grant agreement no 278410.

† A full version of the paper is available at `http://www.cs.huji.ac.il/~ornak/publications/concur17.pdf`.

network and searches for a maximal flow from the source to the target [10, 18]. The problem was first formulated and solved in the 1950's [16, 17]. It has attracted much research on improved algorithms [12, 11, 19, 32] and applications [1].

The maximum-flow problem can be applied in many settings in which something travels along a network. This covers numerous application domains, including traffic in road or rail systems, fluids in pipes, currents in an electrical circuit, packets in a communication network, and many more [1]. Less obvious applications involve flow networks that are constructed in order to model settings with an abstract network, as in the case of scheduling with constraints [1] or elimination in partially completed tournaments [36]. In addition, several classical graph-theory problems can be reduced to the maximum-flow problem. This includes the problem of finding a maximum bipartite matching, minimum path cover, maximum edge-disjoint or vertex-disjoint path, and many more [10, 1]. Variants of the maximum-flow problem can accommodate further settings, like circulation problems [38], multiple source and target vertices, costs for unit flows, multiple commodities, and more [14].

All the above applications reduce the problem at hand to the problem of finding a maximal flow in a network. Often, however, one would like to reason about properties of flow networks that go beyond their maximal flow. This is especially true when the vertices or edges of the network attain information to which the properties can refer. For example, the vertices of a network may be labeled by their security level, and we may want to check whether all legal flow functions are such that the flow in every low-security vertex is at most 20, or check whether there is a flow function in which more than 100 units of flow reach the target and still the flow in every low-security vertex is at most 20. As another example, assume that each vertex in the network is labeled by the service provider that owns it, and we want to find a maximal flow under the constraint that flow travels through vertices owned by at most two providers.

The challenge of reasoning about properties of systems has been extensively studied in the context of formal verification. In particular, in *temporal-logic model checking* [8, 34], we check whether a system has a desired property by translating the system into a labeled state-transition graph, translating the property into a temporal-logic formula, and deciding whether the graph satisfies the formula. Model checking is one of the notable success stories of theoretical computer science, with exciting theoretical research that is being transformed into industrial applications [9, 7]. By viewing networks as labeled state-transition graphs, we can use existing model-checking algorithms and tools in order to reason about the structural properties of networks. We can check, for example, that every path from the source to the target eventually visits a *check-sum* vertex. Most interesting properties of flow networks, however, refer to flows and their values, and not just to the structural properties of the network. Traditional temporal logics do not support the specification and verification of such properties.

We introduce and study *Flow Logics*: modal logics that treat flow functions as explicit first-order objects and enable the specification of rich properties of flow networks. The syntax of our logic BFL* (Branching Flow Logic) is similar to the syntax of the temporal logic CTL*, except that atomic assertions are built from both atomic propositions and *flow propositions*, like $> \gamma$ or $\geq \gamma$, for $\gamma \in \mathbb{N}$, which refer to the value of the flow in a vertex, and that first-order quantification can be applied both to paths and to flow functions. Thus, in addition to the path quantifiers A ("for all paths") and E ("there exists a path") that range over paths, states formulas may contain the flow quantifiers $\mathcal{A}$ ("for all flow functions") and $\mathcal{E}$ ("there exists a flow function"). For example, the BFL* formula $\mathcal{E}((\geq 100) \wedge AG(\mathit{low} \rightarrow (\leq 20)))$ states the property discussed above, namely that there is a flow function in which the value

of the flow is at least 100, and in all paths, the value of flow in vertices with low security is at most 20.

We study the theoretical aspects of BFL* as well as extensions and fragments of it. We demonstrate their applications in reasoning about flow networks, and we examine the complexity of their model-checking problem. Below we briefly survey our results. (1) We show that while maximal flow can always be achieved by integral flows [16], in the richer setting of flow logic, restricting attention to integral flows may change the satisfaction value of formulas. Accordingly, our semantics for BFL* considers two types of flow quantification: one over integral flows and another over *non-integral* ones. (2) We prove that bisimulation [33] is not a suitable equivalence relation for flow logics, which are sensitive to unwinding. We relate this to the usefulness of *past operators* in flow logic, and we study additional aspects of the expressive power of BFL*. (3) We consider extensions of BFL* by path quantifiers that range over paths on which flow travels (rather than over all paths in the network), and by *first-order quantification on flow values*. (4) We study the model-checking complexity of BFL*, its extensions, and some natural fragments. We show that algorithms for temporal-logic model-checking can be extended to handle flow logics, and that the complexity the BFL* model-checking problems is PSPACE-complete. We study also the *network complexity* of the problem, namely the complexity in terms of the network, assuming that the formula is fixed [29, 27], and point to a fragment of BFL* for which the model-checking problem can be solved in polynomial time.

Related Work. There are three types of related works: (1) efforts to generalize the maximal-flow problem to richer settings, (2) extensions of temporal logics by new elements, in particular first-order quantification over new types, and (3) works on logical aspects of networks and their use in formal methods. Below we briefly survey them and their relation to our work.

As discussed early in this section, numerous extensions to the classical maximal-flow problems have been considered. In particular, some works that add constraints on the maximal flow, like capacities on vertices, or lower bounds on the flow along edges. Closest to flow logics are works that refer to labeled flow networks. For example, [20] considers flow networks in which edges are labeled, and the problem of finding a maximal flow with a minimum number of labels. Then, the maximal utilization problem of *capacitated automata* [26] amounts to finding maximal flow in a labeled flow network where flow is constrained to travel only along paths that belong to a given regular language. Our work suggests a formalism that embodies all these extensions, as well as a framework for formally reasoning about many more extensions and settings.

The competence of temporal-logic model checking initiated numerous extensions of temporal logics, aiming to capture richer settings. For example, *real-time* temporal logics include clocks with a real-time domain [2], *epistemic temporal logics* include knowledge operators [21], and *alternating temporal logics* include game modalities [3]. Closest to our work is *strategy logic* [6], where temporal logic is enriched by first-order quantification of strategies in a game. Beyond the theoretical interest in strategy logic, it was proven useful in synthesizing strategies in multi-agent systems and in the solution of rational synthesis [15].

Finally, network verification is an increasingly important topic in the context of protocol verification [23]. Tools that allow verifying properties of network protocols have been developed [40, 31]. These tools support verification of network protocols in the design phase as well as runtime verification [22]. Some of these tools use a query language called *Network Datalog* in order to specify network protocols [30]. Verification of *Software Defined Net-*

works has been studied widely, for example in [24, 28, 5]. Verification of safety properties in networks with finite-state middleboxes was studied in [39]. Network protocols describe forwarding policies for packets, and are thus related to specific flow functions. However, the way traffic is transmitted in these protocols does not correspond to the way flow travels in a flow network. Thus, properties verified in this line of work are different from these we can reason about with flow logic.

Due to the lack of space, some details and proofs are omitted, and can be found in the full version, in the authors' URLs.

2 The Flow Logic BFL*

A *flow network* is $N = \langle AP, V, E, c, \rho, s, T\rangle$, where AP is a set of atomic propositions, V is a set of vertices, $s \in V$ is a source vertex, $T \subseteq V$ is a set of target vertices, $E \subseteq (V \setminus T) \times (V \setminus \{s\})$ is a set of directed edges, $c : E \to \mathbb{N}$ is a capacity function, assigning to each edge an integral amount of flow that the edge can transfer, and $\rho : V \to 2^{AP}$ assigns each vertex $v \in V$ to the set of atomic propositions that are valid in v. Note that no edge enters the source vertex or leaves a target vertex. We assume that all vertices $t \in T$ are reachable from s and that each vertex has at least one target vertex reachable from it. For a vertex $u \in V$, let E^u and E_u be the sets of incoming and outgoing edges to and from u, respectively. That is, $E^u = (V \times \{u\}) \cap E$ and $E_u = (\{u\} \times V) \cap E$.

A *flow* is a function $f : E \to \mathbb{N}$ that describes how flow is directed in N. The capacity of an edge bounds the flow in it, thus for every edge $e \in E$, we have $f(e) \leq c(e)$. All incoming flow must exit a vertex, thus for every vertex $v \in V \setminus (\{s\} \cup T)$, we have $\sum_{e\in E^v} f(e) = \sum_{e\in E_v} f(e)$. We extend f to vertices and use $f(v)$ to denote the flow that travels through v. Thus, for $v \in V\setminus(\{s\}\cup T)$, we define $f(v) = \sum_{e\in E^v} f(e) = \sum_{e\in E_v} f(e)$, for the source vertex s, we define $f(s) = \sum_{e\in E_s} f(e)$, and for a target vertex $t \in T$, we define $f(t) = \sum_{e\in E^t} f(e)$. Note that the preservation of flow in the internal vertices guarantees that $f(s) = \sum_{t\in T} f(t)$, which is the amount of flow that travels from s to all the target vertices together. We say that a flow function f is *maximal* if for every flow function f', we have $f'(s) \leq f(s)$. A maximal flow function can be found in polynomial time [17]. The maximal flow for N is then $f(s)$ for some maximal flow function f.

The logic BFL* is a *Branching Flow Logic* that can specify properties of networks and flows in them. As in CTL*, there are two types of formulas in BFL*: *state formulas*, which describe vertices in a network, and *path formulas*, which describe paths. In addition to the operators in CTL*, the logic BFL* has *flow propositions*, with which one can specify the flow in vertices, and *flow quantifiers*, with which one can quantify flow functions universally or existentially. When flow is not quantified, satisfaction is defined with respect to both a network and a flow function. Formally, given a set AP of atomic propositions, a BFL* state formula is one of the following:

(S1) An atomic proposition $p \in AP$.
(S2) A flow proposition $> \gamma$ or $\geq \gamma$, for an integer $\gamma \in \mathbb{N}$.
(S3) $\neg\varphi_1$ or $\varphi_1 \vee \varphi_2$, for BFL* state formulas φ_1 and φ_2.
(S4) $A\psi$, for a BFL* path formula ψ. A is a path quantifier.
(S5) $\mathcal{A}\varphi$, for a BFL* state formula φ. $\mathcal{A}$ is a flow quantifier.

A BFL* path formula is one of the following:

(P1) A BFL* state formula.
(P2) $\neg\psi_1$ or $\psi_1 \vee \psi_2$, for BFL* path formulas ψ_1 and ψ_2.
(P3) $X\psi_1$ or $\psi_1 U \psi_2$, for BFL* path formulas ψ_1 and ψ_2.

We say that a BFL* formula φ is *closed* if all flow propositions appear in the scope of a flow quantifier. The logic BFL* consists of the set of closed BFL* state formulas. We refer to state formula of the form $\mathcal{A}\varphi$ as a *flow state formula*.

The semantics of BFL* is defined with respect to vertices in a flow network. Before we define the semantics, we need some more definitions and notations. Let $N = \langle AP, V, E, c, \rho, s, T\rangle$. For two vertices u and w in V, a finite sequence $\pi = v_0, v_1, \ldots, v_k \in V^*$ of vertices is a (u, w)-*path* in N if $v_0 = u$, $v_k = w$, and $\langle v_i, v_{i+1}\rangle \in E$ for all $0 \leq i < k$. If $w \in T$, then π is a *target* u-*path*.

State formulas are interpreted with respect to a vertex v in N and a flow function $f : E \to \mathbb{N}$. When the formula is closed, satisfaction is independent of the function f and we omit it. We use $v, f \models \varphi$ to indicate that the vertex v satisfies the state formula φ when the flow function is f. The relation $\models$ is defined inductively as follows.

(S1) For an atomic proposition $p \in AP$, we have that $v, f \models p$ iff $p \in \rho(v)$.
(S2) For $\gamma \in \mathbb{N}$, we have $v, f \models > \gamma$ iff $f(v) > \gamma$ and $v, f \models \geq \gamma$ iff $f(v) \geq \gamma$.
(S3a) $v, f \models \neg\varphi_1$ iff $v, f \not\models \varphi_1$.
(S3b) $v, f \models \varphi_1 \vee \varphi_2$ iff $v, f \models \varphi_1$ or $v, f \models \varphi_2$.
(S4) $v, f \models A\psi$ iff for all target v-paths π, we have that $\pi, f \models \psi$.
(S5) $v, f \models \mathcal{A}\varphi$ iff for all flow functions f', we have $v, f' \models \varphi$.

Path formulas are interpreted with respect to a finite path π in N and a flow function $f : E \to \mathbb{N}$. We use $\pi, f \models \varphi$ to indicate that the path π satisfies the path-flow formula ψ when the flow function is f. The relation $\models$ is defined inductively as follows. Let $\pi = v_0, v_1, \ldots, v_k$. For $0 \leq i \leq k$, we use π^i to denote the suffix of π that starts at v_i, thus $\pi^i = v_i, v_{i+1}, \ldots, v_k$.
(P1) For a state formula φ, we have that $\pi, f \models \varphi$ iff $v_0, f \models \varphi$.
(P2a) $\pi, f \models \neg\psi$ iff $\pi, f \not\models \psi$.
(P2b) $\pi, f \models \psi_1 \vee \psi_2$ iff $\pi, f \models \psi_1$ or $\pi, f \models \psi_2$.
(P3a) $\pi, f \models X\psi_1$ iff $k > 0$ and $\pi^1, f \models \psi_1$.
(P3b) $\pi, f \models \psi_1 U \psi_2$ iff there is $j \leq k$ such that $\pi^j, f \models \psi_2$, and for all $0 \leq i < j$, we have $\pi^i, f \models \psi_1$

For a network N and a closed BFL* formula φ, we say that N satisfies φ, denoted $N \models \varphi$, iff $s \models \varphi$ (note that since φ is closed, we do not specify a flow function).

Additional Boolean connectives and modal operators are defined from $\neg$, $\vee$, X, and U in the usual manner; in particular, $F\psi = \mathbf{true}U\psi$ and $G\psi = \neg F\neg\psi$. We also define dual and abbreviated flow propositions: $< \gamma = \neg(\geq \gamma)$, $\leq \gamma = \neg(> \gamma)$, and $\gamma = (\leq \gamma) \wedge (\geq \gamma)$, a dual path quantifier: $E\psi = \neg A\neg\psi$, and a dual flow quantifier: $\mathcal{E}\varphi = \neg\mathcal{A}\neg\varphi$.

▶ **Example 1.** Consider a network N in which target vertices are labeled by an atomic proposition *target*, and low-security vertices are labeled *red*. The BFL* formula $\mathcal{E}EF(target \wedge 20)$ states that there is a flow in which 20 units reach a target vertex, and the BFL* formula $\mathcal{A}((\geq 20) \to AX(\geq 4))$ states that in all flow functions in which the flow at the source is at least 20, all the successors must have flow of at least 4. Finally, $\mathcal{E}((\geq 100) \wedge AG(red \to (\leq 20)))$ states that there is a flow of at least 100 in which the flow in every low-security vertex is at most 20, whereas $\mathcal{A}((> 200) \to EF(red \wedge (> 20)))$ states that when the flow is above 200, then there must exist a low-security vertex in which the flow is above 20. As an example to a BFL* formula with an alternating nesting of flow quantifiers, consider the formula $\mathcal{E}AG(< 10 \to \mathcal{A} < 15)$, stating that there is a flow such that wherever the flow is below 10, then in every flow it would be below 15. ◀

▶ Remark. Note that while the semantics of CTL* and LTL is defined with respect to infinite trees and paths, path quantification in BFL* ranges over finite paths. We are still going to use techniques and results known for CTL* and LTL in our study. Indeed, for upper bounds, the transition to finite computations only makes the setting simpler. Also, lower-bound proofs for CTL* and LTL are based on an encoding of finite runs of Turing machines, and apply also to finite paths.

Specifying finite paths, we have a choice between weak and strong semantics for the X operator. In the weak semantics, the last vertex in a path satisfies $X\psi$, for all ψ. In particular, it is the only vertex that satisfies $X\mathbf{false}$. In the strong semantics, the last vertex does not satisfy $X\psi$, for all ψ. In particular, it does not satisfy $X\mathbf{true}$. We use the strong semantics.

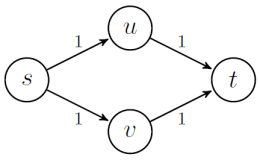
3 Properties of BFL*

3.1 Integral vs. non-integral flow functions

Our semantics of BFL* considers *integral flow functions*: vertices receive integral incoming flow and partition it to integral flows in the outgoing edges. Integral-flow functions arise naturally in settings in which the objects we transfer along the network cannot be partitioned into fractions, as is the case with cars, packets, and more. Sometimes, however, as in the case of liquids, flow can be partitioned arbitrarily. In the traditional maximum-flow problem, it is well known that the maximum flow can be achieved by integral flows [16]. We show that, interestingly, in the richer setting of flow logic, restricting attention to integral flows may change the satisfaction value of formulas.

▶ Proposition 2. Allowing the quantified flow functions in BFL* to get values in $\mathbb{R}$ changes its semantics.

Proof. Consider the network on the right. The BFL* formula $\varphi = \mathcal{E}(1 \wedge AX(> 0))$ states that there is a flow function in which the flow that leaves the source is 1 and the flow of both its successors is strictly positive. It is easy to see that while no integral flow function satisfies the requirement in φ, a flow function in which 1 unit of flow in s is partitioned between u and v does satisfy it. ◀



Proposition 2 suggests that quantification of flow functions that allow non-integral flows may be of interest. In Section 4.3 we discuss such an extension.

3.2 Sensitivity to unwinding

For a network $N = \langle AP, V, E, c, \rho, s, T\rangle$, let N_t be the unwinding of N into a tree. Formally, $N_t = \langle AP, V', E', \rho', s, T'\rangle$, where $V' \subseteq V^*$ is the smallest set such that $s \in V'$, and for all $w \cdot v \in V'$ with $w \in V^*$ and $v \in V \setminus T$, and all $u \in V$ such that $E(v, u)$, we have that $w{\cdot}v{\cdot}u \in V'$, with $\rho'(w{\cdot}v{\cdot}u) = \rho(u)$. Also, $\langle w{\cdot}v, w{\cdot}v{\cdot}u\rangle \in E'$, with $c'(\langle w{\cdot}v, w{\cdot}v{\cdot}u\rangle) = c(\langle v, u\rangle)$. Finally, $T' = V' \cap (V^* \cdot T)$. Note that N_t may be infinite. Indeed, a cycle in N induces infinitely many vertices in N_t.

The temporal logic CTL* is insensitive to unwinding. Indeed, N and N_t are bisimilar, and for every CTL* formula φ, we have $N \models \varphi$ iff $N_t \models \varphi$ [33]. We show that this is not the case for BFL*.

▶ Proposition 3. The value of the maximal flow is sensitive to unwinding.

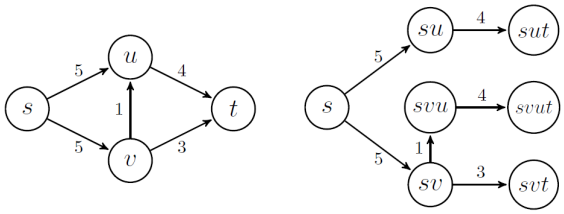


Figure 1 The flow network N and its unwinding N_t.

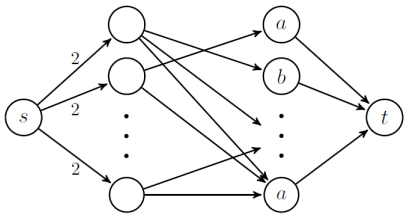


Figure 2 Assigning workers to jobs.

Proof. Consider the network N appearing in Figure 1, and its unwinding N_t which appears in its right. It is easy to see that the value of the maximal flow in N is 7 and the value of the maximal flow from s to T' in N_t is 8. ◀

▶ **Corollary 4.** *The logic BFL⋆ is sensitive to unwinding.*

The sensitivity of BFL⋆ to unwinding suggests that extending BFL⋆ with past operators can increase its expressive power. In Section 4.4, we discuss such an extension.

4 Extensions and Fragments of BFL⋆

In this section we discuss useful extensions and variants of BFL⋆, as well as fragments of it. As we shall show in the sequel, while the extensions come with no computational price, their model checking requires additional techniques.

4.1 Positive path quantification

Consider a network $N = \langle AP, V, E, c, \rho, s, T\rangle$ and a flow function $f : E \to \mathbb{N}$. We say that a path $\pi = v_0, v_1, \ldots, v_k$ is positive if the flow along all the edges in π is positive. Formally, $f(v_i, v_{i+1}) > 0$, for all $0 \leq i < k$. Note that it may be that $f(v_i) > 0$ for all $0 \leq i < k$ and still π is not positive. It is sometimes desirable to restrict the range of path quantification to paths along which flow travels. This is the task of the *positive path quantifier* A^+, with the following semantics (dually, $E^+\psi = \neg A^+\neg\psi$).

- $v, f \models A^+\psi$ iff for all positive target v-paths π, we have that $\pi, f \models \psi$.

▶ **Example 5.** Let W be a set of workers and J be a set of jobs. Each worker $w \in W$ can be assigned to perform jobs from a subset $J_w \subseteq J$. It is required to perform all jobs by assigning exactly one worker to each job and at most two jobs to each worker. This problem can be solved using a flow network. Indeed, a flow of k units in the network described in Figure 2 corresponds to a legal assignment in which k jobs are performed (in the figure,

edges with no specified capacity have capacity 1). Now assume that some jobs should be processed in Location a and the others in Location b. A worker can process jobs only in a single location, a or b. By labeling the job-vertices by their location, the existence of a legal assignment in which k jobs are processed can be expressed by the BFL* formula $\mathcal{E}(k \wedge AX(A^+Xa \vee A^+Xb))$, which uses positive path quantification. ◀

4.2 Maximal flow quantification

It is sometimes desirable to restrict the range of flow quantification to maximal flow functions. This is the task of the *maximal-flow quantifier* $\mathcal{A}^{max}$, with the following semantics (dually, $\mathcal{E}^{max}\varphi = \neg\mathcal{A}^{max}\neg\varphi$).

- $v, f \models \mathcal{A}^{max}\varphi$ iff for all maximal-flow functions f', we have that $v, f' \models \varphi$.

In a similar manner, it is sometimes helpful to relate to the maximal flow in the network. The max-flow constant $\gamma_{max} \in N$ maintains the value of the maximal flow from s to T. We also allow arithmetic operations on γ_{max}.

▶ **Example 6.** Recall the job-assignment problem from Example 5. The formula $\mathcal{E}^{max}(AX(A^+Xa \vee A^+Xb))$ states that the requirements about the locations do not reduce the number of jobs assigned without this requirement. Then, the formula $\mathcal{E}((\geq \gamma_{max} - 4) \wedge AX(A^+Xa \vee A^+Xb))$ states that the requirements about the locations may reduce the number of jobs performed by at most 4. ◀

4.3 Non-integral flow quantification

As discussed in Section 3.1, letting flow quantification range over non-integral flow functions may change the satisfaction value of a BFL* formula. We extend BFL* with a *non-integral flow quantifier* $\mathcal{A}^{\mathbb{R}}$, with the following semantics (dually, $\mathcal{E}^{\mathbb{R}}\varphi = \neg\mathcal{A}^{\mathbb{R}}\neg\varphi$).

- $v, f \models \mathcal{A}^{\mathbb{R}}\varphi$ iff for all flow functions $f' : E \to \mathbb{R}$, we have that $v, f' \models \varphi$.

4.4 Past operators

As discussed in Section 3.2, while temporal logics are insensitive to unwinding, this is not the case for BFL*. Intuitively, this follows from the fact that the flow in a vertex depends on the flow it gets from all its predecessors. This dependency suggests that an explicit reference to predecessors is useful, and motivates the extension of BFL* by past operators.

Adding past to a branching logic, one can choose between a linear-past semantics – one in which past is unique (technically, the semantics is with respect to an unwinding of the network), and a branching-past semantics – one in which all the possible behaviors that lead to present are taken into an account (technically, the semantics is dual to that of future operators, and is defined with respect to the network) [25]. For flow logics, the branching-past approach is the suitable one, and is defined as follows.

For a path $\pi = v_0, v_1, \ldots, v_k \in V^*$, a vertex $v \in V$, and index $0 \leq i \leq k$, we say that π is a source-target (v, i)-path if $v_0 = s$, $v_i = v$, and $v_k \in T$. We add to BFL* two past modal operators, Y ("Yesterday") and S ("Since"), and adjust the semantics as follows. Defining the semantics of logics that refer to the past, the semantics of path formulas is defined with respect to a path and an index in it. We use $\pi, i, f \models \psi$ to indicate that the path π satisfies the path formula ψ from position i when the flow function is f. For state formulas, we adjust the semantics as follows.

(S4) $v, f \models A\psi$ iff for all source-target (v, i)-paths π, we have that $\pi, i, f \models \psi$.

Then, for path formulas, we have the following (the adjustment to refer to the index i in all other modalities is similar).

- $\pi, i, f \models Y\psi_1$ iff $i > 0$ and $\pi, i-1, f \models \psi_1$.
- $\pi, i, f \models \psi_1 S \psi_2$ iff there is $0 \leq j < i$ such that $\pi, j, f \models \psi_2$, and for all $j+1 \leq l \leq i$, we have $\pi, l, f \models \psi_1$.

▶ **Example 7.** Recall the job-assignment problem from Example 5. Assume we want to apply the restriction about the location only to jobs that can be assigned only to workers with no car. Thus, if all the predecessors of a job-vertex are labeled by *car*, then this job can be served in either locations. Using past operators, we can specify this property by $\mathcal{E}(k \wedge AX(A^+X(a \vee AY\, car) \vee A^+X(b \vee AY\, car)))$. ◀

As proven in [25], adding past to CTL* with a branching-past semantics strictly increases its expressive power. The same arguments can be used in order to show that BFL* with past operators is strictly more expressive than BFL*.[1]

4.5 First-Order quantification on flow values

The flow propositions in BFL* include constants. This makes it impossible to relate the flow in different vertices other than specifying all possible constants that satisfy the relation. In *BFL* with quantified flow values* we add flow variables $X = \{x_1, \ldots, x_n\}$ that can be quantified universally or existentially and specify such relations conveniently. We also allow the logic to apply arithmetic operations on the values of variables in X.

For a set of arithmetic operators O (that is, O may include $+, *$, etc.), let BFL$^\star(O)$ be BFL* in which Rule S2 is extended to allow expressions with variables in X constructed by operators in O, and we also allow quantification on the variables in X. Formally, we have the following:

(S2) A flow proposition $> g(x_1, \ldots, x_k)$ or $\geq\ g(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are variables in X and g is an expression obtained from $x_1, \ldots, x_k$ by applying operators in O, possibly using constants in $\mathbb{N}$. We assume that $g : \mathbb{N}^k \to \mathbb{N}$. That is, g leaves us in the domain $\mathbb{N}$.

(S6) $\forall x \varphi$, for $x \in X$ and a BFL$^\star(O)$ formula φ in which x is free.

For a BFL$^\star(O)$ formula φ in which x is a free variable, and a constant $\gamma \in \mathbb{N}$, let $\varphi[x \leftarrow \gamma]$ be the formula obtained by assigning γ to x and replacing expressions by their evaluation. Then, $v, f \models \forall x \varphi'$ iff for all $\gamma \in \mathbb{N}$, we have that $v, f \models \varphi'[x \leftarrow \gamma]$.

▶ **Example 8.** The logic BFL$^\star(\emptyset)$ includes the formula $\mathcal{E}AG\forall x((split \wedge x \wedge > 0) \rightarrow EX(> 0 \wedge < x))$, stating that there is a flow in which all vertices that are labeled *split* and with a positive flow x have a successor in which the flow is positive but strictly smaller than x. Then, BFL*(div) includes the formula $\mathcal{E}AG\forall x(x \rightarrow EX(\geq x \text{ div } 2))$, stating that there is a flow in which all vertices have a successor that has at least half of their flow.

Finally, BFL$^\star(+)$ includes the formula $\exists x \exists y \mathcal{E}^{max} AG(\neg(source \vee target) \rightarrow x \vee y \vee (x+y))$, stating that there are values x and y, such that it is possible to attain the maximal flow by assigning to all vertices, except maybe source and target vertices, values in $\{x, y, x+y\}$. ◀

[1] We note that the result from [25] does not immediately imply the addition of expressive power, as it is based on the fact that only CTL* with past operators is sensitive to unwinding (and, as we prove in Section 3.2, BFL* is sensitive to unwinding). Still, since the specific formula used in [25] in order to prove the sensitivity of CTL* with past to unwinding does not refer to flow, it is easy to see that it has no equivalent BFL* formula.

4.6 Fragments of BFL*

For the temporal logic CTL*, researchers have studied several fragments, most notably LTL and CTL. In this section we define interesting fragments of BFL*.

Flow-CTL* and Flow-LTL

The logics *Flow-CTL** and *Flow-LTL* are extensions of CTL* and LTL in which atomic state formulas may be, in addition to APs, also the flow propositions $> \gamma$ or $\geq \gamma$, for an integer $\gamma \in \mathbb{N}$. Thus, no quantification on flow is allowed, but atomic formulas may refer to flow. The semantics of Flow-CTL* is defined with respect to a network and a flow function, and that of Flow-LTL is defined with respect to a path in a network and a flow function.

Linear Flow Logic

The logic LFL is the fragment of BFL* in which only one external universal path quantification is allowed. Thus, an LFL formula is a BFL* formula of the form $A\psi$, where ψ is generated without rule S4.

Note that while the temporal logic LTL is a "pure linear" logic, in the sense that satisfaction of an LTL formula in a computation of a system is independent of the structure of the system, the semantics of LFL mixes linear and branching semantics. Indeed, while all the paths in N have to satisfy ψ, the context of the system is important. To see this, consider the LFL formula $\varphi = A\mathcal{A}((\geq 10) \rightarrow X(\geq 4))$. The formula states that in all paths, all flow functions in which the flow at the first vertex in the path is at least 10, are such that the flow at the second vertex in the path is at least 4. In order to evaluate the path formula $\mathcal{A}((\geq 10) \rightarrow X(\geq 4))$ in a path π of a network N we need to know the capacity of all the edges from the source of N, and not only the capacity of the first edge in π. For example, φ is satisfied in networks in which there are two successors to the source, each connected by an edge with capacity 4, 5, or 6. Consider now the LFL formula $\varphi' = A\mathcal{E}(10 \wedge X(\geq 4))$. Note that φ' is not equal to the BFL* formula $\theta = \mathcal{E}(10 \wedge AX(\geq 4))$. Indeed, in the latter, the same flow function should satisfy the path formula $X(\geq 4)$ in all paths.

No nesting of flow quantifiers

The logic BFL$_1^\star$ contains formulas that are Boolean combinations of formulas of the form $\mathcal{E}\varphi$ and $\mathcal{A}\varphi$, for a Flow-CTL* formula φ. Of special interest are the following fragments of BFL$_1^\star$:

- $\exists$BFL$_1^\star$ and $\forall$BFL$_1^\star$, where formulas are of the form $\mathcal{E}\varphi$ and $\mathcal{A}\varphi$, respectively, for a Flow-CTL* formula φ.
- $\exists$LFL$_1$ and $\forall$LFL$_1$, where formulas are of the form $\mathcal{E}A\psi$ and $\mathcal{A}A\psi$, respectively, for a Flow-LTL formula ψ, and LFL$_1$, where a formula is a Boolean combination of $\exists$LFL$_1$ and $\forall$LFL$_1$ formulas.

Conjunctive-BFL*

The fragment Conjunctive-BFL* (CBFL*, for short) contains BFL* formulas whose flow state sub-formulas restrict the quantified flow in a conjunctive way. That is, when we "prune" a CBFL* formula into requirements on the network, atomic flow propositions are only conjunctively related. This would have a computational significance in solving the model-checking problem.

Consider an $\exists\text{BFL}^\star_1$ formula $\varphi = \mathcal{E}\theta$. We say that an operator $g \in \{\vee, \wedge, E, A, X, F, G, U\}$ has a *positive polarity* in φ if all the occurrences of g in θ are in a scope of an even number of negations. Dually, g has a *negative polarity* in φ if all its occurrences in θ are in a scope of an odd number of negations.

The logic $\exists\text{CBFL}^\star_1$ is a fragment of $\exists\text{BFL}^\star_1$ in which the only operators with a positive polarity are $\wedge$, A, X, and G, and the only operators with a negative polarity are $\vee$, E, X, and F. Note that U is not allowed, as its semantics involves both conjunctions and disjunctions. Note that by pushing negations inside, we make all operators of a positive polarity; that is, we are left only with $\wedge$, A, X, and G.

Then, since all requirements are universal and conjunctively related, we can push conjunctions outside so that path formulas do not have internal conjunctions – for example, transform $AX(\xi_1 \wedge \xi_2)$ into $AX\xi_1 \wedge AX\xi_2$, and can get rid of universal path quantification that is nested inside another universal path quantification – for example, transform $AXAX\xi_1$ into $AXX\xi$. Finally, since we use the strong semantics to X, we can replace formulas that have X nested inside G by **false**.

The logic $\forall\text{CBFL}^\star_1$ is the dual fragment of $\forall\text{BFL}^\star_1$. In other words, $\mathcal{A}\theta$ is in $\forall\text{CBFL}^\star_1$ iff $\mathcal{E}\neg\theta$ is in $\exists\text{CBFL}^\star_1$. The logic $\text{CBFL}^\star$ is then obtained by going up a hierarchy in which formulas of lower levels serve as atomic propositions in higher levels.

We now define the syntax of $\text{CBFL}^\star$ formally. For simplicity, we define it in a normal form, obtained by applying the rules described above. A $\text{CBFL}^\star_0$ formula is a Boolean assertion over AP. For $i \geq 0$, a $\text{CBFL}^\star_{i+1}$ formula is a Boolean assertion over $\text{CBFL}^\star_i$ formulas and formulas of the form $\mathcal{E}(A\psi_1 \wedge \cdots \wedge A\psi_n)$, where ψ_j is of the form $X^{k_j}\xi_j$ or $X^{k_j}G\xi_j$, where $k_j \geq 0$ and ξ_j is a $\text{CBFL}^\star_i$ formula or a flow proposition (that is, $> \gamma$, $< \gamma$, $\geq \gamma$, or $\leq \gamma$, for an integer $\gamma \in \mathbb{N}$). Then, a $\text{CBFL}^\star$ formula is a $\text{CBFL}^\star_i$ formula for some $i \geq 0$. Note that both $\exists\text{CBFL}^\star_1$ and $\forall\text{CBFL}^\star_1$ are contained in $\text{CBFL}^\star_1$.

▶ **Example 9.** Recall the job-assignment problem from Example 5. The $\text{CBFL}^\star_1$ formula $\mathcal{A}(< 10 \rightarrow EX \leq 0) \wedge \mathcal{E}(15 \wedge AX \geq 1)$ states that if less than 10 jobs are processed, then at least one worker is unemployed, but it is possible to process 15 jobs and let every worker process at least one job.

BFL

The logic BFL is the fragment of $\text{BFL}^\star$ in which every modal operator (X, U) is preceded by a path quantifier. That is, it is the flow counterpart of CTL.

5 Model Checking

In this section we study the model-checking problem for $\text{BFL}^\star$. The problem is decide, given a flow network N and a $\text{BFL}^\star$ formula φ, whether $N \models \varphi$.

▶ **Theorem 10.** *$\text{BFL}^\star$ model checking is PSPACE-complete.*

Proof. Consider a network N and a $\text{BFL}^\star$ formula φ. The idea behind our model-checking procedure is similar to the one that recursively employs LTL model checking in the process of $\text{CTL}^\star$ model checking [13]. Here, however, the setting is more complicated. Indeed, the path formulas in $\text{BFL}^\star$ are not "purely linear", as the flow quantification in them refers to flow in the (branching) network. In addition, while the search for witness paths is restricted to paths in the network, which can be guessed on-the-fly in the case of LTL, here we also search for witness flow functions, which have to be guessed in a global manner.

Let $\{\varphi_1, \ldots, \varphi_k\}$ be the set of flow state formulas in φ. Assume that $\varphi_1, \ldots, \varphi_k$ are ordered so that for all $1 \leq i \leq k$, all the subformulas of φ_i have indices in $\{1, \ldots, i\}$. Our model-checking procedure labels N by new atomic propositions $q_1, \ldots, q_k$ so that for all vertices v and $1 \leq i \leq k$, we have that $v \models q_i$ iff $v \models \varphi_i$ (note that since φ_i is closed, satisfaction is independent of a flow function).

Starting with $i = 1$, we model check φ_i, label N with q_i, and replace the subformula φ_i in φ by q_i. Accordingly, when we handle φ_i, it is an $\exists\text{BFL}_1^\star$ or a $\forall\text{BFL}_1^\star$ formula. That is, it is of the form $\mathcal{E}\xi$ or $\mathcal{A}\xi$, for a Flow-CTL* formula ξ. Assume that $\varphi_i = \mathcal{E}\xi$. We guess a flow function $f : E \to \mathbb{N}$, and perform CTL* model-checking on ξ, evaluating the flow propositions in ξ according to f. Since guessing f requires polynomial space, and CTL* model checking is in PSPACE, so is handling of φ_i and of all the subformulas.

Hardness in PSPACE follows from the hardness of CTL* model checking [37]. ◀

Since BFL* contains CTL*, the lower bound in Theorem 10 is immediate. One may wonder whether reasoning about flow networks without atomic propositions, namely when we specify properties of flow only, is simpler. Theorem 11 below shows that this is not the case. Essentially, the proof follows from our ability to encode assignments to atomic propositions by values of flow.

▶ **Theorem 11.** *BFL* model checking is PSPACE-complete already for $\forall$LFL$_1$ formulas without atomic propositions.*

We also note that when the given formula is in BFL, we cannot avoid the need to guess a flow function, yet once the flow function is guessed, we can verify it in polynomial time. Accordingly, the model-checking problem for BFL is in P^{NP}. We discuss this point further below.

In practice, a network is typically much bigger than its specification, and its size is the computational bottleneck. In temporal-logic model checking, researchers have analyzed the *system complexity* of model-checking algorithms, namely the complexity in terms of the system, assuming the specification is of a fixed length. There, the system complexity of LTL and CTL* is NLOGSPACE-complete [29, 27]. We prove that, unfortunately, this is not the case of BFL*. That is, we prove that while the *network complexity* of the model-checking problem, namely the complexity in terms of the network, does not reach PSPACE, it does require polynomially many calls to an NP oracle. Essentially, each evaluation of a flow quantifier requires such a call. Formally, we have the following.

▶ **Theorem 12.** *The network complexity of BFL* is in Δ_2^P (namely, in P^{NP}).*

Proof. Fixing the length of the formula in the algorithm described in the proof of Theorem 10, we get that k is fixed, and so is the length of each subformula φ_i. Thus, evaluation of φ_i involves a guess of a flow and then model checking of a fixed size Flow-CTL* formula, which can be done in time polynomial in the size of the network. Hence, the algorithm from Theorem 10 combined with an NP oracle gives the required network complexity. ◀

While finding the exact network complexity of model checking BFL* and its fragments is interesting from a complexity-theoretical point of view, it does not contribute much to our story. Here, we prove NP and co-NP hardness holds already for very restricted fragments. As good news, in Section 7 we point that for the conjunctive fragment, model checking can be performed in polynomial time.[2]

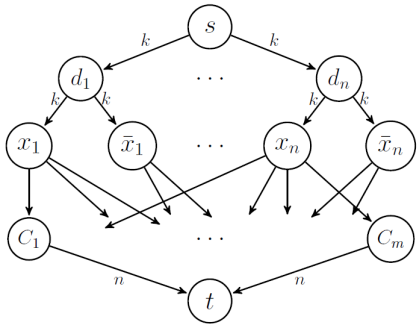
[2] A possible tightening of our analysis is via the complexity class BH, which is based on a Boolean hierarchy over NP. Essentially, it is the smallest class that contains NP and is closed under union,

▶ **Theorem 13.** *The network complexity of $\exists BFL_1^\star$ and $\forall BFL_1^\star$ is NP-complete and co-NP-complete, respectively. Hardness applies already to $\exists LFL_1$ and $\forall LFL_1$ without atomic propositions, and to BFL.*

Proof. For the upper bound, it is easy to see that one step in the algorithm described in the proof of Theorem 10 (that is, evaluating φ_i once all its flow state subformulas have been evaluated), when applied to φ_i of a fixed length is in NP for φ_i of the form $\mathcal{E}\xi$ and in co-NP for φ_i of the form $\mathcal{A}\xi$.

For the lower bound, we prove NP-hardness for $\exists\text{LFL}_1$. Co-NP-hardness for $\forall\text{LFL}_1$ follows by dualization. We describe a reduction from CNF-SAT. Let $\theta = C_1 \wedge \ldots \wedge C_m$ be a CNF formula over the variables $x_1 \ldots x_n$. We assume that every literal in $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$ appears exactly in k clauses in θ. Indeed, every CNF formula can be converted to such a formula in polynomial time and with a polynomial blowup.

We construct a flow network N and an $\exists\text{LFL}_1$ formula $\mathcal{E}A\psi$ such that θ is satisfiable iff $N \models \mathcal{E}A\psi$. The network N is constructed as demonstrated on the right Let $Z = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$. For a literal $z \in Z$ and a clause C_i, the network N contains an edge $\langle z, C_i \rangle$ iff the clause C_i contains the literal z. Thus, each vertex in Z has exactly k outgoing edges. The capacity of each of these edges is 1. The flow-LTL formula $\psi = kn \wedge XX(k \vee 0) \wedge XXX(\geq 1)$. In the full version, we prove that θ is satisfiable iff $N \models \mathcal{E}A\psi$.



Finally, note that ψ does not contain atomic propositions. Also, the same proof holds with the BFL formula $\psi = kn \wedge AXEXk \wedge AXAXAX(\geq 1)$. ◀

5.1 Flow synthesis

In the flow-synthesis problem, we are given a network N and an $\exists\text{BFL}_1^\star$ formula $\mathcal{E}\varphi$, and we have to return a flow function f with which φ is satisfied in N, or declare that no such function exists. The problem is clearly at least as hard as CTL* model checking. Also, by guessing f, its complexity does not go beyond CTL* model-checking complexity. The network complexity of the problem coincides with that of $\exists\text{BFL}_1^\star$ model checking. Thus, we have the following.

▶ **Theorem 14.** *The flow-synthesis problem for $\exists BFL_1^\star$ is PSPACE-complete, and its network complexity is NP-complete.*

6 Model Checking Extensions of BFL*

In Section 4, we define several extensions of BFL*. In this section we study the model-checking complexity for each of the extensions, and show that they do not require an increase

intersection, and complement. The levels of the hierarchy start with $\text{BH}_1 = \text{NP}$, and each level adds internal intersections as well as intersection with a co-NP (even levels) or an NP (odd levels) language [41]. BH is contained in Δ_2^P. It is not hard to prove that the network complexity of the fragment of $\text{BFL}_1^\star$ that contains at most k flow quantifiers is in $\text{BH}_{k+1} \cap \text{co-BH}_{k+1}$. Indeed, the latter contain problems that are decidable in polynomial time with k parallel queries to an NP oracle [4]. A BH_k lower bound can also be shown.

in the complexity. The techniques for handling them are, however, richer: For positive path quantification, we have to refine the network and add a path-predicate that specifies positive flow, in a similar way fairness is handled in temporal logics. For maximal-flow quantification, we have to augment the model-checking algorithm by calls to a procedure that finds the maximal flow. For non-integral flow quantification, we have to reduce the model-checking problem to a solution of a linear-programming system. For past operators, we have to extend the model-checking procedure for CTL* with branching past. Finally, for first-order quantification over flow values, we have to first bound the range of relevant values, and then apply model checking to all relevant values.

Positive path quantification

Given a network N, it is easy to generate a network N' in which we add a vertex in the middle of each edge, and in which the positivity of paths correspond to positive flow in the new intermediate vertices. Formally, assuming that we label the new intermediate vertices by an atomic proposition *edge*, then the BFL* path formula $\xi_{positive} = G(edge \rightarrow > 0)$ characterizes positive paths, and replacing a state formula $A\psi$ by the formula $A(\xi_{positive} \rightarrow \psi)$ restricts the range of path quantification to positive paths. Now, given a BFL* formula φ, it is easy to generate a BFL* formula φ' such that $N \models \varphi$ iff $N' \models \varphi'$. Indeed, we only have to (recursively) modify path formulas so that vertices labeled *edge* are ignored: $X\xi$ is replaced by $XX\xi$, and $\xi_1 U \xi_2$ is replaced by $(\xi_1 \vee edge) U (\xi_2 \wedge \neg edge)$. Hence, the complexity of model-checking is similar to BFL*.

Maximal flow quantification

The maximal flow γ_{max} in a flow network can be found in polynomial time. Our model-checking algorithm for BFL* described in the proof of Theorem 10 handles each flow state subformula $\mathcal{E}\varphi$ by guessing a flow function $f : E \rightarrow \mathbb{N}$ with which the Flow-CTL* formula φ holds. For an $\mathcal{E}^{max}$ quantifier, we can guess only flow functions for which the flow leaving the source vertex is γ_{max}. In addition, after calculating the maximal flow, we can substitute γ_{max}, in formulas that refer to it, by its value. Hence, the complexity of model-checking is similar to that of BFL*.

Non-integral flow quantification

Recall that our BFL* model-checking algorithm handles each flow state subformula $\mathcal{E}\varphi$ by guessing a flow function $f : E \rightarrow \mathbb{N}$ with which the Flow-CTL* formula holds. Moving to non-integral flow functions, the guessed function f should be $f : E \rightarrow \mathbb{R}$, where we cannot bound the size or range of guesses.

Accordingly, in the non-integral case, we guess, for every vertex $v \in V$, an assignment to the flow propositions that appear in φ. Then, we perform two checks. First, that φ is satisfied with the guessed assignment – this is done by CTL* model checking, as in the case of integral flows. Second, that there is a non-integral flow function that satisfies the flow constraints that appear in the vertices. This can be done in polynomial time by solving a system of inequalities [35] (see Lemma 16 for the details in the case of vertex-constrained integral flow functions). Thus, as in the integral case, handling each flow state formula $\mathcal{E}\varphi$ can be done in PSPACE, and so is the complexity of the entire algorithm.

Past operators

Recall that our algorithm reduces BFL* model checking to a sequence of calls to a CTL* model-checking procedure. Starting with a BFL* formula with past operators, the required calls are to a model-checking procedure for CTL* with past. By [25], model checking CTL* with branching past is PSPACE-complete, and thus so is the complexity of our algorithm.

First-Order quantification on flow values

For a flow network $N = \langle AP, V, E, c, \rho, s, T\rangle$, let $C_N = 1 + \Sigma_{e\in E} c(e)$. Thus, for every flow function f for N and for every vertex $v \in V$, we have $f(v) < C_N$. We claim that when we reason about BFL* formulas with quantified flow values, we can restrict attention to values in $\{0, 1, \ldots, C_N\}$.

▶ **Lemma 15.** *Let N be a flow network and let $\theta = \forall x_1 \varphi$ be a BFL$^\star(\{+, *\})$ formula over the variables $X = \{x_1, \ldots, x_n\}$, and without free variables. Then, $N \models \theta$ iff $N \models \varphi[x_1 \leftarrow \gamma]$, for every $0 \leq \gamma \leq C_N$.*

By Lemma 15, the model-checking problem for BFL$^\star(\{+, *\})$ is PSPACE-complete.

7 A Polynomial Fragment

In this section we show that the model-checking problem for CBFL* (see Section 4.6) can be solved in polynomial time.

Our model-checking algorithm reduces the evaluation of a CBFL* formula into a sequence of solutions to the *vertex-constrained flow problem*. In this problem, we are given a flow network $N = \langle AP, V, E, c, \rho, s, T\rangle$ in which each vertex $v \in V$ is attributed by a range $[\gamma_l, \gamma_u] \in \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. The problem is to decide whether there is a flow function $f : E \to \mathbb{R}$ such that for all vertices $v \in V$, we have $\gamma_l \leq f(v) \leq \gamma_u$.

▶ **Lemma 16.** *The vertex-constrained flow problem can be solved in polynomial time. If there is a solution that is a non-integral flow function, then there is also a solution that is an integral flow function, and the algorithm returns such a solution.*

▶ **Theorem 17.** *CBFL* model checking can be solved in polynomial time.*

Proof. Let $N = \langle AP, V, E, c, \rho, s, T\rangle$, and consider a CBFL* formula φ. If φ is in CBFL$^\star_0$, we can clearly label in linear time all the vertices in N by a fresh atomic proposition p_φ that maintains the satisfaction of φ. That is, in all vertices $v \in V$, we have that $p_\varphi \in \rho(v)$ iff $v \models \varphi$. Otherwise, φ is a CBFL$^\star_{i+1}$ formula for some $i \geq 0$. We show how, assuming that the vertices of N are labeled by atomic propositions that maintain satisfaction of the subformulas of φ that are CBFL$^\star_i$ formulas, we can label them, in polynomial time, by a fresh atomic proposition that maintains the satisfaction of φ.

Recall that φ is a Boolean assertion over CBFL$^\star_i$ formulas and flow formulas of the form $\mathcal{E}(A\psi_1 \wedge \cdots \wedge A\psi_n)$, where each ψ_j is of the form $X^{k_j}\xi_j$ or $X^{k_j}G\xi_j$, where $k_j \geq 0$ and ξ_j is a CBFL$^\star_i$ formula or a flow proposition (that is, $> \gamma$, $< \gamma$, $\geq \gamma$, or $\leq \gamma$, for an integer $\gamma \in \mathbb{N}$).

Since CBFL$^\star_i$ subformulas have already been evaluated, we describe how to evaluate subformulas of the form $\theta = \mathcal{E}(A\psi_1 \wedge \cdots \wedge A\psi_n)$. Intuitively, since the formulas in θ include no disjunctions, they impose constraints on the vertices of N in a deterministic manner. These constraints can be checked in polynomial time by solving a vertex-constrained flow problem. Recall that for each $1 \leq j \leq n$, the formula ψ_j is of the form $X^{k_j}\xi_j$ or $X^{k_j}G\xi_j$, for some $k_j \geq 0$, and a CBFL$^\star_i$ formula or a flow proposition ξ_j. In order to evaluate θ in

a vertex $v \in V$, we proceed as follows. For each $1 \leq j \leq n$, the formula ξ_j imposes either a Boolean constraint (in case ξ_j is a CBFL$^\star_i$ formula) or a flow constraint (in case ξ_j is a flow proposition) on a finite subset V^v_j of V. Indeed, if $\psi_j = X^{k_j}\xi_j$, then V^v_j includes all the vertices reachable from v by a path of length k_j, and if $\psi_j = X^{k_j}G\xi_j$, then V^v_j includes all the vertices reachable from v by a path of length at least k_j. We attribute each vertex by the constraints imposes on it by all the conjuncts in θ. If one of the Boolean constrains does not hold, then θ does not hold in v. Otherwise, we obtain a set of flow constraints for each vertex in V. For example, if $\theta = \mathcal{E}(AXXp \wedge AXX > 5 \wedge AG \leq 8)$, then in order to check whether θ holds in s, we assign the flow constraint ≤ 8 to all the vertices reachable from s, and assign the flow constraint > 5 to all the successors of the successors of s. If one of these successors of successors does not satisfy p, we can skip the check for a flow and conclude that s does not satisfy θ. Otherwise, we search for such a flow, as described below.

The flow constrains for a vertex induce a closed, open, or half-closed range. The upper bound in the range may be infinity. For example, the constrains > 6, < 10, ≤ 8 induce the half-closed range $(6, 8]$. Note that it may be that the induced range is empty. For example, the constraints ≤ 6 and > 8 induce an empty range. Then, θ does not hold in v. Since we are interested in integral flows, we can convert all strict bounds to non-strict ones. For example, the range $(6, 8]$ can be converted to $[7, 8]$. Note that since we are interested in integral flow, a non-empty open range may not be satisfiable, and we refer to it as an empty range. For example, the range $(6, 7)$ is empty. Hence, the satisfaction of θ in v is reduced to an instance of the vertex-constrained flow problem. By Lemma 16, deciding whether there is a flow function that satisfies the constraints can be solved in polynomial time. ◀

▶ Remark. Note that the same algorithm can be applied when we consider non-integral flow functions, namely in CBFL* with the $\mathcal{A}^{\mathbb{R}}$ flow quantifier. There, the induced vertex-constrained flow problem may include open boundaries. The solution need not be integral, but can be found in polynomial time by solving a system of inequalities [35].

References

1 R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network flows: Theory, algorithms, and applications*. Prentice Hall Englewood Cliffs, 1993.

2 R. Alur and T.A. Henzinger. A really temporal logic. *Journal of the ACM*, 41(1):181–204, 1994.

3 R. Alur, T.A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49(5):672–713, 2002.

4 R. Beigel. Bounded queries to sat and the boolean hierarchy. *Theoretical Computer Science*, 84(2):199–223, 1991.

5 M. Canini, D. Venzano, P. Perešíni, D. Kostić, and J. Rexford. A nice way to test openflow applications. In *9th USENIX Symposium on Networked Systems Design and Implementation (NSDI 12)*, pages 127–140, 2012.

6 K. Chatterjee, T. A. Henzinger, and N. Piterman. Strategy logic. In *Proc. 18th Int. Conf. on Concurrency Theory*, pages 59–73, 2007.

7 E. Clarke, T.A. Henzinger, and H. Veith. *Handbook of Model Checking*. Elsvier, 2016. Forthcoming.

8 E.M. Clarke and E.A. Emerson. Design and synthesis of synchronization skeletons using branching time temporal logic. In *Proc. Workshop on Logic of Programs*, volume 131 of *Lecture Notes in Computer Science*, pages 52–71. Springer, 1981.

9 E.M. Clarke, O. Grumberg, and D. Peled. *Model Checking*. MIT Press, 1999.

10 T.H. Cormen, C.E. Leiserson, and R.L. Rivest. *Introduction to Algorithms*. MIT Press and McGraw-Hill, 1990.

11 E.A. Dinic. Algorithm for solution of a problem of maximum flow in a network with power estimation. *Soviet Math. Dokl*, 11(5):1277–1280, 1970. English translation by RF. Rinehart.

12 J. Edmonds and R.M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of the ACM*, 19(2):248–264, 1972.

13 E.A. Emerson and C.-L. Lei. Modalities for model checking: Branching time logic strikes back. *Science of Computer Programming*, 8:275–306, 1987.

14 S. Even, A. Itai, and A. Shamir. On the complexity of timetable and multicommodity flow problems. *SIAM Journal on Computing*, 5(4):691–703, 1976.

15 D. Fisman, O. Kupferman, and Y. Lustig. Rational synthesis. In *Proc. 16th Int. Conf. on Tools and Algorithms for the Construction and Analysis of Systems*, volume 6015 of *Lecture Notes in Computer Science*, pages 190–204. Springer, 2010.

16 L.R. Ford and D.R. Fulkerson. Maximal flow through a network. *Canadian journal of Mathematics*, 8(3):399–404, 1956.

17 L.R. Ford and D.R. Fulkerson. *Flows in networks*. Princeton Univ. Press, Princeton, 1962.

18 A.V. Goldberg, É. Tardos, and R.E. Tarjan. Network flow algorithms. Technical report, DTIC Document, 1989.

19 A.V. Goldberg and R.E. Tarjan. A new approach to the maximum-flow problem. *Journal of the ACM*, 35(4):921–940, 1988.

20 D. Granata, R. Cerulli, M.G. Scutellá, and A. Raiconi. Maximum flow problems and an NP-complete variant on edge-labeled graphs. In *Handbook of Combinatorial Optimization*, pages 1913–1948. Springer, 2013.

21 J.Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *Journal of the ACM*, 37(3):549–587, 1990.

22 P. Kazemian, M. Chang, H. Zeng, G. Varghese, N. McKeown, and S. Whyte. Real time network policy checking using header space analysis. In *10th USENIX Symposium on Networked Systems Design and Implementation (NSDI 13)*, pages 99–111, 2013.

23 P. Kazemian, G. Varghese, and N. McKeown. Header space analysis: Static checking for networks. In *9th USENIX Symposium on Networked Systems Design and Implementation (NSDI 12)*, pages 113–126, 2012.

24 A. Khurshid, X. Zou, W. Zhou, M. Caesar, and P.B. Godfrey. Veriflow: verifying network-wide invariants in real time. In *10th USENIX Symposium on Networked Systems Design and Implementation (NSDI 13)*, pages 15–27, 2013.

25 O. Kupferman, A. Pnueli, and M.Y. Vardi. Once and forall. *Journal of Computer and Systems Science*, 78(3):981–996, 2012.

26 O. Kupferman and T. Tamir. Properties and utilization of capacitated automata. In *Proc. 34th Conf. on Foundations of Software Technology and Theoretical Computer Science*, volume 29 of *LIPIcs*, pages 33–44. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2014.

27 O. Kupferman, M.Y. Vardi, and P. Wolper. An automata-theoretic approach to branching-time model checking. *Journal of the ACM*, 47(2):312–360, 2000.

28 M. Kuzniar, P. Peresini, M. Canini, D. Venzano, and D. Kostic. A soft way for openflow switch interoperability testing. In *Proceedings of the 8th international conference on Emerging networking experiments and technologies*, pages 265–276. ACM, 2012.

29 O. Lichtenstein and A. Pnueli. Checking that finite state concurrent programs satisfy their linear specification. In *Proc. 12th ACM Symp. on Principles of Programming Languages*, pages 97–107, 1985.

30 B.T. Loo, T. Condie, M. Garofalakis, D.E. Gay, J.M. Hellerstein, P. Maniatis, R. Ramakrishnan, T. Roscoe, and I. Stoica. Declarative networking: language, execution and

optimization. In *Proceedings of the 2006 ACM SIGMOD international conference on Management of data*, pages 97–108. ACM, 2006.

31 N.P. Lopes, N. Bjørner, P. Godefroid, K. Jayaraman, and G. Varghese. Checking beliefs in dynamic networks. In *12th USENIX Symposium on Networked Systems Design and Implementation (NSDI 15)*, pages 499–512, 2015.

32 A. Madry. Computing maximum flow with augmenting electrical flows. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, pages 593–602. IEEE, 2016.

33 R. Milner. An algebraic definition of simulation between programs. In *Proc. 2nd Int. Joint Conf. on Artificial Intelligence*, pages 481–489. British Computer Society, 1971.

34 J.P. Queille and J. Sifakis. Specification and verification of concurrent systems in Cesar. In *Proc. 9th ACM Symp. on Principles of Programming Languages*, volume 137 of *Lecture Notes in Computer Science*, pages 337–351. Springer, 1982.

35 A. Schrijver. *Combinatorial Optimization*. Springer, 2003.

36 B.L. Schwartz. Possible winners in partially completed tournaments. *SIAM Review*, 8(3):302–308, 1966.

37 A.P. Sistla and E.M. Clarke. The complexity of propositional linear temporal logic. *Journal of the ACM*, 32:733–749, 1985.

38 É. Tardos. A strongly polynomial minimum cost circulation algorithm. *Combinatorica*, 5(3):247–255, 1985.

39 Y. Velner, K. Alpernas, A. Panda, A. Rabinovich, M. Sagiv, S. Shenker, and S. Shoham. Some complexity results for stateful network verification. In *Proc. 22nd Int. Conf. on Tools and Algorithms for the Construction and Analysis of Systems*, pages 811–830. Springer, 2016.

40 A. Wang, P. Basu, B.T. Loo, and O. Sokolsky. Declarative network verification. In *International Symposium on Practical Aspects of Declarative Languages*, pages 61–75. Springer, 2009.

41 G. Wechsung. On the boolean closure of np. In *Proceedings of the International Conference on Fundamentals of Computation Theory*, volume 199 of *Lecture Notes in Computer Science*, pages 485–493. Springer, 1985.

Rule Formats for Nominal Process Calculi*

Luca Aceto[1], Ignacio Fábregas[2], Álvaro García-Pérez[3], Anna Ingólfsdóttir[4], and Yolanda Ortega-Mallén[5]

1 **ICE-TCS, School of Computer Science, Reykjavik University, Iceland**
`luca@ru.is`
2 **ICE-TCS, School of Computer Science, Reykjavik University, Iceland; and Departamento de Sistemas Informáticos y Computación, Universidad Complutense de Madrid, Spain; and IMDEA Software Institute, Madrid,Spain**
`fabregas@ucm.es`
3 **ICE-TCS, School of Computer Science, Reykjavik University, Iceland; and IMDEA Software Institute, Madrid, Spain**
`alvaro.garcia.perez@imdea.org`
4 **ICE-TCS, School of Computer Science, Reykjavik University, Iceland**
`annai@ru.is`
5 **Departamento de Sistemas Informáticos y Computación, Universidad Complutense de Madrid, Spain**
`yolanda@ucm.es`

Abstract

The nominal transition systems (NTSs) of Parrow et al. describe the operational semantics of nominal process calculi. We study NTSs in terms of the nominal residual transition systems (NRTSs) that we introduce. We provide rule formats for the specifications of NRTSs that ensure that the associated NRTS is an NTS and apply them to the operational specification of the early pi-calculus. Our study stems from the recent Nominal SOS of Cimini et al. and from earlier works in nominal sets and nominal logic by Gabbay, Pitts and their collaborators.

1998 ACM Subject Classification D.3.1 Formal Definitions and Theory, F.1.1 Models of Computation, F.1.2 Modes of Computation, F.3.1 Specifying and Verifying and Reasoning about Programs, F.3.2 Semantics of Programming Languages

Keywords and phrases nominal sets, nominal structural operational semantics, process algebra, nominal transition systems, scope opening, rule formats

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.10

1 Introduction

The goal of this paper is to develop the foundations of a framework for studying the meta-theory of structural operational semantics (SOS) [27] for process calculi with names and name-binding operations, such as the π-calculi [29]. To this end, we build on the large body

* Research partially supported by the project Nominal SOS (nr. 141558-051) of the Icelandic Research Fund, the project 001-ABEL-CM-2013 within the NILS Science and Sustainability Programme, the Spanish projects N-Greens Software (S2013/ICE-2731), TRACES (TIN2015-67522-C3-3-R), StrongSoft (TIN2012-39391-C04) and RISCO (TIN2015-71819-P), and the projects RACCOON (H2020-EU 714729) and MATHADOR (COGS 724.464) of the European Research Council, and the Spanish addition to MATHADOR (TIN2016-81699-ERC).

of work on rule formats for SOS, as surveyed in [2, 22], and on the nominal techniques of Gabbay, Pitts and their co-workers [8, 15, 25, 30].

Rule formats provide syntactic templates guaranteeing that the models of the calculi, whose semantics they specify, enjoy some desirable properties. A first design decision that has to be taken in developing a theory of rule formats for a class of languages is therefore the choice of the semantic objects specified by the rules. The target semantic model we adopt in our study is that of *nominal transition systems* (NTSs), which have been introduced by Parrow *et al.* in [23, 24] as a uniform model to describe the operational semantics of a variety of calculi with names and name-binding operations. Based on this choice, a basic sanity criterion for a collection of rules describing the operational semantics of a nominal calculus is that they specify an NTS, and we present a rule format guaranteeing this property (Thm. 28).

As a first stepping stone in our study, we introduce *nominal residual transition systems* (NRTSs), and study NTSs in terms of NRTSs (Section 2). More specifically, NRTSs enjoy one desirable property in the setting of nominal calculi, namely that their transition relation is equivariant (which means that it treats names uniformly). NTSs are NRTSs that, in addition to having an equivariant transition relation, satisfy a property Parrow *et al.* call *alpha-conversion of residuals* (see Def. 3 for the details). The latter property formalises a key aspect of calculi in which names can be scoped to represent local resources. To wit, one crucial feature of the π-calculus is scope opening [21]. Consider a transition $p \xrightarrow{\overline{a}(\nu b)} p'$ in which a process p exports a private/local channel name b along channel a. Since the name b is local, it 'can be subject to alpha-conversion' [23] and the transitions $p \xrightarrow{\overline{a}(\nu c)} p\{b/c\}$ should also be present for each 'fresh name' c.

In contrast to related work [7, 9], our approach uses nominal terms to connect the specification system with the semantic model. This has the advantage of capturing the requirement that transitions be 'up to alpha-equivalence' (typical in nominal calculi) without instrumenting alpha-conversion explicitly in the specification system.

We specify an NRTS by means of a nominal residual transition system specification (NRTSS), which describes the syntax of a nominal calculus in terms of a nominal signature (Section 3) and its semantics by means of a set of inference rules (Section 4). We develop the basic theory of the NRTS/NRTSS framework, building on the nominal algebraic datatypes of Pitts [25] and the nominal rewriting framework of Fernandez and Gabbay [9]. Based on this framework, we provide rule formats [2, 22] for NRTSSs (Section 5) that ensure that the induced transition relation is equivariant (Theorem 22) and enjoys alpha-conversion of residuals (Theorem 28), and is therefore an NTS. Section 6 presents an example of application of our rule formats to the setting of the π-calculus, and Section 7 discusses avenues for future work, as well as related work, and concludes.

2 Preliminaries

Nominal sets

We follow earlier foundational work by Gabbay and Pitts on nominal sets in [17, 25, 26]. We assume a countably infinite set $\mathbb{A}$ of *atoms* and consider Perm $\mathbb{A}$ as the group of *finite permutations of atoms* (hereafter *permutations*) ranged over by π, where we write ι for the *identity*, $\circ$ for *composition* and π^{-1} for the *inverse* of permutation π. We are particularly interested in *transpositions* of two atoms: $(a\,b)$ stands for the permutation that swaps a with b and leaves all other atoms fixed. Every permutation π is equal to the composition of a finite number of transpositions, i.e. $\pi = (a_1\,b_1) \circ \ldots \circ (a_n\,b_n)$ with $n \geq 0$.

An *action* of the group Perm $\mathbb{A}$ on a set S is a binary operation mapping each $\pi \in \text{Perm } \mathbb{A}$ and $s \in S$ to an element $\pi \cdot s \in S$, and satisfying the identity law $\iota \cdot s = s$ and the composition law $(\pi_1 \circ \pi_2) \cdot s = \pi_1 \cdot (\pi_2 \cdot s)$. A Perm $\mathbb{A}$-*set* is a set equipped with an action of Perm $\mathbb{A}$.

We say that a set of atoms A *supports* an object s iff $\pi \cdot s = s$ for every permutation π that leaves each element $a \in A$ invariant. In particular, we are interested in sets all of whose elements have finite support (Def. 2.2 of [25]).

▶ **Definition 1** (Nominal sets). A *nominal set* is a Perm $\mathbb{A}$-set all of whose elements are finitely supported.

For each element s of a nominal set, we write $\text{supp}(s)$ for the least set that supports s, called the *support* of s. (Intuitively, the action of permutations on a set S determines that a finitely supported $s \in S$ only depends on atoms in $\text{supp}(s)$, and no others.) The set $\mathbb{A}$ of atoms is a nominal set when $\pi \cdot a = \pi a$ since $\text{supp}(a) = \{a\}$ for each atom $a \in \mathbb{A}$. The set Perm $\mathbb{A}$ of finite permutations is also a nominal set where the permutation action on permutations is given by conjugation, i.e. $\pi \cdot \pi' = \pi \circ \pi' \circ \pi^{-1}$, and the support of a permutation π is $\text{supp}(\pi) = \{a \mid \pi a \neq a\}$.

Given two Perm $\mathbb{A}$-sets S and T and a function $f : S \to T$, the action of permutation π on function f is given by conjugation, i.e. $(\pi \cdot f)(s) = \pi \cdot f(\pi^{-1} \cdot s)$ for each $s \in S$. We say that a function $f : S \to T$ is *equivariant* iff $\pi \cdot f(s) = f(\pi \cdot s)$ for every $\pi \in \text{Perm } \mathbb{A}$ and every $s \in S$. The intuition is that an equivariant function f is atom-blind, in that f does not treat any atom preferentially. It turns out that a function f is equivariant iff $\text{supp}(f) = \emptyset$ (Rem. 2.13 of [25]). The function supp is equivariant (Prop. 2.11 of [25]).

Let S be a Perm $\mathbb{A}$-set, we write S_{fs} for the nominal set that contains the elements in S that are finitely supported. The *nominal function set* between nominal sets S and T is the nominal set $S \to_{\text{fs}} T$ of finitely supported functions from S to T—be they equivariant or not. Let S_1 and S_2 be nominal sets. The product $S_1 \times S_2$ is a nominal set (Prop. 2.14 of [25]). The permutation action for products is given componentwise (Eq (1.12) of [25]).

An element $s_1 \in S_1$ *is fresh in* $s_2 \in S_2$, written $s_1 \# s_2$, iff $\text{supp}(s_1) \cap \text{supp}(s_2) = \emptyset$. The freshness relation is equivariant (Eq. (3.2) of [25]).

Finally, we consider *atom abstractions* (Section 4 of [25]), which represent alpha-equivalence classes of elements.

▶ **Definition 2** (Atom abstraction). Given a nominal set S, the *atom abstraction* of atom a in element $s \in S$, written $\langle a \rangle s$, is the Perm $\mathbb{A}$-set $\langle a \rangle s = \{(b, (b\,a) \cdot s) \mid b = a \vee b \# s\}$, whose permutation action is $\pi \cdot \langle a \rangle s = \{(\pi \cdot b, \pi \cdot ((b\,a) \cdot s)) \mid \pi \cdot b = \pi \cdot a \vee \pi \cdot b \# \pi \cdot s\}$.

We write $[\mathbb{A}]S$ for the set of *atom abstractions* in elements of S, which is a nominal set (Def. 4.4 of [25]), since $\text{supp}(\langle a \rangle s) = \text{supp}(s) \setminus \{a\}$ for each atom a and element $s \in S$.

Nominal Transition Systems

Nominal transitions systems adopt the state/residual presentation for transitions of [4], where a residual is a pair consisting of an action and a state. In [23], Parrow *et al.* develop modal logics for process algebras à la Hennessy-Milner. Here we are mainly interested in the transition relation and we adapt Definition 1 in [23] by removing the predicates. We write $\mathcal{P}_\omega(\mathbb{A})$ for the *finite power set* of $\mathbb{A}$.

▶ **Definition 3** (Nominal transition system). A *nominal transition system* (NTS) is a quadruple $(S, Act, \text{bn}, \longrightarrow)$ where S and Act are nominal sets of *states* and *actions* respectively, $\text{bn} : Act \to \mathcal{P}_\omega(\mathbb{A})$ is an equivariant function that delivers the *binding names* in an action, and $\longrightarrow \subseteq S \times (Act \times S)$ is an equivariant binary transition relation from states to *residuals*

(we let $Act \times S$ be the set of residuals). The function bn is such that $\mathrm{bn}(\ell) \subseteq \mathrm{supp}(\ell)$ for each $\ell \in Act$. We often write $p \longrightarrow (\ell, p')$ in lieu of $(p, (\ell, p')) \in \longrightarrow$.

Finally, the transition relation $\longrightarrow$ must satisfy *alpha-conversion of residuals*, that is, if $a \in \mathrm{bn}(\ell)$, $b\#(\ell, p')$ and $p \longrightarrow (\ell, p')$ then also $p \longrightarrow ((a\,b) \cdot \ell, (a\,b) \cdot p')$, or equivalently $p \longrightarrow (a\,b) \cdot (\ell, p')$.

We will consider an NTS (without its associated binding-names function bn) as a particular case of a nominal residual transition system, which we introduce next.

▶ **Definition 4** (Nominal residual transition system)**.** A *nominal residual transition system* (NRTS) is a triple $(S, R, \longrightarrow)$ where S and R are nominal sets, and where $\longrightarrow \subseteq S \times R$ is an equivariant binary transition relation. We say S is the set of *states* and R is the set of *residuals*.

The connection between NTSs and NRTSs will be explained in more detail in Section 5.

3 Nominal terms

This section is devoted to the notion of nominal terms, which are syntactic objects that make use of the atom abstractions of Definition 2 and represent terms up to alpha-equivalence. As a first step, we introduce raw terms, devoid of any notion of alpha-equivalence. Our raw terms resemble those from the literature, mainly [8, 9, 25, 30], but with some important differences. In particular, our terms include both variables (i.e. unknowns) and moderated terms (i.e. explicit permutations over raw terms), and we consider atom and abstraction sorts. (The raw terms of [25] do not include moderated terms, and the ones in [9, 30] only consider moderated variables. In [8] the authors consider neither atom nor abstraction sorts.) We also adopt the classic presentation of free algebras and term algebras in [6, 18] in a different way from that in [8, 25]. The raw terms correspond to the standard notion of free algebra over a signature generated by a set of variables. We then adapt the Σ-structures of [8] to our sorting schema. Finally, the nominal terms are the interpretations of the ground terms in the initial Σ-structure; they coincide with the nominal algebraic terms of [25].

▶ **Definition 5** (Nominal signature and nominal sort)**.** A *nominal signature* (or simply a *signature*) Σ is a triple (Δ, A, F) where $\Delta = \{\delta_1, \ldots, \delta_n\}$ is a finite set of *base sorts*, A is a countable set of *atom sorts*, and F is a finite set of *function symbols*. The *nominal sorts* over Δ and A are given by the grammar $\sigma ::= \delta \mid \alpha \mid [\alpha]\sigma \mid \sigma_1 \times \ldots \times \sigma_k$, with $k \geq 0$, $\delta \in \Delta$ and $\alpha \in A$. The sort $[\alpha]\sigma$ is the *abstraction sort*. Symbol $\times$ denotes the *product sort*, which is associative; $\sigma_1 \times \ldots \times \sigma_k$ stands for the sort of the empty product when $k = 0$, which we may write as $\mathbf{1}$. We write S for the set of nominal sorts. We arrange the function symbols in F based on the sort of the data that they produce. We write $f_{ij} \in F$ with $1 \leq i \leq n$ and $1 \leq j \leq m_i$ such that f_{ij} has arity $\sigma_{ij} \to \delta_i$, where δ_i is a base sort.

The theory of nominal sets extends to the case of (countably) many-sorted atoms (see Section 4.7 in [25]). We assume that $\mathbb{A}$ contains a countably infinite collection of atoms a_α, b_α, c_α, ... for each atom sort α such that the sets of atoms $\mathbb{A}_\alpha$ of each sort are mutually disjoint. We write $\mathrm{Perm}_s\,\mathbb{A} = \{\pi \in \mathrm{Perm}\,\mathbb{A} \mid \forall \alpha \in A.\, \forall a \in \mathbb{A}_\alpha.\, \pi\, a \in \mathbb{A}_\alpha\}$ for the subgroup of finite permutations that respect the sorting. The sorted nominal sets are the $\mathrm{Perm}_s\,\mathbb{A}$-sets whose elements are finitely supported. In the sequel we may drop the s subscript in $\mathrm{Perm}_s\,\mathbb{A}$ and omit the 'sorted' epithet from 'sorted nominal sets'.

We let $\mathcal{V}$ be a set that contains a countably infinite collection of *variable names* (variables for short) x_σ, y_σ, z_σ, ... for each sort σ, such that the sets of variables $\mathcal{V}_\sigma$ of each sort are mutually disjoint. We also assume that $\mathcal{V}$ is disjoint from $\mathbb{A}$.

▶ **Definition 6** (Raw terms). Let $\Sigma = (\Delta, A, F)$ be a signature. The set of *raw terms over signature* Σ *and set of variables* $\mathcal{V}$ (*raw terms* for short) is given by the grammar

$$t_\sigma ::= \quad x_\sigma \mid a_\alpha \mid (\pi \bullet t_\sigma)_\sigma \mid ([a_\alpha]t_\sigma)_{[\alpha]\sigma} \mid (t_{\sigma_1}, \ldots, t_{\sigma_k})_{\sigma_1 \times \ldots \times \sigma_k} \mid (f_{ij}(t_{\sigma_{ij}}))_{\delta_i},$$

where term x_σ is a *variable* of sort σ, term a_α is an *atom* of sort α, term $(\pi \bullet t_\sigma)_\sigma$ is a *moderated term* (i.e. the explicit, or delayed, permutation π over term t_σ), term $([a_\alpha]t_\sigma)_{[\alpha]\sigma}$ is the *abstraction of atom* a_α *in term* t_σ, term $(t_{\sigma_1}, \ldots, t_{\sigma_k})_{\sigma_1 \times \ldots \times \sigma_k}$ is the *product of terms* t_{σ_1}, …, t_{σ_k}, and term $(f_{ij}(t_{\sigma_{ij}}))_{\delta_i}$ is the *datum of base sort* δ_i *constructed from term* $t_{\sigma_{ij}}$ *and function symbol* $f_{ij} : \sigma_{ij} \to \delta_i$. When they are clear from the context or immaterial, we leave the arities and sorts implicit and write x, a, $\pi \bullet t$, $[a]t$, $(t_1, \ldots, t_k)$, $f(t)$, etc.

The raw terms are the inhabitants of the carrier of the free algebra over the set of variables $\mathcal{V}$ and over the S-sorted conventional signature that consists of the function symbols in F, together with a constant symbol for each atom a_α, a unary symbol that produces moderated terms for each permutation π and each sort σ, a unary symbol that produces abstractions for each atom a_α and sort σ, and a k-ary symbol that produces a product of sort $\sigma_1 \times \ldots \times \sigma_k$ for each sequence of sorts σ_1, …, σ_k. (See [18] for a classic presentation of term algebras, initial algebra semantics and free algebras.)

We write $\mathbb{T}(\Sigma, \mathcal{V})_\sigma$ for the set of raw terms of sort σ. A raw term t is *ground* iff no variables occur in t. We write $\mathbb{T}(\Sigma)_\sigma$ for the set of ground terms of sort σ. The sets of raw terms (resp. ground terms) of each sort are mutually disjoint as terms carry sort information. Therefore we sometimes identify the family $(\mathbb{T}(\Sigma, \mathcal{V})_\sigma)_{\sigma \in \mathsf{S}}$ of S-indexed raw terms and the family $(\mathbb{T}(\Sigma)_\sigma)_{\sigma \in \mathsf{S}}$ of S-indexed ground terms with their respective ranges $\bigcup_{\sigma \in \mathsf{S}} \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $\bigcup_{\sigma \in \mathsf{S}} \mathbb{T}(\Sigma)_\sigma$, which we abbreviate as $\mathbb{T}(\Sigma, \mathcal{V})$ and $\mathbb{T}(\Sigma)$ respectively.

The set $\mathbb{T}(\Sigma, \mathcal{V})$ of raw terms is a nominal set, with the Perm $\mathbb{A}$-action and the support of a raw term given by:

$$\begin{array}{rcl@{\qquad}rcl}
\pi \cdot x & = & x & \mathrm{supp}(x) & = & \emptyset \\
\pi \cdot a & = & \pi\, a & \mathrm{supp}(a) & = & \{a\} \\
\pi \cdot (\pi_1 \bullet t) & = & (\pi \cdot \pi_1) \bullet (\pi \cdot t) & \mathrm{supp}(\pi \bullet t) & = & \mathrm{supp}(\pi) \cup \mathrm{supp}(t) \\
\pi \cdot [a]t & = & [\pi\, a](\pi \cdot t) & \mathrm{supp}([a](t)) & = & \{a\} \cup \mathrm{supp}(t) \\
\pi \cdot (t_1, \ldots, t_k) & = & (\pi \cdot t_1, \ldots, \pi \cdot t_k) & \mathrm{supp}((t_1, \ldots, t_k)) & = & \mathrm{supp}(t_1) \cup \ldots \cup \mathrm{supp}(t_k) \\
\pi \cdot (f(t)) & = & f(\pi \cdot t), & \mathrm{supp}(f(t)) & = & \mathrm{supp}(t).
\end{array}$$

It is straightforward to check that the permutation action for raw terms is sort-preserving (remember that permutations are also sort-preserving). The set $\mathbb{T}(\Sigma)$ of ground terms is also a nominal set since it is closed with respect to the Perm $\mathbb{A}$-action given above.

▶ **Example 7** (π-calculus). Consider a signature Σ for the π-calculus [7, 29] given by a single atom sort ch of channel names, and base sorts pr and ac for processes and actions respectively. The function symbols (adapted from [29]) are the following:

$$\begin{array}{lll}
F = \{\ \mathit{null} : \mathbf{1} \to \mathsf{pr}, & \mathit{par} : (\mathsf{pr} \times \mathsf{pr}) \to \mathsf{pr}, & \mathit{tauA} : \mathbf{1} \to \mathsf{ac}, \\
\quad \mathit{tau} : \mathsf{pr} \to \mathsf{pr}, & \mathit{sum} : (\mathsf{pr} \times \mathsf{pr}) \to \mathsf{pr}, & \mathit{inA} : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac}, \\
\quad \mathit{in} : (\mathsf{ch} \times [\mathsf{ch}]\mathsf{pr}) \to \mathsf{pr}, & \mathit{rep} : \mathsf{pr} \to \mathsf{pr}, & \mathit{outA} : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac}, \\
\quad \mathit{out} : (\mathsf{ch} \times \mathsf{ch} \times \mathsf{pr}) \to \mathsf{pr}, & \mathit{new} : [\mathsf{ch}]\mathsf{pr} \to \mathsf{pr}, & \mathit{boutA} : (\mathsf{ch} \times \mathsf{ch}) \to \mathsf{ac} \quad \}.
\end{array}$$

Recalling terminology from [7, 29], *null* stands for inaction, $\mathit{tau}(p)$ for the internal action after which process p follows, $\mathit{in}(a, [b]p)$ for the input at channel a where the input name is bound to b in the process p that follows, $\mathit{out}(a, b, p)$ for the output of name b through

channel a after which process p follows, $par(p,q)$ for parallel composition, $sum(p,q)$ for nondeterministic choice, $rep(p)$ for parallel replication, and $new([a]p)$ for the restriction of channel a in process p (a is private in p). Actions and processes belong to different sorts. We use $tauA$, $outA(a,b)$, $inA(a,b)$ and $boutA(a,b)$ respectively for the internal action, the output action, the input action and the bound output action.

The set of terms of the π-calculus corresponds to the subset of ground terms over Σ of sort pr and ac in which no moderated (sub-)terms occur. For instance, the process $(\nu b)(\overline{a}b.0)$ corresponds to the ground term $new([b](out(a,b,null)))$, whose support is $\{a,b\}$. Both free and bound channel names (such as the a and b respectively in the example process) are represented by atoms. The set of ground terms also contains generalised processes and actions with moderated (sub-)terms $\pi \bullet p$, which stand for a delayed permutation π that ought to be applied to a term p, e.g. $new(\pi \bullet ([b](out(a,b,null))))$. ◀

Raw terms allow variables to occur in the place of any ground subterm. The variables represent *unknowns*, and should be mistaken neither with free nor bound channel names. For instance, the raw term $new([b](out(a,b,x)))$ represents a process $(\nu b)(\overline{a}b.P)$ where the x is akin to the meta-variable P, which stands for some unknown process. The process $(\nu b)(\overline{a}b.P)$ unifies with $(\nu b)(\overline{a}b.0)$ by replacing P with 0. In the nominal setting, the raw term $new([b](out(a,b,x)))$ unifies with ground term $new([b](out(a,b,null)))$, by means of a *substitution* φ such that $\varphi(x) = null$. Formally, substitutions are defined below.

▶ **Definition 8** (Substitution). A *substitution* $\varphi : \mathcal{V} \to_{\text{fs}} \mathbb{T}(\Sigma,\mathcal{V})$ is a sort-preserving, finitely supported function from variables to raw terms. The *domain* $\text{dom}(\varphi)$ *of a substitution* φ is the set $\{x \mid \varphi(x) \neq x\}$. A substitution φ is *ground* iff $\varphi(x) \in \mathbb{T}(\Sigma)$ for every variable $x \in \text{dom}(\varphi)$.

The set of substitutions is a nominal set. The *extension to raw terms* $\overline{\varphi}$ *of substitution* φ is the unique homomorphism induced by φ from the free algebra $\mathbb{T}(\Sigma,\mathcal{V})$ to itself, which coincides with the function given by:

$$\begin{array}{rclcrcl} \overline{\varphi}(x) & = & \varphi(x) & \qquad & \overline{\varphi}([a]t) & = & [a](\overline{\varphi}(t)) \\ \overline{\varphi}(a) & = & a & & \overline{\varphi}(t_1,\ldots,t_k) & = & (\overline{\varphi}(t_1),\ldots,\overline{\varphi}(t_k)) \\ \overline{\varphi}(\pi \bullet t) & = & \pi \bullet \overline{\varphi}(t) & & \overline{\varphi}(f(t)) & = & f(\overline{\varphi}(t)). \end{array}$$

Given substitutions φ and γ we write $\varphi \circ \gamma$ for their composition, which is defined as follows: For every variable x, $(\varphi \circ \gamma)(x) = \overline{\varphi}(t)$ where $\gamma(x) = t$. It is straightforward to check that $(\overline{\varphi \circ \gamma})(t) = \overline{\varphi}(\overline{\gamma}(t))$. We note that our definition of substitution is different form those in both [8, 30], where the authors consider a function that performs the delayed permutations of the moderated terms *on-the-fly*.

▶ **Lemma 9** (Extension to raw terms is equivariant). *Let φ be a substitution and π a permutation. Then, $\pi \cdot \overline{\varphi} = \overline{\pi \cdot \varphi}$.*

It is straightforward to check that the support of $\overline{\varphi}$ coincides with the support of φ. By the above lemma, the set of extended substitutions is also a nominal set, since it is closed with respect to the Perm $\mathbb{A}$-action. Hereafter we sometimes write $\varphi(t)$, where t is a raw term, instead of $\overline{\varphi}(t)$. We may also write φ^π instead of $\pi \cdot \overline{\varphi}$ or $\overline{\pi \cdot \varphi}$ for short.

The following result highlights the relation between substitution and the permutation action.

▶ **Lemma 10** (Substitution and permutation action). *Let φ be a substitution, π a permutation and t a raw term. Then, $\pi \cdot \varphi(t) = \varphi^\pi(\pi \cdot t)$.*

Our goal is to give meaning to ground terms in nominal sets. To this end, we need a suitable class of algebraic structures that can be used to give an *interpretation* of those ground terms.

▶ **Definition 11** (Σ-structure). Let $\Sigma = (\Delta, A, F)$ be a signature. A *Σ-structure* M consists of a nominal set $M[\![\sigma]\!]$ for each sort σ defined as follows

$$\begin{array}{rcl} M[\![\alpha]\!] & = & \mathbb{A}_\alpha \\ M[\![[\alpha]\sigma]\!] & = & [\mathbb{A}_\alpha](M[\![\sigma]\!]) \\ M[\![\sigma_1 \times \ldots \times \sigma_k]\!] & = & M[\![\sigma_1]\!] \times \ldots \times M[\![\sigma_k]\!], \end{array}$$

where the $M[\![\delta_i]\!]$ with $\delta_i \in \Delta$ are given, as well as an equivariant function $M[\![f_{ij}]\!] : M[\![\sigma_{ij}]\!] \to M[\![\delta_i]\!]$ for each symbol $(f_{ij})_{\sigma_{ij} \to \delta_i} \in F$.

The notion of Σ-structure adapts that of Σ-structure in [8] to our sorting convention with atom and abstraction sorts. The Σ-structures characterise a range of interpretations of ground terms into elements of nominal sets, such that any sort σ gives rise to the expected nominal set, i.e. atom sorts give rise to sets of atoms, abstraction sorts give rise to sets of atom abstractions, and product sorts give rise to finite products of nominal sets.

Next we define the *interpretation of a ground term in a Σ-structure*, which resembles the *value of a term* in [8].

▶ **Definition 12** (Interpretation of ground terms in a Σ-structure). Let Σ be a signature and M be a Σ-structure. The *interpretation* $M[\![p]\!]$ *of a ground term* p *in* M is given by:

$$\begin{array}{rcl} M[\![a]\!] & = & a \\ M[\![\pi \bullet p]\!] & = & \pi \cdot M[\![p]\!] \\ M[\![[a]p]\!] & = & \langle a \rangle (M[\![p]\!]) \end{array} \qquad \begin{array}{rcl} M[\![(p_1, \ldots, p_k)]\!] & = & (M[\![p_1]\!], \ldots, M[\![p_k]\!]) \\ M[\![f(p)]\!] & = & M[\![f]\!](M[\![p]\!]). \end{array}$$

The next lemma states that interpretation in a Σ-structure is equivariant and highlights the relation between interpretation and moderated terms.

▶ **Lemma 13** (Interpretation and moderated terms). *Let M be a Σ-structure. Interpretation in M is equivariant, that is, $\pi \cdot M[\![p]\!] = M[\![\pi \cdot p]\!]$ for every ground term p and permutation π. Moreover, $M[\![\pi \bullet p]\!] = M[\![\pi \cdot p]\!]$.*

Finally, we introduce the Σ-structure NT, which formalises the set of *nominal terms*.

▶ **Definition 14** (Σ-structure for nominal terms). Let Σ be a signature. The *Σ-structure NT for nominal terms* is given by the least tuple $(NT[\![\delta_1]\!], \ldots, NT[\![\delta_n]\!])$ satisfying

$$NT[\![\delta_i]\!] = NT[\![\sigma_{i1}]\!] + \ldots + NT[\![\sigma_{im_i}]\!] \quad \text{for each base sort } \delta_i \in \Delta, \text{ and}$$

$NT[\![f_{ij}]\!] = \mathrm{inj}_j : NT[\![\sigma_{ij}]\!] \to NT[\![\delta_i]\!]$, for each function symbol $f_{ij} \in F$.

In the conditions above, the 'less than or equal to' relation for tuples is pointwise set inclusion. The $NT[\![f_{ij}]\!]$ is the jth injection of the ith component in $(NT[\![\delta_1]\!], \ldots, NT[\![\delta_n]\!])$.

Nominal terms represent alpha-equivalence classes of raw terms by using the atom abstractions of Definition 2.

▶ **Definition 15** (Nominal terms). Let Σ be a signature. The set $\mathbb{N}(\Sigma)_\sigma$ of *nominal terms over Σ of sort σ* is the domain of interpretation of the ground terms of sort σ in the Σ-structure NT, that is, $\mathbb{N}(\Sigma)_\sigma = NT[\![\sigma]\!]$.

We sometimes write p, ℓ instead of $NT[\![p]\!]$, $NT[\![\ell]\!]$ when it is clear from the context that we are referring to the interpretation into nominal terms of ground terms p and ℓ.

It can be checked that the nominal sets $\mathbb{N}(\Sigma)_\sigma$ coincide (up to isomorphism) with the nominal algebraic datatypes of Definition 8.9 in [25], and therefore by Theorem 8.15 in [25] the nominal terms represent alpha-equivalence classes of raw terms.

4 Specifications of NRTSs

The NRTSs of Definition 4 are meant to be a model of computation for calculi with name-binding operators and state/residual presentation. In this section we present syntactic specifications for NRTSs. We start by defining nominal residual signatures.

▶ **Definition 16** (Nominal residual signature)**.** A *nominal residual signature* (a residual signature for short) is a quintuple $\Sigma = (\Delta, A, \sigma, \rho, F)$ such that (Δ, A, F) is a nominal signature and σ and ρ are distinguished nominal sorts over Δ and A, which we call *state sort* and *residual sort* respectively. We say that $\mathbb{N}(\Sigma)_\sigma$ is the set of *states* and $\mathbb{N}(\Sigma)_\rho$ is the set of *residuals*.

Let $\mathcal{T} = (S, R, \longrightarrow)$ be an NRTS and $\Sigma = (\Delta, A, \sigma, \rho, F)$ be a residual signature. We say that $\mathcal{T}$ is an NRTS *over signature* Σ iff the sets of states S and residuals R coincide with the sets of nominal terms of state sort $\mathbb{N}(\Sigma)_\sigma$ and residual sort $\mathbb{N}(\Sigma)_\rho$ respectively.

Our next goal is to introduce syntactic specifications of NRTSs, which we call nominal residual transition system specifications. To this end, we will make use of residual formulas and freshness assertions over raw terms, which are defined below.

▶ **Definition 17** (Residual formula and freshness assertion)**.** A *residual formula* (a *formula* for short) over a residual signature Σ is a pair (s, r), where $s \in \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $r \in \mathbb{T}(\Sigma, \mathcal{V})_\rho$. We use the more suggestive $s \longrightarrow r$ in lieu of (s, r). A formula $s \longrightarrow r$ is *ground* iff s and r are ground terms.

A *freshness assertion* (an *assertion* for short) over a signature Σ is a pair (a, t) where $a \in \mathbb{A}$ and $t \in \mathbb{T}(\Sigma, \mathcal{V})$. We will write $a \not\# t$ in lieu of (a, t). An assertion is *ground* iff t is a ground term.

▶ Remark. Formulas and assertions are raw syntactic objects, similar to raw terms, which will occur in the rules of the nominal residual transition system specifications to be defined, and whose purpose is to represent respectively transitions and freshness relations involving nominal terms. A formula $s \longrightarrow r$ (resp. an assertion $a \not\# t$) unifies with a ground formula $\varphi(s) \longrightarrow \varphi(r)$ (resp. a ground assertion $a \not\# \varphi(t)$), which in turn represents a transition $NT[\![\varphi(s)]\!] \longrightarrow NT[\![\varphi(r)]\!]$ (resp. a freshness relation $a\#NT[\![\varphi(t)]\!]$). For the assertions, notice how the symbols $\not\#$, $\#$ and $NT[\![\ ]\!]$ interact. The ground assertion $a \not\# [a]a$ represents the freshness relation $a\#NT[\![[a]a]\!]$, which is true. On the other hand, the freshness relation $a\#[a]a$ is false because $a \in \mathrm{supp}([a]a)$.

Permutation action and substitution extend to formulas and assertions in the expected way. Formulas and assertions are elements of nominal sets. Their support is the union of the supports of the raw terms in them, hence we write $\mathrm{supp}(t \longrightarrow t')$ and $\mathrm{supp}(a \not\# t)$. We will also write $b\#(t \longrightarrow t')$ and $b\#(a \not\# t)$ for freshness relations involving formulas and assertions respectively.

▶ **Definition 18** (Nominal residual transition system specification)**.** Let Σ be a residual signature $(\Delta, A, \sigma, \rho, F)$. A *transition rule over* Σ (a *rule*, for short) is of the form

$$\frac{\{u_i \longrightarrow u_i' \mid i \in I\} \quad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow t'},$$

abbreviated as $H, \nabla / t \longrightarrow t'$, where $H = \{u_i \longrightarrow u_i' \mid i \in I\}$ is a finitely supported set of formulas over Σ (we call H the set of *premisses*) and where $\nabla = \{a_j \not\# v_j \mid j \in J\}$ is a finite set of assertions over Σ (we call ∇ the *freshness environment*). We say formula $t \longrightarrow t'$ over Σ is the *conclusion*, where t is the *source* and t' is the *target*. A rule is an *axiom* iff it has an empty set of premisses. Note that axioms might have a non-empty freshness environment.

A *nominal residual transition system specification over* Σ (abbreviated to NRTSS) is a set of transition rules over Σ.

Permutation action and substitution extend to rules in the expected way; they are applied to each of the formulas and freshness assertions in the rule.

Notice that the rules of an NRTSS are elements of a nominal set. The support of a rule $H, \nabla / t \longrightarrow t'$ is the union of the support of H, the support of ∇ and the support of $t \longrightarrow t'$. In the sequel we write $\mathrm{supp}(\mathrm{Ru})$ for the support of rule Ru, and $a\#\mathrm{Ru}$ for a freshness relation involving atom a and rule Ru. Observe that the set H of premisses of a rule may be infinite, but its support must be finite. However, the freshness environment ∇ must be finite in order to make the simplification rules of Definition 23 to follow terminating. These simplification rules will be used in Section 5 to define the rule format in Definition 27.

Let $\mathcal{R}$ be an NRTSS. We say that the formula $s \longrightarrow r$ *unifies* with rule Ru in $\mathcal{R}$ iff Ru has conclusion $t \longrightarrow t'$ and $s \longrightarrow r$ is a substitution instance of $t \longrightarrow t'$. If s and r are ground terms, we also say that transition $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ unifies with Ru.

▶ **Definition 19** (Proof tree)**.** Let Σ be a residual signature and $\mathcal{R}$ be an NRTSS over Σ. A proof tree in $\mathcal{R}$ of a transition $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ is an upwardly branching rooted tree without paths of infinite length whose nodes are labelled by transitions such that

(i) the root is labelled by $NT[\![s]\!] \longrightarrow NT[\![r]\!]$, and

(ii) if $K = \{NT[\![q_i]\!] \longrightarrow NT[\![q_i']\!] \mid i \in I\}$ is the set of labels of the nodes directly above a node with label $NT[\![p]\!] \longrightarrow NT[\![p']\!]$, then there exist a rule

$$\frac{\{u_i \longrightarrow u_i' \mid i \in I\} \qquad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow t'}$$

in $\mathcal{R}$ and a ground substitution φ such that $\varphi(t \longrightarrow t') = p \longrightarrow p'$ and, for each $i \in I$ and for each $j \in J$, $\varphi(u_i \longrightarrow u_i') = q_i \longrightarrow q_i'$ and $a_j \# NT[\![\varphi(v_j)]\!]$ hold.

We say that $NT[\![s]\!] \longrightarrow NT[\![r]\!]$ is *provable* in $\mathcal{R}$ iff it has a proof tree in $\mathcal{R}$. The transition relation specified by $\mathcal{R}$ consists of all the transitions that are provable in $\mathcal{R}$.

The nodes of a proof tree are labelled by transitions, which contain nominal terms (i.e. syntactic objects that use the atom abstractions of Definition 2). The use of nominal terms captures the convention in typical nominal calculi of considering terms 'up to alpha-equivalence'.

The fact that the nodes of a proof tree are labelled by nominal terms is the main difference between our approach and previous work in nominal structural operational semantics [1], nominal rewriting [9, 30] and nominal algebra [15]. In all these works, the 'up-to-alpha-equivalence' transitions are explicitly instrumented within the model of computation by adding to the specification system inference rules that perform alpha-conversion of raw terms.

5 Rule formats for NRTSSs

This section defines two rule formats for NRTSSs that ensure that:

(i) an NRTSS induces an equivariant transition relation, and thus an NRTS of Definition 4;

(ii) an NRTSS induces a transition relation which, together with an equivariant function bn, corresponds to an NTS of Definition 3 [23]. For the latter, we need to ensure that the induced transition relation is equivariant and satisfies *alpha-conversion of residuals* (recall, if $p \longrightarrow (\ell, p')$ is provable in $\mathcal{R}$ and a is in the set of binding names of ℓ, then for every atom b that is fresh in (ℓ, p') the transition $p \longrightarrow (a\,b) \cdot (\ell, p')$ is also provable).

As a first step, we introduce a rule format ensuring equivariance of the induced transition relation.

▶ **Definition 20** (Equivariant format)**.** Let $\mathcal{R}$ be an NRTSS. $\mathcal{R}$ is in *equivariant format* iff the rule $(a\,b) \cdot \textsc{Ru}$ is in $\mathcal{R}$, for every rule $\textsc{Ru}$ in $\mathcal{R}$ and for each $a, b \in \mathbb{A}$.

▶ **Lemma 21.** *Let $\mathcal{R}$ be an NRTSS in equivariant format. For every rule $\textsc{Ru}$ in $\mathcal{R}$ and for every permutation π, the rule $\pi \cdot \textsc{Ru}$ is in $\mathcal{R}$.*

▶ **Theorem 22** (Rule format for NRTSs)**.** *Let $\mathcal{R}$ be an NRTSS. If $\mathcal{R}$ is in equivariant format then $\mathcal{R}$ induces an NRTS.*

Before introducing a rule format ensuring alpha-conversion of residuals, we adapt to our freshness environments the simplification rules and the entailment relation of Definition 10 and Lemma 15 in [9], which we will use in the definition of the rule format.

▶ **Definition 23** (Simplification of freshness environments)**.** Consider a signature Σ. The following rules, where a, b are assumed to be distinct atoms and ∇ is a freshness environment over Σ, define *simplification of freshness environments*:

$$\begin{array}{rcl@{\qquad}rcl}
\{a \fresh b\} \cup \nabla & \Longrightarrow & \nabla & \{a \fresh (p_1, \ldots, p_k)\} \cup \nabla & \Longrightarrow & \{a \fresh p_i, \ldots, a \fresh p_k\} \cup \nabla \\
\{a \fresh \pi \bullet t\} \cup \nabla & \Longrightarrow & \{\pi^{-1} \cdot a \fresh t\} \cup \nabla & \{a \fresh [a]p\} \cup \nabla & \Longrightarrow & \nabla \\
\{a \fresh [b]p\} \cup \nabla & \Longrightarrow & \{a \fresh p\} \cup \nabla & \{a \fresh f(p)\} \cup \nabla & \Longrightarrow & \{a \fresh p\} \cup \nabla.
\end{array}$$

The rules define a reduction relation on freshness environments. We write $\nabla \Longrightarrow \nabla'$ when ∇' is obtained from ∇ by applying one simplification rule, and $\Longrightarrow^*$ for the reflexive and transitive closure of $\Longrightarrow$.

▶ **Lemma 24.** *The relation $\Longrightarrow$ is confluent and terminating.*

A freshness assertion is *reduced* iff it is of the form $a \fresh a$ or $a \fresh x$. We say that $a \fresh a$ is *inconsistent* and $a \fresh x$ is *consistent*. An environment ∇ is *reduced* iff it consists only of reduced assertions. An environment containing a freshness assertion that is not reduced can always be simplified using one of the rules in Definition 23. Therefore, by Lemma 24, an environment ∇ reduces by $\Longrightarrow^*$ to a unique reduced environment, which we call the *normal form* of ∇, written $\langle \nabla \rangle_{nf}$. An environment ∇ is *inconsistent* iff $\langle \nabla \rangle_{nf}$ contains some inconsistent assertion. We say ∇ *entails* ∇' (written $\nabla \vdash \nabla'$) iff either ∇ is an inconsistent environment, or $\langle \nabla' \rangle_{nf} \subseteq \langle \nabla \rangle_{nf}$. We write $\vdash \nabla$ iff $\emptyset \vdash \nabla$.

▶ **Lemma 25.** *Let ∇ be an environment over Σ. Then, for every ground substitution φ, the conjunction of the freshness relations represented by $\varphi(\langle \nabla \rangle_{nf})$ holds iff the conjunction of the freshness relations represented by $\varphi(\nabla)$ hold.*

In particular, if $\vdash \nabla$ then for every ground substitution φ the freshness relations represented by $\varphi(\nabla)$ hold.

We are interested in NTS [23], which consider signatures with base sorts ac and pr, with a single atom sort ch and with source and residual sorts pr and $\mathsf{ac} \times \mathsf{pr}$ respectively. We let Σ_{NTS} be any such signature parametric on a set F of function symbols that we keep implicit. We let $\mathrm{bn} : \mathbb{N}(\Sigma)_{\mathsf{ac}} \to \mathcal{P}_\omega(\mathbb{A}_{\mathsf{ch}})$ be the binding-names function of a given NTS. From now on we restrict the attention to the NTS of [23], and the definitions and results to come apply to NRTS/NRTSS over a signature Σ_{NTS}. We require that the rules of an NRTSS only contain ground actions ℓ and therefore function bn is always defined over $NT[\![\ell]\!]$. (Recall that we write $\mathrm{bn}(\ell)$ instead of $\mathrm{bn}(NT[\![\ell]\!])$ since it is clear in this context that the ℓ stands for a nominal term.) The rule format that we introduce in Definition 27 relies on identifying the rules that give rise to transitions with actions ℓ such that $\mathrm{bn}(\ell)$ is non-empty. To this end, we adapt the notion of strict stratification from [3, 14].

▶ **Definition 26** (Partial strict stratification)**.** Let $\mathcal{R}$ be an NRTSS over a signature Σ_{NTS} and bn be a binding-names function. Let S be a partial map from pairs of ground processes and actions to ordinal numbers. S is a *partial strict stratification of $\mathcal{R}$ with respect to* bn iff

(i) $S(\varphi(t), \ell) \neq \bot$, for every rule in $\mathcal{R}$ with conclusion $t \longrightarrow (\ell, t')$ such that $\mathrm{bn}(\ell)$ is non-empty and for every ground substitution φ, and

(ii) $S(\varphi(u_i), \ell_i) < S(\varphi(t), \ell)$ and $S(\varphi(u_i), \ell_i) \neq \bot$, for every rule in $\mathcal{R}$ with conclusion $t \longrightarrow (\ell, t')$ such that $S(\varphi(t), \ell) \neq \bot$, for every premiss $u_i \longrightarrow (\ell_i, u_i')$ of $\mathcal{R}$ and for every ground substitution φ.

We say a pair (p, ℓ) of ground process and action *has order* $S(p, \ell)$.

The choice of S determines which rules will be considered by the rule format for NRTSSs of Definition 27 below, which guarantees that the induced transition relation satisfies alpha-conversion of residuals and, therefore, the associated transition relation together with function bn are indeed an NTS. We will intend the map S to be such that the only rules whose source and label of the conclusion have defined order are those that may take part in proof trees of transitions with some binding atom in the action.

▶ **Definition 27** (Alpha-conversion-of-residuals format)**.** Let $\mathcal{R}$ be an NRTSS over a signature Σ_{NTS}, bn be a binding-names function and S be a partial strict stratification of $\mathcal{R}$ with respect to bn. Assume that all the actions occurring in the rules of $\mathcal{R}$ are ground. Let

$$\frac{\{u_i \longrightarrow (\ell_i, u_i') \mid i \in I\} \qquad \nabla}{t \longrightarrow (\ell, t')}\ \textsc{Ru}$$

be a rule in $\mathcal{R}$. Let D be the set of variables that occur in the source t of Ru but do not occur in the premisses $u_i \longrightarrow (\ell_i, u_i')$ with $i \in I$, the environment ∇ or the target t' of the rule. The rule Ru is in *alpha-conversion-of-residuals format with respect to S* (*ACR format with respect to S* for short) iff for each ground substitution φ such that $S(\varphi(t), \ell) \neq \bot$, there exists a ground substitution γ such that $\mathrm{dom}(\gamma) \subseteq D$, and for every atom a in the set $\mathbb{A} \setminus \{c \in \mathrm{supp}(t) \mid \langle \{c \# t\} \rangle_{nf} = \emptyset\}$ and for every atom $b \in \mathrm{bn}(\ell)$, the following hold:

(i) $\{a \# t'\} \cup \nabla \vdash \{a \# u_i' \mid i \in I\}$,

(ii) $\{a \# t'\} \cup \nabla \cup \{a \# u_i \mid i \in I\} \vdash \{a \# \gamma(t)\}$, and

(iii) $\nabla \cup \{b \# u_i \mid i \in I \wedge b \in \mathrm{bn}(\ell_i)\} \vdash \{b \# \gamma(t)\}$.

An NRTSS $\mathcal{R}$, together with a binding-names function bn is in *ACR format* iff $\mathcal{R}$ is in equivariant format and there exists a partial strict stratification S such that all the rules in $\mathcal{R}$ are in ACR format with respect to S.

Given a transition $p \longrightarrow (\ell, q)$ that unifies with the conclusion of Ru, the rule format ensures that any atom a fresh in (ℓ, q) is also fresh in p, and also that the binding atom b is fresh in p. We have obtained the constraints of the rule format by considering the variable flow in each node of a proof tree and the freshness relations that we want to ensure. Constraints (i) and (ii) cover the case for the freshness relation $a\#p$ and Constraint (iii) covers the case for the freshness relation $b\#p$. The purpose of substitution γ is to ignore the variables that occur in the source of a rule but are dropped everywhere else in the rule. Constraints (i) and (ii) are not required for atoms a that for sure are fresh in p, and this explains why the a in the rule format ranges over $\mathbb{A} \setminus \{c \in \mathrm{supp}(t) \mid \langle\{c \not\approx t\}\rangle_{nf} = \emptyset\}$. For instance, take rule ResB from Section 6. Condition $\{c \not\approx (boutA(a,b), new([c]y)),\ c \not\approx boutA(a,b)\} \vdash \{c \not\approx (boutA(a,b), y)\}$ does not hold because $c \not\approx [c]y$ does not entail that $c \not\approx y$. However, for a transition $NT[\![new([c]p)]\!] \longrightarrow NT[\![(\ell, new([c]p'))]\!]$, c is fresh in $NT[\![new([c]p)]\!]$ even if c is not fresh in $NT[\![p]\!]$.

▶ **Theorem 28** (Rule format for NTSs). *Let $\mathcal{R}$ be an NRTSS. If $\mathcal{R}$, together with the binding-names function* bn*, is in ACR format then the NRTS induced by $\mathcal{R}$ and* bn *constitute an NTS—that is, the transition relation induced by $\mathcal{R}$ is equivariant and satisfies alpha-conversion of residuals.*

Sketch of the proof. Given a transition $NT[\![\varphi(t)]\!] \longrightarrow NT[\![\varphi((\ell, t'))]\!]$, we first prove the freshness relations $a\#NT[\![\varphi(\gamma(t))]\!]$ and $b\#NT[\![\varphi(\gamma(t))]\!]$. Both relations are proven by induction on $S(\varphi(\gamma(t)), \ell)$, and by analysing the variable flow in the rule unifying with $\varphi(t) \longrightarrow \varphi(\ell, t')$. For the first relation, we assume $a\#NT[\![\varphi(t')]\!]$, use Constraint (i) to prove that $a\#NT[\![\varphi(u'_i)]\!]$ for each target u'_i of a premiss, apply the induction hypothesis to obtain $a\#NT[\![\varphi(\gamma(u_i)]\!]$ for each source of a premiss u_i, and use Constraint (ii) to conclude that $a\#NT[\![\varphi(\gamma(t))]\!]$. For the second relation, the induction hypothesis ensures that $b\#NT[\![\varphi(\gamma(u_i))]\!]$ for each source u_i of a premiss having b as a binding name, and we use Constraint (iii) to conclude that $b\#NT[\![\varphi(\gamma(t))]\!]$. From these two freshness relations it is straightforward to prove that $NT[\![\varphi(t)]\!] \longrightarrow (a\,b) \cdot NT[\![\varphi((\ell, t'))]\!]$ and we are done. ◀

6 Example of application to the early π-calculus

Consider the NRTSS $\mathcal{R}$ for the early π-calculus [21] over a signature Σ_{NTS} where F is the set of function symbols from Example 7. Below we collect an excerpt of the rules, where $a, b, c \in \mathbb{A}_{\mathsf{ch}}$ and ℓ is a ground action:

$$\frac{b \not\approx [c]x}{in(a,[c]x) \longrightarrow (inA(a,b), (c\,b) \bullet x)}\ \textsc{In} \qquad \frac{x \longrightarrow (outA(a,b), y) \qquad b \not\approx a}{new([b]x) \longrightarrow (boutA(a,b), y)}\ \textsc{Open}$$

$$\frac{x_1 \longrightarrow (\ell, y_1)}{sum(x_1,x_2) \longrightarrow (\ell, y_1)}\ \textsc{SumL} \qquad \ell \notin \{boutA(a,b)\}\ \frac{x_1 \longrightarrow (\ell, y_1)}{par(x_1,x_2) \longrightarrow (\ell, (par(y_1,x_2)))}\ \textsc{ParL}$$

$$\frac{x_1 \longrightarrow (boutA(a,b), y_1) \qquad b \not\approx x_2}{par(x_1,x_2) \longrightarrow (boutA(a,b), (par(y_1,x_2)))}\ \textsc{ParResL}$$

$$\frac{}{out(a,b,x) \longrightarrow (outA(a,b), x)}\ \textsc{Out} \qquad \frac{x \longrightarrow (\ell, y)}{rep(x) \longrightarrow (\ell, (par(y, rep(x))))}\ \textsc{Rep}$$

$$\frac{x_1 \longrightarrow (boutA(a,b), y_1) \qquad x_2 \longrightarrow (inA(a,b), y_2) \qquad b \not\approx x_2}{par(x_1,x_2) \longrightarrow (tauA, new([b](par(y_1,y_2))))}\ \textsc{CloseL}$$

An input process $NT[\![in(a,[c]p)]\!]$ can perform a transition to a process $NT[\![(c\,b)\cdot p]\!]$ that is obtained by substituting a channel name b received through channel a for channel name c in p. In the rule In, the moderated term $(c\,b)\bullet x$ needs to be used in order to indicate that permutation $(c\,b)$ will be performed over the term substituted for variable x.

The rule CloseL specifies the interaction of a process such as $NT[\![new([b](out(a,b,p)))]\!]$, which exports a private channel name b through channel a, composed in parallel with an input process such as $NT[\![in(a,[c]q)]\!]$ that reads through channel a. The private name b is exported and the resulting process $NT[\![new([b](par(p,(c\,b)\cdot q)))]\!]$ is the parallel composition of processes p and q where atom b is restricted. For illustration, consider the raw terms $t \equiv new([b](out(a,b,p)))$ and $t' \equiv (boutA(a,b),p)$. The transition $NT[\![t]\!] \longrightarrow NT[\![t']\!]$ is provable in $\mathcal{R}$ by the following proof tree:

$$\dfrac{\dfrac{}{NT[\![out(a,b,p)]\!] \longrightarrow NT[\![(outA(a,b),p)]\!]}\ \textsc{Out}}{NT[\![new([b](out(a,b,p)))]\!] \longrightarrow NT[\![(boutA(a,b),p)]\!]}\ \textsc{Open},\ \text{as } b\#a.$$

Notice that the nodes of the proof tree above are labelled by transitions involving nominal terms. Therefore, if we were to start with the raw term $q \equiv new([c](out(a,c,p)))$—which is alpha-equivalent to t—then the transition $NT[\![q]\!] \longrightarrow NT[\![t']\!]$ would have the same proof tree as above, since $NT[\![t]\!]$ and $NT[\![q]\!]$ are the same nominal term. This contrasts with the related work [7, 9], which considers raw terms in the model of computation and instruments alpha-conversion explicitly in the specification system.

We use the rule format of Definition 27 to show that $\mathcal{R}$, together with equivariant function $\mathrm{bn}(\ell) = \{b \mid \ell = boutA(a,b)\}$ specifies an NTS. We consider the following partial strict stratification:

$$\begin{array}{rcll}
S(out(a,b,p), outA(a,b)) & = & 0 & \\
S(par(p,q),\ell) & = & 1+\max\{S(p,\ell),S(q,\ell)\} & \ell \in \{boutA(a,b), outA(a,b)\} \\
S(sum(p,q),\ell) & = & 1+\max\{S(p,\ell),S(q,\ell)\} & \ell \in \{boutA(a,b), outA(a,b)\} \\
S(rep(p),\ell) & = & 1+S(p,\ell) & \ell \in \{boutA(a,b), outA(a,b)\} \\
S(new([c]p),\ell) & = & 1+S(p,\ell) & \ell \in \{boutA(a,b), outA(a,b)\} \text{ and } c \notin \{a,b\} \\
S(new([b]p), boutA(a,b)) & = & 1+S(p, outA(a,b)) & \\
S(p,\ell) & = & \bot \quad \text{o.w.} &
\end{array}$$

We check that $\mathcal{R}$ is in ACR format as follows. The only rules in $\mathcal{R}$ whose sources and actions unify with pairs of processes and actions that have defined order are Out, Open and ParResL, and the instance of rule ParL where $\ell = outA(a,b)$, and the instances of rules SumL, Rep and Res where $\ell \in \{boutA(a,b), outA(a,b)\}$ (and the corresponding instances of the symmetric versions ParResR, ParR and SumR, which are omitted in the excerpt). We will only check the ACR-format for rules Out, SumL and Open.

For rule Out, we have an empty set of premisses and the set D of atoms that are in $\mathrm{supp}(out(a,b,x))$ but are not in $\mathrm{supp}(outA(a,b),x)$ is empty. Therefore we can do away with substitution γ. There is no atom a such that $\langle\{a \not\# out(a,b,x)\} = \emptyset\rangle_{nf}$ and the set $\mathrm{bn}(outA(a,b))$ is empty. We only need to check that for every atom c, $\{c \not\# (outA(a,b),x)\} \vdash \{c \not\# out(a,b,x)\}$. For atoms $c \in \mathrm{supp}(outA(a,b),x)$ the obligation of the rule format vacuously holds, and therefore it is enough to pick an atom c fresh in the rule and check that $\{c \not\# (outA(a,b),x)\} \vdash \{c \not\# out(a,b,x)\}$, which is straightforward.

For rule SumL, we first check the instance where $\ell = boutA(a,b)$. We have premiss $x_1 \longrightarrow (boutA(a,b), y_1)$ and the set D contains x_2. We pick γ such that $\gamma(x_2) = null$. There is no atom a such that $\langle\{a \not\# sum(x_1,x_2)\}\rangle_{nf} = \emptyset$ and the set $\mathrm{bn}(boutA(a,b))$ contains atom

b. Again, it is enough to pick atom c fresh in the rule and check that

$$\{c \# (boutA(a,b), y_1)\} \vdash \{c \# (boutA(a,b), y_1)\} \quad \text{and}$$
$$\{c \# (boutA(a,b), y_1), c \# x_1\} \vdash \{c \# \gamma(sum(x_1,x_2))\} \quad \text{and}$$
$$\{b \# x_1\} \vdash \{b \# \gamma(sum(x_1,x_2))\},$$

which holds since $\gamma(sum(x_1,x_2)) = sum(x_1, null)$ and $b \# null$ reduces to the empty set.

Now we check the instance where $\ell = outA(a,b)$. We have premiss $x_1 \longrightarrow (outA(a,b), y_1)$ and the set D and the substitution γ are the same as before. There is no atom a such that $\langle\{a \# sum(x_1,x_2)\}\rangle_{nf} = \emptyset$ and the set $\mathrm{bn}(outA(a,b))$ is empty. Again, it is enough to pick atom c fresh in the rule and check that $\{c \# (outA(a,b), y_1)\} \vdash \{c \# (outA(a,b), y_1)\}$ and $\{c \# (outA(a,b), y_1), c \# x_1\} \vdash \{c \# \gamma(sum(x_1,x_2))\}$, which holds as before.

For rule Open the set D is empty and $\langle\{b \# new([b]x)\}\rangle_{nf} = \emptyset$. It is enough to pick atom c fresh in the rule (and therefore different from b) and check that

$$\{c \# (boutA(a,b), y), b \# a\} \vdash \{c \# (boutA(a,b), y)\} \quad \text{and}$$
$$\{c \# (boutA(a,b), y), b \# a, c \# x\} \vdash \{c \# new([b]x))\} \quad \text{and}$$
$$\{b \# x, b \# a\} \vdash \{b \# new([b]x)\},$$

which holds because $b \# new([b]x)$ reduces to the empty set.

Atoms a, b and c in the specification of $\mathcal{R}$ range over $\mathbb{A}_{\mathsf{ch}}$, and thus $\mathcal{R}$ is in equivariant format. Therefore $\mathcal{R}$ is in ARC format. By Theorem 28 the NRTS induced by $\mathcal{R}$, together with function bn, constitute an NTS of Definition 3.

7 Conclusions and future work

The work we have presented in this paper stems from the recently proposed Nominal SOS (NoSOS) framework [7] and from earlier proposals for nominal logic in [8, 15, 30]. It is by no means the only approach studied so far in the literature that aims at a uniform treatment of binders and names in programming and specification languages. Other existing approaches that accommodate variables and binders within the SOS framework are those proposed by Fokkink and Verhoef in [13], by Middelburg in [19, 20], by Bernstein in [5], by Ziegler, Miller and Palamidessi in [31] and by Fiore and Staton in [10] (originally, by Fiore and Turi in [11]). The aim of all of the above-mentioned frameworks is to establish sufficient syntactic conditions guaranteeing the validity of a semantic result (congruence in the case of [5, 10, 19, 31] and conservativity in the case of [13, 20]). In addition, Gabbay and Mathijssen present a nominal axiomatisation of the λ-calculus in [16]. None of these approaches addresses equivariance nor the property of alpha-conversion of residuals in [23].

Our current proposal aims at following closely the spirit of the seminal work on nominal techniques by Gabbay, Pitts and their co-workers, and paves the way for the development of results on rule formats akin to those presented in the aforementioned references. Amongst those, we consider the development of a congruence format for the notion of bisimilarity presented in [23, Def. 2] to be of particular interest. The logical characterisation of bisimilarity given in [23] opens the intriguing possibility of employing the divide-and-congruence approach from [12] to obtain an elegant congruence format and a compositional proof system for the logic.

In the NTSs of Parrow *et al.* [23], scope opening is modelled by the property of alpha-conversion of residuals. We are currently exploring an alternative in which scope opening is encoded by a *residual abstraction* of sort $[\mathsf{ch}](\mathsf{ac} \times \mathsf{pr})$. We have developed mutual, one-to-one translations between the NTSs and the NRTSs with residual abstractions. The generality of

our NRTSs also allows for neat specifications of variants of the π-calculus such as Sangiorgi's internal π-calculus [28].

Developing rule formats for SOS is always the result of a trade-off between ease of application and generality. Our rule format for alpha-conversion of residuals in Definition 27 is no exception and might be generalised in various ways. For instance, the quantification on atom a in conditions (i) and (ii), and the use of substitution γ might be made more general by a finer analysis of the variable flow in a rule. Another generalisation of the rule format would consider possibly open raw actions.

Finally, we are developing rule formats for properties other than alpha-conversion of residuals. One such rule format ensures a *non-dropping property* for NRTSs to the effect that, in each transition, the support of a state is a subset of the support of its derivative.

References

1 L. Aceto, M. Cimini, M. J. Gabbay, A. Ingólfsdóttir, M. R. Mousavi, and M. A. Reniers. Nominal structural operational semantics. In preparation.

2 L. Aceto, W. Fokkink, and C. Verhoef. Structural operational semantics. In *Handbook of Process Algebra*, chapter 3, pages 197–292. Elsevier, 2001.

3 L. Aceto, I. Fábregas, Á. García-Pérez, and A. Ingólfsdóttir. A unified rule format for bounded nondeterminism in SOS with terms as labels. *Journal of Logical and Algebraic Methods in Programming*, 2017. doi:10.1016/j.jlamp.2017.03.002.

4 J. Bengtson and J. Parrow. Formalising the pi-calculus using nominal logic. *Logical Methods in Computer Science*, 5(2), 2009. doi:10.2168/LMCS-5(2:16)2009.

5 K. L. Bernstein. A congruence theorem for structured operational semantics of higher-order languages. In *13th Annual IEEE Symposium on Logic in Computer Science*, pages 153–164. IEEE Computer Society, 1998. doi:10.1109/LICS.1998.705652.

6 S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra: The Millennium Edition*. Springer Verlag, 2000.

7 M. Cimini, M. R. Mousavi, M. A. Reniers, and M. J. Gabbay. Nominal SOS. *Electronic Notes in Theoretical Computer Science*, 286:103–116, 2012. doi:10.1016/j.entcs.2012.08.008.

8 R. Clouston and A. Pitts. Nominal equational logic. *Electronic Notes in Theoretical Computer Science*, 172:223–257, 2007. doi:10.1016/j.entcs.2007.02.009.

9 M. Fernández and M. J. Gabbay. Nominal rewriting. *Information and Computation*, 205(6):917–965, 2007. doi:10.1016/j.ic.2006.12.002.

10 M. P. Fiore and S. Staton. A congruence rule format for name-passing process calculi. *Information and Computation*, 207(2):209–236, 2009. doi:10.1016/j.ic.2007.12.005.

11 M. P. Fiore and D. Turi. Semantics of name and value passing. In *16th Annual IEEE Symposium on Logic in Computer Science*, pages 93–104. IEEE Computer Society, 2001. doi:10.1109/LICS.2001.932486.

12 W. Fokkink, R. J. van Glabbeek, and P. de Wind. Compositionality of Hennessy-Milner logic by structural operational semantics. *Theoretical Computer Science*, 354(3):421–440, 2006. doi:10.1016/j.tcs.2005.11.035.

13 W. Fokkink and C. Verhoef. A conservative look at operational semantics with variable binding. *Information and Computation*, 146(1):24–54, 1998. doi:10.1006/inco.1998.2729.

14 W. Fokkink and T. D. Vu. Structural operational semantics and bounded nondeterminism. *Acta Informatica*, 39(6-7):501–516, 2003. doi:10.1007/s00236-003-0111-1.

15 M. J. Gabbay and A. Mathijssen. Nominal (universal) algebra: Equational logic with names and binding. *Journal of Logic and Computation*, 19(6):1455–1508, 2009. doi:10.1093/logcom/exp033.

16 M. J. Gabbay and A. Mathijssen. A nominal axiomatization of the lambda calculus. *Journal of Logic and Computation*, 20(2):501–531, 2010. doi:10.1093/logcom/exp049.

17 M. J. Gabbay and A. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13(3–5):341–363, 2002. doi:10.1007/s001650200016.

18 J. A. Goguen, J. W. Thatcher, E. G. Wagner, and J. B. Wright. Initial algebra semantics and continuous algebras. *Journal of the ACM*, 24(1):68–95, 1977. doi:10.1145/321992.321997.

19 C. A. Middelburg. Variable binding operators in transition system specifications. *Journal of Logic and Algebraic Programming*, 47(1):15–45, 2001. doi:10.1016/S1567-8326(00)00003-5.

20 C. A. Middelburg. An alternative formulation of operational conservativity with binding terms. *Journal of Logic and Algebraic Programming (JLAP)*, 55(1-2):1–19, 2003. doi:10.1016/S1567-8326(02)00039-5.

21 R. Milner, J. Parrow, and D. Walker. A calculus of mobile processes (Parts I and II). *Information and Computation*, 100(1):1–77, 1992. doi:10.1016/0890-5401(92)90008-4.

22 M. R. Mousavi, M. A. Reniers, and J. F. Groote. SOS formats and meta-theory: 20 years after. *Theoretical Computer Science*, 373(3):238–272, 2007. doi:10.1016/j.tcs.2006.12.019.

23 J. Parrow, J. Borgström, L.-H. Eriksson, R. Gutkovas, and T. Weber. Modal logics for nominal transition systems. In *26th International Conference on Concurrency Theory*, volume 42 of *LIPIcs*, pages 198–211. Schloss Dagstuhl, 2015. doi:10.4230/LIPIcs.CONCUR.2015.198.

24 J. Parrow, T. Weber, J. Borgström, and L.-H. Eriksson. Weak nominal modal logic. In *Formal Techniques for Distributed Objects, Components, and Systems - 37th IFIP WG 6.1 International Conference, FORTE 2017*, volume 10321 of *Lecture Notes in Computer Science*, pages 179–193. Springer, 2017. doi:10.1007/978-3-319-60225-7_13.

25 A. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*. Cambridge University Press, 2013.

26 A. Pitts. Nominal techniques. *ACM SIGLOG News*, 3(1):57–72, 2016. doi:10.1145/2893582.2893594.

27 G. D. Plotkin. A structural approach to operational semantics. *Journal of Logic and Algebraic Programming*, 60-61:17–139, 2004.

28 D. Sangiorgi. pi-Calculus, internal mobility, and agent-passing calculi. *Theoretical Computer Science*, 167(1&2):235–274, 1996. doi:10.1016/0304-3975(96)00075-8.

29 D. Sangiorgi and D. Walker. *The π-calculus — A Theory of Mobile Processes*. Cambridge University Press, 2001.

30 C. Urban, A. Pitts, and M. J. Gabbay. Nominal unification. *Theoretical Computer Science*, 323(1–3):473–497, 2004. doi:10.1016/j.tcs.2004.06.016.

31 A. Ziegler, D. Miller, and C. Palamidessi. A congruence format for name-passing calculi. *Electronic Notes in Theoretical Computer Science*, 156(1):169–189, 2006. doi:10.1016/j.entcs.2005.09.032.

Divergence and Unique Solution of Equations*

Adrien Durier[1], Daniel Hirschkoff[2], and Davide Sangiorgi[3]

1 **Université de Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, France and Università di Bologna and INRIA, Italy**
2 **Université de Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, France**
3 **Università di Bologna and INRIA, Italy**

Abstract

We study proof techniques for bisimilarity based on *unique solution of equations*. We draw inspiration from a result by Roscoe in the denotational setting of CSP and for failure semantics, essentially stating that an equation (or a system of equations) whose infinite unfolding never produces a divergence has the unique-solution property. We transport this result onto the operational setting of CCS and for bisimilarity. We then exploit the operational approach to: refine the theorem, distinguishing between different forms of divergence; derive an abstract formulation of the theorems, on generic LTSs; adapt the theorems to other equivalences such as trace equivalence, and to preorders such as trace inclusion. We compare the resulting techniques to enhancements of the bisimulation proof method (the 'up-to techniques'). Finally, we study the theorems in name-passing calculi such as the asynchronous π-calculus, and revisit the completeness proof of Milner's encoding of the λ-calculus into the π-calculus for Lévy-Longo Trees.

1998 ACM Subject Classification F.3.2 Semantics of Programming Languages

Keywords and phrases Bisimilarity, unique solution of equations, termination, process calculi

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.11

1 Introduction

In this paper we study the technique of *unique solution of equations* for (weak) behavioural relations. We mainly focus on bisimilarity but we also consider other equivalences, such as trace equivalence, as well as preorders such as trace inclusion. Roughly, the technique consists in proving that two tuples of processes are componentwise in a given behavioural relation by establishing that they are solutions of the same system of equations.

In this work, behavioural relations, hence also bisimilarity, are meant to be *weak* because they abstract from internal moves of terms, as opposed to the *strong* relations, which make no distinctions between the internal moves and the external ones (i.e., the interactions with the environment). Weak equivalences are, practically, the most relevant ones: e.g., two equal programs may produce the same result with different numbers of evaluation steps. Further, the problems tackled in this paper only arise in the weak case.

The technique of unique solution has been proposed by Milner in the setting of CCS, and plays a prominent role in proofs of examples in his book [13]. The method is important in verification techniques and tools based on algebraic reasoning [22, 2, 10]. Not all equations have a unique solution: for instance any process trivially satisfies $X = X$. In Milner's theorem [13], uniqueness of solutions is subject to some limitations: the equations must

* This work was supported by Labex MILYON/ANR-10-LABX-0070, and by the project ANR-16-CE25-0011 REPAS.

be 'strongly guarded and sequential', that is, the variables of the equations may only be used underneath a visible prefix and preceded, in the syntax tree, only by the sum and prefix operators. This limits the expressiveness of the technique (since occurrences of other operators above the variables, such as parallel composition and restriction, in general cannot be removed), and its transport onto other languages (e.g., languages for distributed systems or higher-order languages usually do not include the sum operator, which makes the theorem essentially useless). A comparable technique, involving similar limitations, has been proposed by Hoare in his CSP book [12].

Trying to overcome such limitations, a variant of the technique, called *unique solution of contractions* has been proposed [25]. The technique is for behavioural equivalences; however the meaning of 'solution' is defined in terms of the contraction of the chosen equivalence. Contraction is, intuitively, a preorder that conveys an idea of efficiency on processes, where efficiency is measured on the number of internal actions needed to perform a certain activity. The condition for applicability of the technique is, as for Milner's, purely syntactic: each variable in the body of an equation should be underneath a prefix. The technique has two main disadvantages: for proving an equivalence one needs also the theory of the associated contraction preorder; there may be processes for which the technique is not applicable simply because the contraction is strictly finer than the equivalence, and therefore one of the processes fails to be a solution.

In this paper we explore a different approach, inspired by results by Roscoe in CSP [21, 20], essentially stating that a guarded equation (or system of equations) whose infinite unfolding never produces a divergence has the unique-solution property. The theorem is presented, as usual in CSP, with respect to denotational semantics and failure based equivalence [5, 6]. In such a setting, where divergence is catastrophic (e.g., it is the bottom element of the domain), the theorem has an elegant and natural formulation. (Indeed, Roscoe develops a denotational model [20] in which the proof of the theorem is just a few lines.)

We draw inspiration from Roscoe's work to formulate the counterpart of these results in the operational setting of CCS and bisimilarity. In comparison with the denotational CSP proof, the operational CCS proof is more complex. The operational setting offers however a few advantages. First, we can formulate more refined versions of the theorem, in which we distinguish between different forms of divergence. (These refinements would look less natural in the denotational and trace-based setting of CSP, where any divergence causes a process to be considered undefined.) A second and more important advantage comes as a consequence of the flexibility of the operational approach: the unique-solution theorems can be tuned to other behavioural relations (both equivalences and preorders), and to other languages.

To highlight the latter aspect, we present abstract formulations of the theorems, on a generic LTS (i.e., without reference to CCS), where the body of an equation becomes a function on the states of the LTS. The CCS theorems are instances of the abstract formulations. Similarly we can derive analogous theorems for other languages. Indeed we can do so for all languages whose constructs have an operational semantics with rules in the GSOS format [3] (assuming appropriate hypotheses, among which congruence properties). In contrast, the analogous theorems fail for languages whose constructs follow the *tyft/tyxt* [11] format, due to the possibility of rules with a *lookahead*. We also consider extensions of the theorems to name-passing calculi such as the π-calculus. The abstract version of our main unique-solution theorem has been formalised using the Coq proof assistant [8].

Today, for concrete proofs of bisimilarity results, the bisimulation proof method is predominant, also thanks to enhancements of the method provided by the so called 'up-to techniques' [19]. Powerful enhancements are 'up to context', whereby in the derivatives of

two terms a common context can be erased, 'up to expansion', whereby two derivatives can be rewritten using the expansion preorder, and 'up to transitivity', whereby the matching between two derivatives is made with respect to the transitive closure of the candidate relation (rather than the relation alone). Different enhancements can sometimes be combined, though care is needed to preserve soundness. One of the most powerful combinations is Pous *'up to transitivity and context'* technique, which relies on a termination hypothesis. This technique generalises 'up to expansion' and combines it with 'up to context' and 'up to transitivity'. We show that, under an additional side condition, our techniques are at least as powerful as this up-to technique: any up-to relation can be turned into a system of equations of the same size as the up-to relation and that satisfies the hypothesis of our theorems.

An important difference between unique solution of equations and up-to techniques arises in the (asynchronous) π-calculus. In this setting, forms of bisimulation enhancements that involve 'up to context' require closure of the candidate relation under substitutions (or instantiation of the parameters of an abstraction with arbitrary values). It is an open problem whether this closure is necessary in the asynchronous π-calculus, where bisimilarity is closed under substitutions. Our unique-solution techniques are strongly reminiscent of up to context techniques (the body of an equation acts like a context that is erased in a proof using 'up to context'); yet, surprisingly, no closure under substitutions is required.

As an example of application of our techniques in the π-calculus we revisit the completeness part of the proof of full abstraction for the encoding of the λ-calculus into the π-calculus [24, 27] with respect to Levy-Longo Trees (LTs). The proof in [24, 27] uses 'up to expansion and context'. Such up-to techniques seem to be essential: without them, it would be hard even to define the bisimulation candidate. For our proof using unique-solution, there is one equation for each node of a given LT, describing the shape of such node.

Outline of the paper. Section 2 provides background about CCS and behavioural relations. We formulate our main results for CCS in Section 3, and generalise them in an abstract setting in Section 4. Section 5 shows how our results can be applied to the π-calculus.

2 Background

CCS. We assume an infinite set of *names* $a, b, \ldots$ and a set of *constant identifiers* (or simply *constants*) to write recursively defined processes. The special symbol τ does not occur in the names and in the constants. We recall the grammar of CCS:

$$P \;:=\; P_1 \mid P_2 \;\mid\; \Sigma_{i\in I}\mu_i.\,P_i \;\mid\; \boldsymbol{\nu} a\, P \;\mid\; K \qquad\qquad \mu \;:=\; a \;\mid\; \overline{a} \;\mid\; \tau$$

where I is a countable indexing set. We write $\mathbf{0}$ when I is empty, and $P + Q$ for binary sums. Each constant K has a definition $K \stackrel{\triangle}{=} P$. We sometimes omit trailing $\mathbf{0}$, e.g., writing $a \mid b$ for $a.\,\mathbf{0} \mid b.\,\mathbf{0}$. The operational semantics is given by means of an LTS, and is given in Figure 1 (the symmetric versions of the rules `parL` and `comL` have been omitted).

Some standard notations for transitions: $\Longrightarrow$ is the reflexive and transitive closure of $\stackrel{\tau}{\longrightarrow}$, and $\stackrel{\mu}{\Longrightarrow}$ is $\Longrightarrow\stackrel{\mu}{\longrightarrow}\Longrightarrow$ (the composition of the three relations). Moreover, $P \stackrel{\hat{\mu}}{\longrightarrow} P'$ holds if $P \stackrel{\mu}{\longrightarrow} P'$ or ($\mu = \tau$ and $P = P'$); similarly $P \stackrel{\widehat{\mu}}{\Longrightarrow} P'$ holds if $P \stackrel{\mu}{\Longrightarrow} P'$ or ($\mu = \tau$ and $P = P'$). Letters $\mathcal{R}, \mathcal{S}$ range over relations. We use the infix notation for relations, e.g., $P\,\mathcal{R}\,Q$ means that $(P, Q) \in \mathcal{R}$, and denote as $\mathcal{R}\mathcal{S}$ the composition of $\mathcal{R}$ and $\mathcal{S}$. A relation *terminates* if there is no infinite sequence $P_1\mathcal{R}P_2\mathcal{R}\ldots$. We use a tilde to denote a tuple, with countably many elements; thus the tuple may also be infinite. All notations are extended to tuples

$$\texttt{sum}\ \frac{}{\Sigma_{i\in I}\mu_i.\,P_i \xrightarrow{\mu_i} P_i} \qquad \texttt{parL}\ \frac{P \xrightarrow{\mu} P'}{P \mid Q \xrightarrow{\mu} P' \mid Q} \qquad \texttt{comL}\ \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'}$$

$$\texttt{res}\ \frac{P \xrightarrow{\mu} P'}{\boldsymbol{\nu} a\, P \xrightarrow{\mu} \boldsymbol{\nu} a\, P'}\ \mu \neq a, \overline{a} \qquad \texttt{const}\ \frac{P \xrightarrow{\mu} P'}{K \xrightarrow{\mu} P'}\ \text{if } K \stackrel{\triangle}{=} P$$

Figure 1 The LTS for CCS

componentwise; e.g., $\widetilde{P}\ \mathcal{R}\ \widetilde{Q}$ means that $P_i\ \mathcal{R}\ Q_i$, for each component i of the tuples $\widetilde{P}$ and $\widetilde{Q}$. We use $\stackrel{\text{def}}{=}$ for abbreviations; in contrast, $\stackrel{\triangle}{=}$ is used for the definition of constants, and $=$ for syntactic equality and for equations. We focus on *weak* behavioural equivalences, which abstract from the number of internal steps performed.

▶ **Definition 1** (Bisimilarity)**.** A relation $\mathcal{R}$ is a *bisimulation* if, whenever $P\ \mathcal{R}\ Q$, we have:

1. $P \xrightarrow{\mu} P'$ implies that there is Q' such that $Q \stackrel{\hat{\mu}}{\Longrightarrow} Q'$ and $P'\ \mathcal{R}\ Q'$;
2. the converse, on the actions from Q.

P and Q are *bisimilar*, written $P \approx Q$, if $P\ \mathcal{R}\ Q$ for some bisimulation $\mathcal{R}$.

Systems of equations. Unique solution of equations [13] intuitively says that if a context C obeys certain conditions, then all processes P that satisfy the equation $P \approx C[P]$ are bisimilar with each other.

We need variables to write equations. We use capital letters X, Y, Z for these variables and call them *equation variables*. The body of an equation is a CCS expression possibly containing equation variables. We use $E, E', \ldots$ to range over *equation expressions*; these are process expressions that may contain occurrences of variables; that is, the grammar for processes is extended with a production for variables.

▶ **Definition 2.** Assume that, for each i of a countable indexing set I, we have a variable X_i, and an expression E_i, possibly containing some variables. Then $\{X_i = E_i\}_{i\in I}$ (sometimes written $\widetilde{X} = \widetilde{E}$) is a *system of equations*. (There is one equation for each variable X_i.)

$E[\widetilde{P}]$ is the process resulting from E by replacing each variable X_i with the process P_i, assuming $\widetilde{P}$ and $\widetilde{X}$ have the same length. (This is syntactic replacement.) The components of $\widetilde{P}$ need not be different from each other, while this must hold for the variables $\widetilde{X}$.

▶ **Definition 3.** Suppose $\{X_i = E_i\}_{i\in I}$ is a system of equations. We say that:

- $\widetilde{P}$ is a *solution of the system of equations for* $\approx$ if for each i it holds that $P_i \approx E_i[\widetilde{P}]$.
- The system has *a unique solution for* $\approx$ if whenever $\widetilde{P}$ and $\widetilde{Q}$ are both solutions for $\approx$, then $\widetilde{P} \approx \widetilde{Q}$.

For instance, the system $X_1 = a.\,X_2$, $X_2 = b.\,X_1$ has a unique solution, whereas the equations $X = X$, or $X = \tau.\,X$, or $X = a \mid X$ do not.

A system of equations is *guarded* (resp. *strongly guarded*) if each occurrence of a variable in the body of an equation is underneath a prefix (resp. a visible prefix, i.e., different from τ).

Divergences. We consider the standard notion of divergence:

▶ **Definition 4** (Divergence)**.** A process P *diverges* if it can perform an infinite sequence of internal moves, possibly after some visible ones; i.e., there are processes P_i, $i \geq 0$, and some n, such that $P = P_0 \xrightarrow{\mu_0} P_1 \xrightarrow{\mu_1} P_2 \xrightarrow{\mu_2} \ldots$ and for all $i > n$, $\mu_i = \tau$. We call a *divergence of* P the sequence of transitions $\left(P_i \xrightarrow{\mu_i} P_{i+1}\right)_i$.

▶ **Example 5.** The process $L \triangleq a.\boldsymbol{\nu} a\,(L \mid \overline{a})$ diverges, since $L \xrightarrow{a} \boldsymbol{\nu} a\,(L \mid \overline{a})$, and (leaving aside $\mathbf{0}$ and useless restrictions) $\boldsymbol{\nu} a\,(L \mid \overline{a})$ has a τ transition onto itself.

3 Main Results

3.1 Divergences and Unique Solution

This section is devoted to our main results for bisimilarity, in the case of CCS. We need to reason with the unfoldings of the given equation $X = E$: we define the n-th unfolding of E to be E^n; thus E^1 is defined as E, E^2 as $E[E]$, and E^{n+1} as $E^n[E]$. The *infinite* unfolding represents the simplest and most intuitive solution to the equation. In the CCS grammar, such a solution is obtained by turning the equation into a constant definition, namely the constant K_E with $K_E \triangleq E[K_E]$. We call K_E the *syntactic solution of the equation*.

For a system of equations $\widetilde{X} = \widetilde{E}[\widetilde{X}]$, the unfoldings are defined accordingly (where E_i replaces X_i in the unfolding), and the syntactic solutions are defined to be the set of mutually recursive constants $\{K_{\widetilde{E},i} \triangleq E_i[\widetilde{K}_{\widetilde{E}}]\}_i$.

▶ **Theorem 6** (Unique solution)**.** *A guarded system of equations whose syntactic solutions do not diverge has a unique solution for* $\approx$.

We explain the schema of the proof, considering, for simplicity, a single equation $X = E$. We take a solution P of the equation and a transition $P \xrightarrow{\mu} P'$. The goal is to find an n such that $E^n[P]$ can match this transition *without the need of* P; i.e., there is E' with $E^n[P] \stackrel{\hat{\mu}}{\Longrightarrow} E'[P]$, and for any process Q also $E^n[Q] \stackrel{\hat{\mu}}{\Longrightarrow} E'[Q]$ holds.

We look for this n incrementally. If the matching transition $E^m[P] \stackrel{\hat{\mu}}{\Longrightarrow} P_m$ (recall the P is solution) involves some transitions of P, then $E^m[P]$ does not work. We then consider a matching transition emanating from $E^{m+1}[P]$, which starts with the transitions in $E^m[P] \stackrel{\hat{\mu}}{\Longrightarrow} P_m$ that do not involve P. We observe that there are are at least m of these, because P is underneath at least m prefixes in $E^m[P]$.

This procedure necessarily stops: otherwise, we could build an infinite sequence of transitions involving only the unfoldings of E, and with at most one visible transition: this would yield a divergence in the syntactic solution of E. With this construction at hand, given another solution Q of the equation, we construct a bisimulation containing the pair (P, Q).

3.2 Innocuous Divergences

In the remainder of the section we refine Theorem 6 by taking into account only certain forms of divergence. To introduce the idea, consider the equation $X = a.\,X \mid K$, for $K \triangleq \tau.\,K$: the divergences induced by K do not prevent uniqueness of the solution, as any solution P necessarily satisfies $P \approx a.\,P$. Indeed the variable of the equation is strongly guarded and a visible action has to be produced before accessing the variable. These divergences are not dangerous because they do not percolate through the infinite unfolding of the equation; in other words, a finite unfolding may produce the same divergence, therefore it is not necessary to go to the infinite unfolding to diverge. We call such divergences *innocuous*. Formally, these divergences are derived by applying only a finite number of times rule `const` of the LTS (see Figure 1) to the constant that represents the syntactic solution of the equation.

▶ **Definition 7** (Innocuous divergence)**.** Consider a guarded system of equations $\widetilde{X} = \widetilde{E}$ and its syntactic solutions $\widetilde{K}_{\widetilde{E}}$. A divergence of $K_{\widetilde{E},i}$ (for some i) is called *innocuous* when

summing up all usages of rule `const` with one of the $K_{\widetilde{E},j}$s (including $j = i$) *in all derivation proofs of the transitions belonging to the divergence*, we obtain a finite number.

▶ **Theorem 8** (Unique solution with innocuous divergences)**.** *Let $\widetilde{X} = \widetilde{E}$ be a system of guarded equations, and $\widetilde{K}_{\widetilde{E}}$ be its syntactic solutions. If all divergences of any $K_{\widetilde{E},i}$ are innocuous, then $\widetilde{E}$ has a unique solution for $\approx$.*

▶ Remark. The conditions for unique solution in Theorems 6 and 8 are a mixture of syntactic (guardedness) and semantic (divergence-free) conditions. A purely semantic condition can be used if rule `const` of Figure 1 is modified so that the unfolding of a constant yields a τ-transition:

$$\frac{}{K \xrightarrow{\tau} P} \text{ if } K \stackrel{\triangle}{=} P$$

Thus in the theorems the condition that the equations are guarded could be dropped. The resulting theorems would actually be more powerful because they would accept equations not all of which are guarded: it is sufficient that each equation has a finite unfolding that is guarded. For instance the system of equations $X = b \mid Y$, $Y = a.\, X$ would be accepted, although the first equation is not guarded.

The next lemma states a condition to ensure that all divergences produced by a system of equations are innocuous. This condition will be sufficient in all examples in the paper.

▶ **Lemma 9.** *In a system of equations $\widetilde{X} = \widetilde{E}$, suppose for each i there is n_i such that in $E_i^{n_i}$, each variable is underneath a visible prefix (say, a or $\overline{a}$) whose complementary prefix ($\overline{a}$ or a) does not appear in any equation. Then the system has only innocuous divergences.*

3.3 An example: lazy and eager servers

We now show an example of application of our technique, taken from [25]. The example also illustrates the relative strengths of the two unique solution theorems (Theorems 6 and 8).

For the sake of readability, we use a version of CCS with value passing; this could be translated into pure CCS [13]. In a value-passing calculus, $a(x).\, P$ is an input at a in which x is the placeholder for the value received, whereas $\overline{a}\langle n\rangle.\, P$ is an output at a of the value n; and $A\langle n\rangle$ is a parametrised constant. This example consists of two implementations of a server; this server, when interrogated by clients at a channel c, should start a certain interaction protocol with the client, after consulting an auxiliary server A at a.

We consider the two following implementations of this server: the first one, L, is 'lazy', and consults A only *after* a request from a client has been received. In contrast, the other one, E, is 'eager', and consults A *beforehand*, so then to be ready in answering a client:

$$\begin{array}{rcl} L & \stackrel{\triangle}{=} & c(z).\, a(x).\, (L \mid R\langle x, z\rangle) \\ E & \stackrel{\triangle}{=} & a(x).\, c(z).\, (E \mid R\langle x, z\rangle) \end{array} \qquad\qquad A\langle n\rangle \stackrel{\triangle}{=} \overline{a}\langle n\rangle.\, A\langle n+1\rangle$$

Here $R\langle x, z\rangle$ represents the interaction protocol that is started with a client, and can be any process. It may use the values x and z (obtained from the client and the auxiliary server A); the interactions produced may indeed depend on the values x and z. We assume for now that $R\langle x, z\rangle$ may not use channel c and a; that is, the interaction protocol that has been spawned need not come back to the main server or to the auxiliary server. Moreover we assume R may not diverge. We want to prove that the two servers, when composed with A, yield bisimilar processes. We thus define $LS\langle n\rangle \stackrel{\triangle}{=} \boldsymbol{\nu} a\,(A\langle n\rangle \mid L)$ and $ES\langle n\rangle \stackrel{\triangle}{=} \boldsymbol{\nu} a\,(A\langle n\rangle \mid E)$. A proof that $LS\langle n\rangle \approx ES\langle n\rangle$ using the plain bisimulation proof method would be long and

tedious, due to the differences between the lazy and the eager server, and to the fact that R is nearly an arbitrary process.

For a proof using our technique, the equations are: $\{X_n = c(z).(X_{n+1} \mid R\langle n, z\rangle)\}_n$. The proofs that the two servers are solutions can be carried out using a few algebraic laws: expansion law, structural laws for parallel composition and restriction, one τ-law. To apply Theorem 6, we also have to check that the equations may not produce divergences. This check is straightforward, as no silent move may be produced by interactions along c, and any two internal communications at a are separated by a visible input at c. Moreover, by assumption, the protocol R does not produce internal divergences.

If however the hypothesis that R may not diverge is lifted, then Theorem 6 is not applicable anymore, and divergences are possible. However, such divergences are innocuous: the equation need not be unfolded an infinite number of times for the divergence to occur. We can therefore still prove the result, by appealing to the more powerful Theorem 8.

3.4 Comparison with other techniques

Milner's syntactic condition for unique solution of equations essentially allows only equations in which variables are underneath prefixes and sums. The technique is not complete [25]; for instance it cannot handle the server example of Section 3.3.

The technique of 'unique solution of contractions' [25] relies on the theory of an auxiliary preorder (contraction), needed to establish the meaning of 'solution'; and the soundness theorems in [25] use a purely syntactic condition (guarded variables). In contrast, our techniques with equations do not rely on auxiliary relations and their theory, but the soundness theorems use a semantic condition (divergence). The two techniques are incomparable. Considering the server example of Section 3.3, the contraction technique is capable of handling also the case in which the protocol R is freely allowed to make calls back to the main server, including the possibility that, in doing this, infinitely many copies of R are spawned. This possibility is disallowed for us, as it would correspond to a non-innocuous divergence. On the other hand, when using contraction, a solution is evaluated with respect to the contraction preorder, that conveys an idea of efficiency (measured against the number of silent transitions performed). Thus, while two bisimilar processes are solutions of exactly the same set of equations, they need not be solutions of the same contractions. For instance, we can use our techniques to prove that processes $K \stackrel{\triangle}{=} \tau.a.a.K$ and $H \stackrel{\triangle}{=} a.H$ are bisimilar because solutions of the equation $X = a.X$; in contrast, only H is a solution of the corresponding contraction.

Up-to techniques. We compare our unique-solution techniques with one of the most powerful forms of enhancement of the bisimulation proof method, namely Pous 'up to transitivity and context' technique [17]. That technique allows us to use 'up to weak bisimilarity', 'up to transitivity', and 'up to context' techniques together. While 'up to weak bisimilarity' and 'up to transitivity' are known to be unsound techniques [19], here they are combined at the price of a termination hypothesis over a 'control relation', below written $\succ$.

We write $\mathcal{C}(\mathcal{R})$ for the context closure of a relation $\mathcal{R}$ (the set of all pairs $(C[\widetilde{P}], C[\widetilde{Q}])$ with $\widetilde{P}\ \mathcal{R}\ \widetilde{Q}$). Moreover, $\overline{\mathcal{R}}$ stands for $(\approx \cup \mathcal{C}(\mathcal{R}))$, and $\mathcal{R}^+$ for the transitive closure of $\mathcal{R}$.

▶ **Definition 10.** Let $\succ$ be a relation that is transitive, closed under contexts, and such that $\succ$ $(\stackrel{\tau}{\longrightarrow}^+)$ terminates. A relation $\mathcal{R}$ is a *bisimulation up to* $\succ$ *and context* if, whenever $P\ \mathcal{R}\ Q$:

1. if $P \stackrel{\mu}{\longrightarrow} P'$ then $Q \stackrel{\hat{\mu}}{\Longrightarrow} Q'$ for some Q' with $P'\ (\succ \cap \overline{\mathcal{R}})^+\ \mathcal{C}(\mathcal{R})\ \approx\ Q'$;
2. the converse, on the transitions from Q.

If $\mathcal{R}$ is a relation then we can also view $\mathcal{R}$ as an ordered sequence of pairs (e.g., assuming some lexicographical ordering). Then $\mathcal{R}_i$ indicates the tuple obtained by projecting the pairs in $\mathcal{R}$ on the i-th component ($i = 1, 2$). The *size* of a relation is the number of pairs it contains. The size of a system of equations is the number of equations it consists of.

▶ **Theorem 11** (Completeness with respect to up-to techniques)**.** *Suppose $\mathcal{R}$ is a bisimulation up to $\succ$ and context. Then there is a guarded system of equations, of the same size as $\mathcal{R}$, with only innocuous divergences, admitting $\mathcal{R}_1$ and $\mathcal{R}_2$ as solutions.*

In the proof, the equations are defined by exploiting the expansion law in CCS [13]. The proof then involves a rather delicate analysis in which the termination hypothesis is used to prove that the syntactic solutions of this system have only innocuous divergences (they may indeed have divergences). We may then apply Theorem 8 (rather than Theorem 6).

▶ Remark. In [17], the transitive closure $(\succ \cap \overline{\mathcal{R}})^+$ in Definition 10 is actually a *reflexive* and transitive closure. We do not know if relaxing this technical condition breaks Theorem 11.

4 Abstract Formulation

4.1 Abstract LTS and Operators

In this section we propose generalisations of the unique-solution theorems. For this we introduce abstract formulations of them, which are meant to highlight their essential ingredients. When instantiated to the specific case of CCS, such abstract formulations yield the theorems in Section 3. The proofs are adapted from those of the corresponding theorems in Section 3. The results of this section, up to Theorem 15, have been formalised in Coq [8].

The abstract formulation is stated on a generic LTS, that is, a triple $\mathcal{T} = (S, \Lambda, \rightarrow)$ where: S is the set of states; Λ the set of action labels, containing the special label τ accounting for silent actions; $\rightarrow$ is the transition relation. As usual, we write $s_1 \xrightarrow{\mu} s_2$ when $(s_1, \mu, s_2) \in \rightarrow$. The definition of weak bisimilarity $\approx$ is as in Section 2. We omit explicit reference to $\mathcal{T}$ when there is no ambiguity.

We reason about *state functions*, i.e., functions from S to S, and use f, f', g to range over them. We recall that $(f \circ g)(x) = f(g(x))$ for all x. The CCS processes of Section 2 correspond here to the states of an LTS; and CCS contexts correspond to state functions.

▶ **Definition 12** (Autonomy)**.** For state functions f, f' we say that there is an *autonomous μ-transition from f to f'*, written $f \xrightarrow{\mu} f'$, if for all states x it holds that $f(x) \xrightarrow{\mu} f'(x)$.

Likewise, given a set $\mathcal{F}$ of state functions and $f \in \mathcal{F}$, we say that a transition $f(x) \xrightarrow{\mu} y$ is *autonomous on $\mathcal{F}$* if, for some $f' \in \mathcal{F}$ we have $f \xrightarrow{\mu} f'$ and $y = f'(x)$. Moreover, we say that *function f is autonomous on $\mathcal{F}$* if all the transitions emanating from f (that is, all transitions of the form $f(x) \xrightarrow{\mu} y$, for some x, μ, y) are autonomous on $\mathcal{F}$.

When $\mathcal{F}$ is clear, we omit it, and we simply say that a function *is autonomous*.

Thus, f is autonomous on $\mathcal{F}$ if, for some indexing set I, there are μ_i and $f_i \in \mathcal{F}$ such that for all x we have $f(x) \xrightarrow{\mu_i} f_i(x)$, for each i; the set of all transitions emanating from $f(x)$ is precisely $\cup_i \{f(x) \xrightarrow{\mu_i} f_i(x)\}$. Autonomous functions correspond to CCS guarded contexts, which do not need their process argument to perform the first transition. We now formulate conditions under which, intuitively, a state function behaves like a CCS context. Functions satisfying these conditions are called *operators*.

▶ **Definition 13** (Set of operators)**.** Consider an LTS $\mathcal{T}$, and a set $\mathcal{O}$ of functions from S to S. We say that $\mathcal{O}$ is a *set of operators on $\mathcal{T}$* if the following conditions hold:

1. $\mathcal{O}$ contains the identity function;
2. $\mathcal{O}$ is closed by composition (that is, $f \circ g \in \mathcal{O}$ whenever $f, g \in \mathcal{O}$);
3. composition preserves autonomy (i.e., if g is autonomous on $\mathcal{O}$, then so is $f \circ g$);
4. all functions in $\mathcal{O}$ respect $\approx$, i.e., $x \approx y$ and $f \in \mathcal{O}$ imply $f(x) \approx f(y)$.

A 'symmetric variant' of clause 3 always holds: if f is autonomous, then so is $f \circ g$. The autonomous transitions of a set of operators yield an LTS whose states are the operators themselves. Such transitions are of the form $f \xrightarrow{\mu} g$. Where the underlying set $\mathcal{O}$ of operators is clear, we simply call a function belonging to $\mathcal{O}$ an *operator*.

We use state functions to express equations, such as $X = f(X)$. We thus look at conditions under which such an equation has a unique solution (again, the generalisation to a system of equations is easy, using n-ary functions).

Thinking of functions as equation expressions, to formulate our abstract theory about unique solution of equations, we have to define the divergences of finite and infinite unfoldings of state functions. The nth unfolding of f (for $n \geq 1$), f^n, is the function obtained by n applications of f. An operator is *well-behaved* if there is n with f^n autonomous (the well-behaved operators correspond, in CCS, to equations some finite unfolding of which yields a guarded expression). We also have to reason about the infinite unfolding of an equation $X = f(X)$. For this, given a set $\mathcal{O}$ of operators, we consider the infinite terms obtained by infinite compositions of operators in $\mathcal{O}$, that is, the set coinductively defined by the grammar:

$$F := f \circ F \qquad \text{where } f \in \mathcal{O} \text{ (i.e., } f \text{ is a metavariable for the elements in } \mathcal{O}).$$

(We do not need finite compositions, as $\mathcal{O}$ itself is closed under finite compositions.) In particular, we write f^∞ for the infinite term $f \circ f \circ f \circ \ldots$.

We define the autonomous transitions for such infinite terms using the following rules:

$$\frac{g \xrightarrow{\mu} g'}{g \circ F \xrightarrow{\mu} g' \circ F}\ g \text{ autonomous} \qquad \frac{(g \circ f) \circ F \xrightarrow{\mu} F'}{g \circ (f \circ F) \xrightarrow{\mu} F'}\ g \text{ not autonomous}$$

Intuitively a term is 'unfolded', with the second rule, until an autonomous function is uncovered, and then transitions are computed using the first rule (we disallow unnecessary unfoldings; these would complicate our abstract theorems, by adding duplicate transitions, since the transitions of $g \circ f$ duplicate those of g when g is autonomous). An infinite term has no transitions if none of its finite unfoldings ever yields an autonomous function. This situation does not arise for terms of the form f^∞ or $g \circ f^\infty$, where f is well-behaved, which are the terms we are interested in. Note that no infinite term belongs to a set of operators.

▶ **Definition 14** (Operators and divergences)**.** Let f, f', f_i be operators in a set $\mathcal{O}$ of operators, and consider the LTS induced by the autonomous transitions of operators in $\mathcal{O}$. A sequence of transitions $f_1 \xrightarrow{\mu_1} f_2 \xrightarrow{\mu_2} f_3 \ldots$ is a *divergence* if for some $n \geq 1$ we have $\mu_i = \tau$ whenever $i \geq n$. We also say that f_1 *diverges*. We apply these notations and terminology also to infinite terms (such as $f_1 \circ f_2 \circ \ldots$), as expected.

In the remainder of the section we fix a set $\mathcal{O}$ of operators and we only consider autonomous transitions on $\mathcal{O}$. We now state the "abstract version" of Theorem 6 (the proof being similar).

▶ **Theorem 15** (Unique solution, abstract formulation)**.** *Let f be a well-behaved operator on $\mathcal{O}$. If f^∞ does not diverge, then the equation $X = f(X)$ either has no solution or has a unique solution for $\approx$.*

The equation in the statement of the theorem might have no solution at all. For example, consider the LTS $(\mathbb{N}, \{a\}, \rightarrow)$ where for each n we have $n + 1 \xrightarrow{a} n$. The function f with

$f(n) = n + 1$ is an operator of the set $\mathcal{O} = \{f^n\}_{n\in\mathbb{N}}$ (with $f^0 = \mathtt{Id}$, the identity function). The function f is autonomous because, for all n, the only transition of $f(n)$ is $f(n) \xrightarrow{a} n$ (this transition is autonomous because $f \xrightarrow{a} \mathtt{Id}$). A fixpoint of f would be an element x with $x \xrightarrow{a} x$, and there is no such x in the LTS.

Theorem 15 can be refined along the lines of Theorem 8. For this, we have to relate the divergences of any f^n (for $n \geq 1$) to divergences of f^∞, in order to distinguish between innocuous and non-innocuous divergences. To build a divergence of f^∞ from a divergence of f^n, for $n \geq 1$, we need to reason up to (finite) unfoldings of f: thus we set $=_f$ to be the symmetric reflexive transitive closure of the relation that relates g and g' whenever $g = g' \circ f$.

▶ **Lemma 16.** *Consider an autonomous operator f on $\mathcal{O}$ and a divergence of f^n*

$$f^n \xrightarrow{\mu_1} f_1 \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_i} f_i \xrightarrow{\tau} f_{i+1} \xrightarrow{\tau} \ldots$$

This yields a divergence of f^∞: $f^\infty \xrightarrow{\mu_1} g_1 \circ f^\infty \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_i} g_i \circ f^\infty \xrightarrow{\tau} g_{i+1} \circ f^\infty \xrightarrow{\tau} \ldots$ such that for all $i \geq 1$, g_i is an operator and $g_i =_f f_i$.

Given a divergence Δ of f^n, we write Δ^∞ to indicate the divergence of f^∞ obtained from Δ as in Lemma 16. We call a divergence of f^∞ *innocuous* when it can be described in this way, that is, as a divergence Δ^∞ obtained from a divergence Δ of f^n, for some n.

▶ **Theorem 17** (Unique solution with innocuous divergences, abstract formulation)**.** *Let $f \in \mathcal{O}$ be a well-behaved operator. If all divergences of f^∞ are innocuous, then the equation $X = f(X)$ either has no solution or has a unique solution for $\approx$.*

4.2 Reasoning with other Behavioural Relations

Trace-based Equivalences. We can adapt the results of the previous section about bisimilarity to other settings, including both preorders and non-coinductive relations. As an example, we consider trace-based relations. We write $\approx_{tr}$ for trace equivalence, that equates two processes having the same set of (finite) traces. The definitions from previous sections are the same as for $\approx$, replacing $\approx$ with $\approx_{tr}$. All theorems presented for $\approx$ can be adapted to $\approx_{tr}$, and the proofs are similar. As an example, Theorem 17 becomes:

▶ **Theorem 18.** *Let $f \in \mathcal{O}$ be a well-behaved operator. If all divergences of f^∞ are innocuous, then the equation $X = f(X)$ either has no solution or has a unique solution for $\approx_{tr}$.*

In contrast, the theorems fail for *infinitary trace equivalence*, $\approx_{tr}^\infty$ (whereby two processes are equated if they have the same traces, including the infinite ones), for the same reason why the 'unique solution of contraction' technique fails in this case [25]. As a counterexample, we consider equation $X = a + a.X$, whose syntactic solution has no divergences. The process $P \stackrel{\triangle}{=} \sum_{n>0} a^n$ is a solution, yet it is not $\approx_{tr}^\infty$-equivalent to the syntactic solution of the equation, because the syntactic solution has an infinite trace involving a transitions.

Preorders. We show how the theory for equivalences can be transported onto *preorders*. This means moving to *systems of pre-equations*, $\{X_i \leq E_i\}_{i\in I}$. With preorders, our theorems have a different shape: we do not use pre-equations to reason about unique solution – we expect interesting pre-equations to have many solutions, some of which may be incomparable with each other. We rather derive theorems to prove that, in a given preorder, any solution of a pre-equation is below its syntactic solution.

▶ Remark. The opposite direction for pre-equations, namely $\{X_i \geq E_i\}_{i \in I}$ is less interesting. It would mean aiming to prove that the syntactic solution is below other solutions. This is usually a trivial property for a behavioural preorder, without any hypothesis such as autonomy or non divergence (a possible exception is a preorder with infinitary observables, such as infinitary trace inclusion, where the property may fail).

We write $\subseteq_{tr}$ for trace inclusion, $\subseteq_{tr^\infty}$ for infinitary trace inclusion, and $\leq_s$ for weak simulation. These preorders are standard from the literature [28].

In the abstract setting, the body of the pre-equations are functions. Then the theorems give us conditions under which, given a pre-equation $X \leq f(X)$ and a behavioural preorder $\preceq$, a solution r, i.e., a state for which $r \preceq f(r)$ holds, is below the syntactic solution f^∞. We present the counterpart of Theorem 17; other theorems are transported in a similar manner.

▶ **Theorem 19.** *Let $f \in \mathcal{O}$ be a well-behaved operator. If all divergences of f^∞ are innocuous, then, given $\leq \in \{\subseteq_{tr}, \subseteq_{tr^\infty}, \leq_s\}$, whenever $x \leq f(x)$ we also have $x \leq f^\infty$, for any state x.*

Theorem 19 intuitively says that the syntactic solution of a pre-equation is *maximal* among all solutions. Note that, in contrast with equations, Theorem 19 and the theory of pre-equations also work for *infinitary trace inclusion*.

4.3 Rule formats

A way to instantiate the results in Sections 4.1 and 4.2 is to consider *rule formats* [1, 16]. These provide a specification for the form of the SOS rules used to describe the constructs of a language. To fit a rule format into the abstract formulation of the theory from Section 4.1, we view the constructs of a language as functions on the states of the LTS (the processes of the language). One of the most common formats is GSOS [3, 29, 9]. For lack of space we only show the instantiation to GSOS of Theorem 8. In the statement, we consider an extension of a GSOS language with constants, in the same way as they appear in CCS, so that the definitions of 'syntactic solution of equations' and of 'innocuous divergence' can be taken to be the same as for CCS (these are easier to grasp than their formulation in Section 4.1).

▶ **Theorem 20.** *Consider a language whose constructs have SOS rules in the GSOS format and preserve $\approx$, and an equation $X = E$ for the language. If E is autonomous (over the set of functions corresponding to the contexts of the language), and if, in the language extended with constants, the syntactic solution of the equation only has innocuous divergences, then either the equation has no solution or it has a unique solution for $\approx$.*

We briefly discuss the hypotheses in the theorem. Some GSOS rule formats guarantee congruence for weak bisimilarity [29, 9], which allows one to remove the corresponding condition (the condition could actually we weakened, by considering the syntactic positions in which the variables can occur in the equations). Checking the autonomy property is often straightforward; for instance, it holds if, in the body of an equation, all variables are underneath an axiom construct, that is, a construct that (like prefix in CCS) is defined by means of SOS rules in which the set of premises is empty. In the tyft/tyxt formats [11], lookaheads are possible. Lookahead allows one to write rules that 'look into the future' (a transition is allowed if certain sequences of actions are possible); this breaks autonomy (condition 3 of Definition 13), hence Theorem 17 is not applicable.

5 Name Passing: the π-calculus

5.1 Unique solution in the asynchronous π-calculus

In this section, we port our results onto the asynchronous π-calculus, $A\pi$ (addressing the full π-calculus would also be possible, but somewhat more involved). To allow constants (recursive definitions) and equations, we enrich the syntax of $A\pi$ [26] with parametrised processes $(\widetilde{a})\,P$. These and the constants form the set of *abstractions*, ranged over by F, G. We omit the definitions of free and bound names. An expression is *closed* if it does not have free names. Constant definitions are of the form $K \triangleq F$, where F is a closed abstraction. The grammar for processes includes also the application construct $F\langle\widetilde{a}\rangle$, used to instantiate the formal parameters of the abstraction F with the actual parameters $\widetilde{a}$.

The body of an equation is also a closed abstraction, possibly containing equation variables. Since the calculus is polyadic, we rely on a sorting system [15] to avoid disagreements in the arities of the tuples of names carried by a given name, and in the parameters of abstractions and equations. For lack of space, we omit the full grammar and the operational semantics (see [24]). When writing examples, for readability, we assume that the syntax contains the guarded replication operator $!a(\widetilde{b}).\,P$ (it could be encoded, using constants).

In bisimulations or similar coinductive relations for the asynchronous π-calculus, no name instantiation is required in the input clause or elsewhere (provided α-convertible processes are identified); i.e., the *ground* versions of the relations are congruence relations [26]. Similarly, the extension of bisimilarity to abstractions only considers fresh names: $F \approx F'$ if $F\langle\widetilde{a}\rangle \approx F'\langle\widetilde{a}\rangle$ where $\widetilde{a}$ is a tuple of fresh names (as usual, of the appropriate sort).

Theorems 6 and 8 for CCS can be adapted to the asynchronous π-calculus. The definitions concerning transitions and divergences are transported to $A\pi$ as expected. In the case of an abstraction, one first has to instantiate the parameters with fresh names; thus F has a divergence if the process $F\langle\widetilde{a}\rangle$ has a divergence, where $\widetilde{a}$ are fresh names.

▶ **Theorem 21** (Unique solution in Aπ). *A guarded system of equations whose syntactic solutions do not contain divergences has a unique solution for* $\approx$.

▶ **Theorem 22** (Unique solution with innocuous divergences in Aπ). *A guarded system of equations whose syntactic solutions only have innocuous divergences has a unique solution for* $\approx$.

We pointed out in earlier sections on CCS the connection between techniques based on unique solution of equations and 'up to context' enhancements of the bisimulation proof method. The same connection is less immediate in name-passing calculi, where indeed there are noticeable differences. In particular, 'up to context' enhancements for the ground bisimilarity of the π-calculus require closure under name instantiation, even when ground bisimilarity is known to be preserved by substitutions (it is an open problem whether the closure can be lifted). Thus, when comparing two derivatives $C[P]$ and $C[Q]$, in general it is not sufficient that P and Q alone are in the candidate relation: one is required to include also all their closures under name substitutions (or, if the terms in the holes are abstractions, instantiation of their parameters with arbitrary tuples of names). In contrast, the two unique solution theorems above are 'purely ground': $F = (\widetilde{x})P$ is solution of an equation $X = (\widetilde{x})E$ if P and $E\{^F/_X\}$ are ground bisimilar – a single ground instance of the equation is evaluated.

5.2 An application: encoding of the call-by-name λ-calculus

To show an extended application of our techniques for the π-calculus, we revisit the proof of full abstraction for Milner's encoding of the call-by-name (or lazy) λ-calculus into $A\pi$ [14] with respect to Lévy Longo Trees (LTs), precisely the completeness part. We use M, N to range over the set Λ of λ-terms, and x, y, z to range over λ variables. If M is an open λ-term, then either M diverges, or $M \Rightarrow \lambda x.\, M'$, or $M \Rightarrow x\ M_1\ \ldots\ M_n$.

▶ **Definition 23** (Lévy-Longo Tree). The *Lévy-Longo Tree* (LT) of an open λ-term M, written $LT(M)$, is the (possibly infinite) tree defined coinductively as follows.

1. If M diverges, then $LT(M)$ is the tree with a single node labelled $\bot$.
2. If $M \to \lambda x.\, M'$, then $LT(M)$ is the tree with a root labelled with "$\lambda x.$", and $LT(M')$ as its unique descendant.
3. If $M \to x\ M_1\ \ldots\ M_n$, then $LT(M)$ is the tree with a root labelled with "x", and $LT(M_1), \ldots, LT(M_n)$ (in this order) as its n descendants.

LT equality (two λ-terms are identified if their LTs are equal) can also be presented as a bisimilarity (*open bisimilarity*, $\simeq^o$), defined as the largest *open bisimulation*.

▶ **Definition 24.** A relation $\mathcal{R}$ on Λ is an *open bisimulation* if, whenever $M\ \mathcal{R}\ N$:

1. $M \Rightarrow \lambda x.\, M'$ implies $N \Rightarrow \lambda x.\, N'$ with $M'\ \mathcal{R}\ N'$;
2. $M \Rightarrow x\ M_1\ \ldots\ M_n$ with $n \geq 0$ implies $N \Rightarrow x\ N_1\ \ldots\ N_n$ and $M_i\ \mathcal{R}\ N_i$ for all $1 \leq i \leq n$.
3. The converse of clauses 1 and 2 on the challenges from N.

Milner's encoding is defined thus: $[\![\lambda x.\, M]\!] \triangleq (p)\ p(x,q).\ [\![M]\!]\langle q\rangle$ $\quad [\![x]\!] \triangleq (p)\ \overline{x}\langle p\rangle$

$$[\![M\ N]\!] \triangleq (p)\ \boldsymbol{\nu} r, x\ ([\![M]\!]\langle r\rangle \mid \overline{r}\langle x, p\rangle \mid\ !x(q).\, [\![N]\!]\langle q\rangle)$$

The full abstraction theorem for the encoding [24, 27] states that two λ-terms have the same LT iff their encodings into $A\pi$ are weakly bisimilar terms. Full abstraction has two components: soundness, which says that if the encodings are weakly bisimilar then the original terms have the same LT; and completeness, which is the converse direction. The proof [24] first establishes some operational correspondence between the behaviour (visible and silent actions) of λ-terms and of their encodings. Then, exploiting this correspondence, soundness and completeness are proved using the bisimulation proof method. For soundness, this is just open bisimulation (Definition 24). In contrast, completeness involves enhancements of the proof method, notably 'bisimulation up to context and expansion'. As a consequence, the technique requires having developed the basic theory for the expansion preorder (e.g., precongruence properties and basic algebraic laws), and requires an operational correspondence fine enough in order to be able to reason about expansion.

Below we show that, by appealing to unique solution of equations, completeness can be proved by defining an appropriate system of equations, each of which having a simple shape, and without the need for auxiliary preorders. For this, the only results needed are: (i) validity of β-reduction for the encoding (Lemma 25), whose proof is simple and consists in the application of a few algebraic laws (including laws for replication); (ii) the property that if M diverges then $[\![M]\!]\langle p\rangle$ may never produce a visible action [24].

▶ **Lemma 25** (Validity of β-reduction, [24]). *For $M \in \Lambda$, if $M \to M'$ then $[\![M]\!] \approx [\![M']\!]$.*

▶ **Theorem 26** (Completeness, [24]). *For $M, N \in \Lambda$, $LT(M) = LT(N)$ implies $[\![M]\!] \approx [\![N]\!]$.*

Proof. Suppose V and W are two λ-terms with the same LT. We define a system of equations, solutions of which are obtained from the encodings of V and W. We will then use Theorem 22 to deduce $[\![V]\!] \approx [\![W]\!]$. If V and W have the same LT, then there is an open bisimulation $\mathcal{R}$

containing the pair (V, W). The variables of the equations are of the form $X_{M,N}$ for $M \mathcal{R} N$, and there is one equation for each pair in $\mathcal{R}$. We show how the equation for a pair $M \mathcal{R} N$ is built. We assume an ordering of the λ-calculus variables so to be able to view a finite set of variables as a tuple. Thus we write $\widetilde{x}$ for the variables appearing free in M or N.

Essentially, the equations are the translation of the clauses of Definition 24, assuming a generalisation of the encoding to equation variables:

If M, N are both divergent, then the equation is $X_{M,N} = (\widetilde{x}, p)!\tau$.

If M, N satisfy clause 1 of Definition 24, the equation is $X_{M,N} = (\widetilde{x}, p)p(x, q).\, X_{M',N'}\langle \widetilde{y}, q\rangle$, where $\widetilde{y}$ are the free variables in M', N'.

Finally, if M, N satisfy clause 2 of Definition 24, the equation is given by the translation of $x\ X_{M_1,N_1} \dots X_{M_n,N_n}$, which, rearranging restrictions and parallel compositions, is

$$X_{M,N} = (\widetilde{x}, p)(\boldsymbol{\nu} r_0, \dots, r_n\,)(\ \overline{x}\langle r_0\rangle \mid \overline{r_0}\langle r_1, x_1\rangle \mid \dots \mid \overline{r_{n-1}}\langle r_n, x_n\rangle \mid \\ !x_1(q_1).\, X_{M_1,N_1}\langle \widetilde{x_1}, q_1\rangle \mid \dots \mid !x_n(q_n).\, X_{M_n,N_n}\langle \widetilde{x_n}, q_n\rangle)\ .$$

where $\widetilde{x_i}$ are the free variables in M_i, N_i. Applying Lemma 9 (more precisely, its $A\pi$ analogue, with additional reference to the sorting system to handle name instantiation in abstractions and inputs), we show that the equations may only produce innocuous divergences.

Now, for $(M, N) \in \mathcal{R}$, set $F_{M,N}$ to be the abstraction $(\widetilde{x}, p)[\![M]\!]\langle p\rangle$, and similarly $G_{M,N} \triangleq (\widetilde{x}, p)[\![N]\!]\langle p\rangle$. The set of all such abstractions $F_{M,N}$ yields a solution for the system of equations, and the same for the $G_{M,N}$'s. The proof that they are solutions (hence bisimilar) is a consequence of Lemma 25. For instance, for clause 1 of Definition 24, we have:

$$\begin{array}{rcl} F_{M,N} & = & (\widetilde{x}, p)[\![M]\!]\langle p\rangle \\ & \approx & (\widetilde{x}, p)[\![\lambda x.\, M']\!]\langle p\rangle \ \approx\ (\widetilde{x}, p)[\![\lambda x.\, X_{M',N'}]\!]\langle p\rangle\{^{F_{M',N'}}/_{X_{M',N'}}\} \end{array}$$

◀

6 Future Work

We have compared our techniques to one of the most powerful forms of enhancements of the bisimulation proof method, namely Pous 'up to transitivity and context', showing that, up to a technical condition, our techniques are at least as powerful. We believe that also the converse holds, though possibly under different side conditions. We leave a detailed analysis of this comparison, which seems non-trivial, for future work. In this respect, the goal of the work on unique solution of equations is to provide a way of better understanding up-to techniques and to shed light into the conditions for their soundness. Pous technique, for instance, is arguably more complex, both in its definition and its application.

Another aspect in which a deep comparison with up-to techniques might be useful is understanding the need for the closure under substitutions in up to context techniques for name-passing calculi such as the asynchronous π-calculus. Such a closure is rather heavy and is a long-standing open problem. However, currently it is unclear how to formally relate bisimulation enhancements and 'unique solution of equations' in name-passing calculi.

Up-to techniques have been analysed in an abstract setting using lattice theory [18] and category theory [4, 23]. It could be interesting to do the same for the unique-solution techniques, to study their connections with up-to techniques, and which equivalences can be handled (possibly using, or refining, the abstract formulation of Section 4).

In comparison with the enhancements of the bisimulation proof method, the main drawback of the techniques exposed in this paper is the presence of a semantic condition, involving divergence: the unfoldings of the equations should not produce divergences, or

only produce innocuous divergences. A syntactic condition for this has been proposed (Lemma 9). Various techniques for checking divergence exist in the literature, including type-based techniques [30, 7]. However, in general divergence is undecidable, and therefore, the check may sometimes be unfeasible. Nevertheless, the equations that one writes for proofs usually involve forms of 'normalised' processes, and as such they are divergence free (or at most, contain only innocuous divergences). More experiments are needed to validate this claim or to understand how limiting this problem is.

References

1 Luca Aceto, Wan Fokkink, and Chris Verhoef. *Handbook of Process Algebra (J.A. Bergstra and A. Ponse and S.A. Smolka, editors)*, chapter Structural operational semantics. Elsevier Science, 2001.

2 Jos C.M. Baeten, Twan Basten, and Michel A. Reniers. *Process Algebra: Equational Theories of Communicating Processes*. Cambridge University Press, 2010.

3 Bard Bloom, Sorin Istrail, and Albert R. Meyer. Bisimulation can't be traced. In *Conference Record of the Fifteenth Annual ACM Symposium on Principles of Programming Languages*, pages 229–239, 1988.

4 Filippo Bonchi, Daniela Petrisan, Damien Pous, and Jurriaan Rot. A general account of coinduction up-to. *Acta Inf.*, 54(2):127–190, 2017.

5 Stephen D. Brookes, C. A. R. Hoare, and A. W. Roscoe. A theory of communicating sequential processes. *J. ACM*, 31(3):560–599, 1984.

6 Stephen D. Brookes and A. W. Roscoe. An improved failures model for communicating processes. In *Seminar on Concurrency, Carnegie-Mellon University, Pittsburg, PA, USA*, pages 281–305, 1984.

7 Romain Demangeon, Daniel Hirschkoff, and Davide Sangiorgi. Termination in impure concurrent languages. In *CONCUR 2010 - Concurrency Theory, 21th International Conference*, pages 328–342, 2010.

8 Adrien Durier. Divergence and unique solution of equations in an abstract setting, Coq formal proof, 2017. URL: `http://perso.ens-lyon.fr/adrien.durier/uniquesolution/`.

9 Wan Fokkink and Rob J. van Glabbeek. Divide and congruence II: delay and weak bisimilarity. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16*, pages 778–787. ACM, 2016.

10 Jan Friso Groote and Mohammad Reza Mousavi. *Modeling and Analysis of Communicating Systems*. MIT Press, 2014.

11 Jan Friso Groote and Frits W. Vaandrager. Structured operational semantics and bisimulation as a congruence. *Inf. Comput.*, 100(2):202–260, 1992.

12 C. A. R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, 1985.

13 Robin Milner. *Communication and concurrency*. PHI Series in computer science. Prentice Hall, 1989.

14 Robin Milner. Functions as processes. *Mathematical Structures in Computer Science*, 2(2):119–141, 1992.

15 Robin Milner. *Communicating and mobile systems - the Pi-calculus*. Cambridge University Press, 1999.

16 Mohammad Reza Mousavi, Michel A. Reniers, and Jan Friso Groote. SOS formats and meta-theory: 20 years after. *Theor. Comput. Sci.*, 373(3):238–272, 2007.

17 Damien Pous. *Up to techniques for bisimulations*. PhD thesis, ENS Lyon, February 2008.

18 Damien Pous. Coinduction all the way up. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16*, pages 307–316, 2016.

19 Damien Pous and Davide Sangiorgi. *Advanced Topics in Bisimulation and Coinduction (D. Sangiorgi and J. Rutten editors)*, chapter Enhancements of the coinductive proof method. Cambridge University Press, 2011.

20 A. W. Roscoe. An alternative order for the failures model. *J. Log. Comput.*, 2(5):557–577, 1992.

21 A. W. Roscoe. *The Theory and Practice of Concurrency*. Prentice Hall PTR, Upper Saddle River, NJ, USA, 1997.

22 A. W. Roscoe. *Understanding Concurrent Systems*. Springer, 2010.

23 Jurriaan Rot, Marcello M. Bonsangue, and Jan J. M. M. Rutten. Coalgebraic bisimulation-up-to. In *SOFSEM 2013: Theory and Practice of Computer Science, 39th International Conference on Current Trends in Theory and Practice of Computer Science*, pages 369–381, 2013.

24 Davide Sangiorgi. Lazy functions and mobile processes. In G. Plotkin, C. Stirling, and M. Tofte, editors, *Proof, Language and Interaction: Essays in Honour of Robin Milner*. MIT Press, 2000.

25 Davide Sangiorgi. Equations, contractions, and unique solutions. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2015*, pages 421–432, 2015.

26 Davide Sangiorgi and David Walker. *The Pi-Calculus - a theory of mobile processes*. Cambridge University Press, 2001.

27 Davide Sangiorgi and Xian Xu. Trees from functions as processes. In *25th International Conference, CONCUR 2014, Rome, Italy*, LNCS, pages 78–92. Springer Verlag, 2014.

28 Rob J. van Glabbeek. The linear time-branching time spectrum (extended abstract). In *CONCUR '90, Theories of Concurrency: Unification and Extension, Amsterdam, The Netherlands, August 27-30, 1990, Proceedings*, pages 278–297, 1990.

29 Rob J. van Glabbeek. On cool congruence formats for weak bisimulations. In *Theoretical Aspects of Computing - ICTAC 2005, Second International Colloquium*, pages 318–333, 2005.

30 Nobuko Yoshida, Martin Berger, and Kohei Honda. Strong Normalisation in the Pi-Calculus. *Information and Computation*, 191(2):145–202, 2004.

Controlling a Population*

Nathalie Bertrand[1], Miheer Dewaskar[2], Blaise Genest[3], and Hugo Gimbert[4]

1 **Inria, IRISA, Rennes, France**
2 **University of North Carolina, Chapel Hill, USA**
3 **CNRS, IRISA, Rennes, France**
4 **CNRS, LaBRI, Bordeaux, France**

Abstract

We introduce a new setting where a population of agents, each modelled by a finite-state system, are controlled uniformly: the controller applies the same action to every agent. The framework is largely inspired by the control of a biological system, namely a population of yeasts, where the controller may only change the environment common to all cells. We study a synchronisation problem for such populations: no matter how individual agents react to the actions of the controller, the controller aims at driving all agents synchronously to a target state. The agents are naturally represented by a non-deterministic finite state automaton (NFA), the same for every agent, and the whole system is encoded as a 2-player game. The first player (Controller) chooses actions, and the second player (Agents) resolves non-determinism for each agent. The game with m agents is called the m-population game. This gives rise to a parameterized control problem (where control refers to 2 player games), namely the *population control problem*: can Controller control the m-population game for all $m \in \mathbb{N}$ whatever Agents does?

In this paper, we prove that the population control problem is decidable, and it is a EXPTIME-complete problem. As far as we know, this is one of the first results on parameterized control. Our algorithm, not based on cut-off techniques, produces winning strategies which are symbolic, that is, they do not need to count precisely how the population is spread between states. We also show that if there is no winning strategy, then there is a population size M such that Controller wins the m-population game if and only if $m \leq M$. Surprisingly, M can be doubly exponential in the number of states of the NFA, with tight upper and lower bounds.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Model-checking,control,parametric systems

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.12

1 Introduction

Finite-state controllers, implemented by software, find applications in many different domains: telecommunication, planes, etc. There have been many theoretical studies from the model checking community to show that finite-state controllers are sufficient to control systems in idealised settings. Usually, the problem would be modeled as a game: some players model the controller, and some players model the system [5], the game settings (number of players, their power, their observation) depending on the context.

Lately, finite-state controllers have been used to control living organisms, such as a population of yeasts [23]. In this application, microscopy is used to monitor the fluorescence

* This work was partially supported by ANR project STOCH-MC (ANR-13-BS02-0011-01).

level of a population of yeasts, reflecting the concentration of some molecule, which differs from cell to cell. Finite-state systems can model a discretisation of the population of yeasts [23, 3]. A controller decide the frequency and duration of injections of a sorbitol solution, uniform over the yeast population. However, the response of each cell to the osmotic stress induced by sorbitol varies, influencing the concentration of the fluorescent molecule. The objective is to control the population to drive it through a sequence of predetermined fluorescence states.

In this paper, we model this system of yeasts in an *idealised* setting: we require the (perfectly-informed) controller to surely lead synchronously all agents of a population to a state (one of the predetermined fluorescence states). Such a population control problem does not fit in traditional frameworks from the model checking community. We thus introduce the *m-population game*, where a population of m identical agents is controlled uniformly. Each agent is modeled as a nondeterministic finite-state automaton (NFA), the same for each agent. The first player, called Controller, applies the same action, a letter from the NFA alphabet, to every agent. Its opponent, called Agents, chooses the reaction of each individual agent. These reactions can be different due to non determinism. The objective for Controller is to gather all agents synchronously in the target state (which can be a sink state w.l.o.g.), and Agents seeks the opposite objective. While this idealised setting may not be entirely satisfactory, it constitutes a simple setting, as a first step towards more complex settings.

Dealing with large populations *explicitly* is in general intractable due to the state-space explosion problem. We thus consider the associated *symbolic parameterized control problem*, asking to reach the goal independently of the population size. We prove that this problem is decidable. While *parameterized verification* received recently quite some attention (see related work), our results are one of the first on *parameterized control*, as far as we know.

Our results. We first show that considering an infinite population is not equivalent to the parameterized control problem for all non zero integer m: there are cases where Controller cannot control an infinite population but can control every finite population. Solving the ∞-population game reduces to checking a reachability property on the support graph [21], which can be easily done in PSPACE. On the other hand, solving the parameterized control problem requires new proof techniques, data structures and algorithms.

We easily obtain that when the answer to the population control problem is negative, there exists a population size M, called the *cut-off*, such that Controller wins the m-population game if and only if $m \leq M$. Surprisingly, we obtain a lower-bound on the cut-off doubly exponential in the number of states of the NFA. Following usual cut-off techniques would thus yield an inefficient algorithm of complexity at least 2EXPTIME.

To obtain better complexity, we developped new proof techniques (*not* based on cut-off techniques). Using them, we prove that the population control problem is EXPTIME-complete. As a byproduct, we obtain a doubly exponential upper-bound for the cut-off, matching the lower-bound. Our techniques are based on a reduction to a parity game with exponentially many states and a polynomial number of priorities. The parity game gives some insight on the winning strategies of Controller in the m-population games. Controller selects actions based on a set of *transfer graphs*, giving for each current state the set of states at time i from which agent came from, for different values of i. We show that it suffices for Controller to remember at most a quadratic number of such transfer graphs, corresponding to a quadratic number of indices i. If Controller wins this parity game then he can uniformly apply his winning strategy to all m-population games, just keeping track of these transfer graphs, independently of the exact count in each state. If Agents wins the parity game then he also has a uniform winning strategy in m-population games, for m large enough, which consists

in splitting the agents evenly among all transitions of the transfer graphs. Missing proofs are available in the research report [6].

Related work. Parameterized verification of systems with many identical components started with the seminal work of German and Sistla in the early nineties [16], and received recently quite some attention. The decidability and complexity of these problems typically depend on the communication means, and on whether the system contains a leader (following a different template) as exposed in the recent survey [13]. This framework has been extended to timed automata templates [2, 1] and probabilistic systems with Markov decision processes templates [7, 8]. Another line of work considers population protocols [4, 15]. Close in spirit, are broadcast protocols [14], in which one action may move an arbitrary number of agents from one state to another. Our model can be modeled as a subclass of broadcast protocols, where broadcasts emissions are self loops at a unique state, and no other synchronisation allowed. The parameterized reachability question considered for broadcast protocols is trivial in our framework, while our parameterized control question would be undecidable for broadcast protocols. In these different works, components interact directly, while in our work, the interaction is indirect via the common action of the controller. Further, the problems considered in related work are pure verification questions, and do not tackle the difficult issue of synthesising a controller for all instances of a parameterized system, which we do.

There are very few contributions pertaining to parameterized games with more than one player. The most related is [20], which proves decidability of control of mutual exclusion-like protocols in the presence of an unbounded number of agents. Another contribution in that domain is the one of broadcast networks of identical parity games [8]. However, the game is used to solve a verification (reachability) question rather than a parametrized control problem as in our case. Also the roles of the two players are quite different.

The winning condition we are considering is close to *synchronising words*. The original synchronising word problem asks for the existence of a word w and a state q of a *deterministic* finite state automaton, such that no matter the initial state s, reading w from s would lead to state q (see [24] for a survey). Lately, synchronising words have been extended to NFAs [21]. Compared to our settings, the author assumes a possibly infinite population of agents, who could leak arbitrarily often from a state to another. The setting is thus not parametrized, and a usual support arena suffices to obtain a PSPACE algorithm. Synchronisation for probabilistic models [11, 12] have also been considered: the population of agents is not finite nor discrete, but rather continuous, represented as a distribution. The distribution evolves deterministically with the choice of the controller (the probability mass is split according to the probabilities of the transitions), while in our setting, each agent can non deterministically move. In [11], the controller needs to apply the same action whatever the state the agents are in (like our setting), and then the existence of a controller is undecidable. In [12], the controller can choose the action depending on the state each agent is in (unlike our setting), and the existence of a controller reaching uniformly a set of states is PSPACE-complete.

Last, our parameterized control problem can be encoded as a 2-player game on VASS [9], with one counter per state of the NFA: the opponent gets to choose the population size (a counter value), and the move of each agent corresponds to decrementing a counter and incrementing another. Such a reduction yields a symmetrical game on VASS in which both players are allowed to modify the counter values, in order to check that the other player did not cheat. Symmetrical games on VASS are undecidable [9], and their asymmetric variant (in which only one player is allowed to change the counter values) are decidable in 2EXPTIME [19], thus with higher complexity than our specific parameterized control problem.

2 The population control problem

2.1 The m-population game

A nondeterministic finite automaton (NFA for short) is a tuple $\mathcal{A} = (Q, \Sigma, q_0, \Delta)$ with Q a finite set of states, Σ a finite alphabet, $q_0 \in Q$ an initial state, and $\Delta \subseteq Q \times \Sigma \times Q$ the transition relation. We assume throughout the paper that NFAs are complete, that is, $\forall q \in Q, a \in \Sigma\ , \exists p \in Q :\ (q, a, p) \in \Delta$. In the following, incomplete NFAs, especially in figures, have to be understood as completed with a sink state.

For every integer m, we consider a system $\mathcal{A}^m$ with m identical agents $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of the NFA $\mathcal{A}$. The system $\mathcal{A}^m$ is itself an NFA $(Q^m, \Sigma, q_0^m, \Delta^m)$ defined as follows. Formally, states of $\mathcal{A}^m$ are called configurations, and they are tuples $\mathbf{q} = (q_1, \ldots, q_m) \in Q^m$ describing the current state of each agent in the population. We use the shorthand $\mathbf{q}_0[m]$, or simply $\mathbf{q}_0$ when m is clear from context, to denote the initial configuration $(q_0, \ldots, q_0)$ of $\mathcal{A}^m$. Given a target state $f \in Q$, the f-synchronizing configuration is $f^m = (f, \ldots, f)$ in which each agent is in the target state.

The intuitive semantics of $\mathcal{A}^m$ is that at each step, the same action from Σ applies to all agents. The effect of the action however may not be uniform given the nondeterminism present in $\mathcal{A}$: we have $((q_1, \ldots, q_m), a, (q'_1, \ldots, q'_m)) \in \Delta^m$ iff $(q_j, a, q'_j) \in \Delta$ for all $j \leq m$. A (finite or infinite) play in $\mathcal{A}^m$ is an alternating sequence of configurations and actions, starting in the initial configuration: $\pi = \mathbf{q}_0 a_0 \mathbf{q}_1 a_1 \cdots$ such that $(\mathbf{q}_i, a_i, \mathbf{q}_{i+1}) \in \Delta^m$ for all i.

This is the *m-population game* between Controller and Agents, where Controller chooses the actions and Agents chooses how to resolve non-determinism. The objective for Controller is to gather all agents synchronously in f while Agents seeks the opposite objective.

Our parameterized control problem asks whether Controller can win the m-population game for every $m \in \mathbb{N}$. A strategy of Controller in the m-population game is a function mapping finite plays to actions, $\sigma : (Q^m \times \Sigma)^* \times Q^m \to \Sigma$. A play $\pi = \mathbf{q}_0 a_0 \mathbf{q}_1 a_1 \mathbf{q}_2 \cdots$ is said to *respect* σ, or is a *play under* σ, if it satisfies $a_i = \sigma(\mathbf{q}_0 a_0 \mathbf{q}_1 \cdots \mathbf{q}_i)$ for all $i \in \mathbb{N}$. A play $\pi = \mathbf{q}_0 a_0 \mathbf{q}_1 a_1 \mathbf{q}_2 \cdots$ is *winning* if it hits the f-synchronizing configuration, that is $\mathbf{q}_j = f^m$ for some $j \in \mathbb{N}$. Controller wins the m-population game if he has a strategy such that all plays under this strategy are winning. One can assume without loss of generality that f is a sink state. If not, it suffices to add a new action leading tokens from f to the new target sink state ☺ and tokens from other states to a losing sink state ☹. The goal of this paper is to study the following parameterized control problem:

Population control problem
Input: An NFA $\mathcal{A} = (Q, q_0, q_u, \Sigma, \Delta)$ and a target state $f \in Q$.
Output: Yes iff for every integer m Controller wins the m-population game.

For a fixed m, the winner of the m-population game can be determined by solving the underlying reachability game with $|Q|^m$ states, which is intractable for large values of m. On the other hand, the answer to the population control problem gives the winner of the m-population game for arbitrary large values of m. To obtain a decision procedure for this parameterised problem, new data structures and algorithmic tools need to be developed, much more elaborate than the standard algorithm solving reachability games.

▶ **Example 1.** We illustrate the population control problem with the example $\mathcal{A}_{\mathsf{split}}$ on alphabet $\{a, b, \delta\}$ in Figure 1. Here, to represent configurations we use a counting abstraction, and identify $\mathbf{q}$ with the vector (n_0, n_1, n_2, n_3), where n_0 is the number of agents in state q_0, and so on. Under these notations, there is a way to gather agents synchronously to f. We can

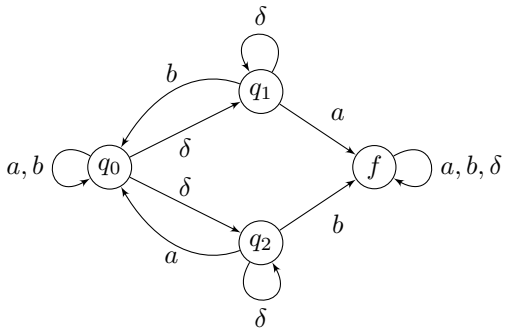


Figure 1 An exemple of NFA: The splitting gadget $\mathcal{A}_{\mathsf{split}}$.

give a symbolic representation of a memoryless winning strategy σ: $\forall k_0, k_1 > 0,\ \forall k_2, k_3 \geq 0,\ \sigma(k_0, 0, 0, k_3) = \delta,\ \sigma(0, k_1, k_2, k_3) = a,\ \sigma(0, 0, k_2, k_3) = b$. Indeed, the number of agents outside f decreases by at least one at every other step. The properties of this example will be detailed later and play a part in proving a lower bound (see Proposition 20).

2.2 Parameterized control and cut-off

A first observation for the population control problem is that $\mathbf{q}_0[m]$, f^m and Q^m are stable under a permutation of coordinates. A consequence is that the m-population game is also symmetric under permutation, and thus the set of winning configurations is symmetric and the winning strategy can be chosen uniformly from symmetric winning configurations. Therefore, if Controller wins the m-population game then he has a positional winning strategy which only counts the number of agents in each state of $\mathcal{A}$ (the counting abstraction used in Example 1).

▶ **Proposition 2.** *Let $m \in \mathbb{N}$. If Controller wins the m-population game, then he wins the m'-population game for every $m' \leq m$.*

The idea to define $\sigma_{m'}$ is to simulate the missing $m-m'$ agents arbitrarily and apply σ_m.

Hence, when the answer to the population control problem is negative, there exists a *cut-off*, that is a value $M \in \mathbb{N}$ such that for every $m < M$, Controller has a winning strategy in $\mathcal{A}^m$, and for every $m \geq M$, he has no winning strategy.

▶ **Example 3.** To illustrate the notion of cut-off, consider the NFA on alphabet $A \cup \{b\}$ from Figure 2. Unspecified transitions lead to a sink state ☹.

The cut-off is $M = |Q| - 2$ in this case. Indeed, we have the following two directions:

On the one hand, for $m < M$, there is a winning strategy σ_m in $\mathcal{A}^m$ to reach f^m, in just two steps. It first plays b, and because $m < M$, in the next configuration, there is at least one state q_i such that no agent is in q_i. It then suffices to play a_i to win.

Now, if $m \geq M$, there is no winning strategy to synchronize in f, since after the first b, agents can be spread so that there is at least one agent in each state q_i. From there, Controller can either play action b and restart the whole game, or play any action a_i, leading at least one agent to the sink state ☹.

2.3 Main results

Our main result is the decidability of the population control problem:

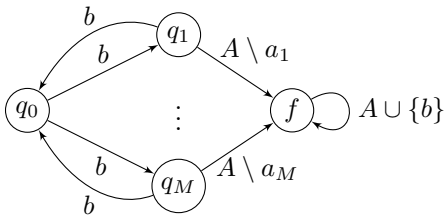


Figure 2 Illustration of the cut-off.

▶ **Theorem 4.** *The population control problem is* EXPTIME*-complete.*

When the answer to the population control problem is positive, there exists a symbolic strategy σ, applicable to all instances $\mathcal{A}^m$, that does not need to count the number of agents in each state. This symbolic strategy requires exponential memory. Otherwise, the cut-off is at most doubly exponential, which is asymptotically tight.

▶ **Theorem 5.** *In case the answer to the population control problem is negative, the cut-off is at most* $\leq 2^{2^{O(|Q|^4)}}$. *There is a family of NFA* $(\mathcal{A}_n)$ *of size* $O(n)$ *and whose cut-off is* 2^{2^n}.

3 The capacity game

The objective of this section is to show that the population control problem is equivalent to solving a game called the *capacity game*. To introduce useful notations, we first recall the population game with infinitely many agents, as studied in [21] (see also [22] p.81).

3.1 The ∞-population game

To study the ∞-population game, the behaviour of infinitely many agents is abstracted into *supports* which keep track of the set of states in which at least one agent is. We thus introduce the *support game*, which relies on the notion of *transfer graphs*. Formally, a transfer graph is a subset of $Q \times Q$ describing how agents are moved during one step. The domain of a transfer graph G is $\mathrm{Dom}(G) = \{q \in Q \mid \exists (q, r) \in G\}$ and its image is $\mathrm{Im}(G) = \{r \in Q \mid \exists (q, r) \in G\}$. Given an NFA $\mathcal{A} = (Q, \Sigma, q_0, \Delta)$ and $a \in \Sigma$, the transfer graph G is compatible with a if for every edge (q, r) of G, $(q, a, r) \in \Delta$. We write $\mathcal{G}$ for the set of transfer graphs.

The *support game* of an NFA $\mathcal{A}$ is a two-player reachability game played by Controller and Agents on the *support arena* as follows. States are supports, *i.e.*, non-empty subsets of Q and the play starts in $\{q_0\}$. The goal support is $\{f\}$. From a support S, first Controller chooses a letter $a \in \Sigma$, then Agents chooses a transfer graph G compatible with a and such that $\mathrm{Dom}(G) = S$, and the next support is $\mathrm{Im}(G)$. A play in the support arena is described by the sequence $\rho = S_0 \xrightarrow{a_1, G_1} S_1 \xrightarrow{a_2, G_2} \dots$ of supports and actions (letters and transfer graphs) of the players. Here, Agents best strategy is to play the maximal graph possible (this is not the case with discrete populations), and we obtain a PSPACE-complete algorithm [21]:

▶ **Proposition 6.** *Controller wins the* ∞*-population game iff he wins the support game.*

This result cannot be used for deciding the population control problem, because Controller might win every m-population game (with $m < \infty$) and at the same time lose the ∞-population game. For that, consider the example from Figure 1. As already shown, Controller wins any m-population game with $m < \infty$. However, Agents can win the ∞-population game by splitting agents from q_0 to both q_1 and q_2 each time Controller plays δ. This way, the sequence of supports is $\{q_0\}\{q_1, q_2\}(\{q_0, f\}\{q_1, q_2, f\})^*$, which never hits $\{f\}$.

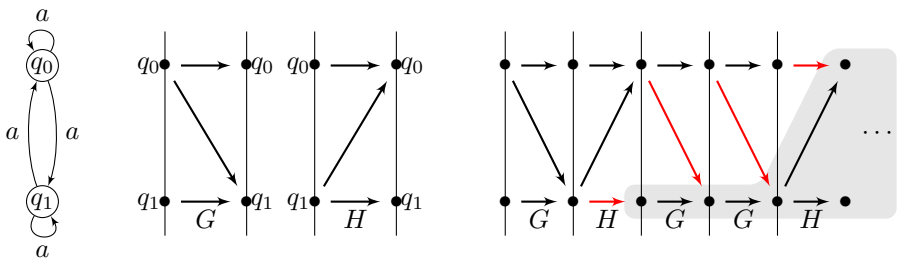


Figure 3 An NFA, two transfer graphs, and a play with finite yet unbounded capacity.

3.2 Realisable plays

Plays of the m-population game (for $m < \infty$) can be abstracted as plays in the support game, by forgetting the identity of agents and keeping only track of edges that are used by at least one agent. Formally, given a play $\pi = \mathbf{q}_0 a_0 \mathbf{q}_1 a_1 \mathbf{q}_2 \cdots$ of the m-population game, define for every integer n, $S_n = \{\mathbf{q}_n[i] \mid 1 \leq i \leq m\}$ and $G_{n+1} = \{(\mathbf{q}_n[i], \mathbf{q}_{n+1}[i]) \mid 1 \leq i \leq m\}$. We denote $\Phi_m(\pi)$ the play $S_0 \xrightarrow{a_1,G_1} S_1 \xrightarrow{a_2,G_2} \dots$ in the support arena, called the projection of π.

Not every play in the support arena can be obtained by projection. This is the reason for introducing the notion of realisable plays:

▶ **Definition 7** (Realisable plays)**.** A play of the support game is *realisable* if there exists $m < \infty$ such that it is the projection by Φ_m of a play in the m-population game.

To characterise realisability, we introduce entries of accumulators:

▶ **Definition 8.** Let $\rho = S_0 \xrightarrow{a_1,G_1} S_1 \xrightarrow{a_2,G_2} \dots$ be a play in the support arena. An *accumulator* of ρ is a sequence $T = (T_j)_{j\in\mathbb{N}}$ such that for every integer j, $T_j \subseteq S_j$, and which is *successor-closed i.e.*, for every $j \in \mathbb{N}$, $(s \in T_j \wedge (s,t) \in G_{j+1}) \implies t \in T_{j+1}$. For every $j \in \mathbb{N}$, an edge $(s,t) \in G_{j+1}$ is an *entry* to T if $s \notin T_j$ and $t \in T_{j+1}$.

▶ **Definition 9** (Plays with finite and bounded capacity)**.** A play has *finite capacity* if all its accumulators have finitely many entries, *infinite capacity* otherwise, and *bounded capacity* if the number of entries of its accumulators is bounded.

Realisability is actually equivalent to *bounded* capacity:

▶ **Lemma 10.** *A play is realisable iff it has bounded capacity.*

An example is given on Figure 3 which represents an NFA, two transfer graphs G and H, and a play $GHG^2HG^3\cdots$. Obviously, this play is not realisable because at least n agents are needed to realise n transfer graphs G in a row: at each G step, at least one agent moves from q_0 to q_1, and no new agent enters q_0. A simple analysis shows that there are only two kinds of non-trivial accumulators $(T_j)_{j\in\mathbb{N}}$ depending on whether their first non-empty T_j is $\{q_0\}$ or $\{q_1\}$. We call these top and bottom accumulators, respectively. All accumulators have finitely many entries, thus the play has finite capacity. However, for every $n \in \mathbb{N}$ there is a bottom accumulator with $2n$ entries. As an example, a bottom accumulator with 4 entries (in red) is depicted on the figure. Therefore, the capacity of this play is not bounded.

3.3 The capacity game

An idea to obtain a game on the support arena equivalent with the population control problem is to make Agents lose whenever the play is not realisable, *i.e.* whenever the play

has unbounded capacity. One issue with (un)bounded capacity is however that it is not a regular property for runs. Hence, it is not easy to use it as a winning condition. On the contrary, *finite* capacity is a regular property. We thus relax (un)bounded capacity using (in)finite capacity and define the corresponding abstraction of the population game:

▶ **Definition 11** (Capacity game)**.** The *capacity game* is the game played on the support arena, where Controller wins a play iff either the play reaches $\{f\}$ or the play has infinite capacity. A player *wins the capacity game* if he has a winning strategy in this game.

We show that this relaxation can be used to decide the population control problem.

▶ **Theorem 12.** *The answer to the population control problem is positive iff Controller wins the capacity game.*

This theorem is a direct corollary of the following proposition:

▶ **Proposition 13.** *Either Controller or Agents wins the capacity game, and the winner has a winning strategy with* finite memory*. In case Controller is the winner of the capacity game, he wins all m-population games, for every integer m. In case Agents wins the capacity game with a strategy with finite memory of size M, he wins the $|Q|^{1+|\mathsf{M}|\cdot 4^{|Q|}}$-population game.*

Proof of first and second assertions. We start with the first assertion. Whether a play has infinite capacity can be verified by a non-deterministic Büchi automaton of size $2^{|Q|}$ on the alphabet of transfer graphs, which guesses an accumulator on the fly and checks that it has infinitely many entries. This Büchi automaton can be determinised into a parity automaton (*e.g.* using Safra's construction) with state space M of size $\mathcal{O}\left(2^{2^{|Q|}}\right)$. The synchronized product of this deterministic parity automaton with the support game produces a parity game which is equivalent with the capacity game, in the sense that, up to unambigous synchronization with the deterministic automaton, plays and strategies in both games are the same and the synchronization preserves winning plays and strategies. Since parity games are determined and positional [25], either Controller or Agents has a positional winning strategy in the parity game, thus either Controller or Agents has a winning strategy with finite memory M in the capacity game.

Let us prove the second assertion. Assuming that Controller wins the capacity game with a strategy σ, he can win any m-population game, $m < \infty$, with the strategy $\sigma_m = \sigma \circ \Phi_m$. The projection $\Phi_m(\pi)$ of every infinite play π respecting σ_m is realisable, thus $\Phi_m(\pi)$ has bounded, hence finite, capacity (Lemma 10). Moreover $\Phi_m(\pi)$ respects σ, and since σ wins the capacity game, $\Phi_m(\pi)$ reaches $\{f\}$. Thus π reaches f^m and σ_m is winning.

The last assertion is proved in [6]. ◀

As consequence of Proposition 13, the population control problem can be decided by explicitely computing the parity game and solving it, in 2EXPTIME. In the next section we will improve this complexity bound to EXPTIME.

We conclude with an example showing that, in general, positional strategies are not sufficient to win the capacity game. Consider the example of Figure 4, where the only way for Controller to win is to reach a support without q_2 and play c. With a memoryless strategy, Controller cannot win the capacity game. There are only two memoryless strategies from support $S = \{q_1, q_2, q_3, q_4\}$. If Controller only plays a from S, the support remains S and the play has bounded capacity. If he only plays b's from S, then Agents can split tokens from q_3 to both q_2, q_4 and the play remains in support S, with bounded capacity. In both cases, the play has finite capacity and Controller loses.

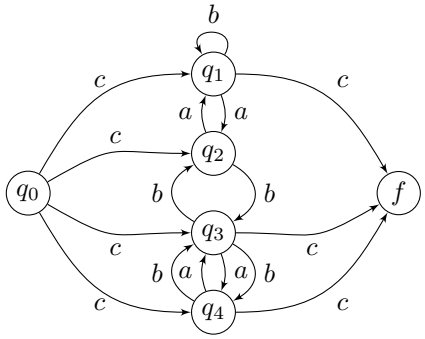


Figure 4 Population game where Controller needs memory to win the associated capacity game.

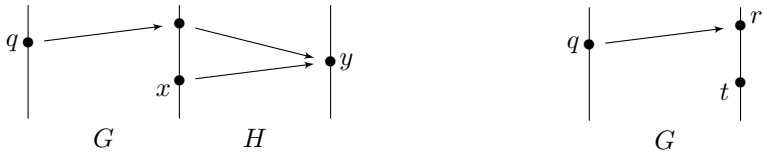


Figure 5 Left: G leaks at H; Right: G separates (r, t).

However, Controller can win the capacity game. His (finite-memory) winning strategy σ consists in first playing c, and then playing alternatively a and b, until the support does not contain $\{q_2\}$, in which case he plays c to win. Two consecutive steps ab send q_2 to q_1, q_1 to q_3, q_3 to q_3, and q_4 to either q_4 or q_2. To prevent Controller from playing c and win, Agents needs to spread from q_4 to both q_4 and q_2 every time ab is played. Consider the accumulator T defined by $T_{2i} = \{q_1, q_2, q_3\}$ and $T_{2i-1} = \{q_1, q_2, q_4\}$ for every $i > 0$. It has an infinite number of entries (from q_4 to T_{2i}). Hence Controller wins if this play is executed. Else, Agents eventually keeps all agents from q_4 in q_4 when ab is played, implying the next support does not contain q_2. Strategy σ is thus a winning strategy for Controller.

4 Solving the capacity game in EXPTIME

To solve efficiently the capacity game, we build an equivalent exponential size parity game with a polynomial number of parities. To do so, we enrich the support arena with a *tracking list* responsible of checking whether the play has finite capacity. The tracking list is a list of transfer graphs, which are used to detect certain patterns called *leaks*.

4.1 Leaking graphs

In order to detect whether a play $\rho = S_0 \xrightarrow{a_1,G_1} S_1 \xrightarrow{a_2,G_2} \dots$ has finite capacity, it is enough to detect *leaking* graphs (characterising entries of accumulators). Further, leaking graphs have special *separation* properties which will allow us to track a small number of graphs. For G, H two graphs, we denote $(a, b) \in G \cdot H$ iff there exists z with $(a, z) \in G$, and $(z, b) \in H$.

▶ **Definition 14** (Leaks and separations). Let G, H be two transfer graphs. We say that G *leaks at* H if there exist states q, x, y with $(q, y) \in G \cdot H$, $(x, y) \in H$ and $(q, x) \notin G$. We say that G *separates* a pair of states (r, t) if there exists $q \in Q$ with $(q, r) \in G$ and $(q, t) \notin G$.

The tracking list will be composed of concatenated graphs *tracking* i of the form $G[i, j] = G_{i+1} \cdots G_j$ relating S_i with S_j: $(s_i, s_j) \in G[i, j]$ if there exists $(s_k)_{i<k<j}$ with $(s_k, s_{k+1}) \in$

G_{k+1} for all $i \leq k \leq j$. Infinite capacity relates to leaks in the following way:

▶ **Lemma 15.** *A play has infinite capacity iff there exists an index i such that $G[i,j]$ leaks at G_{j+1} for infinitely many indices j.*

In this case, we say that index i *leaks infinitely often*. Note that if G separates (r,t), and r,t have a common successor by H, then G leaks at H. To link leaks with separations, we consider for each index k, the pairs of states that have a common successor, in possibly several steps, as expressed by the symmetric relation R_k: $(r,t) \in R_k$ iff there exists $j \geq k$ and $y \in Q$ such that $(r,y) \in G[k,j] \wedge (t,y) \in G[k,j]$.

▶ **Lemma 16.** *For $i < n$ two indices, the following three properties hold:*
1. *If $G[i,n]$ separates $(r,t) \in R_n$, then there exists $m \geq n$ such that $G[i,m]$ leaks at G_{m+1}.*
2. *If index i does not leak infinitely often, then the number of indices j such that $G[i,j]$ separates some $(r,t) \in R_j$ is finite.*
3. *If index i leaks infinitely often, then for all $j > i$, $G[i,j]$ separates some $(r,t) \in R_j$.*

4.2 The tracking list

The *tracking list* exploits the relationship between leaks and separations. It is a list of transfer graphs which altogether separate all possible pairs, and are sufficient to detect when leaks occur. Notice that telling at step j whether the pair (r,t) belongs to R_j cannot be performed by a deterministic automaton. We thus *a priori* have to consider every pair $(r,t) \in Q^2$ for separation. The tracking list $\mathcal{L}_n$ at step n is defined inductively as follows. $\mathcal{L}_0$ is the empty list, and for $n > 0$, the list $\mathcal{L}_n$ is computed in three stages:
1. first, every graph H in the list $\mathcal{L}_{n-1}$ is concatenated with G_n, yielding $H \cdot G_n$;
2. second, G_n is added at the end of the obtained list;
3. last, the list is filtered: a graph H is kept if and only if it separates a pair of states $(p,q) \in Q^2$ which is not separated by any graph that appears earlier in the list.

Because of the third item, there are at most $|Q|^2$ graphs in the tracking list. The list may become empty if no pair of states is separated by any graph, for example if all the graphs are complete. Let $\mathcal{L}_n = \{H_1, \cdots, H_\ell\}$ be the tracking list at step n. Then each transfer graph $H_r \in \mathcal{L}_n$ is of the form $H_r = G[t_r, n]$. We say that r is the *level* of H_r, and t_r the *index tracked* by H_r. Observe that the lower the level of a graph in the list, the smaller the index it tracks. When we consider the sequence of tracking lists $(\mathcal{L}_n)_{n\in\mathbb{N}}$, for every index i, either it eventually stops to be tracked or it is tracked forever from step i, *i.e.* for every $n \geq i$, $G[i,n]$ belongs to $\mathcal{L}_n$. In the latter case, i is said to be *remanent* (because it will never disappear).

Using Lemma 15 and the second and third statements of Lemma 16, we obtain:

▶ **Lemma 17.** *A play has infinite capacity iff there exists an index i such that i is remanent and leaks infinitely often.*

4.3 The parity game

We now describe a parity game $\mathcal{PG}$, which extends the support arena with on-the-fly computation of the tracking list.

Priorities. By convention, lowest priorities are the most important and the odd parity is good for Controller, so Controller wins iff the lim inf of the priorities is odd. With each level $1 \leq r \leq |Q|^2$ of the tracking list are associated two priorities $2r$ and $2r+1$, and on top of that are added priorities 1 and $2|Q|^2+2$, hence the set of all priorities is $\{1, \ldots, 2|Q|^2+2\}$.

When Agents chooses a transition labelled by a transfer graph G, the tracking list is updated with G and the priority of the transition is determined as the smallest among: priority 1 if the support $\{f\}$ has ever been visited, priority $2r+1$ for the smallest r such that H_r (from level r) leaks at G, priority $2r$ for the smallest level r where a graph was removed, and in all other cases priority $2|Q|^2+2$.

States and transitions. $\mathcal{G}^{\leq |Q|^2}$ denotes the set of list of at most $|Q|^2$ transfer graphs.

- States of $\mathcal{PG}$ form a subset of $\{0,1\} \times 2^Q \times \mathcal{G}^{\leq |Q|^2}$, each state being of the form $(b, S, H_1, \dots, H_\ell)$ with $b \in \{0,1\}$ a bit indicating whether a support in $\{f\}$ has been seen, S the current support and $(H_1, \dots, H_\ell)$ the tracking list. The initial state is $(0, \{q_0\}, \emptyset)$.
- Transitions in $\mathcal{PG}$ are all $(b, S, H_1, \dots, H_\ell) \xrightarrow{\mathbf{p},a,G} (b', S', H'_1, \dots, H'_{\ell'})$ where $\mathbf{p}$ is the priority, and such that $S \xrightarrow{a,G} S'$ is a transition of the support arena, and
  1. $(H'_1, \dots, H'_{\ell'})$ is the tracking list obtained by updating the tracking list $(H_1, \dots, H_\ell)$ with G, as explained in subsection 4.2;
  2. if $b = 1$ or if $S' \subseteq F$, then $\mathbf{p} = 1$ and $b' = 1$;
  3. otherwise $b' = 0$. In order to compute the priority $\mathbf{p}$, we let $\mathbf{p}'$ be the smallest level $1 \leq r \leq \ell$ such that H_r leaks at G and $\mathbf{p}' = \ell + 1$ if there is no such level, and we also let $\mathbf{p}''$ as the minimal level $1 \leq r \leq \ell$ such that $H'_r \neq H_r \cdot G$ and $\mathbf{p}'' = \ell + 1$ if there is no such level. Then $\mathbf{p} = \min(2\mathbf{p}' + 1, 2\mathbf{p}'')$.

We are ready to state the main result of this paper, which yields an EXPTIME complexity for the population control problem. This entails the first statement of Theorem 4, and together with Proposition 13, also the first statement of Theorem 5.

▶ **Theorem 18.** *Controller wins the game $\mathcal{PG}$ if and only if Controller wins the capacity game. Solving these games can be done in time $O(2^{(1+|Q|+|Q|^4)(2|Q|^2+2)})$. Strategies with $2^{|Q|^4}$ memory states are sufficient to both Controller and Agents.*

Proof. The state space of parity game $\mathcal{PG}$ is the product of the set of supports with a deterministic automaton computing the tracking list. There is a natural correspondence between plays and strategies in the parity game $\mathcal{PG}$ and in the capacity game.

Controller can win the parity game $\mathcal{PG}$ in two ways: either the play visits the support $\{f\}$, or the priority of the play is $2r+1$ for some level $1 \leq r \leq |Q|^2$. By design of $\mathcal{PG}$, this second possibility occurs iff r is remanent and leaks infinitely often. According to Lemma 17, this occurs if and only if the corresponding play of the capacity game has infinite capacity. Thus Controller wins $\mathcal{PG}$ iff he wins the capacity game.

In the parity game $\mathcal{PG}$, there are at most $2^{1+|Q|}\left(2^{|Q|^2}\right)^{|Q|^2} = 2^{1+|Q|+|Q|^4}$ states and $2|Q|^2+2$ priorities, implying the complexity bound using state-of-the-art algorithms [18]. Actually the complexity is even quasi-polynomial according to the algorithms in [10]. Notice however that this has little impact on the complexity of the population control problem, as the number of priorities is logarithmic in the number of states of our parity game.

Further, it is well known that the winner of a parity game has a positional winning strategy [18]. A *positional* winning strategy σ in the game $\mathcal{PG}$ corresponds to a *finite-memory* winning strategy σ' in the capacity game, whose memory states are the states of $\mathcal{PG}$. Actually in order to play σ', it is enough to remember the tracking list, *i.e.* the third component of the state space of $\mathcal{PG}$. Indeed, the second component, in 2^Q, is redundant with the actual state of the capacity game and the bit in the first component is set to 1 when the play visits $\{f\}$ but in this case the capacity game is won by Controller whatever is played afterwards. Since there at most $2^{|Q|^4}$ different tracking lists, we get the upper bound on the memory. ◀

5 Lower bounds

The proofs of Theorems 4 and 5 are concluded by the proofs of lower bounds.

▶ **Theorem 19.** *The population control problem is* EXPTIME*-hard.*

Proof. We first prove PSPACE-hardness of the population control problem, reducing from the halting problem for polynomial space Turing machines. We then extend the result to obtain the EXPTIME-hardness, by reducing from the halting problem for polynomial space *alternating* Turing machines. Let $\mathcal{M} = (S, \Gamma, T, s_0, s_f)$ be a Turing machine with $\Gamma = \{0, 1\}$ as tape alphabet. By assumption, there exists a polynomial P such that, on initial configuration $x \in \{0,1\}^n$, $\mathcal{M}$ uses at most $P(n)$ tape cells. A transition $t \in T$ is of the form $t = (s, s', b, b', d)$, where s and s' are, respectively, the source and the target control states, b and b' are, respectively, the symbols read from and written on the tape, and $d \in \{\leftarrow, \rightarrow, -\}$ indicates the move of the tape head. From $\mathcal{M}$ and x, we build an NFA $\mathcal{A} = (Q, \Sigma, q_0, \Delta)$ with a distinguished state ☺ such that, $\mathcal{M}$ terminates in s_f on input x if and only if $(\mathcal{A}, ☺)$ is a positive instance of the population control problem.

The high-level description of $\mathcal{A}$ is as follows. States in Q are of several types: contents of the $P(n)$ cells (one state (b, p) per content and per position), position of the tape head (one state p per possible position), control state of the Turing machine (one state s per control state), and three special states, namely an initial state q_0, a sink winning state ☺, and a sink losing state ☹. With each transition $t = (s, s', b, b', d)$ in the Turing machine and each position p of the tape, we associate an action $a_{t,p}$ in $\mathcal{A}$, which simulates the effect of transition t when the head position is p. Thus, on action $a_{t,p}$ there is a transition from the source state s to the target state s', another from the tape head position p to its update according to d, and also from (b, p) to (b', p). Moreover, from head position $q \neq p$, $a_{t,p}$ leads to ☹, so that in any population game, Controller only plays actions associated with the current head position. Similarly from states (b'', p) with $b'' \neq b$, states $s'' \neq s$, action $a_{t,p}$ leads to ☹. Initially, an `init` action is available from q_0 and leads to s_0, to position 0 for the tape head, and to cells (b, p) that encode the initial tape contents on input x. The NFA also has winning actions, that allow one to check that there are no agents in a subset of states, and send the remaining ones to the target ☺. One such action should be played when agents encoding the state of the Turing machine lie in s_f, indicating that $\mathcal{M}$ accepted. Another winning action `win` is played whenever there are not enough agents to encode the initial configuration: Agents needs m to be at least $P(n) + 2$ to fill states corresponding to the initial tape contents ($P(n)$ tokens), the initial control state s_0 and the initial head position. The sink losing state ☹ is used to pinpoint an error in the simulation of $\mathcal{M}$.

Now, in order to encode an alternating Turing machine, we assume that the control states of $\mathcal{M}$ alternate between states of Controller and states of Agents. The NFA $\mathcal{A}$ is extended with a state C, for Controller, and an additional transition labelled `init` from q_0 to C. Assume first, that C contains at most an agent; we will later explain how to impose this. Beyond C, the NFA also contains on state t per transition of $\mathcal{M}$, which will represent that Agents chooses to play transition t. To do so, from state C, for any action $a_{t,p}$, there are transitions to all states t'. From state t, actions of the form $a_{t,p}$ are allowed, leading back to C. That is, actions $a_{t',p}$ with $t' \neq t$ lead from t to the sink losing state ☹. This encodes that Controller must follow the transition t chosen by Agents. To punish Agents in case the current tape contents is not the one expected by the transition $t = (s, s', b, b', d)$ he chooses, there are trashing actions $\texttt{trash}_s$ and $\texttt{trash}_{p,b}$ enabled from state t. Action $\texttt{trash}_s$ leads from t to ☺, and also from s to ☹. Similarly, $\texttt{trash}_{p,b}$ leads from t to ☺ and from any position state $q \neq p$ to ☹, and from (b, p) to ☹. In this way, Agents will not move the token

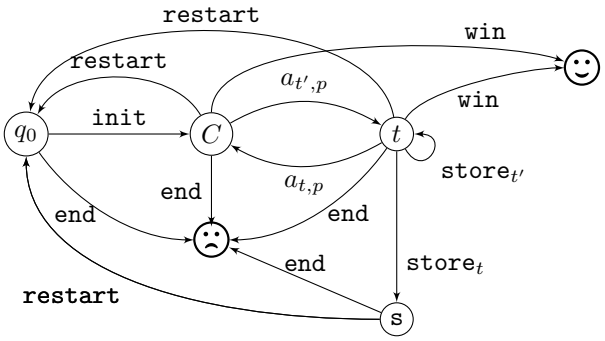


Figure 6 Gadget simulating a single agent in C.

from C to an undesired t. Last, there are transitions on action `end` from state ☺, C and any of the t's to the target state ☺. Moreover, action `end` from any other state (in particular the ones encoding the Turing machine configuration) leads to ☹. This whole construction encodes, assuming that there is a single agent in C after the first transition, that Controller can choose the transition from a Controller state of $\mathcal{M}$, and Agents can choose the transition from an Agents state.

Let us now explain how to deal with the case where Agents places several agents in state C on the initial action `init`, enabling the possibility to later send agents to several t's simultaneously. For that, consider the gadget from Figure 6. We use an extra state `s`, actions $\texttt{store}_t$ for each transition t, and action `restart`. Action $\texttt{store}_t$ leads from t to *store*, and loops on every other state. From all states except ☺ and ☹, action `restart` leads to q_0. Last, the effects of `win` and `end` are modified as follow: `win` leads from (non winning control states) $s \neq s_f$ to ☹ and loops on every other Turing machine , including s_f. It also leads from C and from any t to ☺; `end` goes from q_0, C, the t's and `s` to ☹ (it can be played only if all tokens from Figure 6 are in ☺), and leads all tokens from the Turing machine configuration to ☺.

Assume that input x is not accepted by the alternating Turing machine $\mathcal{M}$, and let m be at least $P(n) + 3$. In the m-population game, Agents has a winning strategy placing initially a single agent in state C. If Controller plays $\texttt{store}_t$ (for some t), either no agents are stored, or the unique agent in C is moved to `s`. Thus Controller cannot play `end` and has no way to lead the agents encoding the Turing machine configuration to ☺, until he plays `restart`, which moves all the agents back to q_0. This shows that $\texttt{store}_t$ is useless to Controller and thus Agents wins.

Conversely, if Controller has a strategy in $\mathcal{M}$ witnessing the acceptance of x, in order to win the m-population game, Agents would need to cheat in the simulation of $\mathcal{M}$ and place at least two agents in C to eventually split them to $t_1, \ldots, t_n$. Then, Controller can play the corresponding actions $\texttt{store}_{t_2}, \ldots, \texttt{store}_{t_n}$ moving all agents (but the ones in t_1) in `s`, after which he plays his winning strategy from t_1 resulting in sending some agents to ☺. Then, Controller plays `restart` and proceeds inductively with strictly less agents from q_0, and eventually plays `end` to win. ◀

Surprisingly, the cut-off can be as high as doubly exponential in the size of the NFA.

▶ **Proposition 20.** *There exists a family of NFA $(\mathcal{A}_n)_{n\in\mathbb{N}}$ such that $|\mathcal{A}_n| = 2n + 7$, and for $M = 2^{2^n+1} + n$, there is no winning strategy in $\mathcal{A}_n^M$ and there is one in $\mathcal{A}_n^{M-1}$.*

Proof. Let $n \in \mathbb{N}$. The NFA $\mathcal{A}_n$ we build is the disjoin union of *two* NFAs with different properties, namely $\mathcal{A}_{\mathsf{split}}, \mathcal{A}_{\mathsf{count},n}$. On the one hand, for $\mathcal{A}_{\mathsf{split}}$, winning the game with m

agents requires $\Theta(\log m)$ steps. On the other hand, $\mathcal{A}_{\mathsf{count},n}$ implements a usual counter over n bits (as used in many different publications), such that Controller can avoid to lose during $O(2^n)$ steps. The combination of these two gadgets ensures a cut-off for $\mathcal{A}_n$ of 2^{2^n}.

Recall Figure 1, which presents the splitting gadget that has the following properties. In $\mathcal{A}^m_{\mathsf{split}}$ with $m \in \mathbb{N}$ agents, $(s1)$ there is a winning strategy ensuring to win in $2\lfloor \log_2 m \rfloor + 2$ steps; $(s2)$ no strategy can ensure to win in less than $2\lfloor \log_2 m \rfloor + 1$ steps.

The counting gadget that implements a counter with states ℓ_i (meaning bit i is 0) and h_i (for bit i is 1) enjoys the following properties: $(c1)$ there is a strategy in $\mathcal{A}_{\mathsf{count},n}$ to ensure avoiding ☹ during 2^n steps, by playing α_i whenever the counter suffix from bit i is $01\cdots1$; $(c2)$ for $m \geq n$, no strategy of $\mathcal{A}^m_{\mathsf{count},n}$ avoid ☹ for 2^n steps.

The two gadgets (splitting and counting) are combined by a new initial state leading by two transitions labeled *init* to the initial states of both NFAs. Actions consist of pairs of actions, one for each gadget: $\Sigma = \{a, b, \delta\} \times \{\alpha_i \mid 1 \leq i \leq n\}$. We add an action $*$ which can be played from any state of $\mathcal{A}_{\mathsf{count},n}$ but ☹, and only from f in $\mathcal{A}_{\mathsf{split}}$, leading to the global target state ☺.

Let $M = 2^{2^n+1} + n$. We deduce that the cut-off is $M - 1$ as follows:

- For M agents, a winning strategy for Agents is to first split n tokens from the initial state to the q_0 of $\mathcal{A}_{\mathsf{count},n}$, in order to fill each l_i with 1 token, and 2^{2^n+1} tokens to the q_0 of $\mathcal{A}_{\mathsf{split}}$. Then Agents splits evenly tokens between q_1, q_2 in $\mathcal{A}_{\mathsf{split}}$. In this way, Controller needs at least $2^n + 1$ steps to reach the final state of $\mathcal{A}_{\mathsf{split}}$ $(s2)$, but Controller reachs ☹ after these $2^n + 1$ steps in $\mathcal{A}_{\mathsf{count},n}$ $(c2)$.
- For $M - 1$ agents, Agents needs to use at least n tokens from the initial state to the q_0 of $\mathcal{A}_{\mathsf{count},n}$, else Controller can win easily. But then there are less than 2^{2^n+1} tokens in the q_0 of $\mathcal{A}_{\mathsf{split}}$. And thus by $(s1)$, Controller can reach f within 2^n steps, after which he still avoids ☹ in $\mathcal{A}_{\mathsf{count},n}$ $(c1)$. And then Controller sends all agents to ☺ using $*$.

Thus, the family $(\mathcal{A}_n)$ of NFA exhibits a doubly exponential cut-off. ◀

6 Discussion

Obtaining an EXPTIME algorithm for the control problem of a population of agents was challenging. We also managed to prove a matching lower-bound. Further, the surprising doubly exponential matching upper and lower bounds on the cut-off imply that the alternative technique, checking that Controller wins all m-population game for m up to the cut-off, is far from being efficient.

The idealised formalism we describe in this paper is not entirely satisfactory: for instance, while each agent can move in a non-deterministic way, unrealistic behaviours can happen, *e.g.* all agents synchronously taking infinitely often the same choice. An almost-sure control problem in a probabilistic formalism should be studied, ruling out such extreme behaviours. As the population is discrete, we may avoid the undecidability that holds for distributions [11] and is inherited from the equivalence with probabilistic automata [17]. Abstracting continuous distributions by a discrete population of arbitrary size could thus be seen as an approximation technique for undecidable formalisms such as probabilistic automata.

Acknowledgements. We are grateful to Gregory Batt for fruitful discussions concerning the biological setting. Thanks to Mahsa Shirmohammadi for interesting discussions.

References

1 Parosh Abdulla, Giorgio Delzanno, Othmane Rezine, Arnaud Sangnier, and Riccardo Traverso. On the verification of timed ad hoc networks. In *Proceedings of Formats'11*, volume 6919 of *Lecture Notes in Computer Science*, pages 256–270. Springer, 2011.

2 Parosh Abdulla and Bengt Jonsson. Model checking of systems with many identical timed processes. *Theoretical Computer Science*, 290(1):241–263, 2003.

3 S. Akshay, Blaise Genest, Bruno Karelovic, and Nikhil Vyas. On regularity of unary probabilistic automata. In *Proceedings of STACS'16*, volume 47 of *Leibniz International Proceedings in Informatics*, pages 8:1–8:14. Leibniz-Zentrum für Informatik, 2016.

4 Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. In *Proceedings of PODC'04*, pages 290–299. ACM, 2004.

5 André Arnold, Aymeric Vincent, and Igor Walukiewicz. Games for synthesis of controllers with partial observation. *Theoretical Computer Science*, 1(303):7–34, 2003.

6 Nathalie Bertrand, Miheer Dewaskar, Blaise Genest, and Hugo Gimbert. Controlling a population. Technical report, HAL, 2017. URL: `https://hal.archives-ouvertes.fr/hal-01558029`.

7 Nathalie Bertrand and Paulin Fournier. Parameterized verification of many identical probabilistic timed processes. In *Proceedings of FSTTCS'13*, volume 24 of *Leibniz International Proceedings in Informatics*, pages 501–513. Leibniz-Zentrum für Informatik, 2013.

8 Nathalie Bertrand, Paulin Fournier, and Arnaud Sangnier. Playing with probabilities in reconfigurable broadcast networks. In *Proceedings of FoSSaCS'14*, volume 8412 of *Lecture Notes in Computer Science*, pages 134–148. Springer, 2014.

9 Tomás Brázdil, Petr Jančar, and Antonín Kučera. Reachability games on extended vector addition systems with states. In *Proceedings of ICALP'10*, volume 6199 of *Lecture Notes in Computer Science*, pages 478–489. Springer, 2010.

10 Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li, and Frank Stephan. Deciding parity games in quasipolynomial time. In *Proceedings of STOCS'17*, pages 252–263. ACM, 2017.

11 Laurent Doyen, Thierry Massart, and Mahsa Shirmohammadi. Infinite synchronizing words for probabilistic automata (erratum). Technical report, CoRR abs/1206.0995, 2012.

12 Laurent Doyen, Thierry Massart, and Mahsa Shirmohammadi. Limit synchronization in Markov decision processes. In *Proceedings of FoSSaCS'14*, volume 8412 of *Lecture Notes in Computer Science*, pages 58–72. Springer, 2014.

13 Javier Esparza. Keeping a crowd safe: On the complexity of parameterized verification (invited talk). In *Proceedings of STACS'14*, volume 25 of *Leibniz International Proceedings in Informatics*, pages 1–10. Leibniz-Zentrum für Informatik, 2014.

14 Javier Esparza, Alain Finkel, and Richard Mayr. On the verification of broadcast protocols. In *Proceedings of LICS'99*, pages 352–359. IEEE Computer Society, 1999.

15 Javier Esparza, Pierre Ganty, Jérôme Leroux, and Rupak Majumdar. Verification of population protocols. In *Proceedings of CONCUR'15*, volume 42 of *Leibniz International Proceedings in Informatics*, pages 470–482. Leibniz-Zentrum für Informatik, 2015.

16 Steven M. German and A. Prasad Sistla. Reasoning about systems with many processes. *J. ACM*, 39(3):675–735, 1992.

17 Hugo Gimbert and Youssouf Oualhadj. Probabilistic automata on finite words: Decidable and undecidable problems. In *Proceedings of ICALP'10*, volume 6199 of *Lecture Notes in Computer Science*, pages 527–538. Springer, 2010.

18 Marcin Jurdzinski. Small progress measures for solving parity games. In *Proceedings of STACS'00*, volume 1770 of *Lecture Notes in Computer Science*, pages 290–301. Springer, 2000.

19 Marcin Jurdziński, Ranko Lazić, and Sylvain Schmitz. Fixed-dimensional energy games are in pseudo polynomial time. In *Proceedings of ICALP'15*, volume 9135 of *Lecture Notes in Computer Science*, pages 260–272. Springer, 2015.

20 Panagiotis Kouvaros and Alessio Lomuscio. Parameterised Model Checking for Alternating-Time Temporal Logic. In *Proceedings of ECAI'16*, volume 285 of *Frontiers in Artificial Intelligence and Applications*, pages 1230–1238. IOS Press, 2016.

21 Pavel Martyugin. Computational complexity of certain problems related to carefully synchronizing words for partial automata and directing words for nondeterministic automata. *Theory of Computing Systems*, 54(2):293–304, 2014.

22 Mahsa Shirmohammadi. *Qualitative analysis of synchronizing probabilistic systems.* PhD thesis, ULB, 2014.

23 Jannis Uhlendorf, Agnès Miermont, Thierry Delaveau, Gilles Charvin, François Fages, Samuel Bottani, Pascal Hersen, and Gregory Batt. In silico control of biomolecular processes. In *Computational Methods in Synthetic Biology*, chapter 13, pages 277–285. Humana Press, Springer, 2015.

24 Mikhail V. Volkov. Synchronizing automata and the Černý conjecture. In *Proceedings of LATA'08*, volume 5196 of *Lecture Notes in Computer Science*, pages 11–27. Springer, 2008.

25 Wieslaw Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theor. Comput. Sci.*, 200(1-2):135–183, 1998. doi:10.1016/S0304-3975(98)00009-7.

The Robot Routing Problem for Collecting Aggregate Stochastic Rewards*

Rayna Dimitrova[1], Ivan Gavran[2], Rupak Majumdar[3], Vinayak S. Prabhu[4], and Sadegh Esmaeil Zadeh Soudjani[5]

1 **Max Planck Institute for Software Systems, Kaiserslautern, Germany**
Rayna@mpi-sws.org
2 **Max Planck Institute for Software Systems, Kaiserslautern, Germany**
Gavran@mpi-sws.org
3 **Max Planck Institute for Software Systems, Kaiserslautern, Germany**
Rupak@mpi-sws.org
4 **Max Planck Institute for Software Systems, Kaiserslautern, Germany**
Vinayak@mpi-sws.org
5 **Max Planck Institute for Software Systems, Kaiserslautern, Germany**
Sadegh@mpi-sws.org

Abstract

We propose a new model for formalizing reward collection problems on graphs with dynamically generated rewards which may appear and disappear based on a stochastic model. The *robot routing problem* is modeled as a graph whose nodes are stochastic processes generating potential rewards over discrete time. The rewards are generated according to the stochastic process, but at each step, an existing reward disappears with a given probability. The edges in the graph encode the (unit-distance) paths between the rewards' locations. On visiting a node, the robot collects the accumulated reward at the node at that time, but traveling between the nodes takes time. The optimization question asks to compute an optimal (or ϵ-optimal) path that maximizes the expected collected rewards.

We consider the finite and infinite-horizon robot routing problems. For finite-horizon, the goal is to maximize the total expected reward, while for infinite horizon we consider limit-average objectives. We study the computational and strategy complexity of these problems, establish NP-lower bounds and show that optimal strategies require memory in general. We also provide an algorithm for computing ϵ-optimal infinite paths for arbitrary $\epsilon > 0$.

1998 ACM Subject Classification I.2.8 Problem Solving, Control Methods, and Search

Keywords and phrases Path Planning, Graph Games, Quantitative Objectives, Discounting

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.13

1 Introduction

Reward collecting problems on metric spaces are at the core of many applications, and studied classically in combinatorial optimization under many well-known monikers: the traveling salesman problem, the knapsack problem, the vehicle routing problem, the orienteering problem, and so on. Typically, these problems model the metric space as a discrete graph whose nodes or edges constitute rewards, either deterministic or stochastic, and ask how to traverse the graph to maximize the collected rewards. In most versions of the problem,

* An extended version of this paper [9] is available at https://arxiv.org/abs/1704.05303.

rewards are either fixed or cumulative. In particular, once a reward appears, it stays there until collection. However, in many applications, existing rewards may disappear (e.g., a customer changing her mind) or have more "value" if they are collected fast.

We introduce the *Robot Routing problem*, which combines the spatial aspects of traveling salesman and other reward collecting problems on graphs with stochastic reward generation and with the possibility that uncollected rewards may disappear at each stage. The robot routing problem consists of a finite graph and a reward process for each node of the graph. The reward process models dynamic requests which appear and disappear. At each (discrete) time point, a new reward is generated for the node according to a stochastic process with expectation λ. However, at each point, a previously generated reward disappears with a fixed probability δ. When the node is visited, the entire reward is collected. The optimization problem for robot routing asks, given a graph and a reward process, what is the optimal (or ϵ-optimal) path a robot should traverse in this graph to maximize the expected reward?

As an illustrating example for our setting, consider a vendor planning her path through a city. At each street corner, and at each time step, a new customer arrives with expectation λ, and an existing customer leaves with probability δ. When the vendor arrives at the corner, she serves all the existing requests at once. We ignore other possible real-world features and behaviors e.g., customers leaving queues if the queue length is long. How should the vendor plan her path? Similar problems can be formulated for traffic pooling [25], for robot control [13], for patrolling [15], and many other scenarios.

Despite the usefulness of robot routing in many scenarios involving dynamic appearance and disappearance of rewards, algorithms for its solution have not, to the best of our knowledge, been studied before. In this paper, we study two optimization problems: the *value computation problem*, that asks for the maximal expected reward over a *finite* or *infinite* horizon, and the *path computation problem*, that asks for a path realizing the optimal (or ϵ-optimal) reward. The key observation to solving these problems is that the reward collection can be formulated as discounted sum problems over an extended graph, using the correspondence between stopping processes and discounted sum games.

For finite horizon robot routing we show that the value *decision* problem (deciding if the maximal expected reward is at least a certain amount) is NP-complete when the horizon bound is given in unary, and the value and optimal path can be computed in exponential time using dynamic programming.

For the infinite horizon problem, where the accumulated reward is defined as the long run average, we show that the value decision problem is NP-hard if the probability of a reward disappearing is more than a threshold dependent on the number of nodes. We show that computing the optimal long run average reward can be reduced to a 1-player mean-payoff game on an *infinite graph*. By solving the mean payoff game on a finite truncation of this graph, we can approximate the solution up to an arbitrary precision. This gives us an algorithm that, for any given ϵ, computes an ϵ-optimal path in time exponential in the size of the original graph and logarithmic in $1/\epsilon$. Unlike finite mean-payoff 2-player games, strategies which generate optimal paths for robot routing even in the 1-player setting can require memory. For the *non-discounted* infinite horizon problem (that is, when rewards do not disappear) we show that the optimal path and value problems are solvable in polynomial time.

Related work. The robot routing problem is similar in nature to a number of other problems studied in robot navigation, vehicle routing, patrolling, and queueing network control, but to the best of our knowledge has not been studied so far.

There exists a plethora of versions of the famous traveling salesman problem (TSP) which explore the trade-off between the cost of the constructed path and its reward. Notable

examples include the orienteering problem [23], in which the number of locations visited in a limited amount of time is to be maximized, vehicle routing with time-windows [17] and deadlines-TSP [2], which impose restrictions or deadlines on when locations should be visited, as well as discounted-reward-TSP [4] in which soft deadlines are implemented by means of discounting. Unlike in our setting, in all these problems, rewards are static, and there is no generation and accumulation of rewards, which is a key feature of our model.

In the dynamic version of vehicle routing [7] and the dynamic traveling repairman problem [3], tasks are dynamically introduced and the objective is to minimize the expected task waiting time. In contrast, we focus on limit-average objectives, which are a classical way to combine rewards over infinite system runs. Patrolling [6] is another graph optimization problem, motivated by operational security. The typical goal in patrolling is to synthesize a strategy for a defender against a single attack at an undetermined time and location, and is thus incomparable to ours. A single-robot multiple-intruders patrolling setting that is close to ours is described in [15], but there again the objective is to merely detect whether there is a visitor/intruder at a given room. Thus, the patrolling environment in [15] is described by the probability of detecting a visitor for each location. On the contrary, our model can capture *counting patrolling problems*, where the robot is required not only to detect the presence of visitors but to register/count as many of them as possible. Another related problem is the information gathering problem [20]. The key difference between the information gathering setting and ours is that [20] assumes that making an observation earlier has bigger value than if a lot of observations have already been made. This restriction on the reward function is *not* present in our model, since the reward value collected when visiting node v at time t (making observation (v, t), in their terms) only depends on the last time when v was previously visited, and not on the rest of the path (the other observations made, in their terms).

Average-energy games [8, 5] are a class of games on finite graphs in which the limit-average objective is defined by a double summation. The setting discussed in [8, 5] considers static edge weights and no discounting. Moreover, the inner sum in an average-energy objective is over the whole prefix so far, while in our setting the inner sum spans from the last to the current visit of the current node, which is a crucial difference between these two settings.

Finally, there is a rich body of work on multi-robot routing [24, 1, 18, 10, 11] which is closely related to our setting. However, the approaches developed there are limited to static tasks with fixed or linearly decreasing rewards. The main focus in the multi-robot setting is the task allocation and coordination between robots, which is a dimension orthogonal to the aggregate reward collection problem which we study.

Markov decision processes (MDP) [19] seem superficially close to our model. In an MDP, the rewards are determined statically as a function of the state and action. In contrast, the dynamic generation and accumulation of rewards in our model, especially the individual discounting of each generated reward, leads to algorithmic differences: for example, while MDPs admit memoryless strategies for long run average objectives, strategies require memory in our setting and there is no obvious reduction to, e.g., an exponentially larger, MDP.

We employed the reward structure of this article in [13] with the goal of synthesizing controllers for reward collecting Markov processes in continuous space. The work [13] is mainly focused on addressing the continuous dynamics of the underlying Markov process where the authors use abstraction techniques [12] to provide approximately optimal controllers with formal guarantees on the performance while maintaining the probabilistic nature of the process. In contrast, we tackle the challenges of this problem with having a deterministic graph as the underlying dynamical model of the robot and study the computational complexity of the proposed algorithms thoroughly.

Contributions. We define a novel optimization problem for formalizing and solving reward collection in a metric space where stochastic rewards appear as well as disappear over time.

- We consider reward-collection problems in a novel model with *dynamic generation* and *accumulation* of rewards, where each reward *can disappear with a given probability.*
- We study the value decision problem, the value computation problem, and the path computation problem over a finite horizon. We show that the value decision problem is NP-complete when the horizon is given in unary. We describe a dynamic programming approach for computing the optimal value and an optimal path in exponential time.
- We study the value decision problem, the value computation problem, and the path computation problem over an infinite horizon. We show that for sufficiently large values of the disappearing factor δ the value decision problem is NP-hard. We provide an algorithm which for any given $\epsilon > 0$, computes an ϵ-optimal path in time exponential in the size of the original graph and logarithmic in $1/\epsilon$. We demonstrate that strategies (in the 1-player robot routing games) which generate infinite-horizon optimal paths can require memory.

2 Problem Formulation

Preliminaries and notation. A finite directed graph $G = (V, E)$ consists of a finite set of nodes V and a set of edges $E \subseteq V \times V$. A path $\pi = v_0, v_1, \ldots$ in G is a finite or infinite sequence of nodes in G, such that $(v_i, v_{i+1}) \in E$ for each $i \geq 0$. We denote with $|\pi| = N$ the length (number of edges) of a finite path $\pi = v_0, v_1, \ldots, v_N$ and write $\pi[i] = v_i$ and $\pi[0 \ldots N] = v_0, v_1, \ldots v_N$. For an infinite path π, we define $|\pi| = \infty$. We also denote the cardinality of a finite set U by $|U|$. We denote by $\mathbb{N} = \{0, 1, ..\}$ and $\mathbb{Z}_+ = \{1, 2, ..\}$ the sets of non-negative and positive integers respectively. We define $\mathbb{Z}[n, m] = \{n, n+1, \ldots, m\}$ for any $n, m \in \mathbb{N}$, $n \leq m$. We denote with $\mathbb{I}(\cdot)$ the indicator function which takes a Boolean-valued expression as its argument and returns 1 if this expression evaluates to true and 0 otherwise.

Problem setting. Fix a graph $G = (V, E)$. We consider a discrete-time setting where at each time step $t \in \mathbb{N}$, at each node $v \in V$ a reward process generates rewards according to some probability distribution. Once generated, each reward at a node decays according to a decaying function. A *reward-collecting* robot starts out at some node $v_0 \in V$ at time $t = 0$, and traverses one edge in E at each time step. Every time the robot arrives at a node $v \in V$, it collects the reward accumulated at v since the last visit to v. Our goal is to compute the maximum expected reward that the robot can possibly collect, and to construct an optimal path for the robot in the graph, i.e., a path whose expected total reward is maximal.

To formalize reward accumulation, we define a function Last_π which (for path π) maps an index $t \leq |\pi|$ and a node $v \in V$ to the length of the path starting at the previous occurrence of v in π till position t; and to $t + 1$ if v does not occur in π before time t:

$$\mathsf{Last}_\pi(t, v) := \min\left(t + 1,\ \{t - j \in \mathbb{N} \mid j < t, \pi[j] = v\}\right).$$

Reward functions. Let $\xi : \Omega \times V \to \mathbb{R}$ be a set of random variables defined on a sample space Ω and indexed by the set of nodes V. Then $\xi(\cdot, v)$, $v \in V$, is a measurable function from Ω to $\mathbb{R}$ that generates a random reward at node v at any time step. Let π be the path in G traversed by the robot. At time t, the position of the robot is the node $\pi[t]$, and the robot collects the uncollected decayed reward generated at node $\pi[t]$ (since its last visit to $\pi[t]$) up till and including time t. Then, the robot traverses the edge $(\pi[t], \pi[t + 1])$, and at time $t + 1$ it collects the rewards at node $\pi[t + 1]$.

The uncollected reward at time t at a node v given a path π traversed by the robot is defined by the random variable

$$\mathsf{acc}_\pi(t,v) \;:=\; \sum_{j=0}^{\mathsf{Last}_\pi(t,v)-1} \gamma(v)^j \xi(w(t-j),v), \quad w(\cdot)\in\Omega.$$

The value $\gamma(v)$ in the above definition is a *discounting factor* that models the probability that a reward at node v survives for one more round, that is, the probability that a given reward instance at node v disappears at any step is $1-\gamma(v)$.

Note that the previous time a reward was collected at node v was at time $t-\mathsf{Last}_\pi(v,t)$, the time node v was last visited before t. Thus $\mathsf{acc}_\pi(t,v)$ corresponds to the rewards generated at node v at times t, $t-1,\ldots,t-\mathsf{Last}_\pi(t,v)+1$, which have decayed by factors of $\gamma(v)^0,\gamma(v)^1,\ldots,\gamma(v)^{\mathsf{Last}_\pi(t,v)-1}$, respectively. When traversing a path π, the robot collects the accumulated reward $\mathsf{acc}_\pi(t,\pi[t])$ at time t at node $\pi[t]$.

We define the *expected finite N-horizon sum reward* for a path π as:

$$r^{(N)}_{\mathsf{sum}}(\pi) \;:=\; \mathbb{E}\left[\sum_{t=0}^{N} \mathsf{acc}_\pi(t,\pi[t])\right].$$

Let $\lambda : V \to \mathbb{R}_{\geq 0}$ be a function that maps each node $v\in V$ to the *expected value of the reward generated at node v* for each time step, $\lambda(v)=\mathbb{E}\left[\xi(\cdot,v)\right]$. We assume that the rewards generated at each node are independent of the agent's move. Thus, the function λ will be sufficient for our study, since we have

$$r^{(N)}_{\mathsf{sum}}(\pi) \;=\; \sum_{t=0}^{N} \mathsf{Eacc}_\pi(t,\pi[t]), \text{ where } \mathsf{Eacc}_\pi(t,v) := \sum_{j=0}^{\mathsf{Last}_\pi(t,v)-1} \gamma(v)^j\lambda(v). \tag{1}$$

For an infinite path π, the *limit-average* expected reward is defined as

$$r_{\mathsf{av}}(\pi) \;=\; \liminf_{N\to\infty} \frac{r^{(N)}_{\mathsf{sum}}(\pi)}{N+1}. \tag{2}$$

The finite and infinite-horizon *reward values* for a node v are defined as the best rewards over all paths originating in v: $R^{(N)}_{\mathsf{sum}}(v)=\sup_\pi\left\{r^{(N)}_{\mathsf{sum}}(\pi)\,|\,\pi[0]=v,|\pi|=N\right\}$ and $R_{\mathsf{av}}(v)=\sup_\pi\left\{r_{\mathsf{av}}(\pi)\,|\,\pi[0]=v,|\pi|=\infty\right\}$, respectively. The choice of limit-average in (2) is due to the unbounded sum reward $r^{(N)}_{\mathsf{sum}}(\pi)$ when N goes to infinity. For a given path π, the sequence $r^{(N)}_{\mathsf{sum}}(\pi)/(N+1)$ in (2) may not converge. Thus we opt for the worst case limiting behavior of the sequence. Alternatively, one may select the best case limiting behavior $\limsup_{N\to\infty}$ in (2) with no substantial change in the results of this paper.

Node-invariant functions λ and γ and definition of cost functions. In the case when the functions λ and γ are constant, we write λ and γ for the respective constants. In this case, the expressions for $r^{(N)}_{\mathsf{sum}}(\pi)$ and $r_{\mathsf{av}}(\pi)$ can be simplified using the identity $1+\gamma+\gamma^2+\cdots+\gamma^{q-1}=\frac{1-\gamma^q}{1-\gamma}$ for $\gamma<1$. Then we have

$$\begin{aligned} r^{(N)}_{\mathsf{sum}}(\pi) &= \sum_{t=0}^{N}\sum_{j=0}^{\mathsf{Last}_\pi(\pi[t],t)-1}\gamma^j\lambda = \lambda\cdot\sum_{t=0}^{N}\left(1+\gamma+\ldots+\gamma^{\mathsf{Last}_\pi(t,\pi[t])-1}\right)\\ &= \lambda\cdot\sum_{t=0}^{N}\frac{1-\gamma^{\mathsf{Last}_\pi(t,\pi[t])}}{1-\gamma} = \frac{(N+1)\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma}\sum_{t=0}^{N}\gamma^{\mathsf{Last}_\pi(t,\pi[t])}. \end{aligned} \tag{3}$$

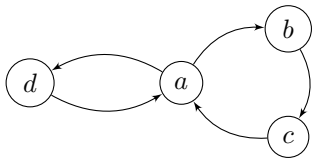


Figure 1 A graph $G_{\mathsf{e}} = (V_{\mathsf{e}}, E_{\mathsf{e}})$ with two simple cycles sharing a single node.

The expression $r_{\mathsf{av}}(\pi)$ can be simplified as:

$$r_{\mathsf{av}}(\pi) = \liminf_{N\to\infty} \frac{1}{N+1} r^{(N)}_{\mathsf{sum}}(\pi) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{t=0}^{N} \gamma^{\mathsf{Last}_\pi(t,\pi[t])}. \tag{4}$$

For the special case $\gamma = 1$ (i.e., when the rewards are not discounted), the expression for the finite-horizon reward is $r^{(N)}_{\mathsf{sum}}(\pi) = \lambda \sum_{t=0}^{N} \mathsf{Last}_\pi(t, \pi[t])$.

We define *cost functions* that map a path π to a real valued finite- or infinite-horizon cost:

$$c^{(N)}_{\mathsf{sum}}(\pi) := \sum_{t=0}^{N} \gamma^{\mathsf{Last}_\pi(\pi[t],t)} \quad \text{and} \quad c_{\mathsf{av}}(\pi) := \limsup_{N\to\infty} \frac{c^{(N)}_{\mathsf{sum}}(\pi)}{N+1}. \tag{5}$$

From Equations (3) and (4), the computation of optimal paths for the reward functions $r^{(N)}_{\mathsf{sum}}$ and r_{av} corresponds to computing paths that minimize the cost functions $c^{(N)}_{\mathsf{sum}}(\pi)$ and $c_{\mathsf{av}}(\pi)$, respectively. Analogously to $R^{(N)}_{\mathsf{sum}}(v)$ and $R_{\mathsf{av}}(v)$, the infimums of the cost functions in (5) over paths are denoted by $C^{(N)}_{\mathsf{sum}}(v)$ and $C_{\mathsf{av}}(v)$ respectively.

▶ **Example 1.** Consider the graph $G_{\mathsf{e}} = (V_{\mathsf{e}}, E_{\mathsf{e}})$ in Figure 1 with $V_{\mathsf{e}} = \{a, b, c, d\}$, which we will use as a running example throughout the paper. The functions λ and γ are constant.

Consider the finite path $\pi_1 = adabcad$. For the occurrences of node a in π_1 we have $\mathsf{Last}_{\pi_1}(0, a) = 1$, $\mathsf{Last}_{\pi_1}(2, a) = 2$, $\mathsf{Last}_{\pi_1}(5, a) = 3$, and similarly for the other nodes in π_1. The reward for π_1 as a function of λ and γ is $r^6_{\mathsf{sum}}(\pi_1) = \frac{7\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma}(\gamma+\gamma^2+\gamma^2+\gamma^4+\gamma^5+\gamma^3+\gamma^5)$ for $\gamma < 1$ and $r^6_{\mathsf{sum}}(\pi_1) = 22\lambda$ for $\gamma = 1$. For the infinite path $\pi_2 = (abc)^\omega$ we have $\mathsf{Last}_{\pi_2}(0, a) = 1$, $\mathsf{Last}_{\pi_2}(1, b) = 2$, $\mathsf{Last}_{\pi_2}(2, c) = 3$ and the value of Last is 3 in all other cases. Thus we have $r_{\mathsf{av}}(\pi_2) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma}\gamma^3$ for $\gamma < 1$ and $r_{\mathsf{av}}(\pi_2) = 3\lambda$ for $\gamma = 1$. Similarly, for $\pi_3 = (abcad)^\omega$ we have $r_{\mathsf{av}}(\pi_3) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma} \cdot \frac{(\gamma^2+\gamma^3+3\gamma^5)}{5}$ for $\gamma < 1$ and $r_{\mathsf{av}}(\pi_3) = 4\lambda$ for $\gamma = 1$.

Problem statements. We investigate optimization and decision problems for finite and infinite-horizon robot routing. The *value computation problems* ask for the computation of $R^{(N)}_{\mathsf{sum}}(v)$ and $R_{\mathsf{av}}(v)$. The corresponding decision problems asks to check if the respective one of these two quantities is greater than or equal to a given threshold $R \in \mathbb{R}$.

▶ **Definition 2** (Value Decision Problems)**.** Given a finite directed graph $G = (V, E)$, an expected reward function $\lambda : V \to \mathbb{R}_{\geq 0}$, a discounting function $\gamma : V \to (0, 1]$, an initial node $v_0 \in V$ and a threshold value $R \in \mathbb{R}$,

- The *finite horizon value decision problem* is to decide, given N, if $R^{(N)}_{\mathsf{sum}}(v_0) \geq R$.
- The *infinite horizon value decision problem* is to decide if $R_{\mathsf{av}}(v_0) \geq R$.

For a finite directed graph $G = (V, E)$, expected reward and discounting functions $\lambda : V \to \mathbb{R}_{\geq 0}$ and $\gamma : V \to (0, 1]$ and $v_0 \in V$, a finite path π is said to be an *optimal path* for time-horizon N if (a) $\pi[0] = v_0$ and $|\pi| = N$, and (b) for every path π' in G with $\pi'[0] = v_0$

and $|\pi'| = N$ it holds that $r^{(N)}_{\mathsf{sum}}(\pi) \geq r^{(N)}_{\mathsf{sum}}(\pi')$. Similarly, an infinite path π is said to be *optimal for the infinite horizon* if $\pi[0] = v_0$ and for every infinite path π' with $\pi'[0] = v_0$ in G it holds that $r_{\mathsf{av}}(\pi) \geq r_{\mathsf{av}}(\pi')$. We can also define corresponding *threshold paths*: given a value R a path π is said to be threshold R-optimal if $r^{(N)}_{\mathsf{sum}}(\pi) \geq R$ or $r_{\mathsf{av}}(\pi) \geq R$, respectively. An ϵ-*optimal* path is one which is $R^{(N)}_{\mathsf{sum}}(v_0) - \epsilon$ or $R_{\mathsf{av}}(v_0) - \epsilon$ threshold optimal (for finite or infinite horizon respectively).

▶ **Example 3.** Consider again the graph G_{e} shown in Figure 1. Examining the expressions computed in Example 1, we have that $r_{\mathsf{av}}(\pi_2) > r_{\mathsf{av}}(\pi_3)$ for $\gamma = 0.1$ and $r_{\mathsf{av}}(\pi_3) > r_{\mathsf{av}}(\pi_2)$ for $\gamma = 0.9$. Thus, in general, the optimal value depends on γ. Due to the structure of the set of infinite paths in G_{e} we can analytically compute the optimal value $R_{\mathsf{av}}(v)$ for each $v \in V_{\mathsf{e}}$ as a function of γ and a corresponding optimal path (see the full version [9] for the proof):

- if $\gamma \in [0, a_1]$, then $R_{\mathsf{av}}(v) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma}\gamma^3$ and the path $(abc)^\omega$ is optimal;
- if $\gamma \in [a_1, a_2]$, then $R_{\mathsf{av}}(v) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma} \cdot \frac{(\gamma^2+4\gamma^3+2\gamma^5+\gamma^8)}{8}$ and $(abcabcad)^\omega$ is optimal;
- if $\gamma \in [a_2, 1]$, then $R_{\mathsf{av}}(v) = \frac{\lambda}{1-\gamma} - \frac{\lambda}{1-\gamma} \cdot \frac{(\gamma^2+\gamma^3+3\gamma^5)}{5}$ and the path $(abcad)^\omega$ is optimal.

The constants $a_1 \approx 0.2587$ and $a_2 \approx 0.2738$ are respectively the unique real roots of polynomials $\gamma^6+2\gamma^3-4\gamma+1$ and $5\gamma^6-14\gamma^3+12\gamma-3$ in the interval $(0, 1)$. Note that for $\gamma = 1$ we have $R_{\mathsf{av}}(v) = 4\lambda$ which is achieved by $(abcad)^\omega$. The path $(abc)(ad)(abc)^2(ad)^2 \ldots (abc)^n(ad)^n \ldots$, which is not ultimately periodic, also achieves the optimal reward.

Paths as strategies. We often refer to infinite paths as resulting from strategies (of the collecting agent). A *strategy* σ in G is a function that maps finite paths $\pi[0 \ldots m]$ to nodes such that if $\sigma(\pi)$ is defined then $(\pi[m], \sigma(\pi)) \in E$. Given an initial node v_0, the strategy σ generates a unique infinite path π, denoted as $\mathsf{outcome}(v_0, \sigma)$. Thus, every infinite path $\pi = v_0, v_1, \ldots$ defines a unique strategy σ_π where $\sigma_\pi(\pi[0 \ldots i]) = v_{i+1}$, and $\sigma_\pi(\epsilon) = v_0$, and σ_π is undefined otherwise. Clearly, $\mathsf{outcome}(v_0, \sigma_\pi) = \pi$. We say a strategy σ is optimal for a path problem if the path $\mathsf{outcome}(v_0, \sigma)$ is optimal. A strategy σ is *memoryless* if for every two paths $\pi'[0 \ldots m'], \pi''[0 \ldots m'']$ for which $\pi'[m'] = \pi''[m'']$, it holds that $\sigma(\pi') = \sigma(\pi'')$. We say that memoryless strategies suffice for the optimal path problem if there always exists a memoryless strategy σ such that $\mathsf{outcome}(v_0, \sigma)$ is an optimal path.

3 Finite Horizon Rewards: Computing $R^{(N)}_{\mathsf{sum}}(v)$

In this section we consider the finite-horizon problems associated with our model. The following theorem summarizes the main results.

▶ **Theorem 4.** *Given $G = (V, E)$, expected reward and discounting functions, node $v \in V$, and horizon $N \in \mathbb{N}$:*

1. *The finite-horizon value decision problem is NP-complete if N is in unary.*
2. *The value $R^{(N)}_{\mathsf{sum}}(v)$ for $v \in V$ is computable in exponential time even if N is in binary.*

Analogous results hold for the related reward problem where in addition to the initial node v, we are also given a destination node v_f, and the objective is to go from v to v_f in at most N steps while maximizing the reward.

The finite-horizon value problem is NP-hard by reduction from the Hamiltonian path problem (see the full version [9]), even in the case of node-invariant λ and γ. Membership in NP in case N is in unary follows from the fact that we can guess a path of length N and check that the reward for that path is at least the desired threshold value.

To prove the second part of the theorem, we construct a finite *augmented weighted graph*.

For simplicity, we give the proof for node-invariant λ and γ, working with the cost functions c_{sum} and c_{av}. The augmented graph construction in the general case is a trivial generalization by changing the weights of the nodes, and the dynamic programming algorithm used for computing the optimal cost values is easily modified to compute the corresponding reward values instead. For $\gamma < 1$ the objective is to minimize $c^{(N)}_{\mathsf{sum}}(\pi) = \sum_{t=0}^{N} \gamma^{\mathsf{Last}_\pi(t,\pi[t])}$ and for $\gamma = 1$ the objective is to maximize $r^{(N)}_{\mathsf{sum}}(\pi) = \lambda \sum_{t=0}^{N} \mathsf{Last}_\pi(t, \pi[t])$ over paths π.

Augmented weighted graph. Given a finite directed graph $G = (V, E)$ we define the *augmented weighted graph* $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ which "encodes" the values $\mathsf{Last}_\pi(t, v)$ for the paths in G explicitly in the augmented graph node. We can assume w.l.o.g. that $V = \{1, 2, \ldots, |V|\}$.

- The set of nodes $\widetilde{V}$ is $V \times \mathbb{Z}_+^{|V|}$ (the set $\widetilde{V}$ is infinite). A node $(v, b_1, b_2, \ldots, b_{|V|}) \in \widetilde{V}$ represents the fact that the current node is v, and that for each node $u \in V$ the last visit to u (before the current time) was b_u time units before the current time.
- The weight of a node $(v, b_1, b_2, \ldots, b_{|V|}) \in \widetilde{V}$ is γ^{b_v}.
- The set of edges $\widetilde{E}$ consists of edges $(v, b_1, b_2, \ldots, b_{|V|}) \;\to\; (v', b'_1, b'_2, \ldots, b'_{|V|})$ such that $(v, v') \in E$; and $b'_v = 1$, and $b'_u = b_u + 1$ for all $u \neq v$.

Let π be a path in G. In the graph $\widetilde{G}$ there exists a unique path $\widetilde{\pi}$ that corresponds to π:

$$\widetilde{\pi} = (\pi[0], 1, 1, \ldots, 1)\,,\ (\pi[1], b^1_1, b^1_2, \ldots, b^1_{|V|})\,,\ (\pi[2], b^2_1, b^2_2, \ldots, b^2_{|V|})\,, \ldots \tag{6}$$

starting from the node $(\pi[0], 1, 1, \ldots, 1)$ such that for all t and for all $v \in V$, we have $\mathsf{Last}_\pi(t, v) = b^t_v$. Dually, for each path $\widetilde{\pi}$ in $\widetilde{G}$ starting from $(v_0, 1, 1, \ldots, 1)$, there exists a unique path π in G from the node v_0 such that $\mathsf{Last}_\pi(t, v) = b^t_v$ for all t and v.

For a path $\widetilde{\pi}$ in the form of (6) let

$$\widetilde{c}^{(N)}_{\mathsf{sum}}(\widetilde{\pi}) := \sum_{t=0}^{N} \gamma^{b^t_{v_t}} \quad \text{and} \quad \widetilde{c}_{\mathsf{av}}(\widetilde{\pi}) := \limsup_{N\to\infty} \frac{1}{N+1} \sum_{t=0}^{N} \gamma^{b^t_{v_t}}.$$ Observe that:

- $\widetilde{c}^{(N)}_{\mathsf{sum}}(\widetilde{\pi})$ is the sum of weights associated with the first $N+1$ nodes of $\widetilde{\pi}$.
- $\widetilde{c}_{\mathsf{av}}(\widetilde{\pi})$ is the limit-average of the weights associated with the nodes of $\widetilde{\pi}$.

Thus, $\widetilde{c}^{(N)}_{\mathsf{sum}}$ and $\widetilde{c}_{\mathsf{av}}$ define the classical total finite sum (shortest paths) and limit average objectives on weighted (infinite) graphs [26]. Additionally, $\widetilde{c}^{(N)}_{\mathsf{sum}}(\widetilde{\pi}) = c^{(N)}_{\mathsf{sum}}(\pi)$, and $\widetilde{c}_{\mathsf{av}}(\widetilde{\pi}) = c_{\mathsf{av}}(\pi)$ where π is the path in G corresponding to the path $\widetilde{\pi}$.

Now, define $\widetilde{C}^{(N)}_{\mathsf{sum}}((v_0, 1, 1, \ldots, 1))$ as the infimum of $\widetilde{c}^{(N)}_{\mathsf{sum}}(\widetilde{\pi})$ over all paths $\widetilde{\pi}$ with $\widetilde{\pi}[0] = (v_0, 1, 1, \ldots, 1)$, and similarly for $\widetilde{C}_{\mathsf{av}}((v_0, 1, 1, \ldots, 1))$. Then it is easy to see that $C^{(N)}_{\mathsf{sum}}(v_0) = \widetilde{C}^{(N)}_{\mathsf{sum}}((v_0, 1, 1, \ldots, 1))$ and $C_{\mathsf{av}}(v_0) = \widetilde{C}_{\mathsf{av}}((v_0, 1, 1, \ldots, 1))$. Thus, we can reduce the optimal path and value problems for G to standard objectives in $\widetilde{G}$. The major difficulty is that $\widetilde{G}$ is infinite. However, note that only the first $N+1$ nodes of $\widetilde{\pi}$ are relevant for the computation of $\widetilde{c}^{(N)}_{\mathsf{sum}}(\widetilde{\pi})$. Thus, the value of $\widetilde{C}^{(N)}_{\mathsf{sum}}((v_0, 1, \ldots, 1))$ can be computed on a *finite* subgraph of $\widetilde{G}$, obtained by considering only the finite subset of nodes $V \times \mathbb{Z}[1, N+1]^{|V|} \subseteq \widetilde{V}$.

For $\gamma < 1$, we obtain the value $\widetilde{C}^{(N)}_{\mathsf{sum}}(\widetilde{\pi})$ by a standard dynamic programming algorithm which computes the shortest path of length N on this finite subgraph starting from the node $(v_0, 1, 1, \ldots, 1)$ (and keeping track of the number of steps). For $\gamma = 1$, where the objective is to maximize $r^{(N)}_{\mathsf{sum}}(\pi) = \lambda \sum_{t=0}^{N} \mathsf{Last}_\pi(t, \pi[t])$ over paths π, we proceed analogously. Note that the subgraph used for the dynamic programming computations is of exponential size in terms of the size of G and the description of N. This gives the desired result in Theorem 4.

4 Infinite Horizon Rewards: Computing $R_{\mathsf{av}}(v)$

Since we consider finite graphs, every infinite path eventually stays in some strongly connected component (SCC). Furthermore, the value of the reward function $r_{\mathsf{av}}(\pi)$ does not change if we alter/remove a finite prefix of the path π. Thus, it suffices to restrict our attention to the SCCs of the graph: the problem of finding an optimal path from a node $v \in V$ reduces to finding the SCC that gives the highest reward among the SCCs reachable from node v. Therefore, we assume that the graph is strongly connected.

4.1 Hardness of Exact $R_{\mathsf{av}}(v)$ Value Computation

Since it is sufficient for the hardness results, we consider node-invariant λ and γ.

Insufficiency of memoryless strategies. Before we turn to the computational hardness of the value decision problem, we look at the *strategy complexity* of the optimal path problem and show that optimal strategies need memory.

▶ **Proposition 5.** *Memoryless strategies do not suffice for the infinite horizon problem.*

Proof. Consider Example 3. A memoryless strategy results in paths which cycle exclusively either in the left cycle, or the right cycle (as from node a, it prescribes a move to either only b or only d). As shown in Example 3, the optimal path for $\gamma \geq a_1$ needs to visit both cycles. Thus, memoryless strategies do not suffice for this example. ◀

For ω-regular objectives, strategies based on *latest visitation records* [14, 22], which depend only on the *order* of the last node visits (*i.e.*, for all node pairs $v_1 \neq v_2 \in V$, whether the last visit of v_1 was before that of v_2 or vice versa) are sufficient. However, we can show that such strategies do not suffice either. To see this, recall the graph in Figure 1 for which the optimal path for $\gamma = 0.26$ is $(abcabcad)^\omega$. Upon visiting node a this strategy chooses one of the nodes b or d depending on the *number* of visits to b since the last occurrence of d. On the other hand, every strategy based *only on the order of last visits* is not able to count the number of visits to b and thus, results in a path that ends up in one of $(abc)^\omega$, or $(ad)^\omega$, or $(abcad)^\omega$, which are not optimal for this γ. The proof is given in the full version [9]. It is open if finite memory strategies are sufficient for the infinite-horizon optimal path problem.

NP-Hardness of the value decision problem. To show NP-hardness of the infinite-horizon value decision problem, we first give bounds on $R_{\mathsf{av}}(v_0)$. The following Lemma proven in the full version [9], establishes these bounds.

▶ **Lemma 6.** *For any graph $G = (V, E)$ and any node $v_0 \in V$, $R_{\mathsf{av}}(v_0)$ is bounded as*

$$\lambda \frac{1-\gamma^p}{1-\gamma} \leq R_{\mathsf{av}}(v_0) \leq \lambda \frac{1-\gamma^{n_{\mathsf{v}}}}{1-\gamma}, \tag{7}$$

where $n_{\mathsf{v}} = |V|$ and p is the length of the longest simple cycle in G.

▶ **Corollary 7.** *If the graph $G = (V, E)$ contains a Hamiltonian cycle, any path $\pi = (v_1 v_2 \dots v_{|V|})^\omega$, with $v_1 v_2 \dots v_{|V|} v_1$ being a Hamiltonian cycle, is optimal and the optimal value of $R_{\mathsf{av}}(v_0)$ is exactly the upper bound in (7).*

The following lemma establishes a relationship between the value of optimal paths and the existence of a Hamiltonian cycle in the graph, and is useful for providing a lower bound on the computational complexity of the value decision problem.

▶ **Lemma 8.** *If the upper bound in* (7) *is achieved with a path* π *for some* $\gamma < 1/|V|$*, then the graph contains a Hamiltonian cycle* ρ *that occurs in* π *infinitely often.*

Proof. The proof is by contradiction. Suppose π does not visit any Hamiltonian cycle infinitely often. Then it visits each such cycle at most a finite number of times. Without loss of generality we can assume that the path doesn't visit any such cycles, since the total number of Hamiltonian cycles is finite. We have for $n_{\mathsf{v}} = |V|$

$$c^{(N)}_{\mathsf{sum}}(\pi) \geq \sum_{t=0}^{N} \gamma^{\mathsf{Last}_\pi(t,\pi[t])} \mathbb{I}(\mathsf{Last}_\pi(t,\pi[t]) < n_{\mathsf{v}}) \geq \gamma^{n_{\mathsf{v}}-1} \sum_{t=0}^{N} \mathbb{I}(\mathsf{Last}_\pi(t,\pi[t]) < n_{\mathsf{v}}).$$

Now let's look at $\pi_{n,n_{\mathsf{v}}} = \pi[n(n+1)\dots(n+n_{\mathsf{v}})]$ a finite sub-path of length n_{v}. There is at least one node repeated in $\pi_{n,n_{\mathsf{v}}}$ since the graph has n_{v} distinct nodes. Note that if $\pi[n] = \pi[n+n_{\mathsf{v}}]$, there must be another repetition due to the lack of Hamiltonian cycles in the path. In either case, there is an $i_n \in \mathbb{Z}[n, n+n_{\mathsf{v}}]$ such that $\mathsf{Last}_\pi(i_n, \pi[i_n]) < n_{\mathsf{v}}$.

$$c^{(N)}_{\mathsf{sum}}(\pi) \geq \gamma^{n_{\mathsf{v}}-1} \sum_{t=0}^{N} \mathbb{I}(\mathsf{Last}_\pi(t,\pi[t]) < n_{\mathsf{v}}) \geq \gamma^{n_{\mathsf{v}}-1} \left\lfloor \frac{N}{n_{\mathsf{v}}} \right\rfloor$$

$$\Rightarrow c_{\mathsf{av}}(\pi) \geq \gamma^{n_{\mathsf{v}}-1} \limsup_{N\to\infty} \frac{1}{N+1} \left\lfloor \frac{N}{n_{\mathsf{v}}} \right\rfloor = \gamma^{n_{\mathsf{v}}-1} \frac{1}{n_{\mathsf{v}}} \Rightarrow \gamma^{n_{\mathsf{v}}} \geq \left(\gamma^{n_{\mathsf{v}}-1}\right)/n_{\mathsf{v}} \;\Rightarrow\; \gamma \geq 1/n_{\mathsf{v}},$$

which is a contradiction with the assumption that $\gamma < 1/n_{\mathsf{v}}$. Then a necessary condition for $c_{\mathsf{av}}(\pi) = \gamma^{n_{\mathsf{v}}}$ with some $\gamma < 1/n_{\mathsf{v}}$ is the existence of a Hamiltonian cycle. ◀

▶ Remark. Following the same reasoning of the above proof it is possible to improve the upper bound in (7) as $R_{\mathsf{av}}(v_0) \leq \lambda\left(1-\gamma^p/n_{\mathsf{v}}\right)/(1-\gamma)$ for small values of γ, where p is the length of the longest simple cycle of the graph.

▶ **Theorem 9.** *The infinite horizon value decision problem is NP-hard for* $\gamma < 1/|V|$*.*

Proof. We reduce the Hamiltonian cycle problem to the infinite horizon optimal path problem. Given a graph $G = (V, E)$, we fix some λ and $\gamma < 1/|V|$. We show that G is Hamiltonian iff $R_{\mathsf{av}}(v_0) \geq \lambda\left(1-\gamma^{|V|}\right)/(1-\gamma)$. If G has a Hamiltonian cycle $v_1 v_2 \dots v_{|V|} v_1$, then the infinite path $\pi = (v_1 v_2 \dots v_{|V|})^\omega$ has reward $r_{\mathsf{av}}(\pi) = \lambda\left(1-\gamma^{|V|}\right)/(1-\gamma)$, for any choice of γ. For the other direction, applying Lemma 8 with $\gamma < 1/|V|$ implies that G is Hamiltonian. ◀

Non-discounted rewards ($\gamma = 1$) and node-invariant function λ. Contrary to the finite-horizon non-discounted case, the infinite-horizon optimal path and value problems for $\gamma = 1$ can be solved in polynomial time. To see this, note that the reward expression $r^{(N)}_{\mathsf{sum}}(\pi)$ can be written as $r^{(N)}_{\mathsf{sum}}(\pi) = \lambda \sum_{v\in V} y(v,\pi,N)$, where $y(v,\pi,N)$ is defined as $y(v,\pi,N) = 1 + \max U$, for $U = \{j \in \mathbb{N} | j \leq N, \pi[j] = v\} \cup \{-1\}$. Then, we can bound the reward by

$$r_{\mathsf{av}}(\pi) \leq \lim_{N\to\infty} \frac{\lambda \sum_{i=1}^{|V(\pi)|}(N+1-i+1)}{N+1} = \lim_{N\to\infty} \frac{\lambda|V(\pi)|(2N-|V(\pi)|+3)}{2(N+1)} = \lambda|\mathsf{Inf}(\pi)|,$$

where $\mathsf{Inf}(\pi)$ is the set of nodes visited in the path π infinitely often. This indicates that the maximum reward is bounded by λ times the maximal size of a reachable SCC in the graph G. This upper bound is also achievable: we can construct an optimal path by finding a maximal SCC reachable from the initial node and a (not necessarily simple) cycle $v_1 \dots v_n v_1$ that visits all the nodes in this SCC. Then, a subset of optimal paths contains paths of the form $\pi_0 \cdot (v_1 \dots v_n)^\omega$, where π_0 is any finite path that reaches v_1. This procedure can be done with a computational time polynomial in the size of G.

▶ **Example 10.** Consider the graph in Figure 1. The optimal reward for the infinite-horizon non-discounted case is 4λ, achievable by the path $\pi = (abcad)^\omega$. Another path which is not ultimately periodic but achieves the same reward $\pi' = (abc)(ad)(abc)^2(ad)^2 \dots (abc)^n(ad)^n \dots$.

Note that for the case $\gamma = 1$ there always exists an ultimately periodic optimal path, such a path is generated by a finite-memory strategy.

4.2 Approximate Computation of $R_{\mathsf{av}}(v)$

In the previous section we discussed how to solve the infinite-horizon value and path computation problems for the non-discounted case. Now we show how the infinite-horizon path and value computation problems for $\gamma < 1$ can be effectively approximated. We first define functions that over and underapproximate $C_{\mathsf{av}}(v)$ (thus also $R_{\mathsf{av}}(v)$) and establish bounds on the error of these approximations. Given an integer $K \in \mathbb{N}$, approximately optimal paths and an associated interval approximating $C_{\mathsf{av}}(v)$ can be computed using a finite augmented graph $\widetilde{G}_K$ based on the augmented graph $\widetilde{G}$ of Section 3. Intuitively, $\widetilde{G}_K$ is obtained from $\widetilde{G}$ by pruning nodes that have a component greater than K in their augmentation. By increasing the value of K, the approximation error can be made arbitrarily small.

We describe the approximation algorithm for node-invariant λ and γ. The results generalize trivially to the case when λ and γ are not node-invariant by choosing K large enough to satisfy the condition that bounds the approximation error for each $\lambda(v)$ and $\gamma(v)$.

Approximate cost functions. Consider the following functions from $V^* \times \mathbb{N}$ to $\mathbb{R}_{\geq 0}$:

$$\bar{c}^{(N)}_{\mathsf{sum}}(\pi, K) := \sum_{t=0}^{N} \gamma^{\min\{K, \mathsf{Last}_\pi(t,\pi[t])\}} \quad \text{and}$$

$$\underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) := \sum_{t=0}^{N} \gamma^{\mathsf{Last}_\pi(t,\pi[t])} \mathbb{I}(\mathsf{Last}_\pi(t, \pi[t]) \leq K).$$

Informally, for $\bar{c}^{(N)}_{\mathsf{sum}}(\pi, K)$, if the last visit to node $\pi[t]$ occurred more than K time units before time t, the cost is γ^K, rather than the original smaller amount $\gamma^{\mathsf{Last}_\pi(t,\pi[t])}$. For $\underline{c}^{(N)}_{\mathsf{sum}}(\pi, K)$, if the last visit to $\pi[t]$ occurred more than K time steps before time t, then the cost is 0. For both, if the last visit to the node $\pi[t]$ occurred less than or equal to K steps before, we pay the actual cost $\gamma^{\mathsf{Last}_\pi(t,\pi[t])}$. The above definition implies that $\underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) \leq c^{(N)}_{\mathsf{sum}}(\pi) \leq \bar{c}^{(N)}_{\mathsf{sum}}(\pi, K)$ for every π. Then we have $\underline{C}(v_0, K) \leq C_{\mathsf{av}}(v_0) \leq \overline{C}(v_0, K)$, where we define

$$\underline{C}(v_0, K) := \inf_{\pi, \pi[0]=v_0} \limsup_{N\to\infty} \frac{\underline{c}^{(N)}_{\mathsf{sum}}(\pi, K)}{N+1} \quad \text{and}$$

$$\overline{C}(v_0, K) := \inf_{\pi, \pi[0]=v_0} \limsup_{N\to\infty} \frac{\bar{c}^{(N)}_{\mathsf{sum}}(\pi, K)}{N+1}.$$

The difference between the upper and lower bounds can be tuned by selecting K:

$$\begin{aligned}
&\bar{c}^{(N)}_{\mathsf{sum}}(\pi, K) - \underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) = \sum_{t=0}^{N} \gamma^K \mathbb{I}(\mathsf{Last}_\pi(t, \pi[t]) \geq K+1) \\
&\implies 0 \leq \bar{c}^{(N)}_{\mathsf{sum}}(\pi, K) - \underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) \leq (N+1)\gamma^K \\
&\implies \underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) \leq \bar{c}^{(N)}_{\mathsf{sum}}(\pi, K) \leq \underline{c}^{(N)}_{\mathsf{sum}}(\pi, K) + (N+1)\gamma^K \\
&\implies \underline{C}(v_0, K) \leq \overline{C}(v_0, K) \leq \underline{C}(v_0, K) + \gamma^K.
\end{aligned}$$

Therefore $C_{\mathsf{av}}(v_0)$ belongs to the interval $[\underline{C}(v_0,K),\overline{C}(v_0,K)] \subset [0,\gamma^K]$ and the length of the interval is at most γ^K. In order to guarantee the total error of $\epsilon > 0$ for the actual reward $R_{\mathsf{av}}(v_0)$[1], we can select $K \in \mathbb{N}$ according to $\frac{\lambda}{1-\gamma}\gamma^K \le \epsilon \Longrightarrow K \ge \ln\left[\frac{\epsilon(1-\gamma)}{\lambda}\right] / \ln\gamma$. The value $C_{\mathsf{av}}(v_0)$ can be computed up to the desired degree of accuracy by computing either $\overline{C}(v_0,K)$ or $\underline{C}(v_0,K)$. Next, we give the procedure for computing $\overline{C}(v_0,K)$ (the procedure for $\underline{C}(v_0,K)$ is similar). It utilizes a finite augmented weighted graph $\widetilde{G}_K$.

Truncated augmented weighted graph $\widetilde{G}_K$. Recall the infinite augmented weighted graph $\widetilde{G}$ from Section 3. We define a truncated version $\widetilde{G}_K = (\widetilde{V}_K, \widetilde{E}_K)$ of $\widetilde{G}$ where we only keep track of $\mathsf{Last}_\pi(t,v)$ values less than or equal to K. For $\widetilde{G}_K$ we define two weight functions $\overline{w}$ and $\underline{w}$, for $\bar{c}^{(N)}_{\mathsf{sum}}$ and $\underline{c}^{(N)}_{\mathsf{sum}}$ respectively.

- The set of nodes $\widetilde{V}_K$ is $V \times \mathbb{Z}[0,K]^{|V|}$.
- For a node $\widetilde{v} = (v, b_1, b_2, \ldots, b_{|V|}) \in \widetilde{V}$,
  - the weight $\overline{\omega}(\widetilde{v})$ is γ^{b_v} if $b_v > 0$ and γ^K otherwise.
  - the weight $\underline{\omega}(\widetilde{v})$ is γ^{b_v} if $b_v > 0$ and 0 otherwise.
- The set of edges $\widetilde{E}_K$ consists of edges $(v, b_1, b_2, \ldots, b_{|V|}) \to (v', b'_1, b'_2, \ldots, b'_{|V|})$ such that $(v \to v') \in E$, $b'_v = 1$, and for $u \neq v$:
  - $b'_u = b_u + 1$ if $b_u > 0$ and $b_u + 1 \le K$,
  - $b'_u = 0$ if $b_u = 0$ or $b_u + 1 > K$.

  Thus, in $\widetilde{G}_K$ we specify two different weights for path π at time step t in the case when the previous visit to $\pi[t]$ in π was more than K time steps ago.

Similarly to the infinite augmented graph we have

- $\bar{c}^{(N)}_{\mathsf{sum}}(\pi,K)$ is the sum of weights assigned by $\overline{w}$ to the first $(N+1)$ nodes of $\widetilde{\pi}$.
- $\underline{c}^{(N)}_{\mathsf{sum}}(\pi,K)$ is the sum of weights assigned by $\underline{w}$ to the first $(N+1)$ nodes of $\widetilde{\pi}$.

It is easy to see that $\overline{C}(v_0,K)$ is the least possible limit-average cost with respect to $\overline{w}$ in $\widetilde{G}_K$ starting from the node $\widetilde{v}_0 = (v_0, 1, 1, \ldots, 1)$. The same holds for $\underline{C}(v_0,K)$ with $\underline{w}$. Below we show how to compute $\overline{C}(v_0,K)$. The case $\underline{C}(v_0,K)$ is analogous, and thus omitted.

Algorithm for computing $\overline{C}(v_0,K)$. We now describe a method to compute $\overline{C}(v_0,K)$ as the least possible limit-average cost in $\widetilde{G}_K$ with respect to $\overline{w}$. It is well-known that this can be reduced to the computation of the minimum cycle mean in the weighted graph [26], which in turn can be done using the algorithm from [16] that we now describe.

As before, first we assume that $\widetilde{G}_K$ is strongly connected. For every $\widetilde{v} \in \widetilde{V}_K$, and every $n \in \mathbb{Z}_+$, we define $W_n(\widetilde{v})$ as the minimum weight of a path of length n from $\widetilde{v}_0$ to $\widetilde{v}$; if no such path exists, then $W_n(\widetilde{v}) = \infty$. The values $W_n(\widetilde{v})$ can be computed by the recurrence

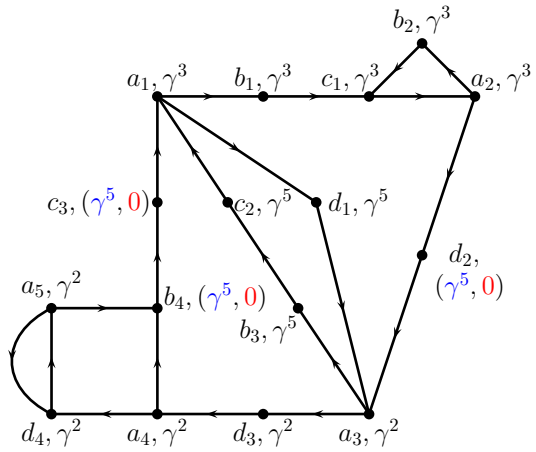
$$W_n(\widetilde{v}) = \min_{(\widetilde{u},\widetilde{v}) \in \widetilde{E}_K} \{W_{n-1}(\widetilde{u}) + \overline{w}(\widetilde{v})\}, \quad n = 1, 2, \ldots, |\widetilde{V}_K|,$$

with the initial conditions $W_0(\widetilde{v}_0) = 0$ and $W_0(\widetilde{v}) = \infty$ for any $\widetilde{v} \neq \widetilde{v}_0$. Then, we can compute

$$\overline{C}(v_0,K) = \min_{\widetilde{v} \in \widetilde{V}_K} \max_{n \in \mathbb{Z}[0,|\widetilde{V}_K|-1]} \left[\frac{W_{|\widetilde{V}_K|}(\widetilde{v}) - W_n(\widetilde{v})}{|\widetilde{V}_K| - n} \right].$$

A cycle $\widetilde{\rho}$ with the computed minimum mean can be extracted by fixing the node $\widetilde{v}$ which achieves the minimum in the above value and the respective path length n and finding a

[1] Since $R_{\mathsf{av}}(v_0)$ is upper bounded by $\lambda/(1-\gamma)$, we assume that the required accuracy ϵ is less than this upper bound.



$a_1 = (a, 3, 2, 1, 4)$ $a_2 = (a, 3, 2, 1, 0)$
$a_3 = (a, 2, 4, 3, 1)$ $a_4 = (a, 2, 0, 5, 1)$
$a_5 = (a, 2, 0, 0, 1)$

$b_1 = (b, 1, 3, 2, 5)$ $b_2 = (b, 1, 3, 2, 0)$
$b_3 = (b, 1, 5, 4, 2)$ $b_4 = (b, 1, 0, 0, 2)$

$c_1 = (c, 2, 1, 3, 0)$ $c_2 = (c, 2, 1, 5, 3)$
$c_3 = (c, 2, 1, 0, 3)$

$d_1 = (d, 1, 3, 2, 5)$ $d_2 = (d, 1, 3, 2, 0)$
$d_3 = (d, 1, 5, 4, 2)$ $d_4 = (d, 1, 0, 0, 2)$

Figure 2 The SCC of the finite augmented graph $\widetilde{G}_5$ for the graph in Figure 1. The node labels are the values of the functions $\overline{w}$ and $\underline{w}$ (in black if $\overline{w} = \underline{w}$, otherwise respectively in blue and red).

minimum-weight path from $\widetilde{v}_0$ to $\widetilde{v}$ and a cycle of length $|\widetilde{V}_K| - n$ within this path. Thus, the path $\widetilde{\pi}$ in $\widetilde{G}_K$ obtained by repeating $\widetilde{\rho}$ infinitely often realizes this value. A path π from v_0 in G with $c_{\mathsf{av}}(\pi) = \overline{C}(v_0, K)$ is obtained from $\widetilde{\pi}$ by projection on V.

In the general case, when $\widetilde{G}_K$ is not strongly connected we have to consider each of its SCCs reachable from $\widetilde{v}_0$, and determine the one with the least minimum cycle mean.

For each SCC with m edges and n nodes the computation of the quantities W requires $O(n \cdot m)$ operations. The computation of the minimum cycle mean for this component requires $O(n^2)$ further operations. Since $n \leq m$ because of the strong connectivity, the overall computation time for the SCC is $O(n \cdot m)$. Finally, the SCCs of $\widetilde{G}_K$ can be computed in time $O(|\widetilde{V}_K| + |\widetilde{E}_K|)$ [21]. This gives us the following result.

▶ **Lemma 11.** *The value $\overline{C}(v_0, K)$ and a path π with limit average cost $c_{\mathsf{av}}(\pi) = \overline{C}(v_0, K)$ can be computed in time polynomial in the size of $\widetilde{G}_K$.*

The same result can be established for the under approximation $\underline{C}(v_0, K)$.

▶ Remark. The number of nodes of $\widetilde{G}_K$ is $|\widetilde{V}_K| = |V| \times (K+1)^{|V|}$. For the approximation procedure described above it suffices to augment the graph with the information about which nodes were visited in the last K steps and in what order. Thus, we can alternatively consider a graph with $|V| \times |V|^K$ nodes in the case when the computed K is smaller than $|V|$.

▶ **Example 12.** Figure 2 shows the SCC of the augmented graph $\widetilde{G}_K$ reachable from initial node $(a, 1, 1, 1, 1)$ for the graph given in Figure 1 and $K = 5$. The nodes in the SCC are denoted by shorthands in the picture, *e.g.*, $a_1 = (a, 3, 2, 1, 4)$. The labels on the nodes correspond to the values of the weight functions $\overline{w}$ and $\underline{w}$. As we can see, $\widetilde{G}_5$ already contains simple cycles $(a_2 b_2 c_2 a_2), (a_3 b_3 c_2 a_1 b_1 c_1 a_2 d_2 a_3), (a_3 b_3 c_2 a_1 d_1 a_3)$, which correspond to the optimal paths presented in Example 3. The outcome of the minimum cycle mean for $\widetilde{G}_5$ will be the same with the two sets of weights only for the first and third interval for γ determined in Example 3, but will be different for the second case in which the term γ^8 is replaced respectively by γ^5 and 0 for the upper and lower bounds.

Theorem 13, a consequence of Lemma 11, states the approximate computation result.

▶ **Theorem 13.** *Given a graph $G = (V, E)$, node $v_0 \in V$, rational values λ and $0 < \gamma < 1$, and error bound $\epsilon > 0$, we can compute in time polynomial in $|V|(K+1)^{|V|}$ for $K =$*

$\ln\left[\frac{\epsilon(1-\gamma)}{\lambda}\right] / \ln\gamma$ *(i.e., exponential in* $|V|$*), infinite paths* π_{under} *and* π^{over} *and values* $r_{\mathsf{av}}(\pi_{\mathsf{under}})$ *and* $r_{\mathsf{av}}(\pi_{\mathsf{over}})$ *such that* $r_{\mathsf{av}}(\pi_{\mathsf{under}}) \leq R_{\mathsf{av}}(v_0) \leq r_{\mathsf{av}}(\pi_{\mathsf{over}})$ *and* $r_{\mathsf{av}}(\pi_{\mathsf{over}}) - r_{\mathsf{av}}(\pi_{\mathsf{under}}) \leq \epsilon$.

4.3 Approximation via Bounded Memory

The algorithm presented earlier is based on an augmentation of the graph with a specific structure updated deterministically and whose size depends on the desired quality of approximation. Furthermore, in this graph there exists a memoryless strategy with approximately optimal reward. We show that this allows us to quantify how far from the optimal reward value is a strategy that is optimal among the ones with bounded memory of fixed size.

First, we give the definition of memory structures. A *memory structure* $\mathcal{M} = (M, m_0, \delta)$ for a graph $G = (V, E)$ consists of a set M, initial memory $m_0 \in M$ and a memory update function $\delta : M \times V \to M$. The memory update function can be extended to $\delta^* : V^* \to M$ by defining $\delta^*(\epsilon) = m_0$ and $\delta^*(\pi \cdot v) = \delta(\delta^*(\pi), v)$. A memory structure $\mathcal{M}$ together with a function $\tau : V \times M \to V$ such that $(v, \tau(v, m)) \in E$ for all $v \in V$ and $m \in M$, and an initial node $v_0 \in V$ define a strategy $\sigma : V^* \to V$ where $\sigma(\epsilon) = v_0$ and $\sigma(\pi \cdot v) = \tau(v, \delta^*(\pi))$. In this case we say that the strategy σ has memory $\mathcal{M}$. Given a bound $B \in \mathbb{N}$ on memory size we define the finite graph $G \times B = (V^{G\times B}, E^{G\times B})$, where $V^{G\times B} = V \times \mathbb{Z}[1, B]$; and $E^{G\times B} = \{((v, i), (v', j)) \in (V \times \mathbb{Z}[1, B]) \times (V \times \mathbb{Z}[1, B]) \mid (v, v') \in E\}$.

Memoryless strategies in this graph precisely correspond to strategies that have memory of size B. More precisely, for each strategy σ in $G = (V, E)$ that has memory $\mathcal{M} = (M, m_0, \delta)$ there exists a memoryless strategy $\sigma_{\mathcal{M}}$ in $G \times |M|$ such that the projection of $\mathsf{outcome}((v_0, m_0), \sigma_{\mathcal{M}})$ on V is $\mathsf{outcome}(v_0, \sigma)$. Conversely, for each memoryless strategy $\sigma_{\mathcal{M}}$ in $G \times B$ there exist a memory structure $\mathcal{M} = (M, m_0, \delta)$ with $|M| = B$ and a strategy σ with memory $\mathcal{M}$ in G such that the projection of $\mathsf{outcome}((v_0, m_0), \sigma_{\mathcal{M}})$ on V is $\mathsf{outcome}(v_0, \sigma)$.

▶ **Example 14.** Recall that in Example 3 we established that an optimal path for $\gamma = 0.26$ is the path $(abcabcad)^\omega$. In the graph $G \times 3$ there exists a simple cycle corresponding to this path, namely $(a,1)(b,1)(c,2)(a,2)(b,2)(c,3)(a,3)(d,1)(a,1)$. Thus, the optimal path $(abcabcad)^\omega$ corresponds to a strategy with memory size of 3.

An optimal strategy among those with memory of a given size B can be computed by inspecting the memoryless strategies in $G \times B$ and selecting the one with maximal reward (these strategies are finitely but exponentially many).

A strategy returned by the approximation algorithm presented earlier uses a memory structure of size $(K+1)^{|V|}$. If $(K+1)^{|V|} \leq B$, then this strategy is isomorphic to some memoryless strategy σ in $G \times B$. Since the reward achieved by the optimal memoryless strategy in $G \times B$ is at least that of σ, its value is at most $\frac{\lambda}{1-\gamma}\gamma^K$ away from $R_{\mathsf{av}}(v_0)$. Now, $(K+1)^{|V|} \leq B$ iff $K \leq \left\lfloor \frac{\ln B}{\ln |V|} \right\rfloor - 1$. Thus, under a memory size B, we are guaranteed to find a strategy which has reward that at most $\frac{\lambda}{1-\gamma}\gamma^{\left\lfloor \frac{\ln B}{\ln |V|} \right\rfloor - 1}$ away from the optimal.

Next we sketch an algorithm for computing optimal B-memory bounded strategies. As mentioned previously, we can restrict our attention to memoryless strategies in $G \times B$. Memoryless strategies in this graph lead to lasso shaped infinite paths, with an initial non-cyclic path followed by a simple cycle which is repeated infinitely often. This means that we can restrict our attention to paths of length $|V| \cdot B$ which complete the lasso. Now, we apply this length bound to restrict our attention to the finite portion of the augmented graph $\widetilde{G}$ from Section 3 that corresponds to path lengths at most $|V| \cdot B$. This finite subgraph contains nodes $V \times \mathbb{N}[1, |V| \cdot B]^{|V|}$. The dynamic programming algorithm is polynomial time on this graph, hence we get a running time which is polynomial in $|V| \cdot (|V| \cdot B)^{|V|}$.

▶ **Theorem 15.** *Given a graph $G = (V, E)$, a node $v_0 \in V$, rational values λ and $0 < \gamma < 1$, and a memory bound $B > 1$, we can compute a B-memory optimal strategy σ with reward r_{av} at most $\frac{\lambda}{1-\gamma}\gamma^{\left\lfloor \frac{\ln B}{\ln |V|} \right\rfloor - 1}$ away from the optimal $R_{\mathsf{av}}(v_0)$ in time polynomial in $|V|^{(|V|+1)} \cdot B^{|V|}$.*

5 Conclusion and Open Problems

We have introduced the robot routing problem, which is a reward collection problem on graphs in which the reward structure combines spatial aspects with dynamic and stochastic reward generation. We have studied the computational complexity of the finite and infinite-horizon versions of the problem, as well as the strategy complexity of the infinite-horizon case. We have shown that optimal strategies need memory in general, and that strategies based on last visitation records do not suffice. However, the important question about whether finite-memory suffices for the infinite-horizon optimal path problem or infinite memory is needed remains open. In case finite-memory suffices, it will be desirable to provide an algorithm for precisely solving the infinite-horizon value problem. So far we have only given methods for approximating the optimal solution.

Another interesting line of future work is the generalization of the model to incorporate timing constraints. More specifically, in this work we have assumed that all the rewards accumulated at a node are instantaneously collected. This assumption is justified by the fact that in many request-serving scenarios the time taken to serve the requests at a given location is negligible compared to the time necessary to travel between the locations. A more precise model, however, would have to incorporate the serving time per node, which would depend on the amount of collected reward. This would imply that the elapsed time becomes a random variable, while in our current model it is deterministic.

Other extensions include the setting where the robot can react to the environment by observing the collected rewards, or observing the accumulated rewards at nodes of the graph, or where there is probabilistic uncertainty in the transitions of the robot in the graph. Finally, the robot routing problem presents new challenges for development of efficient approximation algorithms, which is a major research direction concerning path optimization problems.

References

1 Sofia Amador, Steven Okamoto, and Roie Zivan. Dynamic multi-agent task allocation with spatial and temporal constraints. In *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, pages 1384–1390. AAAI Press, 2014. URL: `http://dl.acm.org/citation.cfm?id=2615731.2616029`.

2 Nikhil Bansal, Avrim Blum, Shuchi Chawla, and Adam Meyerson. Approximation algorithms for deadline-TSP and vehicle routing with time-windows. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing 2004*, pages 166–174. ACM, 2004. `doi:10.1145/1007352.1007385`.

3 Dimitris J. Bertsimas and Garrett J. van Ryzin. A stochastic and dynamic vehicle routing problem in the Euclidean plane. *Operations Research*, 39(4):601–615, 1991. `doi:10.1287/opre.39.4.601`.

4 Avrim Blum, Shuchi Chawla, David R. Karger, Terran Lane, Adam Meyerson, and Maria Minkoff. Approximation algorithms for orienteering and discounted-reward TSP. *SIAM J. Comput.*, 37(2):653–670, 2007. `doi:10.1137/050645464`.

5 Patricia Bouyer, Piotr Hofman, Nicolas Markey, Mickael Randour, and Martin Zimmermann. Bounding average-energy games. In *Foundations of Software Science and Computa-*

tion Structures: 20th International Conference, FOSSACS 2017, pages 179–195. Springer Berlin Heidelberg, 2017. doi:10.1007/978-3-662-54458-7_11.

6 Tomás Brázdil, Petr Hlinený, Antonín Kucera, Vojtech Rehák, and Matús Abaffy. Strategy synthesis in adversarial patrolling games. *CoRR*, abs/1507.03407, 2015.

7 Francesco Bullo, Emilio Frazzoli, Marco Pavone, Ketan Savla, and Stephen L. Smith. Dynamic vehicle routing for robotic systems. *Proceedings of the IEEE*, 99(9):1482–1504, 2011. doi:10.1109/JPROC.2011.2158181.

8 Krishnendu Chatterjee and Vinayak S. Prabhu. Quantitative temporal simulation and refinement distances for timed systems. *IEEE Trans. Automat. Contr.*, 60(9):2291–2306, 2015. doi:10.1109/TAC.2015.2404612.

9 Rayna Dimitrova, Ivan Gavran, Rupak Majumdar, Vinayak S. Prabhu, and Sadegh Esmaeil Zadeh Soudjani. The robot routing problem for collecting aggregate stochastic rewards. *CoRR*, abs/1704.05303, 2017. URL: http://arxiv.org/abs/1704.05303.

10 Ali Ekici, Pinar Keskinocak, and Sven Koenig. Multi-robot routing with linear decreasing rewards over time. In *2009 IEEE International Conference on Robotics and Automation, ICRA 2009*, pages 958–963. IEEE, 2009. doi:10.1109/ROBOT.2009.5152803.

11 Ali Ekici and Anand Retharekar. Multiple agents maximum collection problem with time dependent rewards. *Computers & Industrial Engineering*, 64(4):1009–1018, 2013. doi:10.1016/j.cie.2013.01.010.

12 Sadegh Esmaeil Zadeh Soudjani and Alessandro Abate. Adaptive and sequential gridding procedures for the abstraction and verification of stochastic processes. *SIAM Journal on Applied Dynamical Systems*, 12(2):921–956, 2013. doi:10.1137/120871456.

13 Sadegh Esmaeil Zadeh Soudjani and Rupak Majumdar. Controller synthesis for reward collecting Markov processes in continuous space. In *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, HSCC '17, pages 45–54, New York, NY, USA, 2017. ACM. doi:10.1145/3049797.3049827.

14 Yuri Gurevich and Leo Harrington. Trees, automata, and games. In *Proc. Symposium on Theory of Computing*, pages 60–65. ACM Press, 1982. doi:10.1145/800070.802177.

15 Satoshi Hoshino and Shingo Ugajin. Adaptive patrolling by mobile robot for changing visitor trends. In *2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 104–110, 2016. doi:10.1109/IROS.2016.7759041.

16 Richard M. Karp. A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics*, 23(3):309–311, September 1978. doi:10.1016/0012-365X(78)90011-0.

17 Anthonius W. J. Kolen, Alexander H. G. Rinnooy Kan, and Henricus W. J. M. Trienekens. Vehicle routing with time windows. *Oper. Res.*, 35(2):266–273, April 1987. doi:10.1287/opre.35.2.266.

18 Justin Melvin, Pinar Keskinocak, Sven Koenig, Craig A. Tovey, and Banu Yuksel Ozkaya. Multi-robot routing with rewards and disjoint time windows. In *2007 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 2332–2337. IEEE, 2007. doi:10.1109/IROS.2007.4399625.

19 Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994. doi:10.1002/9780470316887.

20 Ruben Stranders, Enrique Munoz de Cote, Alex Rogers, and Nicholas R. Jennings. Near-optimal continuous patrolling with teams of mobile information gathering agents. *Artificial Intelligence*, 195:63 – 105, 2013. doi:10.1016/j.artint.2012.10.006.

21 Robert Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972. doi:10.1137/0201010.

22 Wolfgang Thomas. On the synthesis of strategies in infinite games. In *STACS 95: Theoretical Aspects of Computer Science*, volume 900 of *Lecture Notes in Computer Science*, pages 1–13. Springer-Verlag, 1995. doi:10.1007/3-540-59042-0_57.

23 Pieter Vansteenwegen, Wouter Souffriau, and Dirk Van Oudheusden. The orienteering problem: A survey. *European Journal of Operational Research*, 209(1):1–10, 2011. doi:10.1016/j.ejor.2010.03.045.

24 Changyun Wei, Koen V. Hindriks, and Catholijn M. Jonker. Dynamic task allocation for multi-robot search and retrieval tasks. *Appl. Intell.*, 45(2):383–401, 2016. doi:10.1007/s10489-016-0771-5.

25 Tichakorn Wongpiromsarn, Alphan Ulusoy, Calin Belta, Emilio Frazzoli, and Daniela Rus. Incremental synthesis of control policies for heterogeneous multi-agent systems with linear temporal logic specifications. In *2013 IEEE International Conference on Robotics and Automation*, pages 5011–5018, 2013. doi:10.1109/ICRA.2013.6631293.

26 Uri Zwick and Mike Paterson. The complexity of mean payoff games on graphs. *Theor. Comput. Sci.*, 158(1&2):343–359, 1996. doi:10.1016/0304-3975(95)00188-3.

Coverability Synthesis in Parametric Petri Nets*

Nicolas David[1], Claude Jard[2], Didier Lime[3], and Olivier H. Roux[4]

1 **University of Nantes, LS2N, Nantes, France**
nicolas.david1@univ-nantes.fr
2 **University of Nantes, LS2N, Nantes, France**
claude.jard@univ-nantes.fr
3 **École Centrale de Nantes, LS2N, Nantes, France**
didier.lime@ec-nantes.fr
4 **École Centrale de Nantes, LS2N, Nantes, France**
olivier-h.roux@ec-nantes.fr

Abstract

We study Parametric Petri Nets (PPNs), i.e., Petri nets for which some arc weights can be parameters. In that setting, we address a problem of parameter synthesis, which consists in computing the exact set of values for the parameters such that a given marking is coverable in the instantiated net.

Since the emptiness of that solution set is already undecidable for general PPNs, we address a special case where parameters are used only as input weights (preT-PPNs), and consequently for which the solution set is downward-closed. To this end, we invoke a result for the representation of upward closed set from Valk and Jantzen. To use this procedure, we show we need to decide universal coverability, that is decide if some marking is coverable for every possible values of the parameters. We therefore provide a proof of its ExpSpace-completeness, thus settling the previously open problem of its decidability.

We also propose an adaptation of this reasoning to the case of parameters used only as output weights (postT-PPNs). In this case, the condition to use this procedure can be reduced to the decidability of the existential coverability, that is decide if there exists values of the parameters making a given marking coverable. This problem is known decidable but we provide here a cleaner proof, providing its ExpSpace-completeness, by reduction to ω-Petri Nets.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Petri net, Parameters, Coverability, Unboundedness, Synthesis

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.14

1 Introduction

The introduction of parameters in models aims to improve genericity. It also allows the designer to leave unspecified aspects, such as those related to the modeling of the environment. This increase in modeling power usually results in greater complexity in the analysis and verification of the model. Beyond verification of properties, the use of parameters opens the way to very relevant issues in design, such as the computation of the parameters values ensuring satisfaction of the expected properties. This is the synthesis problem: given a property, compute the exact set of all parameter values such that, instantiated with these values, the system satisfies this property. This notably permits an estimation of the *robustness*

* This work was partially supported by ANR project PACS (ANR-14-CE28-0002) and Pays de la Loire research project AFSEC.

of a given instance of a model. Indeed, in full knowledge of "good values" for the parameters, we may be able to quantify the distance from a "bad value" providing an idea of how reliable is the system. Thus parameterised systems are of particular interest both in allowing the handling of more realistic classes of models and addressing more realistic verification issues. We therefore address here the case of parameterised concurrent systems modelled as parametric Petri nets.

Related work. The study of parameterised models and more specifically the synthesis has been studied in different parametrics settings. Parameters representing delays in timed systems modeled as timed automata have been particularly studied, but with very few decidability results [1]. Synthesis for such systems is only possible in very particular settings, such as bounded integer parameters computed symbolically in timed automata [15] or integer parameters in timed automata with parameters used only as upper bounds, or only as lower bounds, in timing constraints [4]. We focus here on a different type of parameters which represent discrete values. In Petri nets, this corresponds to parameterising the initial marking [5, 17] and the weights of the arcs in transitions [7]. The latter work deals with two decidability problems induced by the use of parameters: The *existential coverability*: does there exists an integer valuation v on the set of parameters such that m is coverable in the marked Petri net where parameters are replaced by the value given by v? And the *universal coverability*: is m is coverable in such a net for every possible valuation v? Those problems are both undecidable in the most general case, and syntactic subclasses restricting the use of parameters have been introduced, for which the different problems are decidable.

Contributions. We focus on computing the exact solution set to the synthesis problem for coverability in parametric Petri nets, i.e., the set of all parameter values such that in the net instantiated with these values, a given marking is coverable.

The emptiness and universality of the solution set being undecidable in general, computing this set can only be done in a restricted setting. We thus focus on the case when parameters are used only as input weights (preT-PPNs) or only as output weights (postT-PPNs). These assumptions give some structure to the solution set: we prove that it is then downward-closed wrt. the usual order on integer vectors for preT-PPNs, and upward-closed for postT-PPNs.

We show how a procedure by Valk and Jantzen from [20] can be used for computing a finite minimal basis of the solution set for postT-PPNs or its complement for preT-PPNs. This requires deciding universal coverability in preT-PPNs and existential coverability in postT-PPNs. The former is an open problem: our main result is it is ExpSpace-complete, which we prove by considering the more generic property of simultaneous unboundedness studied by Demri in [8]. The latter is known decidable but we provide a cleaner proof and additionally prove its ExpSpace-completeness. These results interestingly allow us to carry over a Rackoff upper bound into this parametric setting.

Finally, we prove that in what is called distinctT-PPNs in [7], i.e., when the set of parameters appearing as input weights, and the set of parameters appearing as output weights are disjoint, the solution set cannot be represented using any formalism for which the emptiness of the intersection with equality constraints is decidable.

Organization of the Paper. Section 2 gives basic notations and recalls useful mathematical results on orders and Petri nets. Section 3 presents Parametric Petri Nets and recalls some decidability problems. There, we also study the structure of the solution sets for preT- and postT-PPNs and show under which condition Valk and Jantzen's algorithm can be used

to construct finite representation of those sets. In section 4 we give our construction for proving the ExpSpace-completeness of the universal coverability for preT-PPNs. Section 5 revisits the proof of the decidability of existential coverability in postT-PPNs and proves its ExpSpace-completeness. We also discuss the case of distinctT-PPNs. Finally, in Section 6, we conclude and present future work. Due to space contraints, proofs are omitted.

2 Background

2.1 Notations

We denote by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers. As usual, $\mathbb{N}_\omega$ is the union $\mathbb{N} \cup \{\omega\}$ where for each $n \in \mathbb{N}$, $n + \omega = \omega$, $\omega - n = \omega$ and $\omega \leq \omega$. Moreover, if $n \in \mathbb{N}$, $n < \omega$. Let X be a finite set. We denote by 2^X the powerset of X and $|X|$ the size of X. If $X \subseteq \mathbb{N}^k$, $\neg X$ denotes its complement in $\mathbb{N}^k$. Given a finite set X, S_X denotes the symmetric group on X (*i.e.* the set of all permutations of elements of X). Given a set X, we define a *linear expression* on X by the following grammar: $\lambda ::= k \mid k * x \mid \lambda + \lambda$ where $k \in \mathbb{Z}$, $x \in X$. We denote by $\mathcal{L}(X)$ the set of linear expressions on X.

Let $V \subseteq \mathbb{N}$, a V-valuation for X is a function from X to V. We denote by V^X the set of V-valuations on X. Considering $v \in V^X$, we write $\texttt{dom}(v)$ the domain of v (X in this case) and $\texttt{im}(v)$ its image. We refer to $\mathbb{N}_\omega$-valuations as *extended* valuations and to $\mathbb{N}$-valuations simply as valuations. The set $V^\emptyset$ is reduced to a singleton $\{\emptyset_V\}$ where $\emptyset_V$ is the empty function. If X is finite, considering some arbitrary order on X, an (extended) valuation can be seen as a vector of size $|X|$. For any subset $X' \subseteq X$ and valuation $v \in V^X$, we define the restriction $v_{|X'}$ of v to X' as the unique V-valuation on X' such that $v_{|X'}(x) = v(x)$ for all $x \in X'$. We extend this notation to sets of valuations: given $Y \subseteq V^X$, $Y_{|X'}$ denotes its projection on X' that is to say $Y_{|X'} = \{v_{|X'} \mid v \in Y\}$. Given a value a of $\mathbb{N}_\omega$, we denote as $\mathbf{a}$ the uniform (extended) valuation that maps every element of X to a. Given an extended valuation v, we write $\omega(v)$ for the subset of X such that $x \in \omega(v)$ iff $v(x) = \omega$. We write $\mathbb{N}(v)$ for the subset of X such that $x \in \mathbb{N}(v)$ iff $v(x) \in \mathbb{N}$.

Given a linear expression λ on X and an extended valuation v on $X' \subseteq X$, $v(\lambda)$ is the linear expression obtained when substituting each element x in X' from λ, by the corresponding value $v(x)$. If $X' = X$ we obtain an element of $\mathbb{N}_\omega$.

Given a set R, finite sets S, A, B such that $S = A \cup B$ and $A \cap B = \emptyset$, and functions $f \in R^A$ and $g \in R^B$, we write $f \cup g \in R^S$ the function defined by $(f \cup g)_{|A} = f$ and $(f \cup g)_{|B} = g$. We call $f \cup g$ the *union* of f and g. Note that given x in A, $y = f(x)$ is called the image of x and when there is no ambiguity (*i.e.* when f is injective), x is called the fiber of y by f.

Finally, let A be an alphabet and A^* be the free monoid over A. Let $w \in A^*$ be a word. We write $|w|$ the length of w. Given $a \in A$, $|w|_a$ is the number of occurrences of a in w. We define ϵ as the identity element of A^*. We write $t \sqsubseteq s$ when t is a prefix of s. We denote by $Pref(L)$ the prefix closure of a langage L, *i.e.* $Pref(L) = \bigcup_{s \in L} \{t \mid t \sqsubseteq s\}$.

2.2 Order

A *quasi order* (qo for short) $\lesssim$ on some set S is a reflexive and transitive binary relation on S. The pair $(S, \lesssim)$ is then called a quasiordered set. For $x, y \in S$ and given a qo $\lesssim$ on S, x and y are said *comparable* if either $x \lesssim y$ or $y \lesssim x$. A relation $<$ is a *strict order* on a set S if it is irreflexive and transitive (which implies asymmetry).

Given any quasi order $\lesssim$ on a set S we can define: (i) a strict order $<$ given by $x < y$ iff $x \lesssim y \wedge \neg(y \lesssim x)$, (ii) an equivalence relation $\sim$ given by $x \sim y$ iff $x \lesssim y \wedge y \lesssim x$, (iii) its dual quasi order $\gtrsim$ given by $y \gtrsim x$ iff $x \lesssim y$. A *well-quasi-ordering* (wqo for short) is a qo $\lesssim$ on a set S such that, for any infinite sequence $s = x_0, x_1, x_2, ...$ in S, there exists indexes $i < j$ with $x_i \lesssim x_j$. Now consider $\leq$, the qo on $\mathbb{N}^k$ component-wise. Formally, for every $x, y \in \mathbb{N}^k$, we write that $x \leq y$ iff for every component i of x and y, $x(i) \leq y(i)$. Dickson's Lemma [9] states that $(\mathbb{N}^k, \leq)$ is a well-quasi-ordered set. Let us also recall the following lemma[1]:

▶ **Lemma 1** ([9]). *Let $p_0, p_1, ..., p_n, ...$ be an infinite sequence of elements of $(\mathbb{N}_\omega)^k$. Then, there is an infinite sequence $p_{i_1}, p_{i_2}, ..., p_{i_n}, ...$ such that $p_{i_1} \leq p_{i_2} \leq ... \leq p_{i_n} \leq ....$ (with $i_1 < i_2 < \cdots < i_n < \ldots$).*

2.3 Downward and Upward closed sets

We reuse definitions and concepts from [10, 11] which are summed up in [12]. An upward closed set of the well quasi ordered set $(\mathbb{N}^k, \leq)$ is a subset U of $\mathbb{N}^k$ such that if $x \in U$, $y \in \mathbb{N}^k$ and $x \leq y$ then $y \in U$. The upward closure of a vector u, written $\uparrow u$ is the set $\{m \in \mathbb{N}^k \mid u \leq m\}$. Given a set U, we write $\uparrow U$ for the upward closure of U, defined as $\uparrow U = \bigcup_{u \in U} \uparrow u$. This implies that $\uparrow U$ is the least upward closed set in which U is included. Any upward closed set U can be represented by a finite set F, called basis, such that $U = \uparrow F$. The minimal elements of F still form a basis of U independently of F. This basis is minimal for inclusion among all bases and is thus called the minimal upward basis of F.

A downward closed set of the well quasi ordered set $(\mathbb{N}^k, \leq)$ is a subset D of $\mathbb{N}^k$ such that if $x \in D$, $y \in \mathbb{N}^k$ and $y \leq x$ then $y \in D$. The downward closure of a vector d, written $\downarrow d$ is the set $\{m \in \mathbb{N}^k \mid m \leq d\}$. Given a set D, we write $\downarrow D$ the downward closure of D, defined as $\downarrow D = \bigcup_{d \in D} \downarrow d$. This implies that $\downarrow D$ is the least downward closed set in which D is included. Moreover, the downward closure of a finite set is finite. To symbolically represent downward closed sets, we use the extension $\mathbb{N}_\omega^k$. The definitions remain otherwise the same. If D is a downward closed set, we can write $D = \mathbb{N}^k \cap \downarrow F$ where F is a finite set of $\mathbb{N}_\omega^k$. We call F a downward basis of D. The maximal elements of F still form a basis of D independently of F. This basis is minimal for the inclusion among all bases and is thus called the minimal downward basis of D.

We also recall results from [3]: the union and the intersection of two upward (resp. downward) closed sets is an upward (resp. downward) closed set. The complement of an upward closed set is a downward closed set and vice-versa. Given the basis of an upward closed set, it is possible to compute the basis of its complement using for instance the procedure suggested in Example 5 of [14], and vice versa by adapting this procedure.

Finally, Valk and Jantzen proposed in [20] a necessary and sufficient condition, recalled in Lemma 2, to ensure that a finite basis of an upward closed set is effectively computable.

▶ **Lemma 2** ([20]). *Given an upward closed set $U \subseteq \mathbb{N}^k$, a finite basis of U is effectively computable iff for each $v \in \mathbb{N}_\omega^k$, the emptiness of $\downarrow v \cap U$ is decidable, which is also equivalent to ask whether for all element $v \in \mathbb{N}_\omega^k$, it is decidable to answer whether $\downarrow v \cap \mathbb{N}^k \subseteq \neg U$.*

[1] The existence of such increasing subsequences can also be used as a definition for wqo, which leads to an equivalent notion. Note that a proof can also be found in [16].

2.4 Petri Nets

▶ **Definition 3** (Marked Petri Net). A Petri net $\mathcal{N}$ is a tuple $(P, T, Pre, Post)$ such that P is a finite set of places of $\mathcal{S}$, T is a finite set of transitions of $\mathcal{S}$, Pre and $Post$ are functions from $P \times T$ to $\mathbb{N}$. A *marking* of $\mathcal{N}$ is an $\mathbb{N}$-valuation on P. A marked Petri Net (PN) is a pair $\mathcal{S} = (\mathcal{N}, m_0)$ where $\mathcal{N}$ is a Petri net and m_0 the initial marking of $\mathcal{N}$.

Given a transition t of T, we define $Pre(t)$ as the univariate function on P at the point t which associates to each p of P the weight $Pre(p, t)$. We define $Post(t)$ in a similar way. A transition $t \in T$ is said *enabled* by a marking m when $m \geq Pre(t)$.

▶ **Definition 4** (PN Semantics). The semantics of a PN is a transition system $\mathcal{S}_{\mathcal{T}} = (Q, q_0, \rightarrow)$ where, $Q = \mathbb{N}^P$, $q_0 = m_0$, $\rightarrow \subseteq Q \times T \times Q$ such that for all $t \in T$, $m \xrightarrow{t} m' \Leftrightarrow m \geq Pre(t)$ and $m' = m - Pre(t) + Post(t)$

This relation can be extended to sequences of transitions as follows: (i) $m \xrightarrow{\epsilon} m'$ if $m = m'$ (ii) $m \xrightarrow{wt} m'$ if $\exists m'', m \xrightarrow{w} m'' \wedge m'' \xrightarrow{t} m'$ where $w \in T^*$ and $t \in T$. We write $\xrightarrow{*}$ the reflexive transitive closure of $\rightarrow$, i.e., $m \xrightarrow{*} m'$ when there exists $w \in T^*$ such that $m \xrightarrow{w} m'$.

▶ **Definition 5** (Reachability). Let $\mathcal{S} = (\mathcal{N}, m_0)$, where $\mathcal{N} = (P, T, Pre, Post)$, a marking m of $\mathbb{N}^P$ is reachable in $\mathcal{S}$ iff $m_0 \xrightarrow{*} m$.

The reachability set $\mathcal{RS}(\mathcal{S})$ of $\mathcal{S}$ is the set of all reachable markings of $\mathcal{S}$.

▶ **Definition 6** (Coverability). Let $\mathcal{S} = (\mathcal{N}, m_0)$, where $\mathcal{N} = (P, T, Pre, Post)$, and m a marking of $\mathbb{N}^P$, m is coverable in $\mathcal{S}$ iff $\exists m' \in \mathcal{RS}(\mathcal{S}), m' \geq m$.

The coverability set $\mathcal{CS}(\mathcal{S})$ of $\mathcal{S}$ is the set of markings coverable in $\mathcal{S}$. Coverability is decidable in marked Petri nets [16]. The coverability set is an over approximation of the reachability set in the sense that $CS(\mathcal{S}) = \downarrow \mathcal{RS}(\mathcal{S})$. Given a marked PN $\mathcal{S}$, and a marking m, we denote by $\mathsf{cov}(\mathcal{S}, m) \in \{True, False\}$ the coverability of m in $\mathcal{S}$. In Petri nets, coverability allows to verify safety properties. We recall that the coverability set of a marked Petri net is computable in the sense that its minimal downward basis is computable (see, *e.g.*, [12]).

3 Monotonicity in Parametric Petri Nets

3.1 Parametric Petri Nets and Parametric Problems

Following [7], we recall the definitions related to marked Parametric Petri Nets (PPNs). We omit the case of parametric initial markings which is a subcase of parametric output weights.

▶ **Definition 7** (Parametric Petri Net). A marked parametric Petri Net (PPN) is a pair $\mathcal{S} = (\mathcal{N}, m_0)$ where $\mathcal{N} = (P, T, Pre, Post, \mathbb{P})$ such that P is a finite set of places of $\mathcal{N}$, T is a finite set of transitions of $\mathcal{N}$, $\mathbb{P}$ is a finite set of parameters of $\mathcal{N}$, Pre and $Post$ are functions from $P \times T$ to $\mathbb{N} \cup \mathbb{P}$, m_0 is the initial marking of $\mathcal{N}$ belonging to $\mathbb{N}^P$.

We define the *parametric transitions* of $\mathcal{S}$, $\Theta \subseteq T$ as the set of transitions with at least one parameter on an input or output arc: $\Theta = \{t \in T \mid \exists p \in P \; s.t. \; Pre(p, t) \in \mathbb{P} \vee Post(p, t) \in \mathbb{P}\}$. We refer to $T \setminus \Theta$ as the set of plain transitions in echo to the notations of [13].

Considering an arbitrary ordering on places, parametric markings can be represented as vectors of linear combinations on the set of parameters *i.e.* from $\mathcal{L}(\mathbb{P})^{|P|}$. Similarly, Pre and $Post$ can be seen as matrices of $(\mathbb{N} \cup \mathbb{P})^{|P| \times |T|}$.

Given a marked PPN $\mathcal{S} = (\mathcal{N}, m_0)$, where $\mathcal{N} = (P, T, Pre, Post, \mathbb{P})$, for any $\mathbb{N}$-valuation v on a subset X of $\mathbb{P}$, we define the v-instance of $\mathcal{S}$ as the marked PPN $v(\mathcal{S}) = (v(\mathcal{N}), m_0)$ where $v(\mathcal{N}) = (P, T, v(Pre), v(Post), \mathbb{P}\backslash X)$. By $v(Pre)$ and $v(Post)$ we denote the functions/matrices obtained by replacing in their entries each parameter λ in $\texttt{dom}(v)$ by $v(\lambda)$. If $X = \mathbb{P}$, $v(\mathcal{N})$ and $v(\mathcal{S})$ are respectively a Petri net and a marked Petri net. We also recall subclasses introduced in [7]. Given a PPN, $\mathcal{N} = (P, T, Pre, Post, \mathbb{P})$, if $Pre \in P \times T \to \mathbb{N}$, we call it a postT-PPN, whereas if $Post \in P \times T \to \mathbb{N}$, we call it a preT-PPN.

Given a PPN $\mathcal{S}$, and a marking m, we define two basic parametric decision problems: Does there exist a valuation v such that m is coverable in $v(\mathcal{S})$ (Existential coverability) ? Is m coverable in $v(\mathcal{S})$ for all valuations v (Universal coverability) ? Those problems were partly studied in [7]. In particular both *Existential coverability* and *Universal coverability* are proved to be undecidable for the generic class of PPNs. In this paper, we are interested in a more general question:

▶ **Definition 8** (*coverability*-Synthesis problem)**.** Compute all the valuations v, such that $\texttt{cov}(v(\mathcal{SP}), m)$ is true.

We call this set of valuations the coverability synthesis set of a marked PPN $\mathcal{S}$ and a marking m, denoted by $\mathcal{CV}(\mathcal{S}, m)$. We also call it the solution set to the synthesis problem.

▶ Remark. From any PN $\mathcal{S}$, we can build a PPN $\mathcal{S}'$ by adding an unused parameter. Then checking existential or universal coverability on $\mathcal{S}'$ is equivalent to checking coverability on $\mathcal{S}$. Those parametric problems are therefore ExpSpace-hard. The same reasoning applies for other properties such as (simultaneous) unboundedness.

3.2 Special Structure of the Coverability Synthesis Set for PreT-PPNs and PostT-PPNs

When we restrict the use of parameters to input arcs, we ensure that any marking coverable in a v-instance remains coverable for any v'-instance such that $v' \leq v$. Intuitively, decreasing the valuation leads to a more permissive firing condition. Symmetrically, when we restrict the use of parameters to output arcs, we ensure that any marking coverable in a v-instance, remains coverable for any v'-instance such that $v \leq v'$. Intuitively, firing the same parametric transition while increasing the valuation leads to greater (and thus more permissive) markings. Those results are formalized in Lemma 9.

▶ **Lemma 9.** *Let $\mathcal{S}_{pre}$ and $\mathcal{S}_{post}$ be respectively a marked preT-PPN and postT-PPN of initial marking m_0 and s_0.*

- *For every transitions sequence w of $\mathcal{S}_{pre}$ and for every valuation v, if $m_0 \xrightarrow{w} m$ in $v(\mathcal{S}_{pre})$, then for every valuation $v' \leq v$, there exists $m' \geq m$ such that $m_0 \xrightarrow{w} m'$ in $v'(\mathcal{S}_{pre})$.*
- *For every transitions sequence w of $\mathcal{S}_{post}$ and for every valuation v, if $s_0 \xrightarrow{w} s$ in $v(\mathcal{S}_{post})$, then for every valuation $v' \geq v$, there exists $s' \geq s$ such that $s_0 \xrightarrow{w} s'$ in $v'(\mathcal{S}_{post})$.*

Note that those properties of monotonicity directly provide a notable structure for the solution set of the synthesis for those two subclasses presented in Corollary 10.

▶ **Corollary 10.** *Given $\mathcal{S}_{pre}$, $\mathcal{S}_{post}$ a marked preT-PPN and a marked postT-PPN respectively and goal markings m and s for each of those nets,*

- *$\mathcal{CV}(\mathcal{S}_{pre}, m)$ is downward closed.*
- *$\mathcal{CV}(\mathcal{S}_{post}, s)$ is upward closed.*

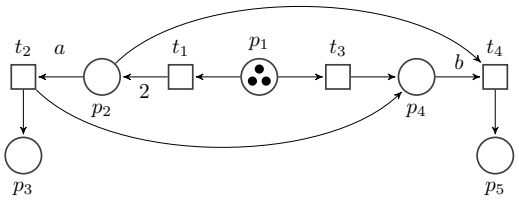


Figure 1 $\mathcal{S}_1$ where $\mathbb{P} = \{a, b\}$.

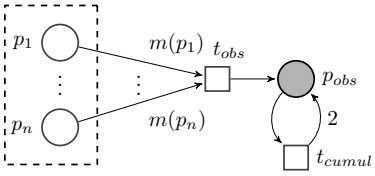


Figure 2 Reduction of coverability to the place boundedness.

3.3 Reduction of Valk and Jantzen Condition for PreT-PPNs and PostT-PPNs

Given a preT-PPN $\mathcal{S}$ and a marking m, one way to compute $\mathcal{CV}(\mathcal{S}, m)$ is thus to find its finite minimal basis. A naive enumeration is not possible however since this set may be infinite. In particular, the strategy that consists in testing for universality and, in the negative case, enumerating until a witness of non coverability is found would in general provide only a subset of $\mathcal{CV}(\mathcal{S}, m)$. In fact, the main difficulty here resides in the fact that the elements of the minimal basis have to be found among the complete lattice induced by $\leq$ on $\mathbb{N}_\omega^{\mathbb{P}}$.

In order to represent a finite basis of a downward closed set of valuations, we need to extend valuations to $\mathbb{N}_\omega$. Given a preT-PPN and an extended valuation v, we extend the predicate $\texttt{cov}(v(\mathcal{S}), m)$ to extended valuations as follows: $\texttt{cov}(v(\mathcal{S}), m) \overset{def}{\Leftrightarrow} \forall v' \in \mathbb{N}^{\omega(v)}$, $\texttt{cov}(v'(v_{|\mathbb{N}(v)}(\mathcal{S})), m)$.

Figure 1 presents a preT-PPN with two parameters a and b. If we consider the valuation v defined by $v(a) = 1$ and $v(b) = \omega$, $\texttt{cov}(v(\mathcal{S}), m)$ is therefore equivalent to the universal coverability of m in $v_{|\{a\}}(\mathcal{S})$ where $v_{|\{a\}}$ is a valuation defined by $v(a) = 1$, that is to say "can we cover m in $\mathbf{1}_{|\{a\}}(\mathcal{S})$ for any value of b ?". Note that this extension of $\texttt{cov}(v(\mathcal{S}), m)$ is consistent with the classic behavior: if $\mathbb{N}(v) = \mathbb{P}$, then $\texttt{cov}(v(\mathcal{S}), m)$ asks the coverability of m in the marked Petri Net $v(\mathcal{S})$.

We recall that in postT-PPNs, universal coverability is true iff $\texttt{cov}(\mathbf{0}(\mathcal{S}), m)$. In a similar manner to preT-PPN, we extend the notation of cov, as follows: given a postT-PPN and an extended valuation v, we extend the predicate $\texttt{cov}(v(\mathcal{S}), m)$ to extended valuations as follows: $\neg\texttt{cov}(v(\mathcal{S}), m) \overset{def}{\Leftrightarrow} \forall v' \in \mathbb{N}^{\omega(v)}$, $\neg\texttt{cov}(v'(v_{|\mathbb{N}(v)}(\mathcal{S})), m)$. This definition is similar to the definition extended for preT-PPNs where coverability has been replaced by non-coverability.

With those extended notations, we now wonder if it is possible to compute a finite basis of $\mathcal{CV}(\mathcal{S}, m)$ where $\mathcal{S}$ is a preT-PPN or a postT-PPN and m a goal marking. To this end, we suggest to use an algorithm by Valk and Jantzen [20] to compute a finite representation of those sets. Nevertheless, to ensure that this algorithm is suitable to our context and that those basis are effectively computable, we need to clarify two points:

- First, this algorithm is used to compute bases of upward closed sets.
- Second, the necessary and sufficient condition recalled in Lemma 2 must be satisfied.

To address the first point, by Corollary 10 notice that in the case of postT-PPNs, the set $\mathcal{CV}(\mathcal{S}, m)$ is upward closed, and the procedure could be applied directly on it. In the case of

preT-PPNs, since $\mathcal{CV}(\mathcal{S}, m)$ is downward closed, we need to consider $\neg\mathcal{CV}(\mathcal{S}, m)$, which is upward closed as recalled in Section 2.3. We also recalled in that section that given a finite basis of an upward closed set, it is possible to compute a finite basis of its complement. It is therefore equivalent to being able to compute a finite basis of the set of valuations for which it is not possible to cover m or to be able to compute a finite basis of the set of valuations for which it is possible to cover m. Thus, a finite basis of $\neg\mathcal{CV}(\mathcal{S}, m)$ is effectively computable iff a finite basis of $\mathcal{CV}(\mathcal{S}, m)$ is computable. Considering this reasoning, we address the second point through the following Lemma:

▶ **Lemma 11.** *The Valk and Jantzen condition can be reduced to the following criteria:*

1. *we can compute a finite representation of the coverability synthesis set in preT-PPNs iff universal coverability is decidable in preT-PPNs*
2. *we can compute a finite representation of the coverability synthesis set in postT-PPNs iff existential coverability is decidable in postT-PPNs*

We now start by focusing on universal coverability in preT-PPNs.

4 Universal Coverability in preT-PPNs

We address the problem of universal coverability through that of the more general universal simultaneous unboundedness. We will prove that both are ExpSpace-complete.

4.1 Simultaneous Unboundedness

▶ **Definition 12** (Simultaneous Unboundedness [8])**.** Given $\mathcal{N} = (P, T, Pre, Post)$, and $\mathcal{S} = (\mathcal{N}, m_0)$, considering a subset $X \subseteq P$, $\mathcal{S}$ is simultaneous X unbounded if for any $B \geq 0$, there is a run w such that $m_0 \xrightarrow{w} m$ and for all $i \in X$, we have $m(i) \geq B$. If X is reduced to a singleton $\{p\}$, $\mathcal{S}$ is said p-unbounded.

▶ Remark. Notice that coverability can be easily reduced to simultaneous unboundedness by the use of an observer as depicted in Figure 2. The transition t_{obs} has an input condition equal to the marking we want to cover m. Its output effect provides a token in a place p_{obs}, that, once is marked, can become unbounded through the firing of t_{cumul}. With this construction, m is coverable in the net iff it is simultaneous p_{obs}-unbounded.

Since there exist polynomial translations from VASS to VAS and PN and from PN to VAS (and VASS) [18, 2], we have the following Theorem, initially stated with VASS in [8].

▶ **Theorem 13** ([8])**.** *Simultaneous unboundedness for PNs is ExpSpace-complete.*

4.2 Notion of Incremental Model

To prove the decidability of universal coverability in preT-PPNs, we will prove the decidability of universal simultaneous unboundedness. We will also prove that the latter belong to ExpSpace. Together with Remark of Section 3.1, we can then conclude that both problems are ExpSpace-complete.

Formally, given a parametric Petri net, and a set of places X, the parametric net is *universally simulatenous X unbounded* iff for every possible valuation v of its parameters, the v-instance of this net is simultaneous X unbounded.

We first show that it is sufficient for a net to be simultaneous unbounded on a set of places in infinitely many instances (under uniform valuations) of the parametric Petri net to be universally simultaneous unbounded on this set of places. Indeed, for any valuation v, we can find a uniform valuation $\mathbf{k}$ such that $v \leq \mathbf{k}$ and apply Lemma 9.

▶ **Lemma 14.** *Given a marking m and a marked preT-PPN $\mathcal{S}$ and X a subset of places of $\mathcal{S}$, the two following propositions are equivalent :*

1. *$(\mathcal{S}, m_0)$ is universally simultaneous X unbounded*
2. *$\{k \in \mathbb{N} \mid (\mathbf{k}(\mathcal{S}), m_0)$ is simultaneous X unbounded$\}$ is infinite*

This Lemma is used for the proof of the upcoming Lemma 15. It is indeed an important result since it reduces the infinite set of valuations over which we should investigate to the infinite set of uniform valuation that is totally ordered *i.e.* two elements of this set are always comparable.

We can now address the problem of universal simultaneous unboundedness. To solve this problem, we reduce it to the existence of a classic Petri net built upon our parametric model satisfying an adequately chosen simultaneous unboundedness property. The classic Petri net is in fact obtained by evaluating a preT-PPN, called incremental net, under the uniform valuation $\mathbf{0}$. The incremental net has a polynomial size in the original preT-PPN and it directly depends on the original preT-PPN and a sequence of distinct parametric transitions. This Section is thus driven by the idea that universal simultaneous X unboundedness on a preT-PPN $\mathcal{S}$ is equivalent to the existence of a sequence σ of distinct parametric transitions of $\mathcal{S}$, such that the incremental model build on $\mathcal{S}$ and σ evaluated under $\mathbf{0}$ satisfies a simultaneous unboundedness property depending on X and σ.

Before providing the theoretical definition, let us consider the main intuition of our construction. If a net is universally simultaneous unbounded on a set of places X, two main cases are possible: we can either find a path such that the places of X are unbounded without using any parametric transition, and then the corresponding run works for any valuation, or we need at least one parametric transition.

In the latter case, since there is an infinite number of valuations and a finite number of parametric transitions, using the pigeonhole principle, there is at least one such transition that must be used as the first parametric transition in the run for an infinite number of valuations. The input places of its parametric arcs are therefore not bounded. Thus, the valuation of the input parametric arcs of this transition is not limiting anymore since we can generate an arbitrary large amount of tokens in the corresponding places. Therefore, we will later evaluate[2] those parameters to 0 in order to perform the verification on a classic Petri net.

Nevertheless, we need to ensure that the set of input places of the parametric arcs are not bounded (without using that transition). This is exactly the goal of this incremental model. Indeed, once fired, we could then consider a new net where the first parametric transition can be involved as well as non parametric transitions and investigate for the newly unbounded places. Either we can unbound the places of the goal set X or we can reuse previous reasoning and choose a new parametric transition that has to be involved in infinitely many instances. What is important to note here is that at each firing of a new parametric transition, that never occurred in the run, we need to ensure that its input places of parametric input arcs were unbounded using only previous transitions of the run and to remember what are the new places that can be unbounded through the use of this new transition. We will now formalize how it is possible to remember the boundedness of the input places of parametric arcs by presenting exhaustively the model of incremental nets.

Given a preT-PPN $\mathcal{N} = (P, T', Pre, Post, \mathbb{P})$ and a partition of its transitions $T' = T \cup \Theta$ between its plain and parametric transition, we denote by $\mathcal{N}_p$ the Petri Net obtained from

[2] Note that any other finite valuation would be suitable since the input places of the parametric arcs are unbounded.

$\mathcal{N}$ by removing all transitions of Θ from $\mathcal{N}$. An example is given in Figure 3. Let us now consider a finite sequence $\sigma \in Pref(S_\Theta)$ where S_Θ is the symmetric group over Θ seen as a language. Let $|T| = m$, $|P| = n$ and $|\sigma| = k$. We define the incremental model $\mathcal{I}$ of $\mathcal{N}$ along σ. We write $\mathcal{I} = \texttt{incr}(\mathcal{N}, \sigma)$ to denote this preT-PPN. This model is illustrated by the example[3] at the right hand side of Figure 3. Its construction consists of the following main steps:

- **(i)** Consider $\mathcal{N}_p$ and k copies of $\mathcal{N}_p$, where to each of those $k+1$ subnets is associated a global lock place, ensuring that exactly one copy is active at any given instant. The copies correspond to the black subnet of this example, whereas the global locks p_0^i's are depicted in blue and dotted arcs.
- **(ii)** Add a copy of the first i transitions of σ to each i^{th} copy of $\mathcal{N}_p$, for $1 \leq i \leq k$.
- **(iii)** Between the $i-1^{th}$ and i^{th} subnets, add a copy of the $i+1^{th}$ transition of σ, depicted in plain green arcs in Figure 3, for $1 \leq i \leq k$. Notice that this copy presents a special behaviour: its input effect impacts the $i-1^{th}$ subnet and its lock p_0^{i-1} whereas its output effect impacts the i^{th} subnet and its lock p_0^i. This ensures that we change of active subnet only after the firing of a the first occurrence of a precise parametric transition.
- **(iv)** Finally, we ensure that given every copy of a transition, including the intermediate copies that allow to change the active copy, it modifies simultaneously the places in the associated copy as explained above, but also all copies of greater index (*i.e.* those that have not been activated yet). Those arcs ensure that every later subnet always has the "same" marking as the active copy. They are depicted by dashed red arcs in Figure 3. Note that we synchronise the different copies and do not merge them because we use them to remember the order in which the different input places to parametric transitions become unbounded.

Let us suppose we evaluated this incremental model in order to perform an execution. At the beginning of any execution, given a precise subnet, let us say the i^{th} subnet, it follows the behavior of the subnets with lower index because of synchronisations introduced in item (iv). Then, once this copy become active, after the firing of a given parametric transition introduced in item (iii), it will dictate the behavior of the global net (and thus of the subnets with greater index through synchronisations). Once the next copy becomes active, our original i^{th} subnet cannot change its state anymore. It is now literally an historic state of the global run of the incremental net.

More formally, the incremental net $\texttt{incr}(\mathcal{N}, \sigma)$ is the preT-PPN $(\mathcal{P}, \mathcal{T}, \textsc{Pre}, \textsc{Post}, \mathbb{P})$ such that $\mathcal{P} = \left\{p_j^i \mid 0 \leq i \leq k \wedge 0 \leq j \leq n\right\}$ where p_0^i represents the lock related to the i^{th} copy of $\mathcal{N}$ whereas p_j^i with $j > 0$ represents the copy of the place p_j of P in the i^{th} subnet and $\mathcal{T} = \left\{t_j^i \mid 0 \leq i \leq k \wedge 1 \leq j \leq m\right\} \cup \left\{\theta_j^i \mid 1 \leq j \leq i \leq k\right\} \cup \left(\cup_{1 \leq i \leq k} \left\{\theta_i^0\right\}\right)$ where t_j^i represents the copy of the transition t_j of T in the i^{th} subnet, θ_j^i represents the copy of the transition θ_j of Θ in the i^{th} subnet, θ_i^0 represents a copy of the transition θ_i from σ which is used to change the active copy (from the $i-1^{th}$ to the i^{th}).

We define with this construction a net mapping to relate places and transitions from both models. Given the two nets $\mathcal{N}$ and $\mathcal{I}$ defined as above, considering previous notations, for each $0 \leq i \leq |\sigma|$ we define the application $f^i_{\mathcal{N}\to\mathcal{I}}$ that links the original net $\mathcal{N}$ to its i^{th} copy in $\mathcal{I}$ (except the corresponding lock). We define $f^i_{\mathcal{N}\to\mathcal{I}} : T \cup \{\theta_j \in \sigma \mid j \leq i\} \cup P \to \mathcal{T} \cup \mathcal{P}$ such that for $t_j \in T$ (resp. $\theta_j \in \sigma$ and $p_j \in P$), $f^i_{\mathcal{N}\to\mathcal{I}}(t_j) = t_j^i$ (resp. $f^i_{\mathcal{N}\to\mathcal{I}}(\theta_j) = \theta_j^i$ and $f^i_{\mathcal{N}\to\mathcal{I}}(p_j) = p_j^i$). We can then define f^{-1} the function that maps components of the copies

[3] The exact meaning of the notations used to refer to the different components of this example will be provided after this informal intuition on the construction.

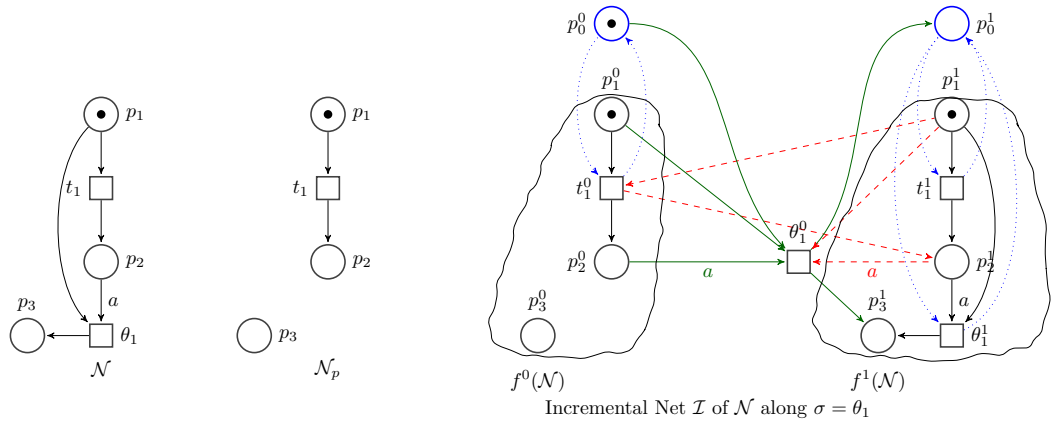


Figure 3 Construction of an Incremental Net3.

of $\mathcal{N}$ in $\mathcal{I}$ to their original fiber by the previous application. Formally, f^{-1} is defined by: $f^{-1} : \cup_{0\leq i\leq k} \mathtt{im}(f^i_{\mathcal{N}\to\mathcal{I}}) \to T \cup P \cup \Theta$ and associates to $t^i_j \in \mathcal{T}$ (resp. $\theta^i_j \in \mathcal{T}$ and $p^i_j \in \mathcal{P}$) $f^{-1}(t^i_j) = t_j$ (resp. $f^{-1}(\theta^i_j) = \theta_j$ and $f^{-1}(p^i_j) = p_j$). Finally, we define the application $h_{\mathcal{I}\to\mathcal{N}} : \cup_{1\leq i\leq k}\{\theta^0_i\} \subseteq \mathcal{T} \to \Theta$ that maps the intermediate parametric transitions between each copies of $\mathcal{N}$ in $\mathcal{I}$, θ^0_i to their original fiber from $\mathcal{N}$ and occuring in σ, that is to say the i^{th} transition of σ.

Those applications allow us to define formally the functions Pre and Post. Given i' and j', let $x^{i'}_{j'}$ denote either t^i_j or θ^i_j from $\left\{t^i_j \mid 0 \leq i \leq k \wedge 1 \leq j \leq m\right\} \cup \left\{\theta^i_j \mid 1 \leq j \leq i \leq k\right\}$ in the following expressions:

- $\textsc{Pre}(p^i_j, x^{i'}_{j'})(resp.\ \textsc{Post}(p^i_j, x^{i'}_{j'})) = \begin{cases} 0 \text{ if } (i < i') \text{ or } (i > i' \text{ and } j = 0) \\ 1 \text{ if } i = i' \text{ and } j = 0 \\ Pre(f^{-1}(p^i_j), f^{-1}(x^{i'}_{j'})) \text{ otherwise} \\ (resp.\ Post(f^{-1}(p^i_j), f^{-1}(x^{i'}_{j'}))) \text{ otherwise} \end{cases}$
- $\textsc{Pre}(p^i_j, \theta^0_{i'}) = \begin{cases} 0 & \text{if } (i+1 < i') \text{ or } (i+1 > i' \text{ and } j = 0) \\ 1 & \text{if } i+1 = i' \text{ and } j = 0 \\ Pre(f^{-1}(p^i_j), h^{-1}(\theta^0_{i'})) & \text{otherwise} \end{cases}$
- $\textsc{Post}(p^i_j, \theta^0_{i'}) = \begin{cases} 0 & \text{if } (i < i') \text{ or } (i > i' \text{ and } j = 0) \\ 1 & \text{if } i = i' \text{ and } j = 0 \\ Post(f^{-1}(p^i_j), h^{-1}(\theta^0_{i'})) & \text{otherwise} \end{cases}$

Given a net $\mathcal{N}$ and the function $f^i_{\mathcal{N}\to\mathcal{I}}$ we extend the definition of $f^i_{\mathcal{N}\to\mathcal{I}}$ to sets by $f^i_{\mathcal{N}\to\mathcal{I}}(X) = \{f^i(x) \mid x \in X\}$ and nets by defining $f^i_{\mathcal{N}\to\mathcal{I}}(\mathcal{N})$ as $(f^i(P), f^i(T), \textsc{Pre}_{f^i(P)\times f^i(T)}, \textsc{Post}_{f^i(P)\times f^i(T)}, \mathbb{P})$. When the context is clear, we omit the subscript $\mathcal{N} \to \mathcal{I}$. As examples, $f^0(\mathcal{N})$ and $f^1(\mathcal{N})$ are provided in Figure 3. Finally, we associate to a marking m of $\mathcal{N}$ the marking $f(m)$ defined by for all p of $\cup_{0\leq i\leq k}(f^i(P))$, $f(m)(p) = m(f^{-1}(p))$. Notice that this ignores the locks introduced in the net. Given the initial marking m_0, we thus define the initial marking of the incremental net μ_0 as $f(m_0)$ for the copies of the places, and 0 in all locks except the first one which receives 1. Formally, $\mu_0(p^i_j) = m_0(p_j)$ if $j \neq 0$, 1 if $i = j = 0$ and 0 otherwise.

The idea behind this construction is double. First, we can enforce the order of the first occurrence of a parametric transition which is dictated by the sequence σ. Second, we can access the exact amount of tokens stored in a place before the firing of the first occurrence of a parametric transition and thus keep an historic of the state of a run, just before the

firing of this parametric transition, through the copies of the original net. Based on those two observations, we will be able to observe if the input places of the parametric arcs of the first occurrence of a parametric transition in a run are bounded or not.

4.3 Complexity of Universal Simultaneous Unboundedness

We will now see that universal simultaneous unboundedness can be reduced to the existence of a sequence σ of distinct parametric transitions such that the incremental net built upon this sequence is simultaneous unbounded on an adequately defined set of places. We first provide the intuition behind this statement before providing its formal version. We must ensure that each input place of a parametric arc of the first occurrence of a parametric transition is unbounded. Based on the previous construction, one can notice that given a parametric transition θ_i occurring in σ, its input places are only impacted by the transitions occurring in the first i copies of the net. Thus, once θ_i is fired in the incremental net, the new feasible transitions will not impact the amount of tokens stored in the i first subnets. We will thus be able to verify if the input places were bounded or not before its firing, by observing the places of the copy that occurs just before the first firing of this transition. For each parametric transition of σ, we should thus verify that the input places in the corresponding copies are unbounded, and finally verify that the places that should be unbounded as part of the original property are indeed unbounded in the last copy of the net and that for each instance of the incremental net under a uniform valuation. Nevertheless, since the corresponding input places of parametric transitions are unbounded, it is sufficient to verify this property for only one instance of the incremental net, and in particular we will later choose the $\mathbf{0}$-instance. Indeed, if such a property is verified for any $\mathbf{k}$-instance, then, it could be verified for any $\mathbf{k'}$-instance (with $k' > k$) by exhibiting the witness run and performing more loops.

▶ **Lemma 15.** *Let $\mathcal{N} = (P, T', Pre, Post, \mathbb{P})$ be a preT-PPN, such that $T' = T \cup \Theta$ where Θ represents the parametric transitions of $\mathcal{N}$ and T its plain transitions. For every a set of places of $X \subseteq P$, the following propositions are equivalent:*

1. *$(\mathcal{N}, m_0)$ is universally simultaneous X unbounded*
2. *$\exists \sigma = t_1, ..., t_l \in Pref(S_\Theta)$, considering the incremental model $\mathcal{I}$ of $\mathcal{N}$ along σ, $\mathcal{I} = \boldsymbol{incr}(\mathcal{N}, \sigma)$, $\exists k \in \mathbb{N}$, $(\mathbf{k}(\mathcal{I}), \mu_0)$ is simultaneous Y unbounded where $Y = f^l_{\mathcal{N}\to\mathcal{I}}(X) \cup \left(\bigcup_{t_i \in \sigma} f^{i-1}_{\mathcal{N}\to\mathcal{I}}(\Pi(t_i))\right)$.*
3. *$\exists \sigma = t_1, ..., t_l \in Pref(S_\Theta)$, considering the incremental model $\mathcal{I}$ of $\mathcal{N}$ along σ, $\mathcal{I} = \boldsymbol{incr}(\mathcal{N}, \sigma)$, $\forall k \in \mathbb{N}$, $(\mathbf{k}(\mathcal{I}), \mu_0)$ is simultaneous Y unbounded where $Y = f^l_{\mathcal{N}\to\mathcal{I}}(X) \cup \left(\bigcup_{t_i \in \sigma} f^{i-1}_{\mathcal{N}\to\mathcal{I}}(\Pi(t_i))\right)$.*

Following the notations, $Pref(S_\Theta)$ corresponds to the finite set of sequences of distinct parametric transitions. Remark that $f^l_{\mathcal{N}\to\mathcal{I}}(X)$ represents the copy of the places of X in the last subnet of $\mathcal{I}$. The set $\bigcup_{t_i \in \sigma} f^{i-1}_{\mathcal{N}\to\mathcal{I}}(\Pi(t_i))$ is a bit more complex: for each transition $t_i \in \sigma$, $\Pi(t_i)$ represents the input places of the parametric arcs. We therefore address here the unboundedness of the copies of those places in the corresponding subnet of the $\mathcal{I}$

Proof. We provide here the sketch of the implication (1) ⇒ (3). The goal is to find the sequence of parametric transitions along which we construct the incremental model seen in Section 4.2. This proof is done by induction on the number of parametric transitions in $\mathcal{N}$.

- In the base case, $\mathcal{N}$ is a PN. Therefore the incremental model considered is isomorphic to $\mathcal{N}$ and the property is immediate.

- In the inductive step, the case where it is possible that the places from X are unbounded without using parametric transitions is straightforward. In the other case, we show that there is a parametric transition θ that can be used as the first parametric transition occurring in a run leading to some simultaneous X unbounded markings in the coverability tree of $(\mathbf{k}(\mathcal{N}), m_0)$ for an infinite number of uniform valuation $\mathbf{k}$.
  1. We then prove that the input places of the parametric input arcs of θ must be unbounded in $(\mathcal{N}_p, m_0)$, that is to say, there is a marking z with some ω's on those components in the basis of the coverability set of this net.
  2. We now consider the net where those places have been removed and the projection of the marking obtained by firing θ from z. In this net θ is then a plain transition. Using the induction assumption, we can build an incremental model $\mathcal{J}$ along a sequence σ' for this reduced net.
  3. The end of this proof consists in building the incremental model $\mathcal{I}$ of $\mathcal{N}$ along the sequence $\theta\sigma'$ and to construct the set Y. There is no particular difficulty in this last point but the construction is a bit involved.

◀

From Lemma 15 we can observe that answering the universal simultaneous X unboundedness on a preT-PPN can be reduced to guessing an element σ of the finite set $Pref(S_\Theta)$ such that $(\mathbf{0}(\mathcal{I}), \mu_0)$ is simultaneous Y unbounded. Since checking simultaneous X unboundedness can be done in ExpSpace as recalled in Theorem 13, we obtain a NExpSpace procedure. Then, a well-known theorem by Savitch [19] stating that there is therefore a ExpSpace deterministic procedure answering this problem and the Remark from Section 3.1 allow us to deduce Theorem 16. Note that the following Corollary directly comes from Theorem 16 and Lemma 11.

▶ **Theorem 16.** *The Universal Simultaneous Unboundedness problem for preT-PPNs is ExpSpace-complete.*

▶ **Corollary 17.** *Given a marked preT-PPN $\mathcal{S}$ and a marking m, we can compute a finite representation of $\mathcal{CV}(\mathcal{S}, m)$.*

5 Synthesis in postT-PPNs and distinctT-PPNs

5.1 Complexity of Existential Coverability in postT-PPNs

We propose here a cleaner proof for the decidability of the existential coverability in postT-PPNs, and provide its complexity. We use a polynomial time transformation[4] from postT-PPN to ωPN (see [13]) which preserves existential coverability and invoke a transformation from ωPN to PN underlined in [13]. First, we recall definitions and results from [13].

▶ **Definition 18** (ω-Petri Net [13])**.** An ω-Petri Net (ωPN) is a tuple $(P, T, Pre, Post)$ where P and T are respectively a finite set of places and transitions. Pre (resp. $Post$) is a function of $P \times T$ to $\mathbb{N}_\omega$ that gives the input (resp. output) effect of a transition t on a place p.

▶ **Definition 19** (ωPN Semantics)**.** Given a marking m, and a transition t such that $m \geq Pre(t)$, firing t from m gives a new marking m' s.t. $\forall p \in P, m'(p) = m(p) - Pre(p,t) + o$

[4] More specifically we obtain an ω-output-PN or ωOPN for short, which corresponds to the natural subclass of ωPNs where $Pre \in P \times T \mapsto \mathbb{N}$.

where $o = Post(t,p)$ if $Post(p,t) \in \mathbb{N}$ and $o \geq 0$ if $Post(p,t) = \omega$. We denote this by $m \xrightarrow{t} m'$. Thus $Post(p,t) = \omega$ means that an arbitrary number of tokens are generated in p.

From this semantics, we can notice that ω's play a role not unlike output parameters, with the crucial difference that their "value" can change along the execution of the net. Let consider a postT-PPN $\mathcal{N}$ and let us associate to this model the ωOPN $\mathcal{N}'$ such that we replace each parametric arc of $\mathcal{N}$ by an ω arc.

▶ **Lemma 20** (postT-PPNs to ωOPNs). *Let $\mathcal{N}$ be a postT-PPN (which involves parameters of a set $\mathbb{P}$) and let $\mathcal{N}'$ be its corresponding ωOPN (with the same set of places and transitions) and let m_0 be their common initial marking. Given a marking $m \in \mathcal{RS}(\mathcal{N}', m_0)$, there exists a valuation v such that there exists a marking $m' \geq m$ with $m' \in \mathcal{RS}(v(\mathcal{N}), m_0)$. Moreover*[5]*, $\cup_{v \in \mathbb{N}^{\mathbb{P}}} \mathcal{RS}(v(\mathcal{N}), m_0) \subseteq \mathcal{RS}(\mathcal{N}', m_0)$.*

We can thus directly deduce Theorem 21 by reducing existential coverability in postT-PPNs to coverability in ωPN which belongs to ExpSpace by [13]. Note that the following Corollary comes from Theorem 21 and Lemma 11.

▶ **Theorem 21** (Complexity of Existential Coverability). *The existential coverability problem on postT-PPNs is ExpSpace-complete.*

▶ **Corollary 22.** *Given a marked postT-PPN $\mathcal{S}$ and a marking m, we can compute a finite representation of $\mathcal{CV}(\mathcal{S}, m)$.*

5.2 Representing the Coverability Synthesis Set for DistinctT-PPNs

Let us finally consider the case of PPNs in which the set of parameters used as input weights, and the set of parameters used as output weights, are disjoint. For this class, called distinctT-PPNs, the emptiness of the solution set to the synthesis problem for coverability is decidable [7]. Interestingly, we can adapt an idea originally used for L/U-automata in [15] to prove that the structure of this set is however much more complex than for preT-PPNs or postT-PPNs. In particular, one cannot represent this set with a finite set, a finite union of downward and/or upward closed sets or a finite union of convex polyhedra.

▶ **Lemma 23.** *If it can be computed, the solution of the synthesis of coverability in distinctT-PPN cannot, in general, be represented using any formalism for which emptiness of the intersection with equality constraints is decidable.*

6 Conclusion

It can be challenging to find meaningful parametric infinite state systems with decidable decision problems. We achieved here to prove a powerful result for two strict syntactical subclasses of parametric Petri nets: interestingly, the set of all valid valuations of parameters, allowing to cover a given marking, is effectively computable for parametric Petri nets where parameters are restricted to only input arcs or only output arcs.

Indeed, we have shown how the computability of the synthesis set for coverability in preT-PPNs and postT-PPNs can be reduced to a decision problem, respectively, universal coverability and existential coverability, which is then used in Valk and Jantzen's procedure.

[5] Notice that this is only an inclusion. Indeed, contrarily to postT-PPNs, in ωPNs, the effect of an arc can change along the same execution.

We proved that these two decision problems are both ExpSpace-complete. Putting the two types of parameters together while forbidding any parameter to be used as both an input and output weight preserves the decidability of the emptiness of the solution set. However, we have proved that, even with this restriction, the solution set can in general not be represented using any formalism for which emptiness of the intersection with equality constraints is decidable, which seems a big restriction in practice.

Future work includes studying (simultaneous) unboundedness for classes other than preT-PPNs that is to say postT-PPNs and distinctT-PPNs. Most problems (such as universal simultaneous unboundedness for postT-PPNs and distincT-PPNs) can be settled easily by adapting the proofs of [7], except for existential simultaneous unboundedness for postT-PPNs. The translation to ωPNs proposed here is not sufficient to conclude its decidability either since we have to ensure that the increasing markings are all reached for a common parameter valuation.

Acknowledgements. We are grateful to anonymous commenters whose helpful remarks have allowed us to improve this paper.

References

1 Rajeev Alur, Thomas A. Henzinger, and Moshe Y. Vardi. Parametric real-time reasoning. In *ACM Symposium on Theory of Computing*, pages 592–601, 1993.

2 Florent Avellaneda and Rémi Morin. Vector Addition Systems with States vs. Petri Nets. report, Laboratoire d'informatique Fondamentale de Marseille - LIF, October 2012.

3 Ahmed Bouajjani and Richard Mayr. Model checking lossy vector addition systems. In *16th Annual Symposium on Theoretical Aspects of Computer (STACS'09)*, pages 323–333, Trier, Germany, March 1999. Springer.

4 Laura Bozzelli and Salvatore La Torre. Decision problems for lower/upper bound parametric timed automata. *Formal Methods in System Design*, 35(2):121–151, 2009.

5 Giovanni Chiola, Susanna Donatelli, and Giuliana Franceschinis. On Parametric PT nets and their Modelling Power. *12th International Conference Application and Theory of Petri Nets (ICATPN'91)*, page 206, 1991.

6 Jean-Michel Couvreur, Denis Poitrenaud, and Pascal Weil. *Branching Processes of General Petri Nets*, pages 129–148. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

7 Nicolas David, Claude Jard, Didier Lime, and Olivier H. Roux. Discrete parameters in Petri nets. In *The 36th International Conference on Application and Theory of Petri Nets and Concurrency (Petri Nets 2015)*, volume 9115 of *LNCS*, pages 137–156. Springer, Brussels, Belgium, June 2015.

8 Stéphane Demri. On selective unboundedness of VASS. *Journal of Computer and System Sciences*, 79(5):689–713, August 2013.

9 Leonard Eugene Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413–422, 1913. URL: `http://www.jstor.org/stable/2370405`.

10 Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part I: Completions. In *26th Annual Symposium on Theoretical Aspects of Computer Science (STACS'09)*, volume 3 of *Leibniz International Proceedings in Informatics*, pages 433–444, Freiburg, Germany, February 2009. Leibniz-Zentrum für Informatik.

11 Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part II: complete WSTS. *Logical Methods in Computer Science*, 8(3), 2012.

12 Alain Finkel and Jérôme Leroux. Recent and simple algorithms for Petri nets. *Software & Systems Modeling*, 14(2):719–725, 2015.

13 Gilles Geeraerts, Alexander Heußner, M. Praveen, and Jean-François Raskin. ω-Petri nets: Algorithms and complexity. *Fundam. Inform.*, 137(1):29–60, 2015.

14 Jean Goubault-Larrecq. On a Generalization of a Result by Valk and Jantzen. Technical report, LSV, 2009.

15 Aleksandra Jovanović, Didier Lime, and Olivier H. Roux. Integer Parameter Synthesis for Real-Time Systems. *IEEE Transactions on Software Engineering*, 41(5):445–461, May 2015.

16 Richard M. Karp and Raymond E. Miller. Parallel program schemata. *Journal of Computer and System Sciences*, 3(2):147–195, 1969.

17 Marco Ajmone Marsan, Gianfranco Balbo, Gianni Conte, Susanna Donatelli, and Giuliana Franceschinis. *Modelling with Generalized Stochastic Petri Nets*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994.

18 Christophe Reutenauer. *The mathematics of Petri nets*. Prentice-Hall, Inc., April 1990.

19 Walter J. Savitch. Relationships between nondeterministic and deterministic tape complexities. *Journal of Computer and System Sciences*, 4(2):177–192, April 1970.

20 Rüdiger Valk and Matthias Jantzen. The residue of vector sets with applications to decidability problems in Petri nets. *Acta Informatica*, 21(6):643–674, 1985.

Divide and Congruence III: Stability & Divergence

Wan Fokkink[1], Rob van Glabbeek[2], and Bas Luttik[3]

1 **Vrije Universiteit Amsterdam, The Netherlands**
2 **Data61, CSIRO, Sydney, Australia and**
University of New South Wales, Sydney, Australia
3 **Eindhoven University of Technology, The Netherlands**

Abstract

In two earlier papers we derived congruence formats for weak semantics on the basis of a decomposition method for modal formulas. The idea is that a congruence format for a semantics must ensure that the formulas in the modal characterisation of this semantics are always decomposed into formulas that are again in this modal characterisation. Here this work is extended with important stability and divergence requirements. Stability refers to the absence of a τ-transition. We show, using the decomposition method, how congruence formats can be relaxed for weak semantics that are stability-respecting. Divergence, which refers to the presence of an infinite sequence of τ-transitions, escapes the inductive decomposition method. We circumvent this problem by proving that a congruence format for a stability-respecting weak semantics is also a congruence format for its divergence-preserving counterpart.

1998 ACM Subject Classification F.3.2 Operational Semantics, F.4.1 Modal Logic

Keywords and phrases Structural Operational Semantics, Weak Semantics, Modal Logic

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.15

1 Introduction

Structural operational semantics generates a labelled transition system, in which states are the closed terms over a signature, and transitions between states carry labels. Transitions are obtained from a transition system specification (TSS), consisting of proof rules called transition rules. States in labelled transition systems can be identified by a wide range of behavioural equivalences, based on e.g. branching structure or decorated versions of execution sequences. Weak semantics, which take into account the internal action τ, are classified in [11]. A significant number of the weak semantics based on a bisimulation relation carry a stability or divergence requirement. Stability refers to the absence of a τ-transition and divergence to the presence of an infinite sequence of τ-transitions.

In general a behavioural equivalence induced by a TSS is not guaranteed to be a congruence, i.e. the equivalence class of a term $f(p_1, \ldots, p_n)$ need not be determined by f and the equivalence classes of its arguments $p_1, \ldots, p_n$. Being a congruence is an important property, for instance in order to fit the equivalence into an axiomatic framework. Respecting stability or preserving divergence sometimes needs to be imposed in order to obtain a congruence relation, for example in case of the priority operator [1].

Modal logic captures observations an experimenter can make during a session with a process. A modal characterisation of an equivalence on processes consists of a class C of modal formulas such that two processes are equivalent if and only if they satisfy the same formulas in C. For instance, Hennessy-Milner logic [17] constitutes a modal characterisation of (strong) bisimilarity. A cornerstone for the current paper is the work in [3] to decompose formulas from Hennessy-Milner logic with respect to a structural operational semantics in

the ntyft format [16] without lookahead. Here the *decomposition* of a modal formula φ w.r.t. a term $t = f(p_1, \ldots, p_n)$ is a selection of n-tuples of modal formulas, one of which needs to be satisfied by the processes p_i in order for t to satisfy φ. Based on this method, formats for behavioural equivalences can be generated from their modal characterisation, ensuring that they are congruences. Such formats help to avoid repetitive congruence proofs and obtain insight into the congruence property. Key idea is that congruence is ensured if formulas from the modal characterisation C of an equivalence are always decomposed into formulas that are again in C. This approach was extended to weak semantics in [10, 7].

Here we expand the latter work to weak semantics that respect stability or preserve divergence. We focus on *branching bisimilarity* and *rooted branching bisimilarity* [15] and consider for each a stability-respecting and two divergence-preserving variants. Divergence-preserving branching bisimilarity [15] is the coarsest congruence relation for the parallel composition operator that only equates processes satisfying the same formulas from the well-known temporal logic CTL* minus the next-time operator X [14]. With regard to stability the expansion is relatively straightforward: we extend the modal characterisation of the semantics with one clause to capture that a semantics is stability-respecting, and study the decomposition of this additional clause. Next we show how the congruence formats for branching bisimilarity and rooted branching bisimilarity from [10] can be relaxed, owing to the extended modal characterisation for stability-respecting branching bisimilarity. Notably, the transition rules for the priority operator are within the more relaxed formats.

The divergence preservation property escapes the inductive decomposition method, as it concerns an infinite sequence of τ-transitions. We overcome this problem by a general framework for lifting congruence formats from a weak semantics $\approx$ to a finer semantics $\sim$. We show four applications of this method. In two cases $\approx$ is stability-respecting and in two cases rooted stability-respecting branching bisimilarity, while in $\sim$ stability is replaced by two different forms of divergence. Hence the congruence format for stability-respecting branching bisimilarity is also applicable to divergence-preserving as well as weakly divergence-preserving branching bisimilarity; and likewise for their rooted counterparts.

2 Preliminaries

2.1 Stability-respecting / divergence-preserving branching bisimilarity

A *labelled transition system (LTS)* is a triple $(\mathbb{P}, Act, \rightarrow)$, with $\mathbb{P}$ a set of *processes*, Act a set of *actions*, and $\rightarrow \subseteq \mathbb{P} \times Act \times \mathbb{P}$. We normally let $Act = A \cup \{\tau\}$ where τ is an *internal action* and A some set of external or observable actions not containing τ. We write A_τ for $A \cup \{\tau\}$. We use p, q to denote processes, α, β, γ for elements of A_τ, and a, b for elements of A. We write $p \xrightarrow{\alpha} q$ for $(p, \alpha, q) \in \rightarrow$, $p \xrightarrow{\alpha}$ for $\exists q \in \mathbb{P} : p \xrightarrow{\alpha} q$, and $p \not\xrightarrow{\alpha}$ for $\neg(p \xrightarrow{\alpha})$. Furthermore, $\stackrel{\epsilon}{\Longrightarrow}$ denotes the transitive-reflexive closure of $\xrightarrow{\tau}$.

▶ **Definition 1.** Let $\mathcal{B} \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation.

- $\mathcal{B}$ is a *branching bisimulation* if $p \mathcal{B} q$ and $p \xrightarrow{\alpha} p'$ implies either $\alpha = \tau$ and $p' \mathcal{B} q$, or $q \stackrel{\epsilon}{\Longrightarrow} q' \xrightarrow{\alpha} q''$ for some q' and q'' with $p \mathcal{B} q'$ and $p' \mathcal{B} q''$.
- $\mathcal{B}$ is *stability-respecting* if $p \mathcal{B} q$ and $p \not\xrightarrow{\tau}$ implies $q \stackrel{\epsilon}{\Longrightarrow} q' \not\xrightarrow{\tau}$ for some q' with $p \mathcal{B} q'$.
- $\mathcal{B}$ is *divergence-preserving* if it satisfies the following condition:

  (D) if $p \mathcal{B} q$ and there is an infinite sequence of processes $(p_k)_{k \in \mathbb{N}}$ with $p = p_0$, $p_k \xrightarrow{\tau} p_{k+1}$ and $p_k \mathcal{B} q$ for all $k \in \mathbb{N}$, then there exists an infinite sequence of processes $(q_\ell)_{\ell \in \mathbb{N}}$ with $q = q_0$, $q_\ell \xrightarrow{\tau} q_{\ell+1}$ for all $\ell \in \mathbb{N}$, and $p_k \mathcal{B} q_\ell$ for all $k, \ell \in \mathbb{N}$.

  The definition of a *weakly divergence-preserving* relation is obtained by omitting the condition "and $p_k \mathcal{B} q_\ell$ for all $k, \ell \in \mathbb{N}$."

Processes p, q are *branching bisimilar*, denoted $p \leftrightarrows_b q$, if there exists a branching bisimulation $\mathcal{B}$ with $p \mathcal{B} q$. They are *stability-respecting*, *divergence-preserving* or *weakly divergence-preserving* branching bisimilar, denoted $p \leftrightarrows_b^s q$, $p \leftrightarrows_b^\Delta q$ or $p \leftrightarrows_b^{\Delta^\top} q$, if moreover $\mathcal{B}$ is stability-respecting, divergence-preserving or weakly divergence-preserving, respectively.

We have $\leftrightarrows_b \supset \leftrightarrows_b^s \supset \leftrightarrows_b^{\Delta^\top} \supset \leftrightarrows_b^\Delta$. The relations $\leftrightarrows_b$, $\leftrightarrows_b^s$, $\leftrightarrows_b^\Delta$ and $\leftrightarrows_b^{\Delta^\top}$ are equivalences [2, 11, 13]. However, they are not *congruences* with respect to most process algebras from the literature, meaning that the equivalence class of a process $f(p_1, \ldots, p_n)$, with f an n-ary function symbol, is not always determined by the equivalence classes of its arguments, i.e. the processes $p_1, \ldots, p_n$. Therefore an additional rootedness condition is imposed.

▶ **Definition 2.** *Rooted* branching bisimilarity, $\leftrightarrows_{rb}$, is the largest symmetric relation on $\mathbb{P}$ with $p \leftrightarrows_{rb} q$ and $p \xrightarrow{\alpha} p'$ implies $q \xrightarrow{\alpha} q'$ for some q' with $p' \leftrightarrows_b q'$. Likewise, *rooted* stability-respecting, divergence-preserving or weakly divergence-preserving branching bisimilarity, denoted by $p \leftrightarrows_{rb}^s q$, $p \leftrightarrows_{rb}^\Delta q$ or $p \leftrightarrows_{rb}^{\Delta^\top} q$, is the largest symmetric relation $\mathcal{R}$ on $\mathbb{P}$ with $p \mathcal{R} q$ and $p \xrightarrow{\alpha} p'$ implies $q \xrightarrow{\alpha} q'$ for some q' with $p' \leftrightarrows_b^s q'$, $p' \leftrightarrows_b^\Delta q'$ or $p' \leftrightarrows_b^{\Delta^\top} q'$.

The various notions of bisimilarity defined above are examples of so-called behavioural equivalences. For a general formulation of the results in Section 4, it is convenient to formally define a notion of behavioural equivalence that includes at least the examples above. Note that a common feature of their definitions is that they associate with every LTS $\mathrm{G} = (\mathbb{P}_\mathrm{G}, Act_\mathrm{G}, \rightarrow_\mathrm{G})$ a binary relation $\sim_\mathrm{G}$. (For instance, in the case of branching bisimilarity, the relation $\sim_\mathrm{G}$ associated with G is defined as the binary relation $\leftrightarrows_b \subseteq \mathbb{P}_\mathrm{G} \times \mathbb{P}_G$ such that $p \leftrightarrows_b q$ if (and only if) there exists a branching bisimulation $\mathcal{B} \subseteq \mathbb{P}_\mathrm{G} \times \mathbb{P}_G$ with $p \mathcal{B} q$.) One may, thus, think of a behavioural equivalence as a family of binary relations indexed by LTSs. It turns out that we need to impose just one extra condition on such families to arrive at a suitable formalisation of the notion of behavioural equivalence. The condition states that the relation associated with the disjoint union of two LTSs restricted to one of the components coincides with the relation associated with that component.

Two LTSs $\mathrm{G} = (\mathbb{P}_\mathrm{G}, Act_\mathrm{G}, \rightarrow_\mathrm{G})$ and $\mathrm{H} = (\mathbb{P}_\mathrm{H}, Act_\mathrm{H}, \rightarrow_\mathrm{H})$ are called *disjoint* if $\mathbb{P}_\mathrm{G} \cap \mathbb{P}_\mathrm{H} = \emptyset$. In that case $\mathrm{G} \uplus \mathrm{H}$ denotes their union $(\mathbb{P}_\mathrm{G} \cup \mathbb{P}_\mathrm{H}, Act_\mathrm{G} \cup Act_\mathrm{H}, \rightarrow_\mathrm{G} \cup \rightarrow_\mathrm{H})$.

A *behavioural equivalence* $\sim$ on LTSs is a family of equivalence relations $\sim_\mathrm{G}$, one for every LTS G, such that for each pair of disjoint LTSs $\mathrm{G} = (\mathbb{P}_\mathrm{G}, Act_\mathrm{G}, \rightarrow_\mathrm{G})$ and H we have $g \sim_\mathrm{G} g' \Leftrightarrow g \sim_{\mathrm{G}\uplus\mathrm{H}} g'$ for any $g, g' \in \mathbb{P}_\mathrm{G}$. The notions of bisimilarity defined above clearly qualify as behavioural equivalences. We write $\sim \subseteq \approx$ iff $\sim_\mathrm{G} \subseteq \approx_\mathrm{G}$ for each LTS G.

2.2 Modal logic

Modal logic formulas express behavioural properties of processes. Following [11], we extend Hennessy-Milner logic [17] with connectives $\langle\epsilon\rangle\varphi$ and $\langle\hat{\tau}\rangle\varphi$, expressing that a process can perform zero or more, respectively zero or one, τ-transitions to a process where φ holds.

▶ **Definition 3.** The class $\mathbb{O}$ of *modal formulas* is defined as follows, where I ranges over all index sets and α over A_τ: $\varphi ::= \bigwedge_{i\in I} \varphi_i \mid \neg\varphi \mid \langle\alpha\rangle\varphi \mid \langle\epsilon\rangle\varphi \mid \langle\hat{\tau}\rangle\varphi$. We write $\top$ for the empty conjunction, $\varphi_1 \wedge \varphi_2$ for $\bigwedge_{i\in\{1,2\}} \varphi_i$, $\varphi\langle\alpha\rangle\varphi'$ for $\varphi \wedge \langle\alpha\rangle\varphi'$, and $\varphi\langle\hat{\tau}\rangle\varphi'$ for $\varphi \wedge \langle\hat{\tau}\rangle\varphi'$.

$p \models \varphi$ denotes that process p satisfies formula φ. The first two operators represent the standard Boolean operators conjunction and negation. By definition, $p \models \langle\alpha\rangle\varphi$ if $p \xrightarrow{\alpha} p'$ for some p' with $p' \models \varphi$, $p \models \langle\epsilon\rangle\varphi$ if $p \stackrel{\epsilon}{\Longrightarrow} p'$ for some p' with $p' \models \varphi$, and $p \models \langle\hat{\tau}\rangle\varphi$ if either $p \models \varphi$ or $p \xrightarrow{\tau} p'$ for some p' with $p' \models \varphi$.

For each $L \subseteq \mathbb{O}$, we write $p \sim_L q$ if p and q satisfy the same formulas in L. We say that L is a *modal characterisation* of some behavioural equivalence $\sim$ if $\sim_L$ coincides with $\sim$. We write $\varphi \equiv \varphi'$ if $p \models \varphi \Leftrightarrow p \models \varphi'$ for all processes p. The class $L^{\equiv}$ denotes the closure of $L \subseteq \mathbb{O}$ under $\equiv$. Trivially, $p \sim_L q \Leftrightarrow p \sim_{L^{\equiv}} q$.

▶ **Definition 4** ([11]). The subclasses $\mathbb{O}_b$ and $\mathbb{O}_{rb}$ of $\mathbb{O}$ are defined as follows, where a ranges over A and α over A_τ:

$$\begin{array}{lllll} \mathbb{O}_b & \varphi & ::= & \bigwedge_{i\in I} \varphi_i \mid \neg\varphi \mid \langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\varphi) \mid \langle\epsilon\rangle(\varphi\langle a\rangle\varphi) & \\ \mathbb{O}_{rb} & \overline{\varphi} & ::= & \bigwedge_{i\in I} \overline{\varphi}_i \mid \neg\overline{\varphi} \mid \langle\alpha\rangle\varphi \mid \varphi & (\varphi \in \mathbb{O}_b) \end{array}$$

$\mathbb{O}_b$ and $\mathbb{O}_{rb}$ are modal characterisations of $\leftrightarrows_b$ and $\leftrightarrows_{rb}$, respectively (see [10]).

The idea behind stability is: (I) if $p \leftrightarrows_b^s q$ and $p \stackrel{\epsilon}{\Longrightarrow} p' \stackrel{\tau}{\not\longrightarrow}$ with $p \leftrightarrows_b^s p'$, then $q \stackrel{\epsilon}{\Longrightarrow} q' \stackrel{\tau}{\not\longrightarrow}$ with $q \leftrightarrows_b^s q'$. In the definition of $\leftrightarrows_b^s$ this was formulated more weakly: (II) if $p \leftrightarrows_b^s q$ and $p \stackrel{\tau}{\not\longrightarrow}$, then $q \stackrel{\epsilon}{\Longrightarrow} q' \stackrel{\tau}{\not\longrightarrow}$ with $q \leftrightarrows_b^s q'$. It is easy to see that formulations (I) and (II) are equivalent. The additional clause in the following modal characterisation of stability-respecting branching bisimilarity is based on (I).

▶ **Definition 5** ([11]). The subclasses $\mathbb{O}_b^s$ and $\mathbb{O}_{rb}^s$ of $\mathbb{O}$ are defined as follows:

$$\begin{array}{lllll} \mathbb{O}_b^s & \varphi & ::= & \bigwedge_{i\in I} \varphi_i \mid \neg\varphi \mid \langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\varphi) \mid \langle\epsilon\rangle(\varphi\langle a\rangle\varphi) \mid \langle\epsilon\rangle(\neg\langle\tau\rangle\top \wedge \overline{\varphi}) & (\overline{\varphi} \in \mathbb{O}_{rb}^s) \\ \mathbb{O}_{rb}^s & \overline{\varphi} & ::= & \bigwedge_{i\in I} \overline{\varphi}_i \mid \neg\overline{\varphi} \mid \langle\alpha\rangle\varphi \mid \varphi & (\varphi \in \mathbb{O}_b^s) \end{array}$$

The additional clause $\langle\epsilon\rangle(\neg\langle\tau\rangle\top \wedge \overline{\varphi})$ in $\mathbb{O}_b^s$ expresses stability. The first part $\langle\epsilon\rangle(\neg\langle\tau\rangle\top \ldots)$ captures $p \stackrel{\epsilon}{\Longrightarrow} p' \stackrel{\tau}{\not\longrightarrow}$, while the second part $\ldots \wedge \overline{\varphi}$ captures the stability-respecting branching bisimulation class of p'. Since $p' \stackrel{\tau}{\not\longrightarrow}$, unrooted and rooted stability-respecting branching bisimilarity coincide for p', which allows us to take the second part from $\mathbb{O}_{rb}^s$.

▶ **Theorem 6.** *$p \leftrightarrows_b^s q \Leftrightarrow p \sim_{\mathbb{O}_b^s} q$ and $p \leftrightarrows_{rb}^s q \Leftrightarrow p \sim_{\mathbb{O}_{rb}^s} q$, for all $p, q \in \mathbb{P}$.*

All proofs omitted from the paper can be found in [8].

2.3 Structural operational semantics

A *signature* is a set Σ of function symbols f with arity $ar(f)$. A function symbol of arity 0 is called a *constant*. Let V be an infinite set of variables; we assume $|\Sigma|, |A| \leq |V|$. A syntactic object is *closed* if it does not contain any variables. The sets $\mathbb{T}(\Sigma)$ and $\mathrm{T}(\Sigma)$ of terms over Σ and V and closed terms over Σ, respectively, are defined as usual; t, u, v, w denote terms, p, q denote closed terms, and $var(t)$ is the set of variables that occur in term t. A substitution σ is a partial function from V to $\mathbb{T}(\Sigma)$. A closed substitution is a total function from V to closed terms. The domain of substitutions is extended to $\mathbb{T}(\Sigma)$ as usual.

Structural operational semantics generates an LTS in which the processes are the closed terms. The labelled transitions between processes are obtained from a transition system specification, which consists of a set of proof rules called transition rules.

A (*positive* or *negative*) *literal* is an expression $t \xrightarrow{\alpha} u$ or $t \stackrel{\alpha}{\not\longrightarrow}$. A *(transition) rule* is of the form $\frac{H}{\lambda}$ with H a set of literals called the *premises*, and λ a literal called the *conclusion*; the terms at the left- and right-hand side of λ are called the *source* and *target*. A rule $\frac{\emptyset}{\lambda}$ is also written λ. A rule is *standard* if it has a positive conclusion. A *transition system specification (TSS)*, written (Σ, Act, R), consists of a signature Σ, a set of actions Act, and a set R of transition rules over Σ. A TSS is *standard* if all its rules are.

A TSS specifies an LTS in which the transitions are the closed positive literals that can be proved using the rules of the TSS. Since rules may have negative premises, consistency is a concern. Literals $t \xrightarrow{\alpha} u$ and $t \stackrel{\alpha}{\not\longrightarrow}$ are said to *deny* each other; a notion of provability

associated with TSSs is *consistent* if it is not possible to prove two literals that deny each other. To arrive at a consistent notion of provability, we proceed in two steps: first we define the notion of an *irredundant proof*, which on a standard TSS does not allow the derivation of negative literals at all, and then arrive at a notion of *well-supported proof* that allows the derivation of negative literals whose denials are manifestly underivable by irredundant proofs. In [12] it was shown that the notion of well-supported provability is consistent.

▶ **Definition 7** ([3]). Let $P = (\Sigma, Act, R)$ be a TSS. An *irredundant proof* from P of a rule $\frac{H}{\lambda}$ is a well-founded tree with the nodes labelled by literals and some of the leaves marked "hypothesis", such that the root has label λ, H is the set of labels of the hypotheses, and if μ is the label of a node that is not a hypothesis and K is the set of labels of the children of this node then $\frac{K}{\mu}$ is a substitution instance of a rule in R. The rule $\frac{H}{\lambda}$ is irredundantly provable from P, notation $P \vdash_{irr} \frac{H}{\lambda}$, if such a proof exists.

▶ **Definition 8** ([12]). Let $P = (\Sigma, Act, R)$ be a standard TSS. A *well-supported proof* from P of a closed literal λ is a well-founded tree with the nodes labelled by closed literals, such that the root is labelled by λ, and if μ is the label of a node and K is the set of labels of the children of this node, then:

1. either μ is positive and $\frac{K}{\mu}$ is a closed substitution instance of a rule in R;
2. or μ is negative and for each set N of closed negative literals with $\frac{N}{\nu}$ irredundantly provable from P and ν a closed positive literal denying μ, a literal in K denies one in N.

$P \vdash_{ws} \lambda$ denotes that a well-supported proof from P of λ exists. A standard TSS P is *complete* if for each p and α, either $P \vdash_{ws} p \stackrel{\alpha}{\not\longrightarrow}$ or $P \vdash_{ws} p \stackrel{\alpha}{\longrightarrow} q$ for some closed term q.

If $P = (\Sigma, Act, R)$ is a complete TSS, then the LTS *associated with* P is $(\mathrm{T}(\Sigma), Act, \rightarrow)$ with $\rightarrow = \{(p, \alpha, q) \mid P \vdash_{ws} p \stackrel{\alpha}{\longrightarrow} q\}$. We do not associate an LTS with an incomplete TSS.

2.4 Congruence formats

Let $P = (\Sigma, Act, R)$ be a transition system specification, and let $\sim_P$ be an equivalence relation defined on the set of closed terms $\mathrm{T}(\Sigma)$. Then $\sim_P$ is a *congruence* for P if, for each $f \in \Sigma$, we have that $p_i \sim_P q_i$ implies $f(p_1, \ldots, p_{ar(f)}) \sim_P f(q_1, \ldots, q_{ar(f)})$. Note that this is the case if for each open term $t \in \mathbb{T}(\Sigma)$ and each pair of closed substitutions $\rho, \rho' : V \to \mathrm{T}(\Sigma)$ we have $(\forall x \in var(t).\ \rho(x) \sim_P \rho'(x)) \Rightarrow \rho(t) \sim_P \rho'(t)$.

Every complete TSS generates an LTS of which the states are the closed terms of the TSS. Thus each behavioural equivalence $\sim$ associates with every TSS P an equivalence $\sim_P$ on its set of closed terms. By a *congruence format* for $\sim$ we mean a class of TSSs such that for every TSS P in the class the equivalence $\sim_P$ is a congruence. Usually, a congruence format is defined by means of a list of syntactic restrictions on the rules of TSSs.

In an *ntytt rule*, right-hand sides of positive premises are distinct variables that do not occur in the source. An ntytt rule is an *ntyxt rule* if its source is a variable, an *ntyft rule* if its source contains exactly one function symbol and no multiple occurrences of variables, and an *nxytt rule* if the left-hand sides of its premises are variables. A variable in a rule is *free* if it occurs neither in the source nor in right-hand sides of premises. A rule has *lookahead* if some variable occurs in the right-hand side of a premise and in the left-hand side of a premise. A rule is *decent* if it has no lookahead and does not contain free variables. Each combination of syntactic restrictions on rules induces a format for TSSs of the same name. For instance, a TSS is in decent ntyft format if it contains decent ntyft rules only. A TSS is in *ready simulation format* if it consists of ntyft and ntyxt rules that have no lookahead.

In congruence formats for weak semantics, lookahead must be forbidden. To see this, consider CCS [18] extended with an operator f defined by $\frac{x \xrightarrow{a} y \quad y \xrightarrow{b} z}{f(x) \xrightarrow{c} z}$. Then $ab\mathbf{0} \leftrightarrows_{rb} a\tau b\mathbf{0}$, whereas $f(ab\mathbf{0}) \not\leftrightarrows_{rb} f(a\tau b\mathbf{0})$. Therefore congruence formats for weak semantics are generally obtained by imposing additional restrictions on the ready simulation format.

2.5 Decomposition of modal formulas

The decomposition method from [10] gives a special treatment to arguments of function symbols that are deemed *patient*; a predicate marks these arguments. Let Γ be a predicate on arguments of function symbols. A standard ntyft rule is a Γ-*patience rule* [6] if it is of the form $\frac{x_i \xrightarrow{\tau} y}{f(x_1,\ldots,x_{ar(f)}) \xrightarrow{\tau} f(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_{ar(f)})}$ with $\Gamma(f,i)$. A TSS is Γ-*patient* if it contains all Γ-patience rules. A standard ntytt rule is Γ-*patient* if it is irredundantly provable from the Γ-patience rules; else it is called Γ-*impatient*. Let Γ be a predicate on $\{(f,i) \mid 1 \leq i \leq ar(f),\ f \in \Sigma\}$. If $\Gamma(f,i)$, then argument i of f is Γ-*liquid*; otherwise it is Γ-*frozen* [3]. An occurrence of x in t is Γ-*liquid* if either $t = x$, or $t = f(t_1,\ldots,t_{ar(f)})$ and the occurrence is Γ-liquid in t_i for a liquid argument i of f; otherwise the occurrence is Γ-*frozen*.

To each term t and formula φ we assign a set $t^{-1}(\varphi)$ of decomposition mappings $\psi : V \to \mathbb{O}$, such that $\rho(t) \models \varphi$ iff there is a $\psi \in t^{-1}(\varphi)$ with $\rho(x) \models \psi(x)$ for all $x \in var(t)$. To define $t^{-1}(\varphi)$, we use a result from [3], where for each standard TSS P in ready simulation format a collection of decent nxytt rules, called P-*ruloids*, is constructed. First, P is converted into a non-standard TSS P^+ with the property that, for all closed literals μ, we have $P \vdash_{ws} \mu$ if and only if μ is irredundantly provable from P^+. The P-ruloids are the decent nxytt rules irredundantly provable from P^+. In [3] is was proved that there is a well-supported proof from P of a transition $\rho(t) \xrightarrow{a} q$, with ρ a closed substitution, if and only if there is a proof of this transition that uses at the root a P-ruloid with source t.

▶ **Definition 9** ([10])**.** Let $P = (\Sigma, A_\tau, R)$ be a Γ-patient standard TSS in ready simulation format. We define $\cdot^{-1} : \mathbb{T}(\Sigma) \times \mathbb{O} \to \mathcal{P}(V \to \mathbb{O})$ as the function that for each $t \in \mathbb{T}(\Sigma)$ and $\varphi \in \mathbb{O}$ returns the smallest set $t^{-1}(\varphi) \in \mathcal{P}(V \to \mathbb{O})$ of decomposition mappings $\psi : V \to \mathbb{O}$ satisfying the following six conditions. Let t denote a univariate term, i.e. without multiple occurrences of the same variable. (Cases 1–5 associate with every univariate term t a set $t^{-1}(\varphi)$. In Case 6, the definition is generalised to terms that are not univariate, using that every term can be obtained by applying a non-injective substitution to a univariate term.)

1. $\psi \in t^{-1}(\bigwedge_{i\in I} \varphi_i)$ iff there are $\psi_i \in t^{-1}(\varphi_i)$ for $i \in I$ with $\psi(x) = \bigwedge_{i\in I} \psi_i(x)$ for $x \in V$;
2. $\psi \in t^{-1}(\neg\varphi)$ iff there is a function $h : t^{-1}(\varphi) \to var(t)$ such that $\psi(x) = \bigwedge_{\chi\in h^{-1}(x)} \neg\chi(x)$ if $x \in var(t)$, and $\psi(x) = \top$ otherwise;
3. $\psi \in t^{-1}(\langle\alpha\rangle\varphi)$ iff there is a P-ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ and a $\chi \in u^{-1}(\varphi)$ such that $\psi(x) = \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y\in H} \langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma}\not\in H} \neg\langle\gamma\rangle\top$ if $x \in var(t)$, and $\psi(x) = \top$ otherwise;
4. $\psi \in t^{-1}(\langle\epsilon\rangle\varphi)$ iff one of the following holds:
   a. either there is a $\chi \in t^{-1}(\varphi)$ such that $\psi(x) = \langle\epsilon\rangle\chi(x)$ if x occurs Γ-liquid in t, and $\psi(x) = \chi(x)$ otherwise;
   b. or there is a Γ-impatient P-ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u^{-1}(\langle\epsilon\rangle\varphi)$ such that $\psi(x) = \top$ if $x \notin var(t)$, $\psi(x) = \langle\epsilon\rangle\Big(\chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y\in H} \langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma}\not\in H} \neg\langle\gamma\rangle\top\Big)$ if x occurs Γ-liquid in t, and $\psi(x) = \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y\in H} \langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma}\not\in H} \neg\langle\gamma\rangle\top$ otherwise;
5. $\psi \in t^{-1}(\langle\hat{\tau}\rangle\varphi)$ iff one of the following holds:
   a. either $\psi \in t^{-1}(\varphi)$;

b. or there is an x_0 that occurs Γ-liquid in t, and a $\chi \in t^{-1}(\varphi)$ such that $\psi(x) = \langle\hat{\tau}\rangle\chi(x)$ if $x = x_0$, and $\psi(x) = \chi(x)$ otherwise;
c. or there is a Γ-impatient P-ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u^{-1}(\varphi)$ such that $\psi(x) = \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H} \langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \not\in H} \neg\langle\gamma\rangle\top$ if $x \in var(t)$, and $\psi(x) = \top$ otherwise;

6. $\psi \in \sigma(t)^{-1}(\varphi)$ for a non-injective substitution $\sigma : var(t) \to V$ iff there is a $\chi \in t^{-1}(\varphi)$ such that $\psi(x) = \bigwedge_{z \in \sigma^{-1}(x)} \chi(z)$ for all $x \in V$.

The following theorem will be the key to the forthcoming congruence results.

▶ **Theorem 10** ([10]). *Let $P = (\Sigma, A_\tau, R)$ be a Γ-patient complete standard TSS in ready simulation format. For each term $t \in \mathbb{T}(\Sigma)$, closed substitution ρ, and $\varphi \in \mathbb{O}$:*

$$\rho(t) \models \varphi \quad \Leftrightarrow \quad \exists \psi \in t^{-1}(\varphi)\ \forall x \in var(t) :\ \rho(x) \models \psi(x).$$

3 Stability-respecting branching bisimilarity as a congruence

We proceed to apply the decomposition method from the previous section to derive congruence formats for stability-respecting branching bisimilarity and rooted stability-respecting branching bisimilarity. The idea behind the construction of these congruence formats is that the format must guarantee that a formula from the characterising logic of the equivalence under consideration is always decomposed into formulas from this same logic (see Proposition 14). This implies the desired congruence results (see Theorem 15 and Theorem 16).

The definitions of the congruence formats for (rooted) stability-respecting branching bisimilarity, below, presuppose two predicates Λ and $\aleph$ on the arguments of the function symbols of a TSS: Λ marks arguments that contain processes that have started executing, while $\aleph$ marks arguments that contain processes that can execute immediately. For example, in process algebra, Λ and $\aleph$ typically hold for the arguments of the merge $t_1 \| t_2$, and for the first argument of sequential composition $t_1 \cdot t_2$, and do not hold for the second argument of sequential composition. Λ does not hold and $\aleph$ holds for the arguments of alternative composition $t_1 + t_2$. We will instantiate Γ (from Section 2.5) with $\aleph \cap \Lambda$.

We define when a standard ntytt rule is rooted stability-respecting branching bisimulation safe, and base the rooted stability-respecting branching bisimulation format on that notion. The stability-respecting branching bisimulation format is defined by adding one additional restriction to its rooted counterpart: Λ holds for all arguments.

▶ **Definition 11.** A standard ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is *rooted stability-respecting branching bisimulation safe* w.r.t. predicates $\aleph$ and Λ if it satisfies the following conditions.

1. Right-hand sides of positive premises occur only Λ-liquid in u.
2. If $x \in var(t)$ occurs only Λ-liquid in t, then x occurs only Λ-liquid in r.
3. If $x \in var(t)$ occurs only $\aleph$-frozen in t, then x occurs only $\aleph$-frozen in H.
4. Suppose x has exactly one $\aleph$-liquid occurrence in t, and this occurrence is also Λ-liquid.
a. If x has an $\aleph$-liquid occurrence in a negative premise in H or more than one $\aleph$-liquid occurrence in the positive premises in H, then there is a premise $v \xrightarrow{\tau} \not\in H$ such that x occurs $\aleph$-liquid in v.
b. If there is a premise $w \xrightarrow{\tau} y$ in H and x occurs $\aleph$-liquid in w, then r is $\aleph \cap \Lambda$-patient.

Conditions 1–3 were copied from the definition of rooted branching bisimulation safeness from [10], and condition 4b is part of condition 4 in that definition. Condition 4a, however, establishes a relaxation of condition 4 from the definition in [10], where it is required that x has at most one $\aleph$-liquid occurrence in H, which must be in a positive premise. Here, owing to stability, we can be more tolerant, as long as $x \xrightarrow{\tau} \not\in$ can be derived. As a consequence

of this relaxation the rule for the priority operator is rooted stability-respecting branching bisimulation safe, while it is not rooted branching bisimulation safe.

▶ **Definition 12.** A standard TSS is in *rooted stability-respecting branching bisimulation format* if it is in ready simulation format and, for some $\aleph$ and Λ, it is $\aleph\cap\Lambda$-patient and its rules are all rooted stability-respecting branching bisimulation safe w.r.t. $\aleph$ and Λ. This TSS is in *stability-respecting branching bisimulation format* if moreover Λ is universal.

Since the definition of modal decomposition is based on the P-ruloids, we must verify that if P is in rooted stability-respecting branching bisimulation format, then P-ruloids are rooted stability-respecting branching bisimulation safe.

▶ **Proposition 13.** *Let P be an $\aleph\cap\Lambda$-patient TSS in ready simulation format, in which each rule is rooted stability-respecting branching bisimulation safe w.r.t. $\aleph$ and Λ. Then each P-ruloid is rooted stability-respecting branching bisimulation safe w.r.t. $\aleph$ and Λ.*

Consider a standard TSS in rooted stability-respecting branching bisimulation format, w.r.t. some $\aleph$ and Λ. Definition 9 yields decomposition mappings $\psi \in t^{-1}(\varphi)$, with $\Gamma := \aleph\cap\Lambda$. We now prove that if $\varphi \in \mathbb{O}_b^s$, then $\psi(x) \in \mathbb{O}_b^{s\equiv}$ if x occurs only Λ-liquid in t. (That is why in the stability-respecting branching bisimulation format, Λ must be universal.) Furthermore, we prove that if $\varphi \in \mathbb{O}_{rb}^s$, then $\psi(x) \in \mathbb{O}_{rb}^{s\equiv}$ for all x.

▶ **Proposition 14.** *Let P be an $\aleph\cap\Lambda$-patient standard TSS in ready simulation format, in which each rule is rooted stability-respecting branching bisimulation safe w.r.t. $\aleph$ and Λ.*
1. *For each term t and x that occurs only Λ-liquid in t: $\varphi \in \mathbb{O}_b^s \Rightarrow \forall \psi \in t^{-1}(\varphi) : \psi(x) \in \mathbb{O}_b^{s\equiv}$.*
2. *For each term t and variable x: $\overline{\varphi} \in \mathbb{O}_{rb}^s \;\Rightarrow\; \forall \psi \in t^{-1}(\overline{\varphi}) : \psi(x) \in \mathbb{O}_{rb}^{s\equiv}$.*

Now the promised congruence results for $\leftrightarrows_b^s$ and $\leftrightarrows_{rb}^s$ can be proved in the same way as their counterparts for $\leftrightarrows_b$ and $\leftrightarrows_{rb}$ in [10].

▶ **Theorem 15.** *Let P be a complete standard TSS in stability-respecting branching bisimulation format. Then $\leftrightarrows_b^s$ is a congruence for P.*

▶ **Theorem 16.** *Let P be a complete standard TSS in rooted stability-respecting branching bisimulation format. Then $\leftrightarrows_{rb}^s$ is a congruence for P.*

The *priority* operator [1] is a unary function the definition of which is based on an ordering $<$ on atomic actions. The term $\Theta(p)$ executes the transitions of the term p, with the restriction that a transition $p \xrightarrow{a} q$ only gives rise to a transition $\Theta(p) \xrightarrow{a} \Theta(q)$ if there does not exist a transition $p \xrightarrow{b} q'$ with $b > a$. This intuition is captured by the rule $\dfrac{x \xrightarrow{a} y \quad x \xrightarrow{b}\!\!\!\!\!\!/\;\;\; \text{for all } b>a}{\Theta(x) \xrightarrow{a} \Theta(y)}$.

In view of the target $\Theta(y)$, by condition 1 of Definition 11, the argument of Θ must be chosen Λ-liquid. And in view of condition 3 of Definition 11, the argument of Θ must be $\aleph$-liquid. The rule above is rooted stability-respecting branching bisimulation safe, if the following condition on the ordering on atomic actions is satisfied: if $b > a$, then $\tau > a$. Namely, this guarantees condition 4a of Definition 11: if there is a premise $x \xrightarrow{b}\!\!\!\!\!\!/\;\;\;$, then there is also a premise $x \xrightarrow{\tau}\!\!\!\!\!\!/\;\;\;$.

▶ **Corollary 17.** *$\leftrightarrows_b^s$ and $\leftrightarrows_{rb}^s$ are congruences for the priority operator.*

The priority operator Θ does not preserve $\leftrightarrows_{rb}$ (cf. [22, pp. 130–132]). So inevitably, as observed in [10], the rule for Θ is not in the rooted branching bisimulation format. Namely, the $\aleph\cap\Lambda$-liquid argument x in the source occurs $\aleph$-liquid in the negative premises, which violates the more restrictive condition 4 of the rooted branching bisimulation format.

4 Divergence-preserving branching bisimilarity as a congruence

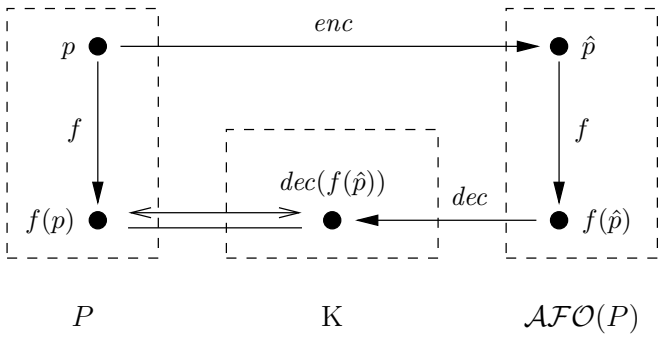
A modal characterisation of divergence-preserving branching bisimilarity is obtained by adding a unary modality Δ to the modal logic for branching bisimilarity: $p \models \Delta\varphi$ if there is an infinite trace $p = p_0 \xrightarrow{\tau} p_1 \xrightarrow{\tau} p_2 \xrightarrow{\tau} \cdots$ such that $p_i \models \varphi$ for all $i \in \mathbb{N}$. Modal formulas $\Delta\varphi$ however elude the inductive decomposition method from Definition 9, because they ask for the existence of an infinite sequence of τ-transitions, as shown by the following example.

▶ **Example 18.** Let $A = \{a_i, b_i \mid i \in \mathbb{N}\} \cup \{c\}$. Parallel composition $\|$ is defined by the rules $\frac{x_1 \xrightarrow{\alpha} y}{x_1 \| x_2 \xrightarrow{\alpha} y \| x_2}$ and $\frac{x_2 \xrightarrow{\alpha} y}{x_1 \| x_2 \xrightarrow{\alpha} x_1 \| y}$, where α ranges over A. We extend its operational semantics with asymmetric communication rules $\frac{x_1 \xrightarrow{a_i} y_1 \quad x_2 \xrightarrow{b_i} y_2}{x_1 \| x_2 \xrightarrow{c} y_1 \| y_2}$ for all $i \in \mathbb{N}$. We define $p_i \xrightarrow{\tau} p_{i+1}$ and $p_i \xrightarrow{a_i} \mathbf{0}$ and $q_i \xrightarrow{\tau} q_{i+1}$ and $q_i \xrightarrow{b_i} \mathbf{0}$ for all $i \in \mathbb{N}$. Then $p_0 \| q_0 \models \Delta(\langle\varepsilon\rangle\langle c\rangle\top)$. There is no obvious way to decompose this into modal properties of its arguments p_0 and q_0.

We circumvent this problem by introducing so-called *abstraction-free* TSSs that allow only patience rules and rules without premises to carry a conclusion with the label τ, and by introducing *oracle transitions* $p \xrightarrow{\omega} \surd$, where the transition label ω reveals some pertinent information on the behaviour of the process p, such as whether p can diverge. On abstraction-free TSSs with appropriate oracle transitions the equivalences $\leftrightarrows_b^s$ and $\leftrightarrows_b^\Delta$ coincide; so there our congruence format for $\leftrightarrows_b^s$ is also a congruence format for $\leftrightarrows_b^\Delta$. We extend this observation to general TSSs by encoding any given TSS into an abstraction-free TSS with oracle transitions, and conclude that on any TSS in the stability-respecting branching bisimulation format $\leftrightarrows_b^\Delta$ is a congruence. The same proof strategy shows that on any such TSS also $\leftrightarrows_b^{\Delta\top}$ is a congruence, and it extends to the rooted case. In Section 4.1 we present this method in more detail. We refrain from introducing the needed machinery, but state exactly which properties of this machinery we require, and prove our main congruence result based on these properties. Instead of dealing with specific equivalences $\leftrightarrows_b^\Delta$ and $\leftrightarrows_b^s$, we work with parametric equivalences $\sim$ and $\approx$ where $\sim$ is finer than $\approx$; they will later be instantiated with $\leftrightarrows_b^\Delta$ and $\leftrightarrows_b^s$. This allows a reuse of our work with $\leftrightarrows_b^{\Delta\top}$ and $\leftrightarrows_b^s$ in the roles of $\sim$ and $\approx$, as well as with $\leftrightarrows_{rb}^{\Delta\top}$ and $\leftrightarrows_{rb}^\Delta$ in the role of $\sim$ and $\leftrightarrows_{rb}^s$ in the role of $\approx$.

4.1 A framework for lifting congruence formats to finer equivalences

Our general proof idea is illustrated below, where P is a TSS in a congruence format for $\approx$. We show that also $\sim$ is a congruence on P. Consider an operator f, for simplicity depicted as unary. Given two closed terms p and q in P with $p \sim q$, we show that $f(p) \sim f(q)$. The picture shows a roundabout trajectory from p to $f(p)$. Imagine a similar trajectory from q to $f(q)$ – not depicted but hovering above the page.



First we apply a transformation $\mathcal{AFO}$ on P, yielding the abstraction-free TSS with oracle transitions $\mathcal{AFO}(P)$, depicted on the right. For each closed term p of P we introduce a constant $\hat{p}$ in $\mathcal{AFO}(P)$, in such away that $p \sim q$ implies $\hat{p} \sim \hat{q}$. Each n-ary operator f of P remains an n-ary operator f of $\mathcal{AFO}(P)$. Since $\sim \subseteq \approx$ we have $\hat{p} \approx \hat{q}$. We argue that if P is within our congruence format for $\approx$, then the TSS $\mathcal{AFO}(P)$ is also within this congruence format, and conclude from $\hat{p} \approx \hat{q}$ that $f(\hat{p}) \approx f(\hat{q})$. An important result, deferred to Section 4.3, is that on $\mathcal{AFO}(P)$ the equivalences $\sim$ and $\approx$ coincide. Hence $f(\hat{p}) \sim f(\hat{q})$.

Finally, we decode the processes $f(\hat{p})$ and $f(\hat{q})$, aiming to return to the LTS generated by P, but actually ending up in another LTS K. Our decoding function *dec* exactly undoes the effects of the encoding *enc*, that sent p to $\hat{p}$, so that $dec(f(\hat{p}))$ is strongly bisimilar with $f(p)$. A crucial property of the function *dec*, also deferred to Section 4.3, is that it is compositional for $\sim$, meaning that from $f(\hat{p}) \sim f(\hat{q})$ we may conclude $dec(f(\hat{p})) \sim dec(f(\hat{q}))$. By imposing the requirement that $\sim$ contains strong bisimilarity ($\leftrightarroweq$), this implies that $f(p) \sim f(q)$.

We now formalise this proof idea. If P is a complete TSS and G is an LTS disjoint from the LTS associated with P, then, in our formalisation, it will sometimes be convenient to write $P \uplus G$ for the disjoint union of the LTSs associated with P and G.

▶ **Theorem 19.** *Let $\sim$ and $\approx$ be behavioural equivalences on LTSs, with $\leftrightarroweq \subseteq \sim \subseteq \approx$. Let $\mathfrak{F}$ be a congruence format for $\approx$, included in the decent ntyft format, and let $\mathcal{AFO}$ be an operation on standard TSSs, where for each TSS $P = (\Sigma, Act, R)$ the signature $\hat{\Sigma}$ of $\mathcal{AFO}(P)$ contains Σ enriched by a fresh constant $\hat{p}$ for each closed term p in $T(\Sigma)$, such that, for each complete standard TSS P in decent ntyft format:*

1. *also $\mathcal{AFO}(P)$ is a complete standard TSS,*
2. *if P is in $\mathfrak{F}$-format then so is $\mathcal{AFO}(P)$,*
3. *$p \sim_P q \;\Rightarrow\; \hat{p} \sim_{\mathcal{AFO}(P)} \hat{q}$,*
4. *$\sim_{\mathcal{AFO}(P)}$ and $\approx_{\mathcal{AFO}(P)}$ coincide, and*

there is an LTS $\mathrm{K} = (\mathbb{P}_\mathrm{K}, Act_\mathrm{K}, \rightarrow_\mathrm{K})$, disjoint from P, and a $dec : \mathrm{T}(\hat{\Sigma}) \to \mathbb{P}_\mathrm{K}$ such that:

5. *$p \sim_{\mathcal{AFO}(P)} q \;\Rightarrow\; dec(p) \sim_\mathrm{K} dec(q)$, and*
6. *$f(p_1, \ldots, p_n) \leftrightarroweq_{P \uplus \mathrm{K}} dec(f(\hat{p}_1, \ldots, \hat{p}_n))$ for any n-ary $f \in \Sigma$ and $p_1, \ldots, p_n \in T(\Sigma)$.*

Then $\mathfrak{F}$ is also a congruence format for $\sim$.

Proof. Let $P = (\Sigma, Act, R)$ be a complete standard TSS in $\mathfrak{F}$-format. We will show that $\sim_P$ is a congruence for P. So let $f \in \Sigma$ be an n-ary function symbol, and let $p_i, q_i \in \mathrm{T}(\Sigma)$ with $p_i \sim_P q_i$ for $i = 1, \ldots, n$. We need to show that $f(p_1, \ldots, p_n) \sim_P f(q_1, \ldots, q_n)$.

By requirements 1 and 2, $\mathcal{AFO}(P)$ is a complete standard TSS in $\mathfrak{F}$-format; hence $\approx_{\mathcal{AFO}(P)}$ is a congruence for $\mathcal{AFO}(P)$. By requirement 3 $\hat{p}_i \sim_{\mathcal{AFO}(P)} \hat{q}_i$ for $i = 1, \ldots, n$. By requirement 4 also $\sim_{\mathcal{AFO}(P)}$ is a congruence for $\mathcal{AFO}(P)$. So $f(\hat{p}_1, \ldots, \hat{p}_n) \sim_{\mathcal{AFO}(P)} f(\hat{q}_1, \ldots, \hat{q}_n)$. Hence, by requirement 5, $dec(f(\hat{p}_1, \ldots, \hat{p}_n)) \sim_\mathrm{K} dec(f(\hat{q}_1, \ldots, \hat{q}_n))$. Therefore, by two applications of requirement 6, and the definition of a behavioural equivalence, $f(p_1, \ldots, p_n) \sim_{P \uplus \mathrm{K}} f(q_1, \ldots, q_n)$ and consequently $f(p_1, \ldots, p_n) \sim_P f(q_1, \ldots, q_n)$. ◀

4.2 Abstraction-freeness

In this section we introduce the machinery needed for Theorem 19, namely the conversion $\mathcal{AFO}$ on TSSs and the function *dec* into the LTS K. We also establish requirements 1 and 6 of Theorem 19, leaving 2–5 to the applications of Theorem 19 in Section 4.3 for specific instances of $\sim$ and $\approx$. We here take the set of actions *Act* used in Section 4.1 to be A_τ.

Again Γ denotes a predicate that marks arguments of function symbols. In [9] we called a standard TSS *abstraction-free* w.r.t. Γ if only its Γ-patience rules carry the label τ in their

conclusion. Here we use a more liberal definition of abstraction-freeness that also allows rules that have no premises, and a conclusion of the form $c \xrightarrow{\tau} d$ for constants c and d.

The next conversion turns a Γ-patient standard TSS $P = (\Sigma, A_\tau, R)$ into a Γ-patient and abstraction-free TSS $\mathcal{AFO}_\Gamma^{O,\zeta}(P)$. It is parameterised by the choice of a fresh set of actions O, so $O \cap A_\tau = \emptyset$, and a partial function $\zeta : \mathrm{T}(\Sigma) \rightharpoonup O$ called an *oracle*. The choice of O and ζ varies for different applications of Theorem 19. This choice will be made in Section 4.3 in such a way that requirements 3 and 4 of Theorem 19 are met, for specific instances of $\sim$ and $\approx$.

▶ **Definition 20.** Given a Γ-patient standard TSS $P = (\Sigma, A_\tau, R)$. Let $\hat{\Sigma}$ be the signature Σ, enriched with a fresh constant $\surd$ and a fresh constant $\hat{p}$ for each closed term $p \in \mathrm{T}(\Sigma)$. Pick a fresh action $\iota \notin A_\tau \cup O$. We define the TSS $\mathcal{AFO}_\Gamma^{O,\zeta}(P)$ as $(\hat{\Sigma}, A_\tau \cup O \cup \{\iota\}, R')$ where the rules in R' are obtained from the rules in R as follows:

1. R_1 is obtained from R by adding for each rule r and each non-empty subset S of positive τ-premises of r, a copy of r in which the labels τ in the premises in S are replaced by ι;
2. R_2 is obtained from R_1 by replacing, in every rule that has a conclusion with the label τ and is not a Γ-patience rule, the τ-label in the conclusion by ι;
3. R_3 is obtained from R_2 by adding the premise $v \xrightarrow{\iota}\!\!\!\!\!\!/\;$ to each rule with a premise $v \xrightarrow{\tau}\!\!\!\!\!\!/\;$;
4. R_4 is obtained from R_3 by the addition of a rule without premises $\hat{p} \xrightarrow{\alpha} \hat{q}$ for each transition $p \xrightarrow{\alpha} q$ *ws*-provable from P;
5. R_5 is obtained from R_4 by adding a rule $\hat{p} \xrightarrow{\zeta(p)} \surd$ for each $p \in \mathrm{T}(\Sigma)$ with $\zeta(p)$ defined;
6. R' adds to R_5 a rule $\dfrac{x_k \xrightarrow{\omega} y}{f(x_1,\ldots,x_n) \xrightarrow{\omega} y}$ for each $\omega \in O$, $f \in \Sigma$ and argument k with $\Gamma(f,k)$.

Step 2 above makes the resulting TSS abstraction-free by renaming τ-labels in conclusions of non-patience rules into ι. To ensure that still the same transitions are derived, modulo the conversion of some τ into ι-labels, step 1 above allows positive premises labelled ι to be used instead of τ in all rules, and step 3 achieves the same purpose for negative premises. These three steps result in a Γ-patient and abstraction-free TSS that could be called $\mathcal{AF}_\Gamma(P)$.

For convenience we consider an auxiliary LTS $\mathrm{G} = (\mathbb{P}_\mathrm{G}, A_\tau, \rightarrow_\mathrm{G})$ with $\mathbb{P}_\mathrm{G} = \{\hat{p} \mid p \in \mathrm{T}(\Sigma)\}$ and $\hat{p} \xrightarrow{\alpha}_\mathrm{G} \hat{q}$ iff $P \vdash_{ws} p \xrightarrow{\alpha} q$, and an auxiliary LTS $\mathrm{H} = (\mathbb{P}_\mathrm{H}, A_\tau \cup O, \rightarrow_\mathrm{H})$ with $\mathbb{P}_\mathrm{H} = \mathbb{P}_\mathrm{G} \cup \{\surd\}$ and $\rightarrow_\mathrm{H} := \rightarrow_\mathrm{G} \cup \{\hat{p} \xrightarrow{\zeta(p)} \surd \mid \zeta(p) \text{ defined}\}$. The LTS G is simply a disjoint copy of the LTS generated by P.

▶ **Lemma 21.** *If $p \sim_P q$ for some $p, q \in T(\Sigma)$, then $\hat{p} \sim_\mathrm{G} \hat{q}$.*

The LTS H adds *oracle transitions* to G. The idea is that $\zeta(p)$ is particular for the $\sim$-equivalence class of $p \in \mathrm{T}(\Sigma)$, which on the one hand ensures that $\hat{p} \sim_\mathrm{H} \hat{q}$ iff $\hat{p} \sim_\mathrm{G} \hat{q}$, and on the other hand enforces that $\sim$ and $\approx$ coincide on H. Namely, if $\hat{p} \approx_\mathrm{H} \hat{q}$ then the oracle action of $\hat{p}$ can be matched by $\hat{q}$ (and vice versa), which implies $\hat{p} \sim_\mathrm{H} \hat{q}$.

Steps 4 and 5 of Definition 20 incorporate the entire LTS H into $\mathcal{AF}_\Gamma(P)$: each state appears as a constant and each transition appears as rule without premises. The operators from Σ can now be applied to arguments of the form $\hat{p}$. Finally, step 6 lets any term $f(x_1, \ldots, x_n)$ inherit the oracle transitions from its Γ-liquid arguments. Steps 4, 5 and 6 preserve abstraction-freeness; for step 4 this uses the relaxed definition of abstraction-freeness that allows to incorporate τ-transitions between constants as rules without premises.

▶ **Example 22.** Let P have the rules

$$\frac{x_1 \xrightarrow{\tau} y}{g(x_1,x_2,x_3) \xrightarrow{\tau} g(y,x_2,x_3)} \qquad \frac{x_1 \xrightarrow{a} y_1 \quad x_1 \xrightarrow{\tau} y_2 \quad x_3 \xrightarrow{\tau} y_3}{g(x_1,x_2,x_3) \xrightarrow{\tau} x_2} \qquad \frac{x_2 \xrightarrow{\tau} y \quad x_3 \xrightarrow{\tau}\!\!\!\!\!\!/\;}{g(x_1,x_2,x_3) \xrightarrow{a} y}$$

where $\Gamma(g,1)$. Then $\mathcal{AFO}_\Gamma^{O,\zeta}(P)$ has the rules

$$\frac{x_1 \xrightarrow{\tau} y}{g(x_1,x_2,x_3) \xrightarrow{\tau} g(y,x_2,x_3)} \qquad \frac{x_1 \xrightarrow{a} y_1 \quad x_1 \xrightarrow{\tau} y_2 \quad x_3 \xrightarrow{\tau} y_3}{g(x_1,x_2,x_3) \xrightarrow{\iota} x_2} \qquad \frac{x_2 \xrightarrow{\tau} y \quad x_3 \not\xrightarrow{\tau} \quad x_3 \not\xrightarrow{\iota}}{g(x_1,x_2,x_3) \xrightarrow{a} y}$$

$$\frac{x_1 \xrightarrow{\iota} y}{g(x_1,x_2,x_3) \xrightarrow{\iota} g(y,x_2,x_3)} \qquad \frac{x_1 \xrightarrow{a} y_1 \quad x_1 \xrightarrow{\iota} y_2 \quad x_3 \xrightarrow{\iota} y_3}{g(x_1,x_2,x_3) \xrightarrow{\iota} x_2} \qquad \frac{x_2 \xrightarrow{\iota} y \quad x_3 \not\xrightarrow{\tau} \quad x_3 \not\xrightarrow{\iota}}{g(x_1,x_2,x_3) \xrightarrow{a} y}$$

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_1 \xrightarrow{\iota} y_2 \quad x_3 \xrightarrow{\tau} y_3}{g(x_1,x_2,x_3) \xrightarrow{\iota} x_2} \qquad \frac{x_1 \xrightarrow{a} y_1 \quad x_1 \xrightarrow{\tau} y_2 \quad x_3 \xrightarrow{\iota} y_3}{g(x_1,x_2,x_3) \xrightarrow{\iota} x_2} \qquad \frac{x_1 \xrightarrow{\omega} y}{g(x_1,x_2,x_3) \xrightarrow{\omega} y}\ (\omega\in O)$$

This illustrates steps 1,2,3,6 of Definition 20. Since there are no closed terms, steps 4,5 are void.

Clearly, for any Γ-patient standard TSS P, the standard TSS $\mathcal{AFO}_\Gamma^{O,\zeta}(P)$ is Γ-patient and abstraction-free w.r.t. Γ. We drop superscripts O and ζ, and $\mathcal{AFO}(P)$ denotes $\mathcal{AFO}_\Gamma(P)$ for the largest Γ for which P is Γ-patient. The signature of $\mathcal{AFO}(P)$ contains the signature of P enriched by a fresh constant $\hat{p}$ for each closed term p in P, as required in Theorem 19.

▶ **Lemma 23.** *Let P be a complete standard TSS in ntyft format. If $\hat{p} \sim_{\mathrm{H}} \hat{q}$ for some $p,q \in T(\Sigma)$, then $\hat{p} \sim_{\mathcal{AFO}(P)} \hat{q}$.*

As an immediate consequence of Lemmas 21 and 23 we have the following corollary.

▶ **Corollary 24.** *Requirement 3 of Theorem 19 is met if $\hat{p} \sim_{\mathrm{G}} \hat{q}$ implies $\hat{p} \sim_{\mathrm{H}} \hat{q}$.* ◀

The inference $\hat{p} \sim_{\mathrm{G}} \hat{q} \Rightarrow \hat{p} \sim_{\mathrm{H}} \hat{q}$ depends on the choice of O and ζ; it is deferred to Section 4.3.

The oracle inheritance rules in $\mathcal{AFO}_\Gamma(P)$ – introduced in step 6 of Definition 20 – ensure that a closed term $f(p_1,\ldots,p_n)$ has an outgoing ω-transition, for $\omega \in O$, iff one of its Γ-liquid arguments p_k has such a transition. Ultimately, all such oracle transitions stem from H. Using that in $\mathcal{AFO}_\Gamma(P)$ any term $p \in \mathrm{T}(\hat{\Sigma})$ can uniquely be written as $\rho(t)$ with $t \in \mathbb{T}(\Sigma)$ and $\rho : var(t) \to \mathbb{P}_{\mathrm{H}}$, this fact can be phrased as follows. Let $t \in \mathbb{T}(\Sigma)$ and $\rho : var(t) \to \mathbb{P}_{\mathrm{H}}$; then $\mathcal{AFO}_\Gamma(P) \vdash_{ws} \rho(t) \xrightarrow{\omega}$ iff t has a Γ-liquid occurrence of a variable x with $\rho(x) \xrightarrow{\omega}_{\mathrm{H}}$. Using this, we now verify requirement 1 of Theorem 19:

▶ **Lemma 25.** *If a standard TSS P in decent ntyft format is complete, then so is $\mathcal{AFO}(P)$.*

The LTS $\mathrm{K} = (\mathbb{P}_{\mathrm{K}}, A_\tau \cup O \cup \{\iota\}, \to_{\mathrm{K}})$ has as states $\mathbb{P}_{\mathrm{K}} = \{dec(p) \mid p \in \mathrm{T}(\hat{\Sigma})\}$, and its transitions are the ones generated by the rules $\frac{x \xrightarrow{\alpha} y}{dec(x) \xrightarrow{\alpha} dec(y)}$ and $\frac{x \xrightarrow{\iota} y}{dec(x) \xrightarrow{\tau} dec(y)}$, where α ranges over A_τ. The operator $dec : \mathrm{T}(\hat{\Sigma}) \to \mathbb{P}_{\mathrm{K}}$ erases all transitions with labels from O and renames labels ι into τ. All other transitions are preserved.

We end this section by verifying requirement 6 of Theorem 19. Intuitively, the behaviour of a process $f(p_1,\ldots,p_n)$ in P is the same as that of $f(\hat{p}_1,\ldots,\hat{p}_n)$ in $\mathcal{AFO}(P)$, except that some τ-transitions of the former are turned into ι-transitions of the latter process, and some oracle transitions may have been added in the latter. Since any rule in $\mathcal{AFO}(P)$ with a conclusion labelled by $A_\tau \cup \{\iota\}$ has positive and negative premises with labels from $A_\tau \cup \{\iota\}$ only, these oracle transitions have no influence on the derivation of any transitions from $\mathcal{AFO}(P)$ with labels in $A_\tau \cup \{\iota\}$. The operator dec removes all oracle transitions and renames ι into τ, thereby returning the behaviour of $f(\hat{p}_1,\ldots,\hat{p}_n)$ to match that of $f(p_1,\ldots,p_n)$ exactly.

▶ **Proposition 26.** *Let P be a complete standard TSS in decent ntyft format. Then $f(p_1,\ldots,p_n) \leftrightarrows_{P \uplus \mathrm{K}} dec(f(\hat{p}_1,\ldots,\hat{p}_n))$ for any n-ary $f \in \Sigma$ and $p_1,\ldots,p_n \in T(\Sigma)$.*

4.3 Application of the framework to divergence-preserving semantics

We apply Theorem 19 to show that the stability-respecting branching bisimulation format and its rooted variant are congruence formats for $\leftrightarrows_b^{\Delta\top}$ and $\leftrightarrows_b^{\Delta}$, and for $\leftrightarrows_{rb}^{\Delta\top}$ and $\leftrightarrows_{rb}^{\Delta}$.

As congruence format in Theorem 19 we take the (rooted) stability-respecting branching bisimulation format intersected with the decent ntyft format. It is straightforward to check that the conversion $\mathcal{AFO}$ on standard TSSs defined in Section 4.2 preserves the (rooted) stability-respecting branching bisimulation format, and thus satisfies requirement 2 of Theorem 19. Here it is important that in step 6 of Definition 20 a term $f(x_1,\ldots,x_n)$ inherits oracle transitions only from its Γ-liquid arguments – else condition 3 of Definition 11 would be violated. Furthermore, condition 4a of Definition 11 is preserved because premises $v\stackrel{\tau}{\not\longrightarrow}$ are kept in place in step 3 of Definition 20; and condition 4b of Definition 11 is preserved because the transformation in Definition 20 does not introduce new positive τ-premises.

We first apply Theorem 19 with $\leftrightarrows_b^{\Delta\top}$ and $\leftrightarrows_b^{s}$ in the roles of $\sim$ and $\approx$. Let us say that a process p in an LTS is *divergent* if there exists an infinite sequence of processes $(p_k)_{k\in\mathbb{N}}$ such that $p_k \stackrel{\tau}{\longrightarrow} p_{k+1}$ for all $k\in\mathbb{N}$, i.e. if $p\models\Delta\top$. In the construction of the LTS H out of G (cf. Section 4.2) we take $O=\{\Delta\top\}$ and let $\zeta(g)=\Delta\top$ iff g is divergent. Thus in H all divergent states of G have a fresh outgoing transition labelled $\Delta\top$.

With this definition of H, we have $\hat{p}\sim_{\mathrm{H}}\hat{q}$ iff $\hat{p}\sim_{\mathrm{G}}\hat{q}$: any weakly divergence-preserving branching bisimulation $\mathcal{B}$ on G relates divergent states with divergent states only, and thus is also a weakly divergence-preserving branching bisimulation on H (adding $\surd\,\mathcal{B}\,\surd$.) Hence, by Corollary 24, requirement 3 of Theorem 19 is satisfied. Requirement 4 is also satisfied:

▶ **Proposition 27.** *On $\mathcal{AFO}(P)$ the equivalences $\leftrightarrows_b^{\Delta\top}$ and $\leftrightarrows_b^{s}$ coincide.*

The next example shows that Proposition 27 would not hold if we had skipped step 6 of Definition 20, inheriting oracle transitions for Γ-liquid arguments, or had not used oracle transitions at all.

▶ **Example 28.** Let $p\in\mathrm{T}(\Sigma)$ have no outgoing transitions, while $q\in\mathrm{T}(\Sigma)$ has only a τ-transition to itself and a τ-transition to p. Then in the LTS G we have $\hat{p}\leftrightarrows_{b\ \mathrm{G}}^{s}\hat{q}$ but $\hat{p}\not\leftrightarrows_{b\ \mathrm{G}}^{\Delta\top}\hat{q}$. After translation to H, the processes $\hat{p}$ and $\hat{q}$ are distinguished by means of oracle transitions, so that we have $\hat{p}\not\leftrightarrows_{b\ \mathrm{H}}^{s}\hat{q}$ and $\hat{p}\not\leftrightarrows_{b\ \mathrm{H}}^{\Delta\top}\hat{q}$. Now let Σ feature a unary operator f with as only rule $\frac{x\stackrel{\tau}{\longrightarrow}y}{f(x)\stackrel{\tau}{\longrightarrow}f(y)}$. If oracle transitions would not be inherited in $\mathcal{AFO}(P)$, then we would have $f(\hat{p})\leftrightarrows_{b\ \mathcal{AFO}(P)}^{s}f(\hat{q})$ but $f(\hat{p})\not\leftrightarrows_{b\ \mathcal{AFO}(P)}^{\Delta\top}f(\hat{q})$.

Also requirement 5 of Theorem 19 holds.

▶ **Proposition 29.** $p\leftrightarrows_{b\ \mathcal{AFO}(P)}^{\Delta\top}q \;\Rightarrow\; dec(p)\leftrightarrows_{b\ \mathrm{K}}^{\Delta\top}dec(q)$.

▶ **Corollary 30.** *The stability-respecting branching bisimulation format intersected with the decent ntyft format is a congruence format for $\leftrightarrows_b^{\Delta\top}$.*

Each standard TSS P in ready simulation format can be converted to a TSS P' in decent ntyft format, preserving the set of ws-provable closed literals [3]. Moreover, if P is in stability-respecting branching bisimulation format, then so is P'. Thus we obtain:

▶ **Theorem 31.** *Let P be a complete standard TSS in stability-respecting branching bisimulation format. Then $\leftrightarrows_b^{\Delta\top}$ is a congruence for P.* □

In a similar fashion it can be proved that the stability-respecting branching bisimulation format is a congruence format for $\leftrightarrows_b^{\Delta}$, and that the rooted stability-respecting branching bisimulation format is a congruence format for $\leftrightarrows_{rb}^{\Delta\top}$ as well as $\leftrightarrows_{rb}^{\Delta}$.

5 Related work

Ulidowski [19, 20, 21] proposed congruence formats, inside GSOS [4], for weak semantics that take into account non-divergence, called *convergence* in [11]. In [19] he introduces the *ISOS* format, and shows that the weak convergent *refusal simulation* preorder is a precongruence for all TSSs in the ISOS format. The GSOS format – in our terminology the *decent nxyft* format – allows only decent ntyft rules with variables as the left-hand sides of premises. The ISOS format is contained in the intersection of the GSOS format and our stability-preserving branching bisimulation format. Its additional restriction is that no variable may occur multiple times as the left-hand side of a positive premise, or both as the left-hand side of a positive premise and in the conclusion of a rule. In [20, 21] he employs *Ordered SOS* (OSOS) TSSs [20]. An OSOS TSS allows no negative premises, but includes priorities between rules: $r < r'$ means that r can only be applied if r' cannot. An OSOS specification can be seen as, or translated into, a GSOS specification with negative premises. Each rule r with exactly one higher-priority rule $r' > r$ is replaced by a number of rules, one for each (positive) premise of r'; in the copy of r, this premise is negated. For a rule r with multiple higher-priority rules r', this replacement is carried out for each such r'.

The `ebo` and `bbo` formats from [20] target convergent *delay* and branching bisimulation equivalence, respectively, whereas the `rebo` and `rbbo` formats from [21] target their rooted counterparts. These rooted formats are more liberal than their unrooted counterparts, and the `(r)bbo` format is more liberal than the `(r)ebo` format. If patience rules are not allowed to have a lower priority than other rules, then the `(r)bbo` format, upon translation from OSOS to GSOS, can be seen as a subformat of our (rooted) stability-respecting branching bisimulation format. Patience rules are in the `(r)bbo` format however, under strict conditions, allowed to be dominated by other rules, which in our setting gives rise to patience rules with negative premises. This is outside the realm of our rooted stability-respecting branching bisimulation format. On the other hand, the TSSs of the process algebra $\mathrm{BPA}_{\varepsilon\delta\tau}$, the binary Kleene star and deadlock testing (see [7]), for which rooted convergent branching bisimulation equivalence is a congruence, are outside `rbbo` but within the rooted stability-respecting branching bisimulation format.

6 Conclusions

We showed how the method from [10] for deriving congruence formats through modal decomposition can be applied to weak semantics that are stability-respecting. We used (rooted and unrooted) stability-respecting branching bisimulation equivalence as a notable example. Moreover, we developed a general method for lifting congruence formats from a weak semantics to a finer semantics, and used it to show that congruence formats for $\leftrightarrows^s_{rb}$ and $\leftrightarrows^s_b$ are also congruence formats for their divergence-preserving counterparts. This research provides a deeper insight into the link between modal logic and congruence formats, and strengthens the framework from [10] for the derivation of congruence formats for weak semantics.

We build on a rich body of earlier work in the realm of structural operational semantics: the notions of well-supported proofs and complete TSSs from [12]; the ntyft/ntyxt format [16, 5]; the transformation to ruloids; and the work on modal decomposition and congruence formats from [3]. In spite of these technicalities, the resulting framework for deriving congruence formats for weak semantics is relatively straightforward. For this one only needs to: (1) provide a modal characterisation of the weak semantics under consideration; (2) study the class of modal formulas that result from decomposing this modal characterisation, and formulate syntactic restrictions on TSSs to bring this class of modal formulas within the

original modal characterisation; and (3) check that these syntactic restrictions are preserved under the transformation to ruloids. Steps (2) and (3) are very similar in structure for different weak semantics, as exemplified by the way we obtained a congruence format for stability-respecting branching bisimulation equivalence. And the resulting congruence formats tend to be more liberal and elegant than existing congruence formats in the literature.

Our intention is to carve out congruence formats for all weak semantics in the spectrum from [11] that have reasonable congruence properties. At first we expected that the current third instalment would allow us to do so. However, it turns out that convergent weak semantics as considered in for instance [20, 21, 23] still need extra work. The modal characterisations of these semantics are three-valued [11], which requires an extension of the modal decomposition technique to a three-valued setting.

References

1 J.C.M. Baeten, J.A. Bergstra, and J.W. Klop. Syntax and defining equations for an interrupt mechanism in process algebra. *Fundamenta Informaticae*, 9(2):127–167, 1986.

2 T. Basten. Branching bisimulation is an equivalence indeed! *Information Processing Letters*, 58(3):141–147, 1996. doi:10.1016/0020-0190(96)00034-8.

3 B. Bloom, W.J. Fokkink, and R.J. van Glabbeek. Precongruence formats for decorated trace semantics. *Transactions on Computational Logic*, 5(1):26–78, 2004. doi:10.1145/963927.963929.

4 B. Bloom, S. Istrail, and A.R. Meyer. Bisimulation can't be traced. *Journal of the ACM*, 42(1):232–268, 1995. doi:10.1145/200836.200876.

5 R. Bol and J.F. Groote. The meaning of negative premises in transition system specifications. *Journal of the ACM*, 43(5):863–914, 1996. doi:10.1145/234752.234756.

6 W.J. Fokkink. Rooted branching bisimulation as a congruence. *Journal of Computer and System Sciences*, 60(1):13–37, 2000. doi:10.1006/jcss.1999.1663.

7 W.J. Fokkink and R.J. van Glabbeek. Divide and congruence II: Delay and weak bisimilarity. In *Proc. LICS 2016*, pages 778–787. ACM/IEEE, 2016. doi:10.1145/2933575.2933590.

8 W.J. Fokkink, R.J. van Glabbeek, and B. Luttik. Divide and congruence III: From decomposition of modal formulas to preservation of stability and divergence. Full version. http://theory.stanford.edu/~rvg/abstracts.html#125.

9 W.J. Fokkink, R.J. van Glabbeek, and P. de Wind. Divide and congruence: From decomposition of modalities to preservation of branching bisimulation. In *Proc. FMCO 2005*, volume 4111 of *LNCS*, pages 195–218, 2006. doi:10.1007/11804192_10.

10 W.J. Fokkink, R.J. van Glabbeek, and P. de Wind. Divide and congruence: From decomposition of modal formulas to preservation of branching and η-bisimilarity. *Information and Computation*, 214:59–85, 2012. doi:10.1016/j.ic.2011.10.011.

11 R.J. van Glabbeek. The linear time-branching time spectrum II: The semantics of sequential systems with silent moves. In *Proc. CONCUR 1993*, volume 715 of *LNCS*, pages 66–81. Springer, 1993. doi:10.1007/3-540-57208-2_6.

12 R.J. van Glabbeek. The meaning of negative premises in transition system specifications II. *Journal of Logic and Algebraic Programming*, 60/61:229–258, 2004. doi:10.1016/j.jlap.2004.03.007.

13 R.J. van Glabbeek, B. Luttik, and N. Trčka. Branching bisimilarity with explicit divergence. *Fundamenta Informaticae*, 93(4):371–392, 2009. doi:10.3233/FI-2009-109.

14 R.J. van Glabbeek, B. Luttik, and N. Trčka. Computation tree logic with deadlock detection. *Logical Methods in Computer Science*, 5(4), 2009.

15 R.J. van Glabbeek and W.P. Weijland. Branching time and abstraction in bisimulation semantics. *Journal of the ACM*, 43(3):555–600, 1996. doi:10.1145/233551.233556.

16 J.F. Groote. Transition system specifications with negative premises. *Theoretical Computer Science*, 118(2):263–299, 1993. doi:10.1016/0304-3975(93)90111-6.

17 M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985. doi:10.1145/2455.2460.

18 R. Milner. *Communication and Concurrency*. Prentice-Hall, 1989.

19 I. Ulidowski. Equivalences on observable processes. In *Proc. LICS 1992*, pages 148–159. IEEE, 1992. doi:10.1109/LICS.1992.185529.

20 I. Ulidowski and I. Phillips. Ordered SOS rules and process languages for branching and eager bisimulations. *Information and Computation*, 178(1):180–213, 2002. doi:10.1006/inco.2002.3161.

21 I. Ulidowski and S. Yuen. Process languages for rooted eager bisimulation. In *Proc. CONCUR 2000*, volume 1877 of *LNCS*, pages 275–289. Springer, 2000. doi:10.1007/3-540-44618-4_21.

22 F.W. Vaandrager. *Algebraic Techniques for Concurrency and their Application*. PhD thesis, University of Amsterdam, 1990.

23 D. Walker. Bisimulation and divergence. *Information and Computation*, 85(2):202–241, 1990. doi:10.1016/0890-5401(90)90048-M.

Checking Linearizability of Concurrent Priority Queues*

Ahmed Bouajjani[1], Constantin Enea[2], and Chao Wang[3]

1 **IRIF, University Paris Diderot, Paris, France**
`abou@irif.fr`
2 **IRIF, University Paris Diderot, Paris, France**
`cenea@irif.fr`
3 **IRIF, University Paris Diderot, Paris, France**
`wangch@irif.fr`

Abstract

Efficient implementations of concurrent objects such as atomic collections are essential to modern computing. Unfortunately their correctness criteria — linearizability with respect to given ADT specifications — are hard to verify. Verifying linearizability is undecidable in general, even on classes of implementations where the usual control-state reachability is decidable. In this work we consider concurrent priority queues which are fundamental to many multi-threaded applications like task scheduling or discrete event simulation, and show that verifying linearizability of such implementations is reducible to control-state reachability. This reduction entails the first decidability results for verifying concurrent priority queues with an unbounded number of threads, and it enables the application of existing safety-verification tools for establishing their correctness.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Concurrency, Linearizability, Model Checking

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.16

1 Introduction

Multithreaded software is typically built with specialized "concurrent objects" like atomic integers, queues, maps, priority queues. These objects' methods are designed to confom to better established sequential specifications, a property known as *linearizability* [14], despite being optimized to avoid blocking and exploit parallelism, e.g., by using machine instructions like compare-and-swap. Intuitively, linearizability asks that every individual operation appears to take place instantaneously at some point between its invocation and its return. Verifying linearizability is intrinsically hard, and undecidable in general [4]. However, recent work [5] has shown that for particular objects, e.g., registers, mutexes, queues, and stacks, the problem of verifying linearizability becomes decidable (for finite-state implementations).

In this paper, we consider another important object, namely the priority queue, which is essential for applications such as task scheduling and discrete event simulation. Numerous implementations have been proposed in the research literature, e.g., [2, 8, 16, 20, 19], and concrete implementations exist in many modern languages like C++ or Java. Priority queues are collections providing *put* and *rm* methods for adding and removing values. Every added value is associated to a priority and a remove operation returns a minimal priority value.

* This work is supported in part by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 678177).

For generality, we consider a partially-ordered set of priorities. Values with incomparable priorities can be removed in any order, and values having the same priority are removed in the FIFO order. Implementations like the PriorityBlockingQueue in Java where same priority values are removed in an arbitrary order can be modeled in our framework by renaming equal priorities to incomparable priorities (while preserving the order constraints).

Compared to previously studied collections like stacks and queues, the main challenge in dealing with priority queues is that the order in which values are removed is not fixed by the happens-before between add/remove operations (e.g., in the case of queues, values are removed in the order in which they were inserted), but by parameters of the *put* operations (the priorities) which come from an unbounded domain. For instance, the sequential behavior $put(a, p_1) \cdot put(b, p_3) \cdot put(c, p_2) \cdot rm(a, p_1) \cdot rm(c, p_2)$ where the priority p_1 is less than p_2 which is less than p_3, is not admitted neither by the regular queue nor the stack.

We give a characterization of concurrent priority queue behaviors violating linearizability in terms of automata. This characterization enables a reduction of checking linearizability for arbitrary implementations to reachability or invariant checking, and implies decidability for checking linearizability of finite-state implementations. While linearizability violations for stacks and queues can be described using finite-state automata [5], the case of priority queues requires *register automata* where registers are used to store and compare priorities.

This characterization is obtained in several steps. We define a recursive procedure that recognizes valid sequential executions, which is then extended to recognize linearizable concurrent executions. Intuitively, for an execution e, this procedure deals with values occurring in e one by one, starting with values of maximal priority (to be removed the latest). For each value x, it checks whether e satisfies some property "local" to that value, i.e., which is agnostic to how the operations adding or removing other values are ordered between them (w.r.t. the happens-before), other than how they are ordered w.r.t. the operations on x. When this property holds, the procedure is applied recursively on the rest of the execution, without the operations on x. This procedure works only for executions where a value is added at most once, but this is not a limitation for *data-independent* implementations whose behavior doesn't depend on the values that are added or removed. In fact, all the implementations that we are aware of are data-independent.

Next, we show that checking whether an execution violates this "local" property for a value x can be done using a class of register automata [15, 9, 18] (transition systems where the states consist of a fixed set of registers that can receive values and be compared). Actually, only two registers are needed: one register r_1 for storing a priority guessed at the initial state, and one register r_2 for reading priorities as they occur in the execution and comparing them with the one stored in r_1. We show that registers storing values added to or removed from the priority queue are not needed, since any data-independent implementation admits a violation to linearizability whenever it admits a violation where the number of values is constant, and at most 4 (the number of priorities can still be unbounded).

The remainder of this article is organized as follows. Section 2 describes the priority queue ADT, lists several semantic properties like data-independence, and recalls the notion of linearizability. Section 3 defines a recursive procedure for checking linearizability of concurrent priority queue behaviors. Section 4 gives an automata characterization of the violations to linearizability, and Section 5 discusses related work. Detailed proofs and constructions can be found in the extended version [7].

2 The Priority Queue ADT

We consider priority queues whose interface contains two methods *put* and *rm* for adding and respectively, removing a value. Each value is assigned with a priority when being added to the data structure (by calling *put*) and the remove method *rm* removes a value with a minimal priority. For generality, we assume that the set of priorities is partially-ordered. Incomparable priorities can be removed in any order. When multiple values are assigned with the same priority, *rm* returns the least recent value. Also, when the set of values stored in the priority queue is empty, *rm* returns the distinguished value *empty*. In this section, we formalize (concurrent) executions and implementations, introduce a set of properties satisfied by all the implementations we are aware of, and recall the standard correctness criterion for concurrent implementations of ADTs known as *linearizability* [14].

2.1 Executions

We fix a (possibly infinite) set $\mathbb{D}$ of data values, a (possibly infinite) set $\mathbb{P}$ of priorities, a partial order $\prec$ among elements in $\mathbb{P}$, and an infinite set $\mathbb{O}$ of operation identifiers. The latter are used to match call and return actions of the same invocation. Call actions $call_o(put, a, p)$ and $call_o(rm, a')$ with $a \in \mathbb{D}$, $a' \in \mathbb{D} \cup \{empty\}$, $p \in \mathbb{P}$, and $o \in \mathbb{O}$, combine a method name and a set of arguments with an operation identifier. The return value of a remove is transformed to an argument value for uniformity [1]. The return actions are denoted in a similar way as $ret_o(put, a, p)$ and respectively, $ret_o(rm, a')$.

An *execution* e is a sequence of call and return actions which satisfy the following well-formedness properties: each return is preceded by a matching call (having the same operation identifier), and each operation identifier is used in at most one call/return. We assume every set of executions is closed under isomorphic renaming of operation identifiers. An $m(a)$-operation in an execution e is an operation identifier o s.t. e contains the actions $call_o(m, a)$ and $ret_o(m, a)$. An execution is called *sequential* when no two operations overlap, i.e., each call action is immediately followed by its matching return action, and *concurrent* otherwise. For readability, we write a sequential execution as a sequence of $put(a, p)$ and $rm(a)$ symbols representing a pair of actions $call_o(put, a, p) \cdot ret_o(put, a, p)$ and $call_o(rm, a) \cdot ret_o(rm, a)$, respectively ($o \in \mathbb{O}$). For example, given two priorities $p_1 \prec p_2$, $put(a, p_2) \cdot put(b, p_1) \cdot rm(b)$ is a sequential execution of the priority queue (rm returns b because it has smaller priority).

We define SeqPQ, the set of sequential priority queue executions, semantically via a labelled transition system (LTS, for short). An LTS is a tuple $A = (Q, \Sigma, \rightarrow, q_0)$, where Q is a set of states, Σ is an alphabet of transition labels, $\rightarrow \subseteq Q \times \Sigma \times Q$ is a transition relation, and q_0 is the initial state. We model the priority queue as an LTS PQ where states are mappings associating priorities in $\mathbb{P}$ with sequences of values in $\mathbb{D}$, representing a snapshot of the priority queue (for each priority, the values are ordered as they were inserted), and the transition labels are $put(a, p)$ and $rm(a)$. Each transition modifies the state as expected. For example, $q_1 \xrightarrow{rm(empty)} q_2$ if $q_1 = q_2$, and q_1 and q_2 map each priority to the empty sequence ϵ. Then, SeqPQ is the set of traces (words) accepted by PQ.

An implementation $\mathcal{I}$ is a set of executions. Implementations represent libraries whose methods are called by external programs. In the remainder of this work, we consider only

[1] Method return values are guessed nondeterministically, and validated at return points. This can be handled using `assume` statements, which only admit executions satisfying a given predicate.

completed executions, where each call action has a corresponding return action. This simplification is sound when the method invocations can always make progress in isolation.

2.2 Semantic Properties of Priority Queues

We define two properties which are important for our results: (1) *data independence* [22, 1] states that priority queue behaviors do not depend on the actual values which are added to the queue, and (2) *closure under projection* [5] states that executions remain valid by removing all the operations adding or removing certain values.

An execution e is *data-differentiated* if every value is added at most once, i.e., for each $d \in \mathbb{D}$, e contains at most one action $call_o(put, d, p)$ with $o \in \mathbb{O}$ and $p \in \mathbb{P}$. Note that this property concerns only values, a data-differentiated execution e may contain more than one value with the same priority. The subset of data-differentiated executions of a set of executions E is denoted by $E_{\neq}$.

A renaming function r is a function from $\mathbb{D}$ to $\mathbb{D}$. Given an execution e, we denote by $r(e)$ the execution obtained from e by replacing every data value x by $r(x)$. Note that r renames only the values and keeps the priorities unchanged. Intuitively, renaming values has no influence on the behavior of the priority queue, contrary to renaming priorities.

▶ **Definition 1.** A set of executions E is *data independent* iff
- for all $e \in E$, there exists $e' \in E_{\neq}$ and a renaming function r, such that $e = r(e')$,
- for all $e \in E$ and for all renamings r, $r(e) \in E$.

The following lemma is a direct consequence of definitions.

▶ **Lemma 2.** SeqPQ *is data independent.*

Beyond sequential executions, every concurrent priority queue implementation that we are aware of is data-independent. From now on, we consider only data-independent implementations. This assumption enables a reduction from checking the correctness of an implementation $\mathcal{I}$ to checking the correctness of its data-differentiated executions in $\mathcal{I}_{\neq}$.

Besides data independence, the sequential executions of the priority queue satisfy the following closure property: an execution remains valid when removing all the operations with an argument in some set of values $D \subseteq \mathbb{D}$ and any $rm(empty)$ operation (since they are read-only and they don't affect the queue's state). To distinguish between different $rm(empty)$ operations while simplifying the technical exposition, we assume that they receive as argument a value, i.e., call actions are of the form $call_o(rm, empty, a)$ for some $a \in \mathbb{D}$. We will make explicit this argument only when needed in our technical development. The projection $e|D$ of an execution e to a set of values $D \subseteq \mathbb{D}$ is obtained from e by erasing all the call/return actions with an argument not in D. We write $e \setminus x$ for the projection $e|_{\mathbb{D}\setminus\{x\}}$. Let $proj(e)$ be the set of all projections of e to a set of values $D \subseteq \mathbb{D}$.

▶ **Lemma 3.** SeqPQ *is closed under projection, i.e.,* $proj(e) \subseteq$ SeqPQ *for each* $e \in$ SeqPQ.

2.3 Linearizability

We recall the notion of *linearizability* [14] which is the *de facto* standard correctness condition for concurrent data structures. Given an execution e, the happen-before relation $<_{hb}$ between operations [2] is defined as follows: $o_1 <_{hb} o_2$, if the return action of o_1 occurs before

[2] In general, we refer to operations using their identifiers.

the call action of o_2 in e. The happens-before relation is an interval order [6]: for distinct o_1, o_2, o_3, o_4, if $o_1 <_{hb} o_2$ and $o_3 <_{hb} o_4$, then either $o_1 <_{hb} o_4$, or $o_3 <_{hb} o_2$. Intuitively, this comes from the fact that concurrent threads share a notion of global time.

Given a (concurrent) execution e and a sequential execution s, we say that e is linearizable w.r.t s, denoted $e \sqsubseteq s$, if there is a bijection $f : O_1 \to O_2$, where O_1 and O_2 are the set of operations of e and s, respectively, such that (1) the call and return actions with identifier o and $f(o)$, respectively, are the same and (2) if $o_1 <_{hb} o_2$, then $f(o_1) <_{hb} f(o_2)$. A (concurrent) execution e is linearizable w.r.t. a set S of sequential executions, denoted $e \sqsubseteq S$, if there exists $s \in S$ such that $e \sqsubseteq s$. A set of concurrent executions E is linearizable w.r.t. S, denoted $E \sqsubseteq S$, if $e \sqsubseteq S$ for all $e \in E$.

The following lemma states that by data-independence, it is enough to consider only data-differentiated executions when checking linearizability. Section 3 will focus on characterizing linearizability for data-differentiated executions.

▶ **Lemma 4.** *A data-independent implementation $\mathcal{I}$ is linearizable w.r.t. a data-independent set S of sequential executions, if and only if $\mathcal{I}_{\neq}$ is linearizable w.r.t. $S_{\neq}$.*

3 Checking Linearizability of Priority Queue Executions

We define a recursive procedure for checking linearizability of a data-differentiated execution w.r.t. SeqPQ. To ease the exposition, Section 3.1 introduces a recursive procedure for checking whether a data-differentiated *sequential* execution is admitted by the priority queue which is then extended to the concurrent case in Section 3.2.

3.1 Characterizing Data-Differentiated Sequential Executions

The recursive procedure *Check-PQ-Seq* outlined in Algorithm 1 checks whether a data-differentiated sequential execution belongs to SeqPQ (i.e., if it is accepted by the LTS PQ). Roughly, it selects one or two operations in the input execution, checks whether their return values are correct by ignoring the order between the other operations other than how they are ordered w.r.t. the selected ones, and calls itself recursively on the execution without the selected operations.

We explain how the procedure works on the following execution:

$$put(c, p_2) \cdot put(a, p_1) \cdot rm(a) \cdot rm(c) \cdot rm(empty) \cdot put(d, p_2) \cdot put(f, p_3) \cdot rm(f) \cdot put(b, p_1) \quad (1)$$

where p_1, p_2, p_3 are priorities such that $p_1 \prec p_2$ and $p_1 \prec p_3$, and p_2 and p_3 are incomparable. Since the $rm(empty)$ operations are read-only (they don't affect the queue's state), they are selected first. An $rm(empty)$-operation o is correct when every $put(x, p)$ operation before o is matched to a $rm(x)$ operation which also occurs before o. This is true in this case for $x \in \{a, c\}$. Thus, the correctness of (1) reduces to the correctness of

$$put(c, p_2) \cdot put(a, p_1) \cdot rm(a) \cdot rm(c) \cdot put(d, p_2) \cdot put(f, p_3) \cdot rm(f) \cdot put(b, p_1) \quad (2)$$

When the execution contains no $rm(empty)$-operation, the procedure selects a put operation adding a value that is not removed and that has a maximal priority. For (2), it selects $put(d, p_2)$ because p_2 is a maximal priority. This operation is correct since d is the last value with priority p_2 in the execution, and the correctness of (2) reduces to the correctness of

$$put(c, p_2) \cdot put(a, p_1) \cdot rm(a) \cdot rm(c) \cdot put(f, p_3) \cdot rm(f) \cdot put(b, p_1) \quad (3)$$

Algorithm 1: *Check-PQ-Seq*

Input: A data-differentiated sequential execution e
Output: true iff $e \in \mathsf{SeqPQ}$

```
1  if e = ε then
2  |  return true;
3  if Has-EmptyRemoves(e) then
4  |  if ∃ o = rm(empty) ∈ e such that EmptyRemove-Seq(e, o) holds then
5  |  |  return Check-PQ-Seq(e \ o);
6  else if Has-UnmatchedMaxPriority(e) then
7  |  if ∃ x ∈ values(e) such that UnmatchedMaxPriority-Seq(e, x) holds then
8  |  |  return Check-PQ-Seq(e \ x);
9  else
10 |  if ∃ x ∈ values(e) such that MatchedMaxPriority-Seq(e, x) holds then
11 |  |  return Check-PQ-Seq(e \ x);
12 |  else
13 |  |  return false;
```

If no operations like above can be found, *Check-PQ-Seq* selects a pair of *put* and *rm* operations adding and removing the same maximal priority value. For (2), it can select $put(c, p_2)$ and $rm(c)$. The value returned by $rm(c)$ is correct if all the values of priority smaller than p_2 added before $rm(c)$ are also removed before $rm(c)$. In this case, a is the only value of priority smaller than p_2 and it satisfies this property. Applying a similar reasoning for all the remaining values, it can be proved that this execution is correct.

Formally, the selected operations depend on the following set of predicates on executions:

$\mathsf{Has\text{-}EmptyRemoves}(e) = \mathsf{true}$ iff e contains a $rm(empty)$-operation

$\mathsf{Has\text{-}UnmatchedMaxPriority}(e) = \mathsf{true}$ iff $p \in \mathit{unmatched\text{-}priorities}(e)$ for a maximal p

where $\mathit{priorities}(e)$, resp., $\mathit{unmatched\text{-}priorities}(e)$, is the set of priorities occurring in *put* operations of e, resp., in *put* operations of e for which there is no *rm* operation removing the same value. We call the latter *unmatched* put operations. A put operation which is not unmatched is called *matched*. For simplicity, we consider the following syntactic sugar $\mathsf{Has\text{-}MatchedMaxPriority}(e) = \neg\mathsf{Has\text{-}EmptyRemoves}(e) \wedge \neg\mathsf{Has\text{-}UnmatchedMaxPriority}(e)$. By an abuse of notation, we assume $\mathsf{Has\text{-}UnmatchedMaxPriority}(e) \Rightarrow \neg\mathsf{Has\text{-}EmptyRemoves}(e)$ (this is sound by the order of the conditionals in *Check-PQ-Seq*).

The predicates defining the correctness of the selected operations are defined as follows:

$\mathsf{EmptyRemove\text{-}Seq}(e, o) = \mathsf{true}$ iff $e = u \cdot o \cdot v$ and $\mathit{matched}(u)$

$\mathsf{UnmatchedMaxPriority\text{-}Seq}(e, x) = \mathsf{true}$ iff $e = u \cdot \mathit{put}(x, p) \cdot v$, $p \not\prec \mathit{priorities}(u \cdot v)$, $p \notin \mathit{priorities}(v)$

$\mathsf{MatchedMaxPriority\text{-}Seq}(e, x) = \mathsf{true}$ iff $e = u \cdot \mathit{put}(x, p) \cdot v \cdot \mathit{rm}(x) \cdot w$, $\mathit{matched}_{\prec}(u \cdot v, p)$, $p \not\preceq \mathit{unmatched\text{-}priorities}(u \cdot v \cdot w)$, $p \not\prec \mathit{priorities}(u \cdot v \cdot w)$, and $p \notin \mathit{priorities}(v \cdot w)$

where $p \prec \mathit{priorities}(e)$ when $p \prec p'$ for some $p' \in \mathit{priorities}(e)$ (and similarly for $p \prec \mathit{unmatched\text{-}priorities}(e)$ or $p \preceq \mathit{unmatched\text{-}priorities}(e)$), $\mathit{matched}_{\prec}(e, p)$ holds when each value with priority strictly smaller than p is removed in e, and $\mathit{matched}(e)$ holds when $\mathit{matched}_{\prec}(e, p)$ holds for each $p \in \mathbb{P}$. Compared to the example presented at the beginning of the section, these predicates take into consideration that multiple values with the same priority are removed in FIFO order: the predicate $\mathsf{MatchedMaxPrioritySeq}(e, x)$ holds when x is the last value with priority p added in e.

When o is an $rm(empty)$-operation, $e \setminus o$ is the maximal subsequence of e which doesn't contain o. For an execution e, $\mathit{values}(e)$ is the set of values in call/return actions of e.

The following lemma states the correctness of *Check-PQ-Seq*.

▶ **Lemma 5.** *Check-PQ-Seq*$(e) = \mathsf{true}$ *iff* $e \in \mathsf{SeqPQ}$*, for every data-differentiated sequential execution* e*.*

3.2 Checking Linearizability of Data-Differentiated Concurrent Executions

The extension of *Check-PQ-Seq* to concurrent executions, checking whether they are linearizable w.r.t. SeqPQ, is obtained by replacing every predicate Γ-Seq with

$$\Gamma\text{-Conc}(e, \alpha) = \mathsf{true} \text{ iff there is a sequential execution } s \text{ such that } e \sqsubseteq s \text{ and } \Gamma\text{-Seq}(s, \alpha)$$

for each $\Gamma \in \{\mathsf{EmptyRemove}, \mathsf{UnmatchedMaxPriority}, \mathsf{MatchedMaxPriority}\}$. The obtained procedure is denoted by *Check-PQ-Conc* (recursive calls are modified accordingly).

The following lemma states the correctness of *Check-PQ-Conc*. Completeness follows easily from the properties of SeqPQ. If *Check-PQ-Conc*$(e) = \mathit{false}$, then there exists a set D of values s.t. either EmptyRemove-Conc$(e|D)$ is false, or UnmatchedMaxPriority-Conc$(e|D, x)$ is false for all the values x of maximal priority that are not removed (and there exists at least one such value), or MatchedMaxPriority-Conc$(e|D, x)$ is false for all the values x of maximal priority (and these values are all removed in $e|D$). It can be easily seen that we get $e|D \not\sqsubseteq \mathsf{SeqPQ}$ in all cases, which by the closure under projection of SeqPQ implies, $e \not\sqsubseteq \mathsf{SeqPQ}$ (since every linearization of e includes as a subsequence a linearization of $e|D$).

▶ **Lemma 6.** *Check-PQ-Conc*$(e) = \mathsf{true}$ *iff* $e \sqsubseteq \mathsf{SeqPQ}$*, for every data-differentiated* e*.*

Proving soundness is highly non-trivial and one of the main technical contributions of this paper. The main technical difficulty is showing that for any execution e, any linearization of $e \setminus x$ for some maximal priority value x can be extended to a linearization of e provided that UnmatchedMaxPriority or MatchedMaxPriority holds (depending on whether there are values with the same priority as x in e which are not removed).

We explain the proof of this property on the execution e in Figure 1(a) where $p_1 \prec p$, $p_1 \prec p_2$, and the predicate Has-MatchedMaxPriority(e) holds. Assume that there exist two sequential executions l and l' such that $e \sqsubseteq l = u \cdot put(x, p) \cdot v \cdot rm(x) \cdot w$, MatchedMaxPriority-Seq$(l, x)$ holds, and $e \setminus x \sqsubseteq l' \in \mathsf{SeqPQ}$. Let $u = \epsilon$, w be any sequence formed of $put(z_2, p_2)$ and $rm(z_1)$ (we distinguish them by adding the suffix "$-w$" to their name, e.g., $rm(z_1) - w$), and v be any sequence containing the remaining operations. In general, the linearization l' can be defined by choosing for each operation, a point in time between its call and return, called *linearization point*. The order between the linearization points defines the sequence l'. Figure 1(a) draws linearization points for the operations in $e \setminus x$ which define l' [3]. We show how to construct a sequence $l'' = l''_1 \cdot put(x, p) \cdot l''_2 \cdot rm(x) \cdot l''_3 \in \mathsf{SeqPQ}$ s.t. $e \sqsubseteq l''$.

- An operation is called p-comparable (resp., p-incomparable) when it receives as argument a value of priority comparable to p (resp., incomparable to p). Defining l''_1, l''_2, and l''_3 as the projection of l' to the set of operations in u, v and w, respectively, leads to a sequence $l'' \notin \mathsf{SeqPQ}$. This is because MatchedMaxPriority-Seq(l, x) imposes no restriction on p-incomparable operations in $u \cdot v$, and the projection of l' to p-incomparable operations in $u \cdot v$ is not in SeqPQ. In this example, this projection is $put(z_1, p_2) \cdot rm(z_2)$.

[3] In general, there may exist multiple ways of choosing linearization points to define the same linearization. Our construction is agnostic to this choice.

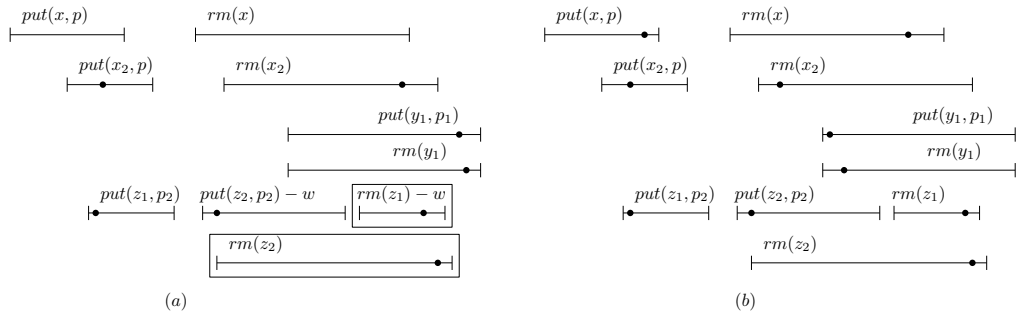


Figure 1 A concurrent execution e exemplifying the soundness of *Check-PQ-Conc*.

- We define the sets of operations U', V' and W' such that l''_1, l''_2 and l''_3 are the projections of l' to U', V', and W', respectively. This is done in two steps:
  1. The first step is to define W'. The p-comparable operations in W' are the same as in w. To identify the p-incomparable operations in W', we search for a p-incomparable operation o which either happens before some p-comparable operation in w, or whose linearization point occurs after $ret(rm, x)$. We add to W' the operation o and all the p-incomparable operations occurring after o in l'. In this example, o is $rm(z_1)$ and the only p-incomparable operation occurring after o in l' is $rm(z_2)$ (they are surrounded by boxes in the figure). In this process, whether a p-incomparable operation is in W' or not only relies on whether it is before or after such an o in l'.
  2. The second step is to define U' and V'. U' contains two kinds of operations: (1) operations whose linearization points are before $ret(put, x, p)$, and (2) other put operations with priority p. V' contains the remaining operations. In this example, U' contains $put(z_1, p_2)$ and $put(x_2, p)$.
- In conclusion, we have that $l''_1 = put(z_1, p_2) \cdot put(x_2, p)$, $l''_2 = put(z_2, p_2) \cdot rm(x_2) \cdot put(y_1, p_1) \cdot rm(y_1)$, and $l''_3 = rm(z_1) \cdot rm(z_2)$. Figure 1(b) draws linearization points for each operation in e defining the linearization l''.

Section 4 introduces a characterization of concurrent priority queue violations using a set of *non-recursive* automata (whose states consist of a fixed number of registers), whose standard synchronized product is equivalent to *Check-PQ-Conc* (modulo a renaming of values which is possible by data-independence). Since SeqPQ is closed under projection (Lemma 3), the recursion in *Check-PQ-Conc* can be eliminated by checking that each projection of a given execution e passes a non-recursive version of *Check-PQ-Conc* where every recursive call **return** *Check-PQ-Conc*(...) is replaced by **return** true. Let *Check-PQ-Conc-NonRec* be the thus obtained procedure.

▶ **Lemma 7.** *Given a data-differentiated execution e, $e \sqsubseteq$ SeqPQ if and only if for each $e' \in proj(e)$, Check-PQ-Conc-NonRec(e') returns* true.

4 Reducing Linearizability of Priority Queues to Reachability

We show that the set of executions for which *Check-PQ-Conc-NonRec* fails on some projection can be described using register automata, modulo a value renaming. Renaming values (which is complete under data independence) allows to simplify the reasoning about projections. W.l.o.g., we assume that all the operations which are not in the projection failing this test use the same distinguished value $\top$, different from those in the projection. Then, it is enough to find an automata characterization of the executions e for which

Check-PQ-Conc-NonRec(e) is false, i.e., for which $\Gamma(e) := \text{Has-}\Gamma(e) \Rightarrow \exists \alpha.\ \Gamma\text{-Conc}(e, \alpha)$ is false, for some $\Gamma \in \{\text{EmptyRemove}, \text{UnmatchedMaxPriority}, \text{MatchedMaxPriority}\}$. Intuitively, $\Gamma(e)$ states that e is linearizable w.r.t. the set of sequential executions described by Γ-Seq (provided that Has-$\Gamma(e)$ holds). Therefore, by an abuse of terminology, an execution e satisfying $\Gamma(e)$ is called *linearizable w.r.t.* Γ, or Γ-*linearizable*. Extending the automaton characterizing non Γ-linearizable executions with self-loops that allow any operation with argument $\top$ results in an automaton satisfying the following property called Γ-*completeness*.

▶ **Definition 8.** For $\Gamma \in \{\text{EmptyRemove}, \text{UnmatchedMaxPriority}, \text{MatchedMaxPriority}\}$, an automaton A is called Γ-*complete* when for each data-independent implementation $\mathcal{I}$:

$A \cap \mathcal{I} \neq \emptyset$ iff there exists $e \in \mathcal{I}$ and $e' \in \mathit{proj}(e)$ such that e' is not Γ-linearizable.

We can show that for any $\Gamma \in \{\text{EmptyRemove}, \text{UnmatchedMaxPriority}, \text{MatchedMaxPriority}\}$ there exists a Γ-complete automaton. For lack of space, we only consider the case $\Gamma = \text{MatchedMaxPriority}$ in Section 4.1. When defining Γ-complete automata, we assume that every implementation $\mathcal{I}$ behaves correctly, i.e., as a FIFO queue, when only values with the same priority are observed. More precisely, we assume that for every execution $e \in \mathcal{I}$ and every priority $p \in \mathbb{P}$, the projection of e to values with priority p is linearizable (w.r.t. SeqPQ). This property can be checked separately using register automata similar to the automata in [5] describing FIFO queue violations. This assumption excludes some obvious violations, such as an $\mathit{rm}(a)$ operation happening before a $\mathit{put}(a, p)$ operation, for some p.

For $\Gamma \in \{\text{UnmatchedMaxPriority}, \text{MatchedMaxPriority}\}$, we consider Γ-complete automata recognizing executions which contain only one maximal priority. This is w.l.o.g. because any data-differentiated execution for which $\Gamma(e)$ is false has such a projection. Formally, given a data-differentiated execution e and p a maximal priority in e, $e|_{\preceq p}$ is the projection of e to the set of values with priorities smaller or equal to p. Then,

▶ **Lemma 9.** *For* $\Gamma \in \{\text{UnmatchedMaxPriority}, \text{MatchedMaxPriority}\}$*, a data-differentiated execution* e *is* Γ*-linearizable iff* $e|_{\preceq p}$ *is* Γ*-linearizable for some maximal priority* p *in* e*.*

Proof. (Sketch) For the "only-if" direction, let e be a data-differentiated execution linearizable w.r.t. $l = u \cdot \mathit{put}(x, p) \cdot v \cdot \mathit{rm}(x) \cdot w$ s.t. MatchedMaxPriority-Seq(l, x) holds. Since the predicate MatchedMaxPriority-Seq(l, x) imposes no restriction on the operations in u, v, and w with priorities incomparable to p, erasing all these operations results in a sequential execution which still satisfies this predicate. Similarly, for $\Gamma = \text{UnmatchedMaxPriority}$.

The "if" direction follows from the fact that if the projection of an execution to a set of operations O_1 has a linearization l_1 and the projection of the same execution to the remaining set of operations has a linearization l_2, then the execution has a linearization which is defined as an interleaving of l_1 and l_2.

Thus, let e be an execution such that $e|_{\preceq p}$ is linearizable w.r.t. $l = u \cdot \mathit{put}(x, p) \cdot v \cdot \mathit{rm}(x) \cdot w$ where MatchedMaxPriority-Seq(l, x) holds. By the property above, we know that e has a linearization $l' = u' \cdot \mathit{put}(x, p) \cdot v' \cdot \mathit{rm}(x) \cdot w'$, such that the projection of l' to values of priority comparable to p is l. Since MatchedMaxPriority-Seq(l, x) doesn't constrain the values of priority incomparable to p, we obtain that MatchedMaxPriority-Seq(l', α) also holds. ◀

The existence of Γ-complete automata enable an effective reduction of checking linearizability of concurrent priority queue implementations to state reachability. Section 4.2 discusses decidability results implied by this reduction.

▶ **Theorem 10.** *Let* $\mathcal{I}$ *be a data-independent implementation. Then, there is a* Γ*-complete automaton* $A(\Gamma)$ *for each* Γ*. Moreover,* $\mathcal{I} \sqsubseteq \text{SeqPQ}$ *iff* $\mathcal{I} \cap A(\Gamma) = \emptyset$ *for all* Γ*.*

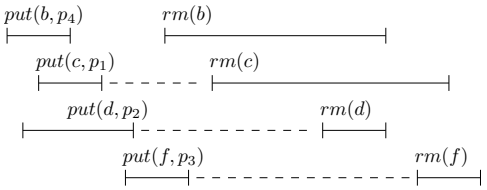


Figure 2 An execution that is not MatchedMaxPriority$^{>}$-linearizable. We represent each operation as a time interval whose left, resp., right, bound corresponds to the call, resp., return action.

4.1 A MatchedMaxPriority-complete automaton

A differentiated execution e is not MatchedMaxPriority-linearizable when all the *put* operations in e using the maximal priority p are matched, and e is not linearizable w.r.t. the set of sequential executions satisfying MatchedMaxPriority-Seq(e, x) for each value x of priority p. We consider two cases depending on whether e contains exactly one value with priority p or at least two values. We denote by MatchedMaxPriority$^{>}$ the strengthening of MatchedMaxPriority with the condition that all the values other than x have a priority strictly smaller than p (corresponding to the first case), and by MatchedMaxPriority$^{=}$ the strengthening of the same formula with the negation of this condition (corresponding to the second case). We use particular instances of register automata [15, 9, 18] whose states include only two registers, one for storing a priority guessed at the initial state, and one for storing the priority of the current action in the execution. The transitions can check equality or the order relation $\prec$ between the values stored in the two registers. Instead of formalizing the full class of register automata, we consider a simpler class which suffices our needs. Thus, we consider a class of labeled transition systems whose states consist of a finite control part and a register r interpreted to elements of $\mathbb{P}$. The transition labels are:

- $r = *$ for storing an arbitrary value to r,
- $call(rm, a)$ and $ret(rm, a)$ for reading call/return actions of a remove,
- $call(put, d, g)$ where $g \in \{= r, \prec r, true\}$ is a guard, for reading a call action $call(put, d, p)$ and checking if p is either equal to or smaller than the value stored in r, or arbitrary,
- $ret(put, d, true)$ for reading a return action $ret(put, d, p)$ for any p.

The set of sequences (executions) accepted by such a transition system is defined as usual.

4.1.1 A MatchedMaxPriority$^{>}$-complete automaton

Figure 2 contains a typical example of an execution e which is not MatchedMaxPriority$^{>}$-linearizable, where $p_1 \prec p_4$, $p_2 \prec p_4$, and $p_3 \prec p_4$. Intuitively, this is a violation because during the whole execution of $rm(b)$, the priority queue stores a smaller priority value (which should be removed before b). To be more precise, we define *the interval of a value* x as the time interval from the return of a put $ret(put, x, p)$ to the call of the matching remove $call(rm, x)$, or to the end of the execution if such a call action doesn't exist. This represents the time interval in which a value is guaranteed to be stored into the priority queue. Concretely, for a standard indexing of actions in an execution, a time interval is a closed interval between the indexes of two actions in the execution. In Figure 2, the interval of each value of priority smaller than p_4 is pictured as a dashed line. There is no sequence l s.t. $e \sqsubseteq l$ and MatchedMaxPriority-Seq(l, b) hold, since each time point from $call(rm, b)$ to $ret(rm, b)$ is included in the interval of a smaller priority value, and $rm(b)$ can't take effect in the interval of a smaller priority value. To formalize this scenario we use the notion of *left-right constraint* defined below.

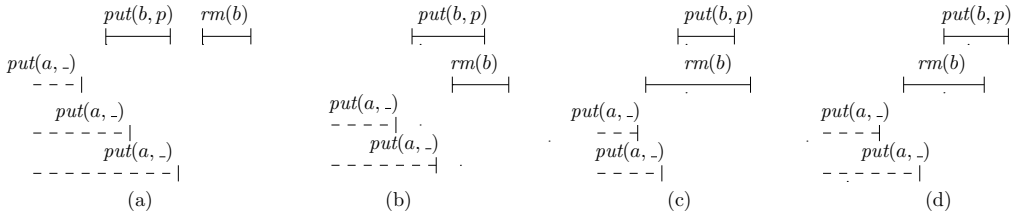


Figure 3 Orderings to be considered when defining a MatchedMaxPriority$^{>}$-complete automaton.

▶ **Definition 11.** Let e be a data-differentiated execution which contains only one maximal priority p, and only one value x of priority p (and no $rm(empty)$ operations). The *left-right constraint of* x is the graph G where:

- the nodes are the values occurring in e,
- there is an edge from d_1 to x, if $put(d_1, \_) <_{hb} put(x, p)$ or $put(d_1, \_) <_{hb} rm(x)$,
- there is an edge from x to d_1, if $rm(x) <_{hb} rm(d_1)$ or $rm(d_1)$ does not exists,
- there is an edge from d_1 to d_2, if $put(d_1, \_) <_{hb} rm(d_2, \_)$.

The execution in Figure 2 is not MatchedMaxPriority$^{>}$-linearizable because the left-right constraint of the maximal priority value b contains the cycle $f \to d \to c \to b \to f$. The presence of such a cycle is equivalent to *non* MatchedMaxPriority$^{>}$-linearizability:

▶ **Lemma 12.** *Let e be a data-differentiated execution such that* Has-MatchedMaxPriority(e) *holds, p is the maximal priority in e, and $put(x, p)$ and $rm(x)$ are only operations with arguments of priority p in e. Then, e is* MatchedMaxPriority*-linearizable iff the left-right constraint of x contains no cycle going through x.*

When the left-right constraint contains a cycle $d_1 \to \ldots \to d_m \to x \to d_1$, for some $d_1,\ldots,d_n \in \mathbb{D}$, we say that x is *covered* by $d_1, \ldots, d_m$. The shape of an execution witnessing such a cycle (i.e., the alternation between call/return actions) can be identified using our class of automata, the only complication being the unbounded number of values $d_1,\ldots,d_n$. However, by data independence, whenever an implementation contains such an execution it also contains an execution where all the values $d_1,\ldots,d_n$ are renamed to the same value a, and x is renamed to b. Therefore, our automata can be defined over a fixed set of values a, b, and $\top$ (recall that $\top$ is used for operations outside of the non-linearizable projection).

To define a MatchedMaxPriority$^{>}$-complete automaton, we need to consider all the possible orders between the call/return actions of the put/rm operations that add and respectively, remove the value b. The case where the put happens-before the remove (as in Figure 2) is pictured in Figure 4. This automaton captures the three possible ways of ordering the first action $ret(put, a, \_)$ w.r.t. the actions with value b, which are pictured in Figure 3(a) (this action cannot occur after $call(rm, b, \_)$ since b must be covered by the a-s). The paths corresponding to these three possible orders are: $q_1 \to q_2 \to q_3 \ldots \to q_7$, $q_1 \to q_2 \to q_3 \ldots \to q_{10}$, and $q_1 \to q_9 \to q_{10} \ldots \to q_7$. Figure 3 lists the four possible orderings of the call/return actions of adding and removing b, and also possible orders of the first $ret(put, a, \_)$ w.r.t the actions with value b. Each such ordering corresponds to an automaton similar to the one in Figure 4, their union defining a MatchedMaxPriority$^{>}$-complete automaton.

4.1.2 A MatchedMaxPriority$^{=}$-complete automaton

When an execution contains at least two values of maximal priority, the acyclicity of the left-right constraints (for all the maximal priority values) is not enough to conclude that

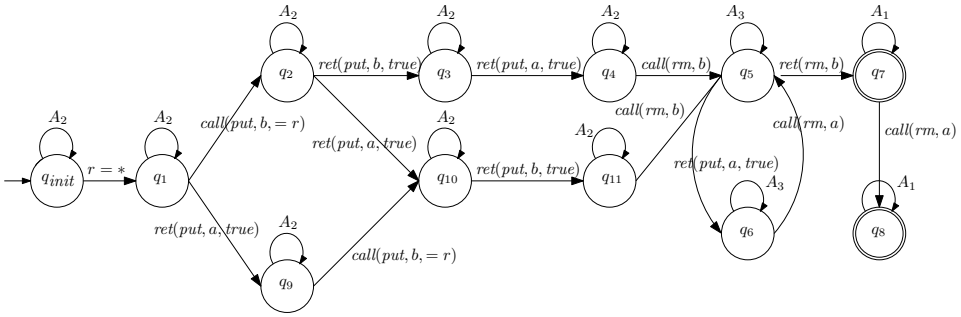


Figure 4 A register automaton capturing the scenario in Figure 3(a). We use the following notations: $A_1 = A \cup \{ret(rm, a)\}$, $A_2 = A \cup \{call(put, a, = r)\}$, $A_3 = A_2 \cup \{ret(rm, a)\}$, where $A = \{call(put, \top, true), ret(put, \top, true), call(rm, \top), ret(rm, \top), call(rm, empty), ret(rm, empty)\}$.

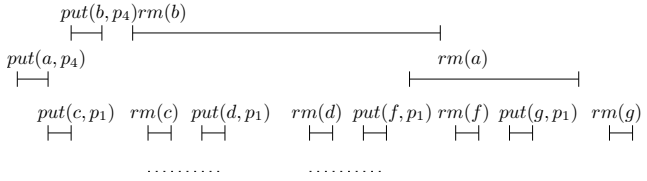


Figure 5 An execution that is not MatchedMaxPriority$^{=}$-linearizable.

the execution is MatchedMaxPriority-linearizable. Intuitively, there may exist a value a which is added before another value b such that all the possible linearization points of $rm(b)$ are disabled by the position of $rm(a)$ in the happens-before. We give an example of such an execution e in Figure 5, where $p_1 \prec p_4$. This execution is not linearizable w.r.t. MatchedMaxPriority (or MatchedMaxPriority$^{=}$) even if neither a nor b are covered by values with smaller priority. Since $put(a, p_4) <_{hb} put(b, p_4)$ and values of the same priority are removed in FIFO order, $rm(a)$ should be linearized before $rm(b)$ (i.e., this execution should be linearizable w.r.t. a sequence where $rm(a)$ occurs before $rm(b)$). Since $rm(b)$ cannot take effect during the interval of a smaller priority value, it could be only linearized in one of the two time intervals pictured with dotted lines in Figure 5. However, each of those time intervals ends before $call(rm, a)$, and thus $rm(a)$ cannot be linearized before $rm(b)$.

To recognize the scenarios in Figure 5, we introduce an order $<_{pb}$ between values which intuitively, can be thought of as "a value a is put before another value b". More precisely, given a data-differentiated execution e and two values a and b of maximal priority, $a <_{pb} b$ if one of the following holds: (1) $put(a, \_) <_{hb} put(b, \_)$, (2) $rm(a) <_{hb} rm(b)$, or (3) $rm(a) <_{hb} put(b, \_)$. Sometimes we use $a <^{A}_{pb} b$, $a <^{B}_{pb} b$, and $a <^{C}_{pb} b$ to explicitly distinguish between these three cases. Let $<^{*}_{pb}$ be the transitive closure of $<_{pb}$.

To define the time intervals in which a remove like $rm(b)$ in Figure 5, can be linearized (outside of intervals of smaller priority values) we use the notion of gap-point. As before, defining time intervals relies on an indexing of actions in an execution, starting with 0.

▶ **Definition 13.** Let e be a data-differentiated execution with only one maximal priority p, and $put(x, p)$ and $rm(x)$ two operations in e. An index $i \in [0, |e| - 1]$ is a *gap-point of* x if i is greater than or equal to the index of both $call(put, x, p)$ and $call(rm, x)$, smaller than the index of $ret(rm, x)$, and not included in the interval of a value with priority smaller than p.

The case of Figure 5 can be formally described as follows: $a <^{*}_{pb} b$ while the right-most gap-point of b is before $call(rm, a)$ or $call(put, a, p_4)$. The following lemma states that these conditions are enough to characterize non-linearizability w.r.t. MatchedMaxPriority$^{=}$.

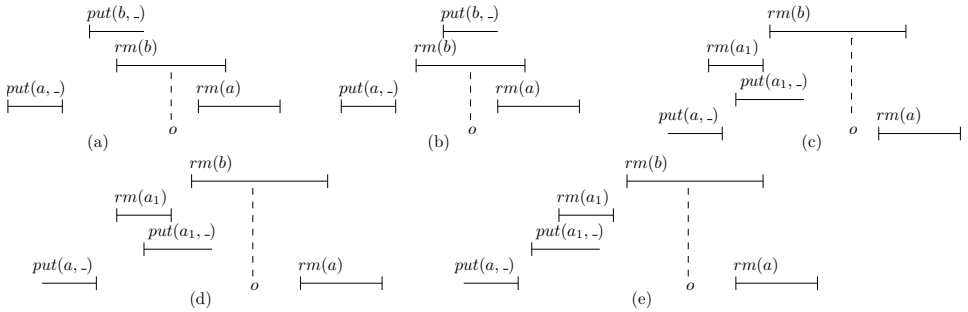


Figure 6 Orderings to be considered when defining a MatchedMaxPriority$^{=}$-complete automaton. We omit the operations which can be ordered arbitrarily, e.g., $put(b)$ in the cases (d) and (e).

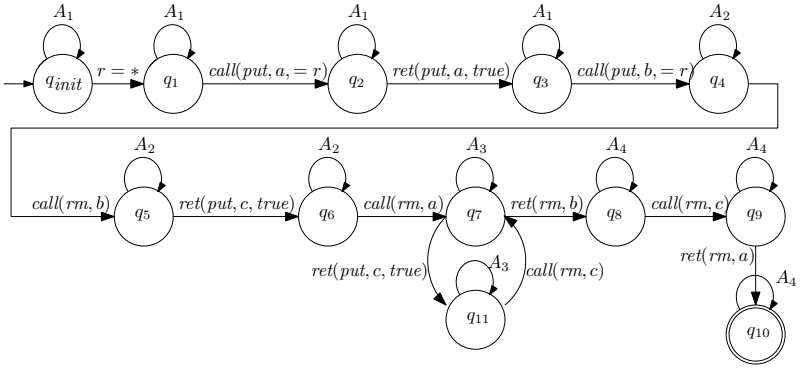


Figure 7 A register automaton for the case in Figure 6(a), where $A_1 = A \cup \{call(put, c, < r)\}$, $A_2 = A_1 \cup \{ret(put, b, = r)\}$, $A_3 = A_2 \cup \{ret(rm, c)\}$, $A_4 = A \cup \{ret(put, b, = r), ret(rm, c)\}$, where $A = \{call(put, \top, true), ret(put, \top, true), call(rm, \top), ret(rm, \top), call(rm, empty), ret(rm, empty)\}$.

▶ **Lemma 14.** *Let e be a data-differentiated execution with only one maximal priority p such that* Has-MatchedMaxPriority(e) *holds. Then, e is not* MatchedMaxPriority$^{=}$*-linearizable iff e contains two values x and y of maximal priority p such that $y <^{*}_{pb} x$, and the rightmost gap-point of x is strictly smaller than the index of $call(put, y, p)$ or $call(rm, y)$.*

The following shows that the number of values needed to witness that $y <^{*}_{pb} x$, for some x and y, is bounded.

▶ **Lemma 15.** *Let e be a data-differentiated execution such that $a <_{pb} a_1 <_{pb} \ldots <_{pb} a_m <_{pb} b$ holds for some set of values a, a_1,...,a_m, b. Then, one of the following holds:*

- $a <^{A}_{pb} b$, $a <^{B}_{pb} b$, *or* $a <^{C}_{pb} b$,
- $a <^{A}_{pb} a_i <^{B}_{pb} b$ *or* $a <^{B}_{pb} a_i <^{A}_{pb} b$, *for some i.*

To characterize violations to MatchedMaxPriority$^{=}$-linearizability, one has to consider all the possible orders between call/return actions of the operations on values a, b, and a_i in Lemma 15, and the right-most gap point of b. Excluding the inconsistent cases, we are left with the five orders in Figure 6, where o denotes the rightmost gap-point of b. For each case, we define an automaton recognizing the induced set of violations. The register automaton for the case in Figure 6(a) is shown in Figure 7. In this case, Lemma 14 is equivalent to the fact that intuitively, the time interval from $call(rm, a)$ to $ret(rm, b)$ is covered by lower priority values (thus, there is no gap-point of b which occurs after $call(rm, a)$). By data-independence, these lower priority values can be renamed to a fixed value c.

4.2 Decidability Result

We describe a class $\mathcal{C}$ of data-independent implementations for which linearizability w.r.t. SeqPQ is decidable. The implementations in $\mathcal{C}$ allow an unbounded number of values but a bounded number of priorities. Each method has a finite set of local variables storing Boolean values or values from $\mathbb{D}$. Methods communicate through a finite number of shared variables interpreted also as Booleans or values from $\mathbb{D}$. To ensure data independence, values in $\mathbb{D}$ may be assigned, but never used in Boolean expressions (e.g., of if-then-else statements). This class captures typical implementations, or finite-state abstractions thereof, e.g., obtained via predicate abstraction. The Γ-complete automata we define use a fixed set $D = \{a, b, c, a_1, \top\}$ of values (a_1 is needed to deal with the second item in Lemma 15). Therefore, for any Γ, $\mathcal{C} \cap A(\Gamma) \neq \emptyset$ iff $\mathcal{C}_D \cap A(\Gamma) \neq \emptyset$, where $\mathcal{C}_D$ is the subset of $\mathcal{C}$ that uses only values in D.

The set of executions $\mathcal{C}_D$ can be represented by a Vector Addition System with States (VASS). Since the set of values and priorities is bounded, each method invocation can be represented by a finite-state automaton (see [4]). For a fixed set of priorities $P \subseteq \mathbb{P}$, the register automata $A(\Gamma)$ can be transformed to finite-state automata (the number of valuations of the registers is bounded). Thus, checking linearizability of an implementation in $\mathcal{C}$ is PSPACE when the number of threads is bounded, and EXPSPACE, otherwise. Moreover, reachability in VASSs can be reduced to checking linearizability of such an implementation. Essentially, given an instance of the VASS reachability problem, one can define a priority queue implementation where the *put* methods behave correctly and additionally, they include the code of the VASS simulation defined in [4], and the *rm* methods behave correctly, except for the moment where the target state is reached, in which case they trigger a linearizability violation by returning an arbitrary value.

▶ **Theorem 16.** *Verifying whether an implementation in $\mathcal{C}$ is linearizable w.r.t.* SeqPQ *is PSPACE-complete for a fixed number of threads, and EXPSPACE-complete otherwise.*

5 Related work

The theoretical limits of checking linearizability have been investigated in previous works. Checking linearizability of a single execution w.r.t. an arbitrary ADT is NP-complete [11] while checking linearizability of all the executions of a finite-state implementation w.r.t. an arbitrary ADT specification (given as a regular language) is EXPSPACE-complete when the number of program threads is bounded [3, 12], and undecidable otherwise [4].

Existing automated methods for proving linearizability of a concurrent object are also based on reductions to safety verification, e.g., [1, 13, 21]. The approach in [21] considers implementations where operations' *linearization points* are manually specified. Essentially, this approach instruments the implementation with ghost variables simulating the ADT specification at linearization points. This approach is incomplete since not all implementations have fixed linearization points. Aspect-oriented proofs [13] reduce linearizability to the verification of four simpler safety properties. This approach has only been applied to queues, and has not produced a fully automated and complete proof technique. The work in [10] proves linearizability of stack implementations with an automated proof assistant. Their approach does not lead to full automation however, e.g., by reduction to safety verification.

Our previous work [5] shows that checking linearizability of finite-state implementations of concurrent queues and stacks is decidable. Roughly, we follow the same schema: the recursive procedure in Section 3.1 is similar to the inductive rules in [5], and its extension to concurrent executions in Section 3.2 corresponds to the notion of step-by-step linearizability in [5]. Although similar in nature, defining these procedures and establishing their

correctness require proof techniques which are specific to the priority queue semantics. The order in which values are removed from a priority queue is encoded in their priorities which come from an unbounded domain, and not in the happens-before order as in the case of stacks and queues. Therefore, the results we introduce in this paper cannot be inferred from those in [5]. At a technical level, characterizing the priority queue violations requires a more expressive class of automata (with registers) than the finite-state automata in [5].

References

1. Parosh Aziz Abdulla, Frédéric Haziza, Lukás Holík, Bengt Jonsson, and Ahmed Rezine. An integrated specification and verification technique for highly concurrent data structures. In Nir Piterman and Scott A. Smolka, editors, *Tools and Algorithms for the Construction and Analysis of Systems - 19th International Conference, TACAS 2013, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2013, Rome, Italy, March 16-24, 2013. Proceedings*, volume 7795 of *Lecture Notes in Computer Science*, pages 324–338. Springer, 2013. doi:10.1007/978-3-642-36742-7_23.
2. Dan Alistarh, Justin Kopinsky, Jerry Li, and Nir Shavit. The spraylist: a scalable relaxed priority queue. In Albert Cohen and David Grove, editors, *Proceedings of the 20th ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming, PPoPP 2015, San Francisco, CA, USA, February 7-11, 2015*, pages 11–20. ACM, 2015. doi:10.1145/2688500.2688523.
3. Rajeev Alur, Kenneth L. McMillan, and Doron A. Peled. Model-checking of correctness conditions for concurrent objects. *Inf. Comput.*, 160(1-2):167–188, 2000. doi:10.1006/inco.1999.2847.
4. Ahmed Bouajjani, Michael Emmi, Constantin Enea, and Jad Hamza. Verifying concurrent programs against sequential specifications. In Matthias Felleisen and Philippa Gardner, editors, *Programming Languages and Systems - 22nd European Symposium on Programming, ESOP 2013, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2013, Rome, Italy, March 16-24, 2013. Proceedings*, volume 7792 of *Lecture Notes in Computer Science*, pages 290–309. Springer, 2013. doi:10.1007/978-3-642-37036-6_17.
5. Ahmed Bouajjani, Michael Emmi, Constantin Enea, and Jad Hamza. On reducing linearizability to state reachability. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II*, volume 9135 of *Lecture Notes in Computer Science*, pages 95–107. Springer, 2015. doi:10.1007/978-3-662-47666-6_8.
6. Ahmed Bouajjani, Michael Emmi, Constantin Enea, and Jad Hamza. Tractable refinement checking for concurrent objects. In Rajamani and Walker [17], pages 651–662. doi:10.1145/2676726.2677002.
7. Ahmed Bouajjani, Constantin Enea, and Chao Wang. Checking linearizability of concurrent priority queues. *Arxiv*, 2017. URL: https://arxiv.org/abs/1707.00639.
8. Irina Calciu, Hammurabi Mendes, and Maurice Herlihy. The adaptive priority queue with elimination and combining. In Fabian Kuhn, editor, *Distributed Computing - 28th International Symposium, DISC 2014, Austin, TX, USA, October 12-15, 2014. Proceedings*, volume 8784 of *Lecture Notes in Computer Science*, pages 406–420. Springer, 2014. doi:10.1007/978-3-662-45174-8_28.
9. Karlis Cerans. Deciding properties of integral relational automata. In Serge Abiteboul and Eli Shamir, editors, *Automata, Languages and Programming, 21st International Colloquium, ICALP94, Jerusalem, Israel, July 11-14, 1994, Proceedings*, volume 820 of *Lecture*

Notes in Computer Science, pages 35–46. Springer, 1994. doi:10.1007/3-540-58201-0_56.

10 Mike Dodds, Andreas Haas, and Christoph M. Kirsch. A scalable, correct time-stamped stack. In Rajamani and Walker [17], pages 233–246. doi:10.1145/2676726.2676963.

11 Phillip B. Gibbons and Ephraim Korach. Testing shared memories. *SIAM J. Comput.*, 26(4):1208–1244, 1997. doi:10.1137/S0097539794279614.

12 Jad Hamza. On the complexity of linearizability. In Ahmed Bouajjani and Hugues Fauconnier, editors, *Networked Systems - Third International Conference, NETYS 2015, Agadir, Morocco, May 13-15, 2015, Revised Selected Papers*, volume 9466 of *Lecture Notes in Computer Science*, pages 308–321. Springer, 2015. doi:10.1007/978-3-319-26850-7_21.

13 Thomas A. Henzinger, Ali Sezgin, and Viktor Vafeiadis. Aspect-oriented linearizability proofs. In Pedro R. D'Argenio and Hernán C. Melgratti, editors, *CONCUR 2013 - Concurrency Theory - 24th International Conference, CONCUR 2013, Buenos Aires, Argentina, August 27-30, 2013. Proceedings*, volume 8052 of *Lecture Notes in Computer Science*, pages 242–256. Springer, 2013. doi:10.1007/978-3-642-40184-8_18.

14 Maurice Herlihy and Jeannette M. Wing. Linearizability: A correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990. doi:10.1145/78969.78972.

15 Michael Kaminski and Nissim Francez. Finite-memory automata. *Theor. Comput. Sci.*, 134(2):329–363, 1994. doi:10.1016/0304-3975(94)90242-9.

16 Jonatan Lindén and Bengt Jonsson. A skiplist-based concurrent priority queue with minimal memory contention. In Roberto Baldoni, Nicolas Nisse, and Maarten van Steen, editors, *Principles of Distributed Systems - 17th International Conference, OPODIS 2013, Nice, France, December 16-18, 2013. Proceedings*, volume 8304 of *Lecture Notes in Computer Science*, pages 206–220. Springer, 2013. doi:10.1007/978-3-319-03850-6_15.

17 Sriram K. Rajamani and David Walker, editors. *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2015, Mumbai, India, January 15-17, 2015*. ACM, 2015.

18 Luc Segoufin and Szymon Toruńczyk. Automata based verification over linearly ordered data domains. In Thomas Schwentick and Christoph Dürr, editors, *28th International Symposium on Theoretical Aspects of Computer Science, STACS 2011, March 10-12, 2011, Dortmund, Germany*, volume 9 of *LIPIcs*, pages 81–92. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2011. doi:10.4230/LIPIcs.STACS.2011.81.

19 Nir Shavit and Itay Lotan. Skiplist-based concurrent priority queues. In *Proceedings of the 14th International Parallel & Distributed Processing Symposium (IPDPS'00), Cancun, Mexico, May 1-5, 2000*, pages 263–268. IEEE Computer Society, 2000. doi:10.1109/IPDPS.2000.845994.

20 Nir Shavit and Asaph Zemach. Scalable concurrent priority queue algorithms. In Brian A. Coan and Jennifer L. Welch, editors, *Proceedings of the Eighteenth Annual ACM Symposium on Principles of Distributed Computing, PODC, '99Atlanta, Georgia, USA, May 3-6, 1999*, pages 113–122. ACM, 1999. doi:10.1145/301308.301339.

21 Viktor Vafeiadis. Automatically proving linearizability. In Tayssir Touili, Byron Cook, and Paul B. Jackson, editors, *Computer Aided Verification, 22nd International Conference, CAV 2010, Edinburgh, UK, July 15-19, 2010. Proceedings*, volume 6174 of *Lecture Notes in Computer Science*, pages 450–464. Springer, 2010. doi:10.1007/978-3-642-14295-6_40.

22 Pierre Wolper. Expressing interesting properties of programs in propositional temporal logic. In *Conference Record of the Thirteenth Annual ACM Symposium on Principles of Programming Languages, St. Petersburg Beach, Florida, USA, January 1986*, pages 184–193. ACM Press, 1986. doi:10.1145/512644.512661.

Nash Equilibrium and Bisimulation Invariance*

Julian Gutierrez[1], Paul Harrenstein[2], Giuseppe Perelli[3], and Michael Wooldridge[4]

1 **Department of Computer Science, University of Oxford, Oxford, United Kingdom**
julian.gutierrez@cs.ox.ac.uk
2 **Department of Computer Science, University of Oxford, Oxford, United Kingdom**
paul.harrenstein@cs.ox.ac.uk
3 **Department of Computer Science, University of Oxford, Oxford, United Kingdom**
giuseppe.perelli@cs.ox.ac.uk
4 **Department of Computer Science, University of Oxford, Oxford, United Kingdom**
mjw@cs.ox.ac.uk

Abstract

Game theory provides a well-established framework for the analysis of concurrent and multi-agent systems. The basic idea is that concurrent processes (agents) can be understood as corresponding to players in a game; plays represent the possible computation runs of the system; and strategies define the behaviour of agents. Typically, strategies are modelled as functions from sequences of system states to player actions. Analysing a system in such a way involves computing the set of (Nash) equilibria in the game. However, we show that, with respect to the above model of strategies—the standard model in the literature—*bisimilarity does not preserve the existence of Nash equilibria*. Thus, two concurrent games which are behaviourally equivalent from a semantic perspective, and which from a logical perspective satisfy the same temporal formulae, nevertheless have fundamentally different properties from a game theoretic perspective. In this paper we explore the issues raised by this discovery, and investigate three models of strategies with respect to which the existence of Nash equilibria is preserved under bisimilarity. We also use some of these models of strategies to provide new semantic foundations for logics for strategic reasoning, and investigate restricted scenarios where bisimilarity can be shown to preserve the existence of Nash equilibria with respect to the conventional model of strategies in the literature.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs, F.4.1 Mathematical logic (Temporal Logic), I.2.11 Distributed AI (Multiagent Systems)

Keywords and phrases Bisimulation, Nash Equilibrium, Multiagent Systems, Strategy Logic

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.17

1 Introduction

The concept of *bisimilarity* plays a central role in both the theory of concurrency [18, 16] and logic [26, 16]. In the context of concurrency, bisimilar systems are regarded as *behaviourally equivalent*—appearing to have the same behaviour when interacting with an

* The authors acknowledge with gratitude the financial support of the ERC Advanced Investigator grant 291528 ("RACE") at Oxford.

arbitrary environment. From a logical/verification perspective, bisimilar systems are known to satisfy the *same temporal logic properties* with respect to languages such as LTL, CTL, or the μ-calculus. These features, in turn, make it possible to verify temporal logic properties of concurrent systems using bisimulation-based approaches. For example, temporal logic model checking techniques may be optimised by applying them to the smallest bisimulation-equivalent model of the system being analysed; or, indeed, to every other model within the system's bisimulation equivalence class. This is possible because the properties that one is interested in checking are *bisimulation invariant*.

Model checking is not the only verification technique that can benefit from bisimulation invariance. Consider abstraction and refinement techniques [9, 10] (where a set of states is either collapsed or broken down in order to build a somewhat simpler set of states); coinduction methods [25] (which can be used to check the correctness of an implementation with respect to a given specification); or reduced BDD representations of a system [7] (where isomorphic, and therefore bisimilar, subgraphs are merged, thereby eliminating part of the initial state space of the system). Bisimulation invariance is therefore a very important concept in the formal analysis and verification of concurrent and multi-agent systems.

Game theory [22] provides another important framework for the analysis of concurrent and multi-agent systems. Within this framework, a concurrent/multi-agent system is viewed as a game, where processes/agents correspond to players, system executions/computation runs to plays, and individual process behaviours are modelled as player strategies, which are used to resolve the possible nondeterministic choices available to each player. In logic and computer science, games have also been extensively used for synthesis and verification. In this case, one is usually focused on two-player zero-sum games where the desired solution concept is given by the existence of winning strategies. Instead, in this paper, we are more interested in the more general framework given by multi-player non-zero-sum games, where the standard solution concept is given by of strategies forming a Nash equilibrium.

A widely-used model for strategies in (concurrent) multi-player games is to view a strategy for a process i as a function f_i which maps finite histories $s_0, s_1, \ldots, s_k$ of system states to actions $f_i(s_0, s_1, \ldots, s_k)$ available to the process/agent/player i at state s_k. In what follows, we use the terms process, agent, and player interchangeably. We refer to this as the conventional model of strategies, as it is the best-known and most widely used model in logic, AI, and computer science (and indeed in extensive form games [22]). For instance, specification languages such as alternating-time temporal logic (ATL [4]), and formal models such as concurrent game structures [4] use this model of strategies. If we model a concurrent/multi-agent system as a game in this way, then the analysis and verification of the system reduces to computing the set of (Nash) equilibria in the associated multi-player game; in some cases, the analysis reduces to the computation of a winning strategy.

Because bisimilar systems are regarded as behaviourally equivalent, and bisimilar systems satisfy the same set of temporal logic properties, it is natural to ask whether the Nash equilibria of bisimilar structures are identical as well; that is, we ask the following question:

Is Nash equilibrium invariant under bisimilarity?

We show that, for the conventional model of strategies, the answer to this question is, in general, 'no'. More specifically, the answer critically depends on precisely how players' strategies are modelled. With the conventional model of strategies, the answer is positive only for some two-player games, but negative in general for games with more than two players. This means, for instance, that, in the general case, bisimulation-based techniques cannot be used when one is also reasoning about the Nash equilibria of concurrent systems that are formally modelled as (concurrent) multi-player games.

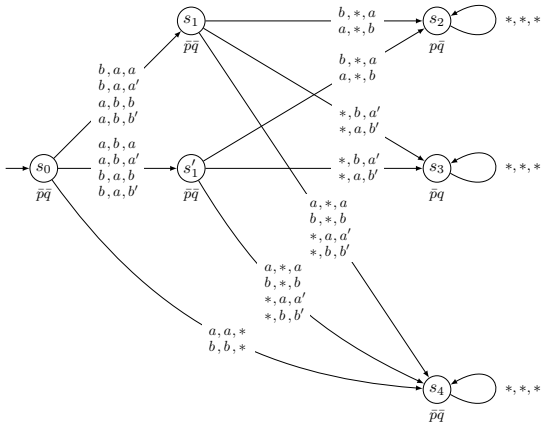


Figure 1 A 3-player game with a Nash equilibrium.

Motivated by this observation—which brings together in a striking way a fundamental concept in game theory and a fundamental concept in logic/concurrency—the purpose of the present paper is to investigate this issue in detail. We present three alternative models of strategies in which Nash equilibria and their existence are preserved under bisimilarity. We also study the above question for special classes of systems, and explore applications to logic. Specifically, we investigate the implications of replacing the conventional model of strategies with some of the models we propose in this paper in logics for strategic reasoning [19, 8], in particular, the semantic implications with respect to Strategy Logic (SL [19]). We also show that, within the conventional model of strategies, Nash equilibrium is preserved by bisimilarity in certain two-player games as well as in the class of concurrent game structures that are induced by iterated Boolean games [14], a logic-based framework for reasoning about the strategic behaviour of different kinds of multi-agent systems [28, 29].

A Motivating Example

So far we have mentioned some cases where one needs or desires a property to be invariant under bisimilarity. However, one may still wonder why it is so important that the particular property of having a Nash equilibrium is preserved under bisimilarity. One reason has its roots in automated formal verification. To illustrate this, imagine that the system of Figure 1 is given as input to a verification tool. It is likely that such a tool will try to perform as many optimisations as possible to the system before any analysis is performed. The simplest of such optimisations—for instance as done by virtually every model checking tool—is to reduce the input system by merging *isomorphic* subtrees (*e.g.*, when generating the ROBDD representation of a system). If such an optimisation is made, the tool will construct the (bisimilar) system in Figure 2. (Observe that the subgraphs rooted at s_1 and s_1' are isomorphic.) However, with respect to the existence of Nash equilibria, such a transformation is unsound in the general case.

For instance, suppose that the system in Figure 1 represents a 3-player game, where each transition is labelled by the choices x, y, z made by player 1, 2, and 3, respectively, an asterisk $*$ being a wildcard for any action for the player in the respective position. Thus, whereas players 1 and 2 can choose to play either a or b at each state, player 3 can choose between a, b, a', or b'. The states are labelled by valuations xy over $\{p, q\}$, where $\bar{x}$ indicates that x is set to false. Assume that player 1 would like p to be true sometime, that player 2

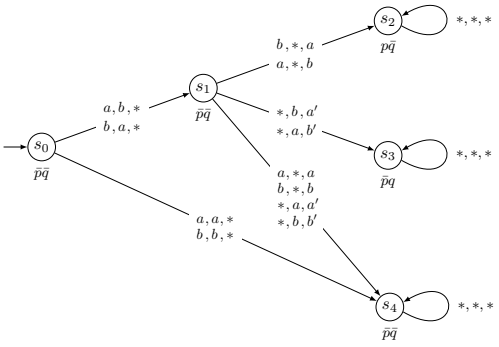


Figure 2 A 3-player game without Nash equilibria.

would like q to be true sometime, and that player 3 desires to prevent both player 1 and player 2 from achieving their goals. Accordingly, their preferences/goals can be formally represented by the LTL formulae $\gamma_1 = \mathsf{F}\, p$, $\gamma_2 = \mathsf{F}\, q$, and $\gamma_3 = \mathsf{G}\, \neg(p \vee q)$, respectively. Informally, $\mathsf{F}\, \varphi$ means "eventually φ holds" and $\mathsf{G}\, \varphi$ means "always φ holds." Moreover, given these players' goals and the conventional model of strategies, we will see later in Section 4.2 that the system in Figure 1 has a Nash equilibrium, whereas no Nash equilibria exists in the (bisimilar) system in Figure 2. This example illustrates a major issue when analysing Nash equilibria in the most widely used models of strategies and games studied in the literature, namely, that even the simplest optimisations commonly used in automated formal verification are not sound with respect to game theoretic analyses.

2 Preliminaries

We begin by introducing the main technical concepts and models used in this paper.

Concurrent Game Structures

We use the model of concurrent game structures, which are well-established in the literature (see, for instance, [4]). A *concurrent game structure (CGS)* is a tuple $M = (\mathrm{Ag}, \mathrm{AP}, \mathrm{Ac}, \mathrm{St}, s^0_M, \lambda, \delta)$, where $\mathrm{Ag} = \{1, \ldots, n\}$ is a set of *players* or agents, AP a set of *propositional variables*, Ac is a set of *actions*, St is a set of *states* containing a unique *initial state* s^0_M. With each player $i \in \mathrm{Ag}$ and each state $s \in \mathrm{St}$, we associate a non-empty set $\mathrm{Ac}_i(s)$ of *feasible* actions that, intuitively, i can perform when in state s. By a *direction* or *decision* we understand a profile of actions $d = (a_1, \ldots, a_n)$ in $\mathrm{Ac} \times \cdots \times \mathrm{Ac}$ and we let Dir denote the set of directions. A direction $d = (a_1, \ldots, a_n)$ is *legal at state* s if $a_i \in \mathrm{Ac}_i(s)$ for all players i. Unless stated otherwise, by "direction" we will henceforth generally mean "legal direction". Furthermore, $\lambda \colon \mathrm{St} \to 2^{\mathrm{AP}}$ is a *labelling function*, associating with every state s a *valuation* $v \in 2^{\mathrm{AP}}$. Finally, δ is a *deterministic transition function*, which associates with each state s and every legal direction $d = (a_1, \ldots, a_n)$ at s a state $\delta(a_1, \ldots, a_n)$. As such δ characterises the behaviour of the system when $d = (a_1, \ldots, a_n)$ is performed at state s.

Computations, Runs, and Traces

The possible behaviours exhibited by a concurrent game structure can be described at at least three different levels of abstraction. Thus, we distinguish between *computations*, *runs*, and *traces*. Computations carry the most information, while traces carry the least, in the

sense that every computation induces a unique run and every run induces a unique trace, but not necessarily the other way round.

A state s' is *accessible* from another state s whenever there is some $d = (a_1, \ldots, a_n)$ such that d is legal at s and $\delta(s, a_1, \ldots, a_n) = s'$. For easy readability we then also write $s \xrightarrow{d} s'$. An *(infinite) computation* is then an infinite sequence of directions $\kappa = d_0, d_1, d_2, \ldots$ such that there are states $s_0, s_1, \ldots$ such that $s_0 = s^0_M$ and $s_0 \xrightarrow{d_0} s_1 \xrightarrow{d_1} s_2 \xrightarrow{d_2} \cdots$. Observe that in every concurrent game model the states $s_0, s_1, \ldots$ in the above definition always exist and are unique. A *finite computation* is finite prefix of a computation κ. We also allow a finite computation to be the empty sequence ϵ of directions. We also use $\delta^*(s, d_0, d_1, \ldots d_k)$ to denote the unique state that is reached from the state s after applying the computation $d_0, d_1, \ldots d_k$.

An *(infinite) run* is an infinite sequence $\rho = s_0, s_1, s_2 \ldots$ of states of sequentially accessible states, with $s_0 = s^0_M$. We say that run $s_0, \ldots, s_k$ is *induced* by computation $d_0, \ldots, d_{k-1}$ if $s_0 \xrightarrow{d_0} s_1 \xrightarrow{d_1} s_2 \xrightarrow{d_2} \cdots$ and $s_0 = s^0_M$. Thus, every computation induces a unique run and every run is induced by at least one computation. A *finite run* is a finite prefix of a run. The sets of infinite and finite runs are denoted by $runs^\omega_M$ and $runs_M$, respectively.

An *(infinite) trace* is a sequence $\tau = v_0, v_1, v_2, \ldots$ of valuations such that there is a run $\rho = s_0, s_1, s_2, \ldots$ in $runs^\omega_M$ such that $v_k = \lambda(s_k)$ for every $k \geq 0$, that is, $\tau = \lambda(s_0), \lambda(s_1), \lambda(s_2), \ldots$. In that case we also say that trace τ is *induced* by run ρ, and if ρ is induced by computation κ, also that τ is induced by κ. By a *finite trace* we mean a finite prefix of a trace. We denote the sets of finite and infinite traces of a concurrent game structure M by $traces_M$ and $traces^\omega_M$, respectively. We use $\rho_M(\kappa)$ to denote the run induced by an infinite computation κ in M, and $\pi_M(\kappa)$, if κ is finite, on the understanding that $\pi_M(\epsilon) = s^0_M$. Also, if $\rho = s_0, s_1, s_2, \ldots$ is a run, by $\tau_M(\rho)$ we denote the trace $\lambda(s_0), \lambda(s_1), \lambda(s_2), \ldots$, and similarly for finite runs $\pi \in runs_M$. Finally, $\tau_M(\rho_M(\kappa))$ is abbreviated as $\tau_M(\kappa)$. When no confusion is likely, we sometimes omit the subscript M and the qualification 'finite'.

Bisimulations and Bisimilarity

One of the most important behavioural/observational equivalences in concurrency is bisimilarity, which is usually defined over Kripke structures or labelled transition systems (see, for instance, [18, 16]). However, the equivalence can be uniformly defined for general concurrent game structures, where decisions/directions play the role of "actions" in transition systems. Formally, let $M = (\mathrm{AP}, \mathrm{Ag}, \mathrm{Ac}, \mathrm{St}, s^0_M, \lambda, \delta)$ and $M' = (\mathrm{AP}, \mathrm{Ag}, \mathrm{Ac}, \mathrm{St}', s^0_{M'}, \lambda', \delta')$ be two concurrent game structures. A *bisimulation*, denoted by $\sim$, between states $s^* \in \mathrm{St}$ and $t^* \in \mathrm{St}'$ is a non-empty binary relation $R \subseteq \mathrm{St} \times \mathrm{St}'$, such that $s^* \; R \; t^*$ and for all $s, s' \in \mathrm{St}$, $t, t' \in \mathrm{St}'$, and $d \in \mathrm{Dir}$:

- $s \; R \; t$ implies $\lambda(s) = \lambda'(t)$,
- $s \; R \; t$ and $s \xrightarrow{d} s'$ implies $t \xrightarrow{d} t''$ for some $t'' \in \mathrm{St}'$ with $s' \; R \; t''$,
- $s \; R \; t$ and $t \xrightarrow{d} t'$ implies $s \xrightarrow{d} s''$ for some $s'' \in \mathrm{St}$ with $s'' \; R \; t'$.

Then, if there is a bisimulation between two states s^* and t^*, we say that they are *bisimilar* and write $s^* \sim t^*$ in such a case. We also say that concurrent game structures M and M' are *bisimilar*—in symbols $M \sim M'$—if $s^0_M \sim s^0_{M'}$. We furthermore say that runs $\rho = s_0, s_1, \ldots$ and $\rho' = s'_0, s'_1, \ldots$ are *statewise bisimilar* (in symbols $\rho \dot{\sim} \rho'$) if $s_k \sim s'_k$ for every $k \geq 0$. Both bisimilarity and statewise bisimilarity are equivalence relations, which is a standard result in the literature (see, for instance, [11, 5, 18]). We, moreover, find that the sets of (finite) computations as well as the sets of (finite) traces of two bisimilar concurrent game structures are *identical*. Moreover, every (finite) computation κ gives rise to statewise bisimilar (finite)

runs and identical (finite) traces in bisimilar concurrent game structures. However, as runs are sequences of states and the states of different concurrent game structures M and M' may be distinct, even if they are bisimilar, no identification of their sets $runs^{\omega}_{M}$ and $runs^{\omega}_{M'}$ of runs can generally be made.

3 Games on Concurrent Game Structures

Concurrent game structures specify the actions the players can take at each state and which states are reached if they all concurrently decide on an action. A full understanding of the system that is being modelled, however, also depends on what goals the players desire to achieve and on what strategies they may adopt in pursuit of these goals. We therefore augment concurrent game structures with preferences and strategies for the players. Thus they define a strategic game and as such they are amenable to game theoretic analysis by standard solution concepts, among which Nash equilibrium is probably the most prominent.

3.1 Strategies and Strategy Profiles

Based on the distinction between computations, runs, and traces, we can also distinguish three types of strategy: computation-based, run-based, and trace-based strategies. The importance of these distinctions is additionally corroborated by Bouyer et al. [6], who show how the specific model of strategies adopted affects the computational complexity of some standard decision problems related to multi-agent systems.

A *computation-based strategy* for a player i in a concurrent game structure M is a function $f_i^{comp}\colon comps_M \to \mathrm{Ac}$ such that $f_i^{comp}(\kappa) \in \mathrm{Ac}_i(s_k)$ for every finite $\kappa \in comps_M$ with $\pi_M(\kappa) = s_0, \ldots, s_k$. Thus, $f_i^{comp}(\epsilon) \in \mathrm{Ac}_i(s^0_M)$, where ϵ is the empty sequence of directions.

Similarly, a *run-based strategy* for player i is a function $f_i^{run}\colon runs_M \to \mathrm{Ac}$ where $f_i^{run}(s_0, \ldots, s_k) \in \mathrm{Ac}_i(s_k)$ for every finite run $(s_0, \ldots, s_k) \in runs$. Finally, a *trace-based strategy* for i is a function $f_i^{trace}\colon traces_M \to \mathrm{Ac}$ such that $f_i^{trace}(\tau) \in \mathrm{Ac}_i(s_k)$ for every trace $\tau \in traces_M$ and every run $\pi = s_0, \ldots, s_k$ such that $\tau = \lambda(s_0), \ldots, \lambda(s_k)$.

A *computation-based strategy profile* is then a tuple $f = (f_1, \ldots, f_n)$ that associates with each player i a computation-based strategy f_i. Run-based and trace-based strategy profiles are defined analogously. Every computation-based strategy profile $f = (f_1, \ldots, f_n)$ induces a unique computation $\kappa_M(f) = d_0, d_1, d_2, \ldots$ in M that is defined inductively such that $d_0 = (f_1(\epsilon), \ldots, f_n(\epsilon))$ and $d_{k+1} = (f_1(d_0, \ldots, d_k), \ldots, f_n(d_0, \ldots, d_k))$, for all $k \in \mathbb{N}$. Similarly, a run-based strategy profile $f = (f_1, \ldots, f_n)$ defines the unique computation $\kappa_M(f) = d_0, d_1, d_2, \ldots$ such that $d_0 = (f_1(s^0_M), \ldots, f_n(s^0_M))$ and $d_{k+1} = (f_1(\pi(d_0, \ldots, d_k)), \ldots, f_n(\pi(d_0, \ldots, d_k))$, for all $k \in \mathbb{N}$. Finally, the computation $\kappa_M(f)$ defined by a trace-based strategy profile f is such that $d_0 = (f_1(\lambda(s^0_M)), \ldots, f_n(\lambda(s^0_M)))$ and $d_{k+1} = (f_1(\tau(d_0, \ldots, d_k)), \ldots, f_n(\tau(d_0, \ldots, d_k))$, for all $k \in \mathbb{N}$. For $f = (f_1, \ldots, f_n)$ a profile of computation-based, run-based, or trace-based strategies, we write with a slight abuse of notation $\rho(f_1, \ldots, f_n)$ for $\rho(\kappa(f_1, \ldots, f_n))$ and $\tau(f_1, \ldots, f_n)$ for $\tau(\rho(f_1, \ldots, f_n))$.

Note that, as the computations and traces of bisimular concurrent games structures coincide, so do the computation-based and trace-based strategies available to the players. Moreover, the computations induced by them will be identical. With the states of bisimilar structures possibly being distinct, however, similar observations do not straightforwardly extend to run-based strategies and strategy profiles.

3.2 Preferences and Goals

We formally specify the preferences of a player i of a concurrent game structure M as a subset Γ_i of *computations*, that is, $\Gamma_i \subseteq comps^{\omega}_M$, and refer to Γ_i as player i's *goal set*. Player i is then understood to (strictly) prefer computations in Γ_i to those not in Γ_i and to be indifferent otherwise. Accordingly, each player's preferences are dichotomous, only distinguishing between the preferred computations in Γ_i and the not preferred ones not in Γ_i. Formally, player i is said to *weakly prefer* computation κ to computation κ' if $\kappa \in \Gamma_i$ whenever $\kappa' \in \Gamma_i$, and to *strictly prefer* κ to κ' if i weakly prefers κ to κ' but not vice versa. If player i weakly prefers κ to κ' and κ' to κ, we say that i is *indifferent* between κ and κ'.

The choice of modelling players' preferences as sets of *computations*—rather than say sets of runs or sets of traces—is for technical convenience and flexibility. Observe that every set of runs is induced by a set of computations, namely the set of computations that give rise to the same runs, and similarly for every set of traces. Thus, we say that a goal set $\Gamma_i \subseteq comps^{\omega}_M$ is *run-based* if for any two computations κ and κ' with $\rho(\kappa) = \rho(\kappa')$ we have that $\kappa \in \Gamma_i$ if and only if $\kappa' \in \Gamma_i$. Similarly, Γ_i is said to be *trace-based* whenever $\tau(\kappa) = \tau(\kappa')$ implies that $\kappa \in \Gamma_i$ if and only if $\kappa' \in \Gamma_i$. Thus, *run-based* goals are *computation-based* goals closed under induced runs, and *trace-based* goals are *computation-based* goals closed under induced traces.

Sometimes—as we did in the example in the introduction—players' goals are specified by *temporal logic formulae*. As the satisfaction of goals only depends on traces, they will directly correspond to trace-based goals, given our formalisation of goals and preferences.

3.3 Games and Nash Equilibrium

With the above definitions in place, we now define a *game on a concurrent game structure* M (also called a *CGS-game*) as a tuple, $G = (M, \Gamma_1, \ldots, \Gamma_n)$, where, for each player i, the goal set $\Gamma_i \subseteq comps^{\omega}_M$ specifies i's dichotomous preferences over the computations in M.

The players of a CGS-game either all play computation-based strategies, all play run-based strategies, or all play trace-based strategies. For each such choice of type of strategies, with the set of players and their preferences specified, every CGS-game defines a strategic game in the standard game theoretic sense. Observe that the set of strategies is infinite in general. Thus the game theoretic solution concept of Nash equilibrium becomes available for the analysis of games on concurrent game structures. If $f = (f_1, \ldots, f_n)$ is a strategy profile and g_i a strategy for player i, we write (f_{-i}, g_i) for the strategy profile $(f_1, \ldots, g_i, \ldots, f_n)$, which is identical to f except that i's strategy is replaced by g_i. Formally, given a CGS-game, we say that a profile $f = (f_1, \ldots, f_n)$ of computation-based strategies is a *Nash equilibrium in computation-based strategies* (or *computation-based equilibrium*) if, for every player i and every computation-based strategy g_i available to i, $\kappa_M(f_{-i}, g_i) \in \Gamma_i$ implies $\kappa_M(f) \in \Gamma_i$. The concepts of *Nash equilibrium in run-based strategies* and *Nash equilibrium in trace-based strategies* are defined analogously, where, importantly, f_i and g_i are required to be of the same type, that is, both run-based or both trace-based. If $\kappa(f) \notin \Gamma_i$ whereas $\kappa(f_{-i}, g_i) \in \Gamma_i$, we also say that player i would like to *deviate* from f_i to g_i. Thus, a run-based profile f is a *run-based equilibrium* whenever no player would like to deviate from it to some *run-based* strategy different from f_i. Similarly, a trace-based profile f is a trace-based equilibrium if no player would prefer to deviate to another *trace-based* strategy.

We furthermore say that a computation κ, run ρ, or a trace τ is *sustained by a Nash equilibrium* $f = (f_1, \ldots, f_n)$ (of any type) whenever $\kappa = \kappa(f)$, $\rho = \rho(f)$, and $\tau = \tau(f)$, respectively. We also refer to a computation, run, or trace that is sustained by a Nash equilibrium as an *equilibrium computation*, *equilibrium run*, and *equilibrium trace*, respectively.

Computation-based equilibrium is a weaker notion than run-based equilibrium, in the sense that, if f is a run-based equilibrium, there is also a corresponding computation-based equilibrium, but not necessarily the other way round. Run-based equilibrium, in turn, is in a similar way a weaker concept than trace-based equilibrium.

4 Invariance of Nash Equilibria under Bisimilarity

From a computational point of view, one may expect games based on bisimilar concurrent game structures and with identical players' preferences to display very similar behaviours, in particular with respect to their Nash equilibria. We find that while this is indeed the case for games with computation-based strategies as well as for games with trace-based strategies, for games with run-based strategies the situation is considerably more complicated. A key observation is that, by contrast to computation-based and trace-based strategies, there need not be a natural *one-to-one* mapping between the sets of run-based strategies in bisimilar concurrent game models. By restricting to so-called bisimulation-invariant run-based strategies, however, we find that order can be restored.

4.1 Computation-based and Trace-based Strategies

If strategies are computation-based, players can have their actions depend on virtually all information that is available in the system. In an important sense full transparency prevails and different actions can be chosen on bisimilar states provided that the computations that led to them are different. As, moreover, the strategies available to player in bisimilar concurrent game structures are identical, we obtain our first main preservation result.

▶ **Theorem 1.** *Let $G = (M, \Gamma_1, \ldots, \Gamma_n)$ and $G' = (M', \Gamma_1, \ldots, \Gamma_n)$ be games on bisimilar concurrent game structures M and M', respectively, and let $f = (f_1, \ldots, f_n)$ be a computation-based profile. Then, f is a Nash equilibrium in computation-based strategies in G if and only if f is a Nash equilibrium in computation-based strategies in G'.*

Since the sets of traces of two bisimilar concurrent game structure coincide, we can also directly compare their trace-based Nash-equilibria. We find that trace-based Nash equilibria are likewise preserved in CGS-games based on bisimilar concurrent game structures.

▶ **Theorem 2.** *Let $G = (M, \Gamma_1, \ldots, \Gamma_n)$ and $G' = (M', \Gamma_1, \ldots, \Gamma_n)$ be games on bisimilar concurrent game structures M and M', respectively, and $f = (f_1, \ldots, f_n)$ be a trace-based strategy profile. Then, f is a Nash equilibrium in trace-based strategies in G if and only if f is a Nash equilibrium in trace-based strategies in G'.*

As an immediate consequence of Theorems 1 and 2, we also find that the properties of computations and traces being sustained by, respectively, a computation-based equilibrium and a trace-based equilibrium are preserved under bisimulation.

4.2 Run-based Strategies

The positive results obtained using computation-based and trace-based strategies (with respect to both computation-based goals and trace-based goals) are now followed by a negative result, already mentioned in the introduction of the paper, which establishes that Nash equilibria in run-based strategies—probably the most widely-used strategy model in logic, computer science, and AI—are not preserved by bisimilarity. Previously we observed that the players' run-based strategies cannot straightforwardly be identified across two

different but bisimilar concurrent game structures. We would therefore have to establish a correspondence between the run-based strategies of different games in an arguably ad hoc way. To cut this Gordian knot, we therefore show the stronger result that the very *existence* of run-based equilibria is not preserved under bisimilarity. That is, there can be two CGS-games, G and G', that are based on bisimilar concurrent game structures but that G possesses a Nash equilibrium and G' does not.

▶ **Theorem 3.** *The existence of run-based Nash equilibria is* not *preserved under bisimilarity. That is, there are games* $(M, \Gamma_1, \ldots, \Gamma_n)$ *and* $(M', \Gamma_1, \ldots, \Gamma_n)$ *on bisimilar concurrent game structures* M *and* M' *such that a run-based equilibrium exists in* G *but not in* G'*.*

To see that the above statement holds, recall the three-player game G_0 on the concurrent game structure M_0 in Figure 1. Assume, as before, that player 1's goal set Γ_1 is given by those computations κ such that $\tau_{M_0}(\kappa) = v_0, v_1, v_2, \ldots$, implies $p \in v_k$ for some $k \geq 0$. Similarly, Γ_2 consists of all computations κ with $\tau_{M_0}(\kappa) = v_0, v_1, v_2, \ldots$ and $q \in v_k$ for some $k \geq 0$ and Γ_3 by those computations κ with $\tau_{M_0}(\kappa) = v_0, v_1, v_2, \ldots$ and $v_k = \emptyset$ for all $k \geq 0$. Recall that, consequently, the preferences of players 1, 2, and 3 are trace-based and can be represented by the LTL formulas $\gamma_1 = \mathsf{F}\, p$, $\gamma_2 = \mathsf{F}\, q$, and $\gamma_3 = \mathsf{G}\, \neg(p \vee q)$, respectively.

Let f_1^* and f_2^* be any run-based strategies for players 1 and 2 such that $f_1^*(s_0) = f_2^*(s_0) = a$. Let, furthermore, player 3's run-based strategy f_3^* be such that $f_3^*(s_0) = a$, $f_3^*(s_0, s_1) = a'$, and $f_3^*(s_0, s_1') = b$. Let $f^* = (f_1^*, f_2^*, f_3^*)$ and observe that $\rho_{M_0}(f^*) = s_0, s_4, s_4, s_4, \ldots$. Accordingly, player 3 has her goal achieved and does not want to deviate from f^*. Players 1 and 2 do not have their goals achieved, but do not want to deviate from f^* either. To see this, let g_1 be any run-based strategy for 1 such that $g_1(s_0) = b$; observe that this is required for any meaningful deviation from f^* by 1. Then $\rho_{M_0}(g_1, f_2^*, f_3^*) = s_0, s_1, s_3, s_3, s_3, \ldots$ or $\rho_{M_0}(g_1, f_2^*, f_3^*) = s_0, s_1, s_4, s_4, s_4, \ldots$, depending on whether $f_2^*(s_0, s_1) = b$ or $f_2^*(s_0, s_1) = a$, respectively. In either case, player 1 does not get her goal achieved and it follows that she does not want to deviate from f^*. An analogous argument—notice that the roles of player 1 and 2 are symmetric—shows that player 2 does not want to deviate from f^* either. We may thus conclude that f^* is a run-based equilibrium in G_0.

Now, consider the game G_1 on concurrent game structure M_1 in Figure 2 with the players' preferences as in M_0. It is easy to check that M_0 and M_1 are bisimilar. Still, there is no run-based equilibrium in G_1. To see this, consider an arbitrary run-based strategy profile $f = (f_1, f_2, f_3)$. First, assume that $\rho_{M_1}(f) = s_0, s_1, s_2, s_2, s_2, \ldots$. Then, player 1 gets his goal achieved and players 2 and 3 do not. If $f_1(s_0, s_1) = a$ then $f_3(s_0, s_1) = b$ and player 3 would like to deviate and play a strategy g_3 with $g_3(s_0, s_1) = a$. On the other hand, if $f_1(s_0, s_1) = b$, player 3 would like to deviate and play a strategy g_3 with $g_3(s_0, s_1) = b$. Player 3 would similarly like to deviate from f if we assume that $\rho_{M_1}(f) = s_0, s_1, s_3, s_3, s_3 \ldots$, in whose case it is player 2 who gets his goal achieved. Now, assume that $\rho_{M_1}(f) = s_0, s_1, s_4, s_4, s_4 \ldots$. In this case player 3 does get her goal achieved, but players 1 and 2 do not. However, player 1 would like to deviate to a strategy g_1 with $g_1(s_0, s_1) = b$ or $g_1(s_0, s_1) = a$, depending on whether $f_3(s_0, s_1) = a$ or $f_3(s_0, s_1) = b$; in a similar fashion, player 2 would like to deviate to a strategy g_2 with $g_2(s_0, s_1) = b$ if $f_1(s_0, s_1) = a'$, and to one with $g_2(s_0, s_1) = a$ if $f_1(s_0, s_1) = b'$. Finally, assume that $\rho_{M_1}(f) = s_0, s_4, s_4, s_4, \ldots$. Then, neither player 1 nor player 2 gets his goal achieved. Now either $f_3(s_0, s_1) \in \{a, b\}$ or $f_3(s_0, s_1) \in \{a', b'\}$. If the former, player 1 would like to deviate to a strategy g_1 with $g_1(s_0) \neq f_1(s_0)$ and $g_1(s_0, s_1) \neq f_3(s_0, s_1)$. If the latter, player 2 would like to deviate to a strategy g_2 with $g_2(s_0) \neq f_2(s_0)$ and either $g_2(s_0, s_1) = b$ if $f_3(s_0, s_1) = a'$ or $g_2(s_0, s_1) = a$ if $f_3(s_0, s_1) = b'$. We can then conclude that the CGS-game G_1 does not have any run-based Nash equilibria.

The main idea behind this counter-example is that in G_0 player 3 can distinguish which player deviates from f^* if the state reached after the first round is not s_4: if that state is s_1, player 1 deviated, otherwise player 2 did. By choosing either a' or b' at s_1, and either a or b at s_1', player 3 can guarantee that neither player 1 nor player 2 gets his goal achieved ("punish" them) and thus deter their deviating from f^*. This possibility to punish deviations from f^* by players 1 and 2 in a single strategy is not available in G_1: choosing from a and b will induce a deviation by player 1, choosing from a' and b' one by player 2.

Observe that the games G_0 and G_1 do *not* constitute a counter-example against the preservation under bisimilarity of either computation-based equilibria or trace-based equilibria. The reasons why such games fail to do so, however, are distinct. For the setting of computation-based strategies, player 3 can still distinguish and "punish" the deviating player in G_1 as (a, b, a) and (b, a, a) are different directions and player 3 can still have his action at s_1 depend on which of these is played at s_0. By contrast, if we assume trace-based strategies, player 3 has to choose the same action at both s_1 and s_1' in G_0. As a consequence, and contrarily to computation-based equilibria, trace-based equilibria exist in neither G_0 nor G_1.[1] Also note that, as the goal sets Γ_1, Γ_2, and Γ_3 are run-based as well as computation-based both in G_1 and G_2, the counter-example still holds if preferences are more fine-grained.

Observe at this point that s_1 and s_1' are bisimilar states. Yet, players are allowed to have run-based strategies (which depend on state histories only) that choose *different* actions at bisimilar states. The above counter-example shows how this relative richness of strategies makes a crucial difference. This raises the question as to whether we actually want players to adopt run-based strategies in which they choose different actions at bisimilar states. This observation leads us to the next section.

4.3 Bisimulation-invariant Run-based Strategies

Bisimilarity formally captures an informal concept of observational indistinguishability on the part of an external observer of the system. The players in a concurrent game structure are in essentially the same situation as an external observer, if they are assumed to be only able to observe the behaviour of the other players, without knowing their internal makeup.

Drawing on this idea of indistinguishability, it is natural to assume that players cannot distinguish statewise bisimilar runs and, as a consequence, can only adopt strategies that choose the same action at runs that are statewise bisimilar. The situation is comparable to the one in extensive games of imperfect information, in which players are required to choose the same action in histories that are in the same information set (cf., for instance, [22, 17]).

To make this idea precise, we say that a run-based strategy f_i is *bisimulation-invariant* if $f_i(\pi) = f_i(\pi')$ for all histories π and π' that are statewise bisimilar. The concept of Nash equilibrium is then similarly restricted to bisimulation-invariant strategies. Formally, a profile $f = (f_1, \ldots, f_n)$ of *bisimulation-invariant* strategies is a *Nash equilibrium in bisimulation-invariant strategies* (or a *bisimulation-invariant equilibrium*) in a game $(M, \Gamma_1, \ldots, \Gamma_n)$ if for all players i and every *bisimulation-invariant* strategy g_i for i, $\tau(f_{-i}, g_i) \in \Gamma_i$ implies $\tau(f) \in \Gamma_i$. In contrast to the situation for general run-based strategies, we find that computations and traces that are sustained by a bisimulation-invariant Nash equilibrium are preserved by bisimulation. Let M and M' be bisimilar concurrent game structures.

[1] Based on a similar example, Almagor et al. [1] also observe that the existence of Nash equilibria in a CGS-game may depend on the type of strategy that is being considered. Note, however, that this is quite different from our observation that Nash equilibria may differ in different but bisimilar CGS-games given a fixed type of strategy, *viz.*, run-based strategies.

Based on the concept of statewise bisimilarity, we associate with every bisimulation-invariant strategy f_i for player i in M, another bisimulation-invariant strategy $\tilde{f}_i$ for player i in M' such that for all $\pi \in runs_{M'}$ and $a \in \mathrm{Ac}$, we have $\tilde{f}_i(\pi) = a$ if $f_i(\pi') = a$ for some $\pi' \in runs_M$ with $\pi \dot{\sim} \pi'$. Transitivity of $\dot{\sim}$ guarantees that $\tilde{f}_i$ is well-defined. Given a profile $f = (f_1, \ldots, f_n)$, let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$. It can then be shown that f and $\tilde{f}$ induce identical computations, that is, $\kappa_M(f) = \kappa_{M'}(\tilde{f})$, which prepares the ground for the following preservation result.

▶ **Theorem 4.** *Let $G = (M, \Gamma_1, \ldots, \Gamma_n)$ and $G' = (M', \Gamma_1, \ldots, \Gamma_n)$ be games on bisimilar concurrent game structures M and M', respectively. Then, profile f is a bisimulation-invariant equilibrium in G if and only if $\tilde{f}$ is a bisimulation-invariant equilibrium in G'.*

As an immediate corollary of Theorem 4, it follows that the property of a computation or trace to be sustained by a bisimulation-invariant equilibria is also preserved under bisimilarity.

4.4 Two-player Games with Run-based Strategies

We now consider the setting of two-player games with run-based strategies and trace-based goals (which include temporal logic goals). This is an important special case, since run-based strategies, as we emphasised in the introduction, are the conventional model of strategies used in logics such as ATL* or Stragegy Logic (SL), as well as in multi-agent systems represented as concurrent game structures [4, 19].

The counterexample against the preservation of existence of Nash equilibria in Section 4.2 involved three players. We find that, if preferences are computation-based, the example can be adapted so as to involve only two players. Hence, we have the following result.

▶ **Theorem 5.** *There are two-player games (M, Γ_1, Γ_2) and (M', Γ_1, Γ_2) on bisimilar concurrent game structures M and M' with Γ_1 and Γ_2 computation-based such that a run-based Nash equilibrium exists in G but not in G'.*

By contrast, if players' preferences are required to be trace-based in any two models to be compared, sustenance of traces by run-based equilibrium—and hence also the existence of Nash equilibria—is preserved under bisimilarity. To establish this, we associate with each run-based profile $f = (f_1, f_2)$ a *bisimulation-invariant* profile $\hat{f} = (\hat{f}_1, \hat{f}_2)$. We then show that f and $\hat{f}$ induce the same traces and that $\hat{f}$ is a bisimulation invariant equilibrium, if f is a run-based equilibrium. We can then leverage Theorem 4, which yields the result.

We only give the definition of $\hat{f}_1$, as the construction of $\hat{f}_2$ is analogous. The key idea is to select for every finite run $\pi = s_0, \ldots, s_k$ a unique representative $\hat{\pi}^{s_0,\ldots,s_k}$ of the equivalence class $[\pi]_{\dot{\sim}}$ of runs statewise bisimilar to π and then define $\hat{f}_1$ such that $\hat{f}_1(\pi) = f_1(\hat{\pi}^{s_0,\ldots,s_k})$. This ensures that $\hat{f}_1$ is bisimulation-invariant. The representative $\hat{\pi}^{s_0,\ldots,s_k}$, however, has to be chosen carefully. Assuming that from every subset of actions we can always select a least element, we define inductively for every finite run $\pi = s_0, \ldots, s_k$ in $runs_M$ another statewise bisimilar finite run $\hat{\pi}^{s_0,\ldots,s_k} = t_0, \ldots, t_k$ in $runs_M$ such that $t_0 = s^0_M$ and, for every $0 \leq m < k$, $t_{m+1} = \delta(t_m, x_1, x_2)$, where x_1 and x_2 are actions available to players 1 and 2 at t_m such that:

- $x_1 = f_1(t_0, \ldots, t_m)$ and $x_2 = f_2(t_0, \ldots, t_m)$, if $\delta(t_m, f_1(t_0, \ldots, t_m), f_2(t_0, \ldots, t_m)) \sim s_{m+1}$,
- $x_1 = f_1(t_0, \ldots, t_m)$ and x_2 is the least action available to player 2 at t_m such that $\delta(t_m, f_1(t_0, \ldots, t_m), x_2) \sim s_{m+1}$, if $\delta(t_m, f_1(t_0, \ldots, t_m), f_2(t_0, \ldots, t_m)) \not\sim s_{m+1}$ and x_2 exists, and
- (x_1, x_2) is lexicographically the least pair of actions available to players 1 and 2 at t_m such that $\delta(t_m, x_1, x_2) \sim s_{m+1}$, otherwise.

By means of an inductive argument it can easily be shown that $s_0, \ldots, s_k \dot{\sim} \pi^{s_0,\ldots,s_k}$, which suffices for $\hat{\pi}^{s_0,\ldots,s_k}$ to be well-defined, that is, that x_1 and x_2 exist for every $0 \leq m < k$. As, $s_0, \ldots, s_k \dot{\sim} t_0, \ldots, t_k$ implies $\hat{\pi}^{s_0,\ldots,s_k} = \hat{\pi}^{t_0,\ldots,t_k}$, it follows that $\hat{f}_1$ is properly defined as a bisimulation-invariant strategy. Importantly, $\rho_M(\hat{f}_1, \hat{f}_2) = \rho_M(f_1, f_2)$, and hence also $\tau_M(\hat{f}_1, \hat{f}_2) = \tau_M(f_1, f_2)$. Moreover, we find by a non-trivial argument that, if $f = (f_1, f_2)$ is a run-based equilibrium, so is $\hat{f} = (\hat{f}_1, \hat{f}_2)$. Finally, as a bisimulation-invariant profile is a bisimulation-invariant equilibrium if and only if it is a run-based equilibrium, we are in a position to leverage Theorem 4 and obtain our result

▶ **Theorem 6.** *Let $G = (M, \Gamma_1, \Gamma_2)$ and $G' = (M', \Gamma_1, \Gamma_2)$ be two two-player games such that Γ_1 and Γ_2 are trace-based in both M and M' and let τ be a trace in $\mathrm{traces}^{\omega}_M$. Then, τ is sustained by a run-based equilibrium in G if and only if τ is sustained by a run-based equilibrium in G'.*

As an immediate consequence of Theorem 6 we also find that the *existence* of Nash equilibria is also preserved in two-player games with trace-based preferences. The case in which players' preferences are run-based, however, we have to leave as an open question.

5 Strategy Logics: New Semantic Foundations

Several logics for strategic reasoning have been proposed in the literature of computer science and AI, such as ATL* [4], Strategy Logic [19, 8], Coalition Logic [23], Coordination Logic [12], Game Logic [24], and Equilibrium Logic [15]. In several cases, the model of strategies that is used is the one that we refer to as run-based in this paper, that is, strategies are functions from finite sequences of states (of some arena) to actions/decisions/choices of players in a given game. As can be seen from our results so far, of the four options we have explored, run-based strategies form the least desirable model of strategies from a semantic point of view since in such a case Nash equilibrium is not preserved under bisimilarity.

This does not necessarily immediately imply that a particular logic with a run-based strategy model is not invariant under bisimilarity. For instance, ATL* is a bisimulation-invariant logic and, as shown in [13] one can reason about Nash equilibrium using ATL* only up-to bisimilarity. A question then remains: whether any of these logics for strategic reasoning becomes invariant under bisimilarity—as explained before, a desirable property—if one changes the model of strategies considered there to, for instance, computation-based or trace-based strategies. We find that this question has a satisfactory positive answer in some cases. In particular, we will consider the above question in the context of Strategy Logic as studied in [19], and in doing so we will provide new semantic foundations for strategy logics.

Let us start by introducing the syntax and semantics under the run-based model of strategies for Strategy Logic (SL [19]) as it has been given in [20]. Syntactically, SL extends LTL with two *strategy quantifiers*, $\langle\langle x \rangle\rangle$ and $[[x]]$, and an *agent binding* operator (i, x), where i is an agent and x is a variable. These operators can be read as *"there exists a strategy x"*, *"for all strategies x"*, and *"bind agent i to the strategy associated with the variable x"*, respectively. SL formulae are inductively built from a set of atomic propositions AP, variables Var, and agents Ag, using the following grammar, where $p \in \mathrm{AP}$, $x \in \mathrm{Var}$, and $i \in \mathrm{Ag}$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{X}\varphi \mid \varphi \mathsf{U} \varphi \mid \langle\langle x \rangle\rangle \varphi \mid [[x]]\varphi \mid (i, x)\varphi.$$

We can now present the semantics of SL formulae, where Str denotes the set of all strategies of some specified type, *i.e.*, computation-based, run-based, or trace-based. Given a concurrent game structure M, for all SL formulae φ, states $s \in \mathrm{St}$ in M, and assignments

$\chi \in \text{Asg} = (\text{Var} \cup \text{Ag}) \to \text{Str}$, mapping variables and agents to strategies, the relation $M, \chi, s \models \varphi$ is defined as follows:

1. For the Boolean and temporal cases, the semantics is standard (see, *e.g.*, [19];
2. For all formulae φ and variables $x \in \text{Var}$ we have:
   a. $M, \chi, s \models \langle\langle x \rangle\rangle \varphi$ if there is a strategy $f \in \text{Str}$ such that $M, \chi[x \mapsto f], s \models \varphi$;
   b. $M, \chi, s \models [[x]]\varphi$ if for all strategies $f \in \text{Str}$ we have that $M, \chi[x \mapsto f], s \models \varphi$.
3. For all $i \in \text{Ag}$ and $x \in \text{Var}$, we have $M, \chi, s \models (i, x)\varphi$ if $M, \chi[i \mapsto \chi(x)], s \models \varphi$.

Intuitively, rules 2a and 2b, respectively, are used to interpret the existential $\langle\langle x \rangle\rangle$ and universal $[[x]]$ quantifiers over strategies, and rule 3 is used to bind an agent to the strategies associated with variable x. All other rules are as in LTL over concurrent game structures. Moreover, for a sentence φ, *i.e.*, a formula with no free variables or agents [19], we say that M satisfies φ, and write $M \models \varphi$, if $M, \emptyset, s_0 \models \varphi$, where $\emptyset$ is the empty assignment.

As it can be seen from its semantics, SL can be interpreted under different models of strategies and goals. As it was originally formulated, SL considers run-based strategies and trace-based preferences/goals [21]. More specifically, the model of goals is a proper subset of the trace-based one, represented by LTL goals over the set AP of variables. Let us fix an interpretation of strategies. We say that SL with that kind of model is invariant under bisimulation if, for all sentences φ and bisimilar models M_1 and M_2, it holds that $M_1 \models \varphi$ iff $M_2 \models \varphi$. In SL, it is possible to represent the existence of a Nash equilibrium in a concurrent game structure [19]. This implies, given Theorem 3, that SL under the standard (run-based model) interpretation is not invariant under bisimulation, as the formula expressing the existence of a Nash equilibrium can distinguish between two bisimilar models.

We now consider SL under the model of computation-based strategies, and find that in such a case SL becomes invariant under bisimilarity. Formally, we have the following result.

▶ **Theorem 7.** *SL with the computation-based model of strategies is invariant under bisimilarity.*

An analogous statement to the above Theorem can also be proved if we consider the model of trace-based strategies, leading to the next result on the semantics of SL.

▶ **Theorem 8.** *SL with the trace-based model of strategies is invariant under bisimilarity.*

6 Concluding Remarks and Related Work

We have shown that in the conventional model of strategies used in logic, computer science, and AI, the existence of a given Nash equilibrium is not preserved under bisimilarity, in particular for multi-player games played over concurrent games structures. With a few examples we also illustrated some of the adverse implications of this result, for instance, in the context of automated formal verification. To resolve this problem, we furthermore investigated alternative models of strategies which exhibit some desirable properties, in particular, *allowing for a formalisation of Nash equilibrium that is invariant under bisimilarity.*

We studied applications of these models and found that through their use, not only Nash equilibria becomes invariant under bisimilarity, but also full logics such as Strategy Logic [21, 8, 19]. This renders it possible to combine commonly used optimisation techniques for model checking with decision procedures for the analysis of Nash equilibria, thus overcoming a critical problem of this kind of logics regarding practical applications via automated verification. Some work also in the intersection between bisimulation equivalences, concurrent game structures, and Nash equilibria is summarised next.

Logics for Strategic Reasoning.

From the current (enormous) set of logics for strategic reasoning in the literature, ATL* [4] and SL [19] stand out, both due to their adoption by a number of practical tools for automated verification and because of their expressive power. On the one hand, ATL* is known to be invariant under bisimilarity using the conventional model of strategies. As such, Nash equilibria can be expressed within ATL* only up to bisimilarity [13]. On the other hand, SL, which is strictly more expressive than ATL*, allows for a simple specification of Nash equilibria, but suffers from not being invariant under bisimilarity with respect to the conventional model of strategies. In this paper, we have put forward a number of solutions to this problem. An additional advantage of replacing the model of strategies for SL (and therefore for concurrent game structures) is that other solution concepts in game theory also become invariant under bisimilarity. For instance, subgame-perfect Nash equilibria and strong Nash equilibria—which are widely used when considering, respectively, dynamic behaviour and cooperative behaviour in multi-agent systems—can also be expressed in SL. Our results therefore imply that these concepts are also invariant under bisimilarity, when considering games over concurrent game structures and goals given by LTL formulae.

Bisimulation Equivalences for Multi-Agent Systems.

Even though bisimilarity is probably the most widely used behavioural equivalence in concurrency, in the context of multi-agent systems other relations may be preferred, for instance, equivalence relations that take a detailed account of the independent interactions and behaviour of individual components in a multi-agent system. In such a setting, "alternating" relations with natural ATL* characterisations have been studied [3, 11]. Our results also apply to such alternating relations. On the one hand, the counter-example shown in Figures 1 and 2 also apply to such alternating (bisimulation) relations. On the other hand, because these alternating relations can be characterised in ATL*, they are at most as strong as bisimilarity. These two facts together imply that Nash equilibria is not preserved by the alternating (bisimulation) equivalence relations in [3, 11] either. Nevertheless, as discussed in [27], the right notion of equivalence for games and their game theoretic solution concepts is most certainly an important and interesting topic of debate.

Computations vs Traces.

A difference between computations and traces is that even though Nash equilibria and their existence are preserved under bisimilarity by three of the four strategy models we have studied, it is not the case that with each strategy model we obtain the same set of Nash equilibria in a given system, or that we can sustain the same set of computations or traces. For instance, consider the games in Figures 1 and 2. As we discussed above, if we consider the model of computation-based strategies and LTL goals (*i.e.*, trace-based goals) as shown in the example, then we obtain two games, each with an associated non-empty set of Nash equilibria, which is preserved by bisimilarity. However, if we consider, instead, the model of trace-based strategies and same LTL goals, then we obtain two games both with empty sets of Nash equilibria—thus, in this case, the non-existence of Nash equilibria is preserved by bisimilarity! To observe this, note that whereas in the case of computation-based strategies player 3 can implement a uniform "punishment" strategy for both player 1 and player 2, in the case of trace-based strategies player 3 cannot do so, even in the game in Figure 1.

Two-Player Games with Trace-Based Goals.

In this paper we also showed that if we consider two-player games together with the conventional model of strategies, the problems that arise with respect to the preservation of Nash equilibria disappear. This is indeed an important finding since most verification games (*e.g.*, model and module checking, synthesis, etc.) can be phrased in terms of two-player games together with temporal logic specifications (*e.g.*, using LTL, CTL, or ATL*). Our results, then, provide proof that, if only two-player games and temporal logic goals are needed, then all equilibrium analyses can be carried out using the conventional model of strategies—along with their associated reasoning tools and verification techniques.

Boolean Game Structures.

Boolean game structures are the special type of concurrent game structure in which each player i has unique control over a subset AP_i of propositional variables and the set $\mathrm{Ac}_i(s)$ of actions available to player i at state s is a non-empty subset of partial valuations in 2^{AP_i}. The key idea is that the choice player i makes with respect to the variables in AP_i at a state s defines the labelling of the subsequent state with respect to AP_i. Formally, for every direction $d' = (a_1, \ldots, a_n)$ in $2^{\mathrm{AP}_1} \times \cdots \times 2^{\mathrm{AP}_n}$ and every state s, it holds that $\delta(s, d') = s'$ implies $\lambda(s') = a_1 \cup \cdots \cup a_n$.

Boolean game structures are a particularly well-behaved class of games, of which *iterated Boolean games* [14] and multi-agent systems modelled using the *Reactive Modules specification language* [2] are special cases. We thus find that, in Boolean game structures, (existence of) Nash equilibrium is invariant under bisimilarity regardless of the model of strategies or goals that one chooses. As in this setting, it is readily shown that if two finite runs are statewise bisimilar, they have to be identical, it immediately follows that run-based strategies are bisimulation-invariant. Hence, by Theorem 4 that (the existence of) run-based equilibria is invariant under bisimulation. For the settings with computation-based and trace-based strategies, Theorems 1 and 2 give the result, respectively.

References

1. S. Almagor, G. Avni, and O. Kupferman. Repairing multi-player games. In L. Aceto and D. de Frutos Escri, editors, *Proceedings of the Twenty-Sixth Annual Conference on Concurrency Theory (CONCUR'15)*, Leibniz International Proceedings in Informatics (LIPIcs), pages 325–339, Madrid, Spain, 2015.
2. R. Alur and T. A. Henzinger. Reactive modules. *Formal Methods in System Design*, 15(1):7–48, 1999.
3. R. Alur, T. A. Henzinger, O. Kupferman, and M. Y. Vardi. Alternating refinement relations. In *Proceedings of the 9th International Conference on Concurrency Theory (CONCUR'98)*, volume 1466 of *LNCS*, pages 163–178. Springer-Verlag, Berlin, Germany, 1998.
4. R. Alur, T.A. Henzinger, and O. Kupferman. Alternating-Time Temporal Logic. *Journal of the ACM*, 49(5):672–713, 2002.
5. C. Baier and J.-P. Katoen. *Principles of Model Checking*. MIT Press, 2008.
6. P. Bouyer, R. Brenguier, N. Markey, and M. Ummels. Nash equilibria in concurrent games with büchi objectives. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'11), Mumbai, India*, pages 375–386, 2011.
7. R. E. Bryant. Symbolic Boolean manipulation with ordered binary-decision diagrams. *ACM Computing Surveys*, 24(3):293–318, 1992.
8. K. Chatterjee, T.A. Henzinger, and N. Piterman. Strategy logic. *Information and Computation*, 208(6):677–693, 2010.

9 Edmund M. Clarke, Orna Grumberg, and David E. Long. Model checking and abstraction. *ACM transactions on Programming Languages and Systems*, 16(5):1512–1542, 1994.

10 P. Cousot and R. Cousot. On abstraction in software verification. In *Forteenth International Conference on Computer Aided Verification (CAV'02)*, volume 2404 of *LNCS*, pages 37–56. Springer, 2002.

11 S. Demri, V. Goranko, and M. Lange. *Temporal Logics in Computer Science: Finite State Systems*, volume 58 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2016.

12 B. Finkbeiner and S. Schewe. Coordination logic. In Anuj Dawar and Helmut Veith, editors, *International Workshop on Computer Science Logic (CSL'10)*, volume 6247 of *LNCS*, pages 305–319. Springer, 2010.

13 J. Gutierrez, P. Harrenstein, and M. Wooldridge. Expressiveness and complexity results for strategic reasoning. In *Proceedings of the Twenty-Sixth Annual Conference on Concurrency Theory (CONCUR'15)*, volume 42 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 268–282, Madrid, Spain, 2015.

14 J. Gutierrez, P. Harrenstein, and M. Wooldridge. Iterated Boolean games. *Information and Computation*, 242:53–79, 2015.

15 J. Gutierrez, P. Harrenstein, and M. Wooldridge. Reasoning about equilibria in game-like concurrent systems. *Annals of Pure and Applied Logic*, 169(2):373–403, 2017.

16 M. Hennessy and R. A. Connolly Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.

17 M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.

18 R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.

19 F. Mogavero, A. Murano, G. Perelli, and M. Y. Vardi. Reasoning about strategies: On the model-checking problem. *ACM Transactions on Computational Logic*, 15(4):1–47, 2014.

20 F. Mogavero, A. Murano, G. Perelli, and M. Y. Vardi. Reasoning about Strategies: On the Satisfiability Problem. *Logical Methods in Computer Science*, 13(1:9), 2017.

21 F. Mogavero, A. Murano, and M. Y. Vardi. Reasoning about strategies. In K. Lodaya and M. Mahajan, editors, *Proceedings of the IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'10)*, volume 8 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 133–144, 2010.

22 M.J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, 1994.

23 M. Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 12(1):149–166, 2002.

24 M. Pauly and R. Parikh. Game logic—an overview. *Studia Logica*, 75(2):165–182, 2003.

25 Davide Sangiorgi. On the origins of bisimulation and coinduction. *ACM Transactions on Programming Languages and Systems*, 31(4):111–151, 2009.

26 J. F. A. K. van Benthem. *Modal Correspondence Theory*. PhD thesis, University of Amsterdam, 1976.

27 J. F. A. K. van Benthem. Extensive games as process models. *Journal of Logic, Language and Information*, 11(3):289–313, 2002.

28 M. Wooldridge, J. Gutierrez, P. Harrenstein, E. Marchioni, G. Perelli, and A. Toumi. Rational verification: From model checking to equilibrium checking. In D. Schuurmans and M. Wellman, editors, *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence (AAAI'16)*, pages 4184–4190, Phoenix, AZ, 2016.

29 M.J. Wooldridge. *Introduction to Multiagent Systems*. John Wiley & Sons, 2001.

Goal-Driven Unfolding of Petri Nets*

Thomas Chatain[1] and Loïc Paulevé[2]

1 **LSV, ENS Cachan – INRIA – CNRS, France**
2 **CNRS, LRI UMR 8623, Univ. Paris-Sud – CNRS, France**

Abstract

Unfoldings provide an efficient way to avoid the state-space explosion due to interleavings of concurrent transitions when exploring the runs of a Petri net. The theory of adequate orders allows one to define finite prefixes of unfoldings which contain all the reachable markings. In this paper we are interested in reachability of a single given marking, called the goal. We propose an algorithm for computing a finite prefix of the unfolding of a 1-safe Petri net that preserves all minimal configurations reaching this goal. Our algorithm combines the unfolding technique with on-the-fly model reduction by static analysis aiming at avoiding the exploration of branches which are not needed for reaching the goal. We present some experimental results.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.2 Modes of Computation, D.2.4 Software/Program Verification, J.3 Life and Medical Sciences

Keywords and phrases model reduction, reachability; concurrency, unfoldings, Petri nets, automata networks

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.18

1 Introduction

Analysing the possible dynamics of a concurrent system expressed as Petri nets can be eased by means of unfoldings and their prefixes which avoid exploring redundant interleaving of transitions.

In this paper, we propose a method which combines the unfolding technique with model reduction in order to explore efficiently and completely the minimal configurations (partially ordered occurrences of transitions) which lead to a given goal marking/marked place. In particular, we aim at ignoring configurations that cannot reach the goal, but also configurations containing transient cycles.

The goal-driven unfolding relies on calling, on the fly, an external model reduction procedure which identifies transitions not part of any minimal configuration for the goal reachability from the current marking. Those useless transitions are then skipped by the unfolding.

We show how model reduction can be applied to the unfolding of a safe Petri net $\mathcal{N}$ in such a way that it preserves minimal configurations. Then we present an algorithm to construct a corresponding goal-driven finite prefix.

We illustrate this procedure on the Petri net of Figure 1. The goal is $\{p'_3, p_5\}$. Notice that only one occurrence of t_3 is needed to reach the goal. So, after the corresponding event, t_3 can be declared useless. Also, after firing t_1, t'_2 is fireable but firing it makes the goal

* This work has been partially funded by ANR-FNR project "AlgoReCell" (ANR-16-CE12-0034); by Labex DigiCosme (project ANR-11-LABEX-0045-DIGICOSME) operated by ANR as part of the program "Investissement d'Avenir" Idex Paris-Saclay (ANR-11-IDEX-0003-02); and by CNRS PEPS INS2I 2017 project "FoRCe".

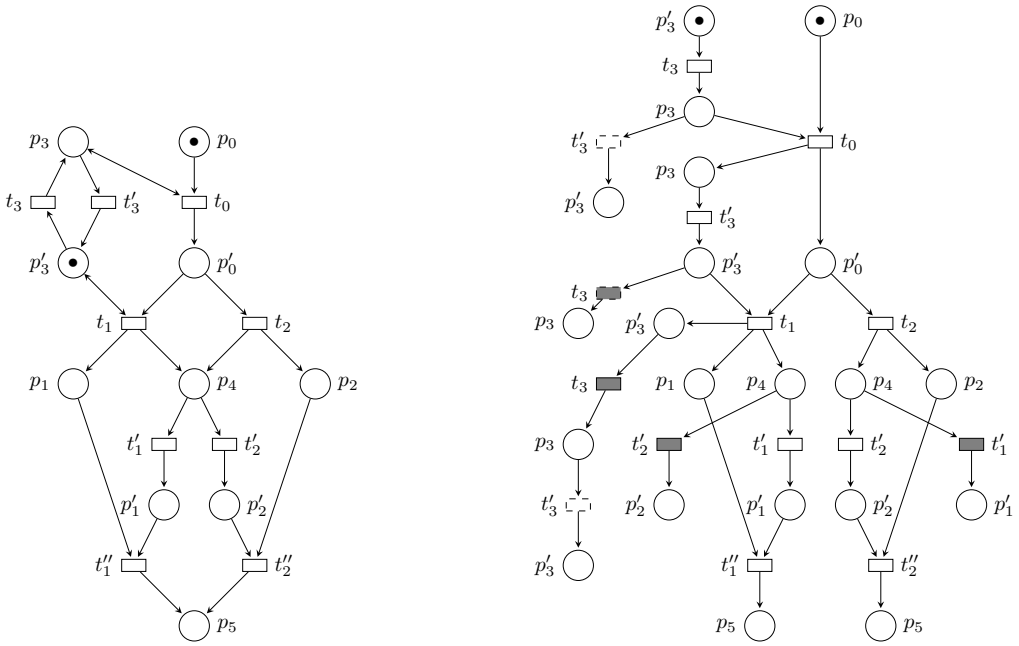


Figure 1 A safe Petri net (left) and a finite complete prefix (right) of its unfolding. Dashed events are flagged as *cut-offs*: the unfolding procedure does not continue beyond them. Events in gray can be declared as useless by the reduction procedure for $\{p'_3, p_5\}$ reachability, and can be skipped during the goal-driven prefix computation.

unreachable. Therefore, a reduction procedure may declare that t'_2 is useless once t_1 has occurred, allowing one to avoid exploring this branch. Symmetrically, t'_1 is useless once t_2 has occurred. It is easy to imagine a larger model where a large piece of behaviour would be reachable from $\{p_1, p'_2, p_3\}$ (but would not allow to reach the goal); or from $\{p_3, p_4\}$ (but would involve transient cycles): the usual complete finite prefix would explore such configurations, while our model reduction can avoid their computation.

The design of the model reduction procedure which identifies useless transitions is out of the scope of the paper. Instead, we consider it as a blackbox, and design our approach assuming the reduction preserves all the minimal (acyclic) sequences of transitions leading to the goal. Moreover, to be of practical interest, the reduction should show a complexity lower than the reachability problem (PSPACE-complete [7]).

As detailed in Section 4.2, skipping transitions declared useless by a reduction procedure involves non-trivial modifications to the algorithm for computing the prefix of the unfolding. Indeed, a particular treatment of cut-offs has to be introduced in order to ensure that the resulting goal-driven prefix includes all the minimal sequences of transitions.

The goal-driven unfolding has practical applications in systems biology [19]. Indeed, numerous dynamical properties relevant for biological networks focus on the reachability of the activity of a particular node in the network, typically a transcription factor known to control a given cellular phenotype. In this perspective, having computational methods that can be tailored for such narrow reachability properties is of practical interest. The *completeness of the minimal sequences of transitions* for the goal reachability is *critical* for several analyses of biological system dynamics. An example is the identification of parts of the network that play a central role to activate a node of interest. By altering such parts (e.g., with mutations) one can expect prevent such an activation [18]. If the analysis considers

only a partial set of minimal sequences, there is no guarantee that the predicted mutations are sufficient to prevent the goal reachability.

We illustrate the benefit of the goal-driven unfolding on models of biological networks by instantiating our method with a reduction procedure introduced in [17].

Related Work

Numerous work address the computation of reachable states in concurrent systems using unfoldings. Several algorithms for checking reachability based on a previously computed finite complete prefix of a Petri net are compared in [12]. Over-approximations of the unfolding (i.e., which contains all the reachable markings, but potentially more) for graph transformation systems are defined in [2].

Despite the negative result [9] which states that depth-first-search strategies are not correct for classical unfolding algorithms, [3] defines *directed unfolding* of Petri nets, which is closely related to our goal-driven unfolding. They rely on a heuristic function (on configurations) to generate an ordering of the events for making a given transition appear as soon as possible during the unfolding. In addition, they can consider heuristic functions to detect configuration from which the goal transition is not reachable. In such a case, no extension will be made to that configuration, which may significantly prune the computed prefix. The major difference with the work presented in the paper is that directed unfolding does not prune transitions leading to spurious transient cycles on the way to the goal. Actually, in their terms, our reduction procedure would not be considered *safely pruning* because we discard (non-minimal) configurations reaching the goal. In a sense, the reduction they achieve on the prefix size corresponds to the extreme case when our external reduction procedure returns the full model if the goal is reachable, and the empty model if not. Indeed, except for the case when the goal is detected as non-reachable, all the other configurations are kept in the directed unfolding, whereas our approach can potentially output a prefix containing only, but all, minimal configurations for the goal reachability.

Less related to our work, static analysis techniques were also used in combination with partial order reductions: [13, 21] rely on an on-the-fly detection of independence relations by static analysis, to improve partial order reductions.

Outline

Section 2 gives the basics of Petri net unfoldings and of their complete finite prefixes. The concepts of minimal configuration and model reduction are introduced in Section 3, and Section 4 details the goal-driven unfolding and prefix with proofs of completeness. Finally, Section 5 applies the goal-driven prefix to actual biological models, and Section 6 concludes this paper.

2 Unfoldings of Petri nets

In this section, we explain the basics of Petri net unfoldings. A more extensive treatment of the theory explained here can be found, e.g., in [8]. Roughly speaking, the unfolding of a Petri net $\mathcal{N}$ is an "acyclic" Petri net $\mathcal{U}$ that has the same behaviours as $\mathcal{N}$ (modulo homomorphism). In general, $\mathcal{U}$ is an infinite net, but if $\mathcal{N}$ is safe, then it is possible [16] to compute a finite prefix $\mathcal{P}$ of $\mathcal{U}$ that is "complete" in the sense that every reachable marking of $\mathcal{N}$ has a reachable counterpart in $\mathcal{P}$. Thus, $\mathcal{P}$ represents the set of reachable markings of $\mathcal{N}$. Figure 1 shows a Petri net and a finite complete prefix of its unfolding.

We now give some technical definitions to introduce unfoldings formally.

▶ **Definition 1** ((Safe) Petri Net). A *(safe) Petri net* is a tuple $\mathcal{N} = \langle P, T, F, M_0 \rangle$ where P and T are sets of *nodes* (called *places* and *transitions* respectively), and $F \subseteq (P \times T) \cup (T \times P)$ is a *flow relation* (whose elements are called *arcs*). A subset $M \subseteq P$ of the places is called a marking, and M_0 is a distinguished *initial marking*.

For any node $x \in P \cup T$, we call *pre-set* of x the set ${}^\bullet x = \{y \in P \cup T \mid (y, x) \in F\}$ and *post-set* of x the set $x^\bullet = \{y \in P \cup T \mid (x, y) \in F\}$. These notations are extended to sets $Y \subseteq P \cup T$, with ${}^\bullet Y = \cup_{x \in Y} {}^\bullet x$ and $Y^\bullet = \cup_{x \in Y} x^\bullet$.

A transition $t \in T$ is *enabled* at a marking M if and only if ${}^\bullet t \subseteq M$. Then t can *fire*, leading to the new marking $M' = (M \setminus {}^\bullet t) \cup t^\bullet$. We write $M \xrightarrow{t} M'$. A *firing sequence* is a (finite or infinite) word $w = t_1 t_2 \dots$ over T such that there exist markings $M_1, M_2, \dots$ such that $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \dots$ For any such firing sequence w, the markings $M_1, M_2, \dots$ are called *reachable markings*.

The Petri nets we consider are said to be *safe* because we will assume that any reachable marking M is such that for any $t \in T$ that can fire from M leading to M', the following property holds: $\forall p \in M \cap M',\ p \in {}^\bullet t \cap t^\bullet \vee p \notin {}^\bullet t \cup t^\bullet$.

Figure 1 (left) shows an example of a safe Petri net. The places are represented by circles and the transitions by rectangles (each one with a label identifying it). The arrows represent the arcs. The initial marking is represented by dots (or tokens) in the marked places.

▶ **Definition 2** (Causality, conflict, concurrency). Let $\mathcal{N} = \langle P, T, F, M_0 \rangle$ be a net and $t, t' \in T$ two transitions of $\mathcal{N}$. We say that t is a *causal predecessor* of t', noted $t < t'$, if there exists a non-empty path of arcs from t to t'. We note $t \leq t'$ if $t < t'$ or $t = t'$. If $t \leq t'$ or $t' \leq t$, then t and t' are said to be *causally related*. The set of causal predecessors of t is denoted $\lfloor t \rfloor$. We write $\lceil t \rceil$ for $\lfloor t \rfloor \cup \{t\}$, which we call the *causal past* of t. Transitions t and t' are *in conflict*, noted $t \# t'$, if there exist $u, v \in T$ such that $u \neq v$, $u \leq t$, $v \leq t'$ and ${}^\bullet u \cap {}^\bullet v \neq \emptyset$. We call t and t' *concurrent*, noted $t \mathrel{co} t'$, if they are neither causally related nor in conflict.

As we said before, an unfolding is an "acyclic" net. This notion of acyclicity is captured by Definition 3. As is convention in the unfolding literature, we shall refer to the places of an occurrence net as *conditions* and to its transitions as *events*. Due to the structural constraints, the firing sequences of occurrence nets have special properties: if some condition c is marked during a run, then the token on c was either present initially or produced by one particular event (the single event in ${}^\bullet c$); moreover, once the token on c is consumed, it can never be replaced by another token, due to the acyclicity constraint on $<$.

▶ **Definition 3** (Occurrence net). An *occurrence net* $\mathcal{O} = \langle C, E, G, C_0 \rangle$ is a Petri net $\langle P, T, F, M_0 \rangle$ with $P = C$, $T = E$, $F = G$, $M_0 = C_0$ for which:

1. The causality relation $<$ is acyclic;
2. $|{}^\bullet p| \leq 1$ for all places p, and $p \in M_0$ iff $|{}^\bullet p| = 0$;
3. for every transition t, $t \# t$ does not hold, and $\{x \mid x \leq t\}$ is finite.

▶ **Definition 4** (Configuration, cut). Let $\mathcal{O} = \langle C, E, G, C_0 \rangle$ be an occurrence net. A set $\mathcal{E} \subseteq E$ is called *configuration* (or *process*) of $\mathcal{O}$ if (i) $\mathcal{E}$ is *causally closed*, i.e. for all $e, e' \in E$ with $e' < e$, if $e \in \mathcal{E}$ then $e' \in \mathcal{E}$; and (ii) $\mathcal{E}$ is conflict-free, i.e. if $e, e' \in \mathcal{E}$, then $\neg(e \# e')$. The *cut* of $\mathcal{E}$, denoted $Cut(\mathcal{E})$, is the set of conditions $(C_0 \cup \mathcal{E}^\bullet) \setminus {}^\bullet \mathcal{E}$.

An occurrence net $\mathcal{O}$ with a net homomorphism h mapping its conditions and events to places and transitions of a net $\mathcal{N}$ is called a *branching process* of $\mathcal{N}$. Intuitively, a configuration of $\mathcal{O}$ is a set of events that can fire during a firing sequence of $\mathcal{N}$, and its cut is the set of conditions marked after that sequence.

2.1 Unfolding

Let $\mathcal{N} = \langle P, T, F, M_0 \rangle$ be a safe Petri net. The unfolding $\mathcal{U} = \langle C, E, G, C_0 \rangle$ of $\mathcal{N}$ is the unique (up to isomorphism) maximal branching process such that the firing sequences and reachable markings of $\mathcal{U}$ represent exactly the firing sequences and reachable markings of $\mathcal{N}$ (modulo h). $\mathcal{U}$ is generally infinite and can be inductively constructed as follows:

1. The conditions C are a subset of $(E \cup \{\bot\}) \times P$. For a condition $c = \langle x, p \rangle$, we will have $x = \bot$ iff $c \in C_0$; otherwise x is the singleton event in ${}^\bullet c$. Moreover, $h(c) = p$. The initial marking C_0 contains one condition $\langle \bot, p \rangle$ per initially marked place p of $\mathcal{N}$.
2. The events E are a subset of $2^C \times T$. More precisely, we have an event $e = \langle C', t \rangle$ for every set $C' \subseteq C$ such that ${}^\bullet c \; co \; {}^\bullet c'$ holds for all $c, c' \in C'$ and $\{ h(c) \mid c \in C' \} = {}^\bullet t$. In this case, we add edges $\langle c, e \rangle$ for each $c \in C'$ (i.e. ${}^\bullet e = C'$), we set $h(e) = t$, and for each $p \in t^\bullet$, we add to C a condition $c = \langle e, p \rangle$, connected by an edge $\langle e, c \rangle$.

Intuitively, a condition $\langle x, p \rangle$ represents the possibility of putting a token onto place p through a particular firing sequence, while an event $\langle C', t \rangle$ represents a possibility of firing transition t in a particular context.

Every firing sequence σ is represented by a configuration of $\mathcal{U}$; we denote this configuration $\mathcal{K}(\sigma)$. Conversely, every configuration $\mathcal{E}$ of $\mathcal{U}$ represents one or several firing sequences ($\mathcal{K}$ is not injective in general); these firing sequences are equivalent up to permutation of concurrent transitions. Their (common) resulting marking corresponds, due to the construction of $\mathcal{U}$, to a reachable marking of $\mathcal{N}$. This marking is defined as $Mark(\mathcal{E}) := \{ h(c) \mid c \in Cut(\mathcal{E}) \}$.

2.2 Finite Complete Prefix

The unfolding $\mathcal{U}$ of a finite safe Petri net $\mathcal{N}$ is infinite in general, but it shows some regularity because $\mathcal{N}$ has finitely many markings and two events e and e' having $Mark(\lceil e \rceil) = Mark(\lceil e' \rceil)$ have isomorphic extensions.

It is known [16, 11] that one can construct a *finite complete prefix* $\mathcal{P}$ of $\mathcal{U}$, i.e. an occurrence net having a causally closed set E' of events of $\mathcal{U}$ which is sufficiently large for satisfying the following: for every reachable marking M of $\mathcal{N}$ there exists a configuration $\mathcal{E}$ of $\mathcal{P}$ such that $Mark(\mathcal{E}) = M$. One can even require that for each transition t of $\mathcal{N}$ enabled in M, there is an event $\langle C, t \rangle \in E'$ enabled in $Cut(\mathcal{E})$.

The idea of the construction is to explore the future of only one among the events e having equal $Mark(\lceil e \rceil)$. The selected event is the one having minimal $\lceil e \rceil$ w.r.t. a so-called *adequate order* on the finite configurations of $\mathcal{U}$. The others are flagged as *cut-offs*; they do not "contribute any new reachable markings". These events are represented by dashed lines in Figure 1.

▶ **Definition 5** (Adequate orders). A strict partial order $\lhd$ on the finite configurations of the unfolding of a safe Petri net $\mathcal{N}$ is called *adequate* if:

- it refines (strict) set inclusion $\subsetneq$, i.e. $C \subsetneq C'$ implies $C \lhd C'$, and
- it is preserved by finite extensions, i.e. for every pair of configurations C, C' such that $Mark(C) = Mark(C')$ and $C \lhd C'$, and for every finite extension D of C, the finite extension D' of C' which is isomorphic to D satisfies $C \uplus D \lhd C' \uplus D'$.

The initial definition of adequate orders [11] also requires that $\lhd$ is well founded, but [6] showed that, for unfoldings of safe Petri nets, well-foundedness is a consequence of the other requirements.

Efficient tools [20, 14] exist for computing finite complete prefixes.

3 Goal-Oriented Model Reduction

The goal-driven unfolding relies on model reduction procedures which preserve minimal firing sequence to reach a given goal g. These reductions aim at removing as many transitions as possible among those that do not participate in any minimal firing sequence. This section details the properties required by our method and introduces several notations used in the rest of the paper.

▶ **Definition 6** (Minimal firing sequence)**.** A firing sequence $t_1 \dots t_n$ of a Petri net $\mathcal{N} = \langle P, T, F, M_0 \rangle$ visiting markings $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \dots \xrightarrow{t_n} M_n$ is said to be *cycling* if it visits the same marking twice, i.e. $M_i = M_j$ for some $0 \leq i < j \leq n$. A *minimal firing sequence* of $\mathcal{N}$ to a goal g is a firing sequence $t_1 \dots t_n$ leading to g for which there exists no different permutation[1] being a cycling firing sequence of $\mathcal{N}$.

For example with Petri net of Figure 1 and considering the goal $\{p'_3, p_5\}$, $t_3t_0t'_3t_2t_3t'_2t''_2t'_3$ is not minimal because its permutation $t_3t_0t'_3t_3t_2t'_2t''_2t'_3$ is also feasible and visits the marking $\{p_3, p'_0\}$ twice. Intuitively, the cycle t'_3t_3 can be removed. The minimal firing sequences of $\mathcal{N}$ to the goal are $t_3t_0t'_3t_1t'_1t''_1$, $t_3t_0t_2t'_3t'_2t''_2$ and their feasible permutations, for instance $t_3t_0t_2t'_2t''_2t'_3$.

▶ Remark. Alternatively, the goal can be seen not as a marking but simply as a set of places to be marked together, possibly with others. Then, one is looking for sequences reaching any marking M with $g \subseteq M$. For minimality, we would then require additionally that no intermediate marking reached before the end of the sequence marks the places in g (and the same for its permutations).

▶ **Definition 7** (Minimal configuration)**.** A *minimal configuration* of a Petri net $\mathcal{N}$ to a goal g is a configuration $E = \mathcal{K}(\sigma)$ for some minimal firing sequence σ of $\mathcal{N}$ to g. Notice that, since all the other σ' such that $E = \mathcal{K}(\sigma')$ are permutations of σ, they are all minimal.

▶ **Lemma 8.** *The goal g is reachable iff it is reachable by a minimal firing sequence (and, consequently, by a minimal configuration).*

Proof. Assume that g is reachable by a non-minimal firing sequence σ. This means that σ has a permutation $t_1 \dots t_n$ which visits the same marking twice, i.e. $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \dots \xrightarrow{t_i} M_i \dots \xrightarrow{t_j} M_j \dots \xrightarrow{t_n} M_n = g$ with $M_i = M_j$ and $i < j$. Then g is also reachable by the strictly shorter sequence $t_1 \dots t_i t_{j+1} \dots t_n$. This operation can be iterated if needed; it always terminates and gives a minimal firing sequence which reaches the goal g. ◀

▶ **Definition 9** (Reduction procedure, useless transitions)**.** A *reduction procedure* `useless-trs` is a function which outputs, for a safe Petri net $\mathcal{N}$ and a goal $g \subseteq P$, a set $\texttt{useless-trs}(\mathcal{N}, g) \subseteq T$ of transitions of $\mathcal{N}$ which do not occur in any *minimal* firing sequence of $\mathcal{N}$ to goal g: for every minimal firing sequence $t_1 \dots t_n$ to goal g, $\texttt{useless-trs}(\mathcal{N}, g) \cap \{t_1, \dots, t_n\} = \emptyset$.

For example, let $\mathcal{N} = \langle P, T, F, M_0 \rangle$ be the Petri net of Figure 1. All the transitions occur in at least one minimal firing sequence to the goal $g = \{p'_3, p_5\}$, so every reduction procedure outputs $\texttt{useless-trs}(\mathcal{N}, g) = \emptyset$. After firing $t_3t_0t'_3$, one reaches marking $\{p'_3, p'_0\}$ from which the only minimal firing sequences to g are $t_1t'_1t''_1$ and $t_2t'_2t''_2$. Hence, a reduction

[1] Contrary to what is common in concurrency theory, we do not necessarily restrict to permutations of independent transitions w.r.t. an independence relation.

procedure called as `useless-trs` $(\langle P, T, F, \{p'_3, p'_0\}\rangle, g)$ may declare t_0, t_3 and t'_3 useless, or any subset of those.

Given a Petri net $\mathcal{N} = \langle P, T, F, M_0\rangle$, $\mathcal{N} \setminus \texttt{useless-trs}(\mathcal{N}, g)$ denotes the reduced model $\langle P, T', F', M_0\rangle$ where $T' = T \setminus \texttt{useless-trs}(\mathcal{N}, g)$ and $F' = F \cap ((P \times T') \cup (T' \times P))$. Property 1 derives from Definition 9 and Lemma 8.

▶ Property 1. Every reduction procedure preserves reachability of the goal: g is reachable in $\mathcal{N}$ iff it is reachable in $\mathcal{N} \setminus \texttt{useless-trs}(\mathcal{N}, g)$.

In the sequel, we aim at iterating the reduction procedures: starting from a model $\mathcal{N} = \langle P, T, F, M_0\rangle$ and a goal g, we will apply the reduction to $\mathcal{N}$, then explore the reduced net $\mathcal{N} \setminus \texttt{useless-trs}(\mathcal{N}, g)$; later on, we will apply again the reduction from a reached state M and compute $\texttt{useless-trs}(\mathcal{N}', g)$ with $\mathcal{N}' = \langle P, T, F, M\rangle \setminus \texttt{useless-trs}(\mathcal{N}, g)$ allowing to explore a further reduced net $\mathcal{N}' \setminus \texttt{useless-trs}(\mathcal{N}', g)$ from M. These iterated calls to the reduction procedure are justified by the following lemma.

▶ **Lemma 10.** *Any minimal sequence in* $\mathcal{N} \setminus \texttt{useless-trs}(\mathcal{N}, g)$ *is minimal in* $\mathcal{N}$.

Proof. Any firing sequence of $\mathcal{N} \setminus \texttt{useless-trs}(\mathcal{N}, g)$ is a firing sequence of $\mathcal{N}$, and the minimality criterion does not depend on the set of transitions in $\mathcal{N}$. ◀

In the remainder of the paper, for a Petri net $\mathcal{N} = \langle P, T, F, M_0\rangle$ and any set $I \subseteq T$ and marking M, we write $\texttt{useless-trs}\,(\mathcal{N}, g, M, I)$ for $\texttt{useless-trs}(\langle P, T, F, M\rangle \setminus I, g) \cup I$.

4 Goal-Driven Unfolding

In this section, we first show that model reduction can be performed during the unfolding of a safe Petri net $\mathcal{N}$ while preserving the minimal configurations to the goal. Next we present an algorithm to construct a finite goal-driven prefix which preserves the reachable markings of the goal-driven unfolding.

4.1 Guiding the Unfolding by a Model Reduction Procedure

The principle of the goal-driven unfolding is that, for some events e in the unfolding (at discretion), a model reduction procedure `useless-trs` is called and the transitions declared useless will not be considered in the future of e. More precisely, the reduction procedure is called on the marking $\mathit{Mark}(\lceil e \rceil)$ of the causal past of e.

Notice that the reduction procedure may already have been used on some events in the causal past of e. Then,

- even if `useless-trs` is not called on e, information about useless transitions inherited from the causal predecessors of e can be used (without calling the model reduction procedure), and this will already prune some branches in the future of e;
- if the reduction procedure is called on $\mathit{Mark}(\lceil e \rceil)$, it can take as input the model already reduced by the transitions declared useless after some event in the causal past of e.

Let $\mathcal{U} = \langle C, E, G, C_0\rangle$ be the full unfolding of a safe Petri net $\mathcal{N}$. Denote E' the set of events on which the reduction procedure is called. The set E' and the reduction procedure define the set of transitions $\mathit{Useless}(e)$ to be ignored in the future of an event $e \in E$. We define $\mathit{Useless}$ inductively as:

$$\mathit{Useless}(e) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{e' \in \lfloor e \rfloor} \mathit{Useless}(e') & \text{if } e \notin E' \\ \texttt{useless-trs}\left(\mathcal{N}, g, \mathit{Mark}(\lceil e \rceil), \bigcup_{e' \in \lfloor e \rfloor} \mathit{Useless}(e')\right) & \text{if } e \in E'. \end{cases}$$

Thus, every event $e = \langle C, t\rangle \in E$ such that $t \in \mathit{Useless}(e')$ for some $e' \in \lfloor e \rfloor$, is discarded from the goal-driven unfolding. Denote E_{Ignored} the set of such events.

It remains to define the goal-driven unfolding as the maximal prefix of the full unfolding $\mathcal{U}$ having no event in E_{Ignored}. Since every discarded event automatically discards all its causal successors, the set of events remaining in the goal-driven unfolding $\mathcal{U}_{\mathrm{gd}}$ is

$$E_{\mathrm{gd}} \stackrel{\mathrm{def}}{=} \{e \in E \mid \lceil e \rceil \cap E_{\mathit{Ignored}} = \emptyset\} \ .$$

Notice that the events and conditions of the goal-driven unfolding as defined above can be constructed inductively following the procedure described in Section 2, enriched so that it attaches the set $\mathit{Useless}(e)$ to every new event e.

▶ **Theorem 11.** *(proof in Appendix A[2]) The goal-driven unfolding $\mathcal{U}_{\mathrm{gd}}$ preserves all minimal configurations from M_0 to the goal.*

A direct corollary is that the goal is reachable in $\mathcal{N}$ iff the goal-driven unfolding contains a configuration which reaches it.

Notice that the precise definition of minimal sequences/configurations is crucial here, and especially the fact that the reduction procedure preserves *all* minimal sequences/configurations. Indeed, imagine a situation where the minimal firing sequences to the goal fire two concurrent transitions t_1 and t_2 and then one out of two possible transitions t_3 and t_4. A reduction procedure which would guarantee only the preservation of *some* minimal firing sequence to the goal could declare t_3 useless when called after the event corresponding to t_1, and declare t_4 useless when called after t_2, thus preventing to reach the goal.

4.2 Goal-Driven Prefix

We now define a finite goal-driven prefix. Our Algorithm 1 relies on the theory of adequate orders [11] developed for unfoldings. Any adequate order on the configurations of the full unfolding can be used, but, since our goal-driven unfolding prunes some branches of the unfolding, we have to adapt the construction.

A prefix $\mathcal{P}$ has the same structure as an unfolding, with an additional field *coff* for the set of cut-off events. As usual, the procedure Putative-GD-Prefix extends iteratively the prefix $\mathcal{P} = \langle C, E, G, C_0, \mathit{coff}\rangle$. An extension is an event $e = \langle C', t\rangle$ with $C' \subseteq C$ s.t. $\forall c, c' \in C',\ c \mathrel{co} c',\ \{h(c) \mid c \in C'\} = {}^\bullet t$, and $\forall \langle e', p\rangle \in C',\ e' \notin \mathit{coff}$. Here the procedure maintains a map Δ of transitions that can be ignored, and considers an extension $e = \langle C', t\rangle$ only if the transition t is not declared useless, i.e., t is absent from $\Delta(c')$ for all pre-conditions $c' \in C'$.

The difficult part is that, when an event e is declared cut-off because $\mathit{Mark}(\lceil e \rceil) = \mathit{Mark}(\lceil e' \rceil)$ for an event $e' \lhd e$, nothing guarantees that the transitions allowed after e are also allowed after e'. Then, e and e' have the same future in the full unfolding, but not necessarily in the goal-driven unfolding.

Figure 2 illustrates this situation. Let the goal be $g = \{p_4, p_3\}$. It can be reached by the firing sequences $a(bb')^*c(bb')^*b$ or $a'b'(bb')^*c(bb')^*b$. Only those who do not take the cycle bb' are minimal, namely acb and $a'b'cb$. Notice that all the transitions participate in at least one minimal firing sequence, so the model $\mathcal{N}$ cannot be reduced from the initial marking (every

[2] Appendices available at `https://hal.archives-ouvertes.fr/hal-01392203/file/godunf.pdf`

Algorithm 1 Algorithm for goal-driven prefix computation.

procedure PUTATIVE-GD-PREFIX($\mathcal{N}, \Delta$) with $\mathcal{N} = \langle P, T, F, M_0 \rangle$
  $\mathcal{P} \leftarrow \langle C \leftarrow \{\langle \bot, p \rangle \mid p \in M_0\}, E \leftarrow \emptyset, G \leftarrow \emptyset, C_0 \leftarrow \{\langle \bot, p \rangle \mid p \in M_0\}, \mathit{coff} \leftarrow \emptyset \rangle$
  repeat
    Let $e = \langle C', t \rangle$ be a $\lhd$-minimal extension of $\mathcal{P}$ s.t. $t \notin \bigcup_{c' \in C'} \Delta(c')$.
    $E \leftarrow E \cup \{e\}$
    $C \leftarrow C \cup \{\langle e, p \rangle \mid p \in t^\bullet\}$
    $G \leftarrow G \cup \{\langle c', e \rangle \mid c' \in C'\} \cup \{\langle e, \langle e, p \rangle \rangle \mid p \in t^\bullet\}$
    if $\exists e' \in E$ s.t. $\mathit{Mark}(\lceil e \rceil) = \mathit{Mark}(\lceil e' \rceil)$ **then**
      $\mathit{coff} \leftarrow \mathit{coff} \cup \{e\}$ */e is a cut-off event/*
    end if
    for all $c \in \{\langle e, p \rangle \mid p \in t^\bullet\}$ s.t. $c \notin \Delta$ **do** */extend Δ with new cond./*
      $\Delta(c) \leftarrow \mathit{Useless}(c, \Delta, \mathcal{P})$
    end for
  until no extension exists
end procedure

procedure POST-$\Delta(\Delta, \mathcal{P})$ with $\mathcal{P} = \langle C, E, G, C_0, \mathit{coff} \rangle$
  $\Delta' \leftarrow \Delta$ */copy map Δ/*
  for $e \in E$ following $\lhd$ order **do**
    for all $c \in e^\bullet$ **do**
      $\Delta'(c) \leftarrow \Delta'(c) \cap \mathit{Useless}(c, \Delta, \mathcal{P})$
    end for
    if $\exists e' \in E \setminus \mathit{coff}$ s.t. $\mathit{Mark}(\lceil e \rceil) = \mathit{Mark}(\lceil e' \rceil)$ **then**
      for all $c' \in \mathit{Cut}(\lceil e' \rceil)$ with $c \in \mathit{Cut}(\lceil e \rceil)$ and $h(c) = h(c')$ **do**
        $\Delta'(c') \leftarrow \Delta'(c') \cap \Delta'(c)$
      end for
    end if
  end for
end procedure

procedure GD-PREFIX($\mathcal{N}$) with $\mathcal{N} = \langle P, T, F, M_0 \rangle$
  $\Delta' \leftarrow \{\langle \bot, p \rangle \mapsto \emptyset \mid p \in M_0\}$
  repeat
    $\Delta \leftarrow \Delta'$ */copy map Δ'/*
    $\Delta, \mathcal{P} \leftarrow$ PUTATIVE-GD-PREFIX($\mathcal{N}, \Delta$) */can add new entries in Δ/*
    $\Delta' \leftarrow$POST-$\Delta(\Delta, \mathcal{P})$
  until $\Delta' = \Delta$
end procedure

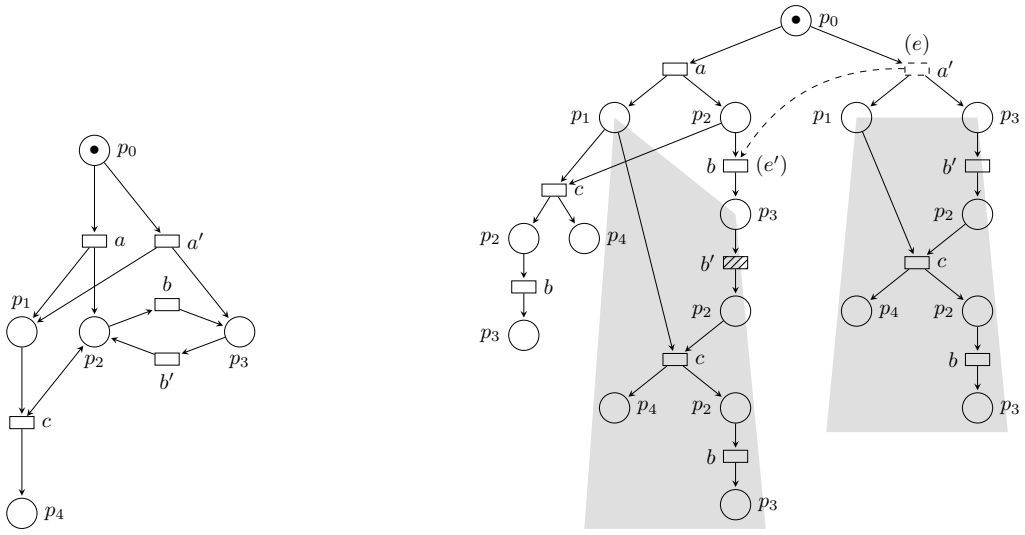


Figure 2 A safe Petri net (left) and one of its branching processes (right). Configurations $\lceil e \rceil$ and $\lceil e' \rceil$ lead to the same marking $\{p_1, p_3\}$ and have isomorphic extensions (in gray). The dashed arrow represents the fact that $\lceil e' \rceil \lhd \lceil e \rceil$. Consequently e is a cut-off.

reduction procedure will output $\texttt{useless-trs}(\mathcal{N}, g) = \emptyset$). On the other hand, if transition a is fired, we reach marking $\{p_1, p_2\}$ from which b', a and a' become useless.

Now, observe the branching process on the right of Figure 2 (it is a prefix of the unfolding $\mathcal{U}$ of $\mathcal{N}$). Notice that the causal past $\lceil e \rceil$ of the event e labeled a' and the causal past of the event e' labeled b reach the same marking $Mark(\lceil e \rceil) = Mark(\lceil e' \rceil) = \{p_1, p_3\}$. Moreover, an adequate order on the configurations of $\mathcal{U}$ may order them as $\lceil e' \rceil \lhd \lceil e \rceil$. Consequently, e is a cut-off and the minimal configuration $\mathcal{K}(a'b'cb)$ is not represented in the finite prefix. Following the idea of the proof of completeness of finite prefixes based on adequate orders, we can indeed shift the extension $b'cb$ of $\lceil e \rceil$ (in gray on the right of Figure 2) to the isomorphic extension of $\lceil e' \rceil$ (also in gray on the figure). We get the configuration $\mathcal{K}(abb'cb)$, which reaches the goal as well. But this configuration is *not* minimal any more because it executes the cycle bb': the marking reached after a is the same as the marking reached after abb'. Actually, the model reduction procedure called from the event labeled a may very well have declared b' useless. Consequently, $\mathcal{K}(abb'cb)$ would not be represented in the prefix. We correct this by allowing after $\lceil e' \rceil$ all the transitions that were allowed after $\lceil e \rceil$.

The difficulty in the definition and in the computation of a finite prefix $\mathcal{P}_{\text{gd}}$ of $\mathcal{U}$ which preserves the markings reachable in $\mathcal{U}_{\text{gd}}$ is to allow in the future of an event e' all the transitions that are useful for at least one of all the configurations which are shifted to $\lceil e' \rceil$ by the mechanics described above. The first answer to this problem is to allow after $\lceil e' \rceil$ all the transitions that were allowed after $\lceil e \rceil$. This solves the problem of an event consuming only post-conditions of e', like the occurrence of b' after e in our example of Figure 2: its corresponding event after e' is now allowed. However, this is not sufficient in general: an event f consuming a post-condition of e may also consume other conditions which are created by events concurrent to $\lceil e \rceil$. Such event f has a corresponding f' in the future of e', consuming conditions which are available after firing a configuration of the form $\lceil e' \rceil \cup \mathcal{E}'$ for some $\mathcal{E}'$ concurrent to $\lceil e' \rceil$. We need the transition $t = h(e) = h(e')$ to be allowed after all the conditions consumed by f'. In the case of a condition $c' \in {}^\bullet f' \setminus \mathit{Cut}(\lceil e' \rceil)$, our procedure ensures this as follows: if it calls the model reduction procedure after the event

${}^\bullet c'$, it also calls it on the marking $Mark(\lceil e' \rceil \cup \lceil {}^\bullet c' \rceil)$ which equals $Mark(\lceil e \rceil \cup \lceil {}^\bullet c \rceil)$. Hence, if t is needed after $\lceil e \rceil \cup \lceil {}^\bullet c \rceil$, it will also be allowed after c'.

Therefore, when applying the reduction procedure after a configuration $\mathcal{E}$, we also take into account a set $Alt(\mathcal{E})$ of alternating configurations defined inductively as:

- $\mathcal{E} \in Alt(\mathcal{E})$
- $\forall \mathcal{E}' \in Alt(\mathcal{E})$, $\forall e, e' \in E$ such that $\lceil e \rceil \rhd \lceil e' \rceil$ and $Mark(\lceil e \rceil) = Mark(\lceil e' \rceil)$, if $Cut(\lceil e' \rceil) \cap \mathcal{E}'^\bullet \neq \emptyset$ and $\lceil e' \rceil \cup \mathcal{E}'$ is conflict free, then $\lceil e' \rceil \cup \mathcal{E}' \in Alt(\mathcal{E})$.

However, in practice, during the computation of the goal-driven prefix, $Alt(\mathcal{E})$ will be computed on the events and configurations derived so far, hence ignoring events later added in the prefix. Also, as explained above, when an event e is stated cut-off because of a $\lhd$-smaller event e', we allow after e' all the transitions allowed after e; but this implies reconsidering some new extensions of e'.

For these reasons, the procedure GD-Prefix($\mathcal{N}$) presented in Algorithm 1 iterates the computation of a putative prefix, progressively refining an over-approximation of transitions to ignore (map Δ), by identifying *a posteriori* the transitions that should *not* have been ignored.

At each iteration, the procedure Putative-GD-Prefix($\mathcal{N}, \Delta$) computes a putative prefix, relying on the previous value of the map Δ of transitions that can be ignored. Essentially, the prefix $\mathcal{P}$ obtained at the first iteration is the naive prefix of $\mathcal{U}_{\text{gd}}$ (prefix without the gray parts on the example of Figure 2).

Once a putative prefix has been computed, we verify *a posteriori* if its related map Δ is correct. This is done by re-computing Δ using the procedure Post-$\Delta(\Delta, \mathcal{P})$, this time taking into account all the events in $\mathcal{P}$ (line 34). By construction, the resulting Δ' can only allow more transitions than Δ. If Δ' differs from Δ, a new putative prefix is computed according to the corrected Δ'.

The procedure Post-$\Delta(\Delta, \mathcal{P})$ takes the $Alt(\lceil e \rceil)$ into account by the way of a modified version of *Useless*(), now defined on conditions rather than on events. Given a condition $c \in e^\bullet$ in a prefix $\mathcal{P}$,

$$Useless(c, \Delta, \mathcal{P}) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{c' \in {}^\bullet e} \Delta(c') & \text{if } e \notin E' \\ \bigcap_{C^* \in Alt(\lceil e \rceil)} \texttt{useless-trs}\left(\mathcal{N}, g, Mark(C^*), \bigcup_{c' \in {}^\bullet e} \Delta(c')\right) & \text{if } e \in E', \end{cases}$$

where E' is the set of events triggering an explicit reduction (Section 4.1).

This iterative construction necessarily terminates (Lemma 12, proof in Appendix B[2]) and converges to a unique finite prefix $\mathcal{P}_{\text{gd}}$. Regarding complexity, putting aside the call to model reduction, whereas all the structures are finite, $Alt(\mathcal{E})$ can have an exponential numbers of configurations due to multiple combinations of configurations sharing an intersection.

▶ **Lemma 12.** *The procedure* GD-Prefix($\mathcal{N}$) *terminates.*

Notice that $\mathcal{P}_{\text{gd}}$ may contain events that are not in E_{gd}. Hence, goal-driven prefix is a prefix of $\mathcal{U}$, but not necessarily a prefix of $\mathcal{U}_{\text{gd}}$. This is the case of the event labeled b' after e', as we discussed above for the example in Figure 2.

Theorem 13 (proof in Appendix C[2]) states completeness of $\mathcal{P}_{\text{gd}}$ w.r.t. minimal configurations. Thus, the goal-driven prefix preserves the reachability of the goal. One can finally remark that, by construction, $\mathcal{P}_{\text{gd}}$ contains at most one non-cutoff event per reachable marking, assuming the adequate order $\lhd$ is total.

▶ **Theorem 13.** *For every configuration $\mathcal{E}$ of $\mathcal{U}_{\text{gd}}$ and for every single-event extension $\{f\}$ of $\mathcal{E}$ such that $\mathcal{E} \cup \{f\}$ is a prefix of a minimal configuration to the goal, there exists*

a configuration $\mathcal{E}'$ in the goal-driven prefix and a single-event extension $\{f'\}$ of $\mathcal{E}'$ with $Mark(\mathcal{E}) = Mark(\mathcal{E}')$ and $h(f) = h(f')$.

Example

Let us consider the Petri net of Figure 2(left) with the goal $\{p_4, p_3\}$.

The goal-driven unfolding can lead to the branching process of Figure 2(right) where the striped transition b' has been removed. Indeed, after transition a, transition b' is declared useless as it is not part of any minimal configuration extending $\mathcal{K}(a)$. Therefore 3 maximal configurations are remaining in the goal-driven unfolding: the two minimal configurations $\mathcal{K}(acb)$ and $\mathcal{K}(a'b'cb)$, and the configuration $\mathcal{K}(ab)$ which does not reach the goal.

The goal-driven prefix can lead to the branching process of Figure 2(right) where the event e is cut-off (because of e'), and therefore its future events are ignored, and where one of the two remaining events firing transition c is declared cut-off (because of the other one). Although the b' transition can be declared useless after $\mathcal{K}(a)$ (and hence $\mathcal{K}(ab)$), the cut-off of e will remove b' from the set of ignored transitions of the conditions matching with p_1 and p_3 on the cut of $\mathcal{K}(ab)$. Therefore, the events and conditions in the left gray area will be added to the prefix, from which all the minimal configurations can be identified.

5 Experiments

In this section, we instantiate our goal-driven unfolding with the goal-oriented model reduction introduced in [17] for automata networks. We compare the size of the complete prefix with the goal-driven prefix on different Petri net models of biological signalling and gene regulatory networks. In general, such networks gather dozens to thousands nodes having sparse interactions (each node is directly influenced by a few other nodes), which call for concurrency-aware approaches to cope with the state space explosion. We took the networks from systems biology literature, specified as Boolean or automata networks: each node is modelled by an automaton, where states model its activity level, most often being binary (active or inactive). The Petri nets are encodings of these automata networks which ensure bisimilarity [5].

5.1 Implementation

In practice, instead of computing putative prefixes from scratch as it is described in Algorithm 1, our implementation for the goal-driven prefix[3] iteratively corrects the putative prefix by propagating transitions missed in the previous iteration. At this stage, it does not use any particular optimization [1], our primary objective being to compare the size of the resulting prefixes. In order to obtain a proper comparison [15], our implementation uses the same arbitrarily-fixed ordering for the complete and goal-driven prefixes extensions.

The computation of `useless-trs` $(\mathcal{N}, g, M, I)$ relies on the goal-oriented reduction of asynchronous automata networks introduced in [17]. This method is based on a static analysis of causal dependencies of transitions and an abstract interpretation of traces which allow to collect all the transitions involved in the minimal configurations to the goal: non-collected transitions can then be ignored. The complexity of the reduction is polynomial with the number of automata and transitions, and exponential with the number of states in individual

[3] Code and models available at `http://loicpauleve.name/godunf.tbz2`

Table 1 Benchmarks of the goal-driven w.r.t. complete prefix of 1-safe Petri nets. For each model, the number of places $|P|$ and transitions $|T|$ is given. The strategy decides when the model reduction should be performed; the number of calls to the reduction procedure is indicated in the column "Nb reductions". Computation times were obtained on an Intel® Core™ i7 3.4GHz CPU with 16GB RAM. N/A: Non Applicable; *: out-of-memory computation (with mole [20], with the same ordering for extensions as our implementation), the indicated prefix size is only a lower bound.

Model	Prefix	Strategy	Prefix size	Time	Nb reductions
RB/E2F	complete	N/A	15,210	24s	N/A
$\|P\| = 80 \quad \|T\| = 54$	goal-driven	always	112	0.5s	136
T-LGL	complete	N/A	>1,900,000*	OT*	N/A
$\|P\| = 98 \quad \|T\| = 159$	goal-driven	always	17	0.3s	17
VPC	complete	N/A	44,500	176s	N/A
$\|P\| = 135 \quad \|T\| = 216$	goal-driven	always	1,827	2h	16,009
		first $1,000$	2,036	60s	1,000
		level ≤ 2	2,400	7s	38

automata (i.e., number of qualitative states of nodes). As shown in [17], the method can lead to drastic model reductions and can be executed in a few hundredths of a second on networks with several hundreds of nodes. Appendix D[2] gives a brief summary of the main principles of the goal-oriented model reduction of [17].

Compared to the results obtained in [17], we expect to obtain much stronger reduction of the dynamics as our goal-driven unfolding applies the reduction "on-the-fly" instead of only at the initial state. To that aim, we will compare the size of the complete prefix, which relies only on [17], with the size of the goal-driven prefix, introduced in this paper.

We applied the goal-driven unfolding to 1-safe Petri net encodings of the automata networks, where there is one place for each local state of each automaton, and a one-to-one relationship between transitions. The places corresponding to states of a same automaton are mutually exclusive by construction. Future work may consider goal-driven unfolding of products of transition systems [10].

The goal-driven prefix we define in this paper supports calling the model reduction procedure at discretion: even if it has a low computational cost, performing the model reduction after each event may turn out to be very time consuming. Our prototype implements simple strategies to decide when the call to the model reduction should be performed: after each event; only for the first n events; and only for events up to a given level in the unfolding.

5.2 Benchmarks

Given a Petri net with an initial marking M_0 and a goal g, we first compute the goal-oriented model reduction from initial marking ($\texttt{useless-trs}\,(\mathcal{N}, g, M_0, \emptyset)$). The resulting net is then given as input to the unfolding, either with the complete finite prefix computation, or with the goal-driven. Therefore, the difference in the size of the prefixes obtained is due only to transition exclusions after at least one event.

Table 1 summarizes the benchmarks between complete and goal-driven prefix on different models of biological networks. The size of a prefix is the number of its non-cutoff events. "RB/E2F" is a model of the cell cycle [4]; "T-LGL" is a model of survival signaling in large granular lymphocyte leukemia [23]; and "VPC" is a model for the specification of vulval precursor cells and cell fusion control in Caenorhabditis elegans [22]. For each model, the initial marking and goal correspond to biological states of interest (checkpoints or

differentiated states). All these models have different network topology and dynamical features, but all include loops, and in general, correspond to automata networks where automata have few internal states, and each transition is conditioned by a few automata compared to the network size.

On these models, the goal-driven prefix shows a significant size reduction, while containing all the minimal configurations. The number of reductions can be larger than the size of the prefix as it accounts for the intermediate putative prefixes (as explained in Section 4.2). For the "VPC" model, we applied several strategies for deciding when the model reduction should be called. In this case, the systematic model reduction led to some re-ordering of the extensions and cut-offs declaration, which required numerous additional calls to the model reduction procedure. This motivates the design of heuristics to estimate when a model reduction should be performed. For the "T-LGL" model, it was impossible to compute the complete finite prefix using only the reduction of [17] on the initial state, whereas the goal-driven cuts most of the configurations and produces a very concise prefix. Such a behaviour can be explained by large transient cycles prior to the goal reachability, which are avoided by the use of model reduction during the prefix computation.

6 Conclusion

We introduced the goal-driven unfolding of safe Petri nets for identifying efficiently *all* the minimal configurations that lead to a given goal. The goal can be a marking of the net, or any partially specified marking, and notably a single marked place. The goal-driven unfolding relies on an external reduction method which identifies transitions that are not part of minimal configuration for the goal reachability. Such useless transitions are then skipped by the unfolding. The computation of a goal-driven prefix requires a particular treatment of cut-offs to ensure that all the markings reachable in the goal-driven unfolding are preserved. The resulting goal-driven prefix contains fewer events prefix than reachable markings, due to the total adequate order, as well as for classical finite complete prefix.

We instantiated our approach with a goal-oriented reduction method for automata networks introduced in [17]. Our experiments on different models of biological systems show a significant reduction of the prefix when driven by the goal. In our framework, the reduction procedure can be applied at discretion, and many possible heuristics could be embedded to decide when the reduction is timely, which impacts both the execution time and the size of the prefix.

Because our method considers the model reduction procedure as a blackbox, it can directly take advantage of any improvements of model reduction procedures which preserve minimal configurations.

Future work will explore the combination with the semi-adequate ordering of configurations of directed unfolding [3] as it may reduce the need for propagating transitions allowed by a cut-off event. Finally, we are considering implementing the goal-driven unfolding within Mole [20].

References

1 Paolo Baldan, Alessandro Bruni, Andrea Corradini, Barbara König, César Rodríguez, and Stefan Schwoon. Efficient unfolding of contextual Petri nets. *TCS*, 449:2–22, 2012. doi:10.1016/j.tcs.2012.04.046.

2 Paolo Baldan, Andrea Corradini, and Barbara König. A static analysis technique for graph transformation systems. In *CONCUR 2001*, volume 2154 of *LNCS*, pages 381–395. Springer, 2001. doi:10.1007/3-540-44685-0_26.

3 Blai Bonet, Patrik Haslum, Sarah L. Hickmott, and Sylvie Thiébaux. Directed unfolding of Petri nets. *Trans. Petri Nets and Other Models of Concurrency*, 1:172–198, 2008. doi:10.1007/978-3-540-89287-8_11.

4 Laurence Calzone, Amélie Gelay, Andrei Zinovyev, François Radvanyi, and Emmanuel Barillot. A comprehensive modular map of molecular interactions in RB/E2F pathway. *Molecular Systems Biology*, 4(1), 2008. doi:10.1038/msb.2008.7.

5 Thomas Chatain, Stefan Haar, Loïg Jezequel, Loïc Paulevé, and Stefan Schwoon. Characterization of reachable attractors using Petri net unfoldings. In *Computational Methods in Systems Biology*, volume 8859 of *LNCS*, pages 129–142. Springer, 2014. doi:10.1007/978-3-319-12982-2_10.

6 Thomas Chatain and Victor Khomenko. On the well-foundedness of adequate orders used for construction of complete unfolding prefixes. *Inf. Process. Lett.*, 104(4):129–136, 2007. doi:10.1016/j.ipl.2007.06.002.

7 Allan Cheng, Javier Esparza, and Jens Palsberg. Complexity results for 1-safe nets. *Theor. Comput. Sci.*, 147(1&2):117–136, 1995. doi:10.1016/0304-3975(94)00231-7.

8 Javier Esparza and Keijo Heljanko. *Unfoldings – A Partial-Order Approach to Model Checking*. Springer, 2008.

9 Javier Esparza, Pradeep Kanade, and Stefan Schwoon. A negative result on depth-first net unfoldings. *STTT*, 10(2):161–166, 2008. doi:10.1007/s10009-007-0030-5.

10 Javier Esparza and Stefan Römer. An unfolding algorithm for synchronous products of transition systems. In *CONCUR*, volume 1664 of *LNCS*, pages 2–20. Springer, 1999. doi:10.1007/3-540-48320-9_2.

11 Javier Esparza, Stefan Römer, and Walter Vogler. An improvement of McMillan's unfolding algorithm. *FMSD*, 20:285–310, 2002. doi:10.1007/3-540-61042-1_40.

12 Javier Esparza and Claus Schröter. Unfolding based algorithms for the reachability problem. *Fundam. Inform.*, 47(3-4):231–245, 2001. URL: http://content.iospress.com/articles/fundamenta-informaticae/fi47-3-4-05.

13 Cormac Flanagan and Patrice Godefroid. Dynamic partial-order reduction for model checking software. In *Proceedings of the 32nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2005*, pages 110–121. ACM, 2005. doi:10.1145/1040305.1040315.

14 Victor Khomenko. Punf. http://homepages.cs.ncl.ac.uk/victor.khomenko/tools/punf/.

15 Victor Khomenko, Maciej Koutny, and Walter Vogler. Canonical prefixes of Petri net unfoldings. *Acta Inf.*, 40(2):95–118, 2003. doi:10.1007/s00236-003-0122-y.

16 Kenneth L. McMillan. Using unfoldings to avoid the state explosion problem in the verification of asynchronous circuits. In *CAV*, pages 164–177, 1992. doi:10.1007/3-540-56496-9_14.

17 Loïc Paulevé. Goal-Oriented Reduction of Automata Networks. In *CMSB 2016 - 14th conference on Computational Methods for Systems Biology*, volume 9859 of *Lecture Notes in Bioinformatics*. Springer, 2016. doi:10.1007/978-3-319-45177-0_16.

18 Loïc Paulevé, Geoffroy Andrieux, and Heinz Koeppl. Under-approximating cut sets for reachability in large scale automata networks. In *CAV*, volume 8044 of *LNCS*, pages 69–84. Springer, 2013. doi:10.1007/978-3-642-39799-8_4.

19 Regina Samaga, Axel Von Kamp, and Steffen Klamt. Computing combinatorial intervention strategies and failure modes in signaling networks. *J. of Computational Biology*, 17(1):39–53, 2010. doi:10.1089/cmb.2009.0121.

20 S. Schwoon. Mole. http://www.lsv.ens-cachan.fr/~schwoon/tools/mole/.

21 Chao Wang, Zijiang Yang, Vineet Kahlon, and Aarti Gupta. Peephole partial order reduction. In *TACAS 2008*, volume 4963 of *LNCS*, pages 382–396. Springer, 2008. doi:10.1007/978-3-540-78800-3_29.

22 Nathan Weinstein and Luis Mendoza. A network model for the specification of vulval precursor cells and cell fusion control in caenorhabditis elegans. *Frontiers in Genetics*, 4(112), 2013. doi:10.3389/fgene.2013.00112.

23 Ranran Zhang, Mithun Vinod Shah, Jun Yang, Susan B Nyland, Xin Liu, Jong K Yun, Réka Albert, and Thomas P Loughran. Network model of survival signaling in large granular lymphocyte leukemia. *PNAS*, 105:16308–13, 2008. doi:10.1073/pnas.0806447105.

Probabilistic Automata of Bounded Ambiguity*

Nathanaël Fijalkow[1], Cristian Riveros[2], and James Worrell[3]

1 **The Alan Turing Institute of Data Science, London, UK and University of Warwick, Coventry, UK**
nfijalkow@turing.ac.uk
2 **Pontificia Universidad Católica de Chile, Santiago, Chile**
cristian.riveros@uc.cl
3 **University of Oxford, Oxford, UK**
james.worrell@cs.ox.ac.uk

Abstract

Probabilistic automata are a computational model introduced by Michael Rabin, extending non-deterministic finite automata with probabilistic transitions. Despite its simplicity, this model is very expressive and many of the associated algorithmic questions are undecidable. In this work we focus on the emptiness problem, which asks whether a given probabilistic automaton accepts some word with probability higher than a given threshold. We consider a natural and well-studied structural restriction on automata, namely the degree of ambiguity, which is defined as the maximum number of accepting runs over all words. We observe that undecidability of the emptiness problem requires infinite ambiguity and so we focus on the case of finitely ambiguous probabilistic automata.

Our main results are to construct efficient algorithms for analysing finitely ambiguous probabilistic automata through a reduction to a multi-objective optimisation problem, called the stochastic path problem. We obtain a polynomial time algorithm for approximating the value of finitely ambiguous probabilistic automata and a quasi-polynomial time algorithm for the emptiness problem for 2-ambiguous probabilistic automata.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Probabilistic Automata, Emptiness Problem, Stochastic Path Problem, Multi-Objective Optimisation Problems

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.19

1 Introduction

Probabilistic automata are a simple and natural extension of non-deterministic automata that were introduced by Rabin [16]. Syntactically, a probabilistic automaton is a non-deterministic finite automaton in which each edge is annotated by a probability. Such an automaton associates to every word a value between 0 and 1, which is the total probability that a run on the word ends in an accepting state. We call this the acceptance probability of the word.

Despite their simplicity, probabilistic automata are very expressive and have been widely studied. Unfortunately the price of this expressiveness is that almost all natural decision problems are undecidable. Consequently, various approaches based on restricting resources such as structure, dimension, or randomness have been studied [5, 8, 7, 4].

* This work was supported by The Alan Turing Institute under the EPSRC grant EP/N510129/1, by the EPSRC grant EP/M012298/1, by the FONDECYT grant 11150653 and the Millennium Nucleus Center for Semantic Web Research under grant NC120004.

In this paper, we look at probabilistic automata of *bounded ambiguity*, where the ambiguity of a word is the number of accepting runs. We say that a probabilistic automaton is f-ambiguous, for a function $f : \mathbb{N} \to \mathbb{N}$, if every word of length n has at most $f(n)$ accepting runs. (Note that ambiguity is a property of the underlying nondeterministic finite automata, and is independent of the transition probabilities.) This restriction has been extensively studied in automata theory; in particular, the landmark paper of Weber and Seidl [18] gives respective structural characterisations of the classes finitely, polynomially, and exponentially ambiguous nondeterministic finite automata, from which polynomial-time algorithms are obtained for deciding membership in each of these classes.

We focus on the most natural and well-studied problem for probabilistic automata, called the *emptiness problem*: given a probabilistic automaton and a threshold, does there exist a word accepted with probability at least a given threshold? Using a classical construction, we first observe that the emptiness problem is already undecidable for polynomially ambiguous probabilistic automata. We are thus led to focus on finitely ambiguous probabilistic automata.

We study the complexity of the emptiness problem on various classes of finitely ambiguous probabilistic automata. For each positive integer k we consider the class of k-ambiguous probabilistic automata, i.e., automata with at most k accepting runs on any word. More generally we fix a polynomial p and consider the class of automata whose ambiguity is at most $p(m)$, where m is the number of states. More generally still, bearing in mind that the ambiguity can be exponential in the number of states, we have the class of all finitely ambiguous automata.

Our main results are as follows. We show that the emptiness problem for finitely ambiguous probabilistic automaton is, respectively:

- in **NEXPTIME** and **PSPACE**-hard for the class of all finitely ambiguous automata;
- **PSPACE**-complete for the class of probabilistic automata with ambiguity bounded by a fixed non-constant polynomial in the number of states.
- in **NP** for the class of k-ambiguous probabilistic automata, for every positive integer k.
- in quasi-polynomial time for the class of 2-ambiguous probabilistic automata.

Naturally associated with the emptiness problem we have the function problem of computing the value of a probabilistic automaton, that is, the supremum over all words of the acceptance probability of a word. Here we show:

- for the class of all finitely ambiguous probabilistic automata, there is no polynomial-time approximation algorithm for the value problem unless $\mathbf{P} = \mathbf{NP}$,
- for the class of k-ambiguous probabilistic automata, the value is approximable up to any multiplicative constant in polynomial time.

The starting point to prove these results is to give an upper bound on the length of a shortest word whose probability exceeds a given threshold. More precisely, we show that for a k-ambiguous probabilistic automaton with n states there is a maximum-probability word of length at most n^k. More generally, we show that for a finitely ambiguous probabilistic automaton with n states, there is a maximum-probability word of length at most $n!$. The latter result easily leads to a **PSPACE** upper bound for the emptiness problem in the case the ambiguity is bounded by a fixed polynomial in the number of states. Most of the remainder of the paper is devoted to the case of k-ambiguous automata for a fixed k.

We give a polynomial-time reduction from the emptiness problem for k-ambiguous probabilistic automata to a multi-objective optimisation problem, which we call the *k-stochastic path problem*. Using this reduction, we obtain a polynomial-time algorithm for approximating the value of a k-ambiguous probabilistic automata, and a quasi-polynomial time algorithm for the emptiness problem of 2-ambiguous probabilistic automata.

2 Preliminaries

Let Σ be a finite alphabet. For any word $w \in \Sigma^*$, we denote by $|w|$ its length. A distribution is a function $\delta : Q \to [0,1]$ such that $\sum_{q \in Q} \delta(q) = 1$. The set of distributions over Q is denoted $\mathcal{D}(Q)$.

Probabilistic automata. A *probabilistic automaton* is a tuple $\mathcal{P} = (Q, q_{in}, \Delta, F)$, where Q is a finite set of states, q_{in} is the initial state, $\Delta : Q \times A \to \mathcal{D}(Q)$ is the transition function, and F is the set of accepting states. Given a word $w = a_1 \cdots a_n$, a run ρ over w is a sequence of states $q_0, q_1, \ldots, q_n$. The probability of such a run is $\mathcal{P}(\rho) = \prod_{\ell \in \{1,\ldots,n\}} \Delta(q_{\ell-1}, a_\ell)(q_\ell)$. We denote by $\mathrm{Run}_{\mathcal{P}}(p \xrightarrow{w} q)$ the set of runs ρ over w starting in p and finishing in q with $\mathcal{P}(\rho) > 0$. The number $\mathcal{P}(p \xrightarrow{w} q)$ is the probability to go from p to q reading w, defined as the sum of the probabilities of its runs, namely:

$$\mathcal{P}(p \xrightarrow{w} q) = \sum_{\rho \in \mathrm{Run}_{\mathcal{P}}(p \xrightarrow{w} q)} \mathcal{P}(\rho).$$

A run ρ is accepting if it starts in q_{in}, satisfies $\mathcal{P}(\rho) > 0$, and finishes in an accepting state, *i.e.* a state in F. We denote by $\mathrm{Run}_{\mathcal{P}}(w)$ the set of accepting runs over w. The *probability* of w over $\mathcal{P}$ is defined as the sum of the probabilities of its accepting runs by:

$$\mathcal{P}(w) = \sum_{\rho \in \mathrm{Run}_{\mathcal{P}}(w)} \mathcal{P}(\rho).$$

Ambiguity. In this paper, we consider different subclasses of probabilistic automata, obtained by restrictions on *ambiguity*. More specifically, we say that:

- $\mathcal{P}$ is *unambiguous* if every word w has at most one accepting run, *i.e.* $|\mathrm{Run}_{\mathcal{P}}(w)| \leq 1$.
- $\mathcal{P}$ is *k-ambiguous* if every word w has at most k accepting runs, *i.e.* $|\mathrm{Run}_{\mathcal{P}}(w)| \leq k$.
- $\mathcal{P}$ is *finitely ambiguous*, if there exists k such that $\mathcal{P}$ is k-ambiguous.
- $\mathcal{P}$ is *polynomially ambiguous*, if there exists a polynomial P such that for every word w, we have $|\mathrm{Run}_{\mathcal{P}}(w)| \leq P(|w|)$.

Depending on whether the polynomial P is linear or quadratic, we say that a polynomial ambiguous automaton $\mathcal{P}$ is linearly ambiguous or quadratically ambiguous, respectively. Note that it is decidable in polynomial time whether a probabilistic automaton $\mathcal{P}$ is unambiguous, finitely ambiguous, or polynomially ambiguous [18]. Furthermore, a consequence of the results of [18] is that an automaton which is not finitely ambiguous is at least linearly ambiguous.

Emptiness problem and value. Let $\mathcal{P}$ be a probabilistic automaton and c a threshold. Following Rabin [16], we define the threshold language induced by $\mathcal{P}$ and c as:

$$L^{>c}(\mathcal{P}) = \{w \in \Sigma^* \mid \mathcal{P}(w) > c\}.$$

The *emptiness problem* asks, given a probabilistic automaton $\mathcal{P}$ and a threshold c, whether the language $L^{>c}(\mathcal{P})$ is non-empty, that is, whether there exists a word w such that $\mathcal{P}(w) > c$.

A related function problem is to compute the value of a probabilistic automaton $\mathcal{P}$, defined by $\mathrm{val}(\mathcal{P}) = \sup_{w \in \Sigma^*} \mathcal{P}(w)$. Note that the emptiness problem is equivalent to asking whether $\mathrm{val}(\mathcal{P}) > c$.

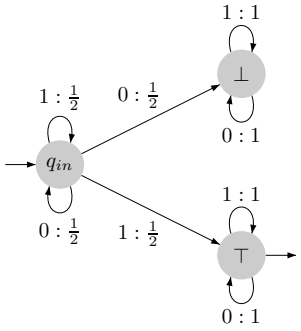


Figure 1 This probabilistic automaton computes bin.

3 Undecidability for Polynomially Ambiguous Probabilistic Automata

In this section, we show undecidability results for polynomially ambiguous probabilistic automata, which justifies the focus of our paper on finitely ambiguous probabilistic automata.

▶ **Theorem 1.** *The emptiness problem is undecidable for quadratically ambiguous probabilistic automata.*

Undecidability of the emptiness problem has long been known for general probabilistic automata [15, 2, 9]. However, the automata involved in the proof have exponential ambiguity. We show that the ideas can be reused to obtain Theorem 1. The key ingredient in the undecidability proof of Bertoni [2] (see [9] for a simple exposition of the ideas) is the construction of a probabilistic automaton computing the value of a rational number given in binary with least significant digit on the left:

$$\mathrm{bin}^R(a_1 \cdots a_n) = \sum_{i=1}^{n} \frac{a_i}{2^{n-i+1}}.$$

The automaton proposed by Bertoni has exponential ambiguity. However, it is possible to construct a linearly ambiguous probabilistic automaton computing the same function but *reversing* the input:

$$\mathrm{bin}(a_1 \cdots a_n) = \sum_{i=1}^{n} \frac{a_i}{2^{i}}.$$

The automaton is represented in Figure 1. Once this automaton is known, it is easy to prove the undecidability of the emptiness problem following the proof scheme of Bertoni, streamlined by Gimbert and Oualhadj [9]:

- First, show that the following problem is undecidable: given a linearly ambiguous probabilistic automaton $\mathcal{P}$, does there exist a word w such that $\mathcal{P}(w) = \frac{1}{2}$? The proof is by reduction from Post's Correspondence Problem (PCP), which can be defined as follows: given a pair of monoid homomorphisms $\varphi_1, \varphi_2 : \Sigma^* \to \{0,1\}^*$, does there exist a word w such that $\varphi_1(w) = \varphi_2(w)$? Using the automaton above computing bin, it is easy to construct a probabilistic automaton $\mathcal{P}$ of linear ambiguity such that $\mathcal{P}(w) - \frac{1}{2} = \frac{1}{2}\left(\mathrm{bin}(\varphi_1(w)) - \mathrm{bin}(\varphi_2(w))\right)$. Since the function bin is (essentially[1]) injective, $\mathcal{P}(w) = \frac{1}{2}$ is equivalent to $\varphi_1(w) = \varphi_2(w)$, proving the correctness of the reduction.

[1] One needs to ensure that the last letter is a 1, which is achieved using a small modification of φ_1 and φ_2

- Second, show that the emptiness problem is undecidable by reduction to the problem above. Given a linearly ambiguous probabilistic automaton $\mathcal{P}$, one can construct a quadratically ambiguous probabilistic automaton $\mathcal{P}'$ such that $\mathcal{P}'(w) = \mathcal{P}(w) \cdot (1 - \mathcal{P}(w))$. Since for x in $[0, 1]$, the following equivalence holds: $x = \frac{1}{2}$ if, and only if, $x \cdot (1 - x) \geq \frac{1}{4}$, the first undecidability result implies the undecidability of the emptiness problem for quadratically ambiguous probabilistic automata.

Given a probabilistic automaton $\mathcal{P}$, we say that a threshold c is isolated if there exists $\varepsilon > 0$ such that for all words w, we have $|\mathcal{P}(w) - c| > \varepsilon$. Rabin [16], proved that if a threshold c is isolated then the corresponding language $L^{\geq c}(\mathcal{P})$ is regular. The isolation problem asks to determine whether a given threshold is isolated for a given automaton. This problem was shown to be undecidable by Bertoni [2]. We can refine the result of [2] to obtain:

▶ **Theorem 2.** *The isolation problem is undecidable for linearly ambiguous probabilistic automata.*

Proof. We construct a reduction from a variant of the Post Correspondence's Problem, called the infinite PCP, and shown to be undecidable in [17]. The problem asks, given two homomorphisms $\varphi_1, \varphi_2 : \Sigma^* \to \{0, 1\}^*$, does there exist an infinite word w in Σ^ω such that $\varphi_1(w) = \varphi_2(w)$ (where φ_1, φ_2 are extended to continuous Σ^ω maps on with respect to product topology). We first observe that equivalently, we ask whether for every $\varepsilon > 0$ there exists w in Σ^* such that $|\mathrm{bin}(\varphi_1(w)) - \mathrm{bin}(\varphi_2(w))| \leq \varepsilon$.

Indeed, if there exists an infinite word w such that $\varphi_1(w) = \varphi_2(w)$, then the sequences obtained by considering the images under φ_1 and φ_2 of prefixes of w have arbitrarily long common prefixes, so the difference of their binary values converges to 0. Conversely, assume that for any $\varepsilon > 0$ there exists a finite word w such that $|\mathrm{bin}(\varphi_1(w)) - \mathrm{bin}(\varphi_2(w))| \leq \varepsilon$, then we construct a solution to the infinite PCP using König's lemma. To this end, for each n let w_n be a finite word such that $|\mathrm{bin}(\varphi_1(w_n)) - \mathrm{bin}(\varphi_2(w_n))| < 2^n$, *i.e.*, such that $\varphi_1(w_n)$ and $\varphi_2(w_n)$ coincide on the first n letters. Applying König's Lemma to the infinite tree defined by the prefix closure of the set $\{w_n \mid n \geq 0\}$ (*i.e.*, each node in the tree is the prefix of some word w_n), there exists an infinite word w such that $\varphi_1(w) = \varphi_2(w)$.

We now construct the reduction from the infinite PCP to the isolation problem for linearly ambiguous probabilistic automata. Given two homomorphisms φ_1 and φ_2 we construct the linearly ambiguous probabilistic automaton $\mathcal{P}$ such that for every w in Σ^*,

$$\mathcal{P}(w) = \frac{1}{2}\left(\mathrm{bin}(\varphi_1(w)) + 1 - \mathrm{bin}(\varphi_2(w))\right).$$

Then[2] for every $\varepsilon > 0$ there exists w in Σ^* such that $|\mathrm{bin}(\varphi_1(w)) - \mathrm{bin}(\varphi_2(w))| \leq \varepsilon$ if, and only if, $|\mathcal{P}(w) - \frac{1}{2}| \leq \varepsilon$. ◀

An automaton is either finitely ambiguous, or at least linearly ambiguous. We proved undecidability results for linearly and quadratically ambiguous automata; the focus of the present paper is on decidability results for finitely ambiguous automata.

4 Decidability and Complexity of Finitely Ambiguous Probabilistic Automata

In this section we study threshold languages and the emptiness problem for finitely ambiguous probabilistic automata. We start by showing regularity of the threshold language $L^{>c}(\mathcal{P})$ for

[2] The same trick as for the proof above is required, to ensure that bin is injective.

a finitely ambiguous probabilistic automaton $\mathcal{P}$ and threshold c. A classical result, due to Rabin [16], shows that the threshold languages need not be regular in general. Unfortunately the proof of regularity, while constructive, is not useful for determining the complexity of the emptiness problem. However we are able to give a direct simple argument that bounds the length of witnesses for the emptiness problem. We then use these bounds to analyse the complexity of the emptiness problem.

▶ **Theorem 3.** *Let $\mathcal{P}$ be a finitely ambiguous probabilistic automaton and c a threshold. Then $L^{>c}(\mathcal{P})$ is a regular language.*

Proof. Consider the set $\mathbb{N}^k$ under the pointwise order. Recall that $I \subseteq \mathbb{N}^k$ is an *ideal* if it is downward closed and directed. Every ideal I has the form

$$I = \{(n_1, \ldots, n_k) \in \mathbb{N}^k : n_{i_1} \leq a_1 \wedge \ldots \wedge n_{i_s} \leq a_s\} \tag{1}$$

for certain indices $1 \leq i_1 < \ldots < i_s \leq k$ and natural numbers $a_1, \ldots, a_s$. From the fact that $\mathbb{N}^k$ is a well-quasi-order it follows that every downward closed subset of $D \subseteq \mathbb{N}^k$ can be written as a finite union of ideals. Indeed such a decomposition can easily be computed from the finite set of minimal elements of $\mathbb{N}^k \setminus D$ [12].

Let $\mathcal{P} = (Q, q_{in}, \Delta, F)$ be a finitely ambiguous probabilistic automaton with transition function $\Delta : Q \times \Sigma \to \mathcal{D}(Q)$. Say that a triple $(p, a, q) \in Q \times \Sigma \times Q$ is an *edge* if $\Delta(p, q)(q) > 0$. Suppose that $\mathcal{P}$ has s edges for some $s \in \mathbb{N}$ and fix a linear ordering on these edges. We say that $m = (m_{i,j}) \in \mathbb{N}^{s \times k}$ is *admissible* for a word $w \in \Sigma^*$ if there exist k distinct accepting runs of $\mathcal{P}$ on w such that $m_{i,j}$ is the number of times that the i-th edge is taken in the j-th accepting run.

For any ideal $I \subseteq \mathbb{N}^{s \times k}$ the set of $w \in \Sigma^*$ such that some $m \in I$ is admissible for w is a regular language. A non-deterministic automaton for this language guesses k distinct accepting runs of $\mathcal{P}$ and counts the number of times each edge is taken on each accepting run, up to a finite threshold N, where N is the largest integer appearing in the description of I in the form (1). It follows that for any downward closed subset $D \subseteq \mathbb{N}^{s \times k}$ the set of $w \in \Sigma^*$ such that some $m \in D$ is admissible for w is a regular language.

Now let $\lambda_1, \ldots, \lambda_s$ be the transition probabilities occuring in $\mathcal{P}$, listed according to the ordering on the edges. Given $k \in \mathbb{N}$, consider the set of tuples

$$S_k \;=\; \left\{ (m_{i,j}) \in \mathbb{N}^{s \times k} \;:\; \sum_{j=1}^{k} \lambda_1^{m_{1,j}} \ldots \lambda_s^{m_{s,j}} > c \right\}.$$

For any word $w \in \Sigma^*$, $w \in L^{>c}(\mathcal{P})$ if and only if there exists some k up to the (finite) degree of ambiguity of $\mathcal{P}$ and some $m \in S_k$ that is admissible for w. Since each set S_k is downwards closed, it follows that $L^{>c}(\mathcal{P})$ is regular. ◀

The threshold language $L^{>c}(\mathcal{P})$ of a finitely ambiguous probabilistic automaton is regular, however, this does not say anything on how to decide efficiently whether $L^{>c}(\mathcal{P})$ is empty or not. Therefore, the next step is to bound the size of a witness word whenever $L^{>c}(\mathcal{P}) \neq \emptyset$. This will lead to upper bounds on the complexity of the emptiness problem restricted to k-ambiguous.

▶ **Lemma 4.** *Let $\mathcal{P}$ be a k-ambiguous probabilistic automaton with n states. For every word w, there exists a word w' of length at most n^k such that $\mathcal{P}(w) \leq \mathcal{P}(w')$.*

This implies that the value of $\mathcal{P}$ is reached by some word of length at most n^k.

Proof. Let $\mathcal{P} = (Q, q_{in}, \Delta, F)$ and suppose that there are exactly k' accepting runs on w for some $k' \leq k$. If w has length strictly greater than $n^{k'}$ then there exists a factorization $w = xyz$ for $x, y, z \in \Sigma^*$, with y non-empty and xz of length at most n^k, such that for each of the accepting runs on w, the infix corresponding to the factor y starts and ends in the same state. Then we have

$$\begin{array}{rcl} \mathcal{P}(w) & = & \sum_{q \in F} \sum_{p \in Q} \mathcal{P}(q_{in} \xrightarrow{x} p)\mathcal{P}(p \xrightarrow{y} p)\mathcal{P}(p \xrightarrow{z} q) \\ & \leq & \sum_{q \in F} \sum_{p \in Q} \mathcal{P}(q_{in} \xrightarrow{x} p)\mathcal{P}(p \xrightarrow{z} q) \\ & = & \mathcal{P}(xz)\,. \end{array}$$

◀

Note that if k is fixed, then the size of a witness for $L^{>c}(\mathcal{P})$ is polynomial in the size of the automaton (i.e. n^k where n is the number of states of $\mathcal{P}$). Unfortunately, it has been shown in [18] that the ambiguity of a finitely ambiguous automata can be exponential in the number of states and, thus, the previous lemma gives a double exponential bound for a witness of $L^{>c}(\mathcal{P})$ when k is not fixed. The next result shows that the size of a witness is at most exponential in the number of states.

▶ **Theorem 5.** *Let $\mathcal{P}$ be a finitely ambiguous probabilistic automaton with n states. For every word w, there exists a word w' of length at most $n!$ such that $\mathcal{P}(w) \leq \mathcal{P}(w')$.*

This implies that the value of $\mathcal{P}$ is reached by some word of length at most $n!$.

Proof. Consider a word $w = a_1 \cdots a_\ell$ of length at least $n!$. For any position i over the runs of $\mathcal{P}$ over w, denote R_i the set of states participating in at least one accepting run over w. Furthermore, we equip R_i with the order defined by $p \leq q$ if $\mathcal{P}(q_0 \xrightarrow{a_1 \cdots a_i} p) \leq \mathcal{P}(q_0 \xrightarrow{a_1 \cdots a_i} q)$, *i.e.* the probability of reading the prefix of w until position i reaches p with smaller probability than q (ties are resolved in a consistent way).

Since w has length at least $n!$, there exist two positions $i < j$ such that the ordered sets R_i and R_j coincide, denoted by R, and there exists a factorization $w = xyz$, with y the word in between positions i and j. Then we look at the runs of y from R to R, and make the following claims:

1. For every $p \in R$, there exists a run over y from p to a state in R.
2. For every $p \in R$, there exists at most one run over y from p to a state in R.
3. For every $p \in R$, we have $\mathcal{P}(q_0 \xrightarrow{uv} p) \leq \mathcal{P}(q_0 \xrightarrow{u} p)$.

The first claim follows from the fact that R is the set of states participating in at least one accepting run over w. For the second claim, if this would not be the case, then the number of runs from R to R would increase unboundedly, contradicting that $\mathcal{P}$ is finitely ambiguous. Then it follows that for any state $p \in R$ there exists a unique run over y from p to some state in R, which is written p'.

For the last item, pick a state $p \in R$ and note that $\mathcal{P}(q_0 \xrightarrow{xy} p') = \mathcal{P}(q_0 \xrightarrow{x} p) \cdot \mathcal{P}(p \xrightarrow{y} p') \leq \mathcal{P}(q_0 \xrightarrow{x} p)$. This reduce the analysis to two cases. On one hand, $p \leq p'$ and then $\mathcal{P}(q_0 \xrightarrow{xy} p) \leq \mathcal{P}(q_0 \xrightarrow{xy} p') \leq \mathcal{P}(q_0 \xrightarrow{x} p)$. On the other hand, $p > p'$ and then there exists a state q in R such that $q \leq p$ and $p \leq q'$. This is because for any state $r \in R$ there exists a unique run over y from r to some state in R. It follows that $\mathcal{P}(q_0 \xrightarrow{xy} p) \leq \mathcal{P}(q_0 \xrightarrow{xy} q') \leq \mathcal{P}(q_0 \xrightarrow{x} q) \leq \mathcal{P}(q_0 \xrightarrow{x} p)$.

Finally, the proof of the theorem follows from the last claim (see Lemma 4). ◀

With the previous bounds in hand, we can study the computational complexity of the emptiness problem for various classes of finitely ambiguous probabilistic automata. For each fixed positive integer k we consider the class of k-ambiguous probabilistic automata. More

generally, we can let the ambiguity of an automaton depend on the number n of states: we consider for each fixed polynomial p the class of all automata that have ambiguity at most $p(n)$. We call this the class of automata of p-bounded ambiguity. More generally still, we have the class of all finitely ambiguous probabilistic automata. (Recall that the ambiguity can be exponential in the number of states in general.)

▶ **Theorem 6.**

- *For each fixed positive integer k, the emptiness problem for the class of k-ambiguous probabilistic automata is in* **NP**.
- *For each fixed polynomial p, the emptiness problem for the class of probabilistic automata with p-bounded ambiguity is in* **PSPACE**. *This problem is* **PSPACE**-*hard already in case $p(n) = n$.*
- *The emptiness problem for the class of finitely ambiguous probabilistic automata is in* **NEXPTIME** *and is* **PSPACE**-*hard.*

Proof. The algorithm for all three cases exploits Lemma 4 and Theorem 5 to guess and check a word witnessing that threshold language is non-empty.

For k-ambiguous we know by Lemma 4 that a witness for checking whether $L^{>c}(\mathcal{P}) \neq \emptyset$ is of polynomial size in $\mathcal{P}$ and, therefore, we can guess a polynomial size word w and check if $\mathcal{P}(w) \geq c$, that is, the problem is in **NP**.

Similarly, for finitely ambiguous we know by Theorem 5 that the witness is of size at most exponential, so we can guess and check if $L^{>c}(\mathcal{P})$ is non-empty in **NEXPTIME**.

To show that emptiness is in **PSPACE** for probabilistic automata of p-bounded ambiguity, one can guess a word w "on the fly" of size exponential and check whether $\mathcal{P}(w) \geq c$. The problem here is that the value $\mathcal{P}(w)$ (written in binary) could be of size exponential in the size of $\mathcal{P}$. To check if $\mathcal{P}(w) \geq c$ with polynomial space one can guess w, and keep a set of counters $\{c_t^i\}$ that stores how many times each transition t is used on the i-th run of $\mathcal{P}$ over w. Since w is of size at most exponential and $\mathcal{P}$ has at most $p(n)$ accepting runs, then we need polynomially many counters, each with at most polynomially many bits, namely, polynomial space to store these counters during the simulation of $\mathcal{P}$ over w. After we conclude guessing w, we can construct a polynomial-size circuit that receives $\{c_i^t\}$ and outputs $\mathcal{P}(w)$. Checking that the value of the circuit is greater or equal than a constant c correspond to decide PosSLP which can be solved in **PSPACE** [1].

Next we consider a fixed polynomial $p(n) = n$, and prove **PSPACE**-hardness of emptiness for the class of probabilistic automata of $p(n)$-bounded ambiguity. The proof is by reduction from the emptiness problem of the intersection of a finite collection of deterministic finite automata: given as input a collection of deterministic finite automata, does there exist a word accepted by each of them? This problem has been shown **PSPACE**-complete in [11]. Given N deterministic automata, we construct a probabilistic automaton $\mathcal{P}$ whose first letter leads with probability $\frac{1}{N}$ to the initial state of each automaton. The probabilistic automaton $\mathcal{P}$ is N-ambiguous (note that N is at most the number of states of $\mathcal{P}$), and there exists a word w such that $\mathcal{P}(w) = 1$ if, and only if, there exists a word accepted by each of the N deterministic automata. ◀

The aim of the last section is to give better algorithms for the k-ambiguous case: in particular, we show that the emptiness problem is in quasi polynomial time for 2-ambiguous probabilistic automata.

5 Algorithms and Approximations for Finitely Ambiguous Probabilistic Automata

This section is devoted to the construction of algorithms for both the emptines problem and approximating the value of finitely ambiguous probabilistic automata. The first step is a reduction to a multi-objective optimisation problem that we call the stochastic path problem. We construct algorithms for this problem relying on recent progress on the literature of multi-objective optimisation problems, and thus obtain algorithms for finitely ambiguous probabilistic automata.

5.1 The Stochastic Path Problem

The stochastic path problem is an optimisation problem on multi-weighted graphs. It is parametrised by a positive integer constant k, giving rise to the k-stochastic path problem, and the bi-stochastic path problem for $k = 2$. An instance is a triple consisting of an acyclic k-weighted graph G and two vertices s and t. A k-weighted graph is given by a set of vertices V of size n and a set of weighted edges $E \subseteq V \times (\mathbb{Q} \cap [0,1])^k \times V$. Note that the same pair of vertices (v, v') can have different edges between them and the weight of an edge is a k-tuple of rational numbers between 0 and 1.

A path π in G is a sequence of consecutive edges, and the set of feasible solutions of the problem are all paths from s to t. We denote by $(p_1(\pi), \ldots, p_k(\pi))$ the component-wise product of the weight vectors along the edges of π, namely, weights are computed multiplicatively along each component. In our applications we think of each component of a weight vector of an edge as the probability of a single event, and each component of a weight vector of a path as the probability of a sequence of events. The value of the path π, denoted by $\mathrm{val}(\pi)$, is obtained by summing each component of the weight vector of the path, namely, $\mathrm{val}(\pi) = \sum_{i=1}^{k} p_i(\pi)$.

As a running example, on the left-hand side of Figure 2 we represent an instance of the bi-stochastic path problem. There are five paths from s to t, and their values are plotted in the right-hand side. For instance, the path s, p, q, t using the left edge from p to q has weight $(.4 \times .9 \times .9,\ .6 \times .1 \times .9) = (.324, .054)$.

The objective of the k-stochastic path problem is to maximize the *sum* of the objective functions, namely, $\mathrm{val}(\pi)$. The value of the above path is $.324 + .054 = .378$. Finally, the decision problem associated with the k-stochastic path problem is the following:

> **The k-stochastic path problem**: given a k-weighted graph G, two vertices s and t and a threshold c in $\mathbb{Q} \cap [0,1]$, does there exist a path π from s to t in G whose value is at least c, *i.e.* such that $\mathrm{val}(\pi) \geq c$?

Towards finding efficient algorithms and approximations of k-ambiguous probabilistic automata, we show a polynomial time reduction from the emptiness problem of k-ambiguous probabilistic automata to the k-stochastic path problem. Intuitively, the reduction consists in constructing the powerset graph of the paths, restricting to at most k paths.

▶ **Lemma 7.** *The emptiness problem of a k-ambiguous probabilistic automaton $\mathcal{P}$ reduces in polynomial time to a k-stochastic path problem $(G_{\mathcal{P}}, s, t)$. In particular, the reduction satisfies that*

1. *for any word w there exists a path π in G from s to t such that $\mathcal{P}(w) \leq val(\pi)$,*
2. *for any path π in G from s to t there exists a word w such that $val(\pi) \leq \mathcal{P}(w)$.*

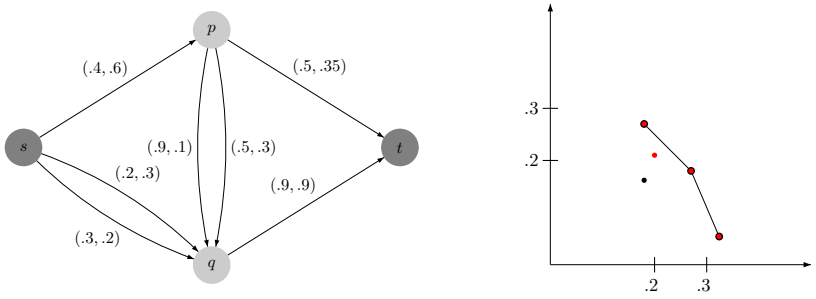


Figure 2 An instance of the bi-stochastic path problem on the left, and the values of all paths from s to t on the right. The four red dots are the Pareto curve, and the three connected red dots the convex Pareto curve.

Proof. Let $\mathcal{P} = (Q, q_{in}, \Delta, F)$ be a k-ambiguous probabilistic automaton with n states. The set of vertices of the k-weighted graph $G_{\mathcal{P}}$ is defined as $Q^k \times \{0, \dots, n^k\} \times \{0,1\}^{k \times k}$ where $\{0,1\}^{k \times k}$ is the set of $k \times k$ matrices over $\{0,1\}$, plus a special source vertex s and a special target vertex t.

Intuitively, being in the vertex $((q_1, \dots, q_k), \ell, M)$ means that we are simulating k runs which are now in the states $(q_1, \dots, q_k)$, that the run so far has length ℓ, and the matrix M indicates which of two k runs are different: $M(i,j) = 1$ if, and only if, the i-th run is different from the j-th run.

The set of edges is defined accordingly to the previous explanation as follows. For the source vertex, there is an edge from s to $((q_{in}, \dots, q_{in}), 0, 0)$ with weight $(1, \dots, 1)$, where 0 is the zero matrix. There is an edge from $((q_1, \dots, q_k), \ell, M)$ to $((q'_1, \dots, q'_k), \ell + 1, M')$ with weight $(p_1, \dots, p_k)$ if there exists a letter a in Σ such that for each $i \in \{1, \dots, k\}$ we have $\Delta(q_i, a)(q'_i) = p_i$, and $M'(i,j) = 1$ if, and only if, $M(i,j) = 1$ or $q'_i \neq q'_j$. Finally, there is an edge from $((q_1, \dots, q_k), \ell, M)$ to t with weight $(p_1, \dots, p_k)$ where for each $i \in \{1, \dots, k\}$ we have $p_i = 1$ if $q_i \in F$ and $M(i,j) = 1$ for every $j < i$, and $p_i = 0$ otherwise. Note that $G_{\mathcal{P}}$ is acyclic and of polynomial size given that k is a fixed value.

We prove the correctness of the construction. Let w be a word. Thanks to Lemma 4, we can assume without loss of generality that w has length at most n^k. Its set of accepting runs induces a path π in $G_{\mathcal{P}}$ from s to t with $\mathrm{val}(\pi) = \mathcal{P}(w)$. Conversely, a path π in $G_{\mathcal{P}}$ from s to t corresponds to a set of accepting runs for some word w with $\mathrm{val}(\pi) \leq \mathcal{P}(w)$. ◀

By the previous result, we can see that the emptiness problem of k-ambiguous probabilistic automata is closely related to the k-stochastic path problem. In the following we use this problem as a proxy to give approximation and efficient algorithms for the emptiness problem.

5.2 Approximating the Value in Polynomial Time

Multi-objective optimisation problems have long been studied; see Papadimitriou and Yannakakis [14] and Diakonikolas and Yannakakis [6] among many others. Since there is typically no single best solution, a natural notion for multi-objective optimisation problems is *Pareto curves*, which is the set of *undominated* solutions. To make things concrete, we illustrate the notion of Pareto curves on the k-stochastic path problem. We fix an instance (G, s, t) of the k-stochastic path problem. A Pareto curve is a set of paths $\mathcal{P}$ such that for every path π, there exists a path π' in $\mathcal{P}$ dominating π, *i.e.* such that for all i in $\{1, \dots, k\}$, we have

$p_i(\pi) \leq p_i(\pi')$. In Figure 2, we can see that the Pareto curve of our running example is given by the four red dots. Here dominating means being to the right and higher, so only one path is dominated by others. Unfortunately, the size of Pareto curves in discrete multi-objective optimisation problems is exponential in the worst case. Hence the introduction of two relaxations: convex and approximate Pareto curves.

A *convex Pareto curve* is a set of paths $\mathcal{C}$ such that for every path π, there exists a family of paths $\pi_1, \ldots, \pi_m \in \mathcal{C}$ such that π is dominated by a convex combination of $\pi_1, \ldots, \pi_m$ in the sense that there exist non-negative coefficients $\lambda_1, \ldots, \lambda_m$ that sum to 1 such that $p_i(\pi) \leq \sum_j \lambda_j p_i(\pi_j)$ for all components i in $\{1, \ldots, k\}$.

Convex Pareto curves have been studied in a general setting by Diakonikolas and Yannakakis [6]. They are in general smaller than Pareto curves, yielding efficient algorithms for convex optimisation problems.

In Figure 2, there exists a convex Pareto curve consisting of only three paths, the fourth one being dominated a convex combination of two other paths. The figure connects the three dots, showing which is sometimes called the Pareto front.

Fix $\varepsilon > 0$, an *ε-Pareto curve* is a set of paths $\mathcal{C}$ such that for every path π, there exists a path π' in $\mathcal{C}$ such that for all i in $\{1, \ldots, k\}$, we have $p_i(\pi) \leq (1 + \varepsilon) \cdot p_i(\pi')$.

The notion of approximate Pareto curves is very appealing in our case for two reasons: first, knowing an approximate Pareto curve usually gives an approximately optimal solution, and second, a very general result of Papadimitriou and Yannakakis [14] shows that in most multi-objective optimisation problems, there exists a polynomially succinct approximate Pareto curve.

The two relaxations are combined to give rise to the notion of *ϵ-convex Pareto curves*: it is a set of paths $\mathcal{C}$ such that for every path π, there exists a family of paths $\pi_1, \ldots, \pi_m \in \mathcal{C}$ such that there exist non-negative coefficients $\lambda_1, \ldots, \lambda_m$ that sum to 1 such that $p_i(\pi) \leq (1 + \epsilon) \sum_j \lambda_j p_i(\pi_j)$ for all components i in $\{1, \ldots, k\}$.

The following result shows the way to find an ϵ-approximation of the value of a k-ambiguous probabilistic automaton $\mathcal{P}$.

▶ **Theorem 8.** *For any fixed k, there exists a polynomial time algorithm which given an instance of the k-stochastic path problem and $\varepsilon > 0$, returns an ε-convex Pareto curve in time polynomial in the instance and $\frac{1}{\varepsilon}$.*

Proof. We rely on general results of Papadimitriou and Yannakakis [14], which give a sufficient condition for the existence of a polynomial time algorithm constructing an ε-convex Pareto curve in time polynomial in the instance and $\frac{1}{\varepsilon}$: it is enough to construct a pseudo-polynomial time algorithm solving the exact version in pseudo-polynomial time. Recall here that an algorithm is pseudo-polynomial if it runs in polynomial time when the numerical inputs are given in unary.

In our case, the exact k-stochastic path problem reads: given an instance (G, s, t) and a value c in $[0, 1] \cap \mathbb{Q}$, does there exist a path π in G from s to t such that $\sum_{i \in \{1,\ldots,k\}} p_i(\pi) = c$? If all transition probabilities are written using B bits, then it is enough to consider paths such that each weight uses at most $|V| \cdot B$ bits with V the set of vertices of G. Hence one can fill in a polynomially large table indexed by $(p, q, p_1, \ldots, p_k)$, which checks for the existence of a path from p to q of weights $(p_1, \ldots, p_k)$ using at most $|V| \cdot B$ bits. ◀

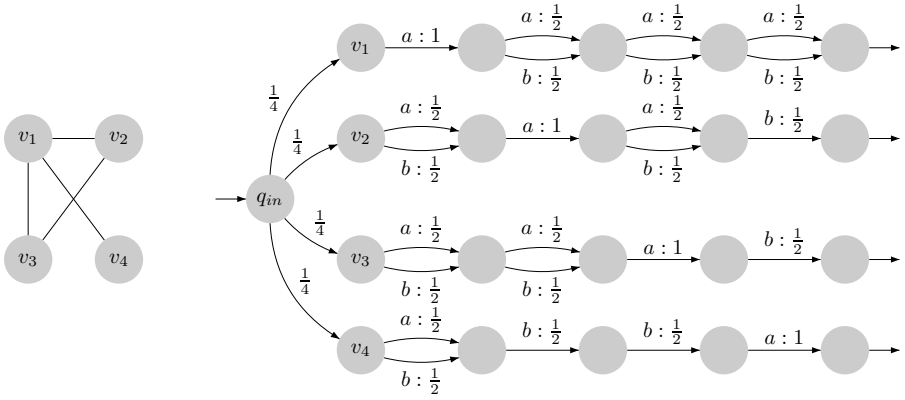


Figure 3 On the left a graph G and on the right the corresponding finitely ambiguous probabilistic automaton $\mathcal{P}$ such that $\text{MaxClique}(G) = 4 \cdot 2^3 \cdot \text{val}(\mathcal{P})$.

Interestingly, the algorithm of Theorem 8 for the k-stochastic path problem yields a polynomial time algorithm to approximate the value of a k-ambiguous probabilistic automaton. Recall that the value of a probabilistic automaton $\mathcal{P}$ is defined by $\text{val}(\mathcal{P}) = \sup_{w \in \Sigma^*} \mathcal{P}(w)$.

▶ **Theorem 9.** *For a fixed k, there exists an algorithm which given a k-ambiguous probabilistic automaton and $\varepsilon > 0$, outputs an ε-approximation of the value in time polynomial in the size of the automaton and $\frac{1}{\varepsilon}$, i.e. a value Output such that*

$$\mathit{Output} \leq \mathit{val}(\mathcal{P}) \leq (1+\varepsilon) \cdot \mathit{Output}.$$

Proof. Given a k-ambiguous probabilistic automaton $\mathcal{P}$, the algorithm for finding an ϵ-approximation of $\text{val}(\mathcal{P})$ is as follows:

1. construct an instance $(G_{\mathcal{P}}, s, t)$ of the k-stochastic path problem using Lemma 7.
2. construct an ε-convex Pareto curve $\mathcal{C}$ for $(G_{\mathcal{P}}, s, t)$ thanks to Theorem 8.
3. output $\text{Output} = \max_{\pi \in \mathcal{C}} \sum_{i \in \{1,\dots,k\}} p_i(\pi)$.

A direct application of Lemma 7 shows that Output is an ϵ-approximation of $\text{val}(\mathcal{P})$. ◀

Can we approximate the value of any finitely ambiguous probabilistic automaton in polynomial time? Unfortunately, by reformulating the hardness result of [13] (that paper uses a different framework), we can give a negative answer to this question, justifying the relevance of Theorem 9.

▶ **Theorem 10** ([13]). *For every $\varepsilon > 0$, there is no polynomial time algorithm computing the value of finitely ambiguous probabilistic automata up to a factor $O(n^{\frac{1}{2}-\varepsilon})$ unless $\mathbf{P} = \mathbf{NP}$.*

The paper [13] constructs a reduction from the size of the maximum clique, for which we know that no polynomial time approximation algorithm exists unless $\mathbf{P} = \mathbf{NP}$. The construction is given in [13] for Hidden Markov models; we illustrate in Figure 3 how to adapt it to probabilistic automata.

Given a graph G with n vertices, we construct a finitely ambiguous probabilistic automaton $\mathcal{P}$ with n^2 states such that for each m smaller than n, the automaton accepts a word with probability at least $\frac{m}{n2^{n-1}}$ if, and only if, the graph contains a clique of size at least m.

We give an intuitive explanation for the construction. A word over $\{a, b\}^*$ represents a set of vertices in the graph; a means in the set and b outside. For instance, the word $aaab$

represents the set of vertices $\{v_1, v_2, v_3\}$, it has probability $\frac{3}{4 \cdot 2^3}$. The automaton $\mathcal{P}$ on input w has n runs, one for each vertex, chosen each with probability $\frac{1}{n}$. Each accepting run has probability $\frac{1}{2^{n-1}}$, and the run corresponding to a vertex v is successful if, and only if, v and all its neighbours belong to the set of vertices represented by the word w. Hence a clique of size m induces a word accepted with probability $\frac{m}{n2^{n-1}}$, and conversely.

Let MaxClique(G) denote the size of the largest clique in G, the equivalence above reads MaxClique$(G) = n2^{n-1}\text{val}(\mathcal{P})$. It follows that a $K(n)$-approximation algorithm for the value of finitely ambiguous probabilistic automata induces a $K(n^2)$-approximation algorithm for the size of the largest clique. It has been proved that MaxClique(G) cannot be approximated within a factor better than $O(n^{1-\varepsilon})$ for every $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$ [19], implying our result.

5.3 A Quasi-Polynomial Time Algorithm for 2-ambiguous Probabilistic Automata

The previous results show that one can always ϵ-approximate the value of k-ambiguous probabilistic automaton. This is however not enough to decide the emptiness problem. On this direction, Theorem 6 shows that for any fixed k the emptiness problem of k-ambiguous probabilistic automata is in **NP**. We show that for $k = 2$ there exists a quasi-polynomial time algorithm for the emptiness problem. For this, we start by showing a quasi-polynomial time algorithm for the bi-stochastic path problem.

▶ **Theorem 11.** *There exists an algorithm which given an instance of the bi-stochastic path problem, returns a convex Pareto curve in quasi-polynomial time.*

The advantage of using $k = 2$ relies on the existence of a quasi-polynomial bound on the size of convex Pareto curves. More precisely, if (G, s, t) is an instance of the bi-stochastic path problem with n vertices, then it can be shown that there exists a convex Pareto curve of size at most $n^{\log(n)}$. This result was proved by Gusfield in his PhD thesis [10], and a matching lower bound was developed by Carstensen [3]. Note that they use a different framework, called parametric optimisation: in the parametric shortest path problem each edge has cost $c + \lambda d$, where λ is a parameter. The length of the shortest path is a piecewise linear concave function of λ, whose pieces correspond to the vertices of the convex Pareto curve for the bi-objective shortest path problem with weights (c, d). It is then easy to obtain an upper bound on the size of convex Pareto curves for the bi-stochastic path problem by reducing it to a bi-objective shortest path problem, mapping the weights (p, q) to $(-\log(p), -\log(q))$. Finally, the upper bound on the size of convex Pareto curves yields a quasi-polynomial time algorithm, by constructing them in a standard divide-and-conquer manner.

The algorithm of Theorem 11 yields a quasi-polynomial time algorithm for the emptiness problem of 2-ambiguous probabilistic automata.

▶ **Theorem 12.** *There exists a quasi-polynomial time algorithm for the emptiness problem of 2-ambiguous probabilistic automata.*

Proof. Given a 2-ambiguous probabilistic automaton $\mathcal{P}$ and a threshold c, the algorithm for deciding the emptiness of $\mathcal{P}$ is the following:

- construct an instance $(G_{\mathcal{P}}, s, t)$ of the bi-stochastic path problem using Lemma 7.
- construct a convex Pareto curve $\mathcal{C}$ for $(G_{\mathcal{P}}, s, t)$ thanks to Theorem 11.
- check whether Output $= \max_{\pi \in \mathcal{C}} \sum_i p_i(\pi) > c$.

A direct application of Lemma 7 shows that Output $= \text{val}(\mathcal{P})$. ◀

We do not know whether there exist quasi-polynomial time algorithms for every $k > 2$, and leave this as an open problem.

References

1. Eric Allender, Peter Bürgisser, Johan Kjeldgaard-Pedersen, and Peter Bro Miltersen. On the complexity of numerical analysis. *SIAM Journal on Computing*, 38(5):1987–2006, 2009.
2. Alberto Bertoni. The solution of problems relative to probabilistic automata in the frame of the formal languages theory. In *GI Jahrestagung*, pages 107–112, 1974.
3. Patricia June Carstensen. *The Complexity of Some Problems in Parametric Linear and Combinatorial Programming*. PhD thesis, University of Michigan, 1983.
4. Rohit Chadha, A. Prasad Sistla, and Mahesh Viswanathan. Emptiness under isolation and the value problem for hierarchical probabilistic automata. In *FoSSaCS*, pages 231–247, 2017.
5. Krishnendu Chatterjee and Mathieu Tracol. Decidable problems for probabilistic automata on infinite words. In *LICS*, pages 185–194, 2012.
6. Ilias Diakonikolas and Mihalis Yannakakis. Succinct approximate convex pareto curves. In *SODA*, pages 74–83, 2008.
7. Nathanaël Fijalkow, Hugo Gimbert, Edon Kelmendi, and Youssouf Oualhadj. Deciding the value 1 problem for probabilistic leaktight automata. *Logical Methods in Computer Science*, 11(2), 2015.
8. Nathanaël Fijalkow, Hugo Gimbert, and Youssouf Oualhadj. Deciding the value 1 problem for probabilistic leaktight automata. In *LICS*, pages 295–304, 2012.
9. Hugo Gimbert and Youssouf Oualhadj. Probabilistic automata on finite words: Decidable and undecidable problems. In *ICALP (2)*, pages 527–538, 2010.
10. Daniel Mier Gusfield. *Sensitivity Analysis for Combinatorial Optimization*. PhD thesis, University of California, Berkeley, 1980.
11. Dexter Kozen. Lower bounds for natural proof systems. In *FOCS*, pages 254–266, 1977.
12. Ranko Lazić and Sylvain Schmitz. The ideal view on Rackoff's coverability technique. In *International Workshop on Reachability Problems*, pages 76–88. Springer, 2015.
13. Rune B. Lyngsø and Christian N. S. Pedersen. The consensus string problem and the complexity of comparing hidden Markov models. *Journal of Computer and System Sciences*, 65(3):545–569, 2002.
14. Christos H. Papadimitriou and Mihalis Yannakakis. On the approximability of trade-offs and optimal access of web sources. In *FOCS*, pages 86–92, 2000.
15. Azaria Paz. *Introduction to Probabilistic Automata*. Academic Press, 1971.
16. Michael O. Rabin. Probabilistic automata. *Information and Control*, 6(3):230–245, 1963.
17. Keijo Ruohonen. Reversible machines and Post's correspondence problem for biprefix morphisms. *Elektronische Informationsverarbeitung und Kybernetik*, 21(12):579–595, 1985.
18. Andreas Weber and Helmut Seidl. On the degree of ambiguity of finite automata. *Theoretical Computer Science*, 88(2):325–349, 1991.
19. David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(1):103–128, 2007.

Two Lower Bounds for BPA*†

Mingzhang Huang[1] and Qiang Yin[2]

1 **BASICS, Shanghai Jiao Tong University, Shanghai, China**
`mingzhanghuang@gmail.com`
2 **BDBC, Beihang University, Beijing, China**
`yinqiang@buaa.edu.cn`

Abstract

Branching bisimilarity of normed Basic Process Algebra (nBPA) was claimed to be EXPTIME-hard in previous papers without any explicit proof. Recently it has been pointed out by Petr Jančar that the claim lacked proper justification. In this paper, we develop a new complete proof for the EXPTIME-hardness of branching bisimilarity of nBPA. We also prove that the associated regularity problem of nBPA is PSPACE-hard. This improves previous P-hard result.

1998 ACM Subject Classification F.4.2 Grammars and Other Rewriting Systems

Keywords and phrases BPA, branching bisimulation, regularity

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.20

1 Introduction

Equivalence checking is a core issue of verification. It asks whether two systems are related by a specific equivalence. Baeten, Bergstra and Klop proved a remarkable result that strong bisimilarity between context free grammars without empty production is decidable [1]. This result was surprising because it seems to contradict the well-known fact that language equivalence between these grammars is undecidable. Context free grammars without empty production can be regarded as normed Basic Process Algebra (nBPA) processes. Moreover, their work extended the decidability result of bisimulation equivalence from finite state systems to infinite state systems. Extensive work has appeared since their inspiring paper, dealing with decidability or complexity issues of checking bisimulation equivalences on various infinite state systems (see a survey [11] and an updated overview [18] on this topic).

The decidability of weak bisimilarity over BPA is one of the central open problems. Although this problem is generally believed to be decidable, so far there is no effective method to handle the difficulties caused by arbitrary silent transitions. We do not know if the weak bisimilarity is decidable or not even for normed BPA. Mayr showed the weak bisimilarity and regularity problem of general BPA are EXPTIME-hard [13]. The regularity problem asks if a BPA process is bisimilar to some (unspecific) finite state process. It is noteworthy that Kiefer showed that the strong bisimilarity problem is already EXPTIME-hard by constructing a reduction from the Hit-or-Run game [10].

The normed case seems easier than the general one. Fu proved that branching bisimilarity, a standard refinement of weak bisimilarity, is decidable on nBPA [5]. He also extended the decidability result to the associated regularity problem. Recently Czerwiński and Jančar

* This work was done in part while Qiang Yin was studying at BASICS, Shanghai Jiao Tong University. The full version is available at `http://basics.sjtu.edu.cn/~yin`.

† Minzhang Huang is supported in part by NSFC (61472239, ANR 61261130589). Qiang Yin is supported in part by Beijing Advanced Innovation Center for Big Data and Brain Computing, Beihang University.

	Weak Bisimilarity	Branching Bisimilarity
Equivalence	EXPTIME-hard [13]	$\in$ EXPTIME [6] **EXPTIME-hard**
Regularity	PSPACE-hard [14, 17]	$\in$ NEXPTIME [4] **PSPACE-hard**

Figure 1 Equivalence checking and regularity checking of nBPA

improved both decidability results to NEXPTIME [4]. He and Huang further showed the branching bisimilarity of nBPA can actually be decided in EXPTIME [6]. However, for the weak bisimilarity of nBPA, there is no upper bound. Stříbrná first gave an NP-hard result by reducing from the Knapsack Problem [20]; Srba then improved it to PSPACE-hard by reducing from QSAT (Quantified SAT) [14]; and the best known lower bound is EXPTIME-hard given by Mayr by a reduction from the acceptance problem of alternating linear-bounded automaton (ALBA) [13]. A natural question is whether some of these lower bounds proofs hold for the branching bisimilarity of nBPA.

The branching bisimilarity of nBPA was claimed to be EXPTIME-hard [5, 6]. Researchers believed that some modifications on Mayr's reduction [13] would transform the weak bisimilarity version reduction into a branching bisimilarity version reduction. However recently Jančar reminded that the claim lacked proper justification [7]. A typical modification of adding silent transition loop to make state bisimilar dose not work. Unfortunately for the EXPTIME-hardness, almost any *small* modification is not a proper reduction from ALBA to the branching bisimilarity. According to the EXPTIME algorithm [6], we know that the main exponential factor is to do with the fact there are exponentially many redundant sets. The redundant set of a process consists of redundant variables that, when placed as a prefix to the process, gives rise to an equivalent process. In Mayr's construction [13], the number of redundant sets is only polynomial. So that the algorithm might perform better under the input of particular constructions. The above modification method does not even work for the NP-hard [20] or PSPACE-hard [14] lower bound construction. The current lower bound of the branching bisimilarity of nBPA is merely P-hard [2]. The same happens to the regularity checking problem of nBPA. The only known lower bound for branching regularity is P-hard [2, 17]. Comparatively, the weak regularity problem is PSPACE-hard [14, 17].

Our Contribution. In this paper we study the lower bounds of the branching bisimilarity and branching regularity problems of normed BPA. The EXPTIME algorithm [6] hints that exponentially many redundant sets lead to exponential running time. We first introduce a novel way to design a structure with exponentially many of redundant sets. Then we use this structure to implement a binary counter and construct a reduction from the Hit-or-Run game [10]. This confirms the EXPTIME-hard lower bound for the branching bisimilarity of nBPA. We also present another reduction from QSAT to branching bisimilarity of nBPA by this structure. Combining with the Srba's reduction from equivalence checking to regularity checking [17], we get a PSPACE-hard lower bound for the branching regularity of nBPA. Figure 1 summarizes the state of the art in equivalence checking and regularity checking with respect to weak and branching bisimilarity of nBPA. The results proved in this paper are marked in boldface.

Organization. Section 2 introduces some basic notions. Section 3 introduces the structure with exponentially many redundant sets. Section 4 proves the EXPTIME-hardness of the

equivalence checking. Section 5 proves the PSPACE-hard lower bound for the regularity checking. Section 6 concludes with some remarks.

2 Preliminaries

A BPA system Γ is a tuple $(\mathcal{V}, \mathcal{A}, \Delta)$, where $\mathcal{V}$ is a finite set of *variables* ranged over by $A, B, C, \ldots, X, Y, Z$; $\mathcal{A}$ is a finite set of *actions* ranged over by λ; and Δ is a finite set of *transition rules*. We use a specific letter τ to denote internal action and use a, b, c, d, e, f, g to range over visible actions from the set $\mathcal{A}\backslash\{\tau\}$. A process defined in Γ is a word $w \in \mathcal{V}^*$. Processes is denoted by $\alpha, \beta, \gamma, \delta, \sigma$. The *nil process* is denoted by a special symbol ϵ. We will use $=$ for the syntactical equality, and $\epsilon\alpha = \alpha\epsilon = \alpha$ by convention. A rule in Δ is in the form $X \xrightarrow{\lambda} \alpha$, where α is a BPA process. The operational semantics of the processes is defined by the following rules.

$$\frac{X \xrightarrow{\lambda} \alpha \in \Delta}{X \xrightarrow{\lambda} \alpha} \qquad \frac{\alpha \xrightarrow{\lambda} \alpha'}{\alpha\beta \xrightarrow{\lambda} \alpha'\beta}$$

We will write $\alpha \longrightarrow \beta$ for $\alpha \xrightarrow{\tau} \beta$ and $\longrightarrow^*$ for the reflexive transitive closure of $\longrightarrow$. A BPA process α is *normed* if $\exists.\lambda_1, \ldots, \lambda_k.\alpha \xrightarrow{\lambda_1} \ldots \xrightarrow{\lambda_k} \epsilon$. A BPA system is *normed* if every variable is normed. We write nBPA for normed BPA. We denote by $\mathcal{V}^0$ the set of variables in $\mathcal{V}$ that can reach ϵ via internal actions alone. For a process α, we use $\textsc{Var}(\alpha)$ to represent the set of variables occurring in α; and we use $|\alpha|$ to denote the size of α, which is defined to be the length of the word α. The size of each rule $X \xrightarrow{\lambda} \alpha$ is defined to be $|\alpha| + 2$. The size of Δ is the sum of the size of all rules. The size of a BPA system $\Gamma = (\mathcal{V}, \mathcal{A}, \Delta)$ is defined by $|\Gamma| = |\mathcal{V}| + |\mathcal{A}| + |\Delta|$.

Bisimulation Equivalence. A symmetric relation $\mathcal{R}$ on BPA processes is a *branching bisimulation* if whenever $\alpha\mathcal{R}\beta$ and $\alpha \xrightarrow{\lambda} \alpha'$ then one of the statements is valid:

- $\lambda = \tau$ and $\alpha'\mathcal{R}\beta$;
- $\beta \longrightarrow^* \beta'' \xrightarrow{\lambda} \beta'$ for some β' and β'' such that $\alpha\mathcal{R}\beta''$ and $\alpha'\mathcal{R}\beta'$.

If we replace the above second item by the following one

- $\beta \longrightarrow^* \gamma_1 \xrightarrow{\lambda} \gamma_2 \longrightarrow^* \beta'$ for some γ_1, γ_2 and β' such that $\alpha'\mathcal{R}\beta'$

then we get the definition of *weak bisimulation*. The largest branching bisimulation, denoted by $\simeq$, is branching bisimilarity; and the largest weak bisimulation, denoted by $\approx$, is weak bisimilarity. It is obvious that both $\simeq$ and $\approx$ are equivalences and are congruences with respect to the composition operator in BPA model. Branching bisimilarity is a refinement of weak bisimilarity, *i.e.* $\simeq \subseteq \approx$. We say α and β are branching bisimilar (weak bisimilar) if $\alpha \simeq \beta$ ($\alpha \approx \beta$). Both branching and weak bisimilarity satisfy a standard property of observational equivalence stated as follows.

▶ **Lemma** (Computation Lemma).

- *If* $\alpha \longrightarrow \alpha_1 \longrightarrow \ldots \longrightarrow \alpha_k$ *and* $\alpha \simeq \alpha_k$, *then for all* $1 \leq i \leq k$ *we have* $\alpha \simeq \alpha_i$.
- *If* $\alpha \longrightarrow \alpha_1 \longrightarrow \ldots \longrightarrow \alpha_k$ *and* $\alpha \approx \alpha_k$, *then for all* $1 \leq i \leq k$ *we have* $\alpha \approx \alpha_i$.

Bisimulation Game. Bisimulation relation has a standard game characterization [21, 19] which is very useful for studying the lower bounds. A branching (resp. weak) bisimulation game is a 2-player game played by *Attacker* and *Defender*. A configuration of the game is pair of processes (α_0, α_1). The game is played in rounds. Each round has 3 steps: (1)

Attacker chooses a move; (2) Defender responds to match Attacker's move; (3) Attacker sets the next round configuration according to Defender's response. One round of *branching bisimulation game* is defined as follows, assuming (β_0, β_1) is the configuration of the current round.

1. Attacker picks up $i \in \{0,1\}$, λ, and β_i' to play $\beta_i \xrightarrow{\lambda} \beta_i'$.
2. Defender responds with $\beta_{1-i} \longrightarrow^* \beta_{1-i}'' \xrightarrow{\lambda} \beta_{1-i}'$ for some β_{1-i}'' and β_{1-i}'. Defender can also play an empty response when $\lambda = \tau$ and we stipulate that $\beta_{1-i}' = \beta_{1-i}$ if Defender plays an empty response.
3. Attacker sets (β_i', β_{1-i}') as the configuration of the next round if Defender plays an empty response; otherwise Attacker sets either (β_i', β_{1-i}') or (β_i, β_{1-i}'') as the configuration of the next round.

A round of *weak bisimulation game* differs from the above one in the last 2 steps.

2. Defender responds with $\beta_{1-i} \longrightarrow^* \xrightarrow{\lambda} \longrightarrow^* \beta_{1-i}'$ for some β_{1-i}'; Defender can also play an empty response when $\lambda = \tau$ and we stipulate that $\beta_{1-i}' = \beta_{1-i}$ in that case.
3. Attacker sets the configuration of the next round to be (β_i', β_{1-i}').

If one player gets stuck, the other one wins. If the game goes on for infinitely many rounds, then Defender wins. We say a player has a *winning strategy*, w.s. for short, if he or she can win no matter how the other one plays. Defender has a w.s. in the branching bisimulation game (α, β) iff $\alpha \simeq \beta$; Defender has a w.s. in the weak bisimulation game (α, β) iff $\alpha \approx \beta$.

Redundant Set. The concept of *redundant set* was first introduced by Fu [5] to show the decidability of branching bisimilarity of nBPA. It also plays an important role in the branching bisimilarity checking algorithms of nBPA [4, 6]. Given a nBPA system $\Gamma = (\mathcal{V}, \mathcal{A}, \Delta)$, the redundant set of α, notation $\text{RD}(\alpha)$, is the set of variables defined by

$$\text{RD}(\alpha) = \{X \in \mathcal{V} \mid X\alpha \simeq \alpha\} \tag{1}$$

It is necessary that $\text{RD}(\alpha) \subseteq \mathcal{V}^0$. Note that not every $R \subseteq \mathcal{V}^0$ can be a redundant set. The problem whether there exists some γ such that $R = \text{RD}(\gamma)$ for a given R is as hard as the branching bisimilarity checking problem [6].

Main Result. A process α is a *finite-state* process if the reachable set $\{\beta \mid \alpha \xrightarrow{\lambda_1} \alpha_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_k} \alpha_k = \beta \text{ and } k \in \mathbb{N}\}$ is finite. Given an equivalence relation $\asymp$, we say that a BPA process α is *regular* with respect to $\asymp$, *i.e.* $\asymp$-REG, if $\alpha \asymp \beta$ for some finite-state process β. Note that α and β can be defined in different systems.

In this paper we are interested in the equivalence checking and regularity checking problems with respect to $\asymp$ on nBPA. They are defined as follows, assuming $\asymp$ is an equivalence relation.

EQUIVALENCE CHECKING WITH RESPECT TO $\asymp$	
Instance:	A nBPA system $(\mathcal{V}, \mathcal{A}, \Delta)$ and two processes α and β.
Question:	$\alpha \asymp \beta$?

REGULARITY CHECKING WITH RESPECT TO $\asymp$	
Instance:	A nBPA system $(\mathcal{V}, \mathcal{A}, \Delta)$ and a process α.
Question:	$\alpha \asymp$-REG?

The following theorem states the two lower bounds proved in this paper.

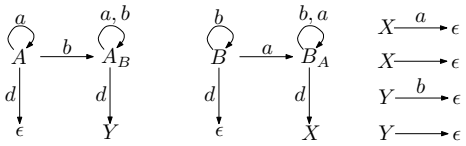
▶ **Theorem 1.** *On nBPA, for every equivalence $\asymp$ such that $\simeq \subseteq \asymp \subseteq \approx$*
1. *equivalence checking with respect to $\asymp$ is EXPTIME-hard; and*
2. *regularity checking with respect to $\asymp$ is PSPACE-hard.*

3 Redundant Sets Construction

According to the branching bisimilarity checking algorithms of nBPA [4, 6], we know that the main exponential factor is exponentially many redundant sets. However there is no obvious evidence for the existence of such large number of redundant sets. Most nBPA systems in the literature have only polynomial ones. In this section, we design a scalable structure with exponentially many redundant sets. More specifically, we will define a nBPA system $\Gamma(i) = (\mathcal{V}(i), \mathcal{A}(i), \Delta(i))$ with parameter i such that (1) $|\Gamma(i)|$ is a polynomial of i; (2) and there are exactly 2^i redundant sets in $\Gamma(i)$. We begin with $\Gamma(2)$ as an example and explain its work mechanism in detail. Then we show how to scale $\Gamma(2)$ to get $\Gamma(i)$ for arbitrary i by incorporating the idea behind $\Gamma(2)$. As an application of our approach, we design an n-bit binary counter that will be used in the sequel based on this structure.

3.1 A Small Example

The definition of $\Gamma(2) = (\mathcal{V}(2), \mathcal{A}(2), \Delta(2))$ is given as follows. $\mathcal{V}(2) = \{X, Y\} \uplus \{A, B\} \uplus \{A_B, B_A\}$, $\mathcal{A}(2) = \{a, b, d, \tau\}$ and the transition rules of $\Delta(2)$ are shown in the following graph.



Every variable is normed and $\mathcal{V}^0(2) = \{X, Y\}$. To see why $\Gamma(2)$ satisfies our requirement, we show that for each $R \in \{\emptyset, \{X\}, \{Y\}, \{X, Y\}\}$, there is some γ such that $\text{RD}(\gamma) = R$. Clearly $\text{RD}(\epsilon) = \emptyset$. It is obvious that $A \not\asymp A_B$ and $B \not\asymp B_A$ due to the d action. Now we have $XA \simeq A$, $YA \not\asymp A$ and $XB \not\asymp B$, $YB \simeq B$, *i.e.*, $\text{RD}(A) = \{X\}$ and $\text{RD}(B) = \{Y\}$. We claim that $\text{RD}(AB) = \text{RD}(BA) = \{X, Y\}$. Moreover, we have the following.

▶ **Claim.** $\text{RD}(A^n B) = \text{RD}(B^n A) = \{X, Y\}$ for all $n \geq 1$.

We first show that $\text{RD}(AB) = \{X, Y\}$. It is clear $X \in \text{RD}(AB)$. It is not so obvious that $Y \in \text{RD}(AB)$. Let us consider the branching bisimulation game of (YAB, AB).

- If Attacker plays $YAB \longrightarrow AB$ or $AB \xrightarrow{\lambda} \alpha$, then Defender plays an empty response or $YAB \longrightarrow AB \xrightarrow{\lambda} \alpha$ respectively. The next round configuration is an identical process pair. Defender wins afterward.
- If Attacker plays $YAB \xrightarrow{b} AB$, then Defender responds with $AB \xrightarrow{b} A_B B$. The game continues from $(AB, A_B B)$.
- At the configuration $(AB, A_B B)$ if Attacker plays an actions a, then Defender responds with the same action and the configuration of next round is still $(AB, A_B B)$. If Attacker plays an action b, then after Defender's response the game reaches the configuration $(A_B B, A_B B)$. Defender wins. Attacker's optimal choice is to play an action d. Defender simply follows the suit and the game reaches the configuration (YB, B).

In a nutshell, from the configuration (YAB, AB) Attacker and Defender's optimal choices will lead the game to the configuration (YB, B), *i.e.*, $YAB \simeq AB$ if and only if $YB \simeq B$. Note that $YB \simeq B$. It follows that $YAB \simeq AB$ and $Y \in \text{RD}(AB)$.

Now let us consider the general case $\text{RD}(A^nB) = \{X, Y\}$ for all $n \geq 1$. Clearly $X \in \text{RD}(A^nB)$. We now show $Y \in \text{RD}(A^nB)$. A key observation of the above game argument is that for each γ we have

$$YA\gamma \simeq A\gamma \iff Y\gamma \simeq \gamma. \tag{2}$$

Repeating (2) for n times we have $YB \simeq B \implies YAB \simeq AB \implies \cdots \implies YA^nB \simeq A^nB$. As a result $Y \in \text{RD}(A^nB)$.

By a similar argument, we can show $\text{RD}(B^nA) = \{X, Y\}$ for all $n \geq 1$.

▶ **Corollary 2.** *For $\gamma \in \{A, B\}^*$ we have (1) $X \in \text{RD}(\gamma)$ iff $A \in \text{VAR}(\gamma)$; (2) $Y \in \text{RD}(\gamma)$ iff $B \in \text{VAR}(\gamma)$.*

3.2 Scalability

We now scale $\Gamma(2)$ to get a nBPA system $\Gamma(i) = (\mathcal{V}(i), \mathcal{A}(i), \Delta(i))$ with parameter i satisfying the following properties.

1. $|\Gamma(i)|$ is a polynomial of i.
2. $|\mathcal{V}(i)^0| = i$ and for each $R \subseteq \mathcal{V}(i)^0$ there is some $\gamma \in (\mathcal{V}(i) \backslash \mathcal{V}(i)^0)^*$ such that $\text{RD}(\gamma) = R$.

Note that the satisfaction of the second property would give rise to exactly 2^i redundant sets in $\Gamma(i)$. Before we give the definition of $\Gamma(i)$, let us take a second look at $\Gamma(2)$. We treat a configuration of the form $(X\gamma, \gamma)$ or $(Y\gamma, \gamma)$ as a redundancy test of X or Y on γ. Intuitively, the variable A has two roles. The first one is to pass the redundancy test of X as for each γ we have $XA\gamma \simeq A\gamma$. The second one is to transfer the redundancy test of Y on $A\gamma$ to the same test on γ as for each γ we have $YA\gamma \simeq A\gamma$ iff $Y\gamma \simeq \gamma$. Similarly, B can pass redundancy test of Y and transfer the redundancy test of X. Thus, we propose the following conditions to meet our requirement. For each $Z \in \mathcal{V}(i)^0$, there is some $A \in \mathcal{V}(i) \backslash \mathcal{V}(i)^0$ so that for each γ it holds that:

(C1) $ZA\gamma \simeq A\gamma$, *i.e.*, A passes the redundancy test of Z; and
(C2) for each $Z' \in \mathcal{V}(i)^0 \backslash \{Z\}$, $Z'A\gamma \simeq A\gamma$ iff $Z'\gamma \simeq \gamma$, *i.e.*, A can transfer the redundancy test of each variable $Z' \in \mathcal{V}(i)^0 \backslash \{Z\}$.

▶ Remark. In order to get a more flexible system to meet other requirements, the condition (C2) can be relaxed as "A transfers the redundancy tests of a portion of $\mathcal{V}(i)^0 \backslash \{Z\}$". More specifically, "$Z'A\gamma \simeq A\gamma$ iff $Z'\gamma \simeq \gamma$" holds for a *subset* of $V(i)^0 \backslash \{Z\}$. We will see an example in Section 3.3.

Comparing $\Gamma(2)$ with the conditions (C1) and (C2), we define $\mathcal{V}(i)$, $\mathcal{A}(i)$ and $\Delta(i)$ as follows.

- $\mathcal{V}(i)^0 = \{Z_1, Z_2, \ldots, Z_i\}$;
- $\mathcal{V}(i) = \mathcal{V}(i)^0 \uplus \{A_1, A_2, \ldots, A_i\} \uplus \{A_{j,k} \mid j \neq k, 1 \leq j, k \leq i\}$;
- $\mathcal{A}(i) = \{a_1, a_2, \ldots, a_i, d, \tau\}$;
- $\Delta(i)$ contains the following rules, where $1 \leq j, k, \ell \leq i$, $j \neq k$, $j \neq \ell$.

$Z_j \xrightarrow{a_j} \epsilon$	$Z_j \longrightarrow \epsilon$	
$A_j \xrightarrow{d} \epsilon$	$A_j \xrightarrow{a_j} A_j$	$A_j \xrightarrow{a_k} A_{j,k}$
$A_{j,k} \xrightarrow{d} Z_k$	$A_{j,k} \xrightarrow{a_j} A_{j,k}$	$A_{j,k} \xrightarrow{a_\ell} A_{j,\ell}$

Clearly $\Gamma(i)$ is of size $O(i^2)$. The following justifies the conditions (C1) and C(2) in $\Gamma(i)$.

▶ **Lemma 3.** *For each γ, $1 \leq j, k \leq i$ and $j \neq k$ it holds that*

1. $Z_j A_j \gamma \simeq A_j \gamma$;
2. $Z_k A_j \gamma \simeq A_j \gamma$ *iff* $Z_k \gamma \simeq \gamma$.

With the help of Lemma 3, we can prove Proposition 4.

▶ **Proposition 4.** $\text{RD}(A_{j_1} A_{j_2} \dots A_{j_n}) = \bigcup_{\ell=1}^{n} \{Z_{j_\ell}\}$.

Proof. If $k \in \bigcup_{\ell=1}^{n} \{j_\ell\}$, we show $Z_k \in \text{RD}(A_{j_1} A_{j_2} \dots A_{j_n})$. Let ℓ' be the least number such that $j_{\ell'} = k$. By (1) of Lemma 3 we have $Z_k A_{j_{\ell'}} \dots A_{j_n} \simeq A_{j_{\ell'}} \dots A_{j_n}$. Now $Z_k A_{j_1} \dots A_{j_n} \simeq A_{j_1} \dots A_{j_n}$ can be derived by repeating (2) of Lemma 3 for $n - \ell'$ times.

If $k \notin \bigcup_{\ell=1}^{n} \{j_\ell\}$, we show $Z_k \notin \text{RD}(A_{j_1} A_{j_2} \dots A_{j_n})$. Note that $Z_k \not\simeq \epsilon$. We are done by repeating (2) of Lemma 3 for n times as $Z_k \not\simeq \epsilon \implies Z_k A_{j_n} \not\simeq A_{j_n} \implies \cdots \implies Z_k A_{j_1} A_{j_2} \dots A_{j_n} \not\simeq A_{j_1} A_{j_2} \dots A_{j_n}$.

◀

3.3 An n-bit Binary Counter

An important application of the redundant sets construction is to implement an n-bit binary counter. The main idea is to use a redundant set of size n to represent the value of an n-bit binary counter. The only challenging part is to define a proper structure of redundant sets so that it is fit to be manipulated by branching bisimulation games. Based on $\Gamma(2n)$, we implement an n-bit binary counter in the nBPA system $\Gamma_0 = (\mathcal{V}_0, \mathcal{A}_0, \Delta_0)$, where

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{V}_0^0 \uplus \mathcal{B} \uplus \mathcal{B}', \\
\mathcal{V}_0^0 &= \{Z_1^0, Z_1^1, Z_2^0, Z_2^1, \dots, Z_n^0, Z_n^1\}, \\
\mathcal{B} &= \{B_i^0, B_i^1 \mid 1 \leq i \leq n\}, \\
\mathcal{B}' &= \{B_i^b(j, b') \mid (i \neq j) \wedge 1 \leq i, j \leq n \wedge b, b' \in \{0, 1\}\}, \\
\mathcal{A}_0 &= \{a_i^0, a_i^1 \mid 1 \leq i \leq n\} \uplus \{d, \tau\}.
\end{aligned}$$

And Δ_0 contains the following rules, where $1 \leq i, j, j' \leq n$, $i \neq j$, $i \neq j'$ and $b, b', b'' \in \{0, 1\}$.

$Z_i^b \xrightarrow{a_i^b} \epsilon$	$Z_i^b \longrightarrow \epsilon$	
$B_i^b \xrightarrow{d} \epsilon$	$B_i^b \xrightarrow{a_i^b} B_i^b$	$B_i^b \xrightarrow{a_j^{b'}} B_i^b(j, b')$,
$B_i^b(j, b') \xrightarrow{d} Z_j^{b'}$	$B_i^b(j, b') \xrightarrow{a_i^b} B_i^b(j, b')$	$B_i^b(j, b') \xrightarrow{a_{j'}^{b''}} B_i^b(j', b'')$

It is clear that $\text{RD}(B_i^b) = \{Z_i^b\}$ for $1 \leq i \leq n$ and $b \in \{0, 1\}$. Intuitively, B_i^b encodes the information that the i-th bit of the counter is b. To understand the structure of redundant sets in Γ_0, let us observe the three roles that B_i^b plays in redundancy tests. Let $\gamma \in \mathcal{B}^*$.

(P1) $Z_i^b B_i^b \gamma \simeq B_i^b \gamma$. This means that B_i^b will pass the redundancy test of Z_i^b.
(P2) $Z_i^{1-b} B_i^b \gamma \not\simeq B_i^b \gamma$. This means that B_i^b will fail the redundancy test of Z_i^{1-b}.
(P3) For $j \neq i$ and $b' \in \{0, 1\}$, $Z_j^{b'} B_i^b \gamma \simeq B_i^b \gamma$ iff $Z_j^{b'} \gamma \simeq \gamma$. This means B_i^b will transfer the redundancy test of $Z_j^{b'}$ to next if $j \neq i$.

Note that the main structure of Γ_0 extends from $\Gamma(2n)$. The difference between Γ_0 and $\Gamma(2n)$ is that Γ_0 satisfies a relaxed version of the condition (C2). The properties of (P1) and (P2) together can be seen as a relaxed version of the condition (C2). The purpose of this design will be clear later. Using the above idea we have the following technical lemma.

▶ **Lemma 5.** *Suppose $\gamma \in \mathcal{B}^*$, the following statements are valid.*

1. $Z_i^b\gamma \simeq \gamma$ *iff there are* γ_1 *and* γ_2 *such that* $\gamma = \gamma_1 B_i^b \gamma_2$ *and* $B_i^{1-b} \notin \textsc{Var}(\gamma_1)$.
2. $Z_i^b\gamma \simeq \gamma$ *implies* $Z_i^{1-b}\gamma \not\simeq \gamma$.

Proof. We only prove (1) here. (2) is a direct consequence of (1).

- ("⇐") Assume w.l.o.g. that $B_i^b \notin \textsc{Var}(\gamma_1)$. Clearly $Z_i^b B_i^b \gamma_2 \simeq B_i^b \gamma_2$. If $\gamma_1 = \epsilon$ we are done; otherwise let $\gamma_1 = B_{i_1}^{b_1} \dots B_{i_k}^{b_k}$. Note that by assumption we have $i_j \neq i$ for all $1 \leq j \leq k$. We are done by repeating (P3) for k times from the equation $Z_i^b B_i^b \gamma_2 \simeq B_i^b \gamma_2$.
- ("⇒") We prove it by contradiction. Suppose there is no γ_1 and γ_2 such that $\gamma = \gamma_1 B_i^b \gamma_2$ and $B_i^{1-b} \notin \textsc{Var}(\gamma_1)$, we show $Z_i^b\gamma \not\simeq \gamma$. There are two cases: (1) $\gamma = \gamma_1 B_i^{1-b}\gamma_2$ and $B_i^0, B_i^1 \notin \textsc{Var}(\gamma_1)$; and (2) $B_i^0, B_i^1 \notin \textsc{Var}(\gamma)$. In the first case $Z_i^b\gamma \not\simeq \gamma$ can be derived from $Z_i^b B_i^{1-b}\gamma_2 \not\simeq B_i^{1-b}\gamma_2$ by repeating (P3) for $|\gamma_1|$ times; in the second case $Z_i^b\gamma \not\simeq \gamma$ can be obtained from $Z_i^b \not\simeq \epsilon$ by repeating (P3) for $|\gamma|$ times.

◀

▶ **Definition 6.** A process $\gamma \in \mathcal{B}^*$ is a *valid encoding* of an n-bit binary counter $b_n b_{n-1} \dots b_1$, notation $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$, if for each $1 \leq i \leq n$ there are γ_i and γ_i' such that $\gamma = \gamma_i B_i^{b_i} \gamma_i'$ and $B_i^{1-b_i} \notin \textsc{Var}(\gamma_i)$.

For a binary counter $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$, we will use $\sharp\gamma$ to denote the value $\sum_{i=1}^n b_i \cdot 2^{i-1}$ in the sequel. One can see Definition 6 as the syntax of an n-bit binary counter in the system Γ_0. This syntax allows us to update a "binary number" in an *overwritten* way. Suppose $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$ and we want to flip the i-th "bit" of γ to get another "binary number" σ. Then by Definition 6 we can simply let $\sigma = B_i^{1-b_i}\gamma$, as one can verify $\sigma \in [\![b_n' b_{n-1}' \dots b_1']\!]$ where $b_i' = 1 - b_i$ and $b_j' = b_j$ for $j \neq i$. By Lemma 5, we give the binary counter a semantic characterization in terms of redundant sets.

▶ **Proposition 7.** *Let* γ *be a process such that* $\gamma \in \mathcal{B}^*$*, we have*

$$\gamma \in [\![b_n b_{n-1} \dots b_1]\!] \iff \textsc{Rd}(\gamma) = \{Z_n^{b_n}, Z_{n-1}^{b_{n-1}}, \dots, Z_1^{b_1}\}. \tag{3}$$

Proposition 7 provides us a way to test a specific "bit" with branching bisimulation games. Suppose $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$ and we want to check whether $b_i = b$. By Proposition 7, we only need to check if Defender has a w.s. in the branching bisimulation game $(Z_i^b\gamma, \gamma)$. The following lemma shows that we can also do bit test by weak bisimulation games.

▶ **Lemma 8.** *Suppose* $\gamma \in \mathcal{B}^*$*, then* $Z_i^b\gamma \simeq \gamma$ *iff* $Z_i^b\gamma \approx \gamma$.

The following Lemma tells us how to test multiple bits. It is a simple consequence when applying Computation Lemma to Proposition 7 and Lemma 8.

▶ **Lemma 9.** *Let* $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$ *and* $\alpha \in \{Z_1^0, Z_1^1, Z_2^0, Z_2^1, \dots, Z_n^0, Z_n^1\}^*$*, then the following statements are valid.*

1. $\alpha\gamma \simeq \gamma$ *iff* $\alpha \in \{Z_1^{b_1}, Z_2^{b_2}, \dots, Z_n^{b_n}\}^*$.
2. $\alpha\gamma \approx \gamma$ *iff* $\alpha \in \{Z_1^{b_1}, Z_2^{b_2}, \dots, Z_n^{b_n}\}^*$.

4 EXPTIME-hardness of Equivalence Checking

In this section, we show that branching bisimilarity on normed BPA is EXPTIME-hard by a reduction from *Hit-or-Run* game [10]. A Hit-or-Run game is a counter game defined by a tuple $\mathcal{G} = (S_0, S_1, \to, s_\vdash, s_\dashv, m_\dashv)$, where $S = S_0 \uplus S_1$ is a finite set of states, $\to \subseteq S \times \mathbb{N} \times (S \cup \{s_\dashv\})$ is a finite set of transition rules, $s_\vdash \in S$ is the initial state, $s_\dashv \notin S$ is the final state, and $m_\dashv \in \mathbb{N}$ is the final value. We use $s \xrightarrow{\ell} t$ to denote $(s, \ell, t) \in \to$ and require that $\ell = 0$ or

$\ell = 2^k$ for some k. For each $s \in S$ there is at least one rule $(s, \ell, t) \in \rightarrow$. A configuration of $\mathcal{G}$ is a pair $(s, m) \in (S \cup \{s_\dashv\}) \times \mathbb{N}$. The game is played by two players, named Player 0 and Player 1. Starting from the initial configuration $(s_\vdash, 0)$, the game $\mathcal{G}$ proceeds in rounds according to the following rule: if the current configuration is $(s, k) \in S_i \times \mathbb{N}$, then Player i chooses a rule of the form $s \xrightarrow{\ell} t$ and the resulting new configuration is $(t, k+\ell)$. If $\mathcal{G}$ reaches $(s_\dashv, m)$ and $m \neq m_\dashv$ then Player 1 wins; if $\mathcal{G}$ reaches the configuration $(s_\dashv, m_\dashv)$ then Player 0 wins; if $\mathcal{G}$ never reaches the final state $s_\dashv$, then Player 0 also wins. As a result, Player 0's goal is to *hit* $(s_\dashv, m_\dashv)$ or *run* from the final state $s_\dashv$. The problem of deciding the winner of Hit-or-Run game with all numbers represented in binary is EXPTIME-complete [9, 10]. The presented version of Hit-or-Run game is due to Kiefer [10]. Kiefer used it to establish the EXPTIME-hardness of strong bisimilarity on general BPA. The main technical result of the section is as follows.

▶ **Proposition 10.** *Given a Hit-or-Run game $\mathcal{G} = (S_0, S_1, \rightarrow, s_\vdash, s_\dashv, m_\dashv)$, a nBPA system $\Gamma_1 = (\mathcal{V}_1, \mathcal{A}_1, \Delta_1)$ and two processes $\xi, \xi' \in \mathcal{V}_1^*$ can be constructed in polynomial time such that*

$$\textit{Player } 0 \textit{ has a w.s. in } \mathcal{G} \iff \xi \simeq \xi' \iff \xi \approx \xi'.$$

Our first lower bound (first item of Theorem 1) is a direct consequence of Proposition 10. Combining with the upper bound from [6], we confirm the complete result.

▶ **Corollary 11.** *Branching bisimilarity checking on nBPA is EXPTIME-complete.*

Now let us fix a Hit-or-Run game $\mathcal{G} = (S_0, S_1, \rightarrow, s_\vdash, s_\dashv, m_\dashv)$ for this section. Let $\mathbf{Op}(s) = \{(\ell, t) \mid (s, \ell, t) \in \rightarrow\}$ and $\mathbf{Op} = \bigcup_{s \in S_0 \uplus S_1} \mathbf{Op}(s)$. We define the nBPA system $\Gamma_1 = (\mathcal{V}_1, \mathcal{A}_1, \Delta_1)$ for Proposition 10 as follows.

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{V}_0 \uplus \mathcal{C} \uplus \mathcal{F} \uplus \mathcal{M}, \\ \mathcal{A}_1 &= \mathcal{A}_0 \uplus \{c, e, f, f', g\} \uplus \{a(\ell, t) \mid (\ell, t) \in \mathbf{Op}\}, \\ \Delta_1 &= \Delta_0 \uplus \Delta_1'. \end{aligned}$$

Γ_1 includes the n-bit counter system Γ_0 as a subsystem and use it represents the counter in the game $\mathcal{G}$. We require that n and $m_\dashv$ satisfy the constrain $n = \lfloor \log_2 m_\dashv \rfloor + 1$. This n-bit counter representation is sufficient for our purpose due to the following observation. When the counter value in $\mathcal{G}$ is greater than $2^n - 1$, Player 0 or Player 1's object is to avoid or respectively to reach the final state $s_\dashv$ and the exact value of the counter no longer matters.

In the following we define the set $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{M}$ and add rules to Δ_1'.

($\mathcal{C}$). The set $\mathcal{C}$ is used to encode the control states of $\mathcal{G}$ and is defined by

$$\mathcal{C} = \{X(s), X'(s), Y(s), Y'(s) \mid s \in S \cup \{s_\dashv\}\}. \tag{4}$$

Basic Idea. Our reduction uses the branching (resp. weak) bisimulation game $\mathcal{G}'$ starting from (ξ, ξ') to mimic the run of $\mathcal{G}$ from $(s_\vdash, 0)$. We imagine that Defender simulates Player 0's performance and Attacker simulates Player 1's performance. For $0 \leq m < 2^n$, let $\mathtt{Bin}(m)$ be the unique n-bit binary representation of m. The reduction will keep the following correspondence between $\mathcal{G}$ and $\mathcal{G}'$. If $\mathcal{G}$ reaches a configuration (s, m) with $m < 2^n$, then $\mathcal{G}'$ can reach a configuration $(X(s)\gamma, X'(s)\gamma)$ for some $\gamma \in [\![\mathtt{Bin}(m)]\!]$ in a reasonable way; if $\mathcal{G}$ reaches (s, m) with $m \geq 2^n$, then $\mathcal{G}'$ can reach $(Y(s)\sigma, Y'(s)\sigma)$ for some $\sigma \in [\![b_n b_{n-1} \ldots b_1]\!]$ in a reasonable way. Intuitively, $Y(s)$ and $Y'(s)$ indicate that the counter of $\mathcal{G}$ overflows.

We do not track the exact value of the counter in that case. The two processes ξ and ξ' for Proposition 10 are defined by

$$\xi = X(s_{\vdash})B_n^0 B_{n-1}^0 \dots B_1^0, \qquad \xi' = X'(s_{\vdash})B_n^0 B_{n-1}^0 \dots B_1^0. \tag{5}$$

Clearly (ξ, ξ') corresponds to the initial configuration $(s_{\vdash}, 0)$ in $\mathcal{G}$.

($\mathcal{F}$). The set $\mathcal{F}$ is used to implement the Defender's Forcing gadgets.

$$\mathcal{F} = \{A(\ell,t), A'(\ell,t), E_s, F_s, E_s(\ell,t), F_s(\ell,t) \mid s \in S_0 \wedge (\ell,t) \in \mathbf{Op}(s)\}. \tag{6}$$

We add the following rules to Δ_1' for the variables in $\mathcal{C} \cup \mathcal{F}$ to simulate the control flow of $\mathcal{G}$.

1. Let $s \in S_1$. In $\mathcal{G}$ Player 1 would choose one pair (ℓ,t) from $\mathbf{Op}(s)$, then correspondingly, rules (a1) and (a2) enable Attacker to choose the next move in $\mathcal{G}'$.
   (a1). $X(s) \xrightarrow{a(\ell,t)} A(\ell,t)$, $X'(s) \xrightarrow{a(\ell,t)} A'(\ell,t)$; $(\ell,t) \in \mathbf{Op}(s)$
   (a2). $Y(s) \xrightarrow{a(\ell,t)} Y(t)$, $Y'(s) \xrightarrow{a(\ell,t)} Y'(t)$. $(\ell,t) \in \mathbf{Op}(s)$
2. Let $s \in S_0$. In $\mathcal{G}$ Player 0 would choose one pair (ℓ,t) from $\mathbf{Op}(s)$. Rules (b1) (b2) and rules (b3) (b4) form two Defender's Forcing gadgets [8], which allow Defender to choose the next move in $\mathcal{G}'$. Let $(\ell,t), (\ell',t') \in \mathbf{Op}(s)$.
   (b1). $X(s) \xrightarrow{c} E_s$, $X(s) \xrightarrow{c} E_s(\ell,t)$, $X'(s) \xrightarrow{c} E_s(\ell,t)$;
   (b2). $E_s \xrightarrow{a(\ell,t)} A(\ell,t)$, $E_s(\ell,t) \xrightarrow{a(\ell,t)} A'(\ell,t)$, $E_s(\ell,t) \xrightarrow{a(\ell',t')} A(\ell',t')$; $((\ell',t') \neq (\ell,t))$
   (b3). $Y(s) \xrightarrow{c} F_s$, $Y(s) \xrightarrow{c} F_s(\ell,t)$, $Y'(s) \xrightarrow{c} F_s(\ell,t)$;
   (b4). $F_s \xrightarrow{a(\ell,t)} Y(t)$, $F_s(\ell,t) \xrightarrow{a(\ell,t)} Y'(t)$, $F_s(\ell,t) \xrightarrow{a(\ell',t')} Y(t')$. $((\ell',t') \neq (\ell,t))$
3. The following two rules for $X(s_{\dashv})$ and $X'(s_{\dashv})$ are used to test the value of counter with respect to $m_{\dashv}$. Let $\mathtt{Bin}(m_{\dashv}) = b_n^{\dashv} b_{n-1}^{\dashv} \dots b_1^{\dashv}$.
   (c). $X(s_{\dashv}) \xrightarrow{f} Z_n^{b_n^{\dashv}} Z_{n-1}^{b_{n-1}^{\dashv}} \dots Z_1^{b_1^{\dashv}}$, $X'(s_{\dashv}) \xrightarrow{f} \epsilon$.
   By Lemma 9, for $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$ we have $X(s_{\dashv})\gamma \simeq X'(s_{\dashv})\gamma$ iff $X(s_{\dashv})\gamma \approx X'(s_{\dashv})\gamma$ iff $\sharp\gamma = m_{\dashv}$.
4. The following two rules are for $Y(s_{\dashv})$ and $Y'(s_{\dashv})$. Player 1 wins if $\mathcal{G}$ reaches a configuration $(s_{\dashv}, m)$ with $m \geq 2^n$. Correspondingly, $\mathcal{G}'$ will reach a configuration $(Y(s_{\dashv})\sigma, Y'(s_{\dashv})\sigma)$ for some $\sigma \in [\![b_n b_{n-1} \dots b_1]\!]$. The following rules enable Attacker to win in this case by performing a special action that Defender can not match.
   (d). $Y(s_{\dashv}) \xrightarrow{f} \epsilon$, $Y'(s_{\dashv}) \xrightarrow{f'} \epsilon$.

($\mathcal{M}$). The set $\mathcal{M}$ is used to initiate the counter update operation and manipulate the n-bit binary counter.

$$\mathcal{M} = \left\{ \begin{array}{l} Add(k,t), Add'(k,t), D(k,t), \\ D(k,t,i), C(k,t,i), C'(k,t,i) \end{array} \;\middle|\; (2^k,t) \in \mathbf{Op} \wedge 0 \leq k < n \right\}. \tag{7}$$

The process pair $(A(\ell,t), A'(\ell,t))$ is used to implement the operation "increasing the counter by ℓ and goto state t". We add the following rules for this pair based on the value of ℓ.

- $A(\ell,t) \xrightarrow{g} X(t)$ and $A'(\ell,t) \xrightarrow{g} X'(t)$ if $\ell = 0$;
- $A(\ell,t) \xrightarrow{g} Y(t)$ and $A'(\ell,t) \xrightarrow{g} Y'(t)$ if $\ell \geq 2^n$;
- $A(\ell,t) \xrightarrow{g} Add(\log \ell, t)$ and $A'(\ell,t) \xrightarrow{g} Add'(\log \ell, t)$ if $0 < \ell < 2^n$.

If $\ell = 0$ or $\ell \geq 2^n$, the counter is either unchanged or overflow. In this case, we can directly switch the control state in $\mathcal{G}'$. If $0 < \ell < 2^n$, $\mathcal{G}'$ use the following mechanism to update the counter.

Binary Counter Manipulation. Suppose we have $\gamma \in \mathcal{B}^*$ representing a counter value, *i.e.* $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$, and want to increase it by 2^k, where $0 \leq k < n$. This operation has two possible outcomes. The counter is either updated to some $\sigma \in [\![b'_n b'_{n-1} \dots b'_1]\!]$ with $\sharp\sigma = \sharp\gamma + 2^k$, or overflow if $\sharp\gamma + 2^k \geq 2^n$. Recall that $\sharp\gamma$ and $\sharp\sigma$ represent the values of γ and σ. A key observation is that we can update γ to σ *locally*. Although there are 2^n many possible values for γ, we can write σ as $\delta\gamma$ for exactly $n-k$ possible δ. Indeed, let $\alpha(k,0)$, $\alpha(k,1) \dots \alpha(k,n-k)$ and $\delta(k,0)$, $\delta(k,1) \dots \delta(k,n-k)$ be the processes defined by

$$\begin{array}{rclcrcl}
\alpha(k,0) & = & Z^0_{k+1}, & & \delta(k,0) & = & B^1_{k+1}; \\
\alpha(k,1) & = & Z^0_{k+2} Z^1_{k+1}, & & \delta(k,1) & = & B^1_{k+2} B^0_{k+1}; \\
 & \vdots & & & & \vdots & \\
\alpha(k,n-k-1) & = & Z^0_n Z^1_{n-1} \dots Z^1_{k+1}, & & \delta(k,n-k-1) & = & B^1_n B^0_{n-1} \dots B^0_{k+1}; \\
\alpha(k,n-k) & = & Z^1_n Z^1_{n-1} \dots Z^1_{k+1}, & & \delta(k,n-k) & = & B^0_n B^0_{n-1} \dots B^0_{k+1}.
\end{array}$$

The set $\{\gamma \mid \gamma \in [\![\mathtt{Bin}(m)]\!], 0 \leq m < 2^n\}$ can be divided into $n-k+1$ classes according to $\alpha(k,0)$, $\alpha(k,1) \dots \alpha(k,n-k)$. Intuitively, each $\alpha(k,i)$ encodes the bits which are flipped when increasing γ by 2^k. Each $\delta(k,i)$ encodes the corresponding effect of that operation. Let $i^*(k)$ be the maximal length of successive bits of 1 starting from b_{k+1} to b_n. Note that $i^*(k) = \Sigma_{i=0}^{n-k-1}(\Pi_{j=0}^{i} b_{k+1+j})$. By Lemma 9, $\gamma \simeq \alpha(k,i)\gamma$ iff $i = i^*(k)$. If $i^*(k) < n-k$, then $\sharp\gamma + 2^k < 2^n$. By the definition of $\delta(k, i^*(k))$ we can let $\sigma = \delta(k, i^*(k))\gamma$ and have $\sharp\sigma = \sharp\gamma + 2^k$. If $i^*(k) = n-k$, then $\sharp\gamma + 2^k \geq 2^n$ and increasing γ by 2^k will cause an overflow.

We now design a branching bisimulation game to simulate the addition operation based on the above idea. The following rules are for $\mathcal{M}$, where $0 \leq i, j \leq n-k$.

$(A1)$.	$Add(k,t) \xrightarrow{c} D(k,t)$	$Add(k,t) \xrightarrow{c} D(k,t,i)$	
$(A2)$.		$Add'(k,t) \xrightarrow{c} D(k,t,i)$	
$(A3)$.	$D(k,t) \xrightarrow{c} C(k,t,i)$		
$(A4)$.	$D(k,t,i) \xrightarrow{c} C'(k,t,i)$	$D(k,t,i) \xrightarrow{c} C(k,t,j)$	$(j \neq i)$
$(A5)$.	$C(k,t,i) \xrightarrow{c} \alpha(k,i)$	$C'(k,t,i) \xrightarrow{c} \epsilon$	
$(A6)$.	$C(k,t,i) \xrightarrow{e} X(t)\delta(k,i)$	$C'(k,t,i) \xrightarrow{e} X'(t)\delta(k,i)$	$(0 \leq i \leq n-k-1)$
$(A7)$.	$C(k,t,n-k) \xrightarrow{e} Y(t)$	$C'(k,t,n-k) \xrightarrow{e} Y'(t)$	

The correctness of the simulation is demonstrated by the following Lemma.

▶ **Lemma 12.** *Suppose $\gamma \in [\![b_n b_{n-1} \dots b_1]\!]$ and $i^*(k) = \Sigma_{i=0}^{n-k-1}(\Pi_{j=0}^{i} b_{k+1+j})$. In the branching bisimulation game starting from $(Add(k,t)\gamma, Add'(k,t)\gamma)$*

- *if $\sharp\gamma + 2^k < 2^n$, then the optimal choices of Attacker and Defender will lead to the game reaching the configuration $(X(t)\delta(k,i^*(k))\gamma, X'(t)\delta(k,i^*(k))\gamma)$ with $\sharp(\delta(k,i^*(k))\gamma) = \sharp\gamma + 2^k$,* i.e., *$Add(k,t)\gamma \simeq Add'(k,t)\gamma$ iff $X(t)\delta(k,i^*(k))\gamma \simeq X'(t)\delta(k,i^*(k))\gamma$;*
- *if $\sharp\gamma + 2^k \geq 2^n$, then the optimal choices of Attacker and Defender will lead to the game reaching the configuration $(Y(t)\gamma, Y'(t)\gamma)$,* i.e., *$Add(k,t)\gamma \simeq Add'(k,t)\gamma$ iff $Y(t)\gamma \simeq Y'(t)\gamma$.*

Proof. Rules (A1) (A2) (A3) (A4) form a classical Defender's Forcing gadget. Defender can use it to force the game from configuration $(Add(k,t)\gamma, Add'(k,t)\gamma)$ to any configuration of the form $(C(k,t,i)\gamma, C'(k,t,i)\gamma)$, where $0 \leq i \leq n-k$. Defender has to play carefully, as at the configuration $(C(k,t,i)\gamma, C'(k,t,i)\gamma)$ Attacker can use rule (A5) to start up bits test by forcing the game to the configuration $(\alpha(k,i)\gamma, \gamma)$. By the definition of $\alpha(k,i)$ and Lemma 9, if $i = i^*(k)$ then Defender can survive the bits test as $\alpha(k,i^*(k))\gamma \simeq \gamma$; otherwise Defender will lose during the bits test as $\alpha(k,i)\gamma \not\approx \gamma$ for $i \neq i^*(k)$. As a result Defender's optimal

move is to force the configuration $(C(k,t,i^*(k))\gamma, C'(k,t,i^*(k))\gamma)$. In that case, Attacker's optimal choice is to use rule (A6) or (A7) to increase the binary number γ by 2^k or flag an overflow error. If $i^*(k) < n-k$, the game reaches $(X(t)\delta(k,i^*(k))\gamma, X'(t)\delta(k,i^*(k))\gamma)$ by rule (A6). As $\delta(k,i^*(k))$ encodes the effect of bits change caused by increasing γ by 2^k, one can verify that $\sharp(\delta(k,i^*(k))\gamma) = \sharp\gamma + 2^k$. If $i^*(k) = n-k$, the game goes to $(Y(t)\gamma, Y'(t)\gamma)$ by rule (A7). ◀

▶ Remark. A process γ cannot perform an immediate internal action if there is no σ such that $\gamma \longrightarrow \sigma$. By construction $X(t)$, $X'(t)$, $Y(t)$ and $Y'(t)$ cannot perform immediate internal actions. As a result we can replace the branching bisimulation game of $(Add(k,t)\gamma, Add'(k,t)\gamma)$ and "$\simeq$" in Lemma 12 with weak bisimulation game $(Add(k,t)\gamma, Add'(k,t)\gamma)$ and "$\approx$".

Lemma 12 promises that the addition operation of the counter is implemented correctly in bisimulation games. We are ready to prove Proposition 10.

Proof of Proposition 10. Suppose $\mathcal{G}$ reaches (s,m) for some $s \in S_0 \uplus S_1$ and $m < 2^n$. The corresponding configuration of $\mathcal{G}'$ is $(X(s)\gamma, X'(s)\gamma)$ for some $\gamma \in [\![\mathtt{Bin}(m)]\!]$. If $s \in S_0$, then Player 0 chooses a rule $s \xrightarrow{\ell} t$ and $\mathcal{G}$ proceeds to $(t, m+\ell)$. We show how the branching bisimulation (resp. weak) bisimulation $\mathcal{G}'$ mimic this behavior while keep the correspondence between $\mathcal{G}$ and $\mathcal{G}'$. We only discuss the case $s \in S_0$ here. The argument for $s \in S_1$ is similar.

First by rules (b1) (b2), Defender forces to the configuration $(A(\ell,t)\gamma, A'(\ell,t)\gamma)$. If $\ell = 0$, then $\mathcal{G}'$ reaches $(X(t)\gamma, X'(t)\gamma)$. If $\ell \geq 2^n$, then $\mathcal{G}'$ reaches $(Y(t)\gamma, Y'(t)\gamma)$. If $0 < \ell < 2^n$, then $\mathcal{G}'$ first reaches $(Add(\log \ell, t)\gamma, Add'(\log \ell, t)\gamma))$. Now the binary counter in $\mathcal{G}'$ will be updated according to ℓ. By Lemma 12, if $m + \ell < 2^n$, then the optimal play of Attacker and Defender will lead to $(X(t)\sigma, X'(t)\sigma)$ with $\sharp\sigma = \sharp\gamma + \ell$. If $m + \ell \geq 2^n$, then the optimal configuration for both Attacker and Defender is $(Y(t)\gamma, Y'(t)\gamma)$.

Once $\mathcal{G}$ reaches a configuration (s', m) with $m \geq 2^n$ and $s' \neq s_\dashv$, $\mathcal{G}'$ reaches $(Y(s')\sigma, Y'(s')\sigma)$ for some σ. By rules (a2) (b3) (b4), $\mathcal{G}'$ will only keep track of the state shift of $\mathcal{G}$ afterward.

If Player 0 has a strategy to hit $(s_\dashv, m_\dashv)$ or run from $s_\dashv$ then Defender can mimic the strategy to push $\mathcal{G}'$ from (ξ, ξ') to a configuration $(X(s_\dashv)\gamma, X'(s_\dashv)\gamma)$ for some $\gamma \in [\![\mathtt{Bin}(m_\dashv)]\!]$ or force $\mathcal{G}'$ to be played infinitely. By rule (c) and Lemma 9, $X(s_\dashv)\gamma \simeq X'(s_\dashv)\gamma$. It follows that $\xi \simeq \xi'$. If Player 1 has a strategy such that no matter how Player 0 chooses, the game will hit some configuration of the form $(s_\dashv, m)$ with $m \neq m_\dashv$. Then Attacker can mimic the strategy to force $\mathcal{G}'$ from (ξ, ξ') to $(X(s_\dashv)\gamma, X'(s_\dashv)\gamma)$ for some $\gamma \in [\![\mathtt{Bin}(m)]\!]$ if $m < 2^n$, or to $(Y(s_\dashv)\sigma, Y'(s_\dashv)\sigma)$ for some σ if $m \geq 2^n$. By rule (c) and Lemma 9, $X(s_\dashv)\gamma \not\approx X'(s_\dashv)\gamma$. By rule (d), $Y(s_\dashv)\sigma \not\approx Y'(s_\dashv)\sigma$. It follows that $\xi \not\approx \xi'$. ◀

5 PSPACE-hardness of Regularity Checking

Srba [17] proved that weak bisimilarity can be reduced to weak regularity under a certain condition. We can verify that his original construction also works for branching regularity.

▶ **Theorem 13** (Srba[17]). *Given a BPA system Γ and two process α and β, one can construct in polynomial time a new BPA system Γ' and a process γ such that (1) γ is $\approx$-REG iff $\alpha \approx \beta$ and both α and β are $\approx$-REG; (2) γ is $\simeq$-REG iff $\alpha \simeq \beta$ and both α and β are $\simeq$-REG; and (3) γ is normed iff α and β are normed.*

By Theorem 13, to get a lower bound of branching regularity on normed BPA we only need to prove a lower bound of branching bisimilarity. Note that we cannot adapt the previous reduction to get an EXPTIME-hardness result for regularity as ξ and ξ' for Proposition 10

are neither $\simeq$-REG nor $\simeq$-REG. Srba proved that weak bisimilarity is PSPACE-hard [14] and the two processes for the construction are $\approx$-REG. This implies that weak regularity of normed BPA is PSPACE-hard. However, the construction in [14] does not work for branching bisimilarity. We can fix this problem by adapting the previous redundant sets construction.

▶ **Proposition 14.** *Given a QSAT formula $\mathfrak{F}$, we can construct a normed BPA system $\Gamma_2 = (\mathcal{V}_2, \mathcal{A}_2, \Delta_2)$ and two normed processes X_1 and X_1' satisfying the following conditions. (1) If $\mathfrak{F}$ is true then $X_1 \simeq X_1'$. (2) If $\mathfrak{F}$ is false then $X_1 \not\approx X_2$. (3) α and β are both $\simeq$-REG.*

Combining Theorem 13 and Proposition 14 we get our second lower bound result (second item of Theorem 1).

Now let us first fix a QSAT formula

$$\mathfrak{F} = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_m \exists y_m . (C_1 \wedge C_2 \wedge \cdots \wedge C_n) \tag{8}$$

where $C_1 \wedge C_2 \wedge \cdots \wedge C_n$ is a conjunctive normal form with boolean variables $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$. Consider the following game interpretation of the QSAT formula $\mathfrak{F}$. There are two players, $\mathbb{X}$ and $\mathbb{Y}$, who are trying to give an assignment in rounds to all the variables $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$. At the i-th round, player $\mathbb{X}$ first assigns a boolean value b_i to x_i and then $\mathbb{Y}$ assigns a boolean value b_i' to y_i. After m rounds we get an assignment $\mathcal{A} = \bigcup_{i=1}^{m} \{x_i \mapsto b_i, y_i \mapsto b_i'\}$. If $\mathcal{A}$ satisfies $C_1 \wedge C_2 \wedge \cdots \wedge C_n$, then $\mathbb{Y}$ wins; otherwise $\mathbb{X}$ wins. It is easy to see that $\mathfrak{F}$ is true iff $\mathbb{Y}$ has a winning strategy. This basic idea of constructing Γ_2 is to design a branching (resp. weak) bisimulation game to mimic the QSAT game on $\mathfrak{F}$. This method resembles the ideas in the previous works [14, 15, 16]. The substantial new ingredient in our construction is $\Gamma(n)$, introduced in Section 3. Γ_2 contains $\Gamma(n)$ as a subsystem and uses it to encode partial assignments in the QSAT game. For $i \in \{1, 2, \ldots, m\}$ and $b \in \{0, 1\}$, let $\alpha(i, b)$ and $\beta(i, b)$ be the processes defined as follows.

- $\alpha(i, b) = A_{i_1} A_{i_2} \ldots A_{i_k}$. If $b = 1$, then $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are all the indices of clauses in $\mathfrak{F}$ that x_i occurs; if $b = 0$, then $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are all the indices of clauses that $\bar{x}_i$ occurs.
- $\beta(i, b) = A_{i_1} A_{i_2} \ldots A_{i_k}$. If $b = 1$, then $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are all the indices of clauses in $\mathfrak{F}$ that y_i occurs; if $b = 0$, then $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are all the indices of clauses that $\bar{y}_i$ occurs.

An assignment $\mathcal{A} = \bigcup_{i=1}^{m} \{x_i \mapsto b_i, y_i \mapsto b_i'\}$ is represented by the process $\gamma(\mathcal{A})$ defined by

$$\gamma(\mathcal{A}) = \beta(m, b_m') \alpha(m, b_m) \ldots \beta(1, b_1') \alpha(1, b_1). \tag{9}$$

Clearly $\mathcal{A}$ satisfies $C_1 \wedge C_2 \wedge \cdots \wedge C_n$ iff $\textsc{Var}(\gamma(\mathcal{A})) = \{A_1, A_2, \ldots, A_n\}$. The following lemma tells us how to test the satisfiability of $\mathcal{A}$ by bisimulation games.

▶ **Lemma 15.** *Suppose $\gamma \in \{A_1, A_2, \ldots, A_n\}^*$. The following statements are equivalent. (1) $Z_1 Z_2 \ldots Z_n \gamma \simeq \gamma$. (2) $Z_1 Z_2 \ldots Z_n \gamma \approx \gamma$. (3) $\textsc{Var}(\gamma) = \{A_1, A_2, \ldots, A_n\}$.*

Now the normed BPA system $\Gamma_2 = (\mathcal{V}_2, \mathcal{A}_2, \Delta_2)$ for Proposition 14 is defined by

$$\begin{aligned} \mathcal{V}_2 &= \mathcal{V}(n) \uplus \{X_i, Y_i, Y_i(1), Y_i(2), Y_i(3) \mid 1 \leq i \leq m\} \uplus \{X_{m+1}, X_{m+1}'\}, \\ \mathcal{A}_2 &= \mathcal{A}(n) \uplus \{c_0, c_1, e\}, \\ \Delta_2 &= \Delta(n) \uplus \Delta_2'. \end{aligned}$$

And Δ_2' contains the following rules, where $1 \leq i \leq m$.

1.	$X_i \xrightarrow{c_0} Y_i\alpha(i,0)$	$X_i \xrightarrow{c_1} Y_i\alpha(i,1)$	
2.	$X_i' \xrightarrow{c_0} Y_i'\alpha(i,0)$	$X_i' \xrightarrow{c_1} Y_i'\alpha(i,1)$	
3.	$Y_i \xrightarrow{e} Y_i(1)$	$Y_i \xrightarrow{e} Y_i(2)$	$Y_i \xrightarrow{e} Y_i(3)$
4.		$Y_i' \xrightarrow{e} Y_i(2)$	$Y_i' \xrightarrow{e} Y_i(3)$
5.	$Y_i(1) \xrightarrow{c_0} X_{i+1}\beta(i,0)$	$Y_i(1) \xrightarrow{c_1} X_{i+1}\beta(i,1)$	
6.	$Y_i(2) \xrightarrow{c_0} X_{i+1}'\beta(i,0)$	$Y_i(2) \xrightarrow{c_1} X_{i+1}\beta(i,1)$	
7.	$Y_i(3) \xrightarrow{c_0} X_{i+1}\beta(i,0)$	$Y_i(3) \xrightarrow{c_1} X_{i+1}'\beta(i,1)$	
8.	$X_{m+1} \xrightarrow{e} Z_1 \dots Z_n$	$X_{m+1}' \xrightarrow{e} \epsilon$	

Proof Sketch of Proposition 14. It is clear that both X_1 and X_1' are $\simeq$-REG. Now consider the branching (resp. weak) bisimulation game starting from (X_1, X_1'). One round of QSAT game on $\mathfrak{F}$ will be simulated by three rounds of branching (resp. weak) bisimulation games. Suppose player $\mathbb{X}$ assigns b_i to x_i and then player $\mathbb{Y}$ assigns b_i' to y_i in the i-th round of QSAT game. Then in the branching (resp. weak) bisimulation game, Attacker uses rule (1) and (2) to push $\alpha(i, b_i)$ to the stack in one round; then in the following two rounds, by Defender's Forcing (rule (3) (4) (5) (6) (7)), Defender pushes $\beta(i, b_i')$ to the stack. In this way, the branching (resp. weak) bisimulation game reaches a configuration in the form $(X_{m+1}\gamma(\mathcal{A}), X_{m+1}'\gamma(\mathcal{A}))$ after $3m$ rounds. Here $\mathcal{A} = \bigcup_{i=1}^{m}\{x_i \mapsto b_i, y_i \mapsto b_i'\}$ is an assignment that $\mathbb{X}$ and $\mathbb{Y}$ generates. It follows from Lemma 15 and rule (8) that if $\mathbb{Y}$ has a w.s. then Defender can ensure that $X_{m+1}\gamma(\mathcal{A}) \simeq X_{m+1}'\gamma(\mathcal{A})$ by the strategy; otherwise if $\mathbb{X}$ has a w.s. then Attacker can use it to reach a pair that guarantee $X_{m+1}\gamma(\mathcal{A}) \not\approx X_{m+1}'\gamma(\mathcal{A})$. ◀

6 Conclusion

The initial motivation of this paper is to confirm the EXPTIME-hardness claim of branching bisimilarity of nBPA [5, 6]. The main contribution of this work is a technique to design a structure with a large number of redundant sets. One can use the binary counter constructed in Section 3.3 to represent the content of the tape of an ALBA. The construction allows to overwrite symbols in cells of the tape and to check their values. This would produce another EXPTIME-hard reduction for the branching bisimilarity of nBPA from the acceptance problem of ALBA . The PSPACE-hard lower bound of branching regularity is a byproduct in the development of this work. There is a gap in the complexity of branching regularity of nBPA. Czerwiński and Jančar proved that it can be decided in NEXPTIME [4]. An EXPTIME algorithm is also feasible based on the exponentially large bisimulation base [6]. Whether there exists a PSPACE algorithm is a natural further question. Another interesting research direction concerns the branching bisimilarity of nBPP (normed Basic Parallel Process), the parallel counterpart of nBPA in the PRS hierarchy [12]. Czerwiński, Hofman and Lasota proved the decidability of this problem [3]. However, the complexity is not clear. The current lower bound is PSPACE-hard [17]. For the branching regularity of nBPP, the lower bound is also PSPACE-hard [17], while the decidability is open. There are huge gaps worth further study. We hope that the technique introduced in this paper could shed some new light on these problems.

Acknowledgments. We thank Prof. Petr Jančar for his kind reminder of the EXPTIME-hardness claim and the members of BASICS for their interest. We thank Prof. Yuxi Fu, Dr. Huan Long and the anonymous reviewers for their comments to improve this paper.

References

1 J. C. M. Baeten, J. A. Bergstra, and J. W. Klop. Decidability of bisimulation equivalence for process generating context-free languages. *J. ACM*, 40(3):653–682, 1993. doi:10.1145/174130.174141.

2 J. Balcázar, J. Gabarró, and M. Sántha. Deciding bisimilarity is P-complete. *Formal aspects of computing*, 4(1):638–648, 1992. doi:10.1007/BF03180566.

3 W. Czerwiński, P. Hofman, and S. Lasota. Decidability of branching bisimulation on normed commutative context-free processes. *Theory of Computing Systems*, 55(1):136–169, 2014. doi:10.1007/s00224-013-9505-9.

4 W. Czerwiński and P. Jančar. Branching bisimilarity of normed BPA processes is in NEXPTIME. In *Proc. LICS 2015*, pages 168–179. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.25.

5 Y. Fu. Checking equality and regularity for normed BPA with silent moves. In *Proc. ICALP'13 (II)*, volume 7966 of *LNCS*, pages 238–249. Springer-Verlag, 2013. doi:10.1007/978-3-642-39212-2_23.

6 C. He and M. Huang. Branching bisimilarity on normed BPA is EXPTIME-complete. In *Proc. LICS 2015*, pages 180–191. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.26.

7 P. Jančar. Branching Bisimilarity of Normed BPA Processes as a Rational Monoid. *arXiv preprint*, 2016. arXiv:1602.05151.

8 P. Jančar and J. Srba. Undecidability of bisimilarity by defender's forcing. *J. ACM*, V:1–26, 2008. doi:10.1145/1326554.1326559.

9 M. Jurdziński, F. Laroussinie, and J. Sproston. Model checking probabilistic timed automata with one or two clocks. In *Proc. TACS'07*, pages 170–184. Springer, 2007. doi:10.1007/978-3-540-71209-1_15.

10 S. Kiefer. BPA bisimilarity is EXPTIME-hard. *Information Processing Letters*, 113(4):101–106, 2013. doi:10.1016/j.ipl.2012.12.004.

11 A. Kučera and P. Jančar. Equivalence-checking with infinite-state systems: Techniques and results. *Theory and Practice of Logic Programming*, 6(3):227–264, 2006. doi:10.1017/S1471068406002651.

12 R. Mayr. Process Rewrite Systems. *Information and Computation*, 156(1):264–286, 2000. doi:10.1006/inco.1999.2826.

13 R. Mayr. Weak bisimilarity and regularity of context-free processes is EXPTIME-hard. *Theoretical Computer Science*, 330(3):553–575, 2005. doi:10.1016/j.tcs.2004.10.008.

14 J. Srba. Applications of the Existential Quantification Technique. In *4th International Workshop on Verification of Infinite-State Systems*, pages 151–152, 2002.

15 J. Srba. Strong bisimilarity and regularity of Basic Parallel Processes is PSPACE-hard. In *Proc. STACS'02*, pages 535–546. Springer, 2002.

16 J. Srba. Strong bisimilarity and regularity of Basic Process Algebra is PSPACE-hard. In *Proc. ICALP'02*, pages 716–727. Springer, 2002.

17 J. Srba. Complexity of weak bisimilarity and regularity for BPA and BPP. *Mathematical Structures in Computer Science*, 13(4):567–587, 2003. doi:10.1017/S0960129503003992.

18 J. Srba. Roadmap of infinite results. *Current Trends In Theoretical Computer Science*, 2(201):337–350, 2004. updated version at http://users-cs.au.dk/srba/roadmap/.

19 C. Stirling. The joys of bisimulation. In *Mathematical Foundations of Computer Science 1998*. Springer Berlin Heidelberg, 1998. doi:10.1007/BFb0055763.

20 J. Stříbrná. Hardness results for weak bisimilarity of simple process algebras. *Electronic Notes in Theoretical Computer Science*, 18:179–190, 1998. doi:10.1016/S1571-0661(05)80259-2.

21 W. Thomas. On the Ehrenfeucht-Fraïssé game in theoretical computer science. In *TAPSOFT'93: Theory and Practice of Software Development*, pages 559–568. Springer Berlin Heidelberg, 1993. doi:10.1007/3-540-56610-4_89.

Infinite-Duration Bidding Games*

Guy Avni[1], Thomas A. Henzinger[2], and Ventsislav Chonev[3]

1 **IST Austria, Klosterneuburg, Austria**
2 **IST Austria, Klosterneuburg, Austria**
3 **Max Planck Institute for Software Systems (MPI-SWS), Saarbrücken, Germany**

Abstract

Two-player games on graphs are widely studied in formal methods as they model the interaction between a system and its environment. The game is played by moving a token throughout a graph to produce an infinite path. There are several common modes to determine how the players move the token through the graph; e.g., in turn-based games the players alternate turns in moving the token. We study the *bidding* mode of moving the token, which, to the best of our knowledge, has never been studied in infinite-duration games. Both players have separate *budgets*, which sum up to 1. In each turn, a bidding takes place. Both players submit bids simultaneously, and a bid is legal if it does not exceed the available budget. The winner of the bidding pays his bid to the other player and moves the token. For reachability objectives, repeated bidding games have been studied and are called *Richman games* [36, 35]. There, a central question is the existence and computation of *threshold* budgets; namely, a value $t \in [0, 1]$ such that if Player 1's budget exceeds t, he can win the game, and if Player 2's budget exceeds $1 - t$, he can win the game. We focus on parity games and mean-payoff games. We show the existence of threshold budgets in these games, and reduce the problem of finding them to Richman games. We also determine the strategy-complexity of an optimal strategy. Our most interesting result shows that memoryless strategies suffice for mean-payoff bidding games.

1998 ACM Subject Classification J.4 Social and Behavioral Sciences, F.1.2 Modes of Computation

Keywords and phrases Bidding games, Richman games, Parity games, Mean-payoff games

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.21

1 Introduction

Two-player infinite-duration games on graphs are an important class of games as they model the interaction of a system and its environment. Questions about automatic synthesis of a reactive system from its specification [40] are reduced to finding a winning strategy for the "system" player in a two-player game. The game is played by placing a token on a vertex in the graph and allowing the players to move it throughout the graph, thus producing an infinite trace. The winner or value of the game is determined according to the trace. There are several common modes to determine how the players move the token that are used to model different types of systems (c.f., [4]). The most well-studied mode is *turn-based*, where the vertices are partitioned between the players and the player who controls the vertex on which the token is placed, moves it. Other modes include *probabilistic* and *concurrent* moves.

* This research was supported in part by the Austrian Science Fund (FWF) under grants S11402-N23 (RiSE/SHiNE) and Z211-N23 (Wittgenstein Award).

We study a different mode of moving, which we refer to as *bidding*, and to the best of our knowledge, has never been studied for infinite-duration games. Both players have *budgets*, where for convenience, we have $B_1 + B_2 = 1$. In each turn a bidding takes place for the right to move the token. The players submit bids simultaneously, where a bid is legal if it does not exceed the available budget. Thus, a bid is a real number in $[0, B_i]$, for $i \in \{1, 2\}$. The player who bids higher pays the other player, and decides where the token moves. Draws can occur and one needs to devise a mechanism for resolving them (e.g., giving advantage to Player 1), and our results do not depend on a specific mechanism.

Bidding arises in many settings and we list several examples below. The players in a two-player game often model concurrent processes. Bidding for moving can model an interaction with a scheduler. The process that wins the bidding gets scheduled and proceeds with its computation. Thus, moving has a cost and processes are interested in moving only when it is critical. When and how much to bid can be seen as quantifying the resources that are needed for a system to achieve its objective, which is an interesting question. Other takes on this problem include reasoning about which input signals need to be read by the system at its different states [20, 2] as well as allowing the system to read chunks of input signals before producing an output signal [28, 27, 33]. Also, our bidding game can model *scrip systems* that use internal currencies for bidding in order to prevent "free riding" [31]. Such systems are successfully used in various settings such as databases [43], group decision making [42], resource allocation, and peer-to-peer networks (see [29] and references therein). Finally, repeated bidding is a form of a sequential auction [37], which is used in many settings including online advertising.

Recall that the winner or value of the game is determined according to the outcome, which is an infinite trace. There are several well-studied objectives in games. The simplest objective is *reachability*, where Player 1 has a target vertex and a trace is winning for him iff it visits the target. Bidding reachability games are equivalent to *Richman games* [36, 35], named after David Richman. Richman games are the first to study the bidding mode of moving. The central question that is studied on Richman games regards a *threshold budget*, which is a function $\textsc{Thresh} : V \to [0, 1]$ such that if Player 1's budget exceeds $\textsc{Thresh}(v)$ at a vertex v, then he has a strategy to win the game. On the other hand, if Player 2's budget exceeds $1 - \textsc{Thresh}(v)$, he can win the game (recall that the budgets add up to 1). In [36, 35], the authors show that threshold budgets exist, are unique, and that finding them is in NP. We slightly improve their result by showing that the problem is in NP and coNP. We illustrate the bidding model and the threshold problem in the following example.

▶ **Example 1.** Consider for example, the bidding reachability game that is depicted in Figure 1. Player 1's goal is to reach t, and Player 2's goal is to prevent this from happening. How much budget suffices for Player 1 to guarantee winning? Clearly, even if Player 1 has all the budget, he cannot win in v_1, thus $\textsc{Thresh}(v_1) = 1$. Similarly, even if Player 2 has all the budget in t, Player 1 has already won, thus $\textsc{Thresh}(t) = 0$. We show a naive solution in which Player 1 wins when his budget exceeds 0.75. Indeed, if Player 1's budget is $0.75 + \epsilon$, for $\epsilon > 0$, then since the budgets add up to 1, Player 2's budget is $0.25 - \epsilon$. In the first turn, Player 1 bids $0.25 + \frac{\epsilon}{2}$ and wins the bidding since Player 2 cannot bid above 0.25. He pays his bid to Player 2 and moves the token to v_2. Thus, at the end of the round, the budgets are $0.5 + \frac{\epsilon}{2}$ and $0.5 - \frac{\epsilon}{2}$ and the token is on v_2. In the second bidding, Player 1 bids all his budget, wins the bidding since Player 2 cannot bid above 0.5, moves the token to t, and wins the game. It turns out that the threshold budgets are lower: it follows from Theorem 3 that they are $\textsc{Thresh}(v_0) = 2/3$ and $\textsc{Thresh}(v_2) = 1/3$. ◀

Figure 1 On the left, a bidding reachability game. On the right, a bidding mean-payoff game where the weights are depicted on the edges.

We introduce and study infinite duration bidding games with richer qualitative objectives as well as quantitative objectives. *Parity games* are an important class of qualitative games as the problem of reactive synthesis from LTL specifications is reduced to a parity game. The vertices in a parity game are labeled by an index in $\{0, \ldots, d\}$, for some $d \in \mathbb{N}$, and an infinite trace is winning for Player 1 iff the parity of the maximal index that is visited infinitely often is odd. The quantitative games we focus on are *mean-payoff* games. An infinite outcome has a value, which can be thought of as the amount of money that Player 1 pays Player 2. Accordingly, we refer to the players in a mean-payoff game as *Maximizer* (Max, for short) and *Minimizer* (Min, for short). The vertices of a mean-payoff game are labeled by values in $\mathbb{Z}$. Consider an infinite trace π. The *energy* of a prefix π^n of length n of π, denoted $E(\pi^n)$, is the sum of the values it traverses. The mean-payoff value of π is $\liminf_{n\to\infty} E(\pi^n)/n$. We are interested in cases where Min can guarantee a non-positive mean-payoff value. It suffices to show that he can guarantee that an infinite outcome π either has infinitely many prefixes with $E(\pi^n) = 0$, or that the energy is bounded, thus there is $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, we have $E(\pi^n) \leq N$. We stress the point that there are two "currencies" in the game: a "monopoly money" that is used to determine who moves the token and which the players do not care about once the game ends, and the values on the vertices, which is the value that Min and Max seek to minimize and maximize, respectively. We illustrate mean-payoff games with the following example.

▶ **Example 2.** Consider the mean-payoff bidding game that is depicted in Figure 1, where for convenience the values are placed on the edges and not on the vertices. We claim that Min has a strategy that guarantees a non-positive mean-payoff value. Without loss of generality, Max always chooses the 1-valued edge. Min's strategy is a *tit-for-tat*-like strategy, and he always takes the (-1)-valued edge. The difficulty is in finidng the right bids. Initially, Min bids 0. Assume Max wins a bidding with $b > 0$. Min will try and *match* this win: he bids b until he wins with it. Let $b_1, \ldots, b_n$ be Max's winning bids before Min wins with b. We call these *un-matched* bids. The next bid Min attempts to match is $b' = \min_{1\leq i\leq n} b_i$; he bids b' until he wins with it, and continues similarly until all bids are matched.

We claim that the tit-for-tat strategy guarantees a non-positive mean-payoff value. Observe first that if a prefix of the outcome has k unmatched bids, then the energy is k. In particular, if all bids are matched, the energy is 0. Suppose Min bids b. We claim that the number of un-matched bids is at most $\lceil 1/b \rceil$. Otherwise, since b is less than all other un-matched bids, Max would need to invest more than a budget of 1. It follows that an infinite outcome that never reaches energy level 0 has bounded energy, thus the mean-payoff value is non-positive. ◀

We study the existence and computation of threshold budgets in parity and mean-payoff bidding games. Also, we determine the strategy complexity that is necessary for winning. Recall that a winning strategy in a game typically corresponds to an implementation of a system. A strategy that uses an unbounded memory, like the tit-for-tat strategy above, is not useful for implementing. Thus, our goal is to find strategies that use little or no memory, which are known as *memoryless* strategies.

We show that parity bidding games are linearly-reducible to Richman games allowing us to obtain all the positive results from these games; threshold budgets exist, are unique, and computing them is no harder than for Richman games, i.e., the problem is in NP and coNP. We find this result quite surprising since for most other modes of moving, parity games are considerably harder than reachability games. The crux of the proof considers bottom strongly-connected components (BSCCs, for short) in the arena, i.e., SCCs with no exiting edges. We show that in a strongly connected bidding parity game, exactly one of the players wins with every initial budget, thus the threshold budgets of the vertices of a BSCC are in $\{0, 1\}$. If the vertex with highest parity in a BSCC is odd, then Player 1 wins, i.e., the threshold budgets are all 0, and otherwise Player 2 wins, i.e., the threshold budgets are all 1. We can thus construct a Richman game by setting the target of Player 1 to the BSCCs that are winning for him and the target of Player 2 to the ones that are winning for him. Moreover, we show that memoryless strategies are sufficient for winning in these games.

We proceed to study mean-payoff bidding games. We adapt the definition of threshold values; we say that $t \in [0, 1]$ is a threshold value for Min if with a budget that exceeds t, Min can guarantee a non-positive mean-payoff value. On the other hand, if Max's budget exceeds $1 - t$, he can guarantee a positive mean-payoff value. We show that threshold values exist and are unique in mean-payoff bidding games. The crux of the existence proof again considers the BSCCs of the game. We show that in a strongly-connected mean-payoff bidding game, the threshold budgets are in $\{0, 1\}$, thus again either Min "wins" or Max "wins" the game. Moreover, this classification can be determined in NP and coNP, thus the complexity of solving bidding mean-payoff games coincides with Richman games. Our results for strongly-connected games are obtained by developing the connection that was observed in [36, 35] between the threshold budget and the reachability probability in a probabilistic model on the same structure as the game. We show a connection between bidding mean-payoff games and *one-counter* 2.5*-player games* [14, 13] to prove the classification of BSCCs. In turn, these games are equivalent to discrete *quasi-birth-death processes* [24] and generalize *solvency games* [11], which can be thought of as a rewarded Markov decision process with a single vertex.

The classification above is existential in nature and does not provide any insight on how a player guarantees a mean-payoff value. Our most technically challenging results concern the constructions strategies for Min and Max. The challenging part of the construction is reasoning about strongly-connected bidding mean-payoff games. Consider a strongly-connected game in which Min can guarantee a non-positive mean-payoff value. The idea of our construction is to tie between changes in Min's budget with changes in the energy; *investing* one unit of budget (with the appropriate normalization) implies a decrease of a unit of energy, and on the other hand, an increase of a unit of energy implies a *gain* of one unit of budget. Since the budgets are bounded by 1, the value cannot increase arbitrarily. Finding the right bids in a general SCC is not trivial, and we find our solution to be surprisingly elegant. The case where Max can guarantee a positive mean-payoff value, is more challenging. Unlike a memoryless strategy for Min, the normalization factor must decrease as the value increases so that Max does not exhaust his budget. We show constant memory strategies in general and identify a fragment in which we show memoryless strategies.

Further bidding games

Variants of bidding games where studied in the past. Already in [35] several variants are studied including a *poorman* version in which the winner of the bidding pays the bank, thus the amount of money in the game decreases as the game proceeds. Motivated by recreational

games, e.g., bidding chess, *discrete bidding games* are studied in [23], where the money is divided into chips, so a bid cannot be arbitrarily small as in the bidding games we study. In *all-pay* bidding games [38], the players all pay their bids to the bank. Non-zero-sum two-player games were recently studied in [30]. They consider a bidding game on a directed acyclic graph. Moving the token throughout the graph is done by means of bidding. The game ends once the token reaches a sink, and each sink is labeled with a pair of payoffs for the two players that do not necessarily sum up to 0. They show existence of *subgame perfect equilibrium* for every initial budget and a polynomial algorithm to compute it.

Due to lack of space, most of the proofs appear in the full version [7].

2 Preliminaries

An *arena* is a pair $\langle G, \alpha \rangle$, where G is a directed graph and α is an objective. A game is played on an arena as follows. A token is placed on a vertex in the arena and the players move it throughout the graph. The *outcome* is an infinite path π. The winner or value is determined according to π and α as we elaborate below. There are several common modes in which the players move the token. In *turn-based games* the vertices are partitioned between the players and the player who controls the vertex on which the token is placed, moves it. Another mode is *probabilistic* choices, where the game can be thought of as a *Markov chain*, thus the edges are labeled with probabilities, and the edge on which the token proceeds is chosen randomly. A combination of these two modes is called *2.5-player* games, where the vertices are partitioned into three sets: Player 1 vertices, Player 2 vertices, and probabilistic vertices. Finally, in *concurrent* games, each player has a possible (typically finite) set of actions he can choose from in a vertex. The players select an action simultaneously, and the choice of actions dictates to which vertex the token moves.

We study a different mode of moving, which we call *bidding*. Both players have budgets, where for convenience, we have $B_1 + B_2 = 1$. In each turn, a bidding takes place to determine who moves the token. Both players submit bids simultaneously, where a bid is a real number in $[0, B_i]$, for $i \in \{1, 2\}$. The player who bids higher pays the other player and decides where the token moves. Note that the sum of budgets always remains 1. While draws can occur, in the questions we study we try avoid the issue of draws.

A *strategy* prescribes to a player which *action* to take in a game, given a finite *history* of the game, where we define these two notions below. In 2.5-player games, histories are paths and actions are vertices. Thus, a strategy for Player i, for $i \in \{1, 2\}$, takes a finite path that ends in a Player i vertex, and prescribes to which vertex the token moves to next. In bidding games, histories and strategies are more complicated as they maintain the information about the bids and winners of the bids. A history is a sequence of the form $v_0, \langle v_1, b_1, i_1 \rangle, \langle v_2, b_2, i_2 \rangle, \ldots, \langle v_k, b_k, i_k \rangle \in V \cdot (V \times [0, 1] \times \{1, 2\})^*$, where, for $j \geq 1$, in the j-th round, the token is placed on vertex v_{j-1}, the winning bid is b_j, and the winner is Player i_j, and Player i_j moves the token to vertex v_j. An action for a player is $\langle b, v \rangle \in ([0, 1] \times V)$, where b is the bid and v is the vertex to move to upon winning. An initial vertex v_0 and strategies f_1 and f_2 for Players 1 and 2, respectively, determine a unique *outcome* π for the game, denoted $out(v_0, f_1, f_2)$, which is an infinite sequence in $V \cdot (V \times [0, 1] \times \{1, 2\})^\omega$. We sometimes abuse notation and refer to $out(v_0, f_1, f_2)$ as a finite prefix of the infinite outcome. We drop v_0 when it is clear from the context. We define the outcome inductively. The first element of the outcome is v_0. Suppose $\pi_1, \ldots, \pi_j$ is defined. The players bids are given by $\langle b_1, v_1 \rangle = f_1(\pi_1, \ldots, \pi_j)$ and $\langle b_2, v_2 \rangle = f_2(\pi_1, \ldots, \pi_j)$. If $b_1 > b_2$, then $\pi_{j+1} = \langle v_1, b_1, 1 \rangle$, and dually when $b_1 < b_2$, we have $\pi_{j+1} = \langle v_2, b_2, 2 \rangle$. We assume there is some tie-breaking

mechanism that determines who the winner is when $b_1 = b_2$, and our results are not affected by what the tie-breaking mechanism is. Consider a finite outcome π. The *payment* of Player 1 in π, denoted $\mathcal{B}_1(\pi)$, is $\sum_{1 \leq j \leq |\pi|} (-1)^{3-i_j} b_j$, and Player 2's payment, denoted $\mathcal{B}_2(\pi)$ is defined similarly. For $i \in \{1, 2\}$, consider an initial budget $B_i^{init} \in [0, 1]$ for Player i. A strategy f is *legal* for Player i with respect to B_i^{init} if for every $v_0 \in V$ and strategy g for the other player, Player i's bid in a finite outcome $\pi = out(v_0, f, g)$ does not surpass his budget. Thus, for $\langle b, v \rangle = f(\pi)$, we have $b \leq B_i^{init} - \mathcal{B}_i(\pi)$.

Richman games and threshold budgets

The simplest qualitative objective is reachability: Player 1 has a target vertex v_R and an infinite outcome is winning for him if it visits v_R. Reachability bidding games are known as Richman games [36, 35]. In Richman games both players have a target, which we denote by v_R and v_S. The game ends once one of the targets is reached. Note that this definition is slightly different from standard reachability games since there, Player 2 has no target and his goal is to keep the game from v_R. Though, we show that for our purposes, since Richman games have no ties, reachability games are equivalent to Richman games (see Lemma 4).

The central question that is studied on bidding games regards a *threshold budget*. A threshold budget is a function $\textsc{Thresh} : V \to [0, 1]$ such that if Player 1's budget exceeds $\textsc{Thresh}(v)$ at a vertex v, then he has a strategy to win the game. On the other hand, if Player 2's budget exceeds $1 - \textsc{Thresh}(v)$, he can win the game. We sometimes use $\textsc{Thresh}_1(v)$ to refer to $\textsc{Thresh}(v)$ and $\textsc{Thresh}_2(v)$ to refer to $1 - \textsc{Thresh}(v)$. We formalize the problem of finding threshold budgets as a decision problem. We define the THRESH-BUDG problem, which takes as input a bidding game $\mathcal{G}$, a vertex v, and a value $t \in [0, 1]$, and the goal is to decide whether $\textsc{Thresh}(v) = t$.

Threshold values are shown to exist in [36] as well as how to compute them. We review briefly their results. Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S \rangle$. We define the *Richman function* as follows. We first define $R(v, i)$, for $i \in \mathbb{N} \cup \{0\}$, where the intuition is that if Player 1's budget exceeds $R(v, i)$, he can win in at most i steps. We define $R(v_R, 0) = 0$ and $R(v, 0) = 1$ for every other vertex $v \in V$. Indeed, Player 1 can win in 0 steps from v_R no matter what his initial budget is, and even if he has all the budget, he cannot win in 0 steps from anywhere else. Consider $i \in \mathbb{N}$ and $v \in V$. We denote by $adj(v) \subseteq V$, the adjacent vertices to v, so $u \in adj(v)$ iff $E(v, u)$. Let v^+ be the vertex that maximizes the expression $\max_{u \in adj(v)} R(u, i-1)$, and let v^- be the vertex that minimizes the expression $\min_{u \in adj(v)} R(u, i-1)$. We define $R(v, i) = \frac{1}{2}\big(R(v^+, i-1) + R(v^-, i-1)\big)$. We define $R(v) = \lim_{i \to \infty} R(v, i)$. The following theorem shows that $R(v)$ equals $\textsc{Thresh}(v)$, and throughout the paper we use them interchangeably. We give the proof of the theorem for completeness.

▶ **Theorem 3.** *[36] For every $v \in V$, we have $\textsc{Thresh}(v) = R(v)$, thus if Player 1's budget at v exceeds $R(v)$, he can win from v, and if Player 2's budget exceeds $1 - R(v)$, he can win from v.*

Proof. We prove for Player 1 and the proof for Player 2 is dual. Let $t \in \mathbb{N}$ be an index such that $B_1^{init} > R(v, t)$. We prove by induction on t that Player 1 wins in at most t steps. The base case is easy. For the inductive step, assume Player 1 has a budget of $R(v, i) + \epsilon$. He bids $b_1 = \frac{1}{2}\big(R(v^+, i-1) - R(v^-, i-1)\big)$. If he wins the bidding, he proceeds to v^- with a budget of $R(v^-, i-1) + \epsilon$. If he loses, then Player 2's bid exceeds b_1 and the worst he can do is move to v^+. But then Player 1's budget is at least $R(v^+, i-1) + \epsilon$. By the induction hypothesis, Player 1 wins in at most $i - 1$ steps from both positions. ◀

We make precise the equivalence between reachability and Richman games.

▶ **Lemma 4.** *Consider a bidding reachability game $\mathcal{G} = \langle V, E, T\rangle$, where $T \subseteq V$ is a target set of vertices for Player 1. Let $S \subseteq V$ be the vertices with no path to T. Consider the Richman game $\mathcal{G}' = \langle V \cup \{v_R, v_S\}, E', v_R, v_S\rangle$, where $E' = E \cup \{\langle v, v_R\rangle : v \in T\} \cup \{\langle v, v_S\rangle : v \in S\}$. For every $v \in V$, the threshold budget of v in $\mathcal{G}$ equals the threshold budget of v in $\mathcal{G}'$.*

Finding threshold budgets

The authors in [35] study the complexity of threshold-budget problem and show that is in NP. They guess, for each vertex v its neighbors v^- and v^+, and devise a linear program with the constraints $R(v) = \frac{1}{2}\big(R(v^-) + R(v^+)\big)$ and, for every neighbor v' of v, we have $R(v^-) \leq R(v') \leq R(v^+)$. The program has a solution iff the guess is correct. They leave open the problem of determining the exact complexity of finding the threshold budgets, and they explicitly state that it is not known whether the problem is in P or NP-hard.

We improve on their result by showing that THRESH-BUDG is in NP and coNP. Our reduction uses an important observation that is made in [36], which will be useful later on. They connect between threshold budgets and reachability probabilities in Markov chains.

▶ Observation 5. Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S\rangle$. Let $M(\mathcal{G})$ be a Markov chain in which for each vertex $v \in V$, the probability of the edges $\langle v, v^+\rangle$ and $\langle v, v^-\rangle$ is $\frac{1}{2}$ and the other outgoing edges from v have probability 0. Then, since $R(v) = \frac{1}{2}\big(R(v^+) + R(v^-)\big)$, in $M(\mathcal{G})$, the probability of reaching v_R from v is $\textsc{Thresh}(v)$.

We reduce THRESH-BUDG to the problem of "solving" a *simple stochastic game* (SSG, for short) [22]. An SSG has two players; one tries to minimize the probability that the target is reached, and the second player tries to minimize it. It is well-known that the game has a *value*, which is the probability of reaching the target when both players play optimally. The problem of finding the value of an SSG is known to be in NP ∩ coNP. The SSG we construct can be seen as a turn-based game in which the player whose turn it is to move is chosen uniformly at random. The details of the proof can be found in the full version.

▶ **Theorem 6.** *THRESH-BUDG for Richman games is in NP ∩ coNP.*

We stress the fact that the strategies in SSGs are very different from bidding games. As mentioned above, there, the strategies only prescribe which vertex to move the token to, whereas in bidding games, a strategy also prescribes what the next bid should be. So, a solution of a Richman game by reducing it to an SSG is existential in nature and does not give insight on the bids a player uses in his winning strategy. We will return to this point later on.

Objectives

We study zero-sum games. The qualitative games we focus on are parity games. A parity game is a triple $\langle V, E, p\rangle$, where $p : V \to \{0, \ldots, d\}$ is a parity function that assigns to each vertex a *parity index*. An infinite outcome is winning for Player 1 iff the maximal index that is visited infinitely often is odd. The quantitative games we focus on are mean-payoff games. A mean-payoff game is $\langle V, E, w\rangle$, where $w : V \to \mathbb{Z}$ is a weight function on the vertices. We often refer to the sum of weights in a path as its *energy*. Consider an infinite outcome $\pi = v_0, \langle v_1, b_1, i_1\rangle, \ldots$. For $n \geq 0$, we use π^n to refer to the prefix of length n of π. The energy of π^n, denoted $E(\pi^n)$, is $\sum_{0 \leq i \leq n-1} w(v_i)$. We define the mean-payoff value of π to be $\liminf_{n\to\infty} \frac{E(\pi^n)}{n}$. The value of π can be thought of as the amount of money Player 1 pays

Player 2. Note that the mean-payoff values do not affect the budgets of the players. That is, the game has two currencies: a "monopoly money" that is used to determine who moves the token and which the players do not care about once the game ends, and the mean-payoff value that is determined according to the weights of the vertices, which is the value that Min and Max seek to minimize and maximize, respectively. Consider a finite outcome π. We use $\mathcal{B}_m(\pi)$ and $\mathcal{B}_M(\pi)$ to denote the sum of payments of Min and Max in the bids. Throughout the paper we use m and M to refer to Min and Max, respectively.

Strategy complexity

Recall that a winning strategy in a two-player game often corresponds to a system implementation. Thus, we often search for strategies that use limited or no memory. That is, we ask whether a player can win even with a *memoryless* strategy, which is a strategy in which the action depends only on the position of the game and not on the history. For example, in turn-based games, for $i \in \{1, 2\}$, a memoryless strategy for Player i prescribes, for each vertex $v \in V_i$, a successor vertex u. It is well known that memoryless strategies are sufficient for winning in a wide variety of games, including turn-based parity games and turn-based mean-payoff games. In Richman games, the threshold budgets tell us who the winner of the game is. But, they do not give insight on how the game is won game, namely what are the bids the winning player bids in order to win. Particularly, when the threshold budgets are 0 as we shall see in Lemmas 7 and 12.

We extend the definition of memoryless strategies to bidding games, though the right definition is not immediate. One can define a memoryless strategy as a function from vertex and budget to action (i.e., bid and vertex) similar to the definition in other games. However, this definition does not preserve the philosophy of implementation with no additional memory. Indeed, recall the proof of Theorem 3. One can define a strategy that, given a vertex $v \in V$ and a budget B, bids according to $R_t(v)$, where t is the minimal index such that $R_t(v) < B$. Clearly, the memory that is needed to implement such a strategy is infinite.

To overcome this issue, we use a different definition. We define a memoryless strategy in a vertex $v \in V$ with initial budget $B \in [0, 1]$ as a pair $\langle u, f_v^B \rangle$, where $u \in adj(v)$ is the vertex to proceed to upon winning and $f_v^B : [0, 1] \to [0, 1]$ is a function that takes the current budget and, in mean-payoff games, also the energy, and returns a bid. We require that f_v^B is *simple*, namely a polynomial or a selection between a constant number of polynomials. For simplicity, we assume a memoryless strategy is generated for an initial vertex with an initial budget, thus there can be different strategies depending where the game starts and with what budget. Also, we call a concatenation of memoryless strategies, a memoryless strategy.

3 Parity Bidding Games

We study threshold budgets in bidding parity games. We first study strongly-connected parity games and show a classification for them; either Player 1 wins with every initial budget or Player 2 wins with every initial budget.

▶ **Lemma 7.** *Consider a strongly-connected parity game $\mathcal{G} = \langle V, E, p \rangle$. There exists $\tau \in \{0, 1\}$ such that for every $v \in V$, we have $R(v) = \tau$. Moreover, we have $\tau = 0$ iff $\max_{v \in V} p(v)$ is odd.*

Proof. The proof relies on the following claim: Player 1 wins a Richman game in which only his target is reachable, with every initial budget. The claim clearly implies the lemma as we

view a strongly-connected bidding parity game as a Richman game in which Player 1 tries to force the game to the vertex with the highest parity index, and Player 2 has no target, thus Player 1 wins with every initial budget. The claim is similar for Player 2. The proof of the claim follows from the fact that the threshold budget of a vertex $v \in V$ is some average between $\text{THRESH}(v_R)$ and $\text{THRESH}(v_S)$, and the average depends on the distances of v to the two targets. When only Player 1's target is reachable, we have $\text{THRESH}(v) = 0$. The details of the proof can be found in the full version. ◀

Consider a bidding parity game $\mathcal{G} = \langle V, E, p \rangle$. Let R and S be the set of vertices in the BSCCs that are winning for Player 1 and Player 2, respectively. Let $\mathcal{G}'$ be the Richman game that is obtained from $\mathcal{G}$ by setting the target of Player 1 to be the vertices in R and the target of Player 2 to be the vertices in S. The following lemma follows from Lemma 7.

▶ **Lemma 8.** *For every $v \in V$, we have that $\text{THRESH}(v)$ in $\mathcal{G}$ equals $\text{THRESH}(v)$ in $\mathcal{G}'$.*

Lemma 8 allows us to obtain the positive results of Richman games in parity bidding games. In the full version, we construct memoryless strategies in Richman games. The idea is to show that a bid at a vertex v of the form $\frac{R(v^+)-R(v^-)}{2} + \epsilon$ guarantees that either v_R is reached within $|V|$ steps, or Player 1's budget increases by a constant. Thus, we have the following.

▶ **Theorem 9.** *The threshold budgets in parity bidding games exist, are unique, THRESH-BUDG is in NP ∩ coNP, and memoryless strategies suffice for winning.*

4 Mean-Payoff Bidding Games

We proceed to study mean-payoff games. We adjust the definition of threshold budgets to the quantitative setting.

▶ **Definition 10.** Consider a mean-payoff bidding game $\mathcal{G} = \langle V, E, w \rangle$. The threshold budget in a vertex $v \in V$, denoted $\text{THRESH}(v)$, is a value $t \in [0, 1]$ such that

1. If Min's budget exceeds t at v, then he can guarantee a non-positive mean-payoff value, and
2. if Max's budget exceeds $1 - t$, then he can guarantee a strictly positive value.

4.1 Solving Bidding Mean-Payoff Games

In this section we solve the problem of finding threshold values in bidding mean-payoff games. Our solution relies on work on probabilistic models, namely *one-counter simple stochastic games* [14, 13], and it is existential in nature. Namely, knowing what the threshold budget is in v does not give much insight on how Min guarantees a non-negative value even if he has sufficient budget, and similarly for Max. Constructing concrete memoryless strategies for the two players is much more challenging and we show constructions in the following sections.

Recall that in bidding parity games, we showed a classification for strongly-connected games; namely, the threshold budgets in all vertices are in $\{0, 1\}$, thus either Player 1 wins with every initial budget or Player 2 wins with every initial budget. We show a similar classification for strongly-connected bidding mean-payoff games: the threshold budgets in all vertices of a strongly-connected bidding mean-payoff game are in $\{0, 1\}$, thus in a strongly-connected bidding mean-payoff game, for every initial energy and every initial budget, either

Min can guarantee a non-positive mean-payoff value or Max can guarantee a positive mean-payoff value. The classification uses a generalization of the Richman function to weighted graphs. Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$ and a vertex $u \in V$. We construct a graph $\mathcal{G}^u = \langle V^u, E^u, w^u\rangle$ by making two copies u_s and u_t of u, where u_s has no incoming edges and u_t has no outgoing edges. Thus, a path from u_s to u_t in $\mathcal{G}^u$ corresponds to a loop in $\mathcal{G}$. Recall that we denote by $w(v)$ the weight of the vertex v.

▶ **Definition 11.** Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$, a vertex $u \in V$. We define the *weighted Richman function* $W : V \to \mathbb{Q}$ first on G^u. We define $W(u_t) = 0$ and for every $v \in (S \setminus \{u_t\})$, we define $W(v) = \frac{1}{2}\big(W(v^+) + W(v^-)\big) + w(v)$. In order to define W on $\mathcal{G}$, we define $W(u)$ to be $W(u_s)$ in $\mathcal{G}^u$.

We use the connection with probabilistic models as in Observation 5 in order to show that W is well defined. We view $\mathcal{G}^u$ as a rewarded Markov chain, in which, for $v \in V$, the outgoing edges from v with positive probability probabilities are $\langle v, v^+\rangle$ and $\langle v, v^-\rangle$, and their probability is $1/2$. The function W coincides with the expected reward of a run that starts and returns to u, which in turn is well-defined since the probability of returning to u is 1.

Similarly to the connection we show in Theorem 6 between Richman values and reachability probabilities in a simple-stochastic game, we prove Lemma 12 by connecting the threshold value in bidding mean-payoff games to the probability that a counter in a one-counter simple-stochastic games reaches value 0. We then use results from [14, 13] on this model to prove the lemma. The proof can be found in the full version.

▶ **Lemma 12.** *Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$. There is $\tau \in \{0, 1\}$ such that for every $v \in V$, we have $\textsc{Thresh}(v) = \tau$. Moreover, we have $\tau = 0$ iff there exists $u \in V$ with $W(u) \leq 0$.*

Lemma 13, which is also helpful in the following sections, shows how to connect the mean-payoff value with the objective of reaching energy 0 or maintaining non-negative energy. Its proof can be found in the full version.

▶ **Lemma 13.** *Consider a strongly-connected bidding mean-payoff game $\mathcal{G}$ and a vertex u in $\mathcal{G}$.*

- *Suppose that for every initial budget and initial energy, Min has a strategy f_m and there is a constant $N \in \mathbb{N}$ such that for every Max strategy f_M, a finite outcome $\pi = out(u, f_m, f_M)$ either reaches energy 0 or the energy is bounded by N throughout π. Then, Min can guarantee a non-positive mean-payoff value in $\mathcal{G}$.*
- *If for every initial budget $B_M^{init} \in [0, 1]$ for Max there exists an initial energy level $n \in \mathbb{N}$ such that Max can guarantee a non-negative energy level in $\mathcal{G}$, then Max can guarantee a positive mean-payoff value in $\mathcal{G}$.*

The proof of the following theorem can be found in the full version. Deciding the classification in Lemma 12 can be done in NP and coNP by guessing the neighbors the vertices and using linear programming, similarly to Richman games. Then, we reduce bidding mean-payoff games to Richman games in a similar way to the proof of Lemma 8 for parity games.

▶ **Theorem 14.** *Threshold budgets exist in bidding mean-payoff games, they are unique, and THRESH-BUDG for bidding mean-payoff games is in NP ∩ coNP.*

4.2 A Memoryless Optimal Strategy for Min

We turn to the more challenging task of finding memoryless strategies for the players, and in this section we focus on constructing a strategy for Min. Theorem 9 and Lemma 12 allow us to focus on strongly-connected bidding mean-payoff games. Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$ that has a vertex $u \in V$ with $W(u) \leq 0$. We construct a Min memoryless strategy that guarantees that for every initial energy and every initial budget, either the energy level reaches 0 or it is bounded. By Lemma 13, this suffices for Min to guarantee a non-positive mean-payoff value in $\mathcal{G}$.

The idea behind our construction is to tie between changes in the energy level and changes of the budget. That is, in order to decrease the energy by one unit, Min needs to *invest* at most one unit of budget (with an appropriate normalization), and when Max increases the energy by one unit, Min's *gain* is at least one unit of budget. Our solution builds on an alternative solution to the two-loop game in Figure 1. This solution is inspired by a similar solution in [35].

▶ **Example 15.** Consider the bidding mean-payoff game that is depicted in Figure 1. We show a Min strategy that guarantees a non-positive mean-payoff value. Consider an initial Min budget of $B_m^{init} \in [0, 1]$ and an initial energy level of $k_I \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $B_m^{init} > \frac{k}{N}$. Min bids $\frac{1}{N}$ and takes the (-1)-weighted edge upon winning. Intuitively, Min invests $\frac{1}{N}$ for every decrease of unit of energy and, since by losing a bidding he gains at least $\frac{1}{N}$, this is also the amount he gains when the energy increases. Formally, it is not hard to show that the following invariant is maintained: if the energy level reaches $k \in \mathbb{N}$, Min's budget is at least $\frac{k}{N}$. Note that the invariant implies that either an energy level of 0 is reached infinitely often, or the energy is bounded by N. Indeed, in order to cross an energy of N, Max would need to invest a budget of more than 1. Lemma 13 implies that the mean-payoff value is non-positive, and we are done. ◀

Extending this result to general strongly connected games is not immediate. Consider a strongly-connected game $\mathcal{G} = \langle V, E, w\rangle$ and a vertex $u \in V$. We would like to maintain the invariant that upon reaching u with energy k, the budget of Min exceeds k/N, for a carefully chosen N. The game in the simple example above has two favorable properties that general SCCs do not necessarily have. First, unlike the game in the example, there can be infinite paths that avoid u, thus Min might need to invest budget in drawing the game back to u. Moreover, different paths from u to itself may have different energy levels, so bidding a uniform value (like the $\frac{1}{N}$ above) is not possible. The solution to these problems is surprisingly elegant and uses the weighted Richman function in Definition 11.

Consider an initial budget of $B_m^{init} \in [0, 1]$ for Min and an initial energy $k_I \in \mathbb{N}$. We describe Min's strategy f_m. At a vertex $v \in V$, Min's bid is $\frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$ and he proceeds to v^- upon winning, where we choose $N \in \mathbb{N}$ in the following. Let w_M be the maximal absolute weighted Richman value in $\mathcal{G}$, thus $w_M = \max_{v\in V} |W(v)|$. Let b_M be the maximal absolute "bid", thus $b_M = \max_{v\in V} |\frac{W(v^+)-W(v^-)}{2}|$. We choose $N \in \mathbb{N}$ such that $B_m^{init} > \frac{k_I+b_M+w_M}{N}$.

In the following lemmas we prove that f_m guarantees that an outcome either reaches energy level 0 or that the energy is bounded, as well as showing that f_m is legal, i.e., that Min always bids less than his budget. The following lemma is the crux of the construction as it connects the weighted Richman function with the change in energy and in budget. Recall that for a finite outcome π the accumulated energy in π is $E(\pi)$ and the payments of Min throughout π is $\mathcal{B}_m(\pi)$.

▶ **Lemma 16.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome that starts in a vertex v and ends in v'. Then, we have $W(v) - W(v') \geq E(\pi) + N \cdot \mathcal{B}_m(\pi)$.*

Proof. We prove by induction on the length of π. In the base case $v = v'$, thus $E(\pi) = \mathcal{B}_m(\pi) = 0$ and the claim is trivial. For the induction step, let b be the winning bid in the first round and let π' be the suffix of π after the first bidding. We distinguish between two cases. In the first case, Min wins the bidding, pays $b = \frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$, and proceeds to v^-. Thus, we have $E(\pi) + N \cdot \mathcal{B}_m(\pi) = w(v) + E(\pi') + N\big(b + \mathcal{B}_m(\pi')\big)$. By the induction hypothesis, we have $E(\pi') + N \cdot \mathcal{B}_m(\pi') \leq W(v^-) - W(v')$, thus $E(\pi) + N \cdot \mathcal{B}_m(\pi) \leq w(v) + \frac{W(v^+)-W(v^-)}{2} + W(v^-) - W(v') = W(v) - W(v')$, and we are done. For the second case, suppose Max wins the bidding. Min's gain is $-b < -\frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$, and Max proceeds to v'' having $W(v'') \leq W(v^+)$. Similar to the previous case, we have $E(\pi) + N \cdot \mathcal{B}_m(\pi) = w(v) + E(\pi') + N\big(-b + B_m(\pi')\big) \leq w(v) - \frac{W(v^-)-W(v^+)}{2} + W(v^+) - W(v') = W(v) - W(v')$, and we are done. ◀

The following corollary of Lemma 16 explains why we refer to our technique as "tying energy and budget". Its proof follows from the fact that $W(u_s) \leq 0$ and $W(u_t) = 0$.

▶ **Corollary 17.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome from u to u. Then, we have $-N \cdot \mathcal{B}_m(\pi) \leq E(\pi)$.*

We formalize the intuition above by means of an invariant that is maintained throughout the outcome. Recall that the game starts from a vertex $u \in V$ with $W(u) \leq 0$, the initial energy is $k_I \in \mathbb{N}$, Min's initial budget is $B_m^{init} \in [0, 1]$, and N is such that $B_m^{init} > \frac{k_I + b_M + w_M}{N}$.

▶ **Lemma 18.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome. Then, when the energy level reaches k, Min's budget is at least $\frac{k+b_M}{N}$.*

Proof. The invariant clearly holds initially. Consider a partition $\pi = \pi_1 \cdot \pi_2$, where π_1 is a maximal prefix of π that ends in u and π_2 starts in u and ends in a vertex $v \in V$. The energy level at the end of π is $k = k_I + E(\pi)$. Recall that $\mathcal{B}_m(\pi)$ is the sum of Min's payments in π, thus his budget at the end of π is $B_m^{init} - \big(\mathcal{B}_m(\pi_1) + \mathcal{B}_m(\pi_2)\big)$. By Corollary 17, we have $-\mathcal{B}_m(\pi_1) \geq \frac{1}{N} E(\pi_1)$ and by Lemma 16, we have $-\mathcal{B}_m(\pi_2) \geq \frac{1}{N}\big(E(\pi_2) - W(u) + W(v)\big) \geq \frac{1}{N}\big(E(\pi_2) - 0 - w_M\big)$. Combining with $B_m^{init} \geq \frac{k_I + b_M + w_M}{N}$, we have that the new budget is at least $\frac{k_I + b_M + w_M}{N} + \frac{E(\pi_1)}{N} + \frac{E(\pi_2) - w_M}{N} = \frac{k + b_M}{N}$, and we are done. ◀

Lemma 18 implies that Min always has sufficient budget to bid according to f_m, thus the strategy is legal. Moreover, since Min's budget cannot exceed 1, Lemma 18 implies that if the energy does not reach 0, then it is bounded by $N - b_M$. Thus, Lemma 13 implies that Min has a memoryless strategy that guarantees a non-positive mean-payoff value in a strongly-connected bidding mean-payoff game having a vertex u with $W(u) \leq 0$. Combining with the memoryless strategy in parity games, we have the following.

▶ **Theorem 19.** *Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ and a vertex $v \in V$. If Min's initial budget exceeds* Thresh(v)*, he has a memoryless strategy that guarantees a non-positive mean-payoff value.*

4.3 A Memoryless Optimal Strategy for Max

The complementary result of the previous section is more involved. Consider a strongly-connected bidding mean-payoff game $\mathcal{G}$ with a vertex u that has $W(u) > 0$. We devise a Max strategy that guarantees a positive mean-payoff value in $\mathcal{G}$. We start with a fragment

of the general case called *recurrent SCCs*, and we generalize our solution later. We say that an SCC $G = \langle V, E\rangle$ is a recurrent, if there is a vertex $u \in V$ such that every cycle in G includes u. We refer to u as the *root* of G.

Intuitively, the construction has two ingredients. First, we develop the idea of tying energy and budget. We construct a Max strategy f_M that guarantees the following: when Max invests a unit of budget (with an appropriate normalization), then the energy increases by at least one unit, and when the energy decreases by one unit, Max's gain is at least $z > 1$ units of budget, where z arises from the game graph. The second ingredient concerns the normalization factor. Recall that in the previous section it was a constant $\frac{1}{N}$. Here on the other hand, it cannot be constant. Indeed, if the normalization does not decrease as the energy increases, Max's budget will eventually run out, which is problematic since with a budget of 1, Min can guarantee reaching energy level 0, no matter how high the energy is. The challenge is to decide when and how to decrease the normalization factor. We split $\mathbb{N}$ into *energy blocks* of size M, for a carefully chosen $M \in \mathbb{N}$. The normalization factor of the bids depends on the block in which the energy is in, and we refer to it as the *currency* of the block. The currency of the n-th block is z^{-n}. Note that the currency of the $(n-1)$-th block is higher by a factor of z from the currency of the n-th block. This is where the first ingredient comes in: investing in the n-th block is done in the currency of the n-th block, whereas gaining in the n-th block is in the higher currency of the $(n-1)$-th block. We switch between the currencies when the energy moves between energy blocks only at the root u of $\mathcal{G}$. This is possible since $\mathcal{G}$ is a recurrent SCC. The mismatch between gaining and investing is handy when switching between currencies as we cannot guarantee that when we reach u the energy is exactly in the boundary of an energy block.

We formalize this intuition. We start by finding an alternative definition for the weighted Richman function. Recall that in order to define W, we constructed a new graph $\mathcal{G}^u$ by splitting u into u_s and u_t. We define the *contribution* of a vertex $v \in V$ to $W(u_s)$, denoted $cont(v)$, as follows. We have $cont(u_s) = 1$. For a vertex $v \in V$, we define $pre(v) = \{v' \in V : v = v'^-$ or $v = v'^+\}$. For $v \in V$, we define $cont(v) = \sum_{v' \in pre(v)} \frac{1}{2} \cdot cont(v')$. The proof of the following lemma uses the connection with probabilistic models, and follows from standard arguments there.

▶ **Lemma 20.** *We have* $W(u) = \sum_{v \in V} \big(cont(v) \cdot w(v)\big)$.

Let $z = \big(\sum_{v:w(v)\geq 0} cont(v) \cdot w(v)\big) \cdot \big(\sum_{v:w(v)<0} cont(v) \cdot |w(v)|\big)^{-1}$. Since $W(u) > 0$, we have $z > 1$. Let $\mathcal{G}^z$ be the game that is obtained from $\mathcal{G}$ by multiplying the negative-weighted vertices by z, thus $\mathcal{G}^z = \langle V, E, w^z\rangle$, where $w^z(v) = w(v)$ if $w(v) \geq 0$ and otherwise $w^z(v) = z \cdot w(v)$. We denote by W^z the weighted threshold budgets in $\mathcal{G}^z$. The following lemma follows immediately from Lemma 20.

▶ **Lemma 21.** *We have* $W^z(u) = 0$.

We define the partition into energy blocks. Let $cycles(u)$ be the set of simple cycles from u to itself and $w_M = \max_{\pi \in cycles(u)} |E(\pi)|$. We choose M such that $M \geq (b_M + 3w_M)/(1 - z^{-1})$, where b_M is the maximal bid as in the previous section. We partition $\mathbb{N}$ into blocks of size M. For $n \geq 1$, we refer to the n-th block as M_n, and we have $M_n = \{M(n-1), M(n-1)+1, \ldots, Mn-1\}$. We use $\beta_n^{\downarrow}$ and $\beta_n^{\uparrow}$ to mark the upper and lower boundaries of M_n, respectively. We use a $M_{\geq n}$ to denote the set $\{M_n, M_{n+1}, \ldots\}$. Consider a finite outcome π that ends in u and let $visit_u(\pi)$ be the set of indices in which π visits u. Let $k_I \in \mathbb{N}$ be an initial energy. We say that π *visits* M_n if $k_I + E(\pi) \in M_n$. We say that π *stays in* M_n starting from an index $1 \leq i \leq |\pi|$ if for all $j \in visit_u(\pi)$ such that $j \geq i$, we have $k_I + E(\pi_1, \ldots, \pi_j) \in M_n$.

We are ready to describe Max's strategy f_M. Suppose the game reaches a vertex v and the energy in the last visit to u was in M_n, for $n \geq 1$. Then, Max bids $z^{-n} \cdot \frac{1}{2}\big(W^z(v^+) - W^z(v^-)\big)$ and proceeds to v^+ upon winning. Note that currency changes occur only in u. Recall that for an outcome π, the sum of payments of Max is $\mathcal{B}_M(\pi)$ and let $E^z(\pi)$ be the change in energy in $\mathcal{G}^z$. The proof of Lemma 16 can easily be adjusted to this setting.

▶ **Lemma 22.** *Consider a Min strategy f_m, and let $\pi = out(f_m, f_M)$ be a finite outcome that starts in v, ends in v', and stays within a block M_n, for $n \geq 1$. We have $W^z(v) - W^z(v') \leq E^z(\pi) - z^n \cdot \mathcal{B}_M(\pi)$. In particular, for $\pi \in cycles(u)$, we have $E^z(\pi) \leq z^n \cdot \mathcal{B}_M(\pi)$.*

We relate between the changes in energy in the two structures. The proof of the following lemma can be found in the full version.

▶ **Lemma 23.** *Consider an outcome $\pi \in cycles(u)$. Then, $E(\pi) \geq E^z(\pi)$ and $E(\pi) \geq zE^z(\pi)$.*

A corollary of Lemmas 22 and 23 is the following. Recall that $\mathcal{B}_M(\pi)$ is the amount that Max pays, thus it is negative when Max gains budget. Intuitively, the corollary states that if the energy increases in M_n, then Max invests in the currency of M_n, and if the energy decreases, he gains in the currency of M_{n-1}.

▶ **Corollary 24.** *Consider a Min strategy f_m, and let $\pi = out(f_m, f_M)$ be a finite outcome such that $\pi \in cycles(u)$. Then, we have $E(\pi) \geq z^n \cdot \mathcal{B}_M(\pi)$ and $zE(\pi) \geq z^n \cdot \mathcal{B}_M(\pi)$.*

Consider an initial Max budget $B_M^{init} \in [0, 1]$. We choose an initial energy $k_I \in \mathbb{N}$ with which f_M guarantees that energy level 0 is never reached. Recall the intuition that increasing the energy by a unit requires an investment of a unit of budget in the right currency. Thus, increasing the energy from the lower boundary $\beta_n^{\downarrow}$ of M_n to its upper boundary $\beta_n^{\uparrow}$, costs $M \cdot z^{-n}$. We use $cost(M_n)$ to refer to $M \cdot z^{-n}$ and $cost(M_{\geq n}) = \sum_{i=n}^{\infty} cost(M_n)$. A first guess for k_I would be $\beta_n^{\downarrow}$ such that $B_M^{init} > cost(M_{\geq n})$. This is almost correct. We need some *wiggle room* to allow for changes in the currency. Let $wiggle = 2w_M + b_M$, where recall that $w_M = \max_{\pi \in cycles(u)} E(\pi)$ and that b_M is the maximal bid. We define k_I to be $\beta_n^{\downarrow}$ such that $B_M^{init} > wiggle \cdot z^{-(n-1)} + cost(M_{\geq n})$ and $\sum_{i=1}^{n} cost(M_i) > 1$, thus Min cannot decrease the energy to 0.

Consider a Min strategy f_m, and let $\pi = out(f_m, f_M)$ be a finite outcome. We partition π into subsequences in which the same currency is used. Let $\pi = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_\ell$ be a partition of π. For $1 \leq i \leq \ell$, we use π^i to refer to the prefix $\pi_1 \cdot \ldots \cdot \pi_i$ of π, and we use $e^i = k_I + E(\pi^i)$ to refer to the energy at the end of π^i. Consider the partition in which, for $1 \leq i \leq \ell$, the prefix π^i visits u and π_i is a maximal subsequence that stays in some energy block.

Suppose π_i stays in M_n. There can be two options; either the energy decreases in π_i, thus the energy before it e^{i-1} is in M_{n+1} and the energy after it e^i is in M_n, or it increases, thus $e^{i-1} \in M_{n-1}$ and $e^i \in M_n$. We then call π^i *decreasing* and *increasing*, respectively. The definition of w_M and the fact that $\mathcal{G}$ is recurrent imply that upon entering M_n, the energy is within w_M of the boundary. Thus, in the case that π^i is decreasing, the energy at the end of π^i is $e^i \geq \beta_n^{\uparrow} - w_M$ and in the case it is increasing, we have $e^i \leq \beta_n^{\downarrow} + w_M$. Let $\ell_0 = 0$, and for $i \geq 1$, let $\ell_i = (\beta_{n+1}^{\downarrow} - w_M) - e^i$ in the first case and $\ell_i = (\beta_n^{\downarrow} + w_M) - e^i$ in the second case. Note that $\ell_i \in \{0, \ldots, 2w_M\}$. In the full version, we prove the following invariant on Max's budget when changing between energy blocks.

▶ **Lemma 25.** *For every $i \geq 0$, suppose π^i ends in M_n. Then, Max's budget is at least $(wiggle + \ell_i) \cdot z^{-(\hat{n}-1)} + cost(M_{\geq \hat{n}})$, where $\hat{n} = n + 1$ if π^i is decreasing and $\hat{n} = n$ if π^i is increasing.*

It is not hard to show that Lemma 25 implies that f_M is legal. That is, consider a finite outcome π that starts immediately after a change in currency. Using Lemma 22, we can prove by induction on the length of π that Max has sufficient budget for bidding. The harder case is when π decreases, and the proof follows from the fact that *wiggle* is in the higher currency of the lower block. Combining Lemma 25 with our choice of the initial energy, we get that the energy never reaches 0 as otherwise Min invests a budget of more than 1. Lemma 13 implies that Max guarantees a positive mean-payoff value in a strongly-connected game.

▶ **Theorem 26.** *Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$ in which all BSCCs are recurrent. For a vertex $v \in V$, if Max has an initial budget that is greater than $1 - \textsc{Thresh}(v)$, he has a memoryless strategy that guarantees a positive mean-payoff value.*

General strongly-connected games

In the full version, we develop a constant-memory strategy for Max that guarantees a positive mean-payoff value. The difficulty lies in coping with outcomes in which the energy forms a sine-like wave on the boundary of an energy block. In recurrent SCCs, we can change currency every time the wave changes block, which does not work in general SCCs as we show in an example. We develop further the two ingredients that are used in f_M. First, recall that investing in an energy block M_n is in the currency of the n-th block, whereas gaining is in the higher currency of the $(n-1)$-th block. In general games, we need a stronger property; investing in M_n is in the lower currency of the $(n+1)$-th block while gaining is still in the higher currency of the $(n-1)$-th block. Next, we differentiate between even blocks, i.e., M_{2n}, and odd blocks, i.e., M_{2n+1}, for some $n \in \mathbb{N}$. When the energy level reaches an even block M_{2n}, the currency used is z^{-n}. In order to determine the currency in the odd blocks, we take the history of the play into account; the currency matches the currency in the last energy block that was visited before entering M_{2n+1}. Hence, we call our strategy a *constant-memory* strategy. The odd blocks serve as "buffers" so that when we change currency, there is a sufficiently large change in energy that in turn implies that Max's budget sufficiently increases compared with the change in energy. Combining with the memoryless strategy in parity games of Theorem 9, we have the following.

▶ **Theorem 27.** *Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$. For a vertex $v \in V$, if Max has an initial budget that is greater than $1 - \textsc{Thresh}(v)$, he has a constant-memoryless strategy that guarantees a positive mean-payoff value.*

5 Discussion and Future Directions

We introduce and study infinite-duration bidding games in which the players bid for the right to move the token. This work belongs to a line of works that transfer concepts and ideas between the areas of formal methods and algorithmic game theory (AGT, for short). Richman games originated in the game theory community in the 90s and recently gained interest by the AGT community [30]. We combine them with the study of infinite-duration games, which is well-studied in the formal methods community. Prior to this work, a series of works focused on applying concepts and ideas from formal methods to *resource-allocation games* [10, 8, 9, 5, 6, 34], which constitutes a well-studied class of games in AGT. More to the formal methods side, there are many works on games that share similar concepts to these that are studied in AGT. For example, logics for reasoning about multi-agent systems [3, 19, 39],

studies of equilibria in games related to synthesis and repair problems [18, 25, 1, 15], and studies of infinite-duration non-zero-sum games [21, 16, 17, 12].

There are several problems we left open as well as plenty of future research directions. We list a handful of them below. We showed that the complexity of THRESH-BUDG is in NP and coNP. We leave open the problem of determining its exact complexity. We conjecture that it is reducible from solving simple stochastic games, which will show that it is as hard as several other problems whose exact complexity is unknown. In this work we focused on parity and mean-payoff games. *Energy games* are games that are played on a weighted graph, where one of the players tries to reach negative energy and the second player tries to prevent it. Note that unlike parity and mean-payoff, the energy objective is not *prefix independent*. We can show that threshold budgets exist in energy games. The complexity of THRESH-BUDG in energy games is interesting and is tied with recent work on optimizing the probability of reaching a destination in a weighted MDP [26, 41]. For acyclic energy bidding games, the problem is PP-hard using a result in [26], and for a single-vertex games the problem is in P using the direct formula of [32]. For general games the problem is open.

Acknowledgments. We thank Petr Novotný for helpful discussions and pointers.

References

1 S. Almagor, G. Avni, and O. Kupferman. Repairing multi-player games. In *Proc. 26th CONCUR*, pages 325–339, 2015.

2 S. Almagor, D. Kuperberg, and O. Kupferman. The sensing cost of monitoring and synthesis. In *Proc. 35th FSTTCS*, pages 380–393, 2015.

3 R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49(5):672–713, 2002.

4 K.R. Apt and E. Grädel. *Lectures in Game Theory for Computer Scientists*. Cambridge University Press, 2011.

5 G. Avni, S. Guha, and O. Kupferman. An abstraction-refinement methodology for reasoning about network games. In *Proc. 26th IJCAI*, 2017.

6 G. Avni, S. Guha, and O. Kupferman. Timed network games. In *Proc. 42nd MFCS*, 2017.

7 G. Avni, T. A. Henzinger, and V. Chonev. Infinite-duration bidding games. *CoRR*, abs/1705.01433, 2017. `https://arxiv.org/abs/1705.01433`.

8 G. Avni, T. A. Henzinger, and O. Kupferman. Dynamic resource allocation games. In *Proc. 9th SAGT*, pages 153–166, 2016.

9 G. Avni, O. Kupferman, and T. Tamir. Congestion games with multisets of resources and applications in synthesis. In *Proc. 35th FSTTCS*, pages 365–379, 2015.

10 G. Avni, O. Kupferman, and T. Tamir. Network-formation games with regular objectives. *Inf. Comput.*, 251:165–178, 2016.

11 N. Berger, N. Kapur, L. J. Schulman, and V. V. Vazirani. Solvency games. In *Proc. 28th FSTTCS*, pages 61–72, 2008.

12 P. Bouyer, R. Brenguier, N. Markey, and M. Ummels. Nash equilibria in concurrent games with büchi objectives. In *Proc. 31st FSTTCS*, pages 375–386, 2011.

13 T. Brázdil, V. Brozek, K. Etessami, and A. Kucera. Approximating the termination value of one-counter mdps and stochastic games. In *Proc. 38th ICALP*, pages 332–343, 2011.

14 T. Brázdil, V. Brozek, K. Etessami, A. Kucera, and D. Wojtczak. One-counter markov decision processes. In *Proc. 21st SODA*, pages 863–874, 2010.

15 R. Brenguier, L. Clemente, P. Hunter, G. A. Pérez, M. Randour, J.-F. Raskin, O. Sankur, and M. Sassolas. Non-zero sum games for reactive synthesis. In *Proc. 10th LATA*, pages 3–23, 2016.

16 T. Brihaye, V. Bruyère, J. De Pril, and H. Gimbert. On subgame perfection in quantitative reachability games. *Logical Methods in Computer Science*, 9(1), 2012.

17 V. Bruyère, N. Meunier, and J.-F. Raskin. Secure equilibria in weighted games. In *Proc. CSL-LICS*, pages 26:1–26:26, 2014.

18 K. Chatterjee, T. A. Henzinger, and M. Jurdzinski. Games with secure equilibria. *Theor. Comput. Sci.*, 365(1-2):67–82, 2006.

19 K. Chatterjee, T. A. Henzinger, and N. Piterman. Strategy logic. *Inf. Comput.*, 208(6):677–693, 2010.

20 K. Chatterjee, R. Majumdar, and T. A. Henzinger. Controller synthesis with budget constraints. In *Proc. 11th HSCC*, pages 72–86, 2008.

21 K. Chatterjee, R. Majumdar, and M. Jurdzinski. On nash equilibria in stochastic games. In *Proc. 13th CSL*, pages 26–40, 2004.

22 A. Condon. On algorithms for simple stochastic games. In *Proc. DIMACS*, pages 51–72, 1990.

23 M. Develin and S. Payne. Discrete bidding games. *The Electronic Journal of Combinatorics*, 17(1):R85, 2010.

24 K. Etessami, D. Wojtczak, and M. Yannakakis. Quasi-birth-death processes, tree-like qbds, probabilistic 1-counter automata, and pushdown systems. *Perform. Eval.*, 67(9):837–857, 2010.

25 D. Fisman, O. Kupferman, and Y. Lustig. Rational synthesis. In *Proc. 16th TACAS*, pages 190–204, 2010.

26 C. Haase and S. Kiefer. The odds of staying on budget. In *Proc. 42nd ICALP*, pages 234–246, 2015.

27 M. Holtmann, L. Kaiser, and W. Thomas. Degrees of lookahead in regular infinite games. *Logical Methods in Computer Science*, 8(3), 2012.

28 F. A. Hosch and L. H. Landweber. Finite delay solutions for sequential conditions. In *ICALP*, pages 45–60, 1972.

29 K. Johnson, D. Simchi-Levi, and P. Sun. Analyzing scrip systems. *Operations Research*, 62(3):524–534, 2014.

30 G. Kalai, R. Meir, and M. Tennenholtz. Bidding games and efficient allocations. In *Proc. 16th EC*, pages 113–130, 2015.

31 I. A. Kash, E. J. Friedman, and J. Y. Halpern. Optimizing scrip systems: crashes, altruists, hoarders, sybils and collusion. *Distributed Computing*, 25(5):335–357, 2012.

32 G. Katriel. Gambler's ruin probability – a general formula. *Statistics and Probability Letters*, 83:2205–2210, 2013.

33 F. Klein and M. Zimmermann. How much lookahead is needed to win infinite games? *Logical Methods in Computer Science*, 12(3), 2016.

34 O. Kupferman and T. Tamir. Hierarchical network formation games. In *Proc. 23rd TACAS*, pages 229–246, 2017.

35 A. J. Lazarus, D. E. Loeb, J. G. Propp, W. R. Stromquist, and D. H. Ullman. Combinatorial games under auction play. *Games and Economic Behavior*, 27(2):229–264, 1999.

36 A. J. Lazarus, D. E. Loeb, J. G. Propp, and D. Ullman. Richman games. 29:439–449, 1996.

37 R. Paes Leme, V. Syrgkanis, and É. Tardos. Sequential auctions and externalities. In *Proc. 23rd SODA*, pages 869–886, 2012.

38 M. Menz, J. Wang, and J. Xie. Discrete all-pay bidding games. *CoRR*, abs/1504.02799, 2015.

39 F. Mogavero, A. Murano, and M. Y. Vardi. Reasoning about strategies. In *Proc. 30th FSTTCS*, pages 133–144, 2010.

40 A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *Proc. 16th POPL*, pages 179–190, 1989.

41 M. Randour, J.-F. Raskin, and O. Sankur. Variations on the stochastic shortest path problem. In *Proc. 16th VMCAI*, pages 1–18, 2015.

42 D. M. Reeves, B. M. Soule, and T. Kasturi. Yootopia! *SIGecom Exchanges*, 6(2):1–26, 2007.

43 M. Stonebraker, P. M. Aoki, W. Litwin, A. Pfeffer, A. Sah, J. Sidell, C. Staelin, and A. Yu. Mariposa: A wide-area distributed database system. *VLDB J.*, 5(1):48–63, 1996.

On the Power of Name-Passing Communication*

Yuxi Fu

BASICS, Shanghai Jiao Tong University, China
fu-yx@cs.sjtu.edu.cn

Abstract

It is shown that generally higher order process calculi cannot be interpreted in name-passing calculi in a robust way.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.2 Modes of Computation

Keywords and phrases Interaction theory, expressiveness, bisimulation

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.22

1 Introduction

Many process calculi have been proposed [19]. It is important to investigate the relative expressiveness between them [29, 25, 18, 20, 23, 10, 8]. It is even more important to study the relative expressiveness between classes of these models [5]. Positive results can tell us which model plays a foundational role in interaction theory and which acts as a prototype in applications. A negative result often reveals a deep distinct property of a model we probably were not aware of beforehand.

There are basically three classes of interactively Turing powerful models characterized by the content of communication. In the name-passing calculi the messages are channels (channel names) [16, 28, 22]. The power of this class of process calculi crucially depends on the ability to pass around private channels. The general view is that the π-calculus is the "λ-calculus" of concurrency theory [15] and is capable of modelling all kinds of phenomena in both theory and application [34]. The second class contains higher order process calculi. The pure process-passing calculus is too weak [5]. The messages in higher order process calculi are normally abstractions [25, 26, 31, 32]. A receiving process can instantiate the parameters of the received abstraction with its own private channels. A seemingly exception is Thomsen's CHOCS [29, 30]. However there is a very strong operator in CHOCS, the so-called relabelling operator [14]. If we think of it the relabelling operator can achieve the effect of instantiations of abstractions. So in CHOCS process-passing is abstraction-passing in disguise. Concerning the relative expressive power of the higher order process calculi, it is a popular belief that they can be completely interpreted in the name-passing calculi. If the messages are neither channels nor abstractions, they can always be coded up by natural numbers. The class of value-passing calculi consists of those models whose contents of communications are numbers [13, 14, 11, 12]. It is proved in [5] that value-passing calculi and name-passing calculi are incompatible in expressive power. This result appears somewhat surprising at first sight. The value-passing calculi turns out to be very expressive. A great deal of higher order communications can be interpreted by value-passing communications [3]. More precisely, if in a higher order process calculus messages may contain neither unbound private channels nor global channels, then the calculus is a submodel of the standard value-passing calculus [3]. The intuition is

* The support from NSFC (61472239, ANR 61261130589) is gratefully acknowledged.

that such messages can be coded up by numbers. Therefore abstraction-passing becomes value-passing. Moreover the Gödel encoding is bijective. So full abstraction poses no problem. What we will show in this short paper, by generalizing the proof in [5], is that even such restricted higher order communications cannot be interpreted by the name-passing calculi in a faithful manner. This result contributes to a better understanding of the relative expressive power of the three classes of interaction models.

Section 2 introduces the criteria for relative expressiveness. Section 3 fixes notations for the π-calculus. Section 4 proves a general negative result about non-interpretability in the π-calculus. Section 5 derives the main result. Section 6 makes some concluding comments.

2 Relative Expressiveness

Given two models $\mathbb{M}, \mathbb{N}$, in what sense is $\mathbb{N}$ at least as expressive as $\mathbb{M}$? Different researchers have different criteria. In logic terms the answer is given by submodel relationship. In what follows we shall motivate a set of criteria that will uncover our definition of submodel relationship.

Generally we understand an interpretation of $\mathbb{M}$ in $\mathbb{N}$ as a binary relation $\propto$ from the set of $\mathbb{M}$-processes to the set of $\mathbb{N}$-processes. The interpretation should be *total* in the sense that for every $\mathbb{M}$-process M there is an $\mathbb{N}$-process N such that $M \propto N$. The interpretation should not introduce extra divergence. There should not be an infinite internal action sequence $N \xrightarrow{\tau} N_0 \xrightarrow{\tau} N_1 \xrightarrow{\tau} N_2 \xrightarrow{\tau} \ldots$ such that $M \propto N_i$ for all $i \in \omega$ if M cannot do any infinite internal action sequence. More formally this energy respect criterion is introduced in [24].

▶ **Definition 1.** The relation $\propto$ is *codivergent* if $M_0 \propto N_0$ implies the following: (i) If there is an infinite internal action sequence $N_0 \xrightarrow{\tau} N_1 \xrightarrow{\tau} N_2 \xrightarrow{\tau} \ldots$, there is some M' such that $M \xrightarrow{\tau} M' \propto N_i$ for some $i > 0$. (ii) If $M_0 \xrightarrow{\tau} M_1 \xrightarrow{\tau} M_2 \xrightarrow{\tau} \ldots$ is an infinite internal action sequence, there is some N' such that $N \xrightarrow{\tau} N'$ and $M_j \propto N'$ for some $j > 0$.

Codivergence is a computational property. The unique equality for computation, the extensional equality, is divergence sensitive. Codivergence is the divergence sensitive property reformulated in the presence of bisimulation. Bisimulation property [15, 21] is in essence also about computations [4]. This is particularly evident in the following definition, where $\Longrightarrow$ is the reflexive and transitive closure of the one step internal transition $\xrightarrow{\tau}$.

▶ **Definition 2.** The relation $\propto$ is a *bisimulation* if the following are valid whenever $M \propto N$:

1. If $M \xrightarrow{\tau} M'$, then
   - **a.** either $N \Longrightarrow N'$ for some N' such that $M \propto N'$ and $M' \propto N'$, or
   - **b.** $N \Longrightarrow N'' \xrightarrow{\tau} N'$ for some N'', N' such that $M \propto N''$ and $M' \propto N'$.
2. If $N \xrightarrow{\tau} N'$, then
   - **a.** either $M \Longrightarrow M'$ for some M' such that $M' \propto N$ and $M' \propto N'$, or
   - **b.** $M \Longrightarrow M'' \xrightarrow{\tau} M'$ for some M'', M' such that $M'' \propto N$ and $M' \propto N'$.

Case (1a) and case (2a) are about *deterministic* computation whereas case (1b) and case (2b) are to do with *nondeterministic* computation. The reader must have noticed that this is the branching bisimulation defined in [33]. The reason for using branching bisimulation is that even if we allow the interpretation N of M to carry out some internal adjustments (actions), we do not allow N to introduce any extra nondeterministic computation step. If $N \xRightarrow{\tau} N'$ for some N' not equivalent to N, then N' cannot be an interpretation of M. This would lead to a contradiction if M cannot do any nondeterministic computation. The problem with the weak bisimulation [14] is that it confuses nondeterministic computation with deterministic computation. This is a confusion that never happens in computation theory.

We also need criteria that take into account of interaction. We assume that all external actions admitted in all our models are carried out at global channels. We say that a process P is *observable* at a global channel a if $P \Longrightarrow P'$ for some P' such that P' may perform an external action at channel a. If N is an interpretation of M then they should have the same capacity to communicate at any particular global channel, hence the following [17].

▶ **Definition 3.** The relation $\propto$ is *equipollent* if $M \propto N$ implies that, for every global channel a, M is observable at a if and only if N is observable at a.

If $M \propto N$ then there should be no difference between M and N detectable by equivalent observers from the respective models. In other words $\propto$ should be closed under the concurrent composition operator. Without this closure property it does not make sense to talk about relative expressiveness for interaction model.

▶ **Definition 4.** The relation $\propto$ is *extensional* if $M \,|\, M' \propto N \,|\, N'$ whenever $M \propto N$ and $M' \propto N'$.

Intuitively $\mathbb{M}$ is a submodel of $\mathbb{N}$ if for each $\mathbb{M}$-process M there is an $\mathbb{N}$-process N that is equal to M as it were. Obviously the equality on $\mathbb{M}$-processes must be a special submodel relationship.

▶ **Definition 5.** The *absolute equality* $=_{\mathbb{M}}$ on $\mathbb{M}$ is the largest binary relation on $\mathbb{M}$-processes that renders true the following statements.

1. It is reflexive.
2. It is extensional, equipollent, codivergent, bisimilar.

We often omit the subscript in $=_{\mathbb{M}}$. It is important that expressiveness and equality are defined in completely the same fashion. Otherwise it would not be a *submodel* relationship, confer Lemma 7.

▶ **Definition 6.** A *subbisimilarity* from $\mathbb{M}$ to $\mathbb{N}$ is a binary relation $\propto$ from $\mathbb{M}$-processes to $\mathbb{N}$-processes such that the following statements are valid.

1. It is total and *semantical*, the latter means that $M' = M \propto N$ implies $M' \propto N$.
2. It is extensional, equipollent, codivergent, bisimilar.

The first condition of Definition 6 is a generalization of the reflexivity condition of Definition 5. Like reflexivity it is a preliminary requirement. The semantical condition asks for nothing more than that the relation $\propto$ should be *syntax independent* when seen as an interpretation from $\mathbb{M}$ to $\mathbb{N}$. We say that $\mathbb{M}$ is a *submodel* of $\mathbb{N}$, notation $\mathbb{M} \sqsubseteq \mathbb{N}$, if there is a subbisimilarity from $\mathbb{M}$ to $\mathbb{N}$. The terminology is enforced by the following full abstraction lemma.

▶ **Lemma 7.** *Suppose $\propto$ is a subbisimilarity from $\mathbb{M}$ to $\mathbb{N}$, and $M \propto N$ and $M' \propto N'$. Then $M = M'$ if and only if $N = N'$.*

Proof. $N =_{\mathbb{N}} N'$ implies $M =_{\mathbb{M}} M'$ since the composition $\propto; =_{\mathbb{N}}; \propto^{-1}$ is a subbisimilarity, where $\propto^{-1}$ is the reverse relation of $\propto$. Conversely if $M =_{\mathbb{M}} M'$ then $M' \propto N$ by the semantical condition. It follows from $M' \propto N'$ and $M' \propto N$ that $N =_{\mathbb{N}} N'$ because $\propto; =_{\mathbb{N}}$ is a subbisimilarity. ◀

In the rest of the paper we say that $P \xrightarrow{\tau} P'$ is a *deterministic* computation step, notation $P \to P'$, if $P' = P$, and that it is a *nondeterministic* computation step, notation $P \xrightarrow{\iota} P'$, if $P' \neq P$. We will write $\to^+$ ($\to^*$) for the (reflexive and) transitive closure of $\to$. A τ-*descendant* of T is a term T' such that $T \Longrightarrow T'$. Every model has an unobservable process $\mathbf{0}$ whose every computation is finite and an unobservable process Ω whose every

computation is infinite. It is easy to see that both $\mathbf{0}$ and Ω can only do deterministic computation. For every subbisimilarity $\propto$ it holds that $Q = \Omega$ whenever $\Omega \propto Q$ or $Q \propto \Omega$. Similarly $Q = \mathbf{0}$ whenever $\mathbf{0} \propto Q$ or $Q \propto \mathbf{0}$.

For more motivations to the absolute equality and the subbismilarity, consult [5].

3 Name-Passing Calculus

We fix in this section the target model [16]. Unlike in the original presentation we introduce syntactical distinction between global channels, private channels and channel variables.

- Let $\mathcal{C}_g$ be the set of global channels. The elements of $\mathcal{C}_g$ are denoted by a, b, c, d, e, f.
- Let $\mathcal{C}_p$ be the set of private channels. The elements of $\mathcal{C}_p$ are denoted by l, m, n, o, p, q.
- Let $\mathcal{C}_v$ be the set of channel variables. The elements of $\mathcal{C}_v$ are denoted by u, v, w, x, y, z.

The set $\mathcal{C}_g \cup \mathcal{C}_p \cup \mathcal{C}_v$ will be ranged over by μ, ν, and the set $\mathcal{C}_g \cup \mathcal{C}_p$ by α, β. The π-terms are constructed from the following grammar.

$$S, T \quad := \quad \sum_{i\in I} \mu(x).T_i \mid \sum_{i\in I} \overline{\mu}\nu_i.T_i \mid S \,|\, T \mid (p)T \mid [\mu{=}\nu]T \mid [\mu{\neq}\nu]T \mid !\mu(x).T.$$

In the above definition I is a finite indexing set. We abbreviate $\sum_{i\in I} \mu(x).T_i$ and $\sum_{i\in I} \overline{\mu}\mu'_i.T_i$ to $\mathbf{0}$ if $I = \emptyset$. If the size of I is one, we get input term $\mu(x).T$ and output term $\overline{\mu}\mu'.T$. We abbreviate $(p)\mu p.T$ to $\mu(p).T$, $a(x).T$ to $a.T$ if x does not appear in T, and $\overline{a}(q).T$ to $\overline{a}.T$ if q does not appear in T. The labeled transition semantics is defined by the following rules.

$$\frac{}{\sum_{i\in I} \alpha(x).T_i \xrightarrow{\alpha\beta} T_i\{\beta/x\}} \qquad \frac{}{\sum_{i\in I} \overline{\alpha}\beta_i.T_i \xrightarrow{\overline{\alpha}\beta_i} T_i} \qquad \frac{T \xrightarrow{\lambda} T' \quad p\notin\lambda}{(p)T \xrightarrow{\lambda} (p)T'} \qquad \frac{T \xrightarrow{\overline{\alpha}p} T'}{(p)T \xrightarrow{\overline{\alpha}(p)} T'}$$

$$\frac{S \xrightarrow{\alpha\beta} S' \quad T \xrightarrow{\overline{\alpha}\beta} T'}{S \,|\, T \xrightarrow{\tau} S' \,|\, T'} \qquad \frac{S \xrightarrow{\alpha p} S' \quad T \xrightarrow{\overline{\alpha}(p)} T'}{S \,|\, T \xrightarrow{\tau} (p)(S' \,|\, T')} \qquad \frac{S \xrightarrow{\lambda} S'}{S \,|\, T \xrightarrow{\lambda} S' \,|\, T}$$

$$\frac{T \xrightarrow{\lambda} T'}{[\alpha{=}\alpha]T \xrightarrow{\lambda} T'} \qquad \frac{T \xrightarrow{\lambda} T'}{[\alpha{\neq}\beta]T \xrightarrow{\lambda} T'} \qquad \frac{}{!\alpha(x).T \xrightarrow{\alpha\beta} T\{\beta/x\} \,|\, !\alpha(x).T}$$

All symmetric rules are omitted. In the third rule $p\notin\lambda$ means that p does not appear in λ. In the transition $T \xrightarrow{\overline{\alpha}(p)} T'$ the action $\overline{\alpha}(p)$ is a bound output action. A π-process is a π-term in which all channel variables are bounded by input prefix operators and all private channels are bounded by scope operators. We write L, M, N, O, P, Q for processes. We write λ_a for an input or output action at channel a and $\overline{\lambda_a}$ for the complementary action. We will find it necessary to use the internal choice $\tau.S + \tau.T$ defined by $(p)(\overline{p} \,|\, p.S \,|\, p.T)$.

The above semantics is called early semantics in literature. There is also a late semantics in which instantiation comes after the commitment of an input action [16]. If we think of bisimulation as a computational property, only early semantics makes sense. However it must be said that, due to the ever presence of name extrusion in π, the late semantics is intrinsic to the π-calculus [27]. We explain the claim by an example. The numbers fetchable at channel α can be defined in π as follows.

$$\begin{aligned} [\![0]\!]_\alpha &\stackrel{\text{def}}{=} \overline{\alpha}(o).\overline{o}(p).\overline{o}(q).\overline{q}, \\ [\![k{+}1]\!]_\alpha &\stackrel{\text{def}}{=} \overline{\alpha}(o).\overline{o}(p).\overline{o}(q).\overline{p}(r).[\![k]\!]_r. \end{aligned}$$

It is simple to define the process $a(x).\mathit{if}\ x{=}0\ \mathit{then}\ P\ \mathit{else}\ Q$ that turns into P if the process has fetched a zero at channel a and turns into Q if the number it imports is not zero. Now

consider $N \stackrel{\text{def}}{=} a(x).\big(\mathit{if}\, x{=}0\ \mathit{then}\, \overline{c}x\ \mathit{else}\, \overline{d}x\big)$, $C \stackrel{\text{def}}{=} !a(x).\overline{c}x$ and $D \stackrel{\text{def}}{=} !a(x).\overline{d}x$. It is easy to see that the equality

$$C \,|\, D \,|\, N \quad = \quad C \,|\, D \tag{1}$$

fails for the π-calculus. The observer O defined in (2) can make a choice between 0 and 1 after it is engaged.

$$O \stackrel{\text{def}}{=} a.b \,|\, \overline{a}(o).\,(\tau.\overline{o}(p).\overline{o}(q).\overline{q} + \tau.\overline{o}(p).\overline{o}(q).\overline{p}(r).[\![0]\!]_r)\,. \tag{2}$$

Let $C \,|\, D \,|\, N \,|\, O \xrightarrow{\tau} C \,|\, D \,|\, (o)(N' \,|\, O')$ be caused by the interaction between N and O. This internal action cannot be bisimulated vacuously by $C \,|\, D \,|\, O$ because $C \,|\, D \,|\, O$ is observable at b whereas $C \,|\, D \,|\, (o)(N' \,|\, O')$ is not. It cannot be bisimulated by $C \,|\, D \,|\, O \xrightarrow{\tau} (o)(C' \,|\, D \,|\, O'')$ caused by the interaction between C and O since $(o)(C' \,|\, D \,|\, O'')$ is not observable at d. Similarly the internal action of $C \,|\, D \,|\, O$ caused by the interaction between D and O does not bisimulate $C \,|\, D \,|\, N \,|\, O \xrightarrow{\tau} C \,|\, D \,|\, (o)(N' \,|\, O')$. It is worth pointing out that (1) fails also for weak bisimilarity.

A lot can happen between the time a π-process starts to fetch a message and the time it sees the full picture of the message. This is the fundamental weakness of name-passing communications. In other type of model there are more efficient implementations of $a(x).\mathit{if}\ x{=}0\ \mathit{then}\ P\ \mathit{else}\ Q$ that render (1) true. This simple example provides all the intuition for the main result of the paper.

The absolute equality for π-calculus can be equivalently defined in terms of input-output behaviours. Let's call a π-term a *quasi π-process* if it does not contain any free channel variables.

▶ **Definition 8.** A codivergent bisimulation $\mathcal{R}$ on quasi π-processes is an *extensional bisimulation* if the following statements are valid for all $\lambda \neq \tau$ whenever $S\mathcal{R}T$:

1. If $S \xrightarrow{\lambda} S'$, then $T \Longrightarrow T'' \xrightarrow{\lambda} T'$ for some T'', T' such that $S\mathcal{R}T''$ and $S'\mathcal{R}T'$.
2. If $T \xrightarrow{\lambda} T'$, then $S \Longrightarrow S'' \xrightarrow{\lambda} S'$ for some S'', S' such that $S''\mathcal{R}T$ and $S'\mathcal{R}T'$.

The *extensional bisimilarity* $\simeq_\pi$ is the largest extensional bisimulation.

Clause (1) and clause (2) of the above definition ensure that equal processes have the same input-output behaviours. The next lemma points out the authoritative role of the absolute equality. It is the minimal equality for interactive objects that subsumes the extensional equality of computation.

▶ **Lemma 9.** *Suppose P, Q are π-processes. Then $P =_\pi Q$ if and only if $P \simeq_\pi Q$.*

Proof. The relation $\simeq_\pi$ is closed under concurrent composition and scope operation. Hence $\simeq_\pi \subseteq =_\pi$. Conversely we show that the relation

$$\left\{ (S,T) \;\middle|\; \begin{array}{l} S \text{ and } T \text{ are quasi } \pi \text{ processes,} \\ (p_1,\ldots,p_n)(\overline{a_1}p_1 \mid \ldots \mid \overline{a_n}p_n \mid S) =_\pi (p_1,\ldots,p_n)(\overline{a_1}p_1 \mid \ldots \mid \overline{a_n}p_n \mid T), \\ p_1,\ldots,p_n \text{ are all the unbound private channels appearing in } S \,|\, T, \\ \text{none of the global channels } a_1,\ldots,a_n \text{ appears in } S \,|\, T. \end{array} \right\}.$$

is an extensional bisimulation. For details see [9]. ◀

4 A General Negative Result

We argue in this section that roughly speaking if a model can release an infinite number of pairwise distinct complete messages, then the model cannot be interpreted in the π-calculus.

Suppose $\mathfrak{D} = \{\lceil 0 \rfloor_a, \lceil 1 \rfloor_a, \lceil 2 \rfloor_a, \ldots, \lceil i \rfloor_a, \ldots\}$ is an infinite set of processes in some model $\mathbb{M}$ such that $i \neq j$ implies

$$\lceil i \rfloor_a \neq \lceil j \rfloor_a. \tag{3}$$

We assume that for every i the process $\lceil i \rfloor_a$ can do one and only one action, which is an output action at channel a, and turns into a process equivalent to $\mathbf{0}$ after the action. In other words,

$$\lceil i \rfloor_a \xrightarrow{\overline{a}(\widehat{i})} = \mathbf{0}, \tag{4}$$

where $\widehat{i}$ is the message released by $\lceil i \rfloor_a$. There is no specific requirement on the messages $\widehat{0}, \widehat{1}, \widehat{2}, \ldots$ apart from that they are pairwise distinct. We require that in $\mathbb{M}$ we can define an absorbing process $\lceil \lambda x.\mathbf{0} \rfloor_a$, a successor process $\lceil \lambda x.x^+ \rfloor_a$, a choice process $\lceil \lambda x.\tau.x^+ + \tau \rfloor_a$, and a test process $\lceil \lambda x.x \overset{?}{>} k \rfloor_a$ for each $k \in \omega$. For each $i \in \omega$ each of the processes can carry out one and only one action. Their operational behaviours are specified as follows:

$$\lceil \lambda x.\mathbf{0} \rfloor_a \xrightarrow{a(\widehat{i})} = \mathbf{0}, \tag{5}$$

$$\lceil \lambda x.x^+ \rfloor_a \xrightarrow{a(\widehat{i})} = \lceil i{+}1 \rfloor_a, \tag{6}$$

$$\lceil \lambda x.\tau.x^+ + \tau \rfloor_a \xrightarrow{a(\widehat{i})} = \tau.\lceil i{+}1 \rfloor_a + \tau, \tag{7}$$

$$\lceil \lambda x.x \overset{?}{>} k \rfloor_a \xrightarrow{a(\widehat{i})} = \mathbf{0}, \quad \text{if } i \leq k, \tag{8}$$

$$\lceil \lambda x.x \overset{?}{>} k \rfloor_a \xrightarrow{a(\widehat{i})} = \lceil i{+}1 \rfloor_a, \quad \text{if } i > k. \tag{9}$$

In $\mathbb{M}$ there is also a replicated form for each of $\lceil \lambda x.\mathbf{0} \rfloor_a$, $\lceil \lambda x.x^+ \rfloor_a$ and $\lceil \lambda x.\tau.x^+ + \tau \rfloor_a$, denoted respectively by $\lceil !\lambda x.\mathbf{0} \rfloor_a$, $\lceil !\lambda x.x^+ \rfloor_a$ and $\lceil !\lambda x.\tau.x^+ + \tau \rfloor_a$. For each $i \in \omega$ their unique actions are described as follows:

$$\lceil !\lambda x.\mathbf{0} \rfloor_a \xrightarrow{a(\widehat{i})} = \lceil !\lambda x.\mathbf{0} \rfloor_a, \tag{10}$$

$$\lceil !\lambda x.x^+ \rfloor_a \xrightarrow{a(\widehat{i})} = \lceil i{+}1 \rfloor_a \mid \lceil !\lambda x.x^+ \rfloor_a, \tag{11}$$

$$\lceil !\lambda x.\tau.x^+ + \tau \rfloor_a \xrightarrow{a(\widehat{i})} = (\tau.\lceil i{+}1 \rfloor_a + \tau) \mid \lceil !\lambda x.\tau.x^+ + \tau \rfloor_a. \tag{12}$$

Let G be the process $\lceil !\lambda x.\mathbf{0} \rfloor_a \mid \lceil !\lambda x.x^+ \rfloor_a \mid \lceil !\lambda x.\tau.x^+ + \tau \rfloor_a$. We assume that in $\mathbb{M}$ the following semantic equality is valid for each $k \in \omega$.

$$\Omega \mid G = \Omega \mid G \mid \lceil \lambda x.x \overset{?}{>} k \rfloor_a. \tag{13}$$

Our assumption on $\mathbb{M}$ is very liberal. It asks for no more than a numerical system that validates (13).

▶ **Theorem 10.** $\mathbb{M} \not\sqsubseteq \pi$.

Proof. Suppose there were a subbisimilarity $\mathfrak{I}$ from the $\mathbb{M}$-processes to the π-processes. In what follows we write $M \mathfrak{I} P \not\to$ for $M \mathfrak{I} P$ and $P \not\to$. Now fix $a \in \mathcal{C}_g$. We fix the notations for the interpretations of the $\mathbb{M}$-processes just described.

- For each $i \in \omega$ let N_i be the π-process such that $\lceil i \rfloor_a \mathfrak{I} N_i \not\to$.
- Let C_0 be the π-process such that $\lceil \lambda x.\mathbf{0} \rfloor_a \mathfrak{I} C_0 \not\to$.
- Let S_0 be the π-process such that $\lceil \lambda x.x^+ \rfloor_a \mathfrak{I} S_0 \not\to$.

- Let E_0 be the π-process such that $\lceil \lambda x.\tau.x^+ + \tau \rfloor_a \,\Im\, E_0 \not\rightarrow$.
- Let C be the π-process such that $\lceil !\lambda x.\mathbf{0} \rfloor_a \,\Im\, C \not\rightarrow$.
- Let S be the π-process such that $\lceil !\lambda x.x^+ \rfloor_a \,\Im\, S \not\rightarrow$.
- Let E be the π-process such that $\lceil !\lambda x.\tau.x^+ + \tau \rfloor_a \,\Im\, E \not\rightarrow$.
- For each $k \in \omega$ let J_k be the π-process such that $\lceil \lambda x.x \stackrel{?}{>} k \rfloor_a \,\Im\, J_k \not\rightarrow$.

It follows from (13) and extensionality that

$$\Omega \,|\, C \,|\, S \,|\, E = \Omega \,|\, C \,|\, S \,|\, E \,|\, J_k. \tag{14}$$

We now derive some properties for the π-processes $N_0, N_1, N_2, N_3, \ldots$.

1. According to codivergence property there is no infinite computation sequence from $S_0 \,|\, N_i$. Suppose $S_0 \,|\, N_i \rightarrow B$. This action must be caused by $S_0 \stackrel{\lambda_a}{\longrightarrow} S'$ and $N_i \stackrel{\overline{\lambda_a}}{\longrightarrow} N'$ for some complementary actions $\lambda_a, \overline{\lambda_a}$ and some π-processes S' and N'. If neither λ_a nor $\overline{\lambda_a}$ is a bound output action then $S_0 \,|\, N_i \stackrel{\lambda_a}{\longrightarrow}\stackrel{\overline{\lambda_a}}{\longrightarrow} B$ and $S_0 \,|\, N_i \stackrel{\overline{\lambda_a}}{\longrightarrow}\stackrel{\lambda_a}{\longrightarrow} B$. It follows that $B \rightarrow^* \stackrel{\lambda_a}{\longrightarrow} \rightarrow^* \stackrel{\overline{\lambda_a}}{\longrightarrow} B' = B$ and $B \rightarrow^* \stackrel{\overline{\lambda_a}}{\longrightarrow} \rightarrow^* \stackrel{\lambda_a}{\longrightarrow} B'' = B$ for some B', B''. Now $B \,|\, B \rightarrow^+ B' \,|\, B'' = B \,|\, B$. We derive by induction that there would be an infinite computation sequence from $S_0 \,|\, N_i \,|\, S_0 \,|\, N_i$, contradicting to the fact that $\lceil \lambda x.x^+ \rfloor_a \,|\, \lceil i \rfloor_a \,|\, \lceil \lambda x.x^+ \rfloor_a \,|\, \lceil i \rfloor_a$ is not divergent. If one of $\lambda_a, \overline{\lambda_a}$ is a bound output action, we also get an infinite sequence of computation by renaming and inserting scope operators in appropriate places. More specifically without loss of generality suppose $\lambda_a = ao$ and $\overline{\lambda_a} = \overline{a}(o)$. Let $S_0 \stackrel{aq}{\longrightarrow} B_1$ and $N_i \stackrel{\overline{a}(p)}{\longrightarrow} B_2$ for fresh private channels p, q. Then

$$B \equiv (o)(B_1\{o/p\} \,|\, B_2\{o/q\}). \tag{15}$$

It follows from $S_0 \,|\, N_i = B$ that there must exist some B' such that

$$S_0 \,|\, N_i \stackrel{aq}{\longrightarrow}\stackrel{\overline{a}(p)}{\longrightarrow} B_1 \,|\, B_2. \tag{16}$$

is bisimulated by

$$B \rightarrow^* \stackrel{aq}{\longrightarrow} \rightarrow^* \stackrel{\overline{a}(p)}{\longrightarrow} B'. \tag{17}$$

Now (15) and (17) imply that (16) can be extended to

$$S_0 \,|\, N_i \stackrel{aq}{\longrightarrow}\stackrel{\overline{a}(p)}{\longrightarrow} B_1 \,|\, B_2 \Longrightarrow \stackrel{\lambda_a^1}{\longrightarrow} \Longrightarrow \stackrel{\overline{\lambda_a^1}}{\longrightarrow} B_1' \,|\, B_2'. \tag{18}$$

for some B_1', B_2' and some $\lambda_a^1, \overline{\lambda_a^1}$. Consequently the bisimulation (17) can be extended to

$$B \rightarrow^* \stackrel{aq}{\longrightarrow} \rightarrow^* \stackrel{\overline{a}(p)}{\longrightarrow} B' \Longrightarrow \stackrel{\lambda_a^1}{\longrightarrow} \Longrightarrow \stackrel{\overline{\lambda_a^1}}{\longrightarrow} B'' \tag{19}$$

for some B''. The extension can be repeated infinitely often. We eventually get an infinite sequence with alternating input-output actions. Similarly we can derive from $S_0 \,|\, N_i \stackrel{\overline{a}(p)}{\longrightarrow}\stackrel{aq}{\longrightarrow} B_1 \,|\, B_2$ an infinite sequence with alternating output-input actions. In this way $S_0 \,|\, N_i \,|\, S_0 \,|\, N_i$ would induce an infinite sequence of computation. This is again a contradiction. We conclude that $S_0 \,|\, N_i \stackrel{\iota}{\longrightarrow} = N_{i+1}$ is essentially the only one-step nondeterministic computation of $S_0 \,|\, N_i$.

2. N_i cannot do both an input action at channel a and an output action at channel a. Otherwise $N_i \,|\, N_i$ would be able to do an interaction, which would be a contradiction. This is because $N_i \,|\, N_i$ cannot perform any nondeterministic computation since $\lceil i \rfloor_a \,|\, \lceil i \rfloor_a$ cannot do any internal action. It cannot do a deterministic computation step since that would induce an infinite sequence of internal actions from $N_i \,|\, N_i \,|\, N_i \,|\, N_i$, like in the previous case. So N_i may perform either an input action or an output action exclusively.

3. If N_i can do an input, respectively output action then N_j can do an input, respectively output action for all $j \in \omega$ since the latter has to interact with C_0.
4. It follows from $\lceil \lambda x.\mathbf{0} \rfloor_a \mid \lceil i \rfloor_a \xrightarrow{\iota} = \mathbf{0}$ that $C_0 \mid N_i \xrightarrow{\iota} = \mathbf{0}$. Suppose $N_i \xrightarrow{\overline{a}c} N'$ and $N_j \xrightarrow{\overline{a}c} N''$ for some N', N''. Clearly $N' = \mathbf{0} = N''$. But then one could derive from $N_i \mid S_0 \xrightarrow{\iota} = N_{i+1}$ and $N_j \mid S_0 \xrightarrow{\iota} = N_{j+1}$ that $N_{i+1} = N_{j+1}$, which implies $i = j$. We conclude that if both N_i and N_j can do free output actions at channel a then the channels they release must be distinct whenever $i \neq j$.
5. According to our assumption we have $S_0 \mid N_i \xrightarrow{\tau} = N_{i+1}$ for all $i \geq 0$. Let's write $P \xrightarrow{\lambda}$ if $P \xrightarrow{\lambda} P'$ for some P'. It should be clear that if $N_1 \xrightarrow{\overline{a}c}$, then $S_0 \mid N_0 \Longrightarrow \xrightarrow{\overline{a}c}$. Similarly if $N_2 \xrightarrow{\overline{a}d}$, then $S_0 \mid N_1 \Longrightarrow \xrightarrow{\overline{a}d}$. Therefore $S_0 \mid S_0 \mid N_0 \Longrightarrow \xrightarrow{\overline{a}d}$. By induction we can prove that if N_i can release a global channel at a then that global channel must appear in $S_0 \mid N_0$. So only a finite number of $N_0, N_1, N_2, \ldots$ can do free output actions. Let h be the least number such that, for every $j \geq h$, N_j does only a bound output action.

We prove the impossibility result by a case analysis on the actions of $N_0, N_1, N_2, N_3, \ldots$.

1. Suppose $k > h$. Let

$$\begin{aligned} N_h &\xrightarrow{\overline{a}(p)} N'_h, \\ N_k &\xrightarrow{\overline{a}(p)} N'_k. \end{aligned}$$

In this case C_0, D_0, E_0, J_h, C, D and E can only do input actions at channel a. Let

$$\begin{aligned} C_0 &\xrightarrow{ap} C_0^p, \\ D_0 &\xrightarrow{ap} S_0^p, \\ E_0 &\xrightarrow{ap} E_0^p, \\ J_h &\xrightarrow{ap} J_p, \\ C &\xrightarrow{ap} C_p = C, \\ D &\xrightarrow{ap} S_p = S_0^p \mid S, \\ E &\xrightarrow{ap} E_p = E_0^p \mid E. \end{aligned}$$

None of C_0, D_0, E_0, J_h, C, D and E may perform any output actions. Otherwise there would be either an infinite deterministic computation or a nondeterministic computation step. It follows from (14) that $\Omega \mid C \mid S \mid E \mid J_h \xrightarrow{ap} \Omega \mid C \mid S \mid E \mid J_p$ must be bisimulated by one of the following.

$$\begin{aligned} \Omega \mid C \mid S \mid E &\xrightarrow{ap} \Omega \mid C_p \mid S \mid E, \\ \Omega \mid C \mid S \mid E &\xrightarrow{ap} \Omega \mid C \mid S_p \mid E, \\ \Omega \mid C \mid S \mid E &\xrightarrow{ap} \Omega \mid C \mid S \mid E_p. \end{aligned}$$

In the first case $\Omega \mid C \mid S \mid E \mid J_h \mid N_k \xrightarrow{\iota} (p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_k)$ should be bisimulated by $\Omega \mid C \mid S \mid E \mid N_k \xrightarrow{\iota} (p)(\Omega \mid C_p \mid S \mid E \mid N'_k)$ due to congruence. This is impossible because

$$\begin{aligned} (p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_k) &= \Omega \mid C \mid S \mid E \mid N_{k+1} \\ &\neq \Omega \mid C \mid S \mid E \\ &= (p)(\Omega \mid C \mid S \mid E \mid N'_k) \\ &= (p)(\Omega \mid C_p \mid S \mid E \mid N'_k) \end{aligned}$$

according to Lemma 7. In the second case $\Omega \mid C \mid S \mid E \mid J_h \mid N_h \xrightarrow{\iota} (p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_h)$ should be bisimulated by $\Omega \mid C \mid S \mid E \mid N_h \xrightarrow{\iota} (p)(\Omega \mid C \mid S_p \mid E \mid N'_h)$. Using again the full abstraction lemma one derives the following contradiction.

$$\begin{aligned}(p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_h) &= \Omega \mid C \mid S \mid E \\ &\neq \Omega \mid C \mid S \mid E \mid N_{h+1} \\ &= (p)(\Omega \mid C \mid S_0^p \mid S \mid E \mid N'_h) \\ &= (p)(\Omega \mid C \mid S_p \mid E \mid N'_h).\end{aligned}$$

In the third case $\Omega \mid C \mid S \mid E \mid J_h \mid N_k \xrightarrow{\iota} (p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_k)$ should be bisimulated by $\Omega \mid C \mid S \mid E \mid N_k \xrightarrow{\iota} (p)(\Omega \mid C \mid S \mid E_p \mid N'_k)$. This is also impossible because $(p)(\Omega \mid C \mid S \mid E_p \mid N'_k) = (p)(\Omega \mid C \mid S \mid E_0^p \mid E \mid N'_k)$ can do a nondeterministic computation step, reaching to a state where N_{k+1} is *not* a concurrent component, while on the other hand $(p)(\Omega \mid C \mid S \mid E \mid J_p \mid N'_k) = \Omega \mid C \mid S \mid E \mid N_{k+1}$.

2. Now suppose the immediate actions of $N_0, N_1, N_2, \ldots$ are input actions. In this case an output action of $\Omega \mid C \mid S \mid E \mid J_h$ induced by J_h, whether it is a free output action or a bound output action, must be bisimulated by an output action of $\Omega \mid C \mid S \mid E$ induced by C or D or E. We can deduce a contradiction as in the first case.

We have proved that there cannot be any subbisimilarity from $\mathbb{M}$ to the π-calculus. ◀

5 Application to Higher Order Process Calculi

We demonstrate in this section that higher order process calculi cannot be interpreted in the π-calculus. Since the π-calculus is complete, it makes sense to focus on complete higher order process calculi. The completeness brings out the simplicity of the counter example and reveals the strength of the negative result.

Intuitively a model is complete if it is Turing powerful in an interactive fashion. A precise definition of completeness is given in terms of the *computability model* denoted by $\mathbb{C}$. The $\mathbb{C}$-processes are generated from the following grammar.

$$P \quad := \quad \mathbf{0} \mid \Omega \mid F_a^b(\mathsf{f}(x)) \mid \overline{a}(i) \mid P \mid P,$$

where f is a computable function and $i \in \omega$. The semantics is defined by the following rules, in which $\mathsf{f}(i)\uparrow$ means that f is undefined on i.

$$\frac{}{\Omega \xrightarrow{\tau} \Omega} \qquad \frac{}{\overline{a}(i) \xrightarrow{\overline{a}(i)} \mathbf{0}} \qquad \frac{}{F_a^b(\mathsf{f}(x)) \xrightarrow{a(i)} \overline{b}(j)}\,\mathsf{f}(i)=j \qquad \frac{}{F_a^b(\mathsf{f}(x)) \xrightarrow{a(i)} \Omega}\,\mathsf{f}(i)\uparrow$$

$$\frac{P \xrightarrow{\lambda} P'}{P \mid Q \xrightarrow{\lambda} P' \mid Q} \qquad \frac{P \xrightarrow{a(i)} P' \quad Q \xrightarrow{\overline{a}(i)} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'}$$

The process Ω diverges. The process $F_a^b(\mathsf{f}(x))$ simulates the input-output behaviour of $\mathsf{f}(x)$ with input channel a and output channel b. The output process $\overline{a}(i)$ releases the number i at channel a. The model $\mathbb{C}$ is the minimal interactive extension of the computable functions. It says nothing about how computations are done. For this reason it is the best model to formalize the notion of completeness for interaction models. See [5] for more discussions and technical backgrounds.

▶ **Definition 11.** An interaction model $\mathbb{M}$ is *complete* if $\mathbb{C} \sqsubseteq \mathbb{M}$.

The completeness of the π-calculus and the completeness of value-passing calculi are proved in [5]. In the same paper it is shown that there is no subbisimilarity from a value-passing calculus to π. The reader should convince herself/himself that the value-passing processes satisfying (5) through (13) are easy to define.

We now convince the reader that the π-calculus cannot in general interpret higher order process calculi in a robust way. As an example we take a look at a particular functional model, denoted by $\mathbb{L}$. This is an abstraction-passing calculus. To define the model we need the syntactical class $\mathcal{V}$ of abstraction variables. The elements of $\mathcal{V}$ are denoted by U, V, W, X, Y, Z. The set of $\mathbb{L}$-*terms* is generated by the following grammar:

$$\begin{aligned} S, T &:= \mathbf{0} \mid \mu(X).T \mid \overline{\mu}[A].T \mid S \,|\, T \mid (p)T \mid A(\mu, \nu), \\ A &:= X \mid \lambda(u, v).T. \end{aligned}$$

An abstraction is either an abstraction variable or of the form $\lambda(u, v).T$. The term $A(\alpha, \beta)$ is an instantiation of A at α, β. We have the grammar equality $(\lambda(u, v).T)(\alpha, \beta) \equiv A\{\alpha/u, \beta/v\}$. We abbreviate $\lambda(u, v).T$ to $\lambda(u).T$ if v does not appear in T and accordingly $A(\alpha, \alpha)$ to $A(\alpha)$. Because we would like to think that the abstraction A in $\overline{\mu}[A].T$ represents a complete message, we impose the following constraint.

(‡) In $\overline{\mu}[A].T$ the abstraction A contains no free channel variables, no unbound private channels and no occurrences of global channel.

The constraint (‡) gives rise to a simpler and weaker model whose algebraic property is easier to work out.

Using the action set $\{\alpha(A), \overline{\alpha}(A) \mid \alpha \in \mathcal{C}_g \cup \mathcal{C}_p\} \cup \{\tau\}$, the simple operational semantics of $\mathbb{L}$ is defined by the following rules. Notice that because of the constraint (‡) on abstractions there is no name extrusion.

$$\frac{}{\alpha(X).T \xrightarrow{\alpha(A)} T\{A/X\}} \qquad \frac{}{\overline{\alpha}[A].T \xrightarrow{\overline{\alpha}(A)} T} \qquad \frac{S \xrightarrow{\lambda} S'}{S \,|\, T \xrightarrow{\lambda} S' \,|\, T} \qquad \frac{S \xrightarrow{\alpha(A)} S' \quad T \xrightarrow{\overline{\alpha}(A)} T'}{S \,|\, T \xrightarrow{\tau} S' \,|\, T'}$$

$$\frac{S \xrightarrow{\lambda} T}{(p)S \xrightarrow{\lambda} (p)T}\ p \notin \lambda$$

The forever diverging process can be defined as follows.

$$\Omega \stackrel{\text{def}}{=} (p)\,(p(X).(Xp \,|\, \overline{p}[X]) \,|\, \overline{p}[\lambda(x).x(X).(Xx \,|\, \overline{x}[X])])\,. \tag{20}$$

To demonstrate the power of $\mathbb{L}$ we provide a direct interpretation of the lazy λ-calculus [1] in $\mathbb{L}$. Given an injective function from the set of λ variables to $\mathcal{V}$, a λ-term M is interpreted as an abstraction by the following structural induction.

$$\begin{aligned} [\![x]\!] &= X, \\ [\![\lambda x.M]\!] &= \lambda(u, v).u(X).\overline{v}[[\![M]\!]], \\ [\![MN]\!] &= \lambda(u, v).(mq)([\![M]\!](m, q) \,|\, \overline{m}[[\![N]\!]].q(Z).Z(u, v)). \end{aligned}$$

The process $[\![M]\!](a, b)$ is a "function" that inputs a "λ-term" at channel a and output the result "λ-term" at channel b. The structural aspect of the interpretation is enforced by the following simple lemma.

▶ **Lemma 12.** $[\![M\{N/X\}]\!] \equiv [\![M]\!]\{[\![N]\!]/X\}$.

Let $=_\beta$ denote the β-conversion. The interpretation is semantically correct, guaranteed by the following facts. In the statement of Lemma 13 we identify $\mathbf{0} \,|\, P$ to P syntactically.

▶ **Lemma 13.** *For closed λ-terms M, N, $M \to N$ if and only if $[\![M]\!](a, b) \to\!\!\to [\![N]\!](a, b)$.*

Proof. The λ-term M must be of the form $(\lambda x.L)M_1M_2\dots M_k$, where the application is associative to the left. The encoding $[\![M]\!](a,b)$ is by definition of the form

$$..(m_2q_2)\,((m_1q_1)([\![\lambda x.L]\!](m_1,q_1)\,|\,\overline{m_1}[[\![M_1]\!]].q_1(Z).Z(m_2,q_2))\,|\,|\,\overline{m_2}[[\![M_2]\!]].q_2(Z).Z(m_3,q_3))\,..$$

which reduces in two deterministic computation steps to $[\![L\{M_1/x\}M_2\dots M_k]\!](a,b) \equiv [\![N]\!](a,b)$ using Lemma 12. The argument is reversible. ◀

▶ **Lemma 14.** *For closed λ-terms M, N, $M =_\beta N$ implies $[\![M]\!](a,b) =_{\mathbb{L}} [\![N]\!](a,b)$.*

Proof. In view of Lemma 13 this is the Church-Rosser property [2]. ◀

It is well-known [2] that there is a λ-term encoding $\lceil 0\rceil, \lceil 1\rceil, \lceil 2\rceil, \dots, \lceil i\rceil, \dots$ of the natural numbers and an encoding $\lceil\_\rceil$ of the recursive functions such that $\lceil \mathsf{f}\rceil\lceil i\rceil \to^+ \lceil \mathsf{f}(i)\rceil$ if $\mathsf{f}(i)$ is defined, and that $\lceil \mathsf{f}\rceil\lceil i\rceil \to\to \dots$ diverges if $\mathsf{f}(i)$ is undefined. Given global channels a, b we define the $\mathbb{L}$-processes

$$\overline{a}(i) \stackrel{\text{def}}{=} \overline{a}[[\![\lceil i\rceil]\!]], \tag{21}$$

$$\lambda_{a,b}(\mathsf{f}) \stackrel{\text{def}}{=} [\![\lceil \mathsf{f}\rceil]\!](a,b). \tag{22}$$

It is straightforward to verify that if $\mathsf{f}(i) = j$ then

$$\lambda_{a,b}(\mathsf{f})\,|\,\overline{a}(i) \xrightarrow{\iota} = \overline{b}(j) \tag{23}$$

and if $\mathsf{f}(i)$ is undefined then

$$\lambda_{a,b}(\mathsf{f})\,|\,\overline{a}(i) \xrightarrow{\iota} = \Omega. \tag{24}$$

The processes defined in (20), (21) and (22) actually provide an encoding of $\mathbb{C}$ into $\mathbb{L}$, hence the following.

▶ **Lemma 15.** $\mathbb{C} \sqsubseteq \mathbb{L}$.

Because of the constraint (‡) the completeness proof given in [32] cannot be carried out in $\mathbb{L}$. The replication operator defined in [32] does not seem to be encodable in $\mathbb{L}$.

To apply the general negative result it suffices to explain how numbers are defined in $\mathbb{L}$. Given an abstraction A we write A^+ for $\lambda(x,y).\overline{x}[A]$. Let

$$\widehat{0} \stackrel{\text{def}}{=} \lambda(x,y).\overline{y}[\mathbf{0}],$$

$$\widehat{k+1} \stackrel{\text{def}}{=} (\widehat{k})^+.$$

The abstractions $\widehat{0}, \widehat{1}, \widehat{2}, \dots$ represent numbers. Using these numbers we can define the following simple processes.

$$
\begin{aligned}
\lceil i \rfloor_a &\stackrel{\text{def}}{=} \overline{a}[\widehat{i}], \\
\lceil \lambda x.\mathbf{0} \rfloor_a &\stackrel{\text{def}}{=} a(X), \\
\lceil \lambda x.x^+ \rfloor_a &\stackrel{\text{def}}{=} a(X).\overline{a}[X^+], \\
\lceil \lambda x.\tau.x^+ + \tau \rfloor_a &\stackrel{\text{def}}{=} a(X).\left(\tau.\overline{a}[X^+] + \tau\right), \\
\lceil \lambda x.x \stackrel{?}{>} k \rfloor_a &\stackrel{\text{def}}{=} a(X).(n_1 o_1)(X(n_1,o_1) \mid n_1(X_1).(n_2 o_2)(X_1(n_2,o_2) \mid \ldots \\
&\qquad \mid n_{k-1}(X_{k-1}).(n_k o_k)(X_{k-1}(n_k,o_k) \mid n_k.\overline{a}[X^+]) \ldots)), \\
\lceil !\lambda x.\mathbf{0} \rfloor_a &\stackrel{\text{def}}{=} (p)\,(a(X).p(Z).(Z(a,p) \mid \overline{p}[Z]) \mid \overline{p}[\lambda(x,z).x(X).z(Z).(Z(x,z) \mid \overline{z}[Z])]), \\
\lceil !\lambda x.x^+ \rfloor_a &\stackrel{\text{def}}{=} (p)(a(X).(\overline{a}[X^+] \mid p(Z).(Z(a,p) \mid \overline{p}[Z])) \\
&\qquad \mid \overline{p}[\lambda(x,z).x(X).(\overline{x}[X^+] \mid z(Z).(Z(x,z) \mid \overline{z}[Z])])), \\
\lceil !\lambda x.\tau.x^+ + \tau \rfloor_a &\stackrel{\text{def}}{=} (p)(a(X).((\tau.\overline{a}[X^+] + \tau) \mid p(Z).(Z(a,p) \mid \overline{p}[Z])) \\
&\qquad \mid \overline{p}[\lambda(x,z).x(X).((\tau.\overline{x}[X^+] + \tau) \mid z(Z).(Z(x,z) \mid \overline{z}[Z])])).
\end{aligned}
$$

The last three processes, $\lceil !\lambda x.\mathbf{0} \rfloor_a$, $\lceil !\lambda x.x^+ \rfloor_a$ and $\lceil !\lambda x.\tau.x^+ + \tau \rfloor_a$, are intuitively the processes $!a(X)$, $!a(X).\overline{a}[X^+]$ and $!a(X).\,(\tau.\overline{a}[X^+] + \tau)$. These replication processes must be implemented in $\mathbb{L}$. It is easy to verify that the processes defined in the above satisfy the necessary requirements stated in (3) through (12).

Let $G \stackrel{\text{def}}{=} \lceil !\lambda x.\mathbf{0} \rfloor_a \mid \lceil !\lambda x.x^+ \rfloor_a \mid \lceil !\lambda x.\tau.x^+ + \tau \rfloor_a$. We now prove (13).

▶ **Lemma 16.** $\Omega \mid G = \Omega \mid G \mid \lceil \lambda x.x \stackrel{?}{>} k \rfloor_a$.

Proof. Consider the following transition

$$\Omega \mid G \mid \lceil \lambda x.x \stackrel{?}{>} k \rfloor_a \stackrel{a(A)}{\longrightarrow} \Omega \mid G \mid J_A, \tag{25}$$

where J_A is the following process

$$(n_1 o_1)(A(n_1,o_1) \mid n_1(X_1).(n_2 o_2)(X_1(n_2,o_2) \mid \ldots \mid (X_{k-1}(n_k,o_k) \mid n_k.\overline{a}[A^+]) \ldots)).$$

The process J_A and all its descendants can either carry out an internal action or enable $\overline{a}[A^+]$. They cannot do any other actions due to the constraint (‡). Let $\mathfrak{J}$ be the set of all terms J' such that $J_A \Longrightarrow J'$. It can be partitioned into three disjoint subsets.

$$
\begin{aligned}
\mathfrak{J}_0 &\stackrel{\text{def}}{=} \{J' \in \mathfrak{J} \mid \text{no } \tau \text{ descendant of } J' \text{ can enable } \overline{a}[A^+]\}, \\
\mathfrak{J}_1 &\stackrel{\text{def}}{=} \{J' \in \mathfrak{J} \mid \text{every } \tau \text{ descendant of } J' \text{ can enable } \overline{a}[A^+] \text{ now or in future}\}, \\
\mathfrak{J}_{\frac{1}{2}} &\stackrel{\text{def}}{=} \mathfrak{J} \setminus (\mathfrak{J}_0 \cup \mathfrak{J}_1).
\end{aligned}
$$

If $J_A \in \mathfrak{J}_0$ then (25) can be bisimulated by $\Omega \mid G \stackrel{a(A)}{\longrightarrow}= \Omega \mid G$ by invoking the component $\lceil !\lambda x.\mathbf{0} \rfloor_a$. Notice that J_A may induce an infinite computation. But this is not a problem in the presence of Ω. If $J_A \in \mathfrak{J}_1$ then (25) can be bisimulated by $\Omega \mid G \stackrel{a(A)}{\longrightarrow}= \Omega \mid G \mid \overline{a}[A^+]$ by invoking the component $\lceil !\lambda x.x^+ \rfloor_a$. If $J_A \in \mathfrak{J}_{\frac{1}{2}}$ then (25) can be bisimulated by $\Omega \mid G \stackrel{a(A)}{\longrightarrow}= \Omega \mid G \mid (\tau.\overline{a}[A^+] + \tau)$ by invoking the component $\lceil \lambda x.\tau.x^+ + \tau \rfloor_a$. We only have to prove that every $J' \in \mathfrak{J}_{\frac{1}{2}}$ renders true the following equality.

$$\Omega \mid (\tau.\overline{a}[A^+] + \tau) = \Omega \mid J'. \tag{26}$$

Suppose $J' \xrightarrow{\tau} J''$. It induces the transition

$$\Omega \,|\, J' \xrightarrow{\tau} \Omega \,|\, J''. \tag{27}$$

If $J'' \in \mathfrak{J}_{\frac{1}{2}}$, then (27) is bisimulated by $\Omega \,|\, (\tau.\overline{a}[A^+] + \tau) \xrightarrow{\tau} \Omega \,|\, (\tau.\overline{a}[A^+] + \tau)$. If $J'' \in \mathfrak{J}_0$, then (27) is bisimulated by $\Omega \,|\, (\tau.\overline{a}[A^+] + \tau) \xrightarrow{\tau} \Omega \,|\, \mathbf{0}$. If $J'' \in \mathfrak{J}_1$, then (27) is bisimulated by $\Omega \,|\, (\tau.\overline{a}[A^+] + \tau) \xrightarrow{\tau} \Omega \,|\, \overline{a}[A^+]$. We are done. ◀

The main result of the paper now follows.

▶ **Theorem 17.** $\mathbb{L} \not\sqsubseteq \pi$.

Theorem 17 provides a seemingly contradictory result to the well-known encoding of higher order π-calculus π^ω in the first order π-calculus [25, 26] and other similar encodings [29, 31]. The higher order process calculi studied in these encodings are both abstraction-passing and complete. The criteria for relative expressiveness adopted in these papers are weaker than the one of this paper. They include extensionality, equipollence and *weak* bisimulation. Some of the encodings satisfy full abstraction property, others satisfy only a weaker form of it. Almost all encodings *op. cit.* satisfy codivergence and branching bisimulation property. As criteria for submodel relationship codivergence and branching bisimulation help to derive reasonable properties about interpretations, which is duly demonstrated in the proof of the general negative result. Without these two criteria we are not able to conclude for example that N_i cannot perform both an input action and an output action.

The source models considered in the above mentioned papers are more powerful than $\mathbb{L}$ in the sense that the latter is restricted by the condition (‡). The equality stated in (13) fails in π^ω since an observer can be powerful enough to detect the presence of the component $\lceil \lambda x.x \stackrel{?}{>} k \rfloor_a$. Thus Theorem 17 does not contradict to the expressiveness results given by the encodings in the afore mentioned papers, at least on the face of it. We would however like to draw reader's attention to two points. Firstly if we impose the restriction (‡) to π^ω we get a variant, denoted by say $\pi^\omega_\emptyset$, that is comparable to $\mathbb{L}$. The proof of Theorem 17 applies to $\pi^\omega_\emptyset$. It follows that $\pi^\omega_\emptyset \not\sqsubseteq \pi$. On the other hand Sangiorgi's encoding of π^ω in π is also an encoding of $\pi^\omega_\emptyset$ in π. What we do not understand at the moment is if the encoding is still fully abstract. Secondly if we drop the condition (‡) on $\mathbb{L}$, we get a model denoted by $\mathbb{H}$. We may ask the question if $\mathbb{H} \not\sqsubseteq \pi$. Notice that $\mathbb{L} \sqsubseteq \mathbb{H}$ implies $\mathbb{H} \not\sqsubseteq \pi$. Notice also that the failure of (13) implies that the identity map from $\mathbb{L}$ to $\mathbb{H}$ is not a subbisimilarity; it does not rule out however that $\mathbb{L} \sqsubseteq \mathbb{H}$. For the same reason $\pi^\omega_\emptyset \sqsubseteq \pi^\omega$ would imply $\pi^\omega \not\sqsubseteq \pi$. We do not expect that either of the questions, "$\mathbb{L} \sqsubseteq \mathbb{H}$?" and "$\pi^\omega_\emptyset \sqsubseteq \pi^\omega$?", is easy to answer. A possible way to attack these problems is to explore the power of universal processes [6]. This is left for future investigation.

The negative result offered by Theorem 17 provides a fresh look at the issue of higher order calculi vs name-passing calculi and forces us to ask some deeper questions.

6 Conclusion

Name-passing calculi are low level models. It should not come as a surprise that neither the value-passing calculi nor the higher order process calculi with complete messages can be interpreted by the π-calculus. The size of a message in the former models is unbounded. In such models in a single interaction a number/abstraction of arbitrary size is passed from one process to another. This is a high level feature. Studies of communication mechanisms at different abstract levels are part of process theory.

Let's summarize what we know about the relative expressiveness of the three classes of model. Now "Name-Passing $\not\sqsubseteq$ Value-Passing $\not\sqsubseteq$ Name-Passing" [5] and for what we know at the moment "Process-Passing $\not\sqsubseteq$ Name-Passing". It is easily seen that "Process-Passing $\not\sqsubseteq$ Value-Passing" because value-passing communications cannot interpret name extrusion. The problem "Value-Passing $\sqsubseteq$ Process-Passing?" deserves attention. The numbers can be coded up by abstractions. The question is how to do it in a bijective way.

Having seen all the negative results, one wonders if there is a communication mechanism that can interpret the name-passing, the value-passing and the abstraction-passing mechanisms. In [7] a universal model $\mathbb{V}$ is defined that can indeed interpret for example the model $\mathbb{L}$, the π-calculus and the value-passing calculus [3]. Such a model, rather than being artificial, is well motivated by the Church-Turing Thesis. It is against this overall picture that the result of this short paper is to be appreciated.

Acknowledgments. We thank the anonymous reviewers for their criticisms and comments. We thank Davide Sangiorgi for his expert advices. We thank BASICS members for their interest.

References

1 S. Abramsky. The Lazy Lambda Calculus. In: Turner, D. Ed., *Declarative Programming*. Addison-Wesley, 65-116, 1988.
2 H. Barendregt. *The Lambda Calculus: Its Syntax and Semantics*. North-Holland, 1984.
3 Y. Fu. The Value-Passing Calculus. In: *Theories of Programming and Formal Methods*, Lecture Notes in Computer Science 8051, 166-195, 2013.
4 Y. Fu. Nondeterministic Structure of Computation. *Mathematical Structures in Computer Science*, 25:1295-1338, 2015.
5 Y. Fu. Theory of Interaction. *Theoretical Computer Science*, 611:1-49, 2016.
6 Y. Fu. The Universal Process. To appear in *Logical Methods in Computer Science*, 2017.
7 Y. Fu. Thesis for Interaction. `http://basics.sjtu.edu.cn/ yuxi/`. 2017.
8 Y. Fu and H. Lu. On the Expressiveness of Interaction. *Theoretical Computer Science*, 411:1387-1451, 2010.
9 Y. Fu and H. Zhu. The Name-Passing Calculus. arXiv:1508.00093, 2015.
10 D. Gorla. Towards a Unified Approach to Encodability and Separation Results for Process Calculi. In: *CONCUR'08*, Lecture Notes in Computer Science 5201, 492-507, 2008.
11 M. Hennessy and A. Ingólfsdóttir. A Theory of Communicating Processes with Value-Passing. *Information and Computation*, 107:202-236, 1993.
12 M. Hennessy and H. Lin. Symbolic Bisimulations. *Theoretical Computer Science*, 138:353-369, 1995.
13 C. Hoare. *Communicating Sequential Processes*. Prentice Hall, 1985.
14 R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.
15 R. Milner. Functions as Processes. *Mathematical Structures in Computer Science*, 2:119-146, 1992.
16 R. Milner, J. Parrow, and D. Walker. A Calculus of Mobile Processes. *Information and Computation*, 100:1-40 (Part I), 41-77 (Part II), 1992.
17 R. Milner and D. Sangiorgi. Barbed Bisimulation. In: *ICALP'92*, Lecture Notes in Computer Science 623, 685-695, 1992.
18 U. Nestmann. What is a Good Encoding of Guarded Choices? *Information and Computation*, 156:287-319, 2000.
19 U. Nestmann. Welcome to the Jungle: A Subjective Guide to Mobile Process Calculi. In: *CONCUR'06*, Lecture Notes in Computer Science 4137, 52-63, 2006.

20 C. Palamidessi. Comparing the Expressive Power of the Synchronous and the Asynchronous π-Calculus. *Mathematical Structures in Computer Science*, 13:685-719, 2003.

21 D. Park. Concurrency and Automata on Infinite Sequences. In: *Theoretical Computer Science*, Lecture Notes in Computer Science 104, 167-183, 1981.

22 J. Parrow. An Introduction to the π-Calculus. In: J. Bergstra, A. Ponse and S. Smolka (Eds.), *Handbook of Process Algebra*. North-Holland, 478-543, 2001.

23 J. Parrow. Expressiveness of Process Algebras. In: *LIX Colloquium'06*, 2006.

24 L. Priese. On the Concept of Simulation in Asynchronous, Concurrent Systems. *Progress in Cybernatics and Systems Research*, 7:85-92, 1978.

25 D. Sangiorgi. Expressing Mobility in Process Algebras: First Order and Higher Order Paradigm. Ph.D. thesis, Department of Computer Science, University of Edinburgh, 1992.

26 D. Sangiorgi. From π-Calculus to Higher Order π-Calculus – and Back. In: *TAPSOFT'93*, Lecture Notes in Computer Science 668, 151-166, 1993.

27 D. Sangiorgi. π-Calculus, Internal Mobility and Agent-Passing Calculi. *Theoretical Computer Science*, 167:235-274, 1996.

28 D. Sangiorgi and D. Walker. *The π Calculus: A Theory of Mobile Processes*. Cambridge University Press, 2001.

29 B. Thomsen. A Calculus of Higher Order Communicating Systems. In: *POPL'89*, 143-154, 1989.

30 B. Thomsen. A Theory of Higher Order Communicating Systems. *Information and Computation*, 116:38-57, 1995.

31 X. Xu. Distinguishing and Relating Higher-Order and First-Order Processes by Expressiveness. *Acta Informatica*, 49:445-484, 2012.

32 X. Xu and Q. Yin and H. Long: On the Computation Power of Name Parameterization in Higher-Order Processes. In: *ICE'15*, 114-127, 2015.

33 R. van Glabbeek and W. Weijland. Branching Time and Abstraction in Bisimulation Semantics. In: *Information Processing'89*, 613-618, 1989.

34 D. Walker. Objects in the π-Calculus. *Information and Computation*, 116:253-271, 1995.

The Power of Convex Algebras*†

Filippo Bonchi[1], Alexandra Silva[2], and Ana Sokolova[3]

1 **CNRS, ENS-Lyon, France**
`filippo.bonchi@ens-lyon.fr`
2 **University College London, UK**
`alexandra.silva@ucl.ac.uk`
3 **University of Salzburg, Austria**
`ana.sokolova@cs.uni-salzburg.at`

Abstract

Probabilistic automata (PA) combine probability and nondeterminism. They can be given different semantics, like strong bisimilarity, convex bisimilarity, or (more recently) distribution bisimilarity. The latter is based on the view of PA as transformers of probability distributions, also called belief states, and promotes distributions to first-class citizens.

We give a coalgebraic account of the latter semantics, and explain the genesis of the belief-state transformer from a PA. To do so, we make explicit the convex algebraic structure present in PA and identify belief-state transformers as transition systems with state space that carries a convex algebra. As a consequence of our abstract approach, we can give a sound proof technique which we call bisimulation up-to convex hull.

1998 ACM Subject Classification F.3 Logics and Meanings of Programs, G.3 Probability and Statistics, F.1.2 Modes of Computation, D.2.4 Software/Program verification

Keywords and phrases belief-state transformers, bisimulation up-to, coalgebra, convex algebra, convex powerset monad, probabilistic automata

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.23

1 Introduction

Probabilistic automata (PA), closely related to Markov decision processes (MDPs), have been used along the years in various areas of verification [40, 37, 38, 2], machine learning [24, 41], and semantics [66, 52]. Recent interest in research around semantics of probabilistic programming languages has led to new insights in connections between category theory, probability theory, and automata [59, 12, 27, 58, 44].

PA have been given various semantics, starting from strong bisimilarity [39], probabilistic (convex) bisimilarity [50, 49], to bisimilarity on distributions [18, 14, 10, 21, 11, 25, 22, 26]. In this last view, probabilistic automata are understood as transformers of belief states, labeled transition systems (LTSs) having as states probability distributions, see e.g. [14, 15, 35, 1, 13, 22, 19]. Checking such equivalence raises a lot of challenges since belief-states are uncountable. Nevertheless, it is decidable [26, 20] with help of convexity. Despite these developments, what remains open is the understanding of the genesis of belief-state transformers and canonicity of distribution bisimilarity, as well as the role of convex algebras.

* Full version available via `http://arxiv.org/abs/1707.02344`.

† This work has been partially supported by the project ANR-16-CE25-0011 REPAS, the ERC Starting Grant ProFoundNet (grant code 679127), and the FWF National Research Network RiSE/SHiNE project S11411-N23.

The theory of coalgebras [30, 46, 31] provides a toolbox for modelling and analysing different types of state machines. In a nutshell, a coalgebra is an arrow $c\colon S \to FS$ for some functor $F\colon \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$. Intuitively, S represents the space of states of the machine, c its transition structure and the functor F its type. Most importantly, every functor gives rise to a canonical notion of behavioural equivalence ($\approx$), a coinductive proof technique and, for finite states machines, a procedure to check $\approx$.

By tuning the parameters $\mathbf{C}$ and F, one can retrieve many existing types of machines and their associated equivalences. For instance, by taking $\mathbf{C} = \mathbf{Sets}$, the category of sets and functions, and $FS = (\mathcal{P}\mathcal{D}S)^L$, the set of functions from L to subsets ($\mathcal{P}$) of probability distributions ($\mathcal{D}$) over S, coalgebras $c\colon S \to FS$ are in one-to-one correspondence with PA with labels in L. Moreover, the associated notion of behavioural equivalence turns out to be the classical strong probabilistic bisimilarity of [39] (see [4, 54] for more details). Recent work [43] shows that, by taking a slightly different functor, forcing the subsets to be convex, one obtains probabilistic (convex) bisimilarity as in [50, 49].

In this paper, we take a coalgebraic outlook at the semantics of probabilistic automata as belief-state transformers: we wish to translate a PA $c\colon S \to (\mathcal{P}\mathcal{D}S)^L$ into a belief state transformer $c^\sharp\colon \mathcal{D}S \to (\mathcal{P}\mathcal{D}S)^L$. Note that the latter is a coalgebra for the functor $FX = (\mathcal{P}X)^L$, i.e., a labeled transition system, since the state space is the set of probability distributions $\mathcal{D}S$. This is reminiscent of the standard determinisation for non-deterministic automata (NDA) seen as coalgebras $c\colon S \to 2 \times (\mathcal{P}S)^L$. The result of the determinisation is a deterministic automaton $c^\sharp\colon \mathcal{P}S \to 2 \times (\mathcal{P}S)^L$ (with state space $\mathcal{P}S$), which is a coalgebra for the functor $FX = 2 \times X^L$. In the case of PA, one lifts the states space to $\mathcal{D}S$, in the one of NDA to $\mathcal{P}S$. From an abstract perspective, both $\mathcal{D}$ and $\mathcal{P}$ are monads, hereafter denoted by $\mathcal{M}$, and both PA and NDA can be regarded as coalgebras of type $c\colon S \to F\mathcal{M}S$.

In [53], a generalised determinisation transforming coalgebras $c\colon S \to F\mathcal{M}S$ into coalgebras $c^\sharp\colon \mathcal{M}S \to F\mathcal{M}S$ was presented. This construction requires the existence of a *lifting* $\overline{F}$ of F to the category of algebras for the monad $\mathcal{M}$. In the case of NDA, the functor $FX = 2 \times X^L$ can be easily lifted to the category of join-semilattices (algebras for $\mathcal{P}$) and, the coalgebra $c^\sharp\colon \mathcal{P}S \to 2 \times (\mathcal{P}S)^L$ resulting from this construction turns out to be exactly the standard determinised automaton. Unfortunately, this is not the case with probabilistic automata: because of the lack of a suitable distributive law of $\mathcal{D}$ over $\mathcal{P}$ [64], it is impossible to suitably lift $FX = (\mathcal{P}X)^L$ to the category of *convex algebras* (algebras for the monad $\mathcal{D}$).

The way out of the impasse consists in defining a powerset-like functor on the category of convex algebras. This is not a lifting but it enjoys enough properties that allow to lift every PA into a labeled transition system on convex algebras. In turn, these can be transformed – without changing the underlying behavioural equivalence – into standard LTSs on **Sets** by simply forgetting the algebraic structure. We show that the result of the whole procedure is exactly the expected belief-state transformer and that the induced notion of behavioural equivalence coincides with a canonical one present in the literature [14, 25, 22, 26].

The analogy with NDA pays back in terms of proof techniques. In [6], Bonchi and Pous introduced an efficient algorithm to check language equivalence of NDA based on coinduction up-to [45]: in a determinised automaton $c^\sharp\colon \mathcal{P}S \to 2 \times (\mathcal{P}S)^L$, language equivalence can be proved by means of bisimulations up-to the structure of join semilattice carried by the state space $\mathcal{P}S$. Algorithmically, this results in an impressive pruning of the search space.

Similarly, in a belief-state transformer $c^\sharp\colon \mathcal{D}S \to (\mathcal{P}\mathcal{D}S)^L$, one can coinductively reason up-to the convex algebraic structure carried by $\mathcal{D}S$. The resulting proof technique, which we call in this paper *bisimulation up-to convex hull*, allows finite relations to witness the equivalence of infinitely many states. More precisely, by exploiting a recent result in convex

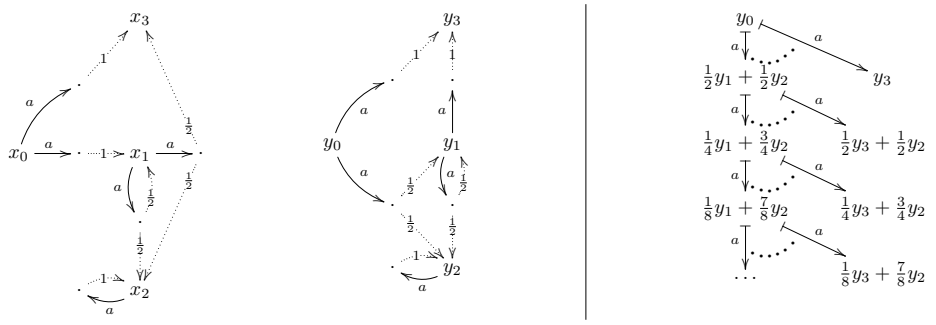


Figure 1 On the left: a PA with set of actions $L = \{a\}$ and set of states $S = \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}$. We depict each transition $s \xrightarrow{a} \zeta$ in two stages: a straight action-labeled arrow from s to $\cdot$ and then several dotted arrows from $\cdot$ to states in S specifing the distribution ζ. On the right: part of the corresponding belief-state transformer. The dots between two arrows $\zeta \overset{a}{\mapsto} \xi_1$ and $\zeta \overset{a}{\mapsto} \xi_2$ denote that ζ can perform infinitely many transitions to states obtained as convex combinations of ξ_1 and ξ_2. For instance $y_0 \overset{a}{\mapsto} \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{2}y_3$.

algebra by Sokolova and Woracek [55], we are able to show that the equivalence of any two belief states can always be proven by means of a *finite* bisimulation up-to.

The paper starts with background on PA (Section 2), convex algebras (Section 3), and coalgebra (Section 4). We provide the PA functor on convex algebras in Section 5. We give the transformation from PA to belief-state transformers in Section 6 and prove the coincidence of the abstract and concrete transformers and semantics. We present bisimulation up-to convex hull in Section 7. Proofs of all results are in the full version.

2 Probabilistic Automata

Probabilistic automata are models of systems that involve both probability and nondeterminism. We start with their definition by Segala and Lynch [50].

▶ **Definition 1.** A probabilistic automaton (PA) is a triple $M = (S, L, \to)$ where S is a set of states, L is a set of actions or action labels, and $\to \subseteq S \times L \times \mathcal{D}(S)$ is the transition relation. As usual, $s \xrightarrow{a} \zeta$ stands for $(s, a, \zeta) \in \to$. ◇

An example is shown on the left of Figure 1. Probabilistic automata can be given different semantics, e.g., (strong probabilistic) bisimilarity [39], convex (probabilistic) bisimilarity [50], and as transformers of belief states [10, 22, 13, 15, 14, 26] whose definitions we present next. For the rest of the section, we fix a PA $M = (S, L, \to)$.

▶ **Definition 2** (Strong Probabilistic Bisimilarity)**.** A relation $R \subseteq S \times S$ is a (strong probabilistic) *bisimulation* if $(s, t) \in R$ implies, for all actions $a \in L$ and all $\xi \in \mathcal{D}(S)$, that

$$s \xrightarrow{a} \xi \Rightarrow \exists \xi' \in \mathcal{D}(S).\ t \xrightarrow{a} \xi' \wedge \xi \equiv_R \xi', \quad \text{and} \quad t \xrightarrow{a} \xi' \Rightarrow \exists \xi \in \mathcal{D}(S).\ s \xrightarrow{a} \xi \wedge \xi \equiv_R \xi'.$$

Here, $\equiv_R \subseteq \mathcal{D}(S) \times \mathcal{D}(S)$ is the lifting of R to distributions, defined by $\xi \equiv_R \xi'$ if and only if there exists a distribution $\nu \in \mathcal{D}(S \times S)$ such that

1. $\sum_{t \in S} \nu(s, t) = \xi(s)$ for any $s \in S$, 2. $\sum_{s \in S} \nu(s, t) = \xi'(t)$ for any $t \in T$, and
3. $\nu(s, t) \neq 0$ implies $(s, t) \in R$.

Two states s and t are (strongly probabilistically) *bisimilar*, notation $s \sim t$, if there exists a (strong probabilistic) bisimulation R with $(s, t) \in R$. ◇

▶ **Definition 3** (Convex Bisimilarity). A relation $R \subseteq S \times S$ is a *convex* (probabilistic) *bisimulation* if $(s,t) \in R$ implies, for all actions $a \in L$ and all $\xi \in \mathcal{D}(S)$, that

$$s \xrightarrow{a} \xi \Rightarrow \exists \xi' \in \mathcal{D}(S).\ t \xrightarrow{a}_c \xi' \wedge \xi \equiv_R \xi', \quad \text{and} \quad t \xrightarrow{a} \xi' \Rightarrow \exists \xi \in \mathcal{D}(S).\ s \xrightarrow{a}_c \xi \wedge \xi \equiv_R \xi'.$$

Here $\to_c$ denotes the convex transition relation, defined as follows: $s \xrightarrow{a}_c \xi$ if and only if $\xi = \sum_{i=1}^n p_i \xi_i$ for some $\xi_i \in \mathcal{D}(S)$ and $p_i \in [0,1]$ satisfying $\sum_{i=1}^n p_i = 1$ and $s \xrightarrow{a} \xi_i$ for $i = 1, \ldots, n$. Two states s and t are *convex bisimilar*, notation $s \sim_c t$, if there exists a convex bisimulation R with $(s,t) \in R$. ◇

Convex bisimilarity is (strong probabilistic) bisimilarity on the "convex closure" of the given PA. More precisely, consider the PA $M_c = (S, L, \to_c)$ in which $s \xrightarrow{a}_c \xi$ whenever $s \in S$ and ξ is in the convex hull (see Section 3 for a definition) of the set $\{\zeta \in \mathcal{D}(S) \mid s \xrightarrow{a} \zeta\}$. Then convex bisimilarity of M is bisimilarity of M_c. Hence, if bisimilarity is the behavioural equivalence of interest, we see that convex semantics arises from a different perspective on the representation of a PA: instead of seeing the given transitions as independent, we look at them as generators of infinitely many transitions in the convex closure.

There is yet another way to understand PA, as belief-state transformers, present but sometimes implicit in [10, 25, 22, 13, 15, 14, 26, 11] to name a few, with behavioural equivalences on distributions. We were particularly inspired by the original work of Deng et al. [13, 15, 14] as well as [26]. Given a PA $M = (S, L, \to)$, consider the labeled transition system $M_{bs} = (\mathcal{D}S, L, \mapsto)$ with states distributions over the original states of M, and transitions $\mapsto \subseteq \mathcal{D}S \times L \times \mathcal{D}S$ defined by

$$\xi \xmapsto{a} \zeta \quad \text{iff} \quad \xi = \sum p_i s_i,\ s_i \xrightarrow{a}_c \xi_i,\ \zeta = \sum p_i \xi_i.$$

We call M_{bs} the belief-state transformer of M. Figure 1, right, displays a part of the belief-state transformer induced by the PA of Figure 1, left. According to this definition, a distribution makes an action step only if all its support states can make the step.

This, and hence the corresponding notion of bisimulation, can vary. For example, in [26] a distribution makes a transition $\xmapsto{a}$ if some of its support states can perform an $\xmapsto{a}$ step. There are several proposed notions of equivalences on distributions [25, 18, 19, 22, 13, 10, 26] that mainly differ in the treatment of termination. See [26] for a detailed comparison.

▶ **Definition 4** (Distribution Bisimilarity). An equivalence $R \subseteq \mathcal{D}S \times \mathcal{D}S$ is a distribution bisimulation of M if and only if it is a bisimulation of the belief-state transformer M_{bs}.

Two distributions ξ and ζ are *distribution bisimilar*, notation $\xi \sim_d \zeta$, if there exists a bisimulation R with $(\xi, \zeta) \in R$. Two states s and t are *distribution bisimilar*, notation $s \sim_d t$, if $\delta_s \sim_d \delta_t$, where δ_x denotes the Dirac distribution with $\delta_x(x) = 1$. ◇

While the foundations of strong probabilistic bisimilarity are well-studied [54, 4, 65] and convex probabilistic bisimilarity was also recently captured coalgebraically [43], the foundations of the semantics of PA as transformers of belief states is not yet explained. One of the goals of the present paper is to show that also that semantics (naturally on distributions [26]) is an instance of generic behavioural equivalence. Note that a (somewhat concrete) proof is given for the bisimilarity of [26] — the authors have proven that their bisimilarity is coalgebraic bisimilarity of a certain coalgebra corresponding to the belief-state transformer. What is missing there, and in all related work, is an explanation of the relationship of the belief-state transformer to the the original PA. Clarifying the foundations of the belief-state transformer and the distribution bisimilarity is our initial motivation.

3 Convex Algebras

By $\mathcal{C}$ we denote the signature of convex algebras

$$\mathcal{C} = \{(p_i)_{i=0}^n \mid n \in \mathbb{N}, p_i \in [0,1], \sum_{i=0}^n p_i = 1\}.$$

The operation symbol $(p_i)_{i=0}^n$ has arity $(n+1)$ and it will be interpreted by a convex combination with coefficients p_i for $i = 0, \ldots, n$. For $p \in [0,1]$ we write $\bar{p} = 1 - p$.

▶ **Definition 5.** A *convex algebra* $\mathbb{X}$ is an algebra with signature $\mathcal{C}$, i.e., a set X together with an operation $\sum_{i=0}^n p_i(-)_i$ for each operational symbol $(p_i)_{i=0}^n \in \mathcal{C}$, such that the following two axioms hold:

- Projection: $\sum_{i=0}^n p_i x_i = x_j$ if $p_j = 1$.
- Barycenter: $\sum_{i=0}^n p_i \left(\sum_{j=0}^m q_{i,j} x_j\right) = \sum_{j=0}^m \left(\sum_{i=0}^n p_i q_{i,j}\right) x_j$.

A convex algebra homomorphism h from $\mathbb{X}$ to $\mathbb{Y}$ is a *convex* (synonymously, *affine*) map, i.e., $h\colon X \to Y$ with the property $h\left(\sum_{i=0}^n p_i x_i\right) = \sum_{i=0}^n p_i h(x_i)$. ◇

▶ Remark 6. Let $\mathbb{X}$ be a convex algebra. Then (for $p_n \neq 1$)

$$\sum_{i=0}^n p_i x_i = \overline{p_n} \left(\sum_{j=0}^{n-1} \frac{p_j}{\overline{p_n}} x_j\right) + p_n x_n \tag{1}$$

Hence, an $(n+1)$-ary convex combination can be written as a binary convex combination using an n-ary convex combination. As a consequence, if X is a set that carries two convex algebras $\mathbb{X}_1$ and $\mathbb{X}_2$ with operations $\sum_{i=0}^n p_i(-)_i$ and $\bigoplus_{i=0}^n p_i(-)_i$, respectively (and binary versions $+$ and $\oplus$, respectively) such that $px + \bar{p}y = px \oplus \bar{p}y$ for all p, x, y, then $\mathbb{X}_1 = \mathbb{X}_2$.

One can also see (1) as a definition, see e.g. [60, Definition 1]. We make the connection explicit with the next proposition, cf. [60, Lemma 1-Lemma 4][1].

▶ **Proposition 7.** *Let X be a set with binary operations $px + \bar{p}y$ for $x, y \in X$ and $p \in (0,1)$. For $x, y, z \in X$ and $p, q \in (0,1)$, assume*

- *Idempotence: $px + \bar{p}x = x$,*
- *Parametric commutativity: $px + \bar{p}y = \bar{p}y + px$,*
- *Parametric associativity: $p(qx + \bar{q}y) + \bar{p}z = pqx + \overline{pq}\left(\frac{p\bar{q}}{\overline{pq}}y + \frac{\bar{p}}{\overline{pq}}z\right)$,*

and define n-ary convex operations by the projection axiom and the formula (1). Then X becomes a convex algebra. ◀

Hence, it suffices to consider binary convex combinations only, whenever more convenient.

▶ **Definition 8.** Let $\mathbb{X}$ be a convex algebra, with carrier X and $C \subseteq X$. C is *convex* if it is the carrier of a subalgebra of $\mathbb{X}$, i.e., if $px + \bar{p}y \in C$ for $x, y \in C$ and $p \in (0,1)$. The *convex hull* of a set $S \subseteq X$, denoted $\mathrm{conv}(S)$, is the smallest convex set that contains S. ◇

Clearly, a set $C \subseteq X$ for X being the carrier of a convex algebra $\mathbb{X}$ is convex if and only if $C = \mathrm{conv}(C)$. Convexity plays an important role in the semantics of probabilistic automata, for example in the definition of convex bisimulation, Definition 3.

[1] Stone's cancellation Postulate V is not used in his Lemma 1-Lemma 4.

4 Coalgebras

In this section, we briefly review some notions from (co)algebra which we will use in the rest of the paper. This section is written for a reader familiar with basic category theory. We have included an expanded version of this section in the full version that also includes basic categorical definitions and more details than what we do here.

Coalgebras provide an abstract framework for state-based systems. Let $\mathbf{C}$ be a base category. A coalgebra is a pair (S, c) of a state space S (object in $\mathbf{C}$) and an arrow $c\colon S \to FS$ in $\mathbf{C}$ where $F\colon \mathbf{C} \to \mathbf{C}$ is a functor that specifies the type of transitions. We will sometimes just say the coalgebra $c\colon S \to FS$, meaning the coalgebra (S, c). A coalgebra homomorphism from a coalgebra (S, c) to a coalgebra (T, d) is an arrow $h\colon S \to T$ in $\mathbf{C}$ that makes the diagram on the right commute. Coalgebras of a functor F and their coalgebra homomorphisms form a category that we denote by $\mathrm{Coalg}_{\mathbf{C}}(F)$. Examples of functors on **Sets** which are of interest to us are:

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ FS & \xrightarrow{Fh} & FT \end{array}$$

1. The constant exponent functor $(-)^L$ for a set L, mapping a set X to the set X^L of all functions from L to X, and a function $f\colon X \to Y$ to $f^L\colon X^L \to Y^L$ with $f^L(g) = f \circ g$.
2. The powerset functor $\mathcal{P}$ mapping a set X to its powerset $\mathcal{P}X = \{S \mid S \subseteq X\}$ and on functions $f\colon X \to Y$ given by direct image: $\mathcal{P}f\colon \mathcal{P}X \to \mathcal{P}Y$, $\mathcal{P}(f)(U) = \{f(u) \mid u \in U\}$.
3. The finitely supported probability distribution functor $\mathcal{D}$ is defined, for a set X and a function $f\colon X \to Y$, as
$$\mathcal{D}X = \{\varphi\colon X \to [0,1] \mid \sum_{x \in X} \varphi(x) = 1, \mathrm{supp}(\varphi) \text{ is finite}\} \qquad \mathcal{D}f(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x).$$
The support set of a distribution $\varphi \in \mathcal{D}X$ is defined as $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$.
4. The functor C [43, 28, 63] maps a set X to the set of all nonempty convex subsets of distributions over X, and a function $f\colon X \to Y$ to the function $\mathcal{P}\mathcal{D}f$.

We will often decompose $\mathcal{P}$ as $\mathcal{P}_{ne} + 1$ where $\mathcal{P}_{ne}$ is the nonempty powerset functor and $(-) + 1$ is the termination functor defined for every set X by $X + 1 = X \cup \{*\}$ with $* \notin X$ and every function $f\colon X \to Y$ by $f + 1(*) = *$ and $f + 1(x) = x$ for $x \in X$.

Coalgebras over a concrete category are equipped with a generic behavioural equivalence, which we define next. Let (S, c) be an F-coalgebra on a concrete category $\mathbf{C}$, with $\mathcal{U}\colon \mathbf{C} \to \mathbf{Sets}$ being the forgetful functor. An equivalence relation $R \subseteq \mathcal{U}S \times \mathcal{U}S$ is a kernel bisimulation (synonymously, a cocongruence) [57, 36, 67] if it is the kernel of a homomorphism, i.e., $R = \ker \mathcal{U}h = \{(s,t) \in \mathcal{U}S \times \mathcal{U}S \mid \mathcal{U}h(s) = \mathcal{U}h(t)\}$ for some coalgebra homomorphism $h\colon (S, c) \to (T, d)$ to some F-coalgebra (T, d). Two states s, t of a coalgebra are behaviourally equivalent notation $s \approx t$ iff there is a kernel bisimulation R with $(s,t) \in R$. A simple but important property is that if there is a functor from one category of coalgebras (over a concrete category) to another that preserves the state space and is identity on morphisms, then this functor preserves behavioural equivalence: if two states are equivalent in a coalgebra of the first category, then they are also equivalent in the image under the functor in the second category.

We are now in position to connect probabilistic automata to coalgebras.

▶ **Proposition 9** ([4, 54]). *A probabilistic automaton $M = (S, L, \to)$ can be identified with a $(\mathcal{P}\mathcal{D})^L$-coalgebra $c_M\colon S \to (\mathcal{P}\mathcal{D}S)^L$ on* **Sets**, *where $s \xrightarrow{a} \xi$ in M iff $\xi \in c_M(s)(a)$ in (S, c_M). Bisimilarity in M equals behavioural equivalence in (S, c_M), i.e., for two states $s, t \in S$ we have $s \sim t \Leftrightarrow s \approx t$.* ◀

It is also possible to provide convex bisimilarity semantics to probabilistic automata via coalgebraic behavioural equivalence, as the next proposition shows.

▶ **Proposition 10** ([43]). *Let $M = (S, L, \to)$ be a probabilistic automaton, and let $(S, \bar{c}_M)$ be a $(C+1)^L$-coalgebra on* **Sets** *defined by $\bar{c}_M(s)(a) = \mathrm{conv}(c_M(s)(a))$ where c_M is as before, if $c_M(s)(a) = \{\xi \mid s \xrightarrow{a} \xi\} \neq \emptyset$; and $\bar{c}_M(s)(a) = *$ if $c_M(s)(a) = \emptyset$. Convex bisimilarity in M equals behavioural equivalence in $(S, \bar{c}_M)$.* ◀

The connection between (S, c_M) and $(S, \bar{c}_M)$ in Proposition 10 is the same as the connection between M and M_c in Section 2. Abstractly, it can be explained using the following well known generic property.

▶ **Lemma 11** ([46, 4]). *Let $\sigma\colon F \Rightarrow G$ be a natural transformation from $F\colon \mathbf{C} \to \mathbf{C}$ to $G\colon \mathbf{C} \to \mathbf{C}$. Then $\mathcal{T}\colon \mathrm{Coalg}_{\mathbf{C}}(F) \to \mathrm{Coalg}_{\mathbf{C}}(G)$ given by $\mathcal{T}(S \xrightarrow{c} FS) = (S \xrightarrow{c} FS \xrightarrow{\sigma_S} GS)$ on objects and identity on morphisms is a functor that preserves behavioural equivalence. If σ is injective, then $\mathcal{T}$ also reflects behavioural equivalence.* ◀

▶ **Example 12.** We have that $\mathrm{conv}\colon \mathcal{P}\mathcal{D} \Rightarrow C+1$ given by $\mathrm{conv}(\emptyset) = *$ and $\mathrm{conv}(X)$ is the already-introduced convex hull for $X \subseteq \mathcal{D}S$, $X \neq \emptyset$ is a natural transformation. Therefore, $\mathrm{conv}^L\colon (\mathcal{P}\mathcal{D})^L \Rightarrow (C+1)^L$ is one as well, defined pointwise. As a consequence from Lemma 11, we get a functor $\mathcal{T}_{\mathrm{conv}}\colon \mathrm{Coalg}_{\mathbf{Sets}}((\mathcal{P}\mathcal{D})^L) \to \mathrm{Coalg}_{\mathbf{Sets}}((C+1)^L)$ and hence bisimilarity implies convex bisimilarity in probabilistic automata.

Also, an injective natural transformation $\iota\colon C+1 \Rightarrow \mathcal{P}\mathcal{D}$ is given by $\iota(X) = X$ and $\iota(*) = \emptyset$ yielding an injective $\chi\colon (C+1)^L \Rightarrow (\mathcal{P}\mathcal{D})^L$. As a consequence, convex bisimilarity coincides with strong bisimilarity on the "convex-closed" probabilistic automaton M_c, i.e., the coalgebra $(S, \bar{c}_M)$ whose transitions are all convex combinations of M-transitions.

4.1 Algebras for a Monad

The behaviour functor F often is, or involves, a monad $\mathcal{M}$, providing certain computational effects, such as partial, non-deterministic, or probabilistic computations.

More precisely, a monad is a functor $\mathcal{M}\colon \mathbf{C} \to \mathbf{C}$ together with two natural transformations: a unit $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow \mathcal{M}$ and multiplication $\mu\colon \mathcal{M}^2 \Rightarrow \mathcal{M}$ that satisfy the laws $\mu \circ \eta_{\mathcal{M}} = id = \mu \circ \mathcal{M}\eta$ and $\mu \circ \mu_{\mathcal{M}} = \mu \circ \mathcal{M}\mu$.

An example that will be pivotal for our exposition is the finitely supported distribution monad. The unit of $\mathcal{D}$ is given by a Dirac distribution $\eta(x) = \delta_x = (x \mapsto 1)$ for $x \in X$ and the multiplication by $\mu(\Phi)(x) = \sum_{\varphi \in \mathrm{supp}(\Phi)} \Phi(\varphi) \cdot \varphi(x)$ for $\Phi \in \mathcal{D}\mathcal{D}X$.

With a monad $\mathcal{M}$ on a category $\mathbf{C}$ one associates the Eilenberg-Moore category $\mathrm{EM}(\mathcal{M})$ of Eilenberg-Moore algebras. Objects of $\mathrm{EM}(\mathcal{M})$ are pairs $\mathbb{A} = (A, a)$ of an object $A \in \mathbf{C}$ and an arrow $a\colon \mathcal{M}A \to A$, satisfying $a \circ \eta = id$ and $a \circ \mathcal{M}a = a \circ \mu$.

A homomorphism from an algebra $\mathbb{A} = (A, a)$ to an algebra $\mathbb{B} = (B, b)$ is a map $h\colon A \to B$ in $\mathbf{C}$ between the underlying objects satisfying $h \circ a = b \circ \mathcal{M}h$.

A category of Eilenberg-Moore algebras which is particularly relevant for our exposition is described in the following proposition. See [61] and [51] for the original result, but also [16, 17] or [29, Theorem 4] where a concrete and simple proof is given.

▶ **Proposition 13** ([61, 16, 17, 29]). *Eilenberg-Moore algebras of the finitely supported distribution monad $\mathcal{D}$ are exactly convex algebras as defined in Section 3. The arrows in the Eilenberg-Moore category $\mathrm{EM}(\mathcal{D})$ are convex algebra homomorphisms.* ◀

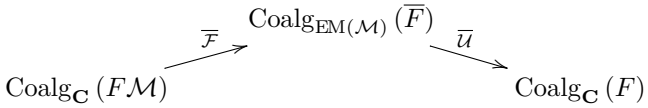
As a consequence, we will interchangeably use the abstract (Eilenberg-Moore algebra) and the concrete definition (convex algebra), whatever is more convenient. For the latter, we also just use binary convex operations, by Proposition 7, whenever more convenient.

4.2 The Generalised Determinisation

We now recall a construction from [53], which serves as source of inspiration for our work.

A functor $\overline{F}\colon \mathrm{EM}(\mathcal{M}) \to \mathrm{EM}(\mathcal{M})$ is said to be a lifting of a functor $F\colon \mathbf{C} \to \mathbf{C}$ if and only if $\mathcal{U} \circ \overline{F} = F \circ \mathcal{U}$. Here, $\mathcal{U}$ is the forgetful functor $\mathcal{U}\colon \mathrm{EM}(\mathcal{M}) \to \mathbf{C}$ mapping an algebra to its carrier. It has a left adjoint $\mathcal{F}$, mapping an object $X \in \mathbf{C}$ to the (free) algebra $(\mathcal{M}X, \mu_X)$. We have that $\mathcal{M} = \mathcal{U} \circ \mathcal{F}$.

Whenever $F\colon \mathbf{C} \to \mathbf{C}$ has a lifting $\overline{F}\colon \mathrm{EM}(\mathcal{M}) \to \mathrm{EM}(\mathcal{M})$, one has the following functors between categories of coalgebras.



The functor $\overline{\mathcal{F}}$ transforms every coalgebra $c\colon S \to F\mathcal{M}S$ over the base category into a coalgebra $c^\sharp\colon \mathcal{F}S \to \overline{F}\mathcal{F}S$. Note that this is a coalgebra on $\mathrm{EM}(\mathcal{M})$: the state space carries an algebra, actually the freely generated one, and $c^\sharp$ is a homomorphism of $\mathcal{M}$-algebras. Intuitively, this amounts to compositionality: like in GSOS specifications, the transitions of a compound state are determined by the transitions of its components.

The functor $\overline{\mathcal{U}}$ simply forgets about the algebraic structure: $c^\sharp$ is mapped into

$$\mathcal{U}c^\sharp\colon \mathcal{M}S = \mathcal{U}\mathcal{F}S \to \mathcal{U}\overline{F}\mathcal{F}S = F\mathcal{U}\mathcal{F}S = F\mathcal{M}S.$$

An important property of $\overline{\mathcal{U}}$ is that it preserves and reflects behavioural equivalence. On the one hand, this fact usually allows to give concrete characterisation of $\approx$ for $\overline{F}$-coalgebras. On the other, it allows, by means of the so-called up-to techniques, to exploit the $\mathcal{M}$-algebraic structure of $\mathcal{F}S$ to check $\approx$ on $\mathcal{U}c^\sharp$.

By taking $F = 2 \times (-)^L$ and $\mathcal{M} = \mathcal{P}$, one transforms $c\colon S \to 2 \times (\mathcal{P}S)^L$ into $\mathcal{U}c^\sharp\colon \mathcal{P}S \to 2 \times (\mathcal{P}S)^L$. The former is a non-deterministic automaton (every c of this type is a pairing $\langle o, t\rangle$ of $o\colon S \to 2$, defining the final states, and $t\colon S \to \mathcal{P}(S)^L$, defining the transition relation) and the latter is a deterministic automaton which has $\mathcal{P}S$ as states space. In [53], see also [32], it is shown that, for a certain choice of the lifting $\overline{F}$, this amounts exactly to the standard determinisation from automata theory. This explains why this construction is called *the generalised determinisation*.

In a sense, this is similar to the translation of probabilistic automata into belief-state transformers that we have seen in Section 2. Indeed, probabilistic automata are coalgebras $c\colon S \to (\mathcal{P}\mathcal{D}S)^L$ and belief state transformers are coalgebras of type $\mathcal{D}S \to (\mathcal{P}\mathcal{D}S)^L$. One would like to take $F = \mathcal{P}^L$ and $\mathcal{M} = \mathcal{D}$ and reuse the above construction but, unfortunately, $\mathcal{P}^L$ does *not* have a *suitable* lifting to $\mathrm{EM}(\mathcal{D})$. This is a consequence of two well known facts: the lack of a *suitable* distributive law $\rho\colon \mathcal{D}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}$ [64] [2] and the one-to-one correspondence between distributive laws and liftings, see e.g. [32]. In the next section, we will nevertheless provide a "powerset-like" functor on $\mathrm{EM}(\mathcal{D})$ that we will exploit then in Section 6 to properly model PA as belief-state transformers.

[2] As shown in [64], there is no distributive law of the powerset monad over the distribution monad. Note that a "trivial" lifting and a corresponding distributive law of the powerset *functor* over the distribution monad exists, it is based on [11] and has been exploited in [32]. However, the corresponding "determinisation" is trivial, in the sense that its distribution bisimilarity coincides with bisimilarity, and it does not correspond to the belief-state transformer.

5 Coalgebras on Convex Algebras

In this section we provide several functors on $\mathrm{EM}(\mathcal{D})$ that will be used in the modelling of probabilistic automata as coalgebras over $\mathrm{EM}(\mathcal{D})$. This will make explicit the implicit algebraic structure (convexity) in probabilistic automata and lead to distribution bisimilarity as natural semantics for probabilistic automata in Section 6.

5.1 Convex Powerset on Convex Algebras

We now define a functor, the (nonempty) convex powerset functor, on $\mathrm{EM}(\mathcal{D})$. Let $\mathbb{A}$ be a convex algebra. We define $\mathcal{P}_{\mathrm{c}}\mathbb{A}$ to be $\mathbb{A}_c = (A_c, a_c)$ where $A_c = \{C \subseteq A \mid C \neq \emptyset,\ C \text{ is convex}\}$ and a_c is the convex algebra structure given by the following *pointwise* binary convex combinations: $pC + \bar{p}D = \{pc + \bar{p}d \mid c \in C, d \in D\}$. It is important that we only allow nonempty convex subsets in the carrier A_c of $\mathcal{P}_{\mathrm{c}}\mathbb{A}$, as otherwise the projection axiom fails.

For convex subsets of a finite dimensional vector space, the pointwise operations are known as the Minkowski addition and are a basic construction in convex geometry, see e.g. [48]. The pointwise way of defining algebras over subsets (carriers of subalgebras) has also been studied in universal algebra, see e.g. [8, 7, 9].

Next, we define $\mathcal{P}_{\mathrm{c}}$ on arrows. For a convex homomorphism $h\colon \mathbb{A} \to \mathbb{B}$, $\mathcal{P}_{\mathrm{c}}h = \mathcal{P}h$. The following property ensures that we are on the right track.

▶ **Proposition 14.** *$\mathcal{P}_{\mathrm{c}}\mathbb{A}$ is a convex algebra. If $h\colon \mathbb{A} \to \mathbb{B}$ is a convex homomorphism, then so is $\mathcal{P}_{\mathrm{c}}h\colon \mathcal{P}_{\mathrm{c}}\mathbb{A} \to \mathcal{P}_{\mathrm{c}}\mathbb{B}$. $\mathcal{P}_{\mathrm{c}}$ is a functor on $\mathrm{EM}(\mathcal{D})$.* ◀

▶ Remark 15. $\mathcal{P}_{\mathrm{c}}$ is not a lifting of C to $\mathrm{EM}(\mathcal{D})$, but it holds that $C = \mathcal{U} \circ \mathcal{P}_{\mathrm{c}} \circ \mathcal{F}$ as illustrated below on the left. $\mathcal{P}_{\mathrm{c}}$ is also not a lifting of $\mathcal{P}_{ne}$, the nonempty powerset functor, but we have an embedding natural transformation $e\colon \mathcal{U} \circ \mathcal{P}_{\mathrm{c}} \Rightarrow \mathcal{P}_{ne} \circ \mathcal{U}$ given by $e(C) = C$, i.e., we are in the situation:

$$\begin{array}{ccc} \mathrm{EM}(\mathcal{D}) & \xrightarrow{\mathcal{P}_{\mathrm{c}}} & \mathrm{EM}(\mathcal{D}) \\ {\scriptstyle\mathcal{F}}\uparrow & & \downarrow{\scriptstyle\mathcal{U}} \\ \mathbf{Sets} & \xrightarrow{C} & \mathbf{Sets} \end{array} \qquad \begin{array}{ccc} \mathrm{EM}(\mathcal{D}) & \xrightarrow{\mathcal{P}_{\mathrm{c}}} & \mathrm{EM}(\mathcal{D}) \\ {\scriptstyle\mathcal{U}}\downarrow & \supseteq & \downarrow{\scriptstyle\mathcal{U}} \\ \mathbf{Sets} & \xrightarrow[\mathcal{P}_{ne}]{} & \mathbf{Sets} \end{array}$$

The right diagram in Remark 15 simply states that every convex subset is a subset, but this fact and the natural transformation e are useful in the sequel. In particular, using e we can show the next result.

▶ **Proposition 16.** *$\mathcal{P}_{\mathrm{c}}$ is a monad on $\mathrm{EM}(\mathcal{D})$, with η and μ as for the powerset monad.* ◀

5.2 Termination on Convex Algebras

The functor $\mathcal{P}_{\mathrm{c}}$ defined in the previous section allows only for nonempty convex subsets. We still miss a way to express termination. The question of termination amounts to the question of extending a convex algebra $\mathbb{A}$ with a single element $*$. This question turns out to be rather involved, beyond the scope of this paper. The answer from [56] is: there are many ways to extend any convex algebra $\mathbb{A}$ with a single element, but there is only one natural functorial way. Somehow now mathematics is forcing us the choice of a specific computational behaviour for termination!

Given a convex algebra $\mathbb{A}$, let $\mathbb{A}+1$ have the carrier $A+\{*\}$ for $* \notin A$ and convex operations given by

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y\,, & x, y \in A, \\ * \qquad\quad , & x = * \text{ or } y = *. \end{cases} \tag{2}$$

Here, the newly added $*$ behaves as a black hole that attracts every other element of the algebra in a convex combination. It is worth to remark that this extension is folklore [23].

▶ **Proposition 17** ([56, 23]). *$\mathbb{A}+1$ as defined above is a convex algebra that extends $\mathbb{A}$ by a single element. The map $h+1$ obtained with the termination functor in* **Sets** *is a convex homomorphism if $h\colon \mathbb{A} \to \mathbb{B}$ is. The assignments $(-)+1$ give a functor on $\mathrm{EM}(\mathcal{D})$.* ◀

We call the functor $(-)+1$ on $\mathrm{EM}(\mathcal{D})$ the termination functor, due to the following.

▶ **Lemma 18.** *The functor $(-)+1$ is a lifting of the termination functor to $\mathrm{EM}(\mathcal{D})$.* ◀

▶ Remark 19. Note that we are abusing notation here: Our termination functor $(-)+1$ on $\mathrm{EM}(\mathcal{D})$ is not the coproduct $(-)+1$ in $\mathrm{EM}(\mathcal{D})$. The coproduct was concretely described in [33, Lemma 4], and the coproduct $\mathbb{X}+1$ has a much larger carrier than $X+1$. Nevertheless, we use the same notation as it is very intuitive and due to Lemma 18.

5.3 Constant Exponent on Convex Algebras

We now show the existence of a constant exponent functor on $\mathrm{EM}(\mathcal{D})$. Let L be a set of labels or actions. Let $\mathbb{A}$ be a convex algebra. Consider $\mathbb{A}^L$ with carrier $A^L = \{f \mid f\colon L \to A\}$ and operations defined (pointwise) by $(pf + \bar{p}g)(l) = pf(l) + \bar{p}g(l)$.

The following property follows directly from the definitions.

▶ **Proposition 20.** *$\mathbb{A}^L$ is a convex algebra. If $h\colon \mathbb{A} \to \mathbb{B}$ is a convex homomorphism, then so is $h^L\colon \mathbb{A}^L \to \mathbb{B}^L$ defined as in* **Sets**. *Hence, $(-)^L$ defined above is a functor on $\mathrm{EM}(\mathcal{D})$.* ◀

We call $(-)^L$ the constant exponent functor on $\mathrm{EM}(\mathcal{D})$. The name and the notation is justified by the following (obvious) property.

▶ **Lemma 21.** *The constant exponent $(-)^L$ on $\mathrm{EM}(\mathcal{D})$ is a lifting of the constant exponent functor $(-)^L$ on* **Sets**. ◀

▶ **Example 22.** Consider a free algebra $\mathcal{F}S = (\mathcal{D}S, \mu)$ of distributions over the set S. By applying first the functor $\mathcal{P}_\mathrm{c}$, then $(-)+1$ and then $(-)^L$, one obtains the algebra

$$(\mathcal{P}_\mathrm{c}\mathcal{F}S + 1)^L = \begin{pmatrix} \mathcal{D}\left((CS+1)^L\right) \\ \downarrow \alpha \\ (CS+1)^L \end{pmatrix}$$

where CS is the set of non-empty convex subsets of distributions over S, and α corresponds to the convex operations[3] $\sum p_i f_i$ defined by

$$\left(\sum p_i f_i\right)(l) = \begin{cases} \{\sum p_i \xi_i \mid \xi_i \in f_i(l)\} & f_i(l) \in CS \text{ for all } i \in \{1,\dots,n\} \\ * & f_i(l) = * \text{ for some } i \in \{1,\dots,n\} \end{cases}$$

[3] In this case, for future reference, it is convenient to spell out the n-ary convex operations.

5.4 Transition Systems on Convex Algebras

We now compose the three functors introduced above to properly model transition systems as coalgebras on $\mathrm{EM}(\mathcal{D})$. The functor that we are interested in is $(\mathcal{P}_\mathrm{c}+1)^L\colon \mathrm{EM}(\mathcal{D}) \to \mathrm{EM}(\mathcal{D})$. A coalgebra $(\mathbb{S}, c)$ for this functor can be thought of as a transition system with labels in L where the state space carries a convex algebra and the transition function $c\colon \mathbb{S} \to (\mathcal{P}_\mathrm{c}\mathbb{S}+1)^L$ is a homomorphism of convex algebras. This property entails compositionality: the transitions of a composite state $px_1 + \bar{p}x_2$ are fully determined by the transitions of its components x_1 and x_2, as shown in the next proposition. We write $x \overset{a}{\mapsto} y$ for $x, y \in S$, the carrier of $\mathbb{S}$ if $y \in c(x)(a)$, and $x \overset{a}{\not\mapsto}$ if $c(x)(a) = *$.

▶ **Proposition 23.** *Let $(\mathbb{S}, c)$ be a $(\mathcal{P}_\mathrm{c}+1)^L$-coalgebra, and let x_1, x_2, y_1, y_2, z be elements of S, the carrier of $\mathbb{S}$. Then, for all $p \in (0,1)$, and $a \in L$*

- *$px_1 + \bar{p}x_2 \overset{a}{\mapsto} z$ iff $z = py_1 + \bar{p}y_2$, $x_1 \overset{a}{\mapsto} y_1$ and $x_2 \overset{a}{\mapsto} y_2$;*
- *$px_1 + \bar{p}x_2 \overset{a}{\not\mapsto}$ iff $x_1 \overset{a}{\not\mapsto} y_1$ or $x_2 \overset{a}{\not\mapsto} y_2$.* ◀

Transition systems on convex algebras are the bridge between PA and LTSs. In the next section we will show that one can transform an arbitrary PA into a $(\mathcal{P}_\mathrm{c}+1)^L$-coalgebra and that, in the latter, behavioural equivalence coincides with the standard notion of bisimilarity for LTSs (Proposition 27).

6 From PA to Belief-State Transformers

Before turning our attention to PA, it is worth to make a further step of abstraction.

Recall from Remark 15 how $\mathcal{P}_\mathrm{c}$ is related to C and $\mathcal{P}_{ne}$. The following definition is the obvious generalisation.

▶ **Definition 24.** Let $\mathcal{M}\colon \mathbf{Sets} \to \mathbf{Sets}$ be a monad and $\mathcal{L}_1, \mathcal{L}_2\colon \mathbf{Sets} \to \mathbf{Sets}$ be two functors. A functor $\mathcal{H}\colon \mathrm{EM}(\mathcal{M}) \to \mathrm{EM}(\mathcal{M})$ is

- a *quasi lifting* of $\mathcal{L}_1$ if the diagram on the left commutes.
- a *lax lifting* of $\mathcal{L}_2$ if there exists an injective natural transformation $e\colon \mathcal{U}\circ\mathcal{H} \Rightarrow \mathcal{L}_2\circ\mathcal{U}$ as depicted on the right.
- an $(\mathcal{L}_1, \mathcal{L}_2)$ *quasi-lax lifting* if it is both a quasi lifting of $\mathcal{L}_1$ and a lax lifting of $\mathcal{L}_2$.

$$\begin{array}{ccc} \mathrm{EM}(\mathcal{M}) & \xrightarrow{\mathcal{H}} & \mathrm{EM}(\mathcal{M}) \\ {\scriptstyle\mathcal{F}}\uparrow & & \downarrow{\scriptstyle\mathcal{U}} \\ \mathbf{Sets} & \xrightarrow{\mathcal{L}_1} & \mathbf{Sets} \end{array} \qquad \begin{array}{ccc} \mathrm{EM}(\mathcal{M}) & \xrightarrow{\mathcal{H}} & \mathrm{EM}(\mathcal{M}) \\ {\scriptstyle\mathcal{U}}\downarrow & \swarrow & \downarrow{\scriptstyle\mathcal{U}} \\ \mathbf{Sets} & \xrightarrow{\mathcal{L}_2} & \mathbf{Sets} \end{array}$$ ◇

So, for instance, $\mathcal{P}_\mathrm{c}$ is a $(C, \mathcal{P}_{ne})$ quasi-lax lifting. From this fact, it follows that $(\mathcal{P}_\mathrm{c}+1)^L$ is a $((C+1)^L, (\mathcal{P}_{ne}+1)^L)$ quasi-lax lifting. Another interesting example is the generalised determinisation (Section 4.2): it is easy to see that $\overline{F}$ is a $(F\mathcal{M}, F)$-quasi-lax lifting. Indeed, like in the generalised powerset construction, one can construct the following functors.

$$\mathrm{Coalg}_{\mathbf{Sets}}(\mathcal{L}_1) \xrightarrow{\overline{\mathcal{F}}} \mathrm{Coalg}_{\mathrm{EM}(\mathcal{M})}(\mathcal{H}) \xrightarrow{\overline{\mathcal{U}}} \mathrm{Coalg}_{\mathbf{Sets}}(\mathcal{L}_2)$$

We first define $\overline{\mathcal{F}}$. Take an $\mathcal{L}_1$-coalgebra (S, c) and recall that $\mathcal{F}S$ is the free algebra $\mu\colon \mathcal{M}\mathcal{M}S \to \mathcal{M}S$. The left diagram in Definition 24 entails that $\mathcal{H}\mathcal{F}S$ is an algebra $\alpha\colon \mathcal{M}\mathcal{L}_1 S \to \mathcal{L}_1 S$. We call $\mathcal{U}c^\sharp$ the composition $\mathcal{U}\mathcal{F}S = \mathcal{M}S \xrightarrow{\mathcal{M}c} \mathcal{M}\mathcal{L}_1 S \xrightarrow{\alpha} \mathcal{L}_1 S = \mathcal{U}\mathcal{H}\mathcal{F}S$. The next lemma shows that $c^\sharp\colon \mathcal{F}S \to \mathcal{H}\mathcal{F}S$ is a map in $\mathrm{EM}(\mathcal{M})$.

▶ **Lemma 25.** *There is a 1-1 correspondence between $\mathcal{L}_1$-coalgebras on* **Sets** *and $\mathcal{H}$-coalgebras on* $\mathrm{EM}(\mathcal{M})$ *with carriers free algebras:*

$$\frac{c\colon S \to \mathcal{L}_1 S \quad \textit{in} \quad \mathbf{Sets}}{c^{\#}\colon \mathcal{F}S \to \mathcal{H}\mathcal{F}S \quad \textit{in} \quad \mathrm{EM}(\mathcal{M})}$$

- *given c, we have $\mathcal{U}c^{\#} = \alpha \circ \mathcal{M}c$ for $\alpha = \mathcal{H}\mathcal{F}S$,*
- *given $c^{\#}$, we have $c = \mathcal{U}c^{\#} \circ \eta$.*

The assignment $\overline{\mathcal{F}}(S,c) = (\mathcal{F}S, c^{\#})$ and $\overline{\mathcal{F}}(h) = \mathcal{M}h$ gives a functor $\overline{\mathcal{F}}\colon \mathrm{Coalg}_{\mathbf{Sets}}(\mathcal{L}_1) \to \mathrm{Coalg}_{\mathrm{EM}(\mathcal{M})}(\mathcal{H})$. ◀

Now we can define $\overline{\mathcal{U}}\colon \mathrm{Coalg}_{\mathrm{EM}(\mathcal{M})}(\mathcal{H}) \to \mathrm{Coalg}_{\mathbf{Sets}}(\mathcal{L}_2)$ as mapping every coalgebra $(\mathbb{S}, c)$ with $c\colon \mathbb{S} \to \mathcal{H}\mathbb{S}$ into

$$\overline{\mathcal{U}}(\mathbb{S},c) = (\mathcal{U}\mathbb{S}, e_{\mathbb{S}} \circ \mathcal{U}c) \quad \text{where} \quad \mathcal{U}\mathbb{S} \xrightarrow{\mathcal{U}c} \mathcal{U}\mathcal{H}\mathbb{S} \xrightarrow{e_{\mathbb{S}}} \mathcal{L}_2\mathcal{U}\mathbb{S}$$

and every coalgebra homomorphism $h\colon (\mathbb{S},c) \to (\mathbb{T},d)$ into $\overline{\mathcal{U}}h = \mathcal{U}h$. Routine computations confirm that $\overline{\mathcal{U}}$ is a functor.

Since $\overline{\mathcal{U}}$ is a functor that keeps the state set constant and is identity on morphisms, every kernel bisimulation on $(\mathbb{S},c)$ is also a kernel bisimulation on $\overline{\mathcal{U}}(\mathbb{S},c)$. The converse is not true in general: a kernel bisimulation R on $\overline{\mathcal{U}}(\mathbb{S},c)$ is a kernel bisimulation on $(\mathbb{S},c)$ only if it is a *congruence* with respect to the algebraic structure of $\mathbb{S}$.

Formally, R is a congruence if and only if the set $\mathcal{U}\mathbb{S}/R$ of equivalence classes of R carries an Eilenberg-Moore algebra and the function $\mathcal{U}[-]_R\colon \mathcal{U}\mathbb{S} \to \mathcal{U}\mathbb{S}/R$ mapping every element of $\mathcal{U}\mathbb{S}$ to its R-equivalence class is an algebra homomorphism.

▶ **Proposition 26.** *The following are equivalent:*

- *R is a kernel bisimulation on $(\mathbb{S},c)$,*
- *R is a congruence of $\mathbb{S}$ and a kernel bisimulation of $\overline{\mathcal{U}}(\mathbb{S},c)$.* ◀

In particular, Proposition 26 and the following result ensure that the functor $\overline{\mathcal{U}}\colon \mathrm{Coalg}_{\mathrm{EM}(\mathcal{D})}(\mathcal{P}_{\mathrm{c}}+1)^L \to \mathrm{Coalg}_{\mathbf{Sets}}\mathcal{P}^L$ preserves and reflect $\approx$.

▶ **Proposition 27.** *Let $(\mathbb{S},c)$ be a $(\mathcal{P}_{\mathrm{c}}+1)^L$-coalgebra. Behavioural equivalence on $\overline{\mathcal{U}}(\mathbb{S},c)$ is a convex congruence*[4]*. Hence, $\overline{\mathcal{U}}$ preserves and reflects behavioural equivalence..* ◀

This means that $\approx$ for $(\mathcal{P}_{\mathrm{c}}+1)^L$-coalgebras, called transition systems on convex algebras in Section 5.4, coincides with the standard notion of bisimilarity for LTSs.

Table 1 summarises all models of PA: from the classical model M being a $\mathcal{P}\mathcal{D}^L$-coalgebra (S, c_M) on **Sets**, via the convex model M_c obtained as $\mathcal{T}_{\mathrm{conv}}(S, c_M)$, to the belief state transformer M_{bs}. The latter coincides with $\overline{\mathcal{U}} \circ \overline{\mathcal{F}} \circ \mathcal{T}_{\mathrm{conv}}(S, c_M)$.

▶ **Theorem 28.** *Let (S, c_M) be a probabilistic automaton. For all $\xi, \zeta \in \mathcal{D}S$,*

$$\xi \sim_d \zeta \quad \Leftrightarrow \quad \xi \approx \zeta \ \ \textit{in} \ \ \overline{\mathcal{U}} \circ \overline{\mathcal{F}} \circ \mathcal{T}_{\mathrm{conv}}(S, c_M).$$ ◀

Hence, distribution bisimilarity is indeed behavioural equivalence on the belief-state transformer and it coincides with standard bisimilarity.

[4] Convex congruences are congruences of convex algebras, see e.g. [55]. They are convex equivalences, i.e., closed under componentwise-defined convex combinations.

Table 1 The three PA models, their corresponding **Sets**-coalgebras, and relations to M.

$M = (S, L, \to)$	$M_c = (S, L, \to_c)$	$M_{bs} = (\mathcal{D}S, L, \mapsto)$
$(S, c_M \colon S \to (\mathcal{P}\mathcal{D})^L)$	$(S, \overline{c}_M \colon S \to (C+1)^L)$	$(\mathcal{D}S, \hat{c}_M \colon \mathcal{D}S \to (\mathcal{P}\mathcal{D}S)^L)$
(S, c_M)	$(S, \overline{c}_M) = \mathcal{T}_{\mathrm{conv}}(S, c_M)$	$(S, \hat{c}_M) = \overline{\mathcal{U}} \circ \overline{\mathcal{F}} \circ \mathcal{T}_{\mathrm{conv}}(S, c_M)$
c_M	$\overline{c}_M = \mathrm{conv}^L \circ c_M$	$\hat{c}_M = (e_{\mathcal{F}S} + 1)^L \circ \mathcal{U}\overline{c}^\sharp_M$

7 Bisimulations Up-To Convex Hull

As we mentioned in Section 4.2, the generalised determinisation allows for the use of up-to techniques [42, 45]. An important example is shown in [6]: given a non-deterministic automaton $c \colon S \to 2 \times \mathcal{P}(S)^L$, one can reason on its determinisation $\mathcal{U}c^\sharp \colon \mathcal{P}(S) \to 2 \times \mathcal{P}(S)^L$ up-to the algebraic structure carried by the state space $\mathcal{P}(S)$. Given a probabilistic automaton $(S, L, \to)$, we would like to exploit the algebraic structure carried by $\mathcal{D}(S)$ to prove properties of the corresponding belief states transformer $(\mathcal{D}(S), L, \mapsto)$. Unfortunately, the lack of a suitable distributive law [64] makes it impossible to reuse the abstract results in [5]. Fortunately, we can redo all the proofs by adapting the theory in [45] to the case of probabilistic automata.

Hereafter we fix a PA $M = (S, L, \to)$ and the corresponding belief states transformer $M_{bs} = (\mathcal{D}(S), L, \mapsto)$. We denote by $Rel_{\mathcal{D}(S)}$ the lattice of relations over $\mathcal{D}(S)$ and define the monotone function $\mathrm{b} \colon Rel_{\mathcal{D}(S)} \to Rel_{\mathcal{D}(S)}$ mapping every relation $R \in Rel_{\mathcal{D}(S)}$ into

$$\begin{aligned} \mathrm{b}(R) \quad ::= \quad \{(\zeta_1, \zeta_2) \quad | \quad \forall a \in L, \quad & \forall \zeta_1' \text{ s.t. } \zeta_1 \overset{a}{\mapsto} \zeta_1', \exists \zeta_2' \text{ s.t. } \zeta_2 \overset{a}{\mapsto} \zeta_2' \text{ and } (\zeta_1', \zeta_2') \in R, \\ & \forall \zeta_1' \text{ s.t. } \zeta_2 \overset{a}{\mapsto} \zeta_2', \exists \zeta_1' \text{ s.t. } \zeta_1 \overset{a}{\mapsto} \zeta_1' \text{ and } (\zeta_1', \zeta_2') \in R\}. \end{aligned}$$

A *bisimulation* is a relation R such that $R \subseteq \mathrm{b}(R)$. Observe that these are just regular bisimulations for labeled transition systems and that the greatest fixpoint of b coincides exactly with $\sim_d$. The *coinduction* principle informs us that to prove that $\zeta_1 \sim_d \zeta_2$ it is enough to exhibit a bisimulation R such that $(\zeta_1, \zeta_2) \in R$.

▶ **Example 29.** Consider the PA in Figure 1 (left) and the belief-state transformer generated by it (right). It is easy to see that the (Dirac distributions of the) states x_2 and y_2 are in $\sim_d$: the relation $\{(x_2, y_2)\}$ is a bisimulation. Also $\{(x_3, y_3)\}$ is a bisimulation: both $x_3 \overset{a}{\not\mapsto}$ and $y_3 \overset{a}{\not\mapsto}$. More generally, for all $\zeta, \xi \in \mathcal{D}(S)$, $p, q \in [0,1]$, $p\zeta + \bar{p}x_3 \sim_d q\xi + \bar{q}y_3$ since both

$$p\zeta + \bar{p}x_3 \overset{a}{\not\mapsto} \text{ and } q\xi + \bar{q}y_3 \overset{a}{\not\mapsto}. \tag{3}$$

Proving that $x_0 \sim_d y_0$ is more complicated. We will show this in Example 32 but, for the time being, observe that one would need an infinite bisimulation containing the following pairs of states.

$$\begin{array}{ccccccccccc} x_0 & \overset{a}{\longmapsto} & x_1 & \overset{a}{\longmapsto} & \frac{1}{2}x_1 + \frac{1}{2}x_2 & \overset{a}{\mapsto} & \frac{1}{4}x_1 + \frac{3}{4}x_2 & \overset{a}{\mapsto} & \dots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ y_0 & \overset{a}{\mapsto} & \frac{1}{2}y_1 + \frac{1}{2}y_2 & \overset{a}{\mapsto} & \frac{1}{4}y_1 + \frac{3}{4}y_2 & \overset{a}{\mapsto} & \frac{1}{8}y_1 + \frac{7}{8}y_2 & \overset{a}{\mapsto} & \dots \end{array}$$

Indeed, all the distributions depicted above have infinitely many possible choices for $\overset{a}{\mapsto}$ but, whenever one of them executes a depicted transition, the corresponding distribution is forced, because of (3), to also choose the depicted transition.

An *up-to technique* is a monotone map $\mathrm{f}\colon Rel_{\mathcal{D}(S)} \to Rel_{\mathcal{D}(S)}$, while a *bisimulation up-to f* is a relation R such that $R \subseteq \mathrm{b}\,\mathrm{f}(R)$. An up-to technique f is said to be *sound* if, for all $R \in Rel_{\mathcal{D}(S)}$, $R \subseteq \mathrm{b}\,\mathrm{f}(R)$ entails that $R \subseteq \sim_d$. It is said to be *compatible* if $\mathrm{f}\,\mathrm{b}(R) \subseteq \mathrm{b}\,\mathrm{f}(R)$. In [45], it is shown that every compatible up-to technique is also sound.

Hereafter we consider the *convex hull* technique $\mathrm{conv}\colon Rel_{\mathcal{D}(S)} \to Rel_{\mathcal{D}(S)}$ mapping every relation $R \in Rel_{\mathcal{D}(S)}$ into its convex hull which, for the sake of clarity, is

$$\mathrm{conv}(R) = \{(p\zeta_1 + \bar{p}\xi_1, p\zeta_2 + \bar{p}\xi_2) \mid (\zeta_1, \zeta_2) \in R,\ (\xi_1, \xi_2) \in R \text{ and } p \in [0,1]\}.$$

▶ **Proposition 30.** conv *is compatible.* ◀

This result has two consequences: First, conv is sound[5] and thus one can prove $\sim_d$ by means of bisimulation up-to conv; Second, conv can be effectively combined with other compatible up-to techniques (for more details see [45] or the full version). In particular, by combining conv with up-to equivalence – which is well known to be compatible – one obtains up-to congruence $\mathrm{cgr}\colon Rel_{\mathcal{D}(S)} \to Rel_{\mathcal{D}(S)}$. This technique maps a relation R into its congruence closure: the smallest relation containing R which is a congruence.

▶ **Proposition 31.** cgr *is compatible.* ◀

Since cgr is compatible and thus sound, we can use bisimulation up-to cgr to check $\sim_d$.

▶ **Example 32.** We can now prove that, in the PA depicted in Figure 1, $x_0 \sim_d y_0$. It is easy to see that the relation $R = \{(x_2, y_2), (x_3, y_3), (x_1, \frac{1}{2}y_1 + \frac{1}{2}y_2), (x_0, y_0)\}$ is a bisimulation up-to cgr: consider $(x_1, \frac{1}{2}y_1 + \frac{1}{2}y_2)$ (the other pairs are trivial) and observe that

$$\begin{array}{ccc} x_1 & \xrightarrow{a} & \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ R & & \mathrm{cgr}(R) \\ \frac{1}{2}y_1 + \frac{1}{2}y_2 & \xrightarrow{a} & \frac{1}{4}y_1 + \frac{3}{4}y_2 \end{array} \qquad \begin{array}{ccc} x_1 & \xrightarrow{a} & \frac{1}{2}x_3 + \frac{1}{2}x_2 \\ R & & \mathrm{cgr}(R) \\ \frac{1}{2}y_1 + \frac{1}{2}y_2 & \xrightarrow{a} & \frac{1}{2}y_3 + \frac{1}{2}y_2 \end{array}$$

Since all the transitions of x_1 and $\frac{1}{2}y_1 + \frac{1}{2}y_2$ are obtained as convex combination of the two above, the arriving states are forced to be in $\mathrm{cgr}(R)$. In symbols, if $x_1 \overset{a}{\mapsto} \zeta = p(\frac{1}{2}x_1 + \frac{1}{2}x_2) + \bar{p}(\frac{1}{2}x_3 + \frac{1}{2}x_2)$, then $\frac{1}{2}y_1 + \frac{1}{2}y_2 \overset{a}{\mapsto} \xi = p(\frac{1}{4}y_1 + \frac{3}{4}y_2) + \bar{p}(\frac{1}{2}y_3 + \frac{1}{2}y_2)$ and $(\zeta, \xi) \in \mathrm{cgr}(R)$.

Recall that in Example 29, we showed that to prove $x_0 \sim_d y_0$ without up-to techniques one would need an infinite bisimulation. Instead, the relation R in Example 32 is a *finite* bisimulation up-to cgr. It turns out that one can always check $\sim_d$ by means of only finite bisimulations up-to. The key to this result is the following theorem, recently proved in [55].

▶ **Theorem 33.** *Congruences of finitely generated convex algebras are finitely generated.* ◀

This result informs us that for a PA with a finite state space S, $\sim_d \subseteq \mathcal{D}(S) \times \mathcal{D}(S)$ is finitely generated (since $\sim_d$ is a congruence, see Proposition 27). In other words there exists a finite relation R such that $\mathrm{cgr}(R) = \sim_d$. Such R is a finite bisimulation up-to cgr:

$$R \subseteq \mathrm{cgr}(R) = \sim_d = \mathrm{b}(\sim_d) = \mathrm{b}(\mathrm{cgr}(R)).$$

▶ **Corollary 34.** *Let $(S, L, \to)$ be a finite PA and let $\zeta_1, \zeta_2 \in \mathcal{D}(S)$ be two distributions such that $\zeta_1 \sim_d \zeta_2$. Then, there exists a finite bisimulation up-to* cgr *R such that $(\zeta_1, \zeta_2) \in R$.* ◀

[5] In [47] a similar up-to technique called "up-to lifting" is defined in the context of probabilistic λ-calculus and proven sound.

8 Conclusions and Future Work

Belief-state transformers and distribution bisimilarity have a strong coalgebraic foundation which leads to a new proof method – bisimulation up-to convex hull. More interestingly, and somewhat surprisingly, proving distribution bisimilarity can be achieved using only *finite* bisimulation up-to witness. This opens exciting new avenues: Corollary 34 gives us hope that bisimulations up-to may play an important role in designing algorithms for automatic equivalence checking of PA, similar to the one played for NDA. Exploring their connections with the algorithms in [26, 20] is our next step.

From a more abstract perspective, our work highlights some limitations of the bialgebraic approach [62, 3, 34]. Despite the fact that our structures are coalgebras on algebras, they are not bialgebras: but still $\approx$ is a congruence and it is amenable to up-to techniques. We believe that *lax* bialgebra may provide some deeper insights.

References

1 Manindra Agrawal, S. Akshay, Blaise Genest, and P. S. Thiagarajan. Approximate verification of the symbolic dynamics of Markov chains. In *Proc LICS'12*, pages 55–64. IEEE, 2012. doi:10.1109/LICS.2012.17.

2 Christel Baier and Joost-Pieter Katoen. *Principles of model checking*. MIT Press, 2008.

3 Falk Bartels. *On generalised coinduction and probabilistic specification formats: distributive laws in coalgebraic modelling*. PhD thesis, Vrije Universiteit, Amsterdam, 2004.

4 Falk Bartels, Ana Sokolova, and Erik de Vink. A hierarchy of probabilistic system types. *Theoretical Computer Science*, 327:3–22, 2004. doi:10.1016/j.tcs.2004.07.019.

5 Filippo Bonchi, Daniela Petrişan, Damien Pous, and Jurriaan Rot. Coinduction up-to in a fibrational setting. In *Proc. CSL-LICS'14*, pages 20:1–20:9. ACM, 2014. doi:10.1145/2603088.2603149.

6 Filippo Bonchi and Damien Pous. Checking nfa equivalence with bisimulations up to congruence. In *Proc. POPL'13*, pages 457–468. ACM, 2013. doi:10.1145/2429069.2429124.

7 Ivica Bošnjak and Rozálija Madarász. On power structures. *Algebra and Discrete Mathematics*, 2:14–35, 2003.

8 Ivica Bošnjak and Rozálija Madarász. Some results on complex algebras of subalgebras. *Novi Sad Journal of Mathematics*, 237(2):231–240, 2007.

9 C. Brink. On power structures. *Algebra Universalis*, 30:177–216, 1993.

10 Pablo Samuel Castro, Prakash Panangaden, and Doina Precup. Equivalence relations in fully and partially observable Markov decision processes. In *Proc. IJCAI'09*, pages 1653–1658, 2009. URL: http://ijcai.org/Proceedings/09/Papers/276.pdf.

11 Silvia Crafa and Francesco Ranzato. A spectrum of behavioral relations over LTSs on probability distributions. In *Proc. CONCUR'11*, pages 124–139. LNCS 6901, 2011. doi:10.1007/978-3-642-23217-6_9.

12 Fredrik Dahlqvist, Vincent Danos, and Ilias Garnier. Robustly parameterised higher-order probabilistic models. In *Proc. CONCUR'16*, pages 23:1–23:15. LIPIcs 50, 2016. doi:10.4230/LIPIcs.CONCUR.2016.23.

13 Yuxin Deng and Matthew Hennessy. On the semantics of Markov automata. *Information and Computation*, 222:139–168, 2013. doi:10.1016/j.ic.2012.10.010.

14 Yuxin Deng, Rob van Glabbeek, Matthew Hennessy, and Carroll Morgan. Characterising testing preorders for finite probabilistic processes. *Logical Methods in Computer Science*, 4(4), 2008. doi:10.2168/LMCS-4(4:4)2008.

15 Yuxin Deng, Rob J. van Glabbeek, Matthew Hennessy, and Carroll Morgan. Testing finitary probabilistic processes. In *Proc. CONCUR'09*, pages 274–288. LNCS 5710, 2009. doi:10.1007/978-3-642-04081-8_19.

16 Ernst-Erich Doberkat. Eilenberg-Moore algebras for stochastic relations. *Information and Computation*, 204(12):1756–1781, 2006. doi:10.1016/j.ic.2006.09.001.

17 Ernst-Erich Doberkat. Erratum and addendum: Eilenberg-Moore algebras for stochastic relations [mr2277336]. *Information and Computation*, 206(12):1476–1484, 2008. doi:10.1016/j.ic.2008.08.002.

18 Laurent Doyen, Thomas A. Henzinger, and Jean-François Raskin. Equivalence of labeled Markov chains. *International Journal of Foundations of Computer Science*, 19(3):549–563, 2008. doi:10.1142/S0129054108005814.

19 Laurent Doyen, Thierry Massart, and Mahsa Shirmohammadi. Limit synchronization in Markov decision processes. In *Proc. FOSSACS'14*, pages 58–72. LNCS 8412, 2014. doi:10.1007/978-3-642-54830-7_4.

20 Christian Eisentraut, Holger Hermanns, Julia Krämer, Andrea Turrini, and Lijun Zhang. Deciding bisimilarities on distributions. In *Proc. QEST'13*, pages 72–88. LNCS 8054, 2013. doi:10.1007/978-3-642-40196-1_6.

21 Christian Eisentraut, Holger Hermanns, and Lijun Zhang. On probabilistic automata in continuous time. In *Proc. LICS'10*, pages 342–351. IEEE, 2010. doi:10.1109/LICS.2010.41.

22 Yuan Feng and Lijun Zhang. When equivalence and bisimulation join forces in probabilistic automata. In *Proc. FM'14*, pages 247–262. LNCS 8442, 2014. doi:10.1007/978-3-319-06410-9_18.

23 Tobias Fritz. Convex spaces I: Definition and Examples, 2015. arXiv:0903.5522v3 [math.MG].

24 Noah D. Goodman, Vikash K. Mansinghka, Daniel M. Roy, Keith Bonawitz, and Joshua B. Tenenbaum. Church: a language for generative models. *CoRR*, abs/1206.3255, 2012. URL: http://arxiv.org/abs/1206.3255.

25 Matthew Hennessy. Exploring probabilistic bisimulations, part I. *Formal Aspects of Computing*, 24(4-6):749–768, 2012. doi:10.1007/s00165-012-0242-7.

26 Holger Hermanns, Jan Krcál, and Jan Kretínský. Probabilistic bisimulation: Naturally on distributions. In *Proc. CONCUR'14*, pages 249–265. LNCS 8704, 2014. doi:10.1007/978-3-662-44584-6_18.

27 Chris Heunen, Ohad Kammar, Sam Staton, and Hongseok Yang. A convenient category for higher-order probability theory. *CoRR*, abs/1701.02547, 2017. URL: http://arxiv.org/abs/1701.02547.

28 Bart Jacobs. Coalgebraic trace semantics for combined possibilitistic and probabilistic systems. *Electronic Notes in Theoretical Computer Science*, 203(5):131–152, 2008. doi:10.1016/j.entcs.2008.05.023.

29 Bart Jacobs. Convexity, duality and effects. In Cristian S. Calude and Vladimiro Sassone, editors, *Theoretical Computer Science - 6th IFIP TC 1/WG 2.2 International Conference, TCS 2010, Held as Part of WCC 2010, Brisbane, Australia, September 20-23, 2010. Proceedings*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 1–19. Springer, 2010. doi:10.1007/978-3-642-15240-5_1.

30 Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*, volume 59 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2016. doi:10.1017/CBO9781316823187.

31 Bart Jacobs and Jan Rutten. A tutorial on (co)algebras and (co)induction. *Bulletin of the EATCS*, 62:222–259, 1997.

32 Bart Jacobs, Alexandra Silva, and Ana Sokolova. Trace semantics via determinization. *Journal of Computer and System Sciences*, 81(5):859–879, 2015. doi:10.1016/j.jcss.2014.12.005.

33 Bart Jacobs, Bas Westerbaan, and Bram Westerbaan. States of convex sets. In *Proc. FOSSACS'15*, pages 87–101. LNCS 9034, 2015. doi:10.1007/978-3-662-46678-0_6.

34 Bartek Klin. Bialgebras for structural operational semantics: An introduction. *Theoretical Computer Science*, 412(38):5043–5069, 2011. doi:10.1016/j.tcs.2011.03.023.

35 Vijay Anand Korthikanti, Mahesh Viswanathan, Gul Agha, and YoungMin Kwon. Reasoning about mdps as transformers of probability distributions. In *Proc. QEST'10*, pages 199–208. IEEE, 2010. doi:10.1109/QEST.2010.35.

36 Alexander Kurz. *Logics for Coalgebras and Applications to Computer Science*. PhD thesis, Ludwig-Maximilians-Universität München, 2000.

37 Marta Z. Kwiatkowska, Gethin Norman, and David Parker. PRISM 4.0: Verification of probabilistic real-time systems. In *Proc. CAV'11*, pages 585–591. LNCS 6806, 2011. doi:10.1007/978-3-642-22110-1_47.

38 Marta Z. Kwiatkowska, Gethin Norman, Roberto Segala, and Jeremy Sproston. Automatic verification of real-time systems with discrete probability distributions. *Theoretical Computer Science*, 282(1):101–150, 2002. doi:10.1016/S0304-3975(01)00046-9.

39 Kim G. Larsen and Arne Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991. doi:10.1016/0890-5401(91)90030-6.

40 Axel Legay, Andrzej S. Murawski, Joël Ouaknine, and James Worrell. On automated verification of probabilistic programs. In *Proc. TACAS'08*, pages 173–187. LNCS 4963, 2008. doi:10.1007/978-3-540-78800-3_13.

41 Hua Mao, Yingke Chen, Manfred Jaeger, Thomas D. Nielsen, Kim G. Larsen, and Brian Nielsen. Learning deterministic probabilistic automata from a model checking perspective. *Machine Learning*, 105(2):255–299, 2016. doi:10.1007/s10994-016-5565-9.

42 Robin Milner. *Communication and concurrency*, volume 84 of *PHI Series in computer science*. Prentice Hall, 1989.

43 Matteo Mio. Upper-expectation bisimilarity and łukasiewicz μ-calculus. In *Proc. FOSSACS'14*, pages 335–350. LNCS 8412, 2014. doi:10.1007/978-3-642-54830-7_22.

44 Michael W. Mislove. Discrete random variables over domains, revisited. In *Concurrency, Security, and Puzzles - Essays Dedicated to Andrew William Roscoe on the Occasion of His 60th Birthday*, pages 185–202. LNCS 10160, 2017. doi:10.1007/978-3-319-51046-0_10.

45 Damien Pous and Davide Sangiorgi. Enhancements of the coinductive proof method. In Davide Sangiorgi and Jan Rutten, editors, *Advanced Topics in Bisimulation and Coinduction*. Cambridge University Press, 2011.

46 Jan Rutten. Universal coalgebra: A theory of systems. *Theoretical Computer Science*, 249:3–80, 2000. doi:10.1016/S0304-3975(00)00056-6.

47 Davide Sangiorgi and Valeria Vignudelli. Environmental bisimulations for probabilistic higher-order languages. In *Proc. POPL'16*, pages 595–607. ACM, 2016. doi:10.1145/2837614.2837651.

48 Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1993. doi:10.1017/CBO9780511526282.

49 Roberto Segala. *Modeling and verification of randomized distributed real-time systems*. PhD thesis, MIT, 1995.

50 Roberto Segala and Nancy Lynch. Probabilistic simulations for probabilistic processes. In *Proc. CONCUR'94*, pages 481–496. LNCS 836, 1994. doi:10.1007/978-3-540-48654-1_35.

51 Zbigniew Semadeni. *Monads and their Eilenberg-Moore algebras in functional analysis*. Queen's University, Kingston, Ont., 1973. Queen's Papers in Pure and Applied Mathematics, No. 33.

52 Falak Sher and Joost-Pieter Katoen. Compositional abstraction techniques for probabilistic automata. In *Proc. TCS'12*, pages 325–341. LNCS 7604, 2012. doi:10.1007/978-3-642-33475-7_23.

53 Alexandra Silva, Filippo Bonchi, Marcello Bonsangue, and Jan Rutten. Generalizing the powerset construction, coalgebraically. In *Proc. FSTTCS'10*, pages 272–283. LIPIcs 8, 2010. doi:10.4230/LIPIcs.FSTTCS.2010.272.

54 Ana Sokolova. Probabilistic systems coalgebraically: A survey. *Theoretical Computer Science*, 412(38):5095–5110, 2011. doi:10.1016/j.tcs.2011.05.008.

55 Ana Sokolova and Harald Woracek. Congruences of convex algebras. *Journal of Pure and Applied Algebra*, 219(8):3110–3148, 2015. doi:10.1016/j.jpaa.2014.10.005.

56 Ana Sokolova and Harald Woracek. Termination in convex sets of distributions. In *Proc. CALCO '17*. LIPIcs, 2017. To appear.

57 Sam Staton. Relating coalgebraic notions of bisimulation. *Logical Methods in Computer Science*, 7(1), 2011. doi:10.2168/LMCS-7(1:13)2011.

58 Sam Staton. Commutative semantics for probabilistic programming. In *Proc. ESOP'17*, pages 855–879. LNCS 10201, 2017. doi:10.1007/978-3-662-54434-1_32.

59 Sam Staton, Hongseok Yang, Frank Wood, Chris Heunen, and Ohad Kammar. Semantics for probabilistic programming: higher-order functions, continuous distributions, and soft constraints. In *Proc. LICS'16*, pages 525–534. ACM, 2016. doi:10.1145/2933575.2935313.

60 M.H. Stone. Postulates for the barycentric calculus. *Annali di Matematica Pura ed Applicata. Serie Quarta*, 29:25–30, 1949. doi:10.1007/BF02413910.

61 T. Świrszcz. Monadic functors and convexity. *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques*, 22:39–42, 1974.

62 Daniele Turi and Gordon D. Plotkin. Towards a mathematical operational semantics. In *Proc. LICS'97*, pages 280–291. IEEE, 1997. doi:10.1109/LICS.1997.614955.

63 Daniele Varacca. *Probability, Nondeterminism and Concurrency: Two Denotational Models for Probabilistic Computation*. PhD thesis, Univ. Aarhus, 2003. BRICS Dissertation Series, DS-03-14.

64 Daniele Varacca and Glynn Winskel. Distributing probability over non-determinism. *Mathematical Structures in Computer Science*, 16(1):87–113, 2006. doi:10.1017/S0960129505005074.

65 Erik de Vink and Jan Rutten. Bisimulation for probabilistic transition systems: a coalgebraic approach. *Theoretical Computer Science*, 221:271–293, 1999. doi:10.1016/S0304-3975(99)00035-3.

66 Ralf Wimmer, Nils Jansen, Andreas Vorpahl, Erika Ábrahám, Joost-Pieter Katoen, and Bernd Becker. High-level counterexamples for probabilistic automata. *Logical Methods in Computer Science*, 11(1), 2015. doi:10.2168/LMCS-11(1:15)2015.

67 Uwe Wolter. On corelations, cokernels, and coequations. *Electronic Notes in Theoretical Computer Science*, 33, 2000. doi:10.1016/S1571-0661(05)80355-X.

Refinement for Signal Flow Graphs

Filippo Bonchi[*1], **Joshua Holland**[2], **Dusko Pavlovic**[†3], **and Paweł Sobociński**[‡4]

1 **Ecole Normale Supérieure, Lyon, France**
2 **University of Southampton, Southampton, UK**
3 **University of Hawaii at Manoa, Honolulu, Hawaii, US**
4 **University of Southampton, Southampton, UK**

Abstract

The symmetric monoidal theory of Interacting Hopf Algebras provides a sound and complete axiomatisation for *linear relations* over a given field. As is the case for ordinary relations, linear relations have a natural order that coincides with *inclusion*. In this paper, we give a presentation for this ordering by extending the theory of Interacting Hopf Algebras with a single additional inequation. We show that the extended theory gives rise to an abelian bicategory—a concept due to Carboni and Walters—and highlight similarities with the algebra of relations. Most importantly, the ordering leads to a well-behaved notion of *refinement* for signal flow graphs.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs, F.3.2 Semantics of Programming Languages

Keywords and phrases Signal flow graphs, refinement, operational semantics, string diagrams, symmetric monoidal inequality theory

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.24

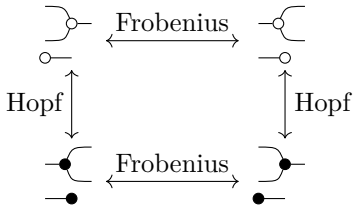
1 Introduction

Signal Flow Graphs (SFGs) were introduced in the 1940s by Shannon [19] as a formal circuit model of a class of simple analog computing machines. They are a common abstraction in control theory and signal processing, used for modelling physical systems and their controllers. Nowadays, cyber-physical systems are modelled and simulated in graphical environments such as Simulink and Modelica that can be seen as great-grandchildren of SFGs.

Their ubiquity is merited because SFGs serve *both* as processors of analogue signals (analytic functions) in continuous time, and as stream transducers in discrete time. The latter makes them amenable to techniques developed by computer scientists for programming language semantics. For instance Rutten [18] showed that coinduction, just as in process algebra, provides a useful proof principle for SFGs. Another example is the *signal flow calculus* [6] where SFGs are represented using string diagrammatic syntax equipped with both a structural operational semantics and a denotational semantics in terms of *linear relations*. Most importantly, denotational equality, which by full abstraction [6] coincides with observational equivalence (trace equivalence), enjoys a sound and complete axiomatization [4]. The same equational theory was independently proposed by Baez and Erbele [3].

* Partially supported by the project ANR-16-CE25-0011 REPAS and AFOSR.
† Partially supported by NSF and AFOSR.
‡ Partially supported by AFOSR.

The axiomatisation of [4, 3]—a *symmetric monoidal theory* (SMT) whose terms are typically rendered graphically as string diagrams—is the starting point of the present work. We adopt the terminology of [5]: the theory of *interacting Hopf algebras* over a ring R, denoted $\mathbb{IH}_{\mathsf{R}}$, consists of a pair of monoids (distinguished by black and white colouring) and a pair of comonoids (again, black and white).



(1)

These black-white (co)monoids satisfy the equations of Frobenius and Hopf algebras, individually recalled in Examples 3 and 4, as illustrated in the schematic to the right.

A theorem in [4] states that $\mathbb{IH}_{\mathsf{R}}$ is a presentation for $\mathbf{LinRel}_{\mathsf{k}}$ the category with arrows *linear relations* (a.k.a. *additive relations*) over k, the field of fractions of R: relations that are also linear subspaces. This paved the way for an equational study of elementary linear algebra by means of string diagrams that, in [22], became *graphical linear algebra*.

Like relations, linear relations are equipped with an ordering that plays a pivotal role in many applications. It is therefore worth seeing $\mathbf{LinRel}_{\mathsf{k}}$ not as a mere category but rather as a *poset enriched* category. In this work, we provide a presentation for the underlying posetal structure of $\mathbf{LinRel}_{\mathsf{k}}$. Our main result states that it is enough to add a *single inequation*

(2)

to the equational theory of $\mathbb{IH}_{\mathsf{R}}$ in order to obtain a sound and complete axiomatization of the ordering between the arrows of $\mathbf{LinRel}_{\mathsf{k}}$. Viewed as linear relations, (2) says that the unique zero-dimensional subspace $\{0\}$ of k, considered as a vector space over itself, is a subset of the unique one-dimensional subspace. Of course, the reverse inequality does not hold.

The focus on the order sheds lights on some interesting properties of $\mathbb{IH}_{\mathsf{R}}$. We show that $\mathbb{IH}_{\mathsf{R}}$ forms an abelian bicategory [11, Def. 5.1] and that it supports operations akin to the algebra of relations [14]. Moreover, the order resolves a mystery surrounding the equational theory of interacting Hopf algebras. The system summarised in (1) is symmetric. There is no difference, equationally, between the white (co)monoid and the black (co)monoid, in spite their different meaning as linear relations: the white is the additive structure, while the black is "copying", e.g.  is typically the diagonal relation. Crucially, (2) breaks this symmetry.

When $\mathsf{R} = \mathsf{k}[x]$ (the ring of polynomials with indeterminate x and coefficients from field k), $\mathbb{IH}_{\mathsf{R}}$ provides a sound and complete axiomatisation for SFGs [5]. The addition of (2) gives a sound and complete axiomatisation for what we call *refinement* of SFGs.

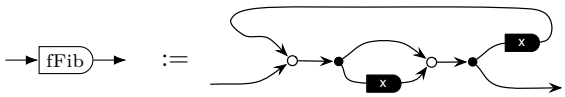
1.1 Structure of the paper

The problem of refinement of SFGs is informally explained with an example in Section 2. In Section 3 we recall the basic concepts of SMTs and, in Section 4, the theory of Interacting Hopf Algebras. In Section 5 we extend the concept of monoidal theory to handle inequations. Section 6 is devoted to proving our main result and, in Section 7, we shed light on the algebraic structure of the resulting theory, drawing parallels with relational algebra. Finally, in Section 8 we return to our motivating problem and discuss related work in Section 9.

2 Fibonacci's rabbits and guinea pigs

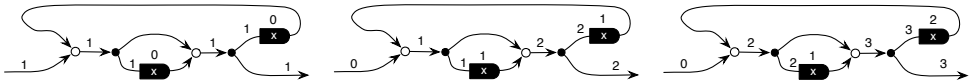
A signal flow graph of sort (m, n), using the discrete semantics, is a stream transducer that takes m input streams and produces n output streams. For example, consider the $(1, 1)$ SFG

below, which implements the well-known Fibonacci recurrence relation:



(3)

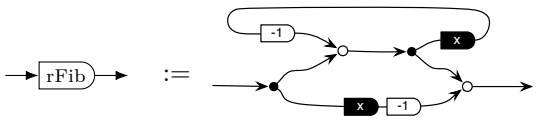
The white circles are adders (two inputs and one output), the black circles are duplicators (one input and two outputs). The 'x' gates are delays, or one-state buffers, which we assume to be initialised with zero. Given the sequence of inputs 1; 0; 0; 0; 0; 0; ..., the output is the Fibonacci sequence 1; 2; 3; 5; 8; 13; .... We illustrate the first few steps below: the state of each delay is illustrated by the number above it, the remaining numbers keep track of the value on each wire at each iteration. A formal operational semantics is recalled in Section 8.



(4)

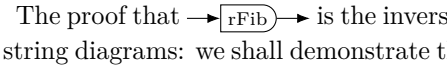
This output—according to the Fibonacci's rule [20] for rabbit reproduction—is the total number of rabbit pairs in each month, starting with a pair of rabbits (the first input is 1), and subsequently not adding nor taking away pairs (all further inputs are 0). Other inputs are possible, in this sense generalising Fibonacci; e.g. adding a pair for two months and taking away two every third month (input 1; 1; −2; 1; 1; −2; ...) yields (1; 3; 3; 5; 10; 14; ...).

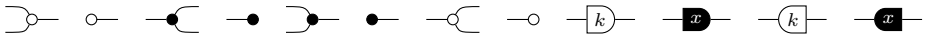
A *trace* of an (m, n) SFG c is a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha}$ is an m-tuple and $\boldsymbol{\beta}$ is the output n-tuple produced by c on $\boldsymbol{\alpha}$: e.g. (3) has (1; 0; 0; ... , 1; 2; 3; ...) and (1; 1; −2; ... , 1; 3; 3; ...) as traces. The *behaviour* of a signal flow graph is the set of all its traces. Note that behaviour is a functional *relation* on streams; in particular, if an SFG is invertible then its inverse has the opposite relation as behaviour. Here (3) *is* invertible and has the following inverse, where the '−1 gates', instances of amplifiers, multiply their input by −1:



(5)

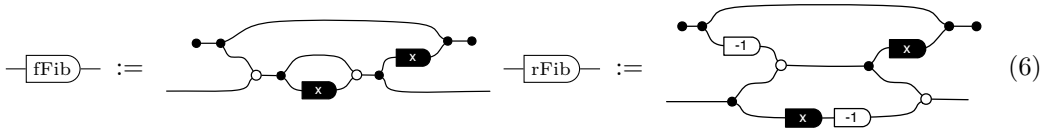
The SFG above thus solves the toy *sustainable rabbit farming problem*: how many rabbits must the farmer buy and sell in each month to maintain, say, four pairs in her rabbit pen? The answer is obtained by using 4; 4; 4; 4; 4; ... as input to (5), resulting in 4; −4; 0; −4; 0; ...: i.e. four pairs bought in the first month and, subsequently, four pairs sold every 2nd month.

The proof that  is the inverse of consists of an algebraic manipulation of string diagrams: we shall demonstrate this below, after a brief discussion of the mathematics behind the approach. As explained in the Introduction, the theory $\mathbb{IH}_{\mathsf{R}}$ of Interacting Hopf Algebras characterises linear relations, and an example of such a relation is the behaviour of any signal flow graph when $\mathsf{R} = \mathsf{k}[x]$. This algebraic theory is not a classical (finite product) algebraic theory but a symmetric monoidal theory. This means replacing traditional tree-like syntax with string diagrams. Concretely, $\mathbb{IH}_{\mathsf{k}[x]}$ involves two commutative monoids and two commutative comonoids, meaning that string diagrams are built up from the following:



where k ranges over the coefficients in k. The different colouring given to the indeterminate x is inspired by its special semantic role in SFGs.

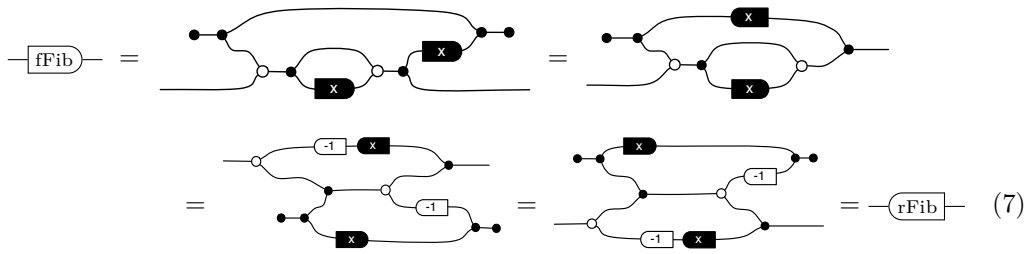
The Fibonacci SFG (3), and its inverse (5)—as string diagrams—look as follows:



(6)

Note the similarity with SFGs (3) and (5) – the string diagram, roughly speaking, is obtained by forgetting the direction of flow—i.e. erasing the arrowheads—and by replacing U-turn wires with so-called 'cups' and 'caps', formed by composing multiplication with counit, and unit with comultiplication. We use notation —fFib—, —rFib— to emphasise that (undirected) string diagrams have been obtained by translating SFGs that have a left-to-right signal flow.

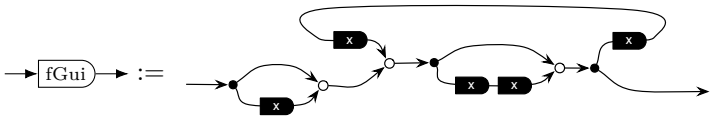
Using the equational theory of [4, 3] we show—by diagrammatic reasoning, i.e. algebraic manipulation of string diagrams—that the inverse of the Fibonacci SFG is as claimed:



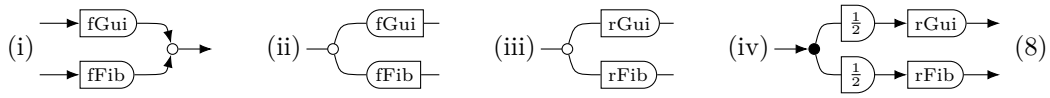
(7)

Note that —fFib— is *equal* to the mirror image of its inverse —rFib—: this, operationally speaking, means simply that the behaviours of →fFib→ and →rFib→ are opposites as relations. This worked example demonstrates the power of equational reasoning in $\mathbb{IH}_{\mathsf{k}[x]}$.

Let us now consider a more interesting variant of the sustainable farming problem. Suppose that the rabbit farmer also keeps guinea pigs, which, for the sake of this paper, gestate twice as long as rabbits. The SFG for guinea pigs is the following:



For instance, on input 1; 0; 0; 0; 0; 0; 0;... the output is 1; 1; 2; 2; 3; 3; 5;.... The combined rabbit and guinea pig pen has SFG (8(i)), with two inputs and one output. The inputs mean buying or selling a species; the output is the total number of animals in the pen.



(8)

We now pose the *sustainable farming problem*: how many rabbits and guinea pigs must the farmer buy and sell in order to keep the total population fixed? To solve the problem, we draw the string diagram that serves as the *specification* of the inverse of (8(i)), as in (8(ii)). Now —fFib— and —fGui— are invertible so we can replace (8(ii)) with the equal diagram (8(iii)).

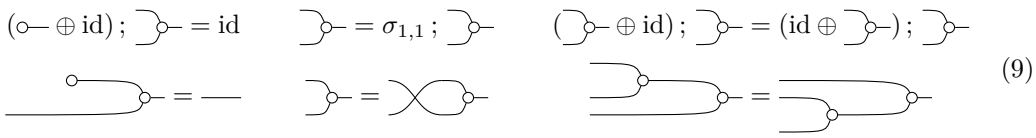
This is as far as we can go in our quest for a SFG, since unfortunately, the specification is not functional: there is no unique solution to the sustainable farming problem. One simple solution is to divide the pen in half and limit the species separately. This, as a SFG, is (8(iv)). As string diagrams (8(iii)) and (8(iv)) are *not* equal. Yet every trace of the second is a trace of the first; this is an example of *refinement*. In this paper, we extend $\mathbb{IH}$ and characterise this refinement relation. Indeed, we will see that, as string diagrams, (8(iv)) $\leq$ (8(iii)).

3 Symmetric Monoidal Theories and props

A *symmetric monoidal theory* (SMT) (Σ, E) consists of a set Σ of *generators* $o : m \to n$, each with an *arity* m and *coarity* n $(m, n \in \mathbb{N})$, along with a set E of equations, which are pairs $(t_1, t_2 : m \to n)$ of Σ-terms; t_1 and t_2 must have the same arity and coarity. A Σ-*term* is constructed inductively from generators in Σ, together with the identity $\mathrm{id} : 1 \to 1$ and the symmetry $\sigma_{1,1} : 2 \to 2$, using composition ; and monoidal product $\oplus$. Given Σ-terms $t : k \to l$, $u : l \to m$ and $v : m \to n$, we construct Σ-terms $t\,;u : k \to m$ and $t \oplus v : k + m \to l + n$.

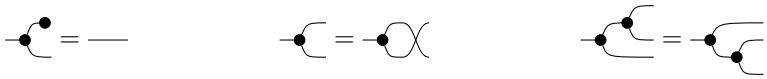
Σ-terms are rendered as diagrams and are considered up to the laws of symmetric strict monoidal categories. Analogously to SFGs, generators are drawn as "circuit components" with dangling wires. The identity is drawn — and the symmetry $\times$. Composition of terms is placing them side-by-side and joining the wires. The monoidal product $\oplus$ is stacking terms on top of each other, as in the following examples. Next we introduce some important SMTs, which are used as building blocks to construct the full SMT for reasoning about SFGs.

▶ **Example 1** (The SMT (Σ_M, E_M) of commutative monoids). Σ_M contains two generators: *multiplication* $\curlyvee : 2 \to 1$ and *unit* $\circ — : 0 \to 1$. E_M contains three equations; we show them both with composition and product explicitly and as diagrams.



(9)

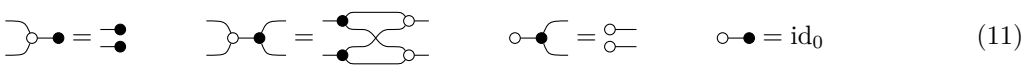
▶ **Example 2** (The SMT (Σ_C, E_C) of commutative comonoids). Again there are two generators, but this time mirrored: there is a *comultiplication* $\curlywedge : 1 \to 2$ and a *counit* $—● : 1 \to 0$. The equations are the following, this time given only in the diagrammatic form:



(10)

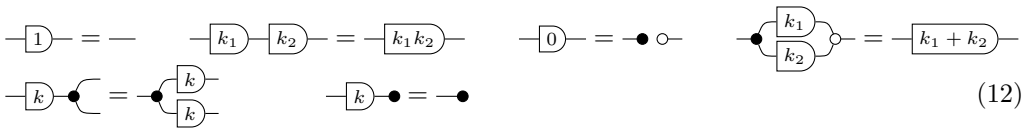
Hopf and Frobenius (bi)monoids are two important ways that monoids and comonoids interact; both are needed for the theories we define in this paper.

▶ **Example 3** (R Hopf monoids). The generators of the SMT are simply those in $\Sigma_M \cup \Sigma_C$, and to the equations in E_M and E_C we add the following *bimonoid laws*:



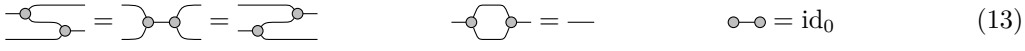
(11)

For a commutative ring R, we need to add generators $—\boxed{k}—$ for every $k \in \mathsf{R}$ and stipulate



(12)

▶ **Example 4** ((extra special) Frobenius monoids). The generators are $\Sigma_M \cup \Sigma_C$. Keeping (1) in mind, we colour all the generators in grey, which will later be instantiated as either black or white. Our equations are now the Frobenius law, together with the special and extra equations. We often call the latter the "bone" equation, due to its appearance when drawn.



(13)

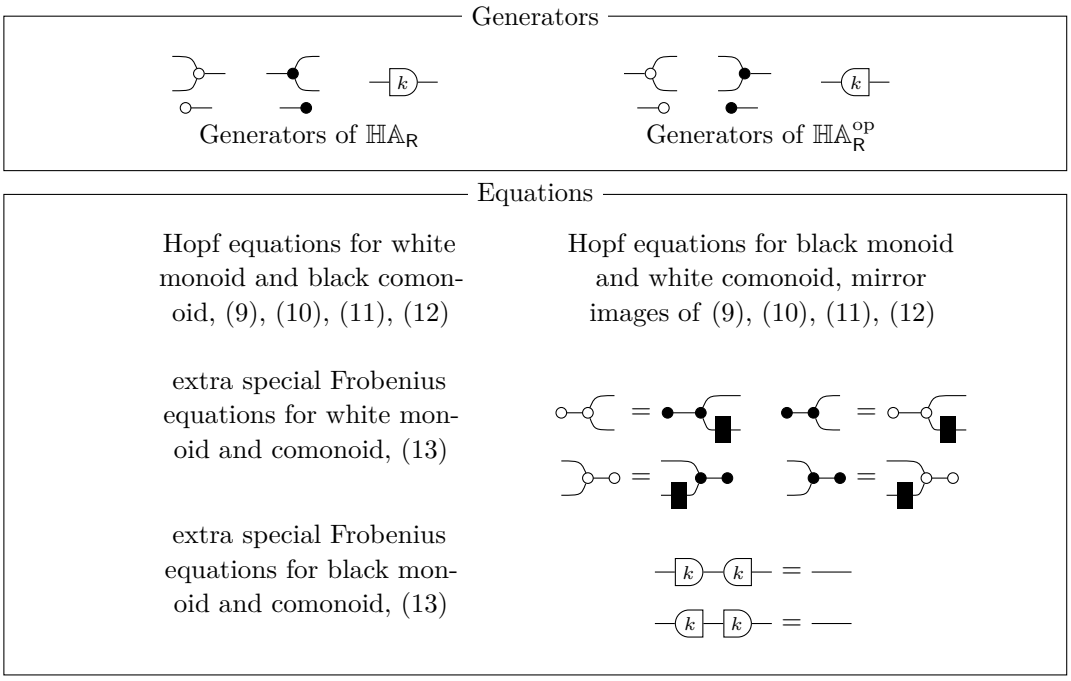


Figure 1 The presentation of $\mathbb{IH}_{\mathsf{R}}$. k takes all values in $\mathsf{R} \setminus \{0\}$. — is the *antipode*, which is defined to be either of —[−1]— or —(−1)—; they can be shown to be equal. See [7] for more details.

We can obtain a symmetric monoidal category from an SMT (Σ, E) as follows:

- objects are natural numbers
- arrows $m \to n$ are Σ-terms $m \to n$ modulo the laws of symmetric monoidal categories and the (smallest congruence containing) the equations $t_1 = t_2$ for each pair $(t_1, t_2) \in E$

Such a category is a special type of symmetric monoidal category called a prop.

▶ **Definition 5.** A *prop* (product and permutation category) is a strict symmetric monoidal category with objects $\mathbb{N}$, where $m \oplus n := m + n$. A homomorphism is an identity-on-objects symmetric monoidal functor, giving a category **PROP**.

▶ **Example 6.** Given a commutative ring R, the prop $\mathbf{Mat}_{\mathsf{R}}$ of matrices over R has as arrows $m \to n$ the $n \times m$ matrices, composition ; is matrix multiplication and $A \oplus B$ is the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. In [16, 7] it is shown that $\mathbf{Mat}_{\mathsf{R}}$ is isomorphic to the prop $\mathbb{HA}_{\mathsf{R}}$ arising from the SMT of Hopf monoids over R (Example 3). The isomorphism $\mathcal{S}' : \mathbb{HA}_{\mathsf{R}} \to \mathbf{Mat}_{\mathsf{R}}$ maps

$$\mathcal{S}'\left(\begin{array}{c}\includegraphics[height=1.2em]{}\end{array}\right) \mapsto \begin{pmatrix}1\\1\end{pmatrix} \quad \mathcal{S}'\left(\begin{array}{c}\includegraphics[height=1.2em]{}\end{array}\right) \mapsto ¡ \quad \mathcal{S}'\left(\begin{array}{c}\includegraphics[height=1.2em]{}\end{array}\right) \mapsto \begin{pmatrix}k\end{pmatrix} \quad \mathcal{S}'\left(\begin{array}{c}\includegraphics[height=1.2em]{}\end{array}\right) \mapsto \begin{pmatrix}1 & 1\end{pmatrix} \quad \mathcal{S}'\left(\begin{array}{c}\includegraphics[height=1.2em]{}\end{array}\right) \mapsto ! \qquad (14)$$

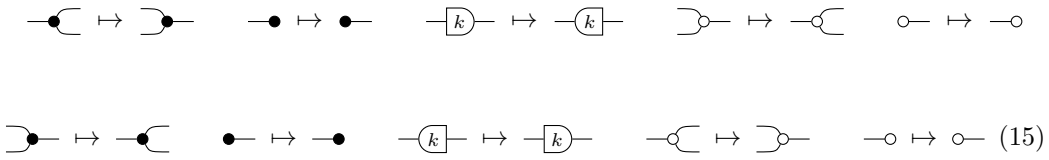
where $! : 0 \to 1$ and $¡ : 1 \to 0$ are given by the universal property of 0 in $\mathbf{Mat}_{\mathsf{R}}$.

Observe (14) defines $\mathcal{S}'$ for all arrows A of $\mathbb{HA}_{\mathsf{R}}$. More generally, to specify a homomorphism from a prop obtained from an SMT (Σ, E), it is enough to define it on the generators in Σ, and check that the equations E hold in the image. We shall often use this argument.

4 Interacting Hopf Algebras

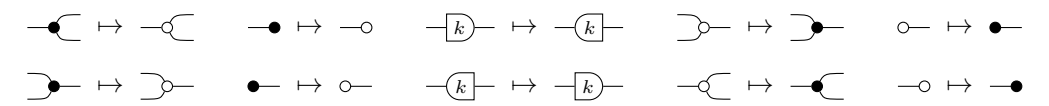
Zanasi with the first and third authors introduced the SMT of Interacting Hopf Algebras [5, 7] as a foundation for SFGs [4, 6, 8]. We recall the equational theory in Fig. 1 where R is a

fixed principal ideal domain, and denote the resulting prop by $\mathbb{IH}_{\mathsf{R}}$. As illustrated in (1), the theory features monoids and comonoids that interact either as extra special Frobenius monoids (Example 4) or as Hopf monoids (Example 3). One remarkable feature of this equational theory is that it contains *two* symmetries: (i) that the mirror image ($\dagger$) of any equation is an equation, and (ii) is that the photographic negative ($\circ$, white $\leftrightarrow$ black) of any equation is an equation. Formally, define prop morphisms $(-)^{\dagger} \colon \mathbb{IH}_{\mathsf{R}}^{\mathrm{op}} \to \mathbb{IH}_{\mathsf{R}}$ mapping

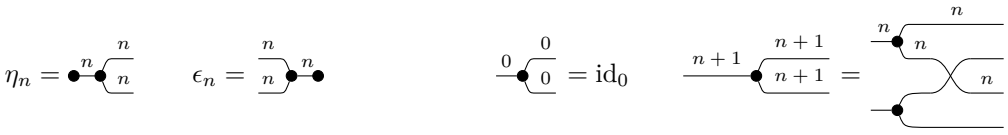


(15)

and $(-)^{\circ} \colon \mathbb{IH}_{\mathsf{R}} \to \mathbb{IH}_{\mathsf{R}}$ mapping

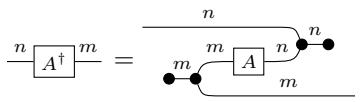


The morphism $(-)^{\dagger}$ is related to the self-dual compact closed structure [15] of $\mathbb{IH}_{\mathsf{R}}$ defined for each $n \in \mathbb{N}$ by assigning $\eta_n : 0 \to n + n$ (cap) and $\epsilon_n : n + n \to 0$ (cup) the string diagrams:

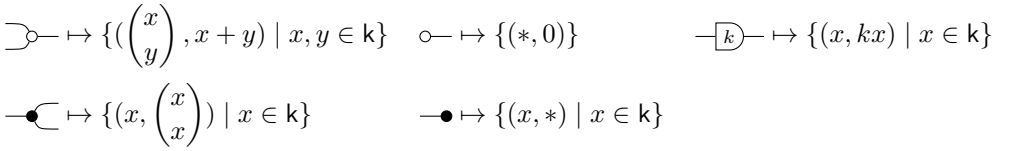


▶ **Remark 7.** *Above we used* *for the n-fold monoidal product of* , *for* id_n *and* *is inductively defined above, and similarly for*  *and*  . *The same convention will be used for the white structure. We shall omit the labels when there is no risk of ambiguity.*

The contravariant morphism induced by the compact closed structure coincides with $(-)^{\dagger}$, as defined in (15), that is for all $A : m \to n$:



Consider the prop $\mathbf{LinRel}_{\mathsf{k}}$ of linear relations over k, the field of fractions of R. An arrow $m \to n$ is a linear subspace of $\mathsf{k}^m \times \mathsf{k}^n$, ; is relational composition and $\oplus$ is direct sum. Also $\mathbf{LinRel}_{\mathsf{k}}$ has a self-dual compact closed structure: the induced contravariant morphism $(-)^{\dagger} \colon \mathbf{LinRel}_{\mathsf{k}}{}^{\mathrm{op}} \to \mathbf{LinRel}_{\mathsf{k}}$ maps a linear relation $R \subseteq \mathsf{k}^m \times \mathsf{k}^n$ to its opposite $R^{\dagger} \subseteq \mathsf{k}^n \times \mathsf{k}^m$. Let $*$ be the unique element of the 0 dimensional vector space k^0. Then



defines a unique prop morphism $\mathcal{S} : \mathbb{IH}_{\mathsf{R}} \to \mathbf{LinRel}_{\mathsf{k}}$ that preserves $(-)^{\dagger}$. For instance,

$$\mathcal{S}(\bullet\!-) = \mathcal{S}(-\!\bullet^{\dagger}) = (\mathcal{S}(-\!\bullet))^{\dagger} = \{\, (*, x) \,|\, x \in \mathsf{k} \,\}.$$

▶ **Theorem 8** ([7]). $\mathcal{S} : \mathbb{IH}_{\mathsf{R}} \to \mathbf{LinRel}_{\mathsf{k}}$ *is an isomorphism.*

Let us explain the relationship of $\mathcal{S}$ with $\mathcal{S}'$ from Example 6. Observe that any $A : m \to n$ in $\mathbb{IH}_{\mathsf{R}}$ built out of the leftmost five generators of Fig. 1 (drawn ) is also in $\mathbb{HA}_{\mathsf{R}}$ and, similarly, any term built of the five rightmost generators () is in $\mathbb{HA}_{\mathsf{R}}^{\mathrm{op}}$. Indeed, we have prop embeddings $\mathbb{HA}_{\mathsf{R}} \to \mathbb{IH}_{\mathsf{R}} \leftarrow \mathbb{HA}_{\mathsf{R}}^{\mathrm{op}}$. Similarly, there are embeddings $\mathbf{Mat}_{\mathsf{R}} \to \mathbf{LinRel}_{\mathsf{k}} \leftarrow {\mathbf{Mat}_{\mathsf{R}}}^{\mathrm{op}}$ mapping a matrix to its graph, and the following commutes [7]:

$$\begin{array}{ccccc} \mathbb{HA}_{\mathsf{R}} & \longrightarrow & \mathbb{IH}_{\mathsf{R}} & \longleftarrow & \mathbb{HA}_{\mathsf{R}}^{\mathrm{op}} \\ {\scriptstyle \mathcal{S}'}\downarrow & & \downarrow{\scriptstyle \mathcal{S}} & & \downarrow{\scriptstyle \mathcal{S}'^{\mathrm{op}}} \\ \mathbf{Mat}_{\mathsf{R}} & \longrightarrow & \mathbf{LinRel}_{\mathsf{k}} & \longleftarrow & {\mathbf{Mat}_{\mathsf{R}}}^{\mathrm{op}} \end{array}$$

The following result informs us that every arrow of $\mathbb{IH}_{\mathsf{R}}$ can be written in *span form*.

▶ **Lemma 9** ([7]). *For all*  *in* $\mathbb{IH}_{\mathsf{R}}$ *there exist* $k \in \mathbb{N}$,  *and*  *such that* = .

Moreover, the following property of $\mathbb{HA}_{\mathsf{R}}$ also holds in $\mathbb{IH}_{\mathsf{R}}$.

▶ **Lemma 10** ([7]). *For all* , = *and* = .

5 Symmetric Monoidal Inequality Theories and Ordered Props

We reviewed the construction of props from SMTs in the previous section. In order to capture *inequalities* of terms, however, we need a new notion. We thus introduce the concept of a *Symmetric Monoidal Inequality Theory* (SMIT), which allows the specification of a partial order on terms built out of generators, analogously to how SMTs specify equivalence relations.

▶ **Definition 11** (Symmetric Monoidal Inequality Theory). A SMIT is a pair (Σ, I). As for SMTs, Σ is a collection of generators $o : m \to n$, and I is a set of pairs (t_1, t_2) of Σ-terms with the same (co)arity, but we now think of them as representing *inequalities*. That is, where before the interpretation of a pair (t_1, t_2) was that $t_1 = t_2$, we now stipulate that $t_1 \leq t_2$.

Set I leads to a preorder on terms by reflexive and transitive closure. A partial order arises through anti-symmetry: t_1 and t_2 are equated when $t_1 \leq t_2$ and $t_2 \leq t_1$. We will use $\leq_I$, or $\leq$ when I is clear from context, and write the corresponding equivalence as equality. The equivalence classes are the arrows of a 2-category $\mathbb{T}_{(\Sigma,I)}$ that we call an *ordered prop*.

▶ **Definition 12** (Ordered Prop). A *2-prop* is a strict symmetric monoidal 2-category whose objects are natural numbers and monoidal product on objects is addition. An *ordered prop* is a 2-prop which is locally posetal, that is, where every hom-category is a poset – i.e. there is at most one 2-cell ($\leq$) between any two arrows. Together with ordered prop morphisms (identity-on-objects strict monoidal 2-functors) we have a category **OrdPROP**.

Since ordered props are a kind of 2-category, in any ordered prop, for all f, f', g, g', we have:

$$\text{if } f \leq f' \text{ and } g \leq g' \text{then } f \,;\, g \leq f' \,;\, g' \tag{16}$$

$$\text{if } f \leq f' \text{ and } g \leq g' \text{then } f \oplus g \leq f' \oplus g'. \tag{17}$$

▶ **Example 13.** The prop $\mathbf{LinRel}_{\mathsf{k}}$ of linear relations has a partial order on arrows given by inclusion as subspaces. It is straightforward to check (16) and (17). The prop $\mathbf{Mat}_{\mathsf{R}}$ of matrices (Example 6) can be also regarded as an ordered prop with discrete order.

Returning to SMITs, the arrows of $\mathbb{T}_{(\Sigma,I)}$ are the equivalence classes of Σ-terms; since arities and coarities are respected, we may use the partial order $\leq_I$ to define the 2-cells in the hom-category $\mathbb{T}_{(\Sigma,I)}(m,n)$. It follows that $\mathbb{T}_{(\Sigma,I)}$ is an ordered prop. Note that SMITs are a generalisation of SMTs. First, any prop can be made into a (discrete) ordered prop by adding identity 2-cells. This gives an embedding (a faithful homomorphism of ordered pros: a strict identity-on-objects symmetric monoidal 2-functor) $\mathbf{PROP} \hookrightarrow \mathbf{OrdPROP}$.

▶ **Remark 14.** *Any SMT (Σ, E) can considered as a SMIT by taking the symmetric closure of E: i.e. $I = E \cup E^{op}$. Then the image of the prop generated by the SMT (Σ, E) under the embedding is isomorphic to the ordered prop given by SMIT (Σ, I).*

6 Presenting the 2-dimensional structure of $\mathbf{LinRel}_\mathsf{k}$

In this section we prove our main theorem. We extend the SMT of Interacting Hopf Algebras to a SMIT and the isomorphism of Theorem 8 to a 2-isomorphism of ordered props. In other words, we characterise the subset order of linear relations (Example 13).

The symmetry discussed in Section 4 is broken in the ordered setting. Indeed, to get from the SMT to a SMIT we follow the procedure of Remark 14 and add *just one inequality* (2): —○ ≤ —●. Interpreted as linear relations (via $\mathcal{S} : \mathbb{IH}_\mathsf{R} \to \mathbf{LinRel}_\mathsf{k}$), (2) says that the unique 0-dimensional subspace $\{0\}$ of k considered as a k-vector space is included in the unique 1-dimensional subspace, i.e. k itself. This is, of course, a strict inclusion.

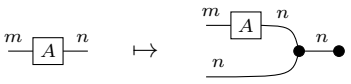
▶ **Theorem 15.** $\mathbb{IH}_\mathsf{R} \cong \mathbf{LinRel}_\mathsf{k}$ *as ordered props.*

For the proof we need to recall some elementary linear algebra. Regarding an $m \times n$ matrix A as a list of its column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, the *span* of A ($\mathrm{Sp}(A)$) is the linear subspace of k^m with elements linear combinations $\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_n\mathbf{a}_n$, $\lambda_i \in \mathsf{k}$. The following is a well-known fact of linear algebra (see, e.g. [2, Proposition 2.13]).

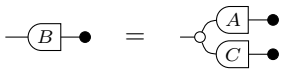
▶ **Lemma 16.** *Suppose that for some $m \times n$ matrix A we have $\mathrm{Sp}(A) \subseteq V$. Then there exists $m \times n'$ matrix C such that $V = \mathrm{Sp}(A, C)$.*

Proof of Theorem 15. Inequation (2) is clearly sound, we thus only have to show completeness; that is (2) suffices to account for any inclusion between *arbitrary* linear relations.

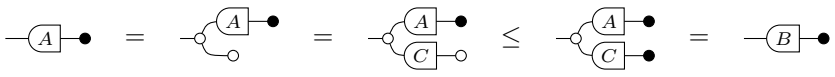
Let therefore $A, B : m \to n$ be arrows of $\mathbb{IH}_\mathsf{R}$ such that $\mathcal{S}(A) \subseteq \mathcal{S}(B) \subseteq \mathsf{k}^m \times \mathsf{k}^n$. Now $\mathsf{k}^m \times \mathsf{k}^n \cong \mathsf{k}^{m+n} \times \mathsf{k}^0 \cong \mathsf{k}^{m+n}$; this, diagrammatically, means the following manipulation:



Using this "rewiring" argument we may assume w.l.o.g. that $A, B : m \to 0$. Further, using Lemma 9, we may assume that A, B consist only of the rightmost five generators in Figure 1: indeed, —□— = —□—□— = —□— by Lemmas 9 and 10. It is thus harmless to consider A and B as matrices, and our initial assumption means that $\mathrm{Sp}(A) \subseteq \mathrm{Sp}(B)$. By the conclusion of Lemma 16, there exists C such that $\mathrm{Sp}(B) = \mathrm{Sp}(A, C)$. Diagrammatically (via $\mathcal{S}$), this gives the following, where for readability we omit decorating the wires:



But we have

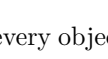


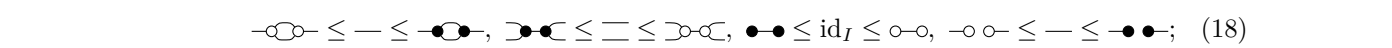
showing that $A \leq B$ is derivable from (2). ◀

While we have shown that (2) suffices to characterise inclusions between subspaces, it is convenient to identify some structural properties that our inequational theory satisfies. By doing so, we are building up a toolbox—useful for reasoning in applications—of principles for reasoning about the structure of the order between linear relations.

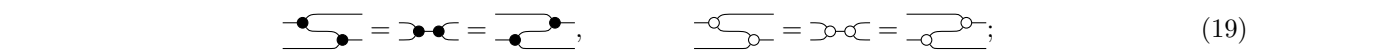
Below we use the notion of *adjunction* in an ordered prop: arrow $f : m \to n$ has a *right adjoint* if there exists $g : n \to m$ such that $\mathrm{id}_m \leq f\,;\, g$ and $g\,;\, f \leq \mathrm{id}_n$, in which case we write $f \dashv g$. Right adjoints, if they exist, are unique: if also $f \dashv g'$ then $g = g'$.

▶ **Definition 17.** An abelian bicategory [11] **A** is a (loc. posetal) monoidal bicategory where:

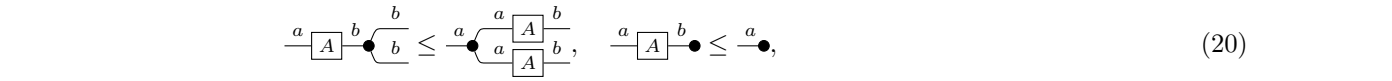
(i) every object a is a commutative comonoid () with right adjoints  , , and a commutative monoid ( ) with right adjoints , . This translates to the following (labelling on the wires omitted for clarity):

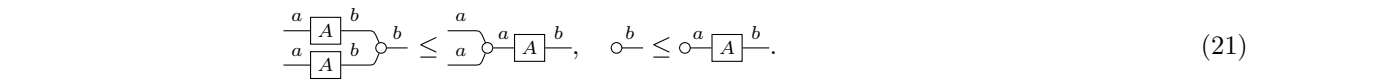

(18)

(ii) ( ) and ( ) with their right adjoints satisfy the Frobenius equations:


(19)

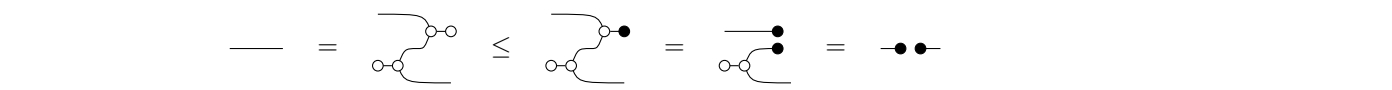
(iii) every arrow  is a lax (,)-comonoid homomorphism and a lax (,)-monoid homomorphism:


(20)

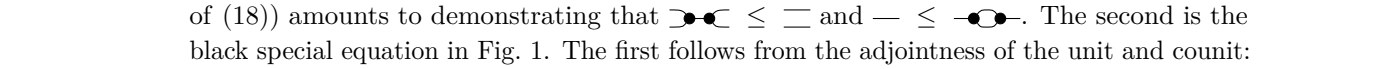

(21)

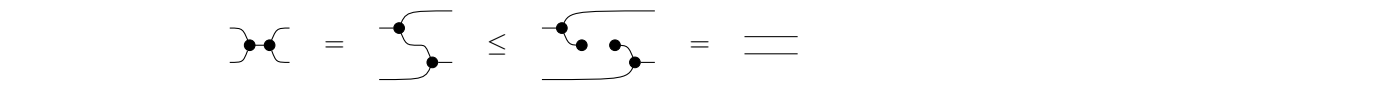
A more concise definition is: **A** and $\mathbf{A}^{\mathrm{op}}$ are both bicategories of relations in the sense of [11].

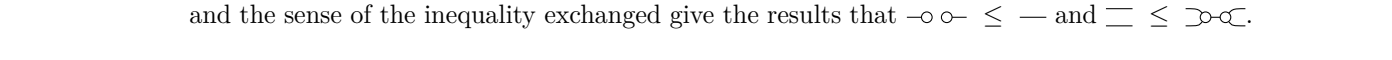
Below, we show that, as an ordered prop, $\mathbb{IH}_{\mathsf{R}}$ is an abelian bicategory. For each object $n \in \mathbb{N}$, comonoids and monoid structures are defined inductively as in Remark 7. A straightforward induction generalises the Frobenius equations for all n in (19), given that they are present for $n = 1$ in Fig. 1. Next we tackle the case of the black units (the rightmost two black inequations of (18)). The unit of the adjunction is a 2-cell witnessing $\mathrm{id}_0 \leq$  and these terms are equated in Fig. 1. It remains to show existence of the counit. For $n = 1$:



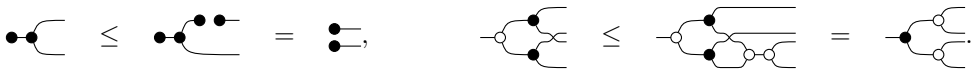
The above argument easily generalises to all n.

Showing adjointness for the black comultiplication (the leftmost two black inequations of (18)) amounts to demonstrating that  and . The second is the black special equation in Fig. 1. The first follows from the adjointness of the unit and counit:



For the white case in (18), the inequations are opposite. The same proofs with colours and the sense of the inequality exchanged give the results that  and .

Now, to show that all arrows are lax comonoid homomorphisms, it is enough to check that each of the generators obeys the conditions of (20). Several of these are in fact equalities. The derivations for the two interesting cases are given below:



Again, the white case (21) is symmetric.

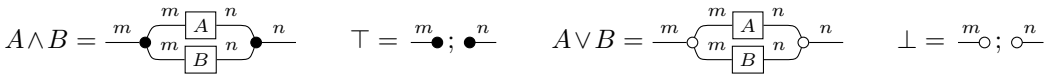
7 The 2-dimensional algebra of Linear Relations

The structure identified in the previous sections enables us to highlight some interesting properties of the ordering $\leq$. We start with a few elementary, but useful observations.

▶ **Lemma 18.** *For all $A, B \in \mathbb{IH}_{\mathsf{R}}(m,n)$, the following hold:*
(a) *If $A \leq B$, then $A^\dagger \leq B^\dagger$;*
(b) $(A^\dagger)^\dagger = A = (A^\circ)^\circ$*;*
(c) *If $A \leq B$, then $A^\circ \geq B^\circ$;*
(d) $(A^\dagger)^\circ = (A^\circ)^\dagger$.

Proof. For all equations $A = B$ in $\mathbb{IH}_{\mathsf{R}}$ (Fig. 1), one has $A^\dagger = B^\dagger$ and $A^\circ = B^\circ$. For the only inequation of $\mathbb{IH}_{\mathsf{R}}$, $\multimap \leq \bullet\!\!-$, we clearly have $\multimap^\dagger \leq \bullet\!\!-^\dagger$ and $\multimap^\circ \geq \bullet\!\!-^\circ$: this implies (a) and (c). The proofs of (b) and (d) are inductions on the definitions of $(-)^\dagger$ and $(-)^\circ$. ◀

We proceed by showing that every homset $\mathbb{IH}_{\mathsf{R}}(m,n)$ carries a lattice structure. Given two arrows $A, B \colon m \to n$ we define $A \wedge B$, $\top$, $A \vee B$ and $\bot$ as follows.



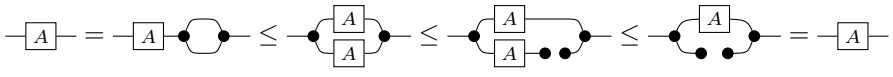
▶ **Lemma 19.** *For all $A, B \in \mathbb{IH}_{\mathsf{R}}(m,n)$, the following hold:*
(a) $(A \vee B)^\dagger = A^\dagger \vee B^\dagger$ *and* $\bot^\dagger = \bot$*;*
(b) $(A \wedge B)^\dagger = A^\dagger \wedge B^\dagger$ *and* $\top^\dagger = \top$*;*
(c) $(A \vee B)^\circ = A^\circ \wedge B^\circ$ *and* $\bot^\circ = \top$*;*
(d) $(A \wedge B)^\circ = A^\circ \vee B^\circ$ *and* $\top^\circ = \bot$.

Proof. Trivial by unfolding the definitions. ◀

▶ **Theorem 20.** *The operations $(\vee, \bot, \wedge, \top)$ define a lattice structure on every homset $\mathbb{IH}_{\mathsf{R}}(m,n)$ wrt the ordering $\leq$.*

Proof. First observe that for all $A \colon m \to n$, we have, by the rightmost inequations in (18) and (20), that $A \leq A\,;\top \leq \top$. By Lemmas 18(b) and 19(d), one deduces $A \geq \bot$.

Now, it is enough to check that both $(\mathbb{IH}_{\mathsf{R}}(m,n), \wedge, \top)$ and $(\mathbb{IH}_{\mathsf{R}}(m,n), \vee, \bot)$ are commutative and idempotent monoids. Observe that if this holds for $(\mathbb{IH}_{\mathsf{R}}(m,n), \wedge, \top)$ then, by Lemmas 18(b) and 19(d), it holds also for $(\mathbb{IH}_{\mathsf{R}}(m,n), \vee, \bot)$. The fact that $(\mathbb{IH}_{\mathsf{R}}(m,n), \wedge, \top)$ is a commutative monoid follows immediately from the fact that ( ,) and ( , ) are commutative (co)monoids. For idempotency we observe the following chain of inequalities.



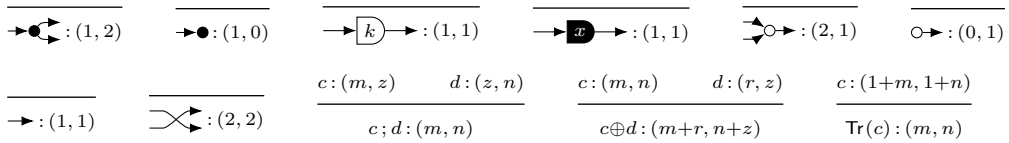


Figure 2 Sort inference rules.

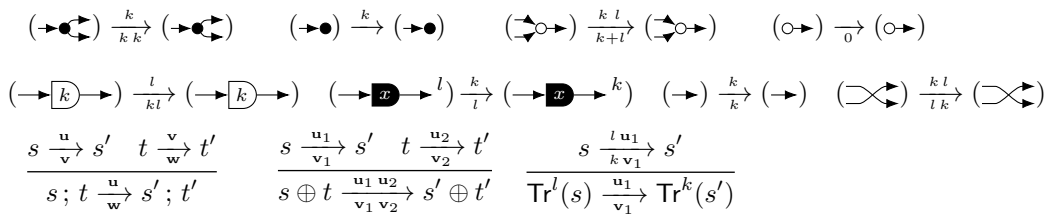
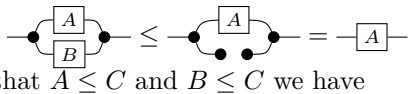
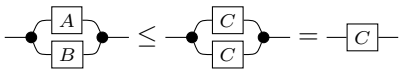


Figure 3 Structural rules for operational semantics, with k, l ranging over k and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ vectors of elements of k of the appropriate size.

Now to see that $\wedge$ defines a meet, note that $A \wedge B \leq A$:  and, by a symmetric argument, $A \wedge B \leq B$. Assuming that $A \leq C$ and $B \leq C$ we have



The argument showing that $\vee$ defines a join is again symmetric. ◀

▶ **Lemma 21.** *For all $A, B, C \in \mathbb{IH}_{\mathsf{R}}(m, n)$, the following hold:*

(a) $C\,;\,(A \wedge B) \leq (C\,;\,A) \wedge (C\,;\,B)$ *and* $C\,;\,\top \leq \top$*;*
(b) $(A \wedge B)\,;\,C \leq (A\,;\,C) \wedge (B\,;\,C)$ *and* $\top\,;\,C \leq \top$*;*
(c) $(A \vee B)\,;\,C \geq (A\,;\,C) \vee (B\,;\,C)$ *and* $\bot\,;\,C \leq \bot$*;*
(d) $C\,;\,(A \vee B) \geq (C\,;\,A) \vee (C\,;\,B)$ *and* $C\,;\,\bot \leq \bot$*.*

Proof. Part (a) follows from (20). With (a), Lemmas 18(a) and 19(b) imply (b). With (b), Lemmas 18(b) and 19(d) imply (c). With (c), Lemmas 18(a) and 19(a) imply (d). ◀

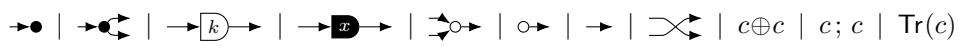
To summarise, we have a well-behaved set of operations

$$\top,\quad \bot,\quad \wedge,\quad \vee,\quad (-)^{\dagger},\quad (-)^{\circ},\quad ;\,,\quad \mathrm{id}$$

which, because of its similarity to the algebra of relations [14], we call *the algebra of linear relations*. Observe that the operations $\top$, $\wedge$, $(-)^{\dagger}$, $;$ and *id* have exactly the same meaning: full relation, intersection, inverse, composition and identity relation. Instead $\bot$, $\vee$, $(-)^{\circ}$ which in the algebra of relations denote, respectively, empty relation, union and complement, do not coincide. The reason is that, in general, these operations cannot be defined on linear relations: e.g., the union of two linear relations may not be linear.

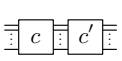
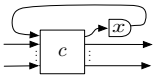
8 Back to Signal Flow Graphs

We recall from [6] the *Directed Signal Flow Calculus*. The syntax is given by the grammar below where k ranges over a fixed field k.

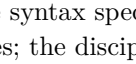
$c ::=$  $\mid c \oplus c \mid c\,;c \mid \mathsf{Tr}(c)$

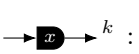
A *sort* is a pair (m, n), with $m, n \in \mathbb{N}$. We shall consider only terms that are sortable, according to the rules of Fig. 2. An inductive argument confirms uniqueness of sorts: if $c : (m, n)$ and $c : (m', n')$ then $m = m'$ and $n = n'$. We will refer to sortable terms as *circuits* since, intuitively, a term $c : (m, n)$ represents a circuit with m inputs on the left and n outputs on the right.

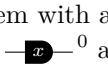
In the syntax specification we used a graphical rendering of the components: we will seldom write terms in the traditional way and instead represent them as diagrams:

$c\,;c'$ is drawn  $c \oplus c'$ is drawn  $\mathsf{Tr}(c)$ is drawn  .

The graphical notation identifies some syntactically different terms, e.g. $(c_1 \oplus c_2)\,;(d_1 \oplus d_2)$ and $(c_1\,;d_1) \oplus (c_2\,;d_2)$. This is harmless (see Remark 1 in [6]) since such circuits are observationally equivalent wrt the operational semantics that we introduce next.

The operational semantics interprets terms as stream transducers. The wires carry elements of a field k that enter and exit through input and output ports. Formally, it is a transition system that has *augmented* circuits as states: each *delay* component () and each *guarded feedback* (Tr) are assigned some value $k \in \mathsf{k}$. States are obtained by replacing delays and feedbacks in the syntax specification with  and Tr^k for each $k \in \mathsf{k}$. We only consider sortable states; the discipline is obtained by adding the following to Fig. 2.

 : $(1, 1)$ and $\dfrac{c : (1+m, 1+n)}{\mathsf{Tr}^k(c) : (m, n)}$

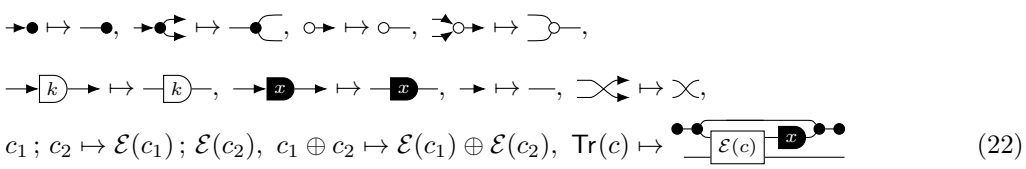
The structural rules for operational semantics are given in Fig. 3 where we use strings of length n to represents vectors in k^n. If state $s : (m, n)$ is the source of a transition $\xrightarrow[\mathbf{w}]{\mathbf{v}} t$ then t is also a state with sort (m, n) and $\mathbf{v}$ and $\mathbf{w}$ are strings (vectors) in k^m and k^n, respectively. Intuitively, $s \xrightarrow[\mathbf{w}]{\mathbf{v}} t$ means that s can become t in one step whenever it inputs $\mathbf{v}$ on the m ports on the left and outputs $\mathbf{w}$ on the n ports on the right. Each circuit c then yields a transition system with a chosen *initial state* s_0 of c, obtained by replacing delays and feedbacks in c with  and Tr^0: this means that we only consider executions where the registers are initialised with 0. A different semantics is considered in [13] where registers can be initialised with arbitrary values.

A *computation* of a circuit c, is an infinite path $s_0 \xrightarrow[\mathbf{v}_0]{\mathbf{u}_0} s_1 \xrightarrow[\mathbf{v}_1]{\mathbf{u}_1} \dots$ in the transition system of c, starting from its initial state s_0. When c has sort (m, n), each $\mathbf{u}_i$ and $\mathbf{v}_i$ are strings over k, say $k_{i1} \dots k_{im}$ and $l_{i1} \dots l_{in}$, respectively. The *trace*, also called *trajectory*, of this computation is then a pair of vectors $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$ where $\alpha_j = k_{0j}k_{1j}\dots$ and $\beta_j = l_{0j}l_{1j}\dots$. For example, consider the (1,1) circuit in (3): the first three steps of an infinite computation are illustrated in (4). The trace for this computation is thus $(1;0;0;0;\dots\ ,\ 1;2;3;5;\dots)$.

We write $it(c)$ for the set of traces, and this is our notion of *observable behaviour*. Two circuits c and d are *observationally equivalent*, written $c \sim d$, iff $it(c) = it(d)$.

A few considerations are in order about the role of the *denotational semantics* given in [6]. The signal flow calculus is canonical in the sense that it enjoys a Kleene-like theorem [8, Thorem 7.4]: it denotes all and only the *rational functions* on streams (see [18] and [17]). Moreover the denotational semantics is fully abstract with respect to the observational equivalence (Corollary 2 and Proposition 4 in [6]) and, from this correspondence, a sound and complete axiomatization for $\sim$ follows. We focus on some technical details of this axiomatisation below.

The idea is to translate circuits of sort (m, n) into arrows $m \to n$ of $\mathbb{IH}_{\mathsf{k}[x]}$, where $\mathsf{k}[x]$ is the principal ideal domain of polynomials with indeterminate x and coefficients from k. Intuitively, the inductively defined translation $\mathcal{E}$ "erases directions" from the wires:



(22)

where $-\boxed{k}-$ and $-\blacksquare x-$, in $\mathbb{IH}_{\mathsf{k}[x]}$, correspond to polynomials k and x in $\mathsf{k}[x]$.

For an example consider the circuits $\to\boxed{\text{fFib}}\to$ and $\to\boxed{\text{rFib}}\to$ in (3) and (5): the corresponding string diagrams $\mathcal{E}(\to\boxed{\text{fFib}}\to)$ and $\mathcal{E}(\to\boxed{\text{fGui}}\to)$ are shown in (6).

The following ensures that the theory of Fig. 1 is sound and complete for trace equivalence.

▶ **Theorem 22** ([6]). *$c \sim d$ iff $\mathcal{E}(c) = \mathcal{E}(d)$, for all circuits c, d.*

To prove equivalence of signal flow calculus terms it is thus enough to view them as string diagrams in $\mathbb{IH}_{\mathsf{k}[x]}$ by forgetting flow direction, and use the equational theory of Fig. 1.

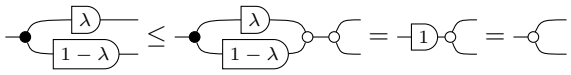
The reader may wonder why we introduced the directed signal flow calculus rather then using string diagrams directly. The reason is that string diagrams of $\mathbb{IH}_{\mathsf{k}[x]}$ are undirected and flow directionality is essential to execute them (see Remark 2 in [6]). String diagrams, however, *do* provide a useful language to reason about signal flow graphs. For instance, using the algebra of linear relations from Section 7, the *opposite* of an arbitrary circuit c can be specified by the string diagram $\mathcal{E}(c)^\dagger$.

It is therefore natural to think of string diagrams in $\mathbb{IH}_{\mathsf{k}[x]}$ as *specifications* and of circuits in the directed signal flow calculus as *implementations*. More formally, we say that a specification A (an arrow of $\mathbb{IH}_{\mathsf{k}[x]}$) *refines* a specification B whenever $A \leq B$ and we say that a circuit c *implements* a specification A whenever $\mathcal{E}(c)$ refines A, i.e., $\mathcal{E}(c) \leq A$.

▶ **Remark 23.** *One could have defined $c \precsim d$ iff $it(c) \subseteq it(d)$ but this notion would collapse to $\sim$, since the observational behaviour of any circuit, which we have defined as a relation, is actually the graph of a function. To see this, note that the operational semantics of Fig. 3 is deterministic: given any state s and transitions $s \xrightarrow[\mathbf{v}]{\mathbf{u}}$, $s \xrightarrow[\mathbf{v}']{\mathbf{u}}$, it follows that $\mathbf{v} = \mathbf{v}'$. A similar, but non-deterministic, semantics subsuming that of Fig. 3 was given in [6, Fig. 2] for arbitrary string diagrams. In fact, losing direction of signal flow makes the definition simpler, since the feedback becomes expressible in terms of the more basic components and does not thus need a separate structural rule. It is the possibly non-deterministic nature of string-diagrams-as-SFG-specifications that makes the refinement relation interesting.*

We now return to the motivating example of Section 2. The fact that the circuit in (5) solves the sustainable rabbit farming problem is witnessed by the fact that it is an implementation of $\mathcal{E}(\to\boxed{\text{fFib}}\to)^\dagger$. Here, since the behaviour of $\to\boxed{\text{fFib}}\to$ is invertible,

there is an equivalence: see the derivation in (7). Instead, the sustainable farming problem for rabbits and guinea pigs cannot be solved by equational reasoning since the combined SFG (8(ii)) (henceforth $\rightarrow\boxed{\text{comb}}\rightarrow$) is not invertible. To prove that SFG (8(iv)) (henceforth $\rightarrow\boxed{\text{sol}}\rightarrow$) is a solution, we should show that it implements $\mathcal{E}(\rightarrow\boxed{\text{comb}}\rightarrow)^\dagger$, namely we should check that $\mathcal{E}(\rightarrow\boxed{\text{comb}}\rightarrow) \leq \mathcal{E}(\rightarrow\boxed{\text{comb}}\rightarrow)^\dagger$. It follows from the general fact shown below; taking $\lambda = \frac{1}{2}$ gives the claimed solution.



9 Related work

Although we concentrated on the discrete semantics, signal flow graphs also have a continuous incarnation where delays act as integrators; for this reason they are a useful foundational model in signal processing and control theory: as a consequence, for computer scientists [1] they are also important as models of cyber-physical systems that can be analysed and verified in concert with discrete models. For example, in *loc. cit.* the authors study SFGs with the aid of block diagrams that are closely related to the Signal Flow Calculus of Section 8.

The operation (22) of passing from the directed calculus to string diagrams by "erasing arrowheads" is similar in spirit to the ideas of Willems [23], who argued that concepts of input and output are inherently non-compositional, complicate the mathematics, and—perhaps most importantly—do not actually exist in the underlying physical reality. It is this realisation that gives rise to the equational theory of Interacting Hopf Algebras. Moreover, Baez and Erbele [3] prove that the same equational theory is suitable for the continuous behaviour. Remarkably similar symmetric monoidal theories appear in concurrency [9, 10, 21] and quantum computing [12]. None of these works, however, investigates the underlying posetal structure. We believe that the structure of cartesian and abelian bicategories [11] may be successfully exploited in those fields.

References

1 Rob Arthan, Ursula Martin, and Paulo Oliva. A Hoare logic for linear systems. *Form Asp Comp*, 25:345–363, 2013.

2 Sheldon Axler. *Linear Algebra Done Right*. Springer, 2nd edition, 1997.

3 John C. Baez and Jason Erbele. Categories in control. Technical report, arXiv:1405.6881, 2014.

4 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. A categorical semantics of signal flow graphs. In *Concurrency Theory - 25th International Conference, (CONCUR 2014)*, volume 8704 of *LNCS*, pages 435–450. Springer, 2014. doi:10.1007/978-3-662-44584-6_30.

5 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. Interacting bialgebras are Frobenius. In *Foundations of Software Science and Computation Structures - 17th International Conference, (FOSSACS 2014)*, number 8412 in LNCS. Springer, 2014.

6 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. Full Abstraction for Signal Flow Graphs. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages - POPL '15*, pages 515–526, New York, New York, USA, 2015. ACM Press. doi:10.1145/2676726.2676993.

7 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. Interacting Hopf algebras. *Journal of Pure and Applied Algebra*, 221(1):144–184, mar 2017. doi:10.1016/j.jpaa.2016.06.002.

8 Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. The Calculus of Signal Flow Diagrams I: Linear relations on streams. *Information and Computation*, 252(v):2–29, feb 2017. doi:10.1016/j.ic.2016.03.002.

9 Roberto Bruni, Ivan Lanese, and Ugo Montanari. A basic algebra of stateless connectors. *Theor. Comput. Sci.*, 366:98–120, 2006.

10 Roberto Bruni, Hernán C. Melgratti, Ugo Montanari, and Paweł Sobociński. Connector algebras for C/E and P/T nets' interactions. *Log. Meth. Comput. Sci.*, 9(3), 2013. doi:10.2168/LMCS-9(3:16)2013.

11 Aurelio Carboni and Robert Frank Carslaw Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49(1–2):11–32, nov 1987. doi:10.1016/0022-4049(87)90121-6.

12 Bob Coecke and Ross Duncan. Interacting quantum observables. In *ICALP'08*, pages 298–310, 2008.

13 Brendan Fong. The Algebra of Open and Interconnected Systems. *arXiv.org*, page 230, 2016. URL: http://arxiv.org/abs/1609.05382.

14 Bjarni Jónsson and Alfred Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, 54(1):80–80, 1948.

15 Gregory M Kelly and Miguel L Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

16 Yves Lafont. Towards an algebraic theory of boolean circuits. *J Pure Appl Alg*, 184:257–310, 2003.

17 Bhagwandas Pannalal Lathi. *Signal processing and linear systems*. Oxford university press New York, 1998.

18 Jan J M M Rutten. A tutorial on coinductive stream calculus and signal flow graphs. *Theoretical Computer Science*, 343(3):443–481, oct 2005. doi:10.1016/j.tcs.2005.06.019.

19 Claude E. Shannon. The theory and design of linear differential equation machines. Technical report, National Defence Research Council, 1942.

20 Laurence Sigler. *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. Springer, 2002.

21 Paweł Sobociński. Nets, relations and linking diagrams. In *Algebra and Coalgebra in Computer Science - 5th International Conference, (CALCO 2013)*, volume 8089 of *LNCS*, pages 282–298. Springer, 2013.

22 Paweł Sobociński. Graphical linear algebra. (blog series), 2015. URL: https://GraphicalLinearAlgebra.net.

23 Jan C. Willems. The behavioural approach to open and interconnected systems. *IEEE Contr. Syst. Mag.*, 27:46–99, 2007.

Brzozowski Goes Concurrent – A Kleene Theorem for Pomset Languages*

Tobias Kappé[1], Paul Brunet[2], Bas Luttik[3], Alexandra Silva[4], and Fabio Zanasi[5]

1 **University College London, London, United Kingdom**
`tkappe@cs.ucl.ac.uk`
2 **University College London, London, United Kingdom**
3 **Eindhoven University of Technology, Eindhoven, The Netherlands**
4 **University College London, London, United Kingdom**
5 **University College London, London, United Kingdom**

Abstract

Concurrent Kleene Algebra (CKA) is a mathematical formalism to study programs that exhibit concurrent behaviour. As with previous extensions of Kleene Algebra, characterizing the free model is crucial in order to develop the foundations of the theory and potential applications. For CKA, this has been an open question for a few years and this paper makes an important step towards an answer. We present a new automaton model and a Kleene-like theorem that relates a relaxed version of CKA to series-parallel pomset languages, which are a natural candidate for the free model. There are two substantial differences with previous work: from expressions to automata, we use Brzozowski derivatives, which enable a direct construction of the automaton; from automata to expressions, we provide a syntactic characterization of the automata that denote valid CKA behaviours.

1998 ACM Subject Classification D.1.3 Parallel Programming, F.1.1 Models of Computation, F.1.2 Modes of Computation, F.4.3 Formal Languages

Keywords and phrases Kleene theorem, Series-rational expressions, Automata, Brzozowski derivatives, Concurrency, Pomsets

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.25

1 Introduction

In their CONCUR'09 paper [5], Hoare, Möller, Struth, and Wehrman introduced Concurrent Kleene Algebra (CKA) as a suitable mathematical framework to study concurrent programs, in the hope of achieving the same elegance that Kozen did when using Kleene Algebra (and extensions) to provide a verification platform for sequential programs.

CKA is a seemingly simple extension of Kleene Algebra (KA): it adds a parallel operator that allows to specify concurrent behaviours compositionally. However, extending the existing KA toolkit – importantly, completeness and decidability results – turns out to be challenging. A fundamental missing ingredient is a characterization of the free model for CKA. This is in striking contrast with KA, where these topics are well understood. Several authors [6, 8] have conjectured the free model to be series-parallel pomset languages – a generalization of regular languages to sets of partially ordered words.

* This work was partially supported by the ERC Starting Grant ProFoundNet (grant code 679127).

In KA, Kleene's theorem provided a pillar for developing the toolkit and axiomatization [13], and, by extension, characterizing the free model. In this light, we pursue a Kleene Theorem for CKA. Specifically, we study series-rational expressions, with a denotational model in terms of pomset languages. Our main contribution is a Kleene Theorem for series-rational expressions, based on constructions faithfully translating between the denotational model and a newly defined operational model, which we call *pomset automata*. In a nutshell, these are finite-state automata in which computations from a certain state s may branch into parallel threads that contribute to the language of s whenever they both reach a final state.

We are not the first to attempt such a Kleene theorem. However, earlier works [16, 8] fall short of giving a precise correspondence between the denotational and operational models, due to the lack of a suitable automata restriction ensuring that only valid behaviours are accepted. We overcome this situation by introducing a generalization of Brzozowski derivatives [3] in the translation from expressions to automata. This guides us to a *syntactic* restriction on automata (rather than the *semantic* condition put forward in previous works), which guarantees the existence of a reverse construction, from automata to expressions. Moreover, following the Brzozowski route allows us to bypass a Thompson-like construction [19], avoiding the introduction of ϵ-transitions and non-determinism present in the aforementioned works.

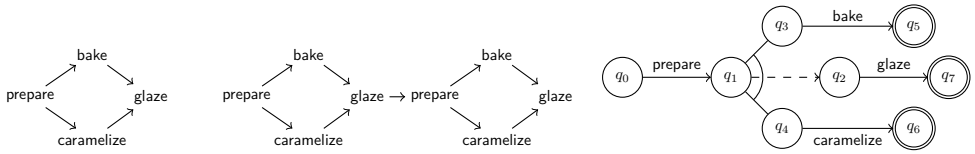
Since series-parallel expressions do not include the parallel analogue of the Kleene star (the "parallel star"), and our denotational model is not sound for the exchange law (which governs the interaction between sequential and parallel composition), our contribution is most accurately described as an operational model for *weak Bi-Kleene Algebra*. We leave it to future work to extend our construction to work with a denotational model that is sound for the exchange law (thus moving to *weak Concurrent Kleene Algebra*), as well as add the parallel star operator (arriving at Concurrent Kleene Algebra proper).

The remainder of this paper is organized as follows. In Section 2, we introduce the necessary notation. In Section 3, we introduce our automaton model as well as some notable subclasses of automata. In Section 4, we discuss how to translate a series-rational expression to a semantically equivalent pomset automaton, while in Section 5 we show how to translate a suitably restricted class of pomset automata to series-rational expressions. We contrast results with earlier work in Section 6. Directions for further work in are listed in Section 7.

To save space, proofs of lemmas are omitted from this paper. For a discussion that includes a proof for every lemma, we refer to the extended version of this paper [9].

2 Preliminaries

Let S be a set; we write 2^S for the set of all subsets of S, and $\binom{S}{2}$ for the set of *multisets* over S of size two. An element of $\binom{S}{2}$ containing $s_1, s_2 \in S$ is written $\{\!\{s_1, s_2\}\!\}$; note that $\{\!\{s_1, s_2\}\!\} = \{\!\{s_2, s_1\}\!\}$, and that s_1 may be the same as s_2. We use the symbols ϕ and ψ to denote multisets. If S and I are sets, and for every $i \in I$ there exists an $s_i \in S$, we call $(s_i)_{i \in I}$ an *I-indexed family over S*. We say that a relation $\prec \subseteq S \times S$ is a *strict order* on S if it is irreflexive and transitive. We refer to $\prec$ as *well-founded* if there are no infinite descending $\prec$-chains, i.e., no family $(s_n)_{n \in \mathbb{N}}$ over S such that $\forall n \in \mathbb{N}, s_{n+1} \prec s_n$. Throughout the paper we fix a finite set Σ called the *alphabet*, whose elements are symbols usually denoted with a and b. Lastly, if $\rightarrow \subseteq X \times Y \times Z$ is a ternary relation, we write $x \xrightarrow{y} z$ instead of $\langle x, y, z\rangle \in \rightarrow$.



(a) Diagram for C. **(b)** Diagram for $C \cdot C$. **(c)** PA accepting C.

Figure 1 Hasse diagrams for pomsets and a pomset automaton accepting one.

2.1 Pomsets

Partially-ordered multisets, or *pomsets* [4] for short, generalise words to a setting where events (elements from Σ) may take place not just sequentially, but also in parallel.

▶ **Definition 2.1.** A *labelled poset* is a tuple $\langle U, \leq_U, \lambda_U \rangle$ consisting of a *carrier* set U, a partial order $\leq_U$ on U and a *labelling* function $\lambda_U : U \to \Sigma$. A *labelled poset isomorphism* is a bijection between poset carriers that bijectively preserves the labels and the ordering. A *pomset* is an isomorphism class of labelled posets; equivalently, it is a labelled poset up-to bijective renaming of elements in U. We write 1 for the empty pomset, Pom_Σ for the set of all pomsets and Pom_Σ^+ for the set of all the non-empty pomsets.

For instance, suppose a recipe for caramel-glazed cookies tells us to *prepare* cookie dough, *bake* cookies in the oven, *caramelize* sugar, and *glaze* the finished cookies. Here, *prepare* precedes *bake* and *caramelize*, while *glaze* succeeds both. A pomset representing this process could be $\langle C, \leq_C, \lambda_C \rangle$, where $C = \{a, b, c, \}$ and $\leq_C$ is such that $a \leq_C b \leq_C d$ and $a \leq_C c \leq_C d$, and $\lambda_C(a) = \mathit{prepare}$, $\lambda_C(b) = \mathit{bake}$, $\lambda_C(c) = \mathit{caramelize}$, $\lambda_C(d) = \mathit{glaze}$.

Note that words are just finite pomsets with a total order. We will sometimes use $a \in \Sigma$ to refer to the pomset with a single point labelled a (and the obvious order); such a pomset is called *primitive*. A pomset can be represented as a Hasse diagram, where nodes have labels in Σ. For instance, the Hasse diagram for the pomset C above is drawn in Figure 1a.

To simplify notation, we refer to a pomset by the carrier U of a labelled poset $\langle U, \leq_U, \lambda_U \rangle$ in its isomorphism class. We use the symbols U, V, W and X to denote pomsets. Pomsets being isomorphism classes, the content of the carrier of the chosen representative is of very little importance; it is the order and labelling that matters. For this reason, we tacitly assume that whenever we have two pomsets, we pick representatives that have disjoint carrier sets.

▶ **Definition 2.2.** The *width* of a pomset U, denoted $\|U\|$, is the size of the largest antichain in U with respect to $\leq_U$, i.e., the maximum $n \in \mathbb{N}$ such that there exist $u_1, u_2, \ldots, u_n \in U$ that are not related by $\leq_U$.

The pomsets we work with in this paper have a finite carrier. As a result, $\|U\|$ is always defined. For instance, the width of the pomset C above is 2, because the nodes (ii) and (iii) are an antichain of size 2, and there is no antichain of size 3.

▶ **Definition 2.3.** Let U and V be pomsets. The *sequential composition* of U and V, denoted $U \cdot V$, is the pomset $\langle U \cup V, \leq_U \cup \leq_V \cup (U \times V), \lambda_U \cup \lambda_V \rangle$. The *parallel composition* of U and V, denoted $U \parallel V$, is the pomset $\langle U \cup V, \leq_U \cup \leq_V, \lambda_U \cup \lambda_V \rangle$. Here, $\lambda_U \cup \lambda_V$ is the function from $U \cup V$ to Σ that agrees with λ_U on U, and with λ_V on V.

Note that 1 is the unit for both sequential and parallel composition. Sequential composition forces the events in the left pomset to be ordered before those in the right pomset. An example, describing the pomset $C \cdot C$, is depicted in Figure 1b.

▶ **Definition 2.4.** The set of *series-parallel* pomsets, $\mathsf{Pom}^{\mathsf{sp}}_\Sigma$, is the smallest set that includes the empty and primitive pomsets and is closed under sequential and parallel composition.

In this paper we will be mostly concerned with series-parallel pomsets. For inductive reasoning about them, it is useful to record the following lemma.

▶ **Lemma 2.5.** *Let $U \in \mathsf{Pom}^{\mathsf{sp}}_\Sigma$. If U is non-empty, then* exactly one *of the following is true: $U = a$ for some $a \in \Sigma$, or $U = V \cdot W$ for non-empty $V, W \in \mathsf{Pom}^{\mathsf{sp}}_\Sigma$, strictly smaller than U, or $U = V \parallel W$ for non-empty $V, W \in \mathsf{Pom}^{\mathsf{sp}}_\Sigma$, strictly smaller than U.*

2.2 Pomset languages

If a sequential program can exhibit multiple traces, we can group the words that represent these traces into a set called a *language*. By analogy, we can group the pomsets that represent the traces that arise from a parallel program into a set, which we refer to as a *pomset language*. Pomset languages are denoted by the symbols $\mathcal{U}$ and $\mathcal{V}$.

For instance, suppose that the recipe for glazed cookies may has an optional fifth step where chocolate sprinkles are spread over the cookies. In that case, there are *two* pomsets that describe a trace arising from the recipe, C^+ and C^-, either with or without the chocolate sprinkles. The pomset language $\mathcal{C} = \{C^-, C^+\}$ describes the new recipe.

▶ **Definition 2.6.** Let $\mathcal{U}$ be a pomset language. $\mathcal{U}$ has *bounded width* if there is $n \in \mathbb{N}$ such that for all $U \in \mathcal{U}$ we have $\|U\| \leq n$. The minimal such n is the *width* of $\mathcal{U}$, written $\|\mathcal{U}\|$.

The pomset languages considered in this paper have bounded width, and hence $\|\mathcal{U}\|$ is always defined. For instance, the width of $\mathcal{C}$ is 2, because the width of both C^+ and C^- is 2.

The sequential and parallel compositions of pomsets can be lifted to pomset languages. We also define a Kleene closure operator, similar to the one defined on languages of words.

▶ **Definition 2.7.** Let $\mathcal{U}$ and $\mathcal{V}$ be pomset languages. We define:

$$\mathcal{U} \cdot \mathcal{V} = \{U \cdot V : U \in \mathcal{U}, V \in \mathcal{V}\} \qquad \mathcal{U} \parallel \mathcal{V} = \{U \parallel V : U \in \mathcal{U}, V \in \mathcal{V}\} \qquad \mathcal{U}^* = \bigcup_{n \in \mathbb{N}} \mathcal{U}^n$$

Where $\mathcal{U}^0 = \{1\}$, and $\mathcal{U}^{n+1} = \mathcal{U} \cdot \mathcal{U}^n$ for all $n \in \mathbb{N}$.

Kleene closure models indefinite repetition. For instance, if our cookie recipe has a final step "repeat until enough cookies have been made", the pomset language $\mathcal{C}^*$ represents all possible traces of repetitions of the recipe; e.g., $C^+ \cdot C^+ \cdot C^- \in \mathcal{C}^*$ is the trace where first two batches of sprinkled cookies are made, followed by one without sprinkles.

2.3 Series-rational expressions

Just like a rational expression can be used to describe a regular structure of sequential events, a series-rational expression can be used to describe a regular structure of possibly parallel events. Series-rational expressions are rational expressions with parallel composition.

▶ **Definition 2.8.** The *series-rational expressions*, denoted $\mathcal{T}_\Sigma$, are formed by the grammar

$$e, f ::= 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e \parallel f \mid e^* .$$

We use the symbols d, e, f, g and h to denote series-rational expressions.

The semantics of a series-rational expression is given by a pomset language.

▶ **Definition 2.9.** The function $[\![-]\!] : \mathcal{T}_\Sigma \to 2^{\mathsf{Pom}_\Sigma}$ is defined inductively, as follows:

$$[\![0]\!] = \emptyset \qquad [\![a]\!] = \{a\} \qquad [\![1]\!] = \{1\} \qquad [\![e^*]\!] = [\![e]\!]^*$$

$$[\![e + f]\!] = [\![e]\!] \cup [\![f]\!] \qquad [\![e \cdot f]\!] = [\![e]\!] \cdot [\![f]\!] \qquad [\![e \parallel f]\!] = [\![e]\!] \parallel [\![f]\!]$$

If $\mathcal{U} \in 2^{\mathsf{Pom}_\Sigma}$ such that $\mathcal{U} = [\![e]\!]$ for some $e \in \mathcal{T}_\Sigma$, then $\mathcal{U}$ is a *series-rational language*.

To illustrate, consider the pomset language $\mathcal{C}^* = \{C^+, C^-\}^*$, which describes the possible traces arising from indefinitely repeating the cookie recipe, optionally adding chocolate sprinkles at every repetition. We can describe the pomset language $\{C^-\}$ with the series-rational expression $c^- = \mathsf{prepare} \cdot (\mathsf{bake} \parallel \mathsf{caramelize}) \cdot \mathsf{glaze}$, and $\{C^+\}$ by $c^+ = c^- \cdot \mathsf{sprinkle}$, which yields the series-rational expression $c = c^- + c^+$ for $\mathcal{C}$. By construction, $[\![c^*]\!] = \mathcal{C}^*$.

2.4 Additive congruence

The following congruence on series-rational expressions will be instrumental in analyzing the automaton we introduce in Section 4, and for restricting said automaton to be finite in Section 4.5.

▶ **Definition 2.10.** We define $\simeq$ as the smallest congruence on $\mathcal{T}_\Sigma$ such that:

$$e_1 + 0 \simeq e_1 \qquad e_1 + e_1 \simeq e_1 \qquad e_1 + e_2 \simeq e_2 + e_1 \qquad e_1 + (e_2 + e_3) \simeq (e_1 + e_2) + e_3$$

$$0 \cdot e_1 \simeq 0 \qquad e_1 \cdot 0 \simeq 0 \qquad 0 \parallel e \simeq 0 \qquad e \parallel 0 \simeq 0$$

When $\{\!\{g, h\}\!\}, \{\!\{g', h'\}\!\} \in \binom{\mathcal{T}_\Sigma}{2}$ such that $g \simeq g'$ and $h \simeq h'$, we write $\{\!\{g, h\}\!\} \simeq \{\!\{g', h'\}\!\}$.

Thus, when we claim that $e \simeq e'$, we say that e is equal to e', modulo associativity, commutativity and idempotence of $+$, as well as its unit 0, and possibly annihilation of sequential and parallel composition by 0. Moreover, this congruence is sound with respect to the semantics, and it identifies all expressions that have an empty denotational semantics.

▶ **Lemma 2.11.** *Let $e, f \in \mathcal{T}_\Sigma$. If $e \simeq f$, then $[\![e]\!] = [\![f]\!]$. Also, $e \simeq 0$ if and only if $[\![e]\!] = \emptyset$.*

There is a simple linear time decision procedure to test whether two expressions are congruent. This justifies our using this relation to build finite automata later on. As a by-product, we get that the emptiness problem for series-rational expressions is linear time decidable.

3 Pomset Automata

We are now ready to describe an automaton model that recognises series-rational languages.

▶ **Definition 3.1.** A *pomset automaton (PA)* is a tuple $\langle Q, \delta, \gamma, F \rangle$ where Q is a set of *states*, with $F \subseteq Q$ the *accepting states*, $\delta : Q \times \Sigma \to Q$ is a function called the *sequential transition function*, $\gamma : Q \times \binom{Q}{2} \to Q$ is a function called the *parallel transition function*.

Note that we do not fix an initial state. As a result, a PA does not define a single pomset language but rather a mapping from its states to pomset languages. The language of a state is defined in terms of a trace relation that involves the transitions of both δ and γ. Here, δ plays the same role as in classic finite automata: given a state and a symbol, it returns the new state after reading that symbol. The function γ warrants a bit more explanation. Given a state q and a binary multiset of states $\{\!\{r, s\}\!\}$, γ tells us the state that is reached after reading two input streams in parallel starting at states r and s, and having both "subprocesses" reach an accepting state. The precise meaning is given in Definition 3.2 below.

▶ **Definition 3.2.** $\to_A \subseteq Q \times \mathsf{Pom}^+_\Sigma \times Q$ is the smallest relation[1] satisfying the rules

$$\frac{}{q \xrightarrow{a}_A \delta(q,a)} \qquad \frac{q \xrightarrow{U}_A q'' \qquad q'' \xrightarrow{V}_A q'}{q \xrightarrow{U \cdot V}_A q'} \qquad \frac{r \xrightarrow{U}_A r' \in F \qquad s \xrightarrow{V}_A s' \in F}{q \xrightarrow{U \parallel V}_A \gamma(q, \{\!|r,s|\!\})}$$

We also define $\twoheadrightarrow_A \subseteq Q \times \mathsf{Pom}_\Sigma \times Q$ by $q \xrightarrow{U}\!\!\!\!\twoheadrightarrow_A q'$ if and only if $q' = q$ and $U = 1$, or $q \xrightarrow{U}_A q'$. The *language* of A at $q \in Q$, denoted $L_A(q)$, is the set $\{U : \exists q' \in F.\ q \xrightarrow{U}\!\!\!\!\twoheadrightarrow_A q'\}$. We say that A *accepts* the language $\mathcal{U}$ if there exists a $q \in Q$ such that $L_A(q) = \mathcal{U}$.

Intuitively, γ ensures that when a process forks at state q into subprocesses starting at r and s, if each of those reaches an accepting state, then the processes can join at $\gamma(q, \{\!|r,s|\!\})$.

We purposefully omit the empty pomset 1 as a label in $\to_A$; doing so would open up the possibility of having traces of the form $q \xrightarrow{1}_A q'$ with $q \neq q'$ (i.e., "silent transitions" or "ϵ-transitions") for example by defining $\gamma(q, \{\!|r,s|\!\}) = q'$ for some $r, s \in F$. Avoiding transitions of this kind allows us to prove claims about $\to_A$ by induction on the pomset size, and leverage Lemma 2.5 in the process to disambiguate between the rules that apply. By extension, we can prove claims about $\twoheadrightarrow_A$ and L_A by treating $U = 1$ as a special case.

For the remainder of this section, we fix a PA $A = \langle Q, \delta, \gamma, F\rangle$, and a state $q \in Q$. To simplify matters later on, we assume that A has a state $\bot \in Q - F$ such that, for every $a \in \Sigma$, it holds that $\delta(\bot, a) = \bot$ and, for every $\phi \in \binom{Q}{2}$, it holds that $\gamma(\bot, \phi) = \bot$. Such a *sink state* is particularly useful when defining γ: for a fixed $q \in Q$ not all $\{\!|r,s|\!\} \in \binom{Q}{2}$ may give a value of $\gamma(q, \{\!|r,s|\!\})$ that contributes to the language accepted by q. In such cases, we can define $\gamma(q, \{\!|r,s|\!\}) = \bot$. Alternatively, we could have allowed γ to be a partial function; we chose γ as a total function so as not to clutter the definition of derivatives in Section 4.

We draw a PA in a way similar to finite automata: each state (except $\bot$) is a vertex, and accepting states are marked by a double border. To represent sequential transitions, we draw labelled edges; for instance, in Figure 1c, $\delta(q_0, \mathsf{prepare}) = q_1$. To represent parallel transitions, we draw hyper-edges; for instance, in Figure 1c, $\gamma(q_1, \{\!|q_3, q_4|\!\}) = q_2$. To avoid clutter, we do not draw either of these edges types the target state is $\bot$. It is not hard to verify that the pomset C of the earlier example is accepted by the PA in Figure 1c.

In principle, the state space of a PA can be infinite; we use this in Section 4 to define a PA that has all possible series-rational expressions as states. It is however also useful to know when we can prune an infinite PA into a finite PA while preserving the languages of the retained states. In Section 5, we use this to translate the PA to a series-rational expression.

Note that it is not sufficient to talk about reachable states, i.e., states that appear in the target of some trace; we must also include states that are "meaningful" starting points for subprocesses. To do this, we first need a handle on these starting points. Specifically, we are interested in the states where the eventual join of the states yields a state that contributes to the behaviour of the PA, and the states may join again, because they are not the sink state. This is captured in the definition below.

▶ **Definition 3.3.** The *support* of q, written $\pi_A(q)$, is $\{\{\!|r,s|\!\} \in \binom{Q}{2} : \gamma(q, \{\!|r,s|\!\}), r, s \neq \bot\}$.

We can now talk about subsets of states of an automaton that are closed, in the sense that the relevant part of a transition function has input and output confined to this set. As a result, we can confine the structure of a given PA to a closed set.

[1] The relation $\to_A$ should not be thought of as deterministic; for fixed $q \in Q$ and $U \in \mathsf{Pom}^+_\Sigma$, there may be multiple distinct $q' \in Q$ such that $q \xrightarrow{U}_A q'$ – see the extended version [9] for additional information.

▶ **Definition 3.4.** A set of states $Q' \subseteq Q$ is *closed* when the following rules are satisfied

$$\frac{}{\bot \in Q'} \qquad \frac{q \in Q' \qquad a \in \Sigma}{\delta(q,a) \in Q'} \qquad \frac{q \in Q' \qquad \phi \in \pi_A(q)}{\gamma(q,\phi) \in Q'} \qquad \frac{q \in Q' \qquad \{\!|r,s|\!\} \in \pi_A(q)}{r,s \in Q'}$$

If Q' is closed, the *generated sub-PA* of A induced by Q', denoted $A\restriction_{Q'}$, is the tuple $\langle Q', \delta\restriction_{Q'}, \gamma\restriction_{Q'}, Q' \cap F\rangle$ where $\delta\restriction_{Q'}$ and $\gamma\restriction_{Q'}$ are the restrictions of δ and γ to Q'.

Because the relevant parts of the transition functions are preserved, it is not surprising that the language of a state in a generated sub-PA coincides with the language of that state in the original PA.

▶ **Lemma 3.5.** *Let $Q' \subseteq Q$ be closed. If $q \in Q'$, then $L_{A\restriction_{Q'}}(q) = L_A(q)$.*

We now work out how to find a closed subset of states that contains a particular state. The first step is to characterize the states reachable from q by means of transitions.

▶ **Definition 3.6.** The *reach* of q, written $\rho_A(q)$, is the smallest set satisfying the rules

$$\frac{}{q \in \rho_A(q)} \qquad \frac{q' \in \rho_A(q) \qquad a \in \Sigma}{\delta(q',a) \in \rho_A(q)} \qquad \frac{q' \in \rho_A(q) \qquad \phi \in \pi_A(q)}{\gamma(q',\phi) \in \rho_A(q)}$$

The reach of a state is closely connected to the states that can be reached from q through the trace relation of the automaton, in the following way:

▶ **Lemma 3.7.** *The set $\rho_A(q) \cup \{\bot\}$ contains $\{q' \in Q : \exists U \in \mathsf{Pom}^{+}_{\Sigma}.\ q \xrightarrow{U}_A q'\} \cup \{q\}$.*

Note that $\rho_A(q) \cup \{\bot\}$ is not necessarily closed: we also need the states required by the fourth rule of closure in Definition 3.4. Thus, if we want to "close" $\rho_A(q) \cup \{\bot\}$ by adding the support of its contents, we need to find closed sets of states that contain branching points. In order to do this inductively, we propose the following subclass of PAs.

▶ **Definition 3.8.** We say that A is *fork-acyclic* if there exists a *fork hierarchy*, which is a strict order $\prec_A \subseteq Q \times Q$ such that the following rules are satisfied.

$$\frac{\{\!|r,s|\!\} \in \pi_A(q)}{r,s \prec_A q} \qquad \frac{a \in \Sigma \qquad r \prec_A \delta(q,a)}{r \prec_A q} \qquad \frac{\phi \in \pi_A(q) \qquad r \prec_A \gamma(q,\phi)}{r \prec_A q}$$

The fork hierarchy is connected with the reach of a state in the following way.

▶ **Lemma 3.9.** *Let $q', r \in Q$. If A is fork-acyclic, $q' \in \rho_A(q)$ and $r \prec_A q'$, then $r \prec_A q$.*

The term *fork-acyclic* has been used in literature for similar automata [16, 7]. However, in op. cit., it is defined in terms of the traces that arise from the transition structure of the automaton. In contrast, our definition is purely syntactic: it imposes an order on states such that forks cannot be nested. To show that, as in [16], our definition implies that languages of the PA have bounded width, we present the following lemma. Since the state space of a PA can be infinite, we additionally require that the fork hierarchy is well-founded.

▶ **Lemma 3.10.** *If A is fork-acyclic and $\prec_A$ is well-founded then $L_A(q)$ is of finite width.*

We introduce the notion of a *bounded PA*, which is sufficient to guarantee the existence of a closed, finite subset containing a given state, even when the PA has infinitely many states.

▶ **Definition 3.11.** Let A be fork-acyclic. We say that A is *bounded* if $\prec_A$ is well-founded, and for all $q \in Q$, both $\pi_A(q)$ and $\rho_A(q)$ are finite.

▶ **Theorem 3.12.** *If A is bounded, then for every state q of A there exists a finite set of states $Q_q \subseteq Q$ that is closed and contains q.*

Proof. The proof proceeds by $\prec_A$-induction; this is sound, because $\prec_A$ is well-founded.

Suppose the claim holds for all $r \in Q$ with $r \prec_A q$. If $q' \in \rho_A(q)$ and $\{\!|r, s|\!\} \in \pi_A(q')$, then $r \prec_A q'$ and thus $r \prec_A q$ by Lemma 3.9; by induction we obtain for every such r a finite set of states $Q_r \subseteq Q$ that is closed and contains r. We choose:

$$Q_q = \{\bot\} \cup \rho_A(q) \cup \bigcup \{Q_r : q' \in \rho_A(q),\ \{\!|r, s|\!\} \in \pi_A(q')\}\,.$$

This set is finite because $\rho_A(q)$ and $\pi_A(q')$ are finite for all $q, q' \in Q$ since A is bounded. To see that Q_q is closed, it suffices to show that the last rule of closure holds for $q' \in \rho_A(q)$; it does, since if $q' \in \rho_A(q)$ and $\{\!|r, s|\!\} \in \pi_A(q')$, then $r \in Q_r$ and $s \in Q_s$, thus $r, s \in Q_q$. ◀

4 Expressions to automata

We now turn our attention to the task of translating a series-rational expression e into a PA that accepts $[\![e]\!]$. We employ Brzozowski's method [3] to construct a single *syntactic PA* where every series-rational expression is a state accepting exactly its denotational semantics. To this end we must define which expressions are accepting, and how the sequential and parallel transition functions transform states – what are, in Brzozowski's vocabulary, their sequential and parallel derivatives?

We start with the accepting states. In Brzozowski's construction, a rational expression is accepting if its denotational semantics includes the empty word. Analogously, a series-rational expression is accepting if its denotational semantics includes the empty pomset.

▶ **Definition 4.1.** We define the set F_Σ to be the smallest subset of $\mathcal{T}_\Sigma$ satisfying the rules:

$$\frac{}{1 \in F_\Sigma} \qquad \frac{e \in F_\Sigma \qquad f \in \mathcal{T}_\Sigma}{e + f, f + e \in F_\Sigma} \qquad \frac{e, f \in F_\Sigma}{e \cdot f, f \cdot e \in F_\Sigma} \qquad \frac{e, f \in F_\Sigma}{e \parallel f, f \parallel e \in F_\Sigma} \qquad \frac{e \in \mathcal{T}_\Sigma}{e^* \in F_\Sigma}$$

It is not hard to see that $e \in F_\Sigma$ if and only if $1 \in [\![e]\!]$. We use $e \star f$ as a shorthand for f if $e \in F_\Sigma$, and 0 otherwise. For an equation $\mathcal{E}$, we write $[\mathcal{E}]$ as a shorthand for 1 if $\mathcal{E}$ holds, and 0 otherwise. We now define sequential and parallel derivatives:

▶ **Definition 4.2.** We define the function $\delta_\Sigma : \mathcal{T}_\Sigma \times \Sigma \to \mathcal{T}_\Sigma$ as follows:

$$\delta_\Sigma(0, a) = 0 \qquad \delta_\Sigma(1, a) = 0 \qquad \delta_\Sigma(b, a) = [a = b] \qquad \delta_\Sigma(e^*, a) = \delta_\Sigma(e, a) \cdot e^*$$

$$\delta_\Sigma(e + f, a) = \delta_\Sigma(e, a) + \delta_\Sigma(f, a) \qquad \delta_\Sigma(e \cdot f, a) = \delta_\Sigma(e, a) \cdot f + e \star \delta_\Sigma(f, a)$$

$$\delta_\Sigma(e \parallel f, a) = e \star \delta_\Sigma(f, a) + f \star \delta_\Sigma(e, a)$$

Furthermore, the function $\gamma_\Sigma : \mathcal{T}_\Sigma \times \binom{\mathcal{T}_\Sigma}{2} \to \mathcal{T}_\Sigma$ is defined as follows:

$$\gamma_\Sigma(0, \phi) = 0 \qquad \gamma_\Sigma(1, \phi) = 0 \qquad \gamma_\Sigma(b, \phi) = 0 \qquad \gamma_\Sigma(e^*, \phi) = \gamma_\Sigma(e, \phi) \cdot e^*$$

$$\gamma_\Sigma(e + f, \phi) = \gamma_\Sigma(e, \phi) + \gamma_\Sigma(f, \phi) \qquad \gamma_\Sigma(e \cdot f, \phi) = \gamma_\Sigma(e, \phi) \cdot f + e \star \gamma_\Sigma(f, \phi)$$

$$\gamma_\Sigma(e \parallel f, \phi) = [\phi \simeq \{\!|e, f|\!\}] + e \star \gamma_\Sigma(f, \phi) + f \star \gamma_\Sigma(e, \phi)$$

The definition of δ_Σ coincides with Brzozowski's derivative on rational expressions. The definition of γ_Σ mimics the definition of δ_Σ on non-parallel terms except $b \in \Sigma$.

The definition of γ_Σ on parallel terms includes (in the first term) the possibility that the starting states provided to the parallel transition function are (congruent to) the operands of the parallel, in which case the target join state is the accepting state 1. The other two terms (as well as the definition of δ_Σ on a parallel term) account for the fact that if $1 \in [\![e]\!]$, then $[\![f]\!] \subseteq [\![e \parallel f]\!]$. Since we do not allow traces labelled with the empty pomset, traces that originate from these operands are thus lifted to the composition when necessary.

▶ **Definition 4.3.** The *syntactic PA* is the PA $A_\Sigma = \langle \mathcal{T}_\Sigma, \delta_\Sigma, \gamma_\Sigma, F_\Sigma \rangle$.

We use L_Σ as a shorthand for L_{A_Σ}, and $\to_\Sigma$ ($\twoheadrightarrow_\Sigma$) as a shorthand for $\to_{A_\Sigma}$ ($\twoheadrightarrow_{A_\Sigma}$).

The remainder of this section is devoted to showing that if $e \in \mathcal{T}_\Sigma$, then $L_\Sigma(e) = [\![e]\!]$.

4.1 Traces of congruent states

In the analysis of the syntactic trace relation $\to_\Sigma$, we often encounter sums of terms. To work with these, it is useful to identify terms modulo $\simeq$. In this section, we establish that such an identification is in fact sound, in the sense that if two expressions are related by $\simeq$, then the languages accepted by the states representing those expressions are also identical.

In the first step towards this goal, we show that F_Σ is well-defined with respect to $\simeq$.

▶ **Lemma 4.4.** *Let $e, f \in \mathcal{T}_\Sigma$ be such that $e \simeq f$. Then $e \in F_\Sigma$ if and only if $f \in F_\Sigma$.*

Also, δ_Σ and γ_Σ are well-defined with respect to $\simeq$, in the following sense:

▶ **Lemma 4.5.** *Let $e, f \in \mathcal{T}_\Sigma$ such that $e \simeq f$. If $a \in \Sigma$, then $\delta_\Sigma(e, a) \simeq \delta_\Sigma(f, a)$. Moreover, if $\phi = \{\!\{g, h\}\!\} \in \binom{\mathcal{T}_\Sigma}{2}$ with $g, h \not\simeq 0$, then $\gamma_\Sigma(e, \phi) \simeq \gamma_\Sigma(f, \phi)$, and if $\psi \in \binom{\mathcal{T}_\Sigma}{2}$ with $\phi \simeq \psi$, then $\gamma_\Sigma(e, \phi) = \gamma_\Sigma(e, \psi)$.*

With these lemmas in hand, we can show that $\simeq$ is a "bisimulation" with respect to $\to_\Sigma$.

▶ **Lemma 4.6.** *Let $e, f \in \mathcal{T}_\Sigma$ be such that $e \simeq f$. If $e \xrightarrow{U}_\Sigma e'$, then there exists an $f' \in \mathcal{T}_\Sigma$ such that $f \xrightarrow{U}_\Sigma f'$ and $e' \simeq f'$.*

Let I be a finite set, and let $(e_i)_{i \in I}$ be an I-indexed family of terms. In the sequel, we treat $\sum_{i \in I} e_i$ as a term, where the e_i are summed in some arbitrary order or bracketing. The lemmas above guarantee that the precise choice of representing this sum as a term makes no matter with regard to the traces allowed.

4.2 Trace deconstruction

We proceed with a series of lemmas that characterise reachable states in the syntactic PA. More precisely, we show that the expressions reachable from some expression e can be written as sums of expressions reachable from subexpressions of e. For this reason, we refer to these observations as *trace deconstruction lemmas*: they deconstruct a trace of an expression into traces of "smaller" expressions. The purpose of these lemmas is twofold; in Section 4.4, they are used to characterise the languages of expressions as they appear in the syntactic PA, while in Section 4.5 they allow us to bound the reach of an expression.

We start by analysing the traces that originate in base terms, such as 0, 1, or $a \in \Sigma$.

▶ **Lemma 4.7.** *Let $e, e' \in \mathcal{T}_\Sigma$ and $U \in \mathsf{Pom}^+_\Sigma$ such that $e \xrightarrow{U}_\Sigma e'$. If $e \in \{0, 1\}$, then $e' = 0$. Furthermore, if $e = b \in \Sigma$, then either $e' = 1$ and $U = b$, or $e' = 0$.*

Note, however, that 0 and 1 are not indistinguishable, for $0 \notin F_\Sigma$ while $1 \in F_\Sigma$.

We also consider the traces that originate in a sum of terms. The intuition here is that the input is processed by both terms simultaneously, and thus the target state must be the sum of the states that are the result of processing the input for each term individually.

▶ **Lemma 4.8.** *Let $e_1, e_2 \in \mathcal{T}_\Sigma$ and $U \in \mathsf{Pom}^+_\Sigma$. If $e_1 + e_2 \xrightarrow{U}_\Sigma e'$, then there exist $e'_1, e'_2 \in \mathcal{T}_\Sigma$ such that $e' = e'_1 + e'_2$, and $e_1 \xrightarrow{U}_\Sigma e'_1$ and $e_2 \xrightarrow{U}_\Sigma e'_2$.*

We now consider the traces starting in a sequential composition. The intuition here is that the syntactic PA must first proceed through the left operand, before it can proceed to process the right operand. Thus, either the pomset is processed by the left operand entirely, or we should be able to split the pomset in two sequential parts: the first part is processed by the left operand, and the second by the right operand.

▶ **Lemma 4.9.** *Let $e_1, e_2 \in \mathcal{T}_\Sigma$ and $U \in \mathsf{Pom}^+_\Sigma$ be such that $e_1 \cdot e_2 \xrightarrow{U}_\Sigma f$. There exist an $f' \in \mathcal{T}_\Sigma$ and a finite set I, as well as I-indexed families $(f'_i)_{i\in I}$ over F_Σ and $(f_i)_{i\in I}$ over $\mathcal{T}_\Sigma$, and I-indexed families $(U'_i)_{i\in I}$, $(U_i)_{i\in I}$ over Pom^+_Σ, such that:*

- *$f \simeq f' \cdot e_2 + \sum_{i\in I} f_i$ and $e_1 \xrightarrow{U}_\Sigma f'$, and*
- *for all $i \in I$, $e_1 \xrightarrow{U'_i}\!\!\!\!\rightarrow_\Sigma f'_i$, $e_2 \xrightarrow{U_i}_\Sigma f_i$, and $U = U'_i \cdot U_i$.*

The next deconstruction lemma concerns traces originating in a parallel composition. Intuitively, the syntactic PA either processes parallel components of the pomset, or processes according to one operand, provided that the other operand allows immediate acceptance.

▶ **Lemma 4.10.** *If $e_1 \parallel e_2 \xrightarrow{U}_\Sigma f$, then there exist $f_1, f_2, f_3 \in \mathcal{T}_\Sigma$, such that*

- *$f \simeq f_1 + f_2 + f_3$,*
- *either $f_1 = 0$, or $e_2 \in F_\Sigma$ and $e_1 \xrightarrow{U}_\Sigma f_1$,*
- *either $f_2 = 0$, or $e_1 \in F_\Sigma$ and $e_2 \xrightarrow{U}_\Sigma f_2$, and*
- *either $f_3 = 0$, or $f_3 = 1$ and there exist $f'_1, f'_2 \in F_\Sigma$ and $U_1, U_2 \in \mathsf{Pom}^+_\Sigma$ such that $U = U_1 \parallel U_2$ and $e_1 \xrightarrow{U_1}_\Sigma f'_1$ and $e_2 \xrightarrow{U_2}_\Sigma f'_2$.*

Finally, we analyse the reachable states of an expression of the form e^*. The intuition here is that, starting in e^*, the PA can iterate traces originating in e indefinitely. The trace should thus be sequentially decomposable, with each component the label of a trace originating in e. Furthermore, all but the last target state of these traces should be accepting.

▶ **Lemma 4.11.** *If $e^* \xrightarrow{U}_\Sigma f$, then there exists a finite set I and an I-indexed family of finite sets $(J_i)_{i\in I}$, as well as I-indexed families $(f_i)_{i\in I}$ over $\mathcal{T}_\Sigma$ and $(U_i)_{i\in I}$ over Pom^+_Σ, and for all $i \in I$ also J_i-indexed families $(f_{i,j})_{j\in J_i}$ over F_Σ and $(U_{i,j})_{j\in J_i}$ over Pom^+_Σ, such that $f \simeq \sum_{i\in I} f_i \cdot e^*$, and for all $i \in I$:*

- *$e \xrightarrow{U_i}_\Sigma f_i$,*
- *for all $j \in J_i$ we have that $e \xrightarrow{U_{i,j}}_\Sigma f_{i,j}$, and*
- *$U = U'_i \cdot U_i$, where U'_i is some concatenation of all $U_{i,j}$ for all $j \in J_i$.*

4.3 Trace construction

In the above, we learned how to deconstruct traces in the syntactic PA. To verify that the state in the syntactic PA associated with a series-rational expression e indeed accepts the series-rational pomset language $[\![e]\!]$, we also need to show the converse, that is, how to *construct* traces in the syntactic PA from smaller traces. In this context it is often useful to work with the preorder obtained from $\simeq$.

▶ **Definition 4.12.** The relation $\lesssim \subseteq \mathcal{T}_\Sigma \times \mathcal{T}_\Sigma$ is defined by $e \lesssim f$ if and only if $e + f \simeq f$.

The intuition to $e \lesssim f$ is that e consists of one or more terms that also appear in f, up to $\simeq$.

In analogy to Lemma 4.6, we show that $\lesssim$ is a "simulation" with respect to traces.

▶ **Lemma 4.13.** *Let $e, e', f \in \mathcal{T}_\Sigma$ be such that $e \lesssim f$. If $e \xrightarrow{U}_\Sigma e'$, then there exists an $f' \in \mathcal{T}_\Sigma$ such that $f \xrightarrow{U}_\Sigma f'$ and $e' \lesssim f'$. Furthermore, if $e \in F_\Sigma$, then $f \in F_\Sigma$.*

The following lemma tells us that we can create a trace labelled with the concatenation of the labels of two smaller traces, and starting in the sequential composition of the original starting states, provided that the first trace ends in an accepting state. Furthermore, the target state of the newly constructed trace contains the target state of the second trace.

▶ **Lemma 4.14.** *Let $e_1, e_2, f_2 \in \mathcal{T}_\Sigma$ and $f_1 \in F_\Sigma$. If $U, V \in \mathsf{Pom}^+_\Sigma$ are such that $e_1 \xrightarrow{U}_\Sigma f_1$ and $e_2 \xrightarrow{V}_\Sigma f_2$, then there exists an $f \in \mathcal{T}_\Sigma$ such that $e_1 \cdot e_2 \xrightarrow{U \cdot V}_\Sigma f$ with $f_2 \lesssim f$.*

We can also construct traces that start in a parallel composition. One way is to construct traces that start in each operand and reach an accepting state; we obtain a trace in their parallel composition almost trivially. If one of the operands is accepting, we can also construct a single trace that starts in the other operand and obtain a trace with the same label starting in the parallel construction. In both cases, we describe the target of the new trace using $\lesssim$.

▶ **Lemma 4.15.** *Let $e_1, e_2 \in \mathcal{T}_\Sigma$. The following hold:*

- *If $f_1, f_2 \in F_\Sigma$ and $U, V \in \mathsf{Pom}^+_\Sigma$ are such that $e_1 \xrightarrow{U}_\Sigma f_1$ and $e_2 \xrightarrow{V}_\Sigma f_2$, then there exists an $f \in \mathcal{T}_\Sigma$ such that $e_1 \parallel e_2 \xrightarrow{U \parallel V}_\Sigma f$ with $1 \lesssim f$.*
- *If $e_2 \in F_\Sigma$ (respectively $e_1 \in F_\Sigma$), and $f' \in \mathcal{T}_\Sigma$ and $U \in \mathsf{Pom}^+_\Sigma$ are such that $e_1 \xrightarrow{U}_\Sigma f'$ (respectively $e_2 \xrightarrow{U}_\Sigma f'$), then there exists an $f \in \mathcal{T}_\Sigma$ such that $e_1 \parallel e_2 \xrightarrow{U}_\Sigma f$ with $f' \lesssim f$.*

Lastly, we present a trace construction lemma to obtain traces originating in expressions of the form e^*. The idea here is that, given a finite number of traces that originate in e, where all (but possibly one) have an accepting state as their target, we can construct a trace originating in e^*, with a concatenation of the labels of the input traces as its label.

▶ **Lemma 4.16.** *Let $e, f_1, f_2, \ldots, f_n \in \mathcal{T}_\Sigma$ (with $n > 0$) be such that $f_1, f_2, \ldots, f_{n-1} \in F_\Sigma$. Also, let $U, U_1, U_2, \ldots, U_n \in \mathsf{Pom}^+_\Sigma$ be such that $U = U_1 \cdot U_2 \cdots U_n$. If for all $i \leq n$ it holds that $e \xrightarrow{U_i}_\Sigma f_i$, then there exists an $f \in \mathcal{T}_\Sigma$ such that $e^* \xrightarrow{U}_\Sigma f$, with $f_n \cdot e^* \lesssim f$.*

4.4 Soundness for the syntactic PA

With trace deconstruction and construction lemmas in our toolbox, we are ready to show that the syntactic PA indeed captures series-rational languages.

First, note that L_Σ can be seen as a function from $\mathcal{T}_\Sigma$ to Pom_Σ, like $[\![-]\!]$. To establish equality between L_Σ and $[\![-]\!]$, we first show that L_Σ enjoys the same homomorphic equalities as those in the definition of the semantic map, i.e., that $L_\Sigma(e)$ can be expressed in terms of L_Σ applied to subexpressions of e. The proofs of the equalities below follow a similar pattern: for the inclusion from left to right we use trace deconstruction lemmas to obtain traces for the component expressions, while for the inclusion from right to left we use trace construction lemmas to build traces for the composed expressions given the traces of the component expressions. We treat the case for the empty pomset separately almost everywhere.

▶ **Lemma 4.17.** *Let $e_1, e_2 \in \mathcal{T}_\Sigma$, and $a \in \Sigma$. The following equalities hold:*

$$L_\Sigma(0) = \emptyset \qquad L_\Sigma(1) = \{1\} \qquad L_\Sigma(a) = \{a\} \qquad L_\Sigma(e_1 + e_2) = L_\Sigma(e_1) \cup L_\Sigma(e_2)$$

$$L_\Sigma(e_1 \cdot e_2) = L_\Sigma(e_1) \cdot L_\Sigma(e_2) \qquad L_\Sigma(e_1 \parallel e_2) = L_\Sigma(e_1) \parallel L_\Sigma(e_2) \qquad L_\Sigma(e_1^*) = L_\Sigma(e_1)^*$$

It is now easy to establish that the Brzozowski construction for the syntactic PA is sound with respect to the denotational semantics of series-rational expressions.

▶ **Theorem 4.18.** *Let $e \in \mathcal{T}_\Sigma$. Then $L_\Sigma(e) = [\![e]\!]$.*

Proof. The proof proceeds by induction on e. In the base, $e = 0$, $e = 1$ or $e = a$ for some $a \in \Sigma$. In all cases, $L_\Sigma(e) = [\![e]\!]$ by Lemma 4.17. For the inductive step, there are four cases to consider: either $e = e_1 + e_2$, $e = e_1 \cdot e_2$, $e = e_1 \parallel e_2$ or $e = e_1^*$. In all cases, the claim follows from the induction hypothesis and the definition of $[\![-]\!]$, combined with Lemma 4.17. ◀

4.5 Bounding the syntactic PA

Ideally, we would like to obtain a single PA with finitely many states that recognizes $[\![e]\!]$ for a given $e \in \mathcal{T}_\Sigma$. Unfortunately, the syntactic PA is not bounded, and thus Theorem 3.12 does not apply. For instance, the requirement that $\rho_\Sigma(e)$ be finite for $e \in \mathcal{T}_\Sigma$ fails; consider the family of distinct terms $(e_n)_{n\in\mathbb{N}}$ defined by $e_0 = 1 \cdot a^*$ and $e_{n+1} = 0 \cdot a^* + e_n$ for $n \in \mathbb{N}$; it is not hard to show that $e_n \in \rho_\Sigma(a^*)$ for $n \in \mathbb{N}$, and thus conclude that $\rho_\Sigma(a^*)$ is infinite. We remedy this problem by quotienting the state space of the syntactic PA by congruence.

In what follows, we write $[e]$ for the congruence class of $e \in \mathcal{T}_\Sigma$ modulo $\simeq$, i.e., the set of all $e' \in \mathcal{T}_\Sigma$ such that $e \simeq e'$. We furthermore write $\mathcal{Q}_\Sigma$ for the set of all congruence classes of expressions in $\mathcal{T}_\Sigma$. We now leverage Lemma 4.5 to define a transition structure on $\mathcal{Q}_\Sigma$.

To save space, this section only summarizes the main stepping stones towards finding a finite PA for an expression; for a full proof, we refer to the full version of this paper [9].

▶ **Definition 4.19.** We define $\delta_\simeq : \mathcal{Q}_\Sigma \times \Sigma \to \mathcal{Q}_\Sigma$ and $\gamma_\simeq : \mathcal{Q}_\Sigma \times \binom{\mathcal{Q}_\Sigma}{2} \to \mathcal{Q}_\Sigma$ as

$$\delta_\simeq([e], a) = [\delta_\Sigma(e, a)] \qquad \gamma_\simeq([e], \{\!\{[f], [g]\}\!\}) = \begin{cases} [0] & f \simeq 0 \text{ or } g \simeq 0 \\ [\gamma_\Sigma(e, \{\!\{g, h\}\!\})] & \text{otherwise} \end{cases}$$

Furthermore, the set $F_\simeq$ is defined to be $\{[e] : e \in F_\Sigma\}$. The *quotiented syntactic PA* is the PA $A_\simeq = \langle \mathcal{Q}_\Sigma, \delta_\simeq, \gamma_\simeq, F_\simeq \rangle$.

Note that, by virtue of Lemma 4.5 and Lemma 4.4, we have that $\delta_\simeq$ and $\gamma_\simeq$, as well as $F_\simeq$, are well-defined. As before, we abbreviate subscripts, for example by writing $\to_\simeq$ rather than $\to_{A_\simeq}$, and $L_\simeq$ rather than $L_{A_\simeq}$. Of course, we also want the quotiented syntactic PA to accept the same languages as the syntactic PA. This turns out to be the case.

▶ **Theorem 4.20.** *Let $e \in \mathcal{T}_\Sigma$. Then $L_\Sigma(e) = L_\simeq([e])$.*

Furthermore, the quotiented syntactic PA is sufficiently restricted to show the following:

▶ **Theorem 4.21.** *The quotiented syntactic PA is fork-acyclic and bounded.*

The desired result then follows from the above, Lemma 3.5 and Theorem 3.12.

▶ **Corollary 4.22.** *Let $e \in \mathcal{T}_\Sigma$. There exists a finite PA A_e that accepts $[\![e]\!]$.*

5 Automata to expressions

To associate with every state q in a bounded PA $A = \langle Q, \delta, \gamma, F \rangle$ a series-rational expression e_q such that $[\![e_q]\!] = L_A(q)$, we modify the procedure for associating a rational expression with a state in a finite automaton described in [12]. The modification consists of adding

parallel terms to the expression associated with q whenever a fork in q contributes to its language, i.e., whenever $\{\!|r, s|\!\} \in \pi_A(q)$.

In view of the special treatment of 1 in the semantics of PAs, it is convenient to first define expressions e_q^+ with the property that $[\![e_q^+]\!] = L_A(q) - \{1\}$; then we can define e_q by $e_q = e_q^+ + [q \in F]$. The definition of e_q^+ proceeds by induction on the well-founded partial order $\prec_A$ associated with a bounded PA. That is, when defining e_q^+ we assume the existence of expressions $e_{q'}^+$ for all $q' \in Q$ such that $q' \prec_A q$.

First, however, we shall define auxiliary expressions $e_{qq'}^{Q'}$ for suitable choices of $Q' \subseteq Q$ and of $q, q' \in Q$. Intuitively, $e_{qq'}^{Q'}$ denotes the pomset language characterizing all paths from q to q' with all intermediate states in Q'; e_q^+ can then be defined as the summation of all $e_{qq'}^{\rho_A(q)}$ with $q' \in F \cap \rho_A(q)$.

▶ **Definition 5.1.** Let Q' be a finite subset of Q, and assume that for all $r \in Q$ such that $r \prec_A q$ for some $q \in Q'$ there exists a series-rational expression $e_r^+ \in \mathcal{T}_\Sigma$ such that $[\![e_r^+]\!] = L_A(q) - \{1\}$. For all $Q'' \subseteq Q'$ and $q, q' \in Q'$, we define a series-rational expression $e_{qq'}^{Q''}$ by induction on the size of Q'', as follows:

1. If $Q'' = \emptyset$, then let $\widetilde{\Sigma} = \{a \in \Sigma : q' = \delta(q, a)\}$, and let $\widetilde{Q} = \{\phi \in \pi_A(q) : \gamma(q, \phi) = q'\}$. We define
$$e_{qq'}^{Q''} = \sum_{a \in \widetilde{\Sigma}} a + \sum_{\{\!|r,s|\!\} \in \widetilde{Q}} e_r^+ \parallel e_s^+ \ .$$
2. Otherwise, we choose a $q'' \in Q''$ and define
$$e_{qq'}^{Q''} = e_{qq'}^{Q''-\{q''\}} + e_{qq''}^{Q''-\{q''\}} \cdot (e_{q''q''}^{Q''-\{q''\}})^* \cdot e_{q''q'}^{Q''-\{q''\}} \ .$$

Note that e_r^+ and e_s^+, appearing in the first clause of the definition of $e_{qq'}^{Q''}$, exist by assumption, for by fork-acyclicity we have that $r, s \prec_A q \in Q'$.

▶ **Theorem 5.2.** *Let Q' be a finite subset of Q and assume that for all $r \in Q$ such that $r \prec_A q$ for some $q \in Q'$ there exists a series-rational expression $e_r^+ \in \mathcal{T}_\Sigma$ with $[\![e_r^+]\!] = L_A(q) - \{1\}$. For all $q, q' \in Q'$, for all $Q'' \subseteq Q'$, and for all $U \in \mathsf{Pom}_\Sigma^+$, we have that $q \xrightarrow{U}_A q'$ according to some path that only visits states in Q'' if, and only if, $U \in \left[\!\!\left[e_{qq'}^{Q''} \right]\!\!\right]$.*

Using the auxiliary expressions $e_{qq'}^{Q''}$, we can now associate series-rational expressions $e_q, e_q^+ \in \mathcal{T}_\Sigma$ with every $q \in Q$, defining e_q^+ by $e_q^+ = \sum_{q' \in \rho_A(q) \cap F} e_{qq'}^{\rho_A(q)}$ and $e_q = e_q^+ + [q \in F]$. Note that $q \in \rho_A(q)$ and, by Lemma 3.9, for all $q' \in Q$ such that $q' \prec_A q''$ for some $q'' \in \rho_A(q)$ we have $q' \prec_A q$, and hence there exists, by induction, a series-rational expression $e_{q'} \in \mathcal{T}_\Sigma$ such that $[\![e_{q'}]\!] = L_A(q')$. So the expressions $e_{qq'}^{\rho_A(q)}$ are, indeed, defined in Definition 5.1.

▶ **Corollary 5.3.** *For every state $q \in Q$ we have $[\![e_q^+]\!] = L_A(q) - \{1\}$ and $[\![e_q]\!] = L_A(q)$.*

6 Discussion

Another automaton formalism for pomsets, *branching automata*, was proposed by Lodaya and Weil [15, 16]. Branching automata define the states where parallelism can start (*fork*) or end (*join*) in two relations; pomset automata condense this information in a single function. Lodaya and Weil also provided a translation of series-parallel expressions to branching automata, based on Thompson's construction [19], which relies on the fact that

their automata encode transitions non-deterministically, i.e., as *relations*. Our Brzozowski-style [3] translation, in contrast, directly constructs transition *functions* from the expressions. Lastly, their translation of branching automata to series-parallel expressions is only sound for a *semantically* restricted class of automata, whereas our restriction is *syntactic*.

Jipsen and Moshier [8] provided an alternative formulation of the automata proposed by Lodaya and Weil, also called *branching automata*. Their method to encode parallelism in these branching automata is conceptually dual to pomset automata: branching automata distinguish based on the target states of traces to determine the join state, whereas pomset automata distinguish based on the origin states of traces. The translations of series-parallel expressions to branching automata and vice versa suffer from the same shortcomings as those by Lodaya and Weil, i.e., transition relations rather than functions and a semantic restriction on automata for the translation of automata to expressions.

Lodaya and Weil observed [16] that the behaviour of their automata corresponds to 1-safe Petri nets. Since the behavior of their branching automata can be matched with our (bounded, fork-acyclic) pomset automata, we believe that 1-safe Petri nets also correspond to our automata. We opted to treat semantics of series-rational expressions in terms of automata instead of Petri nets to find more opportunities to extend to a coalgebraic treatment. While the present paper does not reach this goal, we believe that our formulation in terms of states and transition functions offers some hope of getting there.

Prisacariu introduced *Synchronous Kleene Algebra* (SKA) [17], extending Kleene Algebra with a *synchronous composition* operator. SKA differs from our model in that it assumes that all basic actions are performed in unit time, and that actors responsible for individual actions never idle. In contrast, our (weak BKA-like) model makes no synchrony assumptions: expressions can be composed in parallel, and the relative timing of basic actions within those expressions is irrelevant for the semantics. Prisacariu axiomatized SKA and extended it to *Synchronous Kleene Algebra with Tests* (SKAT); others [2] proposed Brzozowski-style derivatives of SKA expressions and used them to test equivalence of SKA(T) expressions.

7 Further work

We plan to extend our results to semantics of series-parallel expressions in terms of downward-closed pomset languages, i.e., sets of pomsets that are closed under Gischer's subsumption order [4]. Such an extension would correspond to adding the weak exchange law (which relates sequential and parallel compositions), and thus yields an operational model for weak CKA. We conjecture that no change to the automaton model is necessary to accommodate this generalization, just like Struth and Laurence suspect that the downward-closed semantics of series-parallel expressions can be captured by their non-downward closed semantics.

Our series-rational expressions do not include the parallel analogue of the Kleene star (sometimes called "parallel star", or "replication"). Future work could look into extending derivatives to include this operator, and relaxing fork-acyclicity to allow recovering expressions that include the parallel star from an automaton that satisfies this weaker restriction.

A classic result by Kozen [11] axiomatizes language equivalence of rational expressions using Kleene's theorem [10] and the uniqueness of minimal finite automata; consequently, the free model for KA can also be characterized in terms of rational languages. It would be interesting to see if the same technique can be used (based on pomset automata) to show that the axioms of weak Bi-Kleene Algebra are a complete axiomatization of pomset language equivalence of series-rational expressions, and thus characterise the free weak Bi-Kleene Algebra (or even the free weak CKA) in terms of series-rational pomset languages. Although an such a result was recently published [14], it does not rely on an automaton model.

Brzozowski derivatives for classic rational expressions induce a coalgebra on rational expressions that corresponds to a finite automaton. We aim to study series-rational expressions coalgebraically. The first step would be to find the coalgebraic analogue of pomset automata such that language acceptance is characterized by the homomorphism into the final coalgebra. Ideally, such a view of pomset automata would give rise to a decision procedure for equivalence of series-rational expressions based on coalgebraic bisimulation-up-to [18].

Rational expressions can be extended with tests to reason about imperative programs equationally [13]. In the same vein, one can extend series-rational expressions with tests [7, 8] to reason about parallel imperative programs equationally. We are particularly interested in employing such an extension to extend the network specification language NetKAT [1] with primitives for concurrency so as to model and reason about concurrency within networks.

Acknowledgements. We thank the anonymous reviewers for their insightful comments.

References

1 Carolyn Jane Anderson, Nate Foster, Arjun Guha, Jean-Baptiste Jeannin, Dexter Kozen, Cole Schlesinger, and David Walker. NetKAT: semantic foundations for networks. In *Proc. Principles of Programming Languages (POPL)*, pages 113–126, 2014. doi:10.1145/2535838.2535862.

2 Sabine Broda, Sílvia Cavadas, Miguel Ferreira, and Nelma Moreira. Deciding synchronous Kleene algebra with derivatives. In *Proc. Implementation and Application of Automata (CIAA)*, pages 49–62, 2015. doi:10.1007/978-3-319-22360-5_5.

3 Janusz A. Brzozowski. Derivatives of regular expressions. *J. ACM*, 11(4):481–494, 1964. doi:10.1145/321239.321249.

4 Jay L. Gischer. The equational theory of pomsets. *Theor. Comput. Sci.*, 61:199–224, 1988. doi:10.1016/0304-3975(88)90124-7.

5 C. A. R. Hoare, Bernhard Möller, Georg Struth, and Ian Wehrman. Concurrent Kleene Algebra. In *Proc. Concurrency Theory (CONCUR)*, pages 399–414, 2009. doi:10.1007/978-3-642-04081-8_27.

6 Tony Hoare, Stephan van Staden, Bernhard Möller, Georg Struth, and Huibiao Zhu. Developments in Concurrent Kleene Algebra. *J. Log. Algebr. Meth. Program.*, 85(4):617–636, 2016. doi:10.1016/j.jlamp.2015.09.012.

7 Peter Jipsen. Concurrent Kleene Algebra with tests. In *Proc. Relational and Algebraic Methods in Computer Science (RAMiCS) 2014*, pages 37–48, 2014. doi:10.1007/978-3-319-06251-8_3.

8 Peter Jipsen and M. Andrew Moshier. Concurrent Kleene Algebra with tests and branching automata. *J. Log. Algebr. Meth. Program.*, 85(4):637–652, 2016. doi:10.1016/j.jlamp.2015.12.005.

9 Tobias Kappé, Paul Brunet, Bas Luttik, Alexandra Silva, and Fabio Zanasi. Brzozowski goes concurrent – a Kleene theorem for pomset languages. *CoRR*, abs/1704.07199, 2017. URL: http://arxiv.org/abs/1704.07199.

10 Stephen C. Kleene. Representation of events in nerve nets and finite automata. *Automata Studies*, pages 3–41, 1956.

11 Dexter Kozen. A completeness theorem for Kleene Algebras and the algebra of regular events. *Inf. Comput.*, 110(2):366–390, 1994. doi:10.1006/inco.1994.1037.

12 Dexter Kozen. *Automata and computability.* Undergraduate texts in computer science. Springer, 1997.

13 Dexter Kozen. Kleene Algebra with tests. *ACM Trans. Program. Lang. Syst.*, 19(3):427–443, 1997. doi:10.1145/256167.256195.

14 Michael R. Laurence and Georg Struth. Completeness theorems for Bi-Kleene Algebras and series-parallel rational pomset languages. In *Proc. Relational and Algebraic Methods in Computer Science (RAMiCS)*, pages 65–82, 2014. doi:10.1007/978-3-319-06251-8_5.

15 Kamal Lodaya and Pascal Weil. Series-parallel posets: Algebra, automata and languages. In *Proc. Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 555–565, 1998. doi:10.1007/BFb0028590.

16 Kamal Lodaya and Pascal Weil. Series-parallel languages and the bounded-width property. *Theoretical Computer Science*, 237(1):347–380, 2000. doi:10.1016/S0304-3975(00)00031-1.

17 Cristian Prisacariu. Synchronous Kleene algebra. *J. Log. Algebr. Program.*, 79(7):608–635, 2010. doi:10.1016/j.jlap.2010.07.009.

18 Jurriaan Rot, Marcello M. Bonsangue, and Jan J. M. M. Rutten. Coalgebraic bisimulation-up-to. In *Proc. Current Trends in Theory and Practice of Computer Science (SOFSEM)*, pages 369–381, 2013. doi:10.1007/978-3-642-35843-2_32.

19 Ken Thompson. Regular expression search algorithm. *Commun. ACM*, 11(6):419–422, 1968. doi:10.1145/363347.363387.

Algebraic Laws for Weak Consistency

Andrea Cerone[1], Alexey Gotsman[2], and Hongseok Yang[3]

1 **Imperial College London, UK**
a.cerone@imperial.ac.uk
2 **IMDEA Software Institute, Madrid, Spain**
alexey.gotsman@imdea.org
3 **University of Oxford, UK**
hongseok.yang@cs.ox.ac.uk

Abstract

Modern distributed systems often rely on so called weakly consistent databases, which achieve scalability by weakening consistency guarantees of distributed transaction processing. The semantics of such databases have been formalised in two different styles, one based on abstract executions and the other based on dependency graphs. The choice between these styles has been made according to intended applications. The former has been used for specifying and verifying the implementation of the databases, while the latter for proving properties of client programs of the databases. In this paper, we present a set of novel algebraic laws (inequalities) that connect these two styles of specifications. The laws relate binary relations used in a specification based on abstract executions to those used in a specification based on dependency graphs. We then show that this algebraic connection gives rise to so called robustness criteria: conditions which ensure that a client program of a weakly consistent database does not exhibit anomalous behaviours due to weak consistency. These criteria make it easy to reason about these client programs, and may become a basis for dynamic or static program analyses. For a certain class of consistency models specifications, we prove a full abstraction result that connects the two styles of specifications.

1998 ACM Subject Classification C.2.4 Distributed Databases

Keywords and phrases Weak Consistency Models, Distributed Databases, Dependency Graphs

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.26

1 Introduction

Modern distributed systems often rely on databases that achieve scalability by weakening consistency guarantees of distributed transaction processing. These databases are said to implement weak consistency models. Such weakly consistent databases allow for faster transaction processing, but exhibit anomalous behaviours, which do not arise under a database with a strong consistency guarantee, such as serialisability. Two important problems for the weakly consistent databases are: (i) to find elegant formal specifications of their consistency models and to prove that these specifications are correctly implemented by protocols used in the databases; (ii) to develop effective reasoning techniques for applications running on top of such databases. These problems have been tackled by using two different formalisms, which model the run-time behaviours of weakly consistent databases differently.

When the goal is to verify the correctness of a protocol implementing a weak consistency model, the run-time behaviour of a distributed database is often described in terms of *abstract executions* [13], which abstract away low-level implementation details of the database (Section 2). An example of abstract execution is depicted in Figure 1; ignore the bold edges for the moment. It comprises four transactions, T_0, T_1, T_2, and S; transaction T_0 initializes

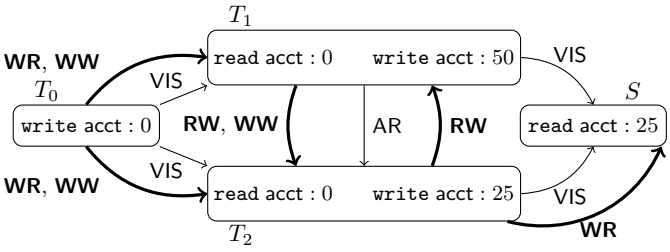


Figure 1 An example of abstract execution and of dependency graph.

the value of an object acct to 0; transactions T_1 and T_2 increment the value of acct by 50 and 25, respectively, after reading its initial value; transaction S reads the value of acct. In this abstract execution, both the updates of T_1 and T_2 are **VIS**ible to transaction S, as witnessed by the two VIS-labelled edges: $T_1 \xrightarrow{\mathsf{VIS}} S$ and $T_2 \xrightarrow{\mathsf{VIS}} S$.

On the other hand, the update of T_1 is not visible to T_2, and vice versa, as indicated by the absence of an edge labelled with VIS between these transactions. Intuitively, the absence of such an edge means that T_1 and T_2 are executed concurrently. Because S sees T_1 and T_2, as indicated by VIS-labelled edges from T_1 and T_2 to S, the result of reading the value of acct in S must be one of the values written by T_1 and T_2. However, because these transactions are concurrent, there is a race, or *conflict*, between them. The AR-labelled edge connecting T_1 to T_2, is used to **AR**bitrate the conflict: it states that the update of T_1 is older than the one of T_2, hence the query of acct in S returns the value written by the latter.

The style of specifications of consistency models in terms of abstract executions can be given by imposing constraints over the relations VIS, AR (Section 2.1). A set of transactions $\mathcal{T} = \{T_1, T_2, \cdots\}$, called a *history*, is allowed by a consistency model specification if it is possible to exhibit two witness relations VIS, AR over $\mathcal{T}$ such that the resulting abstract execution satisfies the constraints imposed by the specification. For example, *serialisability* can be specified by requiring that the relation VIS should be a strict total order. The set of transactions $\{T_0, T_1, T_2, S\}$ from Figure 1 is not serialisable: it is not possible to choose a relation VIS such that the resulting abstract execution relates the transactions T_1, T_2 and the results of read operations are consistent with visible updates.

Specifications of consistency models using abstract executions have been used in the work on proving the correctness of protocols implementing weak consistency models, as well as on justifying operational, implementation-dependent descriptions of these models [11, 12, 13, 15].

The second formalism used to define weak consistency models is based on the notion of *dependency graphs* [1], and it has been used for proving properties of client programs running on top of a weakly consistent database. Dependency graphs capture the data dependencies of transactions at run-time (Section 3); the transactions $\{T_0, T_1, T_2, S\}$ depicted above, together with the bold edges but without normal edges, constitute an example of dependency graph. The edge $T_2 \xrightarrow{\mathsf{WR(acct)}} S$[1] denotes a *write-read dependency*. It means that the read of acct in transaction S returns the value written by transaction T_2, and the edges $T_0 \xrightarrow{\mathsf{WR(acct)}} T_1$ and $T_0 \xrightarrow{\mathsf{WR(acct)}} T_2$ mean something similar. The edge $T_1 \xrightarrow{\mathsf{WW(acct)}} T_2$ denotes a *write-write dependency*, and says that the write to acct in T_2 supersedes the write to the same object in

[1] For simplicity, references to the object acct have been removed from the dependencies of Figure 1.

T_1. The remaining edges $T_1 \xrightarrow{\mathsf{RW(acct)}} T_2$ and $T_2 \xrightarrow{\mathsf{RW(acct)}} T_1$ express *anti-dependencies*. The former means that T_1 reads a value for object acct which is older than the value written by T_2.

When using dependency graphs, consistency models are specified as sets of transactions for which there exist $\mathsf{WR}, \mathsf{WW}, \mathsf{RW}$ relations that satisfy certain properties, usually stated as particular relations being acyclic [7, 16]; for example, serialisability can be specified by requiring that dependency graphs are acyclic. Because dependencies of transactions can be over-approximated at the compilation time, specifications of consistency models in terms of dependency graphs have been widely used for manually or automatically reasoning about properties of client programs of weakly consistent databases [19, 27]. They have also been used in the complexity and undecidability results for verifying implementations of consistency models [9].

Our ultimate aim is to reveal a deep connection between these two styles of specifying weak consistency models, which was hinted at for specific consistent models in the literature. Such a connection would, for instance, give us a systematic way to derive a specification of a weak consistency model based on dependency graphs from the specification based on abstract executions, while ensuring that the original and the derived specifications are equivalent in a sense. In doing so, it would enable us to prove properties about client programs of a weakly consistent database using techniques based on dependency graphs [9, 16, 18] even when the consistency model of the database is specified in terms of abstract executions.

In this paper, we present our first step towards this ultimate aim. First, we observe that each abstract execution determines an underlying dependency graph. Then we study the connection between these two structures at an algebraic level. We propose a set of algebraic laws, parametric in the specification of a consistency model to which the original abstract execution belongs (Section 4). These laws can be used to derive properties of the form $R_{\mathsf{G}} \subseteq R_{\mathsf{A}}$: here R_{G} is an expression from the Kleene Algebra with Tests [22] whose ground terms are run-time dependencies of transactions, and tests are properties over transactions. The relation R_{A} is one of the fundamental relations of abstract executions: VIS, AR, or a novel relation $\overline{\mathsf{VIS}^{-1}}$ that we call *anti-visibility*, defined as $\overline{\mathsf{VIS}^{-1}} = \{(T, S) \mid \neg(S \xrightarrow{\mathsf{VIS}} T)\}$. Some of the algebraic laws that we propose show that there is a direct connection between each kind of dependencies and the relations of abstract executions: $\mathsf{WR} \subseteq \mathsf{VIS}, \mathsf{WW} \subseteq \mathsf{AR}$, and $\mathsf{RW} \subseteq \overline{\mathsf{VIS}^{-1}}$. The other laws capture the connection between the relations of abstract executions $\mathsf{VIS}, \mathsf{AR}$, and $\overline{\mathsf{VIS}^{-1}}$. The exact nature of this connection depends on the specification of the consistency model of the considered abstract execution.

We are particularly interested in deriving properties of the form $R_{\mathsf{G}} \subseteq \mathsf{AR}$. Properties of this form give rise to so called robustness criteria for client programs, conditions ensuring that a program only exhibits serialisable behaviours even when it runs under a weak consistency model [7, 10, 19]. Because AR is a total order, this implies that R_{G} must be acyclic, hence all cycles must be in the complement of R_{G}. We can then check for the absence of such *critical* cycles at compile time: because dependency graphs of serialisable databases are always acyclic, this ensures that said application only exhibits serialisable behaviours.

As another contribution we show that, for a relevant class of consistency models, our algebraic laws can be used to derive properties which are not only necessary, but also sufficient, for dependency graphs in such models (Section 5).

2 Abstract Executions

We consider a database storing objects in $\mathsf{Obj} = \{x, y, \cdots\}$, which for simplicity we assume to be integer-valued. Client programs can interact with the database by executing operations from a set Op, grouped inside *transactions*. We leave the set Op unspecified, apart from requiring that it contains read and write operations over objects: $\{\mathtt{write}(x, n), \mathtt{read}(x, n) \mid x \in \mathsf{Obj}, n \in \mathbb{N}\} \subseteq \mathsf{Op}$.

Histories. To specify a consistency model, we first define the set of all client-database interactions allowed by the model. We start by introducing (run-time) *transactions* and *histories*, which record such interactions in a single computation. Transactions are elements from a set $\mathbb{T} = \{T, S, \cdots\}$; the operations executed by transactions are given by a function $\mathsf{behav} : \mathbb{T} \to 2^{\mathsf{Op}}$, which maps a transaction T to a set of operations that are performed by the transaction and can be observed by other transactions. We often abuse notations and just write $o \in T$ (or $T \ni o$) instead of $o \in \mathsf{behav}(T)$. We adopt similar conventions for $\mathcal{O} \subseteq \mathsf{behav}(T)$ and $\mathcal{O} = \mathsf{behav}(T)$ where $\mathcal{O}$ is a subset of operations.

We assume that transactions enjoy *atomic visibility* : for each object x, (i) a transaction S never observes two different writes to x from a single transaction T and (ii) it never reads two different values of x. Formally, the requirements are that if $T \ni (\mathtt{write}\ x : n)$ and $T \ni (\mathtt{write}\ x : m)$, or $T \ni (\mathtt{read}\ x : n)$ and $T \ni (\mathtt{read}\ x : m)$, then $n = m$. Our treatment of atomic visibility is taken from our previous work on transactional consistency models [15]. Atomic visibility is guaranteed by many consistency models [5, 19, 28]. We point out that although we focus on transactions in distributed systems in the paper, our results apply to weak shared-memory models [4]; there a transaction T is the singleton set of a read operation ($T = \{\mathtt{read}\ x : n\}$), that of a write operation ($T = \{\mathtt{write}\ x : n\}$), or the set of read and write representing a *compare and set* operation ($T = \{\mathtt{read}\ x : n,\ \mathtt{write}\ x : m\}$).

For each object x, we let $\mathsf{Writes}_x := \{T \mid \exists n.\ (\mathtt{write}\ x : n) \in T\}$ and $\mathsf{Reads}_x := \{T \mid \exists n,\ (\mathtt{read}\ x : n) \in \mathcal{T}\}$ be the sets of transactions that write to and read from x, respectively.

▶ **Definition 1.** A history $\mathcal{T}$ is a finite set of transactions $\{T_1, T_2, \cdots, T_n\}$.

Consistency Models. A consistency model Γ is a set of histories that may arise when client programs interact with the database. To define Γ formally, we augment histories with two relations, called *visibility* and *arbitration*.

▶ **Definition 2.** An *abstract execution* $\mathcal{X}$ is a tuple $(\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$ where $\mathcal{T}$ is a history and $\mathsf{VIS}, \mathsf{AR} \subseteq (\mathcal{T} \times \mathcal{T})$ are relations on transactions such that $\mathsf{VIS} \subseteq \mathsf{AR}$ and AR is a strict total order[2].

We often write $T \xrightarrow{\mathsf{VIS}} S$ for $(T, S) \in \mathsf{VIS}$, and similarly for other relations. For each abstract execution $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$, we let $\mathcal{T}_{\mathcal{X}} := \mathcal{T}$, $\mathsf{VIS}_{\mathcal{X}} := \mathsf{VIS}$, and $\mathsf{AR}_{\mathcal{X}} := \mathsf{AR}$.

In an abstract execution $\mathcal{X}$, $T \xrightarrow{\mathsf{VIS}_{\mathcal{X}}} S$ means that the read operations in S may depend on the updates of T, while $T \xrightarrow{\mathsf{AR}_{\mathcal{X}}} S$ means that the update operations of S supersede those performed by T. Naturally, one would expect that the value fetched by read operations in a transaction T is the most up-to-date one among all the values written by transactions visible to T. For simplicity, we assume that such a transaction always exists.

[2] A relation $R \subseteq \mathcal{T} \times \mathcal{T}$ is a strict (partial) order if it is transitive and irreflexive; it is total if for any $T, S \in \mathcal{T}$, either $T = S$, $(T, S) \in R$ or $(S, T) \in R$.

▶ **Definition 3.** An abstract execution $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$ respects the *Last Write Win* (**LWW**) policy, if for all $T \in \mathcal{T}$ such that $T \ni (\texttt{read}\ x : n)$, the set $\mathcal{T}' := \left(\mathsf{VIS}^{-1}(T) \cap \mathsf{Writes}_x\right)$ is not empty, and $\max_{\mathsf{AR}}(\mathcal{T}') \ni (\texttt{write}\ x : n)$, where $\max_{\mathsf{AR}}(\mathcal{T}')$ is the AR-supremum of $\mathcal{T}'$.

▶ **Definition 4.** An abstract execution $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$ *respects causality* if VIS is transitive. Any abstract execution that respects both causality and the LWW policy is said to be *valid*.

We always assume an abstract execution to be valid, unless otherwise stated. Causality is respected by all abstract executions allowed by several interesting consistency models. They also simplify the mathematical development of our results. In [17, Appendix B], we explain how our results can be generalised for consistency models that do not respect causality. We also discuss how the model can be generalised to account for sessions and session guarantees [29].

We can specify a consistency model using abstract executions in two steps. First, we identify properties on abstract executions, or *axioms*, that formally express an informal consistency guarantee, and form a set with the abstract executions satisfying the properties. Next, we project abstract executions in this set to underlying histories, and define a consistency model Γ to be the set of resulting histories.

Abstract executions hide low-level operational details of the interaction between client programs and weakly consistent databases. This benefit has been exploited for proving that such databases implement intended consistency models [11, 12, 13, 15, 20].

2.1 Specification of Weak Consistency Models

In this section we introduce a simple framework for specifying consistency models using the style of specification discussed above. In our framework, axioms of consistency models relate the visibility and arbitration relations via inequalities of the form $R_1 \;;\; \mathsf{AR}_{\mathcal{X}} \;;\; R_2 \subseteq \mathsf{VIS}_{\mathcal{X}}$, where R_1 and R_2 are particular relations over transactions, and $\mathcal{X}$ is an abstract execution. As we will explain later, axioms of this form establish a necessary condition for two transactions in an abstract execution $\mathcal{X}$ to be related by $\mathsf{VIS}_{\mathcal{X}}$, i.e. they cannot be executed concurrently. Despite its simplicity, the framework is expressive enough to capture several consistency models for distributed databases [15, 23]; as we will show in Section 4, one of the benefits of this simplicity is that we can infer robustness criteria of consistency models in a systematic way.

As we will see, the relations R_1, R_2 in axioms of the form above, may depend on the visibility relation of the abstract execution $\mathcal{X}$. To define such relations, we introduce the notion of *specification function*.

▶ **Definition 5.** A function $\rho : 2^{(\mathbb{T}\times\mathbb{T})} \to 2^{(\mathbb{T}\times\mathbb{T})}$ is a *specification function* if for every history $\mathcal{T}$ and relation $R \subseteq \mathcal{T} \times \mathcal{T}$, then $\rho(R) = \rho(\mathcal{T} \times \mathcal{T}) \cap R?$. Here $R?$ is the reflexive closure of R. A *consistency guarantee*, or simply *guarantee*, is a pair of specification functions (ρ, π).

Definition 5 ensures that specification functions are defined locally: for any $R_1, R_2 \subseteq \mathcal{T} \times \mathcal{T}$, $\rho(R_1 \cup R_2) = \rho(R_1) \cup \rho(R_2)$, and in particular for any $R \subseteq \mathcal{T} \times \mathcal{T}$, $\rho(R) = \left(\bigcup_{T,S\in\mathcal{T}} \rho(\{(T,S)\})\right) \cap R?$. The reflexive closure in Definition 5 is needed because we will always apply specification functions to irreflexive relations (namely, the visibility relation of abstract executions), although the result of this application need not be irreflexive. For example, $\rho_{\mathsf{Id}}(R) := \mathsf{Id}$, where Id is the identity function, is a valid specification function.

Each consistency guarantee (ρ, π) defines, for each abstract execution $\mathcal{X}$, an axiom of the form $\rho(\mathsf{VIS}_{\mathcal{X}}) \;;\; \mathsf{AR}_{\mathcal{X}} \;;\; \pi(\mathsf{VIS}_{\mathcal{X}}) \subseteq \mathsf{VIS}_{\mathcal{X}}$: if this axiom is satisfied by $\mathcal{X}$, we say that $\mathcal{X}$

Function		Definition	Guarantee	Associated Axiom
$\rho_{\mathsf{Id}}(R)$	=	Id	$(\rho_{\mathsf{Id}}, \rho_{\mathsf{Id}})$	$\mathsf{AR} \subseteq \mathsf{VIS}$
$\rho_{\mathsf{SI}}(R)$	=	$R \backslash \mathsf{Id}$	$(\rho_{\mathsf{Id}}, \rho_{\mathsf{SI}})$	$\mathsf{AR} ; \mathsf{VIS} \subseteq \mathsf{VIS}$
$\rho_x(R)$	=	$[\mathsf{Writes}_x]$	(ρ_x, ρ_x)	$[\mathsf{Writes}_x] ; \mathsf{AR} ; [\mathsf{Writes}_x] \subseteq \mathsf{VIS}$
$\rho_S(R)$	=	$[\mathtt{SerTx}]$	(ρ_S, ρ_S)	$[\mathtt{SerTx}] ; \mathsf{AR} ; [\mathtt{SerTx}] \subseteq \mathsf{VIS}$

Figure 2 Some Specification Functions and Consistency Guarantees.

satisfies the consistency guarantee (ρ, π). Consistency guarantees impose a condition on when two transactions T, S in an abstract execution $\mathcal{X}$ are not allowed to execute concurrently, i.e. they must be related by a $\mathsf{VIS}_{\mathcal{X}}$ edge. By definition, in abstract executions visibility edges cannot contradict arbitration edges, hence it is only natural that the order in which the transactions T, S above are executed is determined by the arbitration order: in fact, the definition of specification function ensures that $\rho(\mathsf{VIS}_{\mathcal{X}}) \subseteq \mathsf{VIS}_{\mathcal{X}}?$ and $\pi(\mathsf{VIS}_{\mathcal{X}}) \subseteq \mathsf{VIS}_{\mathcal{X}}?$, so that $(\rho(\mathsf{VIS}_{\mathcal{X}}) ; \mathsf{AR}_{\mathcal{X}} ; \pi(\mathsf{VIS}_{\mathcal{X}})) \subseteq \mathsf{AR}_{\mathcal{X}}$ for all abstract executions $\mathcal{X}$.

▶ **Definition 6.** A *consistency model specification* Σ or *x-specification* is a set of consistency guarantees $\{(\rho_i, \pi_i)\}_{i \in I}$ for some index set I.

We define $\mathsf{Executions}(\Sigma)$ to be the set of valid abstract executions that satisfy all the consistency guarantees of Σ. We let $\mathsf{modelOf}(\Sigma) := \{\mathcal{T}_{\mathcal{X}} \mid \mathcal{X} \in \mathsf{Executions}(\Sigma)\}$.

Examples of Consistency Model Specifications

Figure 2 shows several examples of specification functions and consistency guarantees. In the figure we use the relations $[\mathcal{T}] := \{(T, T) \mid T \in \mathcal{T}\}$ and $[o] := \{(T, T) \mid T \ni o\}$ for $\mathcal{T} \subseteq \mathbb{T}$ and $o \in \mathsf{Op}$. The guarantees in the figure can be composed together to specify, among others, several of the consistency models considered in [15]: we give some examples of them below. Each of these consistency models allows different kinds of anomalies: due to lack of space, these are illustrated in [17, Appendix A].

Causal Consistency [24]. This is the weakest consistency model we consider. It is specified by $\Sigma_{\mathsf{CC}} = \varnothing$. In this case, all abstract executions in $\mathsf{Executions}(\Sigma_{\mathsf{CC}})$ respect causality. The execution in Figure 1 is an example in $\mathsf{Executions}(\Sigma_{\mathsf{CC}})$.

Red-Blue Consistency [23]. This model extends causal consistency by marking a subset of transactions as serialisable, and ensuring that no two such transactions appear to execute concurrently. We model red-blue consistency via the x-specification $\Sigma_{\mathsf{RB}} = \{(\rho_S, \rho_S)\}$. In the definition of ρ_S, an element $\mathtt{SerTx} \in \mathsf{Op}$ is used to mark transactions as serialisable, and the specification requires that in every execution $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{RB}})$, any two transactions $T, S \ni \mathtt{SerTx}$ in $\mathcal{X}$ be compared by $\mathsf{VIS}_{\mathcal{X}}$. The abstract execution from Figure 1 is included in $\mathsf{Executions}(\Sigma_{\mathsf{RB}})$, but if it were modified so that transactions T_1, T_2 were marked as serialisable, then the result would not belong to $\mathsf{Executions}(\Sigma_{\mathsf{RB}})$.

Parallel Snapshot Isolation (PSI) [26, 28]. This model strengthens causal consistency by enforcing the *Write Conflict Detection* property: transactions writing to one same object do not execute concurrently. We let $\Sigma_{\mathsf{PSI}} = \{(\rho_x, \rho_x)\}_{x \in \mathsf{Obj}}$: every execution $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{PSI}})$ satisfies the inequality $([\mathsf{Writes}_x] ; \mathsf{AR}_{\mathcal{X}} ; [\mathsf{Writes}_x]) \subseteq \mathsf{VIS}_{\mathcal{X}}$, for all $x \in \mathsf{Obj}$.

Snapshot Isolation (SI) [6]. This consistency model strengthens PSI by requiring that, in executions, the set of transactions visible to any transaction T is a prefix of the arbitration relation. Formally, we let $\Sigma_{\mathsf{SI}} = \Sigma_{\mathsf{PSI}} \cup \{(\rho_{\mathsf{Id}}, \rho_{\mathsf{SI}})\}$. The consistency guarantee $(\rho_{\mathsf{Id}}, \rho_{\mathsf{SI}})$ ensures that any abstract execution $\mathcal{X} \in \mathsf{Executions}(\mathsf{SI})$ satisfies the property $(\mathsf{AR}_{\mathcal{X}} ; \mathsf{VIS}_{\mathcal{X}}) \subseteq \mathsf{VIS}_{\mathcal{X}}$[3].

Similarly to what we did to specify Red-Blue consistency, we can strengthen SI by allowing the possibility to mark transactions as serialisable. The resulting x-specification is $\Sigma_{\mathsf{SI+SER}} = \Sigma_{\mathsf{SI}} \cup \{(\rho_S, \rho_S)\}$. This x-specification captures a fragment of Microsoft SQL server, which allows the user to select the consistency model at which a transaction should run [25].

Serialisability. Executions in this consistency model require the visibility relation to be total. This can be formalised via the x-specification $\Sigma_{\mathsf{SER}} := \{(\rho_{\mathsf{Id}}, \rho_{\mathsf{Id}})\}$. Any $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{SER}})$ is such that $\mathsf{AR}_{\mathcal{X}} \subseteq \mathsf{VIS}_{\mathcal{X}}$, thus enforcing $\mathsf{VIS}_{\mathcal{X}}$ to be a strict total order.

3 Dependency Graphs

We present another style of specification for consistency models based on dependency graphs, introduced in [1]. These are structures that capture the data-dependencies between transactions accessing one same object. Such dependencies can be over approximated at compilation time. For this reason, they have found use in static analysis [7, 16, 18, 19] for programs running under a weak consistency model.

▶ **Definition 7.** A ***dependency graph*** is a tuple $\mathcal{G} = (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW})$, where $\mathcal{T}$ is a history and

1. $\mathsf{WR} : \mathsf{Obj} \to 2^{\mathcal{T}\times\mathcal{T}}$ is such that:
   - **(a)** $\forall T, S \in \mathcal{T}. \forall x. T \xrightarrow{\mathsf{WR}(x)} S \implies T \neq S \wedge \exists n. (T \ni \mathtt{write}\ x : n) \wedge (S \ni \mathtt{read}\ x : n)$,
   - **(b)** $\forall S \in \mathcal{T}. \forall x. (S \ni \mathtt{read}\ x : n) \implies \exists T. T \xrightarrow{\mathsf{WR}(x)} S$,
   - **(c)** $\forall T, T', S \in \mathcal{T}. \forall x. (T \xrightarrow{\mathsf{WR}(x)} S \wedge T' \xrightarrow{\mathsf{WR}(x)} S) \implies T = T'$;
2. $\mathsf{WW} : \mathsf{Obj} \to 2^{\mathcal{T}\times\mathcal{T}}$ is such that for every $x \in \mathsf{Obj}$, $\mathsf{WW}(x)$ is a strict, total order over Writes_x;
3. $\mathsf{RW} : \mathsf{Obj} \to 2^{\mathcal{T}\times\mathcal{T}}$ is such that $S \xrightarrow{\mathsf{RW}(x)} T$ iff $S \neq T$ and $\exists T'. T' \xrightarrow{\mathsf{WR}(x)} S \wedge T' \xrightarrow{\mathsf{WW}(x)} T$.

Given a dependency graph $\mathcal{G} = (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW})$, we let $\mathcal{T}_{\mathcal{G}} := \mathcal{T}$, $\mathsf{WR}_{\mathcal{G}} := \mathsf{WR}$, $\mathsf{WW}_{\mathcal{G}} := \mathsf{WW}$, $\mathsf{RW}_{\mathcal{G}} := \mathsf{RW}$. The set of all dependency graphs is denoted as Graphs. Sometimes, we commit an abuse of notation and use the symbol WR to denote the relation $\bigcup_{x\in\mathsf{Obj}} \mathsf{WR}(x)$, and similarly for WW and RW. The actual meaning of WR will always be clear from the context.

Let $\mathcal{G} \in \mathsf{Graphs}$. The *write-read dependency* $T \xrightarrow{\mathsf{WR}_{\mathcal{G}}(x)} S$ means that S reads the value of object x that has been written by T. By Definition 7, for any transaction $S \in \mathsf{Reads}_x$ there exists exactly one transaction T such that $T \xrightarrow{\mathsf{WR}_{\mathcal{G}}(x)} S$. The relation $\mathsf{WW}_{\mathcal{G}}(x)$ establishes a total order in which updates over object x are executed by transactions; its elements are called *write-write dependencies*. Edges in the relation $\mathsf{RW}_{\mathcal{G}}(x)$ take the name of *anti-dependencies*. $T \xrightarrow{\mathsf{RW}_{\mathcal{G}}(x)} S$ means that transaction T fetches some value for object x, but this is later updated by S. Given an abstract execution $\mathcal{X}$, we can extract a dependency graph $\mathsf{graph}(\mathcal{X})$ such that $\mathcal{T}_{\mathsf{graph}(\mathcal{X})} = \mathcal{T}_{\mathcal{X}}$.

[3] To be precise, the property induced by the guarantee $(\rho_{\mathsf{Id}}, \rho_{\mathsf{SI}})$ is $(\mathsf{AR}_{\mathcal{X}} ; (\mathsf{VIS}_{\mathcal{X}} \backslash \mathsf{Id})) \subseteq \mathsf{AR}_{\mathcal{X}}$. However, since $\mathsf{VIS}_{\mathcal{X}}$ is an irreflexive relation, $\mathsf{VIS}_{\mathcal{X}} \backslash \mathsf{Id} = \mathsf{VIS}_{\mathcal{X}}$. Also, note that $\rho(R) = R$ is not a specification function, so we cannot replace the guarantee $(\rho_{\mathsf{Id}}, \rho_{\mathsf{SI}})$ with $(\rho_{\mathsf{Id}}, \rho)$.

▶ **Definition 8.** Let $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$ be an execution. For $x \in \mathsf{Obj}$, we define $\mathsf{graph}(\mathcal{X}) = (\mathcal{T}, \mathsf{WR}_{\mathcal{X}}, \mathsf{WW}_{\mathcal{X}}, \mathsf{RW}_{\mathcal{X}})$, where:

1. $T \xrightarrow{\mathsf{WR}_{\mathcal{X}}(x)} S \iff (S \ni \mathtt{read}\ x : \_) \wedge T = \max_{\mathsf{AR}}(\mathsf{VIS}^{-1}(S) \cap \mathsf{Writes}_x)$;
2. $T \xrightarrow{\mathsf{WW}_{\mathcal{X}}(x)} S \iff T \xrightarrow{\mathsf{AR}} S \wedge T, S \in \mathsf{Writes}_x$;
3. $T \xrightarrow{\mathsf{RW}_{\mathcal{X}}(x)} S \iff S \neq T \wedge (\exists T'.\, T' \xrightarrow{\mathsf{WR}_{\mathcal{X}}(x)} T \wedge T' \xrightarrow{\mathsf{WW}_{\mathcal{X}}(x)} S)$.

▶ **Proposition 9.** *For any valid abstract execution $\mathcal{X}$, $\mathsf{graph}(\mathcal{X})$ is a dependency graph.*

Specification of Consistency Models using Dependency Graphs. We interpret a dependency graph $\mathcal{G}$ as a labelled graph whose vertices are transactions in $\mathcal{T}_x$, and whose edges are pairs of the form $T \xrightarrow{R} S$, where $R \in \{\mathsf{WR}_{\mathcal{G}}(x), \mathsf{WW}_{\mathcal{G}}(x)_{\mathcal{G}}, \mathsf{RW}_{\mathcal{G}}(x) \mid x \in \mathsf{Obj}\}$. To specify a consistency model, we employ a two-steps approach. We first identify one or more conditions to be satisfied by dependency graphs. Such conditions require cycles of a certain form not to appear in a dependency graph. Then we define a consistency model by projecting the set of dependency graphs satisfying the imposed conditions into the underlying histories. This style of specification is reminiscent of the one used in the CAT [4] language for formalising weak memory models. In the following we treat the relations $\mathsf{WR}_{\mathcal{G}}(x), \mathsf{WW}_{\mathcal{G}}(x), \mathsf{RW}_{\mathcal{G}}(x)$ both as set-theoretic relations, and as edges of a labelled graph.

▶ **Definition 10.** A *dependency graph based specification*, or simply g-specification, is a set $\Delta = \{\delta_1, \cdots, \delta_n\}$, where for each $i \in \{1, \cdots, n\}$, δ_i is a function of type $\mathsf{Graphs} \to 2^{(\mathbb{T}\times\mathbb{T})}$ and satisfies $\delta_i(\mathcal{G}) \subseteq (\mathsf{WR}_{\mathcal{G}} \cup \mathsf{WW}_{\mathcal{G}} \cup \mathsf{RW}_{\mathcal{G}})^*$ for every $\mathcal{G} \in \mathsf{Graphs}$.

Given a g-specification Δ, we define $\mathsf{Graphs}(\Delta) = \{\mathcal{G} \in \mathsf{Graphs} \mid \forall \delta \in \Delta.\, \delta(\mathcal{G}) \cap \mathsf{Id} = \varnothing\}$, and we let $\mathsf{modelOf}(\Delta) = \{\mathcal{T} \mid \exists \mathsf{WR}, \mathsf{WW}, \mathsf{RW}.\, (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW}) \in \mathsf{Graphs}(\Delta)\}$.

The requirement imposed over the functions $\delta_1, \cdots, \delta_n$ ensures that, whenever $(T, S) \in \delta_i(\mathcal{G})$, for some dependency graph $\mathcal{G}$, then there exists a path in $\mathcal{G}$, that connects T to S. For $\Delta = \{\delta_i\}_{i=1}^n$ and $\mathcal{G} \in \mathsf{Graphs}$, the requirement that $\delta_i(\mathcal{G}) \cap \mathsf{Id} = \varnothing$ means that $\mathcal{G}$ does not contain any cycle $T_0 \xrightarrow{R_0} T_1 \xrightarrow{R_1} \cdots \xrightarrow{R_{n-1}} T_n$, such that $T_0 = T_n$, and $(R_0 ; \cdots ; R_{n-1}) \subseteq \delta_i(\mathcal{G})$.

Examples of g-specifications of consistency models. Below we give some examples of g-specifications for the consistency models presented in Section 2.

▶ **Theorem 11.**

1. *An execution $\mathcal{X}$ is serialisable iff $\mathsf{graph}(\mathcal{X})$ does not contain any cycle. That is, $\mathsf{modelOf}(\Sigma_{\mathsf{SER}}) = \mathsf{modelOf}(\{\delta_{\mathsf{SER}}\})$, where $\delta_{\mathsf{SER}}(\mathcal{G}) = (\mathsf{WR}_{\mathcal{G}} \cup \mathsf{WW}_{\mathcal{G}} \cup \mathsf{RW}_{\mathcal{G}})^+$.*
2. *An execution $\mathcal{X}$ is allowed by snapshot isolation iff $\mathsf{graph}(\mathcal{X})$ only admits cycles with at least two consecutive anti-dependency edge. That is, $\mathsf{modelOf}(\Sigma_{\mathsf{SI}}) = \mathsf{modelOf}(\{\delta_{\mathsf{SI}}\})$, where $\delta_{\mathsf{SI}}(\mathcal{G}) = ((\mathsf{WR}_{\mathcal{G}} \cup \mathsf{WW}_{\mathcal{G}}) ; \mathsf{RW}_{\mathcal{G}}?)^+$.*
3. *An execution $\mathcal{X}$ is allowed by parallel snapshot isolation iff $\mathsf{graph}(\mathcal{X})$ has no cycle where all anti-dependency edges are over the same object. Let $\delta_{\mathsf{PSI}_0}(\mathcal{G}) = (\mathsf{WR}_{\mathcal{G}} \cup \mathsf{WW}_{\mathcal{G}})^+$, $\delta_{\mathsf{PSI}(x)}(\mathcal{G}) = (\bigcup_{x\in\mathsf{Obj}}(\mathsf{WR}_{\mathcal{G}} \cup \mathsf{WW}_{\mathcal{G}})^* ; \mathsf{RW}_{\mathcal{G}}(x))^+$, and define $\Delta_{\mathsf{PSI}} = \{\delta_{\mathsf{PSI}_0}\} \cup \{\delta_{\mathsf{PSI}(x)} \mid x \in \mathsf{Obj}\}$. Then, $\mathsf{modelOf}(\Sigma_{\mathsf{PSI}}) = \mathsf{modelOf}(\Delta_{\mathsf{PSI}})$.*

Theorem 11(1) was proved in [1]. The only if condition of Theorem 11(2) was proved in [19]; we proved the if condition of Theorem 11(2) in [16]. Theorem 11(3) improves on the specification we gave for PSI in [16]; the latter does not have any constraints on the objects to which anti-dependencies refer to. We outline the proof of Theorem 11(3) in Section 5.

(a) Algebraic laws for sets of transactions

(a.1) $[\mathcal{T}'] \subseteq \mathsf{Id}$ **(a.2)** $[\mathcal{T}_1 \cap \mathcal{T}_2] = [\mathcal{T}_1] \,;\, [\mathcal{T}_2]$
(a.3) $(R_1 \,;\, [\mathcal{T}']) \cap R_2 = (R_1 \cap R_2) \,;\, [\mathcal{T}']$
(a.4) $([\mathcal{T}'] \,;\, R_1) \cap R_2 = [\mathcal{T}'] \,;\, (R \cap R_2)$

(b) Algebraic laws for (anti-)dependencies

(b.1) $\mathsf{WR}(x) \subseteq [\mathsf{Writes}_x] \,;\, \mathsf{WR}(x) \,;\, [\mathsf{Reads}_x]$
(b.2) $\mathsf{WW}(x) \subseteq [\mathsf{Writes}_x] \,;\, \mathsf{WW}(x) \,;\, [\mathsf{Writes}_x]$
(b.3) $\mathsf{RW}(x) \subseteq [\mathsf{Reads}_x] \,;\, \mathsf{RW}(x) \,;\, [\mathsf{Writes}_x]$
(b.4) $\mathsf{WR}(x) \subseteq \mathsf{WR}(x) \backslash \mathsf{Id}$
(b.5) $\mathsf{WW}(x) \subseteq \mathsf{WW}(x) \backslash \mathsf{Id}$
(b.6) $\mathsf{RW}(x) \subseteq \mathsf{RW}(x) \backslash \mathsf{Id}$

(c) Algebraic laws for abstract Executions

(c.1) $\mathsf{WR}(x) \subseteq \mathsf{VIS}$ **(c.2)** $\mathsf{WW}(x) \subseteq \mathsf{AR}$
(c.3) $\mathsf{RW}(x) \subseteq \overline{\mathsf{VIS}^{-1}}$ **(c.4)** $\mathsf{VIS}^+ \subseteq \mathsf{VIS}$
(c.5) $\mathsf{AR}^+ \subseteq \mathsf{AR}$ **(c.6)** $\mathsf{VIS} \subseteq \mathsf{AR}$
(c.7) $[\mathsf{Writes}_x] \,;\, \mathsf{VIS} \,;\, \mathsf{RW}(x) \subseteq \mathsf{AR}$
(c.8) $\mathsf{VIS} \,;\, \overline{\mathsf{VIS}^{-1}} \subseteq \overline{\mathsf{VIS}^{-1}}$
(c.9) $\overline{\mathsf{VIS}^{-1}} \,;\, \mathsf{VIS} \subseteq \overline{\mathsf{VIS}^{-1}}$
(c.10) $(\overline{\mathsf{VIS}^{-1}} \,;\, \mathsf{VIS}) \cap \mathsf{Id} \subseteq \varnothing$
(c.11) $(\mathsf{VIS} \,;\, \overline{\mathsf{VIS}^{-1}}) \cap \mathsf{Id} \subseteq \varnothing$
(c.12) $\mathsf{AR} \cap \mathsf{Id} \subseteq \varnothing$

(d) Algebraic laws induced by the consistency guarantee (ρ, π)

(d.1) $\rho(\mathsf{VIS}) \,;\, \mathsf{AR} \,;\, \pi(\mathsf{VIS}) \subseteq \mathsf{VIS}$ **(d.2)** $(\pi(\mathsf{VIS}) \,;\, \overline{\mathsf{VIS}^{-1}} \,;\, \rho(\mathsf{VIS})) \backslash \mathsf{Id} \subseteq \mathsf{AR}$
(d.3) $(\mathsf{AR} \,;\, \pi(\mathsf{VIS}) \,;\, \overline{\mathsf{VIS}^{-1}}) \cap \rho(\mathcal{T} \times \mathcal{T})^{-1} \subseteq \overline{\mathsf{VIS}^{-1}}$
(d.4) $(\overline{\mathsf{VIS}^{-1}} \,;\, \rho(\mathsf{VIS}) \,;\, \mathsf{AR}) \cap \pi(\mathcal{T} \times \mathcal{T})^{-1} \subseteq \overline{\mathsf{VIS}^{-1}}$

Figure 3 Algebraic laws satisfied by an abstract execution $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$. Here $\mathsf{graph}(\mathcal{X}) = (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW})$. The inequalities in part **(d)** are valid under the assumption that $\mathcal{X} \in \mathsf{Executions}(\{(\rho, \pi)\})$.

4 Algebraic Laws for Weak Consistency

Having two different styles for specifying consistency models gives rise to the following problems:

Weak Correspondence Problem. Given a x-specification Σ, determine a non-trivial g-specification Δ which over-approximates Σ, that is such that $\mathsf{modelOf}(\Sigma) \subseteq \mathsf{modelOf}(\Delta)$.

Strong Correspondence Problem. Given a x-specification Σ, determine an equivalent g-specification Δ, that is such that $\mathsf{modelOf}(\Sigma) = \mathsf{modelOf}(\Delta)$.

We first focus on the weak correspondence problem, and we discuss the strong correspondence problem in Section 5. This problem is not only of theoretical interest. Determining a g-specification Δ that over-approximates a x-specification Σ corresponds to establishing one or more conditions satisfied by all cycles of dependency graphs from the set $\{\mathsf{graph}(\mathcal{X}) \mid \mathcal{X} \in \mathsf{Executions}(\Sigma)\}$. Cycles in a dependency graph that respect such a condition are called Σ-*critical* (or simply critical), and graphs that admit a non-Σ-critical cycle cannot be obtained from abstract executions in $\mathsf{Executions}(\Sigma)$. One can ensure that an application running under the model Σ is *robust*, i.e. it only produces serialisable behaviours, by checking for the absence of Σ-critical cycles at static time [7, 19]. Robustness of an application can also be checked at run-time, by incrementally constructing the dependency graph of executions, and detecting the presence of Σ-critical cycles [31].

General Methodology. Let Σ be a given x-specification. We tackle the weak correspondence problem in two steps.

First, we identify a set of inequalities that hold for all the executions $\mathcal{X}$ satisfying consistency guarantees (ρ, π) in Σ. There are two kinds of such inequalities. The first are the inequalities in Figure 3, and the second the inequalities corresponding to the axioms of the Kleene Algebra $(2^{\mathbb{T}\times\mathbb{T}}, \varnothing, \mathsf{Id}, \cup, ;, \cdot^*)$ and the Boolean algebra $(2^{\mathbb{T}\times\mathbb{T}}, \varnothing, \mathbb{T} \times \mathbb{T}, \cup, \cap, \bar{\cdot})$. The exact meaning of the inequalities in Figure 3 is discussed later in this section.

Second, we exploit our inequalities to derive other inequalities of the form $R_{\mathcal{X}} \subseteq \mathsf{AR}_{\mathcal{X}}$ for every $\mathcal{X} \in \mathsf{Executions}(\Sigma)$. Here $R_{\mathcal{X}}$ is a relation built from dependencies in $\mathsf{graph}(\mathcal{X})$, i.e. $R_{\mathcal{X}} \subseteq (\mathsf{WR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}})^*$. Because $\mathsf{AR}_{\mathcal{X}}$ is acyclic (that is $\mathsf{AR}_{\mathcal{X}}^+ \cap \mathsf{Id} \subseteq \varnothing$), we may conclude that $R_{\mathcal{X}}$ is acyclic for any $\mathcal{X} \in \mathsf{Executions}(\Sigma)$. In particular, we have that $\mathsf{modelOf}(\Sigma) \subseteq \mathsf{modelOf}(\{\delta\})$, where δ is a function that maps, for every abstract execution $\mathcal{X}$, the dependency graph $\mathsf{graph}(\mathcal{X})$ into the relation $R_{\mathcal{X}}$.

Some of the inequalities we develop, namely those in Figure 3**(d)**, are parametric in the consistency guarantee (ρ, π). As a consequence, our approach can be specialised to any consistency model that is captured by our framework. To show its applicability, we derive critical cycles for several of the consistency models that we have presented.

Presentation of the Laws. Let $\mathcal{X} = (\mathcal{T}, \mathsf{VIS}, \mathsf{AR})$, and $\mathsf{graph}(\mathcal{X}) = (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW})$. We now explain the inequalities in Figure 3. Among these, the inequalities in Figures 3**(a)** and **(b)** should be self-explanatory.

Let us discuss the inequalities of Figure 3**(c)**. The inequalities **(c.1)**, **(c.2)** and **(c.3)** relate dependencies to either basic or derived relations of abstract executions. Dependencies of the form WR, WW are included in the relations $\mathsf{VIS}, \mathsf{AR}$, respectively, as established by inequalities **(c.1)** and **(c.2)**. The inequality **(c.3)**, which we prove presently, is non-standard. It relates anti-dependencies to a novel *anti-visibility* relation $\overline{\mathsf{VIS}^{-1}}$, defined as $T \xrightarrow{\overline{\mathsf{VIS}^{-1}}} S$ iff $\neg(S \xrightarrow{\mathsf{VIS}} T)$. In words, S is *anti-visible* to T if T does not observe the effects of S. As we will explain later, anti-visibility plays a fundamental role in the development of our laws.

Proof of Inequality (c.3). Suppose $T \xrightarrow{\mathsf{RW}(x)} S$ for some object $x \in \mathsf{Obj}$. By definition, $T \neq S$, and there exists a transaction T' such that $T' \xrightarrow{\mathsf{WR}(x)} T$ and $T' \xrightarrow{\mathsf{WW}(x)} S$. In particular, $T' \xrightarrow{\mathsf{VIS}} T$ and $T' \xrightarrow{\mathsf{AR}} S$ by the inequalities **(c.1)** and **(c.2)**, respectively. Now, if it were $S \xrightarrow{\mathsf{VIS}} T$, then we would have that T' is not the AR-supremum of the set of transactions visible to T, and writing to object x. But this contradicts the definition of $\mathsf{graph}(\mathcal{X})$, and the edge $T' \xrightarrow{\mathsf{WR}(x)} T$. Therefore, $T \xrightarrow{\overline{\mathsf{VIS}^{-1}}} S$. ◀

Another non-trivial inequality is **(c.7)** in Figure 3**(c)**. It says that if a transaction T reads a value for an object x that is later updated by another transaction S ($T \xrightarrow{\mathsf{RW}} S$), then the update of S is more recent (i.e. it follows in arbitration) than all the updates to x seen by T. We prove it in [17, Appendix C]. The other inequalities in Figure 3**(c)** are self explanatory.

The inequalities in Figure 3**(d)** are specific to a consistency guarantee (ρ, π), and hold for an execution $\mathcal{X}$ when the execution satisfies (ρ, π). The inequality **(d.1)** is just the definition of consistency guarantee. The next inequality **(d.2)** is where the novel anti-visibility relation, introduced previously, comes into play. While the consistency guarantee (ρ, π) expresses when arbitration induces transactions related by visibility, the inequality **(d.2)** expresses when anti-visibility induces transactions related by arbitration. To emphasise this correspondence, we call the inequality **(d.2)** *co-axiom* induced by (ρ, π). Later in this section, we show how by exploiting the co-axiom induced by several consistency guarantees, we can derive critical cycles of several consistency models.

Proof of Inequality (d.2). Assume $\mathcal{X} \in \mathsf{Executions}(\{(\rho, \pi)\})$. Let $T, T', S', S \in \mathcal{T}$ be such that $T \neq S$, $T \xrightarrow{\pi(\mathsf{VIS})} T' \xrightarrow{\overline{\mathsf{VIS}^{-1}}} S' \xrightarrow{\rho(\mathsf{VIS})} S$. Because AR is total, either $S \xrightarrow{\mathsf{AR}} T$ or $T \xrightarrow{\mathsf{AR}} S$. However, the former case is not possible. If so, we would have $S' \xrightarrow{\rho(\mathsf{VIS})} S \xrightarrow{\mathsf{AR}} T \xrightarrow{\pi(\mathsf{VIS})} T'$.

because $\mathcal{X} \in \mathsf{Executions}(\{(\rho, \pi)\})$, by the inequality **(d.1)**, it would follow that $S' \xrightarrow{\mathsf{VIS}} T'$, contradicting the assumption that $T' \xrightarrow{\overline{\mathsf{VIS}^{-1}}} S'$. Therefore, it has to be $T \xrightarrow{\mathsf{AR}} S$. ◀

The last inequalities **(d.3)** and **(d.4)** in Figure 3**(d)** show that anti-visibility edges of $\mathcal{X}$ are also induced by the consistency guarantee (ρ, π). We prove them formally in [17, Appendix C], where we also illustrate some of their applications.

Applications. We employ the algebraic laws of Figure 3 to derive Σ-critical cycles for arbitrary x-specifications, using the methodology explained previously: given a x-specification Σ and an abstract execution $\mathcal{X}$, we characterise a subset of $\mathsf{AR}_{\mathcal{X}}$ as a relation R_{G} built from the dependencies in $\mathsf{graph}(\mathcal{X})$ and relations of the form $[o]$, where $o \in \mathsf{Op}$. Because $R_{\mathsf{G}} \subseteq \mathsf{AR}_{\mathcal{X}}$, we conclude that R_{G} is acyclic.

The inequalities **(c.1)**, **(c.6)** and **(c.2)** ensure that we can always include write-read and write-write dependencies in the relation R_{G} above. Because of inequalities **(c.3)** and **(d.2)** (among others), we can include in R_{G} also relations that involve anti-dependencies. The following result shows how this methodology can be applied to serialisability. We use the notation $R_1 \overset{(\mathbf{eq})}{\subseteq} R_2$ to denote that the inequality $R_1 \subseteq R_2$ follows from (**eq**).

▶ **Theorem 12.** *For all $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{SER}})$, the relation $(\mathsf{WR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}})$ is acyclic.*

Proof. Recall that $\Sigma_{\mathsf{SER}} = \{(\rho_{\mathsf{Id}}, \rho_{\mathsf{Id}})\}$, where $\rho_{\mathsf{Id}}(\_) = \mathsf{Id}$. We have

$$\mathsf{RW}_{\mathcal{X}} \overset{\mathbf{(b.6)}}{\subseteq} \mathsf{RW}_{\mathcal{X}} \backslash \mathsf{Id} \overset{\mathbf{(c.3)}}{\subseteq} \overline{\mathsf{VIS}_{\mathcal{X}}^{-1}} \backslash \mathsf{Id} = (\rho_{\mathsf{Id}}(\mathsf{VIS}_{\mathcal{X}}) \,;\, \overline{\mathsf{VIS}_{\mathcal{X}}^{-1}} \,;\, \rho_{\mathsf{Id}}(\mathsf{VIS}_{\mathcal{X}})) \backslash \mathsf{Id} \overset{\mathbf{(d.2)}}{\subseteq} \mathsf{AR}_{\mathcal{X}} \tag{1}$$

$$(\mathsf{WR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}}) \overset{\mathbf{(c.1,c.6)}}{\subseteq} (\mathsf{AR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}}) \overset{\mathbf{(c.2)}}{\subseteq} (\mathsf{AR}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}}) \overset{(1)}{\subseteq} \mathsf{AR}_{\mathcal{X}} \tag{2}$$

$$(\mathsf{WR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \mathsf{RW}_{\mathcal{X}})^+ \cap \mathsf{Id} \overset{(2)}{\subseteq} \mathsf{AR}_{\mathcal{X}}^+ \cap \mathsf{Id} \overset{\mathbf{(c.5)}}{\subseteq} \mathsf{AR}_{\mathcal{X}} \cap \mathsf{Id} \overset{\mathbf{(c.12)}}{\subseteq} \varnothing.$$ ◀

Along the lines of the proof of Theorem 12, we can characterise Σ-critical cycles for an arbitrary x-specification Σ. Below, we show how to apply our methodology to derive Σ_{RB}-critical cycles.

▶ **Theorem 13.** *Let $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{RB}})$. Say that a $\mathsf{RW}_{\mathcal{X}}$ edge in a cycle of $\mathsf{graph}(\mathcal{X})$ is* protected *if its endpoints are connected to serialisable transactions via a sequence of $\mathsf{WR}_{\mathcal{X}}$ edges. Then all cycles in $\mathsf{graph}(\mathcal{X})$ have at least one unprotected $\mathsf{RW}_{\mathcal{X}}$ edge. Formally, let $\Vdash \mathsf{RW}_{\mathcal{X}} \dashv\!\Vdash$ be $([\mathtt{SerTx}] \,;\, (\mathsf{WR}_{\mathcal{X}})^* \,;\, \mathsf{RW}_{\mathcal{X}} \,;\, (\mathsf{WR}_{\mathcal{X}})^* \,;\, [\mathtt{SerTx}])$. Then $(\mathsf{WR}_{\mathcal{X}} \cup \mathsf{WW}_{\mathcal{X}} \cup \Vdash \mathsf{RW}_{\mathcal{X}} \dashv\!\Vdash)$ is acyclic.*

$$\begin{cases} \mathsf{WR} \subseteq X_V \quad (\text{V1}) & X_V \;;\; X_V \subseteq X_V \quad (\text{V2}) & \bigcup_{\{x \mid (\rho_x, \rho_x) \in \Sigma\}} \mathsf{WW}(x) \subseteq X_V \quad (\text{V3}) \\ & & \rho(X_V) \;;\; X_A \;;\; \pi(X_V) \subseteq X_V \quad (\text{V4}) \\ \mathsf{WW} \subseteq X_A \quad (\text{A1}) & X_V \subseteq X_A \quad (\text{A2}) & \bigcup_{x \in \mathsf{Obj}} ([\mathsf{Writes}_x] \;;\; X_V \;;\; \mathsf{RW}(x)) \subseteq X_A \quad (\text{A3}) \\ & X_A \;;\; X_A \subseteq X_A \quad (\text{A4}) & (\pi(X_V) \;;\; X_N \;;\; \rho(X_V)) \backslash \mathsf{Id} \subseteq X_A \quad (\text{A5}) \\ \mathsf{RW} \subseteq X_N \quad (\text{N1}) & X_V \;;\; X_N \subseteq X_N \quad (\text{N2}) & X_N \;;\; X_V \subseteq X_N \quad (\text{N3}) \end{cases}$$

Figure 4 The system of inequalities $\mathsf{System}_\Sigma(\mathcal{G})$ for the simple consistency model Σ and the dependency graph $\mathcal{G} = (\mathcal{T}, \mathsf{WR}, \mathsf{WW}, \mathsf{RW})$.

Proof. It suffices to prove that $\Vdash \mathsf{RW}_\mathcal{X} \dashv\Vdash \subseteq \mathsf{AR}_\mathcal{X}$. The rest of the proof is similar to the one of Theorem 12. We recall that $\Sigma_{\mathsf{RB}} = \{(\rho_S, \rho_S)\}$, where $\rho_S(\_) = [\mathtt{SerTx}]$.

$$\mathsf{WR}^*_\mathcal{X} \;;\; \mathsf{RW}_\mathcal{X} \;;\; \mathsf{WR}^*_\mathcal{X} \overset{\textbf{(c.1,c.4)}}{\subseteq} \mathsf{VIS}_\mathcal{X}? \;;\; \mathsf{RW}_\mathcal{X} \;;\; \mathsf{VIS}_\mathcal{X}? \overset{\textbf{(b.6)}}{\subseteq} \mathsf{VIS}_\mathcal{X}? \;;\; (\mathsf{RW}_\mathcal{X} \backslash \mathsf{Id}) \;;\; \mathsf{VIS}_\mathcal{X}? \overset{\textbf{(c.3)}}{\subseteq}$$

$$\mathsf{VIS}_\mathcal{X}? \;;\; (\overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id}) \;;\; \mathsf{VIS}_\mathcal{X}? \subseteq ((\overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id}) \cup (\mathsf{VIS}_\mathcal{X} \;;\; \overline{\mathsf{VIS}^{-1}_\mathcal{X}})) \;;\; \mathsf{VIS}_\mathcal{X}? \overset{\textbf{(c.11)}}{\subseteq}$$

$$((\overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id}) \cup (\mathsf{VIS}_\mathcal{X} \;;\; \overline{\mathsf{VIS}^{-1}_\mathcal{X}}) \backslash \mathsf{Id}) \;;\; \mathsf{VIS}_\mathcal{X}? \overset{\textbf{(c.8)}}{\subseteq} (\overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id}) \;;\; \mathsf{VIS}_\mathcal{X}? \overset{\textbf{(c.10,c.9)}}{\subseteq} \overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id} \quad (3)$$

$$[\mathtt{SerTx}] \;;\; (\overline{\mathsf{VIS}^{-1}_\mathcal{X}} \backslash \mathsf{Id}) \;;\; [\mathtt{SerTx}] \overset{\textbf{(a.3,a.4)}}{=} ([\mathtt{SerTx}] \;;\; \overline{\mathsf{VIS}^{-1}_\mathcal{X}} \;;\; [\mathtt{SerTx}]) \backslash \mathsf{Id} =$$

$$(\rho_S(\mathsf{VIS}_\mathcal{X}) \;;\; \overline{\mathsf{VIS}^{-1}_\mathcal{X}} \;;\; \rho_S(\mathsf{VIS}_\mathcal{X})) \backslash \mathsf{Id} \overset{\textbf{(d.2)}}{\subseteq} \mathsf{AR}_\mathcal{X} \quad (4)$$

$$\Vdash \mathsf{RW}_\mathcal{X} \dashv\Vdash = [\mathtt{SerTx}] \;;\; \mathsf{WR}^*_\mathcal{X} \;;\; \mathsf{RW}_\mathcal{X} \;;\; \mathsf{WR}^*_\mathcal{X} \;;\; [\mathtt{SerTx}] \overset{(3,4)}{\subseteq} \mathsf{AR}_\mathcal{X}. \quad \blacktriangleleft$$

We remark that our characterisation of Σ_{RB}-critical cycle cannot be compared to the one given in [7]. In [17, Appendix C] we show how our methodology can be applied to give a characterisation of Σ_{RB}-critical cycles that is stronger than both the one presented in Theorem 13 and the one given in [7]. We also employ our proof technique to prove both known and new derivations of critical cycles for other x-specifications.

5 Characterisation of Simple Consistency Models

We now turn our attention to the *Strong Correspondence Problem* presented in Section 4. Given a x-specification $\Sigma = \{(\rho_1, \pi_1), \cdots, (\rho_n, \pi_n)\}$ and a dependency graph $\mathcal{G}$, we want to find a sufficient and necessary condition for determining whether $\mathcal{G} = \mathsf{graph}(\mathcal{X})$ for some $\mathcal{X} \in \mathsf{Executions}(\Sigma)$.

In this section we propose a proof technique for solving the strong correspondence problem. This technique applies to a particular class of x-specifications, which we call *simple* x-specifications. This class includes several of the consistency models we have presented.

Characterisation of Simple x-specifications. Recall that for each $x \in \mathsf{Obj}$, the function ρ_x of an abstract execution $\mathcal{X}$ is defined as $\rho_x(\_) = [\mathsf{Writes}_x]$, and the associated axiom is $[\mathsf{Writes}_x] \;;\; \mathsf{AR}_\mathcal{X} \;;\; [\mathsf{Writes}_x] \subseteq \mathsf{VIS}_\mathcal{X}$.

▶ **Definition 14.** A x-specification Σ is *simple* if there exists a consistency guarantee (ρ, π) such that $\Sigma \subseteq \{(\rho, \pi)\} \cup \{(\rho_x, \rho_x)\}_{x \in \mathsf{Obj}}$.

That is, a simple x-specification Σ contains at most one consistency guarantee, beside those of the form (ρ_x, ρ_x) which express the write-conflict detection for some object $x \in \mathsf{Obj}$. Among the x-specifications that we have presented in this paper, the only non-simple one is $\Sigma_{\mathsf{SI+SER}}$.

For simple x-specifications, it is possible to solve the strong correspondence problem. Fix a simple x-specification $\Sigma \subseteq \{(\rho, \pi)\} \cup \{(\rho_x, \rho_x) \mid x \in \mathsf{Obj}\}$ and a dependency graph $\mathcal{G}$. We define a system of inequalities $\mathsf{System}_\Sigma(\mathcal{G})$ in three unknowns X_V, X_A and X_N, and depicted in Figure 4 (the inequalities (V4) and (A5) are included in the system if and only if $(\rho, \pi) \in \Sigma$). These unknowns correspond to subsets of the visibility, arbitration and anti-visibility relations of the abstract execution $\mathcal{X} \in \mathsf{Executions}(\Sigma)$, with underlying dependency graph $\mathcal{G}$, that we wish to find. Note that each one of the inequalities of $\mathsf{System}_\Sigma(\mathcal{G})$, with the exception of (V3), follows the structure of one of the algebraic laws from Figure 3. We prove that, in order to ensure that the abstract execution $\mathcal{X}$ exists, it is sufficient to find a solution of $\mathsf{System}_\Sigma(\mathcal{G})$ whose X_A-component is acyclic. In particular, this is true if and only if the X_A-component of the smallest solution[4] of $\mathsf{System}_\Sigma(\mathcal{G})$ is acyclic.

▶ **Theorem 15.**

Soundness: *for any* $\mathcal{X} \in \mathsf{Executions}(\Sigma)$ *such that* $\mathsf{graph}(\mathcal{X}) = \mathcal{G}$*, the triple* $(X_V = \mathsf{VIS}_\mathcal{X}, X_A = \mathsf{AR}_\mathcal{X}, X_N = \overline{\mathsf{VIS}_\mathcal{X}^{-1}})$ *is a solution of* $\mathsf{System}_\Sigma(\mathcal{G})$*,*

Completeness: *Let* $(X_V = \mathsf{VIS}_0, X_A = \mathsf{AR}_0, X_N = \mathsf{AntiVIS}_0)$ *be the smallest solution of* $\mathsf{System}_\Sigma(\mathcal{G})$*. If* AR_0 *is acyclic, then there exists an abstract execution* $\mathcal{X}$ *such that* $\mathcal{X} \in \mathsf{Executions}(\Sigma)$ *and* $\mathsf{graph}(\mathcal{X}) = \mathcal{G}$*.* ◀

Note that the relation AR_0 need not to be total in the completeness direction of Theorem 15.

Before discussing the proof of Theorem 15, we show how it can be used to prove the equivalence of a x-specification and a g-specification. We give a proof of Theorem 11(3). Theorems 11(1) and 11(2) can be proved similarly, and their proof is given in [17, Appendix D].

Proof Sketch of Theorem 11(3). Recall that $\Delta_{\mathsf{PSI}} = \{\delta_{\mathsf{PSI}_0}\} \cup \{\delta_{\mathsf{PSI}(x)}(\mathcal{G}) \mid x \in \mathsf{Obj}\}$, where $\delta_{\mathsf{PSI}_0}(\mathcal{G}) = (\mathsf{WR}_\mathcal{G} \cup \mathsf{WW}_\mathcal{G})^+$, $\delta_{\mathsf{PSI}(x)}(\mathcal{G}) = ((\mathsf{WR}_\mathcal{G} \cup \mathsf{WW}_\mathcal{G})^* \,;\, \mathsf{RW}_\mathcal{G}(x))^+$. In [17, Appendix D] we prove that $\mathsf{Graphs}(\Delta_{\mathsf{PSI}}) = \mathsf{Graphs}(\{\delta_{\mathsf{PSI}}\})$, where

$$\delta_{\mathsf{PSI}}(\mathcal{G}) = (\mathsf{WR}_\mathcal{G} \cup \mathsf{WW}_\mathcal{G})^+ \cup \bigcup_{x \in \mathsf{Obj}} ([\mathsf{Writes}_x] \,;\, (\mathsf{WR}_\mathcal{G} \cup \mathsf{WW}_\mathcal{G})^* \,;\, \mathsf{RW}_\mathcal{G}(x))^+ .$$

Therefore, it suffices to prove that $\mathsf{modelOf}(\Sigma_{\mathsf{PSI}}) = \mathsf{modelOf}(\{\delta_{\mathsf{PSI}}\})$:

$\mathsf{modelOf}(\Sigma_{\mathsf{PSI}}) \subseteq \mathsf{modelOf}(\{\delta_{\mathsf{PSI}}\})$: given $\mathcal{X} \in \mathsf{Executions}(\Sigma_{\mathsf{PSI}})$, and let $\mathcal{G} := \mathsf{graph}(\mathcal{X})$, we need to show that $\delta_{\mathsf{PSI}}(\mathcal{G}) \cap \mathsf{Id} = \varnothing$. The proof follows the style of Theorems 12 and 13; details can be found in [17, Appendix C],

$\mathsf{modelOf}(\{\delta_{\mathsf{PSI}}\}) \subseteq \mathsf{modelOf}(\Sigma_{\mathsf{PSI}})$: given $\mathcal{G} \in \mathsf{Graphs}(\{\delta_{\mathsf{PSI}}\})$, let $\mathsf{VIS}_\mathcal{G} = (\mathsf{WR} \cup \mathsf{WW})^+$; It is immediate to prove that the triple $(X_V = \mathsf{VIS}_\mathcal{G}, X_A = \delta_{\mathsf{PSI}}(\mathcal{G}), X_N = \mathsf{VIS}_\mathcal{G}? \,;\, \mathsf{RW} \,;\, \mathsf{VIS}_\mathcal{G}?)$ is a solution of $\mathsf{System}_{\Sigma_{\mathsf{PSI}}}(\mathcal{G})$. Because $\delta_{\mathsf{PSI}}(\mathcal{G})$ is acyclic, if we take the smallest solution $(X_V = \_, X_A = \mathsf{AR}_\mathcal{G}, X_N = \_)$ of $\mathsf{System}_\Sigma(\mathcal{G})$, then $\mathsf{AR}_\mathcal{G} \subseteq \delta_{\mathsf{PSI}}(\mathcal{G})$, hence $\mathsf{AR}_\mathcal{G}$ is acyclic. By Theorem 15, there exists an abstract execution $\mathcal{X} \in \mathsf{Executions}(\mathsf{PSI})$ such that $\mathsf{graph}(\mathcal{X}) = \mathcal{G}$, and in particular $\mathcal{T}_\mathcal{X} = \mathcal{T}_\mathcal{G}$. ◀

[4] A solution $(X_V = \mathsf{VIS}, X_A = \mathsf{AR}, X_N = \mathsf{AntiVIS})$ is smaller than another one $(X_V = \mathsf{VIS}', X_A = \mathsf{AR}', X_N = \mathsf{AntiVIS}')$ iff $\mathsf{VIS} \subseteq \mathsf{VIS}', \mathsf{AR} \subseteq \mathsf{AR}'$ and $\mathsf{AntiVIS} \subseteq \mathsf{AntiVIS}'$.

We now turn our attention to the proof of Theorem 15. The proof of the soundness direction is straightforward.

Proof of Theorem 15 (Soundness). Let $\mathcal{X} \in \mathsf{Executions}(\Sigma)$, and define $\mathcal{G} := \mathsf{graph}(\mathcal{X})$. To show that the triple $(X_V = \mathsf{VIS}_{\mathcal{X}}, X_A = \mathsf{AR}_{\mathcal{X}}, X_N = \overline{\mathsf{VIS}_{\mathcal{X}}^{-1}})$ is a solution of $\mathsf{System}_\Sigma(\mathcal{G})$, we need to show that all the inequalities from said system are satisfied, when the unknowns X_A, X_V, X_N are replaced with $\mathsf{VIS}_{\mathcal{X}}, \mathsf{AR}_{\mathcal{X}}, \overline{\mathsf{VIS}_{\mathcal{X}}^{-1}}$, respectively. In practice, all the inequalities, with the exception of (V3), follow from the algebraic laws of Figure 3. Let us prove that (V3) is also valid: for any $(\rho_x, \rho_x) \in \Sigma$ we have that

$$\mathsf{WW}_{\mathcal{X}}(x) \overset{\textbf{(b.2)}}{=} [\mathsf{Writes}_x] \,;\, \mathsf{WW}_{\mathcal{X}}(x) \,;\, [\mathsf{Writes}_x] \overset{\textbf{(c.2)}}{\subseteq} [\mathsf{Writes}_x] \,;\, \mathsf{AR}_{\mathcal{X}} \,;\, [\mathsf{Writes}_x] \overset{\textbf{(d.1)}}{\subseteq} \mathsf{VIS}_{\mathcal{X}}.$$

◀

The proof of the completeness direction of Theorem 15 is much less straightforward. Let $(X_V = \mathsf{VIS}_0, X_A = \mathsf{AR}_0, X_N = \mathsf{AntiVIS}_0)$ be the smallest solution of $\mathsf{System}_\Sigma(\mathcal{G})$. Assume that AR_0 is acyclic. The challenge is that of constructing a valid abstract execution $\mathcal{X}$, i.e. whose arbitration order is total, from the dependencies in $\mathcal{G}$, that is included in $\mathsf{Executions}(\Sigma)$. We do this incrementally: at intermediate stages of the construction we get structures similar to abstract executions, but where the arbitration order can be partial.

▶ **Definition 16.** A *pre-execution* $\mathcal{P} = (\mathcal{T}_{\mathcal{G}}, \mathsf{VIS}, \mathsf{AR})$ is a tuple that satisfies all the constraints of abstract executions, except that AR is not necessarily total, although AR is still required to be total over the set Writes_x for every object x.

The notation adopted for abstract executions naturally extends to pre-executions; also, for any pre-execution $\mathcal{P}$, $\mathsf{graph}(\mathcal{P})$ is a well-defined dependency graph. Given a x-specification Σ, we let $\mathsf{PreExecutions}(\Sigma)$ be the set of all valid pre-executions that satisfy all the consistency guarantees in Σ.

$\mathsf{System}_\Sigma(\mathcal{G})$ is defined so that all of its solutions whose X_A-component is acyclic induce a valid pre-execution in $\mathsf{PreExecutions}(\Sigma)$ with underlying dependency graph $\mathcal{G}$.

▶ **Proposition 17.** *Let* $(X_V = \mathsf{VIS}', X_A = \mathsf{AR}', X_N = \mathsf{AntiVIS}')$ *be a solution to* $\mathsf{System}_\Sigma(\mathcal{G})$. *If* $\mathsf{AR}' \cap \mathsf{Id} = \varnothing$, *then* $\mathcal{P} = (\mathcal{T}_{\mathcal{G}}, \mathsf{VIS}', \mathsf{AR}') \in \mathsf{PreExecutions}(\Sigma)$; *moreover,* $\mathsf{graph}(\mathcal{P}) = \mathcal{G}$.

Proof Sketch. The inequalities (A1), (A2) and (A4) together with the assumption that AR_0 is acyclic, ensure that $\mathcal{P}$ is a pre-execution. In particular, (A1) ensures that AR_0 is a total relation over the set Writes_x, for any $x \in \mathsf{Obj}$. As we explain in [17, Appendix D], the inequalities (V1), (A1) and (A3) enforce the Last Write Wins policy (Definition 3). The inequality (V2) mandates that $\mathcal{P}$ respects causality. Finally, the inequalities (V3) and (V4) ensure that all the consistency guarantees in Σ are satisfied by $\mathcal{P}$. ◀

In particular, the smallest solution $(X_V = \mathsf{VIS}_0, X_A = \mathsf{AR}_0, X_N = \mathsf{AntiVIS}_0)$ of $\mathsf{System}_\Sigma(\mathcal{G})$ induces the pre-execution $(\mathcal{T}_{\mathcal{G}}, \mathsf{VIS}_0, \mathsf{AR}_0) \in \mathsf{PreExecutions}(\Sigma)$.

To construct an abstract execution $\mathcal{X} \in \mathsf{Executions}(\Sigma)$, with $\mathsf{graph}(\mathcal{X}) = \mathcal{G}$, we define a finite chain of pre-executions $\{\mathcal{P}_i,\}_{i=0}^n$, $n \geqslant 0$, as follows: **(i)** let $\mathcal{P}_0 := (\mathcal{T}_{\mathcal{G}}, \mathsf{VIS}_0, \mathsf{AR}_0)$; **(ii)** given $\mathcal{P}_i$, $i \geqslant 0$, choose two different transactions $T_i, S_i \in \mathcal{T}_{\mathcal{G}}$ (if any) that are not related by AR_i, compute the smallest solution $(X_V = \mathsf{VIS}_{i+1}, X_A = \mathsf{AR}_{i+1}, X_N = \_)$ such that $\mathsf{AR}_{i+1} \supseteq \mathsf{AR}_i \cup \{(T_i, S_i)\}$, and let $\mathcal{P}_{i+1} := (\mathcal{T}_{\mathcal{G}}, \mathsf{VIS}_{i+1}, \mathsf{AR}_{i+1})$; **(iii)** if the transactions $T_i, S_i \in \mathcal{T}_{\mathcal{G}}$ from the previous step do not exist, then let $n := i$ and terminate the construction. Because we are assuming that $\mathcal{T}_{\mathcal{G}}$ is finite, the construction of $\{\mathcal{P}_0, \cdots, \mathcal{P}_n\}$ always terminates.

To prove the completeness direction of Theorem 15, we show that all of the pre-executions $\{\mathcal{P}_0, \cdots, \mathcal{P}_n\}$ in the construction outlined above are included in $\mathsf{PreExecutions}(\Sigma)$; then, because in $\mathcal{P}_n = (\mathcal{T}_\mathcal{G}, \mathsf{VIS}_n, \mathsf{AR}_n)$ all transactions are related by AR_n, we may conclude that AR_n is total, and $\mathcal{P}_n \in \mathsf{Executions}(\Sigma)$. According to Proposition 17, it suffices to show that each of the relations $\mathsf{AR}_i, i = 0, \cdots, n$ is acyclic. However, this is not completely trivial, because of how AR_{i+1} is defined: adding one edge (T_i, S_i) in AR_{i+1} may cause more edges to be included in VIS_{i+1}, due to the inequality (V4). This in turn leads to including more edges in AR_{i+1}, thus augmenting the risk of having a cycle in AR_{i+1}.

In practice, the definition of $\mathsf{System}_\Sigma(\mathcal{G})$ ensures that this scenario does not occur.

▶ **Proposition 18.** *For* $i = 0, \cdots, n-1$, *let* $\Delta\mathsf{AR}_i := \mathsf{AR}_i? \;;\; \{(T_i, S_i)\} \;;\; \mathsf{AR}_i?$. *Then* $\mathsf{AR}_{i+1} = \mathsf{AR}_i \cup \Delta\mathsf{AR}_i$.

▶ **Corollary 19.** *For* $i = 0, \cdots, n-1$, *if* $\mathsf{AR}_i \cap \mathsf{Id} = \varnothing$, *then* $\mathsf{AR}_{i+1} \cap \mathsf{Id} = \varnothing$.

Proof. Because $\mathsf{AR}_i \cap \mathsf{Id} = \varnothing$ by hypothesis, by Proposition 18 we only need to show that $\Delta\mathsf{AR}_i \cap \mathsf{Id} = \varnothing$. If $(T, T) \in \Delta\mathsf{AR}_i$ for some $T \in \mathcal{T}_\mathcal{G}$, then it must be $T \xrightarrow{\mathsf{AR}_i?} T_i$ and $S_i \xrightarrow{\mathsf{AR}_i?} T$. It follows that $S_i \xrightarrow{\mathsf{AR}_i?} T_i$. But this contradicts the hypothesis that AR_i does not relate transactions T_i and S_i. Therefore, $(T, T) \notin \Delta\mathsf{AR}_i$ for any $T \in \mathcal{T}_\mathcal{G}$, i.e. $\Delta\mathsf{AR}_i \cap \mathsf{Id} = \varnothing$. ◀

We have now everything in place to prove Theorem 15.

Proof of Theorem 15 (Completeness). Let $\mathcal{G}$ be a dependency graph, and define the chain of pre-executions $\mathcal{P}_0 = (\mathcal{T}_\mathcal{G}, \mathsf{VIS}_0, \mathsf{AR}_0), \cdots, \mathcal{P}_n = (\mathcal{T}_\mathcal{G}, \mathsf{VIS}_n, \mathsf{AR}_n)$ as described above. We show that for any $i = 0, \cdots, n$, $\mathcal{P}_i \in \mathsf{PreExecutions}(\Sigma)$, and $\mathsf{graph}(\mathcal{P}_i) = \mathcal{G}$. Because AR_n is a total order, this implies that $\mathcal{P}_n \in \mathsf{Executions}(\Sigma)$, and $\mathsf{graph}(\mathcal{P}_n) = \mathcal{G}$, as we wanted to prove. The proof is by induction on n.

Case $i = 0$: observe that the triple $(X_V = \mathsf{VIS}_0, X_A = \mathsf{AR}_0, X_N = \_)$ corresponds to the smallest solution of $\mathsf{System}_\Sigma(\mathcal{G})$, hence AR_0 is acyclic by hypothesis. It follows from Proposition 17 that $\mathcal{P}_0 \in \mathsf{PreExecutions}(\Sigma)$, and $\mathsf{graph}(\mathcal{P}_0) = \mathcal{G}$,

Case $i > 0$: assume that $i \leqslant n$; then $i - 1 < n$, and by induction hypothesis $\mathcal{P}_{i-1} \in \mathsf{PreExecutions}(\Sigma)$. In particular, the relation AR_{i-1} is acyclic; by Corollary 19 we obtain that AR_i is acyclic. Finally, recall that the triple $(X_V = \mathsf{VIS}_i, X_A = \mathsf{AR}_i, X_N = \_)$ is a solution of $\mathsf{System}_\Sigma(\mathcal{G})$ by construction. It follows from Proposition 17 that $\mathcal{P}_i \in \mathsf{PreExecutions}(\Sigma)$, and $\mathsf{graph}(\mathcal{P}_i) = \mathcal{G}$. ◀

6 Conclusion

We have explored the connection between two different styles of specifications for weak consistency models at an algebraic level. We have proposed several laws which we applied to devise several robustness criteria for consistency models. To the best of our knowledge, this is the first generic proof technique for proving robustness criteria of weak consistency models. We have shown that, for a particular class of consistency models, our algebraic approach leads to a precise characterisation of consistency models in terms of dependency graphs.

Related Work. Abstract executions have been introduced by Burckhardt in [12] to model the behaviour of eventually consistent data-stores; They have been used to capture the behaviour of replicated data types (Gotsman et al. [13]), geo-replicated databases (Cerone et al. [15]) and non-transactional distributed storage systems (Viotti et al. [30]).

Dependency graphs have been introduced by Adya [1]; they have been used since to reason about programs running under weak consistency models. Bernardi et al., used dependency graphs to derive robustness criteria of several consistency models [7], including PSI and red-blue; in contrast with our work, the proofs there contained do not rely on a general technique. Brutschy et al. generalised the notion of dependency graphs to replicated data types, and proposed a robustness criterion for eventual consistency [10].

Weak consistency also arises in the context of shared memory systems [4]. Alglave et al., proposed the CAT language for specifying weak memory models in [4], which also specifies weak memory models as a set of irreflexive relations over data-dependencies of executions. Castellan [14], and Jeffrey et al. [21], proposed different formalisations of weak memory models via event structures. The problem of checking the robustness of applications has also been addressed for weak memory models [2, 3, 8].

The strong correspondence problem (Section 5) is also highlighted by Bouajjani et al. in [9]: there the authors emphasize the need for general techniques to identify all the *bad patterns* that can arise in dependency-graphs like structures. We solved the strong correspondence problem for SI in [16].

References

1 Atul Adya. Weak consistency: A generalized theory and optimistic implementations for distributed transactions. PhD thesis, MIT, 1999.

2 Jade Alglave, Daniel Kroening, Vincent Nimal, and Daniel Poetzl. Don't sit on the fence: A static analysis approach to automatic fence insertion. *ACM Transactions on Programming Languages Systems*, 39(2):6:1–6:38, 2017.

3 Jade Alglave and Luc Maranget. Stability in weak memory models. In *International Confence on Computer Aided Verification (CAV)*, pages 50–66, 2011.

4 Jade Alglave, Luc Maranget, and Michael Tautschnig. Herding cats: Modelling, simulation, testing, and data mining for weak memory. *ACM Transactions on Programming Languages Systems*, 36(2):7:1–7:74, 2014.

5 Peter Bailis, Alan Fekete, Ali Ghodsi, Joseph M. Hellerstein, and Ion Stoica. Scalable atomic visibility with RAMP transactions. In *2014 ACM SIGMOD International Conference on Management of Data (SIGMOD)*, pages 27–38, 2014.

6 Hal Berenson, Phil Bernstein, Jim Gray, Jim Melton, Elizabeth O'Neil, and Patrick O'Neil. A critique of ANSI SQL isolation levels. In *1995 ACM SIGMOD international conference on Management of data (SIGMOD)*, pages 1–10, 1995.

7 Giovanni Bernardi and Alexey Gotsman. Robustness against consistency models with atomic visibility. In *27th International Conference on Concurrency Theory (CONCUR)*, pages 7:1–7:15, 2016. doi:10.4230/LIPIcs.CONCUR.2016.7.

8 Ahmed Bouajjani, Egor Derevenetc, and Roland Meyer. Checking and enforcing robustness against TSO. In *23rd European Symposium on Programming (ESOP)*, pages 533–553, 2013.

9 Ahmed Bouajjani, Constantin Enea, Rachid Guerraoui, and Jad Hamza. On verifying causal consistency. In *44th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL)*, pages 626–638, 2017.

10 Lucas Brutschy, Dimitar Dimitrov, Peter Müller, and Martin Vechev. Serializability for eventual consistency: Criterion, analysis and applications. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL)*. ACM, January 2017.

11 Sebastian Burckhardt. Principles of eventual consistency. *Foundations and Trends in Programming Languages*, 1(1-2):1–150, 2014. doi:10.1561/2500000011.

12 Sebastian Burckhardt, Manuel Fahndrich, Daan Leijen, and Mooly Sagiv. Eventually consistent transactions. In *22nd European Symposium on Programming (ESOP)*, page 67–86, 2012. URL: `https://www.microsoft.com/en-us/research/publication/eventually-consistent-transactions/`.

13 Sebastian Burckhardt, Alexey Gotsman, Hongseok Yang, and Marek Zawirski. Replicated data types: specification, verification, optimality. In *41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)*, pages 271–284, 2014.

14 Simon Castellan. Weak memory models using event structures. In *Vingt-septièmes Journées Francophones des Langages Applicatifs (JFLA 2016)*, 2016.

15 Andrea Cerone, Giovanni Bernardi, and Alexey Gotsman. A framework for transactional consistency models with atomic visibility. In *26th International Conference on Concurrency Theory (CONCUR)*, pages 58–71. Dagstuhl, 2015.

16 Andrea Cerone and Alexey Gotsman. Analysing snapshot isolation. In *2016 ACM Symposium on Principles of Distributed Computing (PODC)*, pages 55–64, 2016.

17 Andrea Cerone, Alexey Gotsman, and Hongseok Yang. Algebraic laws for weak consistency (extended version). URL: `https://arxiv.org/abs/1702.06028`.

18 Andrea Cerone, Alexey Gotsman, and Hongseok Yang. Transaction chopping for parallel snapshot isolation. In *29th International Symposium on Distributed Computing (DISC)*, pages 388–404, 2015.

19 Alan Fekete, Dimitrios Liarokapis, Elizabeth O'Neil, Patrick O'Neil, and Dennis Shasha. Making snapshot isolation serializable. *ACM Transactions on Database Systems*, 30(2):492–528, 2005.

20 Alexey Gotsman and Hongseok Yang. Composite replicated data types. In Jan Vitek, editor, *24th European Symposium on Programming (ESOP)*, pages 585–609, 2015.

21 Alan Jeffrey and James Riely. On thin air reads towards an event structures model of relaxed memory. In *31st ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 759–767, 2016.

22 Dexter Kozen and Frederick Smith. Kleene algebra with tests: Completeness and decidability. In *10th International Workshop on Computer Science Logic (CSL)*, pages 244–259. Springer-Verlag, 1996.

23 Cheng Li, Daniel Porto, Allen Clement, Johannes Gehrke, Nuno Preguiça, and Rodrigo Rodrigues. Making geo-replicated systems fast as possible, consistent when necessary. In *10th USENIX Symposium on Operating Systems Design and Implementation (OSDI)*, pages 265–278, 2012.

24 Wyatt Lloyd, Michael J. Freedman, Michael Kaminsky, and David G. Andersen. Don't settle for eventual: scalable causal consistency for wide-area storage with COPS. In *23rd ACM Symposium on Operating Systems Principles (SOSP)*, pages 401–416, 2011.

25 Microsoft SQL server documentation, Set Transaction Isolation Level. URL: `https://docs.microsoft.com/en-us/sql/t-sql/statements/set-transaction-isolation-level-transact-sql`.

26 M. Saeida Ardekani, P. Sutra, and M. Shapiro. Non-monotonic snapshot isolation: Scalable and strong consistency for geo-replicated transactional systems. In *32nd International Symposium on Reliable Distributed Systems (SRDS)*, pages 163–172, 2013.

27 D. Shasha, F. Llirbat, E. Simon, and P. Valduriez. Transaction chopping: Algorithms and performance studies. *ACM Trans. Database Syst.*, 20(3):325–363, 1995.

28 Y. Sovran, R. Power, M. K. Aguilera, and J. Li. Transactional storage for geo-replicated systems. In *23rd ACM Symposium on Operating Systems Principles (SOSP)*, pages 385–400, 2011.

29 Douglas B Terry, Alan J Demers, Karin Petersen, Mike J Spreitzer, Marvin M Theimer, and Brent B Welch. Session guarantees for weakly consistent replicated data. In *3rd*

International Conference on Parallel and Distributed Information Systems (PDIS), pages 140–149. IEEE, 1994.

30 Paolo Viotti and Marko Vukolić. Consistency in non-transactional distributed storage systems. *ACM Computing Surveys*, 49(1):19:1–19:34, 2016. doi:10.1145/2926965.

31 Kamal Zellag and Bettina Kemme. Consistency anomalies in multi-tier architectures: Automatic detection and prevention. *The VLDB Journal*, 23(1):147–172, 2014.

Algorithms to Compute Probabilistic Bisimilarity Distances for Labelled Markov Chains*

Qiyi Tang[1] and Franck van Breugel[2]

1 **DisCoVeri Group, Department of Electrical Engineering and Computer Science, York University, Toronto**
2 **DisCoVeri Group, Department of Electrical Engineering and Computer Science, York University, Toronto**

Abstract

In the late nineties, Desharnais, Gupta, Jagadeesan and Panangaden presented probabilistic bisimilarity distances on the states of a labelled Markov chain. This provided a quantitative generalisation of probabilistic bisimilarity introduced by Larsen and Skou a decade earlier. In the last decade, several algorithms to approximate and compute these probabilistic bisimilarity distances have been put forward. In this paper, we correct, improve and generalise some of these algorithms. Furthermore, we compare their performance experimentally.

1998 ACM Subject Classification D.2.4 Software/Program Verification, F.1.1 Models of Computation, G.3 Probability and Statistics

Keywords and phrases labelled Markov chain, probabilistic bisimilarity, pseudometric, policy iteration

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.27

1 Introduction

For the last three and a half decades behavioural equivalences such as bisimilarity, due to Milner [22] and Park [25], have been a cornerstone of concurrency theory. As this theory matured, more detailed models of concurrent systems have been developed, such as models with probabilities. For these enriched models, behavioural equivalences were proposed, such as probabilistic bisimilarity due to Larsen and Skou [21].

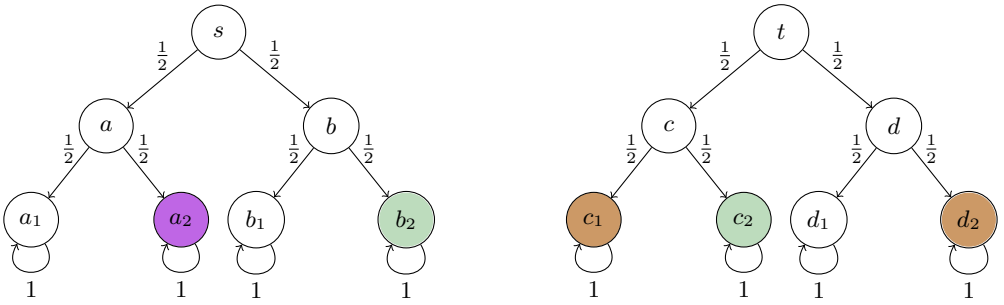
Although these behavioural equivalences allow us to reason about the behaviour of these models, they have one drawback: they are not robust. That is, small changes in the probabilities may cause equivalent states to becomes inequivalent or vice versa. Since the probabilities are usually obtained experimentally and, therefore are often just an approximation, this lack of robustness is a serious limitation of behavioural equivalences for these models with probabilities. This lack of robustness was first pointed out by Giacalone, Jou and Smolka [13]. They suggested generalising behavioural equivalences, which assign to each pair of states a Boolean, to behavioural pseudometrics, which assign to each pair of states a nonnegative real number.

The distance of a pair of states provides a quantitative measure of their behavioural similarity. The smaller this distance, the more alike the states behave. Distance zero captures that the states behave exactly the same, that is, they are behaviourally equivalent.

* Part of this work was done while the second author was visiting the Simons Institute for the Theory of Computing. This research was also supported by the Natural Sciences and Engineering Research Council of Canada.

For a more detailed discussion of the merits of behavioural pseudometrics, we refer the reader to, for example, [24, Chapter 8].

Labelled Markov chains are often used to model systems with probabilistic behaviour. An example of such a Markov chain is depicted below. In a labelled Markov chain, each state has a label. In the example below, the label is represented by the colour of the state. These labels are used to capture that particular properties of interest hold in some states and do not hold in other states.



Several behavioural pseudometrics for labelled Markov chains have been proposed, the probabilistic bisimilarity pseudometric due to Desharnais, Gupta, Jagadeesan and Panangaden [12] being the most notable one. In this paper, we focus on this probabilistic bisimilarity pseudometric. Some of the probabilistic bisimilarity distances for the above labelled Markov chain can be found in the table below. For some historical background on this behavioural pseudometric we refer the reader to [6, Section 1].

	s	t	a	b	c	d
s	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$
t	$\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$
a	$\frac{3}{4}$	$\frac{3}{4}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
b	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
c	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$
d	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

In order to exploit behavioural pseudometrics such as the probabilistic bisimilarity pseudometric, it is essential to be able to approximate or compute these behavioural distances. The first algorithm to approximate these distances was presented by Van Breugel, Sharma and Worrell in [5]. In their algorithm, the distance between states s and t, denoted $\delta(s,t)$, is computed as follows. Since $\delta(s,t) < q$, for some rational q, can be expressed in the existential fragment of the first order theory over the reals as shown by Van Breugel et al., and this theory is decidable as shown by Tarski [29], one can use binary search to approximate $\delta(s,t)$. The satisfiability problem for the existential fragment of the first order theory over the reals can be solved in polynomial space [7].

Subsequently, Chen, Van Breugel and Worrell [8] presented a polynomial time algorithm to compute the distances. They showed that the distances are rational and that those distances can be computed by means of Khachiyan's ellipsoid method [17]. In particular, they showed that the distance function can be expressed as the solution of a linear program. In this case, the separation algorithm, which is an integral part of the ellipsoid method, boils down to solving a minimum cost flow problem. The network simplex algorithm solves the latter problem in polynomial time [23].

It is well known that the ellipsoid method is simpler when the feasible region is bounded and full-dimensional (see, for example, [26, Chapter 13]). In Section 4 we show that the feasible region of the linear program used by Chen et al. to compute the probabilistic bisimilarity distances is bounded and full-dimensional. As a consequence, we can use the simpler version of the ellipsoid method to compute the probabilistic bisimilarity distances.

Bacci, Bacci, Larsen and Mardare [1] put forward yet another algorithm to compute the probabilistic bisimilarity distances. Their algorithm can be viewed as a basic algorithm, enhanced with an optimization. The key idea behind this optimization is to compute the distances on-the-fly. We will come back to this optimization later, but we will first focus on the basic algorithm.

In [28], we show that a slight modification of the basic algorithm of Bacci et al. can be seen as an incarnation of simple policy iteration, also known as sequential policy iteration. In particular, we construct a transformation mapping each labelled Markov chain to a simple stochastic game, a simplification of Shapley's stochastic games [27] due to Condon [9]. The vertices of the simple stochastic game represent pairs of states of the labelled Markov chain. Furthermore, the probabilistic bisimilarity distance of two states of the labelled Markov chain is shown to be equal to the value of the corresponding vertex of the simple stochastic game.

A variety of algorithms has been developed to compute the value function of a simple stochastic game. Several of these algorithms use policy iteration. As long as there exists a choice in the strategy, also known as a policy, of a player that is not locally optimal, switch that choice for one that is locally optimal. Hoffman and Karp [15] introduced a policy iteration algorithm for stochastic games in which all non-optimal choices are switched in each iteration. Condon [10] presented a similar algorithm, known as simple policy iteration, that switches only one non-optimal choice per iteration.

In Section 5 we present our simple policy iteration algorithm. Furthermore, we modify this algorithm to mimic general policy iteration a la Karp and Hoffman. We prove both algorithms correct. In Section 6 we consider the on-the-fly optimization due to Bacci et al. [1]. This optimization boils down to considering partial policies. That is, a player only makes choices for an appropriate subset of the vertices. We call it partial policy iteration. As we will show, Bacci et al. do not always consider partial policies that are defined for sufficiently many vertices. We propose a modification of their algorithm and prove it correct. A partial variant that mimics general policy iteration can be developed and proved correct similarly.

In [28], we show that the number of iterations of the simple policy iteration algorithm is at least exponential in the number of states of the labelled Markov chain. In Section 7 we prove that this is also the case for the simple partial policy iteration algorithm, even if we are only interested in the distance of a single pair of states.

The algorithms considered in this paper are the one which applies the first order theory over the reals, the ellipsoid method, simple policy iteration, general policy iteration, simple partial policy iteration and general partial policy iteration. To compare the performance of these six algorithms to approximate and compute probabilistic bisimilarity distances for labelled Markov chain, we ran several experiments. All six algorithms were implemented in Java. These implementations were run on a number of labelled Markov chains, which model well-known randomized algorithms and were obtained from examples of probabilistic model checkers such as PRISM [20] and jpf-probabilistic [31]. These experiments and the results are discussed in Section 8.

The contributions of this paper are the following. In Section 4 we show that the feasible region of the linear programming of Chen et al. describing the probabilistic bisimilarity dis-

tances is bounded and full-dimensional. As a consequence, we can resort to a simpler version of the ellipsoid method to compute the distances. We study the on-the-fly optimization of the algorithm of Bacci et al. in Section 6 and show that it does not always consider sufficiently many states. We modify their optimization and prove our modification correct. We consider this our main contribution. In Section 7 we prove an exponential lower bound for the simple partial policy iteration algorithm. Finally, by means of experiments we compare the performance of the different algorithms to approximate and compute the probabilistic bisimilarity distances in Section 8.

2 Labelled Markov Chains and Probabilistic Bisimilarity

We start by reviewing the model of interest, labelled Markov chains, and its most well known behavioural equivalence, probabilistic bisimilarity due to Larsen and Skou [21]. We denote the set of probability distributions on a finite set S by $\mathit{Distr}(S)$.

▶ **Definition 1.** A labelled Markov chain is a tuple $\langle S, L, \tau, \ell \rangle$ consisting of
- a finite set S of states,
- a finite set L of labels,
- a transition function $\tau : S \to \mathit{Distr}(S)$, and
- a labelling function $\ell : S \to L$.

We restrict our attention to labelled Markov chains with rational transition probabilities. For the remainder of this paper, we fix such a labelled Markov chain. In order to characterize probabilistic bisimilarity, we first introduce the notion of a coupling of probability distributions.

▶ **Definition 2.** Let $\mu, \nu \in \mathit{Distr}(S)$. The set $\Omega(\mu, \nu)$ of couplings[1] of μ and ν is defined by

$$\Omega(\mu,\nu) = \left\{ \omega \in \mathit{Distr}(S \times S) \;\middle|\; \sum_{t \in S} \omega(s,t) = \mu(s) \wedge \sum_{s \in S} \omega(s,t) = \nu(t) \right\}.$$

The set $\Omega(\mu, \nu)$ is a convex polytope. We denote its vertices by $V(\Omega(\mu, \nu))$. Generally, the set $\Omega(\mu, \nu)$ is infinite, but the set $V(\Omega(\mu, \nu))$ is finite (see, for example, [18, page 259]). The following characterization of probabilistic bisimilarity, in terms of couplings, is due to Jonsson and Larsen [16, Theorem 4.6]. For a discussion of the notion of a coupling we refer the reader to, for example, [4].

▶ **Definition 3.** An equivalence relation $R \subseteq S \times S$ is a probabilistic bisimulation if for all $(s, t) \in R$, $\ell(s) = \ell(t)$ and there exists $\omega \in \Omega(\tau(s), \tau(t))$ such that $\mathrm{support}(\omega) \subseteq R$, where $\mathrm{support}(\omega) = \{ (u, v) \in S \times S \mid \omega(u, v) > 0 \}$. Probabilistic bisimilarity, denoted $\sim$, is the largest probabilistic bisimulation.

3 Probabilistic Bisimilarity Distances

The probabilistic bisimilarity pseudometric of Desharnais et al. [12] maps each pair of states of a labelled Markov chain to a distance, which an element of the unit interval $[0, 1]$. Hence, the pseudometric is a function from $S \times S$ to $[0, 1]$, that is, an element of $[0, 1]^{S \times S}$. Such a function is a pseudometric if it satisfies the following three properties: for all $s, t, u \in S$,

[1] In the literature, different terminology is used. For example, in [1] these are called matchings.

$d(s,s) = 0$, $d(s,t) = d(t,s)$ and $d(s,u) \leq d(s,t) + d(t,u)$. As we will discuss below, the probabilistic bisimilarity pseudometric can be defined as a fixed point of the following function.

▶ **Definition 4.** The function $\Delta : [0,1]^{S\times S} \to [0,1]^{S\times S}$ is defined by

$$\Delta(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \min\limits_{\omega \in \Omega(\tau(s),\tau(t))} \sum\limits_{u,v\in S} \omega(u,v)\, d(u,v) & \text{otherwise} \end{cases}$$

To define the probabilistic bisimilarity pseudometric as a fixed point of Δ we allude to the Knaster-Tarski fixed point theorem [30].

▶ **Theorem 5.** *Let X be a complete lattice and $f : X \to X$ a monotone function.*
(a) *The least fixed point of f is $\bigsqcap\{\, x \in X \mid f(x) \sqsubseteq x \,\}$.*
(b) *The greatest fixed point of f is $\bigsqcup\{\, x \in X \mid x \sqsubseteq f(x) \,\}$.*

For background material on order theory we refer the reader to, for example, [11]. To apply the above theorem, we need to define an order on $[0,1]^{S\times S}$. For d, $e \in [0,1]^{S\times S}$ we write $d \sqsubseteq e$ if $d(s,t) \leq e(s,t)$ for all s, $t \in S$. The set $[0,1]^{S\times S}$ endowed with the order $\sqsubseteq$ forms a complete lattice. Since Δ is a monotone function, we can conclude from the above Knaster-Tarski fixed point theorem that Δ has a least fixed point. We denote this fixed point by δ, which is a pseudometric. This is the probabilistic bisimilarity pseudometric introduced by Desharnais et al.

The probabilistic bisimilarity pseudometric δ provides a quantitative generalisation of probabilistic bisimilarity as captured by the following result which can be found in [12, Theorem 1].

▶ **Theorem 6.** *For all s, $t \in S$, $s \sim t$ if and only if $\delta(s,t) = 0$.*

We conclude this section with an alternative characterization of the probabilistic bisimilarity pseudometric δ which is a small variation on the characterization that can be found in [8, Theorem 8]. It provides the basis for most of the algorithms we will discuss later in this paper.

Theorem 6 tells us that δ assigns to each state pair in $S^2_0 = \{\, (s,t) \in S \times S \mid s \sim t \,\}$ the distance zero. From the definition of Δ we can conclude that δ assigns to each state pair in $S^2_1 = \{\, (s,t) \in S \times S \mid \ell(s) \neq \ell(t) \,\}$ the distance one. Hence, it remains to compute the probabilistic bisimilarity distance for $S^2_? = \{\, (s,t) \in S \times S \mid \ell(s) = \ell(t) \wedge s \not\sim t \,\}$. Note that S^2_0, S^2_1 and $S^2_?$ form a partition of $S \times S$. As we mentioned in the introduction, our algorithms use policy iteration to compute these remaining distances.

▶ **Definition 7.** For a labelled Markov chain $\langle S, L, \tau, \ell \rangle$, the set $\mathcal{T}$ of total policies[2] is defined by

$$\mathcal{T} = \{\, T \in S^2_? \to \mathit{Distr}(S \times S) \mid \forall (s,t) \in S^2_? : T(s,t) \in V(\Omega(\tau(s), \tau(t))) \,\}.$$

Restricting to vertices in the above definition amounts to no loss of generality, since Theorem 9 holds also for total policies that map to arbitrary elements of the convex polytope $\Omega(\tau(s), \tau(t))$, rather than just its vertices (see [1, Remark 13]). For each $T \in \mathcal{T}$, the pair $\langle S \times S, T \rangle$ can be viewed as a Markov chain, by extending T such that

$$T(s,t)(u,v) = \begin{cases} 1 & \text{if } (u,v) = (s,t) \\ 0 & \text{otherwise} \end{cases}$$

for all $(s,t) \in (S \times S) \setminus S^2_?$.

[2] In the literature, different terminology is used. For example, in [1] these are called couplings.

▶ **Definition 8.** Let $T \in \mathcal{T}$. The function $\Gamma^T : [0,1]^{S\times S} \to [0,1]^{S\times S}$ is defined by

$$\Gamma^T(d)(s,t) = \begin{cases} 0 & \text{if } s \sim t \\ 1 & \text{if } \ell(s) \neq \ell(t) \\ \sum_{u,v\in S} T(s,t)(u,v)\, d(u,v) & \text{otherwise} \end{cases}$$

Since $[0,1]^{S\times S}$ is a complete lattice and Γ^T is a monotone function [1], we can conclude from the Knaster-Tarski fixed point theorem that Γ^T has a least fixed point. We denote this fixed point by γ^T. Note that $\gamma^T(s,t)$ captures the probability of reaching any (u,v) with $\ell(u) \neq \ell(v)$ from (s,t) in the Markov chain $\langle S \times S, T\rangle$ (see, for example, [3, Section 10.1.1] for a discussion of reachability probabilities). The probabilistic bisimilarity pseudometric δ can be characterized as follows.

▶ **Theorem 9.** $\delta = \min_{T\in\mathcal{T}} \gamma^T$.

4 The Ellipsoid Method

As shown by Chen, Van Breugel and Worrell [8], the probabilistic bisimilarity distances can be computed in polynomial time. In particular, they show that the distances can be obtained by solving a linear programming problem by means of Khachiyan's ellipsoid method [17]. It is well known that the method is simpler when the feasible region is bounded and full-dimensional (see, for example, [26, Chapter 13]). Below, we will show that the feasible region is bounded and full-dimensional in the setting of Chen et al. This feasible region is defined by

$$d(s,t) \geq 0 \tag{1}$$
$$d(s,t) \leq 1 \tag{2}$$
$$d(s,t) \leq \sum_{(u,v)\in S_?^2} d(u,v)\,\pi(u,v) + \sum_{(u,v)\in S_1^2} \pi(u,v) \tag{3}$$

for all $(s,t) \in S_?^2$ and $\pi \in V(\Omega(\tau(s),\tau(t)))$. Note that we need to decide probabilistic bisimilarity in order to define $S_?^2$ and, hence, the feasible region.

We restrict our attention to the case that the set $S_?^2$ is nonempty. Otherwise, there are no distances to compute. Obviously, the feasible region is bounded by $[0,1]^{S_?^2}$. A feasible region is full-dimensional if and only if there are no implicit inequalities (see, for example, [26, page 101]). An inequality defining the feasible region is implicit if the inequality is actually an equality for each point of the feasible region (see, for example, [26, page 99] for a formal definition).

▶ **Proposition 10.** *1. For all $(s,t) \in S_?^2$, there exists $d \in S_?^2 \to \mathbb{R}$ satisfying (1)–(3) such that $d(s,t) > 0$.*
2. For all $(s,t) \in S_?^2$, there exists $d \in S_?^2 \to \mathbb{R}$ satisfying (1)–(3) such that $d(s,t) < 1$.
3. For all $(s,t) \in S_?^2$ and $\pi \in V(\Omega(\tau(s),\tau(t)))$, there exists $d \in S_?^2 \to \mathbb{R}$ satisfying (1)–(3) such that

$$d(s,t) < \sum_{(u,v)\in S_?^2} \pi(u,v)\, d(u,v) + \sum_{(u,v)\in S_1^2} \pi(u,v)$$

▶ **Theorem 11.** *The feasible region defined by (1)–(3) is full-dimensional.*

Recall that we need to precompute probabilistic bisimilarity to define the feasible region. If, instead, we were not to do so, then we would have to replace (3) with

$$d(s,t) \quad \leq \quad \sum_{(u,v)\in S^2\setminus S_1^2} d(u,v)\,\pi(u,v) + \sum_{(u,v)\in S_1^2} \pi(u,v) \tag{4}$$

Now consider the labelled Markov chain consisting of a single state s. In that case the above inequality amounts to $d(s,s) \leq d(s,s)$ which is implicit and, hence, the feasible region is not full-dimensional.

5 Policy Iteration

In this section we present our modification of the basic algorithm of Bacci et al. [1]. Our optimization will be discussed in Section 6. Before we can present our algorithm, we need one more ingredient.

▶ **Definition 12.** The function $\Lambda : [0,1]^{S\times S} \to [0,1]^{S\times S}$ is defined by

$$\Lambda(d)(s,t) = \begin{cases} 0 & \text{if } s \sim t \\ \Delta(d)(s,t) & \text{otherwise} \end{cases}$$

The following result is proved in [8, Proposition 17]. This result will be instrumental in several proofs later in this paper.

▶ **Lemma 13.** *δ is the unique fixed point of Λ.*

Now that we have all the ingredients, we can present our modification of the basic algorithm by Bacci et al. to compute δ. This modification was first presented in [28].

1 **for each** $(s,t) \in S_?^2$
2 $\quad T(s,t) \leftarrow$ **an element of** $V(\Omega(\tau(s),\tau(t)))$
3 **while** $\exists (s,t) \in S_?^2 : \Lambda(\gamma^T)(s,t) < \gamma^T(s,t)$
4 $\quad T(s,t) \leftarrow \underset{\omega\in V(\Omega(\tau(s),\tau(t)))}{\arg\min} \sum_{u,v\in S} \omega(u,v)\,\gamma^T(u,v)$

The only difference with the algorithm of Bacci et al. is that we use Λ instead of Δ in line 3. In line 2, for each $(s,t) \in S_?^2$, a vertex of $\Omega(\tau(s),\tau(t))$ can be easily obtained in polynomial time by using, for example, Hitchcock's north west corner rule [14]. As we already mentioned, the fixed point γ^T, used in line 3 and 4, captures reachability probabilities and it is well known that these can be computed in polynomial time by solving a system of linear equations (see, for example, [3, Section 10.1.1]). Let $(s,t) \in S_?^2$. To compute $\Lambda(\gamma^T)(s,t)$ in line 3 we first have to decide probabilistic bisimilarity. This can be done in polynomial time as has been shown by Baier in [2]. Computing $\Lambda(\gamma^T)(s,t)$ boils down to solving a minimum cost network flow problem, where ω represents the flow and γ^T captures the cost. This problem can be solved in polynomial time using, for example, Orlin's network simplex algorithm [23]. This algorithm not only computes the minimum cost but also a vertex ω of $\Omega(\tau(s),\tau(t))$ at which that minimum cost is attained, which we use in line 4.

First, we prove the partial correctness of the above algorithm, that is, if the algorithm terminates then it computes the probabilistic bisimilarity distances. Hence, we have to show that at termination γ^T captures δ. Our proofs here are different from the ones presented in [1, 28] as we will use them as a stepping stone towards proving the partial policy version correct.

▶ **Theorem 14.** *For all $T \in \mathcal{T}$, if $\gamma^T(s,t) \leq \Lambda(\gamma^T)(s,t)$ for all $(s,t) \in S_?^2$, then $\gamma^T = \delta$.*

As we already mentioned earlier, for each $(s,t) \in S_?^2$, the set $V(\Omega(\tau(s),\tau(t)))$ is finite. Since we restrict our attention to labelled Markov chains with finitely many states, the set $S_?^2$ is finite as well. Therefore, the set $\mathcal{T}$ is finite. To prove that the loop terminates we show that T becomes smaller in every iteration.

▶ **Definition 15.** The order $\prec$ on $\mathcal{T}$ is defined by $T \prec U$ if $\gamma^T \sqsubset \gamma^U$.

To relate the value of T at the beginning of the loop with its value at the end of the loop, we introduce the following notation.

▶ **Definition 16.** Let $T \in \mathcal{T}$, $(s,t) \in S_?^2$ and $\omega \in V(\Omega(\tau(s),\tau(t)))$. The function $T[(s,t)/\omega] : S_?^2 \to \mathit{Distr}(S \times S)$ is defined by

$$T[(s,t)/\omega](u,v) = \begin{cases} \omega & \text{if } (u,v) = (s,t) \\ T(u,v) & \text{otherwise} \end{cases}$$

Clearly, $T[(s,t)/\omega] \in \mathcal{T}$. Next, we show that T indeed becomes smaller in every iteration of the loop, and as a consequence the loop terminates.

▶ **Theorem 17.** *For all $T \in \mathcal{T}$ and $(s,t) \in S_?^2$, if $\Lambda(\gamma^T)(s,t) < \gamma^T(s,t)$, then $T[(s,t)/\pi] \prec T$, where $\pi = \operatorname{arg\,min}_{\omega \in V(\Omega(\tau(s),\tau(t)))} \sum_{u,v \in S} \omega(u,v)\, \gamma^T(u,v)$.*

In the above simple policy iteration algorithm, in each iteration of the loop the policy is adjusted for a single state pair (s,t) which is locally non-optimal, that is, $\Lambda(\gamma^T)(s,t) < \gamma^T(s,t)$. In our general policy iteration algorithm, the policy is updated for all state pairs which are locally non-optimal.

```
1 for each (s,t) ∈ S_?^2
2     T(s,t) ← an element of V(Ω(τ(s),τ(t)))
3 while ∃(s,t) ∈ S_?^2 : Λ(γ^T)(s,t) < γ^T(s,t)
4     U ← T
5     for each (s,t) ∈ S_?^2 such that Λ(γ^U)(s,t) < γ^U(s,t)
6         T(s,t) ← arg min_{ω ∈ V(Ω(τ(s),τ(t)))} Σ_{u,v∈S} ω(u,v) γ^U(u,v)
```

The proof of partial correctness is the same as for the simple policy iteration algorithm. To prove termination, we slightly generalise Theorem 17.

6 Partial Policy Iteration

As proposed by Bacci et al. in [1], if we are only interested in the probabilistic bisimilarity distances of some of the state pairs, then we may be able to cut down on the number of state pairs for which we need to compute the probabilistic bisimilarity distance. Instead of total policies, we use partial ones. Hence, we generalise the set of total policies $\mathcal{T}$ of Definition 7 as follows. We denote the set of partial functions from $S_?^2$ to $\mathit{Distr}(S \times S)$ by $S_?^2 \nrightarrow \mathit{Distr}(S \times S)$ and the domain of such a function P by $\mathrm{dom}(P)$.

▶ **Definition 18.** For a labelled Markov chain $\langle S, L, \tau, \ell \rangle$, the set $\mathcal{P}$ of partial policies is defined by

$$\mathcal{P} = \{\, P \in S_?^2 \nrightarrow \mathit{Distr}(S \times S) \mid \forall (s,t) \in \mathrm{dom}(P) : P(s,t) \in V(\Omega(\tau(s),\tau(t))) \,\}.$$

Recall that S^2_0, S^2_1 and $S^2_?$ form a partition of $S \times S$. For a given $P \in \mathcal{P}$, S^2_0, S^2_1, $S^2_? \setminus \mathrm{dom}(P)$ and $S^2_? \cap \mathrm{dom}(P)$ form a partition of $S \times S$ as well. This partition is used to generalise the function Γ^T of Definition 8 to the partial setting.

▶ **Definition 19.** Let $P \in \mathcal{P}$. The function $\Theta^P : [0,1]^{S\times S} \to [0,1]^{S\times S}$ is defined by

$$\Theta^P(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ 0 & \text{if } s \sim t \vee (\ell(s) = \ell(t) \wedge s \not\sim t \wedge (s,t) \notin \mathrm{dom}(P)) \\ \sum\limits_{u,v\in S} P(s,t)(u,v)\, d(u,v) & \text{otherwise} \end{cases}$$

Since $[0,1]^{S\times S}$ is a complete lattice and Θ^P is a monotone function, we can conclude from the Knaster-Tarski fixed point theorem that Θ^P has a least fixed point. We denote this fixed point by θ^P.

The set $Q \subseteq S^2_?$ contains those pairs of states for which we want to compute their distances. Recall that for pairs of states in $(S \times S) \setminus S^2_? = S^2_0 \cup S^2_1$ we can obtain their distances by deciding probabilistic bisimilarity and comparing their labels.

```
1  P ← the partial function with empty domain
2  for each (s,t) ∈ Q
3      P(s,t) ← an element of V(Ω(τ(s),τ(t)))
4      expand(P,s,t)
5  while ∃(s,t) ∈ dom(P) : Λ(θ^P)(s,t) < θ^P(s,t)
6      P(s,t) ← arg min_{ω∈V(Ω(τ(s),τ(t)))} Σ_{u,v∈S} ω(u,v) θ^P(u,v)
7      expand(P,s,t)
```

Let $P \in \mathcal{P}$ and $(s,t) \in S^2_?$. The recursive function expand(P,s,t) is defined as follows.

```
8  while ∃(u,v) ∈ support(P(s,t)) ∩ S²_? : (u,v) ∉ dom(P)
9      P(u,v) ← an element of V(Ω(τ(u),τ(v)))
10     expand(P,u,v)
```

To prove properties of this recursive function, we introduce the following predicate.

▶ **Definition 20.** Let $P \in \mathcal{P}$ and $X \subseteq S^2_?$. The predicate $F(P,X)$ is defined by

$$F(P,X) = \forall (s,t) \in \mathrm{dom}(P) \setminus X : \mathrm{support}(P(s,t)) \cap S^2_? \subseteq \mathrm{dom}(P).$$

Roughly, this predicate $F(P,X)$ captures that P is fully defined when we exclude X from its domain. Let $P \in \mathcal{P}$ and $(s,t) \in S^2_?$. Next, we prove that for expand(P,s,t) the precondition $F(P,X)$ implies the postcondition $F(P, X \setminus \{(s,t)\})$. First, we observe that $F(P,X)$ is a loop invariant. At the start of line 10, we have that $F(P, X \cup \{(u,v)\})$. Hence, at the end of line 10 we have $F(P,X)$. To conclude that the loop terminates, we observe that the finite set $\mathrm{dom}(P) \setminus \mathrm{support}(P(s,t))$ becomes smaller in every iteration. Note that expand does not give rise to infinite recursion since for each recursive call the finite set $S^2_? \setminus \mathrm{dom}(P)$ becomes smaller. At the end of the loop we have $F(P,X)$ and $(u,v) \in \mathrm{dom}(P)$ for all $(u,v) \in \mathrm{support}(P(s,t)) \cap S^2_?$, that is, $\mathrm{support}(P(s,t)) \cap S^2_? \subseteq \mathrm{dom}(P)$. Therefore, $F(P, X \setminus \{(s,t)\})$.

The proof of partial correctness of this simple partial policy iteration algorithm is similar to the partial correctness proof provided in Section 5. If the above algorithm terminates, then we have that

$$F(P,\emptyset) \wedge Q \subseteq \mathrm{dom}(P) \wedge \forall (s,t) \in \mathrm{dom}(P) : \theta^P(s,t) \leq \Lambda(\theta^P)(s,t)$$

at termination. As we will show next, from the above we can conclude that θ^P and δ coincide on Q and, hence, θ^P contains the probabilistic bisimilarity distances of Q.

▶ **Theorem 21.** *For all $P \in \mathcal{P}$, if $F(P, \emptyset)$ and $\theta^P(s,t) \leq \Lambda(\theta^P)(s,t)$ for all $(s,t) \in \mathrm{dom}(P)$, then $\theta^P(s,t) = \delta(s,t)$ for all $(s,t) \in \mathrm{dom}(P)$.*

It remains to prove that the simple partial policy iteration algorithm terminates. As we already discussed above, the recursive function expand terminates. Hence, we are left to show that the loop of line 5–7 terminates as well. We prove this by showing that in each iteration of the loop $\langle S^2_? \setminus \mathrm{dom}(P), P\rangle$ becomes smaller. These pairs are ordered lexicographically, with the first component ordered by $\subset$ and the second component ordered by $\prec$, as introduced in Definition 15. Assume that P is updated for (s,t) in line 6 of the current iteration of the loop. We distinguish two cases. If $(u,v) \notin \mathrm{dom}(P)$ for some $(u,v) \in \mathrm{support}(P(s,t)) \cap S^2_?$, then expand$(P,s,t)$ in line 7 will assign a value to $P(u,v)$ in line 9 of the expand function. As a consequence, $\mathrm{dom}(P)$ becomes bigger and, hence, $S^2_? \setminus \mathrm{dom}(P)$ becomes smaller, that is, the first component becomes smaller. Note that in this case P may not become smaller as $\theta^P(u,v)$ was zero and may have become positive. Otherwise, $\mathrm{support}(P(s,t)) \cap S^2_? \subseteq \mathrm{dom}(P)$. In that case, the expand function does not perform any assignments to P and, therefore, $\mathrm{dom}(P)$ stays the same. Thus, the first component stays the same. Furthermore, the iteration changes the partial policy from P to $P[(s,t)/\pi]$ (cf. Definition 16). As we show next, in this case the second component, that is, the partial policy, becomes smaller.

▶ **Theorem 22.** *For all $P \in \mathcal{P}$ and $(s,t) \in \mathrm{dom}(P)$, if $\Lambda(\theta^P)(s,t) < \theta^P(s,t)$, then $P[(s,t)/\pi] \prec P$, where $\pi = \underset{\omega \in V(\Omega(\tau(s),\tau(t)))}{\arg\min} \sum_{u,v \in S} \omega(u,v)\, \theta^P(u,v)$.*

The algorithm of Bacci et al. differs in three major ways from our simple partial policy iteration algorithm. First of all, as we already mentioned in Section 5, they use Δ instead of Λ in line 5. In [28, Theorem 8] we give an example different from the one presented below which shows that Λ is essential for computing the distances correctly. Secondly, in line 5 they consider only those state pairs (s,t) that are reachable from state pairs in Q in the Markov chain $\langle S \times S, P\rangle$, instead of those in $\mathrm{dom}(P)$. But, as we show below, as a result they do not always correctly compute the distances. Thirdly, they assign one to $\theta^P(s,t)$ when $(s,t) \notin \mathrm{dom}(P)$ whereas we assign zero. An example similar to the one presented below can be used to show that this also gives to computing the distances incorrectly.

We conclude this section with an example that shows that Bacci et al. do not always consider partial policies that are defined for sufficiently many state pairs. Consider the labelled Markov chain presented in the introduction. Assume that we are only interested in the probabilistic bisimilarity distance between the states s and t. That is, $Q = \{(s,t)\}$. After executing line 1–4 of the simple partial policy iteration algorithm, we may end up with the partial policy P defined by

$$P(s,t) = \tfrac{1}{2}\mathrm{Dir}_{(a,d)} + \tfrac{1}{2}\mathrm{Dir}_{(b,c)} \quad P(a,d) = \tfrac{1}{2}\mathrm{Dir}_{(a_1,d_2)} + \tfrac{1}{2}\mathrm{Dir}_{(a_2,d_1)}$$
$$P(b,c) = \tfrac{1}{2}\mathrm{Dir}_{(b_1,c_2)} + \tfrac{1}{2}\mathrm{Dir}_{(b_2,c_1)}$$

where the Dirac distribution $\mathrm{Dir}_{(u,v)}$ is defined by

$$\mathrm{Dir}_{(u,v)}(x,y) = \begin{cases} 1 & \text{if } (x,y) = (u,v) \\ 0 & \text{otherwise} \end{cases}$$

At this point, we have $\theta^P(s,t) = 1$, $\theta^P(a,d) = 1$ and $\theta^P(b,c) = 1$. Note that (s,t) is not locally optimal, that is, $\Lambda(\theta^P)(s,t) < \theta^P(s,t)$. We update P by setting $P(s,t) = \frac{1}{2}\mathrm{Dir}_{(a,c)} +$

$\frac{1}{2}\mathrm{Dir}_{(b,d)}$. The expand function on line 7 of the simple partial policy iteration algorithm may give rise to

$$P(a,c) = \tfrac{1}{2}\mathrm{Dir}_{(a_1,c_1)} + \tfrac{1}{2}\mathrm{Dir}_{(a_2,c_2)} \quad P(b,d) = \tfrac{1}{2}\mathrm{Dir}_{(b_1,d_1)} + \tfrac{1}{2}\mathrm{Dir}_{(b_2,d_2)}$$

At this point, we have $\theta^P(s,t) = \frac{3}{4}$, $\theta^P(a,c) = 1$ and $\theta^P(b,d) = \frac{1}{2}$. Since in their algorithm, Bacci et al. only check local optimality for all state pairs reachable from (s,t) in the Markov chain $\langle S \times S, P\rangle$, that is, for (s,t), (a,c) and (b,d), and all three are locally optimal, their algorithm terminates at this point. Our algorithm checks for local optimality for all state pairs in $\mathrm{dom}(P)$. Since neither (a,d) nor (b,c) are locally optimal, our algorithm continues. We update P by setting

$$P(a,d) = \tfrac{1}{2}\mathrm{Dir}_{(a_1,d_1)} + \tfrac{1}{2}\mathrm{Dir}_{(a_2,d_2)} \quad P(b,c) = \tfrac{1}{2}\mathrm{Dir}_{(b_1,c_1)} + \tfrac{1}{2}\mathrm{Dir}_{(b_2,c_2)}$$

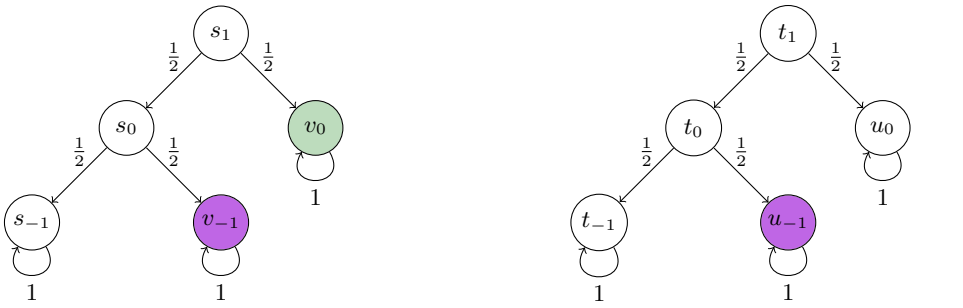
At this point, we have $\theta^P(s,t) = \frac{3}{4}$, $\theta^P(a,d) = \frac{1}{2}$ and $\theta^P(b,c) = \frac{1}{2}$. Since (s,t) is not locally optimal any more, we update P by setting $P(s,t) = \frac{1}{2}\mathrm{Dir}_{(a,d)} + \frac{1}{2}\mathrm{Dir}_{(b,c)}$. This results in $\theta^P(s,t) = \frac{1}{2}$ which is the probabilistic bisimilarity distance of (s,t).

Also the general policy iteration algorithm, which we presented at the end of Section 5, can be generalised to use partial policies instead of total ones.

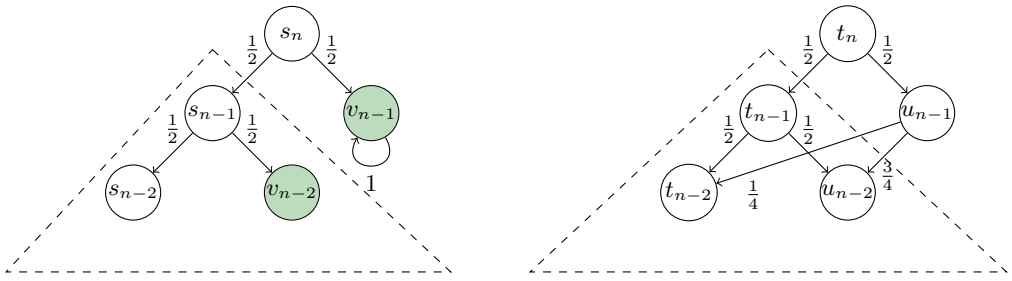
7 An Exponential Lower Bound

Below, we analyze the worst case running time of the simple partial policy iteration algorithm. We show that it is at least exponential in the number of states of the labelled Markov chain.

▶ **Definition 23.** For $n \in \mathbb{N}$, the labelled Markov chain $\mathcal{M}_n$ is defined as follows by induction on n. The labelled Markov chain $\mathcal{M}_0$ is defined as



If $n > 0$ then the labelled Markov chain $\mathcal{M}_n$ is defined as



where the two dashed triangles together represent the labelled Markov chain $\mathcal{M}_{n-1}$.

The above labelled Markov chain $\mathcal{M}_n$ has $4n+10$ states and $7n+14$ transitions. Next, we show that it may take at least $2^{n+1}-1$ iterations of the simple partial policy iteration algorithm to compute the distance of $s_n$and t_n in $\mathcal{M}_n$.

The partial policy iteration algorithm contains some nondeterminism. In particular, in line 3 and 9, an element of $V(\Omega(\tau(s),\tau(t)))$ and $V(\Omega(\tau(u),\tau(v)))$ is chosen. Furthermore, in line 5 a state pair $(s,t) \in \mathrm{dom}(P)$ with $\Lambda(\theta^P)(s,t) < \theta^P(s,t)$ is selected. Note that, for all $1 \leq i \leq n$,

$$V(\Omega(\tau(s_i),\tau(t_i))) = \{\tfrac{1}{2}\mathrm{Dir}_{(s_{i-1},t_{i-1})} + \tfrac{1}{2}\mathrm{Dir}_{(v_{i-1},u_{i-1})}, \tfrac{1}{2}\mathrm{Dir}_{(s_{i-1},u_{i-1})} + \tfrac{1}{2}\mathrm{Dir}_{(v_{i-1},t_{i-1})}\}.$$

Also,

$$V(\Omega(\tau(s_0),\tau(t_0))) = \{\tfrac{1}{2}\mathrm{Dir}_{(s_{-1},u_{-1})} + \tfrac{1}{2}\mathrm{Dir}_{(v_{-1},t_{-1})}, \tfrac{1}{2}\mathrm{Dir}_{(s_{-1},t_{-1})} + \tfrac{1}{2}\mathrm{Dir}_{(v_{-1},u_{-1})}\}.$$

Furthermore, for all $1 \leq i < n$,

$$\begin{aligned} V(\Omega(\tau(s_i),\tau(u_i))) \quad = \quad & \{\tfrac{1}{2}\mathrm{Dir}_{(v_{i-1},u_{i-1})} + \tfrac{1}{4}\mathrm{Dir}_{(s_{i-1},t_{i-1})} + \tfrac{1}{4}\mathrm{Dir}_{(s_{i-1},u_{i-1})}, \\ & \tfrac{1}{2}\mathrm{Dir}_{(s_{i-1},u_{i-1})} + \tfrac{1}{4}\mathrm{Dir}_{(v_{i-1},u_{i-1})} + \tfrac{1}{4}\mathrm{Dir}_{(v_{i-1},t_{i-1})}\}. \end{aligned}$$

To realize the exponential lower bound, in line 3 and 9 we choose the first element of the above sets and in line 5 we select the (s_i,t_i) with maximal index i.

▶ **Theorem 24.** *For each $n \in \mathbb{N}$, there exists a labelled Markov chain of size $O(n)$ and a singleton set Q such that simple partial policy iteration takes $\Omega(2^n)$ iterations to compute the distances for the state pair in Q.*

8 Experimental Results

Next we present results from experiments comparing the performance of six different algorithms: the algorithm described in the introduction by Van Breugel et al. [5] based on the first order theory over the reals, the algorithm by Chen et al. [8] based on the ellipsoid method, our simple policy iteration algorithm, our general policy iteration algorithm, our simple partial policy iteration algorithm and our general partial policy iteration algorithm. We implemented all the algorithms in Java. In our implementations we do not use arbitrary precision arithmetic. These implementations were run on a number of labelled Markov chains. These labelled Markov chains model well-known randomized algorithms and were obtained from examples of probabilistic model checkers such as PRISM [20] and jpf-probabilistic [31].

For each labelled Markov chain, we executed the code ten times. The first few executions were discarded to account for the "warm-up" time that the Java virtual machine needs to perform just-in-time compilation and optimization. For the remaining runs the average running time and the standard deviation were computed for each labelled Markov chain.

Since the distance function is symmetric and the distance from a state to itself is zero, we only need to compute the distance of $\frac{n^2-n}{2}$ state pairs for a labelled Markov chain with n states. Recall that the policy iteration algorithms precompute distances for state pairs that are probabilistic bisimilar or have different labels. Hence, only state pairs with the same label that are not probabilistic bisimilar are considered. In general, the partial policy iteration algorithms compute the distances for even fewer state pairs. For our experiments, we report for each policy iteration algorithm how many state pairs it considers and how many iterations it takes.

The first example we consider is a version of quicksort in which the pivot is chosen randomly. An implementation of this algorithm is part of jpf-probabilistic and this tool can be used to obtain the corresponding labelled Markov chain. The size of the labelled Markov chain grows exponentially in the size of the input, which is the list to be sorted. For example, lists of size 4, 5 and 6 give rise to labelled Markov chains with 10, 28 and 82 states, respectively.

The first order theory over the reals algorithm can only handle labelled Markov chains with a handful of states. We ran the algorithm on the labelled Markov chain with 10 states. It did not terminate within three days. The ellipsoid method takes on average 73 seconds. For the partial algorithms we compute the distance for a single pair of states. Of the 45 state pairs, the policy iteration algorithms consider 8 state pairs and the partial policy iteration algorithms only 3 state pairs. All policy iteration algorithms take less than 45 milliseconds. That makes the ellipsoid method three orders of magnitude slower than our policy iteration algorithms for this small example.

We did not run the first order theory over the reals algorithm on the randomized quicksort example with 28 states. The ellipsoid method takes more than 43 hours, making it five orders of magnitude slower than the policy iteration algorithms, which take less than a dozen seconds. For the example with 82 states, of the 3,321 state pairs the policy iteration algorithms consider 870 states and the partial policy iteration algorithms only 13 states.

Algorithm	List size	Running time	Standard deviation	State pairs	Iterations
Simple	4	42.75 ms	5.96 ms	8	2
General	4	27.93 ms	4.48 ms	8	2
Simple partial	4	12.23 ms	2.84 ms	3	2
General partial	4	8.84 ms	2.25 ms	3	1
Simple	5	12.00 s	19.38 ms	99	13
General	5	4.58 s	20.39 ms	99	13
Simple partial	5	51.58 ms	2.51 ms	3	0
General partial	5	81.62 ms	5.67 ms	3	1
Simple	6	15.61 h	4.00 m	870	159
General	6	47.30 m	6.03 s	870	154
Simple partial	6	24.48 s	162.91 ms	13	3
General partial	6	53.63 s	203.58 ms	13	4

In [19], Knuth and Yao show how to model a die by means of a fair coin. An implementation of their algorithm is part of PRISM. The resulting labelled Markov chain has 13 states.

Algorithm	Running time	Standard deviation	State pairs	Iterations
Simple	266.45 ms	22.40 ms	29	2
General	155.05 ms	6.59 ms	29	2
Simple partial	52.73 ms	3.69 ms	7	2
General partial	39.57 ms	0.83 ms	7	3

In the next experiment, we model two dice, one using only a fair coin and the other one using a biased coin with probability 0.6 for heads and 0.4 for tails. The goal is to compute the probabilistic bisimilarity distance between the two dice. The resulting labelled Markov chain has 20 states.

Algorithm	Running time	Standard deviation	State pairs	Iterations
Simple	10.99 s	69.67 ms	91	51
General	1.51 s	20.45 ms	91	51
Simple partial	1.28 s	12.00 ms	21	17
General partial	0.55 s	19.83 ms	21	18

It can be seen from the above experiments that the simple policy iteration algorithm is often slower than the general policy iteration algorithm. Moreover, if we are only interested in the probabilistic bisimilarity distances between a few states, the partial policy iteration algorithms are much more efficient as only part of the labelled Markov chain is considered and only the distances of related pairs of states are computed.

9 Conclusion

Although behavioural equivalences like probabilistic bisimilarity are not robust, they still play a pivotal role when computing their quantitative generalisations. For example, as we have seen in Section 4, the feasible region is defined in terms of probabilistic bisimilarity. The policies used in the policy iteration algorithms are only defined for states that are not probabilistic bisimilar. Hence, in both cases one first has to decide probabilistic bisimilarity. Deciding probabilistic bisimilarity takes only a fraction of the time to compute the probabilistic bisimilarity distances. For all the examples discussed in Section 8, deciding probabilistic bisimilarity takes less than 50 milliseconds.

The probabilistic bisimilarity pseudometric of Desharnais et al. also has discounted variants (see [12] for details). Bacci et al. consider the undiscounted version, as we do in this paper, as well as the discounted variants in [1]. All the results presented in this paper carry over to the discounted setting.

In [28] we prove an exponential lower bound for the simple policy iteration algorithm. In this paper, we present an exponential lower bound for our simple partial policy iteration algorithm. It is still open whether there exists an exponential lower bound for our general policy iteration algorithm.

Acknowledgements. The authors are thankful to all the referees of this paper for their constructive feedback.

References

1. Giorgio Bacci, Giovanni Bacci, Kim Larsen, and Radu Mardare. On-the-fly exact computation of bisimilarity distances. In Nir Piterman and Scott Smolka, editors, *Proceedings of the 19th International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, volume 7795 of *Lecture Notes in Computer Science*, pages 1–15, Rome, Italy, March 2013. Springer-Verlag.
2. Christel Baier. Polynomial time algorithms for testing probabilistic bisimulation and simulation. In Rajeev Alur and Thomas Henzinger, editors, *Proceedings of the 8th International Conference on Computer Aided Verification*, volume 1102 of *Lecture Notes in Computer Science*, pages 50–61, New Brunswick, NJ, USA, July/August 1996. Springer-Verlag.
3. Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. The MIT Press, Cambridge, MA, USA, 2008.
4. Franck van Breugel. Probabilistic bisimilarity distances. To appear in ACM SIGLOG News, 2017.
5. Franck van Breugel, Babita Sharma, and James Worrell. Approximating a behavioural pseudometric without discount. *Logical Methods in Computer Science*, 4(2), April 2008.
6. Franck van Breugel and James Worrell. The complexity of computing a bisimilarity pseudometric on probabilistic automata. In Franck van Breugel, Elham Kashefi, Catuscia Palamidessi, and Jan Rutten, editors, *Horizons of the Mind – A Tribute to Prakash Panangaden*, volume 8464 of *Lecture Notes in Computer Science*, pages 191–213. Springer-Verlag, 2014.
7. John Canny. Some algebraic and geometric computations in PSPACE. In Janos Simon, editor, *Proceedings of the 20th Annual ACM Symposium on Theory of Computing*, pages 460–467, Chicago, IL, USA, May 1988. ACM.
8. Di Chen, Franck van Breugel, and James Worrell. On the complexity of computing probabilistic bisimilarity. In Lars Birkedal, editor, *Proceedings of the 15th International Conference on Foundations of Software Science and Computational Structures*, volume 7213 of *Lecture Notes in Computer Science*, pages 437–451, Tallinn, Estonia, March/April 2012. Springer-Verlag.
9. Anne Condon. The complexity of stochastic games. *Information and Computation*, 96(2):203–224, February 1992.
10. Anne Condon. On algorithms for simple stochastic games. In Jin-Yi Cai, editor, *Advances in Computational Complexity Theory*, volume 13 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 51–73. American Mathematical Society, 1993.
11. Brian Davey and Hilary Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, United Kingdom, 2002.
12. Josée Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden. Metrics for labeled Markov systems. In Jos Baeten and Sjouke Mauw, editors, *Proceedings of the 10th International Conference on Concurrency Theory*, volume 1664 of *Lecture Notes in Computer Science*, pages 258–273, Eindhoven, The Netherlands, August 1999. Springer-Verlag.
13. Alessandro Giacalone, Chi-Chang Jou, and Scott Smolka. Algebraic reasoning for probabilistic concurrent systems. In *Proceedings of the IFIP WG 2.2/2.3 Working Conference on Programming Concepts and Methods*, pages 443–458, Sea of Gallilee, Israel, April 1990. North-Holland.
14. Frank Hitchcock. The distribution of a product from several sources to numerous localities. *Studies in Applied Mathematics*, 20(1/4):224–230, April 1941.
15. Alan Hoffman and Richard Karp. On nonterminating stochastic games. *Management Science*, 12(5):359–370, January 1966.

16 Bengt Jonsson and Kim Larsen. Specification and refinement of probabilistic processes. In *Proceedings of the 6th Annual Symposium on Logic in Computer Science*, pages 266–277, Amsterdam, The Netherlands, July 1991. IEEE.

17 Leonid Khachiyan. A polynomial algorithm in linear programming. *Soviet Mathematics Doklady*, 20(1):191–194, 1979.

18 Viktor Klee and Christoph Witzgall. Facets and vertices of transportation polytopes. In George B. Dantzig and Arthur F. Veinott, editors, *Proceedings of 5th Summer Seminar on the Mathematis of the Decision Sciences*, volume 11 of *Lectures in Applied Mathematics*, pages 257–282, Stanford, CA, USA, July/August 1967. AMS.

19 Donald E. Knuth and Andrew C. Yao. The complexity of nonuniform random number generation. In J.F. Traub, editor, *Proceedings of a Symposium on New Directions and Recent Results in Algorithms and Complexity*, pages 375–428, Pittsburgh, April 1976. Academic Press.

20 Marta Kwiatkowska, Gethin Norman, and David Parker. Prism 4.0: Verification of probabilistic real-time systems. In Ganesh Gopalakrishnan and Shaz Qadeer, editors, *Proceedings of the 23rd International Conference on Computer Aided Verification*, volume 6806 of *Lecture Notes in Computer Science*, pages 585–591, Snowbird, UT, USA, July 2011. Springer-Verlag.

21 Kim Larsen and Arne Skou:. Bisimulation through probabilistic testing. In *Proceedings of the 16th Annual ACM Symposium on Principles of Programming Languages*, pages 344–352, Austin, TX, USA, January 1989. ACM.

22 Robin Milner. *A Calculus of Communicating Systems*, volume 92 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, Germany, 1980.

23 James Orlin. A polynomial time primal network simplex algorithm for minimum cost flows. *Mathematical Programming*, 78(2):109–129, August 1997.

24 Prakash Panangaden. *Labelled Markov Processes*. Imperial College Press, London, United Kingdom, 2009.

25 David Park. Concurrency and automata on infinite sequences. In Peter Deussen, editor, *Proceedings of 5th GI-Conference on Theoretical Computer Science*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183, Karlsruhe, Germany, March 1981. Springer-Verlag.

26 Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Chichester, UK, 1986.

27 Lloyd Shapley. Stochastic games. *Proceedings of the Academy of Sciences*, 39(10):1095–1100, October 1953.

28 Qiyi Tang and Franck van Breugel. Computing probabilistic bisimilarity distances via policy iteration. In Josée Desharnais and Radha Jagadeesan, editors, *Proceedings of the 27th International Conference on Concurrency Theory*, volume 59 of *Leibniz International Proceedings in Informatics*, pages 22:1–22:15, Quebec City, QC, Canada, August 2016. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

29 Alfred Tarski. *A decision method for elementary algebra and geometry*. University of California Press, Berkeley, CA, USA, 1951.

30 Alfred Tarski. A lattice-theoretic fixed point theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, June 1955.

31 Xin Zhang and Franck van Breugel. Model checking randomized algorithms with Java PathFinder. In *Proceedings of the 7th International Conference on the Quantitative Evaluation of Systems*, pages 157–158, Williamsburg, VA, USA, September 2010. IEEE.

On Decidability of Concurrent Kleene Algebra*†

Paul Brunet[1], Damien Pous[2], and Georg Struth[3]

1 **Univ. Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, France**
2 **Univ. Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, France**
3 **Department of Computer Science, The University of Sheffield, UK**

Abstract

Concurrent Kleene algebras support equational reasoning about computing systems with concurrent behaviours. Their natural semantics is given by series(-parallel) rational pomset languages, a standard true concurrency semantics, which is often associated with processes of Petri nets. We use constructions on Petri nets to provide two decision procedures for such pomset languages motivated by the equational and the refinement theory of concurrent Kleene algebra. The contribution to the first problem lies in a much simpler algorithm and an ExpSpace complexity bound. Decidability of the second, more interesting problem is new and, in fact, ExpSpace-complete.

1998 ACM Subject Classification F.3.1 Specifying, verifying, and reasoning about programs

Keywords and phrases Concurrent Kleene algebra, series-parallel pomsets, Petri nets

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.28

1 Introduction

Kleene algebras axiomatise the equational theory of rational expressions. Their canonical models are rational languages and their equational theories correspond to rational expression equivalence [12, 11, 1, 2]. Deciding identities in Kleene algebras is therefore PSpace-complete [17] by standard automata constructions. Variants of Kleene algebras provide simple algebraic semantics for while-programs, and, in particular, decision procedures for these.

Pomset languages [6], on the other hand, are a widely studied model of true concurrency in which words are generalised from linear orders to partial ones. Recent applications can be found, for instance, in weak memory model verification [9]. Algebras for pomsets have been proposed first by Gischer [5] and more recently, as concurrent Kleene algebra (CKA), by Hoare et al. [8], with the aim of extending the pleasant properties of Kleene algebras into concurrency. Yet much less is known about their structure.

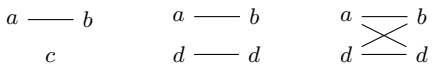
Formally, CKAs are structures $(K, +, \cdot, \|, ^\star, ^{(\star)}, 0, 1)$ that consist of a Kleene algebra $(K, +, \cdot, ^\star, 0, 1)$ and a commutative Kleene algebra $(K, +, \|, ^{(\star)}, 0, 1)$, and satisfy the weak interchange law defined below. Commutative Kleene algebras axiomatise rational commutative expression equivalence, which is decidable [4] and coNExp-complete [7]. In applications of CKA, the elements of K are typically actions of a system: The operation $+$ models nondeterministic choices, $\cdot$ and $\|$ sequential and parallel compositions, 1 the ineffective action, and 0 the abortive one. The sequential star * models the finite sequential iteration of actions

* An extended version of this abstract, including proofs, is available on HAL [3].

† This work was supported by the European Research Council (ERC) under the Horizon 2020 programme (CoVeCe, grant agreement No 678157) and the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007).

in terms of a least fixpoint, the parallel star $^{(\star)}$ their finite parallel iteration. It can be interpreted as the unbounded spawning of parallel processes.

Closed terms in the language of Kleene algebra correspond to rational expressions; their interpretation as word languages is standard. The extension to parallelism, hence to action-labelled partial orders and pomsets, is best explained by example. The expression $(a \cdot b) \parallel c$, for instance, is represented by the first of the following pomsets.



Execution time—the order of the poset—is indicated by lines proceeding from left to right in this implicitly directed graph. Sequential composition thus orders actions, whereas parallel composition leaves them unordered. By analogy to word languages, expressions involving $+$ or the stars require interpretations by sets of pomsets, that is, pomset languages. The expression $(a \cdot b) \parallel (c + (d \cdot d))$, for instance, denotes the language formed by the first two of the pomsets above. The third of the above pomsets is denoted by $(a \parallel d) \cdot (b \parallel d)$. It is obviously "more sequential" than the pomset to its left, which is denoted by $(a \cdot b) \parallel (d \cdot d)$. A corresponding refinement order, which compares degrees of sequentiality, has been defined on pomsets (as the smoother-than relation) by Grabowski [6]. It is isomorphic to the inclusion order on refinement-closed pomset languages and induces a (refinement) order on CKA expressions. Gischer [5] has shown that this order is characterised precisely by the inequality $(a \parallel c) \cdot (b \parallel d) \leq (a \cdot b) \parallel (c \cdot d)$ on CKA expressions (without the stars). This weak interchange law is also one of the standard CKA axioms.

Pomset languages are typically infinite when expressions contain stars. In addition, the width of individual pomsets can be unbounded when parallel stars occur; this star is therefore often omitted [15]. Furthermore, CKA expressions generate subclasses of pomset languages. Those generated by expressions over the full CKA signature are called *series-parallel-rational* (spr-languages), those generated by using a signature without the parallel star are called *series-rational* (sr-languages). Expressions are named accordingly. All pomsets occurring in spr- or sr-languages, which are built inductively from singleton pomsets by sequential and parallel compositions, are series-parallel or, equivalently, free of N-shape subpomsets [19, 6].

Equivalence of spr-expressions, as induced by spr-language identity, is decidable and can be axiomatised by any set of axioms for Kleene algebras plus those for commutative Kleene algebras [13], but a reasonable upper complexity bound has not been established. In the context of CKA with the interchange axiom, completeness or decidability of the refinement of spr-expressions, or even sr-expressions, as induced by inclusion of refinement-closed spr- or sr-languages, remains open. These questions are of obvious interest for comparing concurrent systems with respect to their degree of sequentiality or linearisability

Our first contribution consists in a simple new algorithm and a first complexity bound for sr-expression equivalence. First, using a construction similar to Thompson's [18] and Grabowski's [6], we show that every sr-language is the pomset trace language of a safe labelled Petri net. Using a result by Jategaonkar and Meyer on pomset languages of Petri nets [10], it then follows that sr-expression equivalence is in ExpSpace (Theorem 5).

Our second, more interesting contribution is a proof that sr-expression refinement is ExpSpace-complete (Theorem 26). Note that sr-expression equivalence is sr-expression refinement in both directions. This result requires comparing runs in Petri nets up-to Grabowski's refinement order, using the freedom provided by this formalism to reorder transitions, and a schedule for constructing a comparison function in a canonical way. Preservation of sequentiality or causality in this construction is somewhat intricate: it

requires tracking the history and relationships between *loci* (Section 5.2 and 5.3). The Petri net approach seems natural once more due to the correspondence between nets and pomset languages, and our previous construction. Hardness of sr-expression refinement follows from a reduction from the equivalence problem for regular expressions with a shuffle operation [16], using results by Grabowski that relate pomset and shuffle languages.

2 Preliminary definitions

2.1 Pomsets

We fix a finite alphabet Σ. A *labelled poset* is a triple $\langle X, \leq, \lambda\rangle$ where X is a finite carrier set, $\leq$ is a partial order on X and the map $\lambda : X \to \Sigma$ labels every element in X with a letter in Σ. A *(labelled poset) morphism* is a function between labelled posets that preserves the order and the labels. A *pomset* is an isomorphism class of labelled posets; it is a labelled poset up-to bijective renaming of the elements in X. We represent pomsets as graphs that are implicitly directed from left to right. The vertices, which are the elements of the pomset, are labelled by λ; those edges that can be deduced by transitivity and reflexivity are omitted.

We define the following pomsets and operations on pomsets:

- The *empty pomset*, denoted by P_0, is defined as $\langle \emptyset, \emptyset, [] \rangle$ ([] denoting the empty function);
- for $a \in \Sigma$, the *singleton pomset* P_a is $\langle \{\bullet\}, \{\langle \bullet, \bullet \rangle\}, [\bullet \mapsto a] \rangle$;
- for pomsets $P_1 = \langle X_1, \leq_1, \lambda_1\rangle$ and $P_2 = \langle X_2, \leq_2, \lambda_2\rangle$ with $X_1 \cap X_2 = \emptyset$; the *parallel product of* P_1 *and* P_2 is the pomset obtained by putting them side by side:

$$P_1 \parallel P_2 \triangleq \langle X_1 \cup X_2, \leq_1 \cup \leq_2, \lambda_1 \cup \lambda_2 \rangle \quad ;$$

- the *sequential product* of P_1 and P_2 is the pomset obtained by further declaring all elements of P_1 as smaller than those of P_2:

$$P_1 ; P_2 \triangleq \langle X_1 \cup X_2, \leq_1 \cup \leq_2 \cup X_1 \times X_2, \lambda_1 \cup \lambda_2 \rangle \quad .$$

Pomset P_1 *refines* P_2, written $P_1 \sqsubseteq P_2$, if there exists a bijective morphism $\varphi : X_2 \to X_1$. By definition, therefore,

- $\forall x \in X_2, \lambda_1(\varphi(x)) = \lambda_2(x)$, i.e., the bijection preserves labels; and
- $\forall x, y \in X_2, x \leq_2 y \Rightarrow \varphi(x) \leq_1 \varphi(y)$, i.e., the morphism preserves edges in P_2.

The relation $\sqsubseteq$ is a partial order on pomsets. We write ${}^{\sqsubseteq}S$ for the downward closure of a set S of pomsets with respect to it: ${}^{\sqsubseteq}S \triangleq \{P \mid \exists Q : P \sqsubseteq Q,\ Q \in S\}$. We then extend the refinement order to a preorder on sets of pomsets: $S \sqsubseteq S' \triangleq S \subseteq {}^{\sqsubseteq}S'$. (This definition is equivalent to $S \sqsubseteq S' \triangleq {}^{\sqsubseteq}S \subseteq {}^{\sqsubseteq}S'$.)

2.2 Expressions and pomset languages

A *series-rational expression*, or more briefly *expression*, is a term derived from the following syntax. The set of expressions over the alphabet Σ is written $\mathrm{Rat}^{\parallel}\langle\Sigma\rangle$.

$$e, f ::= e + f \ \mid\ e \cdot f \ \mid\ e \parallel f \ \mid\ e^\star \ \mid\ 0 \ \mid\ 1 \ \mid\ a \qquad (a \in \Sigma)$$

The *language* of an expression is the set of pomsets defined inductively as follows:

$$[\![1]\!] \triangleq \{P_0\} \quad [\![e \cdot f]\!] \triangleq \{P ; Q \mid P \in [\![e]\!],\ Q \in [\![f]\!]\} \quad [\![e \parallel f]\!] \triangleq \{P \parallel Q \mid P \in [\![e]\!],\ Q \in [\![f]\!]\}\,.$$

$$[\![0]\!] \triangleq \emptyset \quad [\![e + f]\!] \triangleq [\![e]\!] \cup [\![f]\!] \quad [\![e^\star]\!] \triangleq \bigcup_{n \in \mathbb{N}} \{P_1 ; \ldots ; P_n \mid \forall i \leq n, P_i \in [\![e]\!]\} \quad [\![a]\!] \triangleq \{P_a\}\,.$$

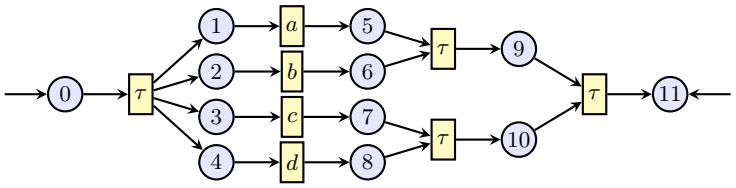
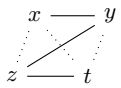


Figure 1 Example of labelled safe Petri net.

A set of pomsets is called *(series-)rational* if it is the language of some expression. It is called *downward-closed rational* if it is the downward-closure of a rational language.

Note that due to the structure of expressions, the pomsets we consider are always *series-parallel*: they are built from trivial pomsets by using sequential and parallel compositions. Valdes et al. proved that this property is equivalent to *N-freeness* [19, 6]: whenever there are four distinct elements x, y, z, t such that $x \leq y$, $z \leq y$, and $z \leq t$, then either $z \leq x$, $t \leq y$, or $x \leq t$.



In the present work we are interested in the following two decision problems.

▶ **Definition 1.** Given two expressions e, f, the problem biKA(e, f) asks if $[\![e]\!] \subseteq [\![f]\!]$.

▶ **Definition 2.** Given two expressions e, f, the problem CKA(e, f) asks if $[\![e]\!] \sqsubseteq [\![f]\!]$.

The first problem, biKA(e, f), asks essentially about equivalence of the sr-expressions e and f. As outlined in the introduction, axioms for Kleene algebras plus those for commutative Kleene algebras (here in fact commutative idempotent semirings without a parallel star) are complete w.r.t. this equivalence. The second one, CKA(e, f), asks whether e is a refinement of f, which relates to CKA with the interchange law, yet again without a parallel star. We conjecture that the aforementioned axioms together with weak interchange are complete for this semantics, but this problem remains open, to the best of our knowledge.

2.3 Labelled safe Petri nets

We now define *labelled safe Petri nets*—the machines that we use to recognise rational pomset languages. We write $\wp_+(X)$ for the set of non-empty subsets of a set X.

A *labelled Petri net* is a tuple $\mathcal{N} = \langle \mathcal{P}, T, p_{in}, p_{fin} \rangle$ where:

- $\mathcal{P}$ is a finite set of *places*;
- $T \subseteq \wp_+(\mathcal{P}) \times (\Sigma \cup \{\tau\}) \times \wp_+(\mathcal{P})$ is a set of *labelled transitions*;
- $p_\in \in \mathcal{P}$ is the *initial place*;
- $p_{fin} \in \mathcal{P}$ is the *final place*.

If $t = \langle P, x, P' \rangle$ is a transition, then P is its *input set*, written ${}^\bullet t$, x is its *label*, written $\ell(t)$, and P' is its *output set*, written $t^\bullet$. Transitions labelled with τ are called *silent*; the others are called *visible*. Without loss of generality, we may restrict ourselves to Petri nets where all inputs and outputs of visible transitions are singleton sets. An example of such a Petri net is displayed in Figure 1.

A *configuration* is a set of places. A transition t is *enabled* from a configuration C if ${}^\bullet t \subseteq C$. Whenever t is enabled in C, then firing this transition leads to the configuration

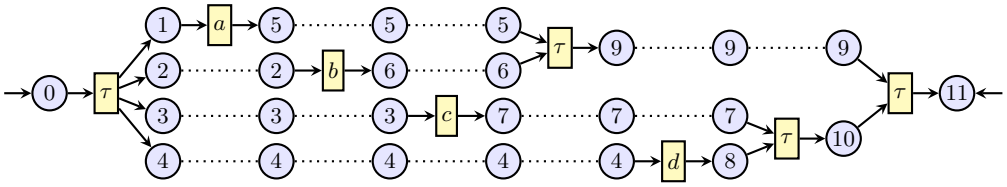


Figure 2 An accepting run of the Petri net in Figure 1.

$C' \triangleq (C \setminus {}^\bullet t) \cup t^\bullet$, and we write $C \xrightarrow{t}_{\mathcal{N}} C'$. A *run* from C_0 to C_n is a sequence $t_1; \ldots; t_n$ such that there exists configurations $C_1, \ldots, C_{n-1}$ such that

$$C_0 \xrightarrow{t_1}_{\mathcal{N}} C_1 \xrightarrow{t_2}_{\mathcal{N}} \cdots C_{n-1} \xrightarrow{t_n}_{\mathcal{N}} C_n \ .$$

We write $C_0 \xrightarrow{t_1;\ldots;t_n}_{\mathcal{N}} C_n$ in this case. If $C_0 = \{p_{in}\}$, then the run is *initial*, if $C_n = \{p_{fin}\}$, then it is *final*, and if both conditions hold, it is *accepting*. Finally, a configuration C is *reachable* if some initial run ends in C.

Figure 2 shows an example of an accepting run. In this representation, columns of circular nodes denote the successive configurations C_i. We draw the transition t_i as a rectangular node between C_{i-1} and C_i, drawing directed edges from its inputs in C_{i-1} to its node, and to its outputs in C_i. The remaining places, those in $C_{i-1} \setminus {}^\bullet t_i$ that happen again in $C_i \setminus t_i^\bullet$, are linked with dotted lines.

A Petri net is *safe* if $(C \setminus {}^\bullet t) \cap t^\bullet = \emptyset$ holds for every reachable configuration C and every transition t enabled in C. In other words, there is always at most one token in every place of a safe Petri net. This justifies our use of sets rather than multisets for configurations a posteriori: we shall only use safe Petri nets.

The *transition automaton* $\mathcal{A}(\mathcal{N})$ of a Petri net $\mathcal{N}$ is a non-deterministic finite state automaton over the alphabet of transitions of $\mathcal{N}$; its states are configurations of $\mathcal{N}$ (i.e. subsets of $\mathcal{P}$), its initial state is $\{p_{in}\}$, its only final state is $\{p_{fin}\}$, and its transitions are the triples $\langle C, t, C' \rangle$ such that $C \xrightarrow{t}_{\mathcal{N}} C'$. Writing $\mathcal{L}(\mathcal{B})$ for the usual word language of an automaton $\mathcal{B}$, the transition automaton is defined so that we have

$$\mathcal{L}(\mathcal{A}(\mathcal{N})) \triangleq \left\{ t_1 \ldots t_n \;\middle|\; \{p_{in}\} \xrightarrow{t_1;\ldots;t_n}_{\mathcal{N}} \{p_{fin}\} \right\}.$$

2.4 Language of a Petri net

Let $R = C_0 \xrightarrow{t_1;\ldots;t_n}_{\mathcal{N}} C_n$ be a run in a Petri net $\mathcal{N}$. We define the *immediate causality relation* $\rightarrow_R \subseteq [1..n] \times [1..n]$ as

$$i \rightarrow_R j \triangleq i < j \wedge \left(\exists p \in t_i^\bullet \cap {}^\bullet t_j : \forall k,\ i < k < j \Rightarrow p \notin {}^\bullet t_k\right) \ .$$

The *causality relation* $\leq_R$ is the reflexive transitive closure of $\rightarrow_R$. Intuitively, $i \leq_R j$ holds if t_j cannot be fired in a subrun of R without firing t_i.

For each run one can define three kinds of traces [10]. For the run fromFigure 2, these are shown in Figure 3.

- The *graph-trace* $\mathcal{G}(R)$ of R is the graph $\langle [1..n], \rightarrow_R \rangle$.
- The *transition-pomset* R is the pomset $\mathbb{T}(R) \triangleq \langle [1..n], \leq_R, \lambda_R \rangle$, where $\lambda_R(i) \triangleq \ell(t_i)$.
- The *pomset-trace* $\mathbb{P}(R)$, of R is the restriction of $\mathbb{T}(R)$ to the set $\{i \mid \ell(t_i) \in \Sigma\}$ of visible actions.

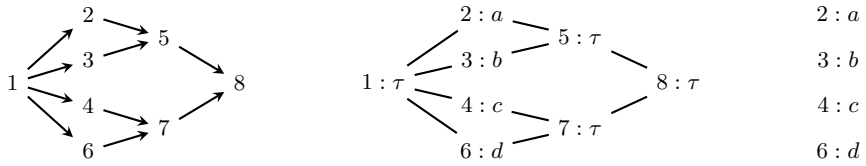


Figure 3 Graph-trace, transition-pomset, and pomset-trace of the run from Figure 2.

The *pomset language* of a Petri net $\mathcal{N}$ is the set $[\![\mathcal{N}]\!]$ of pomset-traces of accepting runs in $\mathcal{N}$. Moreover, we call a run R is *series-parallel* if its graph-trace is series-parallel. Note that this is strictly stronger than requiring that its pomset-trace be series-parallel.

The run in Figure 2 is series-parallel.

3 Reading a pomset in a Petri net

This section describes an operational way of reading and recognising pomsets with Petri nets, as one might read and recognise a word with a finite state automaton. It is independent from the rest of the paper, but might provide insight into the algorithm we develop below to compare languages of nets. Indeed, the guiding intuition behind this algorithm will be to read a net in another net.

Let $\mathcal{N} = \langle \mathcal{P}, T, p_{in}, p_{fin} \rangle$ be a safe labelled Petri net, $P = \langle X, \leq, \lambda \rangle$ a pomset, and $R = C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \cdots C_{n-1} \xrightarrow{t_n} C_n$ a run in $\mathcal{N}$. A *reading of P in $\mathcal{N}$ along R* is a sequence $\langle \rho_0, X_0 \rangle, \ldots, \langle \rho_n, X_n \rangle$ such that:

1. for every $0 \leqslant i \leqslant n$, $X_i \subseteq X$ and ρ_i is a map from C_i to $\wp(X_i)$;
2. for every $0 \leqslant i < n$,
   a. if $\ell(t_{i+1}) \in \Sigma$, and if p_0, p_1 are respectively the input and output places of t_{i+1}, there is an element $x \in \rho_i(p_0)$ such that $\lambda(x) = \ell(t_{i+1})$ and:

$$X_{i+1} = X_i \setminus \{x\}\,; \qquad \rho_{i+1}(p) = \begin{cases} \{y \in X_{i+1} \mid x \leq y\} & \text{if } p = p_1 \\ \rho_{i+1}(p) = \rho_i(p) \setminus \{x\} & \text{otherwise.} \end{cases}$$

   b. if $\ell(t_{i+1}) = \tau$, then

$$X_{i+1} = X_i; \qquad \rho_{i+1}(p) = \begin{cases} \bigcup_{q \in {}^\bullet t_{i+1}} \rho_i(q) & \text{if } p \in t_{i+1}^\bullet \\ \rho_i(p) & \text{otherwise.} \end{cases}$$

The reading is *initial* if $C_0 = \{p_{in}\}$ and $\rho_0(p_{in}) = X_0 = X$. The reading is *final* if $C_n = \{p_{fin}\}$ and $X_n = \emptyset$. The reading is *accepting* if it is both initial and final. P is *accepted* by $\mathcal{N}$ if there is an accepting reading of P in $\mathcal{N}$. The *language recognised* by $\mathcal{N}$ is the set of pomsets accepted by $\mathcal{N}$. It should not be confused with the pomset language of $\mathcal{N}$, as defined above.

▶ Remark. Notice that, if R is accepting, the existence of an accepting reading of P along R can be tested by a simple history-independent non-deterministic algorithm. We start with $X_0 = X$ and $\rho_0 = [p_{in} \mapsto X]$. At step $i+1$ we use condition 2b to compute ρ_{i+1} and X_{i+1} if t_{i+1} is silent,. If t_{i+1} is visible and there is no $x \in \rho_i({}^\bullet t_{i+1})$ such that $\lambda(x) = \ell(t_{i+1})$, then we conclude that there are no readings of P along R. Otherwise, we non-deterministically choose an appropriate x and use condition 2a to compute ρ_{i+1} and X_{i+1}. If this yields $X_n = \emptyset$ we have obtained an accepting reading, otherwise we can conclude that there are no such readings.

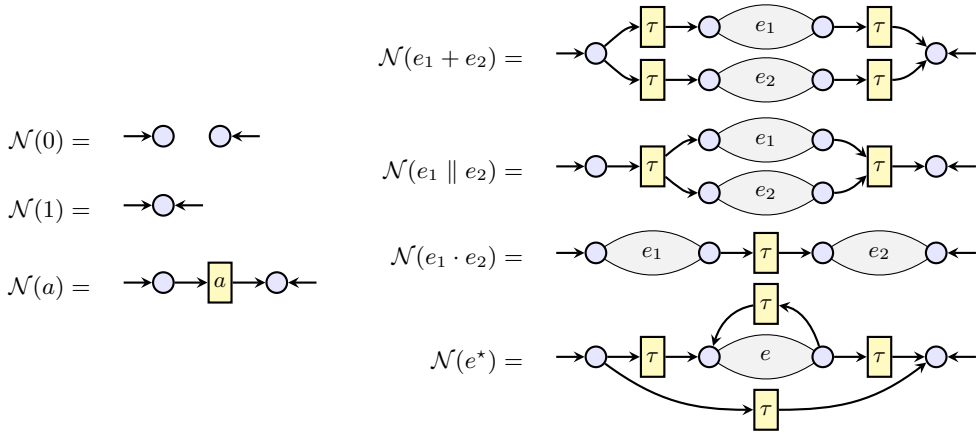
▶ **Lemma 3.** *If R is accepting, there is an accepting reading of P along R if and only if $P \sqsubseteq \mathbb{P}(R)$.*

Proof. See [3]. ◀

▶ **Corollary 4.** *The language recognised by $\mathcal{N}$ is $^{\sqsubseteq}[\![\mathcal{N}]\!]$.*

4 Rational Petri nets

This section shows that every rational pomset language is the pomset language of a (safe labelled) Petri net. To this end, we recursively associate with every expression e a Petri net $\mathcal{N}(e)$ such that $[\![\mathcal{N}(e)]\!] = [\![e]\!]$. Moreover, all accepting runs of this Petri net turn out to be series-parallel. The construction poses no difficulty; it is a simple adaptation of Thompson's construction for rational word languages [18], and an extension of a previous construction by Grabowski [6] for safe Petri nets and pomset languages. We only present this construction graphically here.



This construction yields decidability of biKA in exponential space. Indeed we may build the Petri nets $\mathcal{N}(e)$ and $\mathcal{N}(f)$ from the expressions e and f (these are linear in the size of e and f) and use Jategaonkar and Meyer's result [10] that testing containment of pomset-trace languages of two Petri nets is an ExpSpace-complete problem.

▶ **Theorem 5.** *The problem* biKA *lies in the class* ExpSpace.

▶ **Proposition 6.** *The language recognised by $\mathcal{N}(e)$ is $^{\sqsubseteq}[\![e]\!]$.*

Proof. By construction we have $[\![\mathcal{N}(e)]\!] = [\![e]\!]$. We conclude using Corollary 4. ◀

5 Comparing Petri nets modulo refinement

Next we show how to compare Petri nets modulo refinement. Thanks to the previous construction, this leads to decidability of the problem CKA. We fix two Petri nets $\mathcal{N}_1$ and $\mathcal{N}_2$ for this section and the following one. Our goal is to check whether $[\![\mathcal{N}_1]\!] \sqsubseteq [\![\mathcal{N}_2]\!]$, i.e., whether for each run $R_1 \in \mathcal{L}(\mathcal{N}_1)$, there exists a corresponding run $R_2 \in \mathcal{L}(\mathcal{N}_2)$ such that $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$.

The first difficulty is that we may have to reorder runs in $\mathcal{N}_2$: due to concurrency, transitions might be triggered in different orders and still yield the same pomset.

5.1 Reordering runs

Let $R = t_1; \ldots; t_n$ be a run from C_0 to C_n. Let π be a permutation of $[1..n]$. The action of π on R is defined as $\pi R \triangleq t_{\pi(1)}; \ldots; t_{\pi(n)}$. The permutation π is *compatible* with R if it is order-preserving:

$$\forall i, j,\ i \leq_R j \Rightarrow \pi(i) \leqslant \pi(j).$$

▶ **Lemma 7.** *If π is compatible with R, then $C_0 \xrightarrow{\pi R}_{\mathcal{N}} C_n$, $\mathcal{G}(R) = \mathcal{G}(\pi R)$, and*

$$i \leq_R j \Leftrightarrow \pi(i) \leq_{\pi R} \pi(j).$$

Proof. We can exchange two successive transitions that are not causally linked without changing the graph (up-to isomorphism). We repeat this process until we obtain πR. ◀

Accordingly, we say that a run R' is *equivalent to a run* R if $R' = \pi R$ for some compatible permutation π.

Another important notion for the completeness of the method we propose is that of an *economical* run: a run that fires its silent transitions as late as possible.

▶ **Definition 8.** A run $t_1; \ldots; t_n$ is *economical* if for all $i < j$, if $t_i, \ldots, t_{j-1}$ are silent transitions and t_j is a visible transition, then $i \leq_R j$.

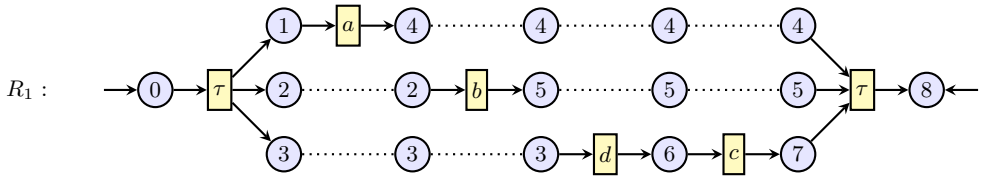
The run in Figure 2 is not economical: the fifth transition is a silent one, but it is not causally related to the next transition, which is visible. We will see that it can be reordered into an equivalent economical run (Proposition 10 and Example 11 below.)

Even more importantly when comparing two runs, we need to ensure that the visible transitions are fired in the same order.

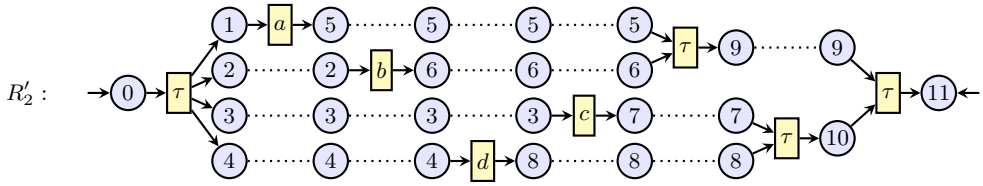
▶ **Definition 9.** Given a run R_1 in $\mathcal{N}_1$ and a run R_2 in $\mathcal{N}_2$ such that $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$, we say that R_2 *follows* R_1 if the subsumption is witnessed by a bijection φ such that for every two visible indices i, j in R_2 we have $i < j \Leftrightarrow \varphi(i) < \varphi(j)$.

▶ **Proposition 10.** *Let R_1 and R_2 be series-parallel runs in $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. If $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$ then there exists an economical and series-parallel run R_2' in $\mathcal{N}_2$ that follows R_1 and is equivalent to R_2.*

▶ **Example 11.** Consider the run R_2 from Figure 2 and the following run R_1:



The pomset of R_1 is $\mathbb{P}(R_1) = P_a \parallel P_b \parallel (P_d; P_c)$, and we may check that $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$. To transform R_2 into a run that is economical and follows R_1, we must (1) exchange the transitions labelled with c and d; and (2) delay the silent transition in the middle of R_2 until all visible transitions have been fired. Doing so, we get the following run R_2', which follows R_1 and is equivalent to R_2.



5.2 Loci

In this more technical section, we define *loci*, as a way to lift the causality relation between transitions to places in the successive configurations of the runs. Let R be a run, with $R = C_0 \xrightarrow{t_1} C_1 \cdots C_{n-1} \xrightarrow{t_n} C_n$, $\mathbb{T}(R) = \langle [1..n], \leq_R, \lambda \rangle$. A *locus* denotes a pair $\langle p, i \rangle$ where $0 \leqslant i \leqslant n$ and $p \in C_i$. In some sense, loci are places with a time index. In the previous pictures those are the numbered circles: p is the number in the circle (the name of the place), and i is the index of its column. Formally, the set of loci of the run R is $\bigcup_{0 \leqslant i \leqslant n} C_i \times \{i\}$. We generate an equivalence relation $\approx_R$ on loci using the following rule:

$$p \notin {}^\bullet t_i \Rightarrow \langle p, i \rangle \approx_R \langle p, i-1 \rangle .$$

Graphically, equivalent loci are linked with dotted lines. Equivalences classes with respect to $\approx_R$ are thus places with a time interval, rather than a single index. The *source* of a locus $\langle p, i \rangle$ is the smallest index of its equivalence class:

$$src\,\langle p, i \rangle \triangleq \min \{ j \leqslant i \mid \langle p, i \rangle \approx_R \langle p, j \rangle \} \quad .$$

Now we define a preorder $\precsim_R$, generated by the following rules:

$$p, q \in {}^\bullet t_i \times t_i^\bullet \Rightarrow \langle p, i-1 \rangle \precsim_R \langle q, i \rangle \quad , \qquad \langle p, i \rangle \approx_R \langle q, j \rangle \Rightarrow \langle p, i \rangle \precsim_R \langle q, j \rangle \quad .$$

Note that two loci in the same configuration are always incomparable. Finally, we inductively define the set of indices of *predecessors* of a locus:

- $pred\,\langle p, 0 \rangle \triangleq \emptyset$.
- if $p \in t_i^\bullet$, then $pred\,\langle p, i \rangle \triangleq \{i\} \cup \bigcup_{q \in {}^\bullet t_i} pred\,\langle q, i-1 \rangle$.
- if $p \notin t_i^\bullet$, then $pred\,\langle p, i \rangle \triangleq pred\,\langle p, i-1 \rangle$.

The *set of visible predecessors of a locus*, written $vpred\,\langle p, i \rangle$, is the subset of indices of visible transitions in $pred\,\langle p, i \rangle$.

▶ **Lemma 12.** *The following properties hold:*

$$\forall i, \forall p \in t_i^\bullet,\ pred\,\langle p, i \rangle = \{ j \mid j \leq_R i \} , \tag{1}$$
$$\forall i, \forall p \in C_i,\ pred\,\langle p, i \rangle = \{ j \mid j \leq_R src\,\langle p, i \rangle \} , \tag{2}$$
$$\forall i, j, p, q,\ \langle p, i \rangle \approx_R \langle q, j \rangle \Rightarrow pred\,\langle p, i \rangle = pred\,\langle q, j \rangle , \tag{3}$$
$$\forall i, j, p, q,\ \langle p, i \rangle \precsim_R \langle q, j \rangle \Rightarrow pred\,\langle p, i \rangle \subseteq pred\,\langle q, j \rangle , \tag{4}$$
$$\forall i, j, p, \forall q \in t_j^\bullet,\ j \in vpred\,\langle p, i \rangle \Rightarrow \langle q, j \rangle \precsim_R \langle p, i \rangle , \tag{5}$$
$$\forall i \leqslant j,\ \exists p \in C_j : i \in pred\,\langle p, j \rangle . \tag{6}$$

Notice that we managed to lift the causality relation $\leq_R$ to the level of loci: this lemma implies that if $i \neq j$, $p \in t_i^\bullet$ and $q \in t_j^\bullet$, then $i \leq_R j$ if and only if $\langle p, i \rangle \precsim_R \langle q, j \rangle$.

5.3 Schedules

We compare runs using the following notion of *schedule*, where we interleave two runs in such a way that they synchronise on visible transitions.

▶ **Definition 13.** Let R_1 and R_2 be two runs with $R_i = C_0^i \xrightarrow{t_1^i} C_1^i \cdots C_{n_i-1}^i \xrightarrow{t_{n_i}^i} C_{n_i}^i$ for $i \in \{1, 2\}$. An *N-schedule* from R_1 to R_2 is a function $\eta : [0..N] \to [0..n_1] \times [0..n_2]$ such that

- $\eta(0) = \langle 0, 0 \rangle$ and $\eta(N) = \langle n_1, n_2 \rangle$;
- if $\eta(k) = \langle i, j \rangle$, then either

1. t^1_{i+1} is a silent transition and $\eta(k+1) = \langle i+1, j\rangle$, or
2. t^2_{j+1} is a silent transition and $\eta(k+1) = \langle i, j+1\rangle$, or
3. t^1_{i+1} and t^2_{j+1} are visible, $\ell\left(t^1_{i+1}\right) = \ell\left(t^2_{j+1}\right)$ and $\eta(k+1) = \langle i+1, j+1\rangle$.

Note that a schedule constructs a bijection between the visible transitions of R_1 and those of R_2. Indeed, each of these transitions must be fired synchronously and agree on labels (case 3). Furthermore, since a schedule starts with $\eta(0) = \langle 0, 0\rangle$ and ends with $\eta(N) = \langle n_1, n_2\rangle$, every transition in both runs must be fired. This means there is a label-preserving bijection φ, called *the bijection induced by* η, from the visible transitions of R_2 to those of R_1 that satisfies $i < j$ if and only if $\varphi(i) < \varphi(j)$.

The notion of schedule is still very weak: there are schedules from one run to another whenever they have the same visible transitions, in the same order. Causality between those transitions is not taken into account. We fix this with the following technical definition. Intuitively, we keep track of the history and relationships between the loci of the configurations, in order to ensure that the causality relation in the presumably smaller run R_1 refines that of R_2.

▶ **Definition 14.** For each N-schedule we define the following sequence of binary relations $(\prec_k)_{k\in[0..N]}$, where $\prec_k \subseteq C^1_i \times C^2_j$ when $\eta(k) = \langle i, j\rangle$, by induction:

- $\prec_0 = C^1_0 \times C^2_0$;
- if $\eta(k) = \langle i, j\rangle$, then

1. if $\eta(k+1) = \langle i+1, j\rangle$, we set $p \prec_{k+1} q \triangleq \begin{cases} p \prec_k q & \text{if } p \notin (t^1_{i+1})^\bullet, \\ \exists p' \in {}^\bullet t^1_{i+1}, p' \prec_k q & \text{otherwise.} \end{cases}$

2. if $\eta(k+1) = \langle i, j+1\rangle$, we set $p \prec_{k+1} q \triangleq \begin{cases} p \prec_k q & \text{if } q \notin (t^2_{j+1})^\bullet, \\ \forall q' \in {}^\bullet t^2_{j+1} : p \prec_k q' & \text{otherwise.} \end{cases}$

3. otherwise, let $t^1_{i+1} = \langle \{p_0\}, a, \{p_1\}\rangle$ and $t^2_{j+1} = \langle \{q_0\}, a, \{q_1\}\rangle$; we set

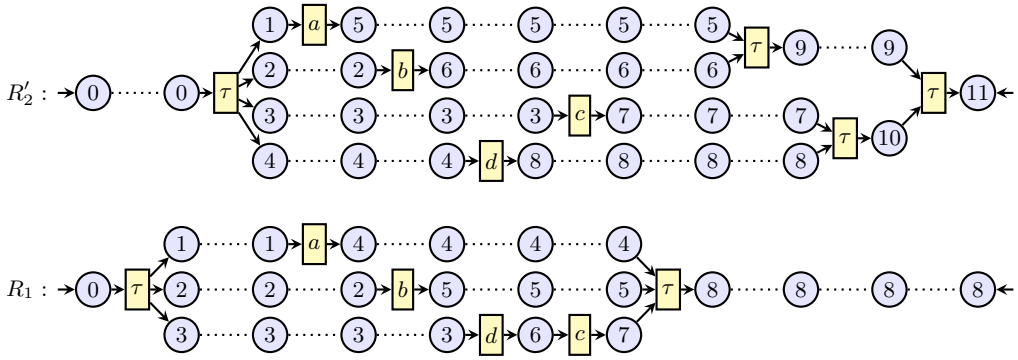
$$p \prec_{k+1} q \triangleq \begin{cases} p_0 \prec_k q \text{ or } (q = q_1 \text{ and } p_0 \prec_k q_0) & \text{if } p = p_1 \\ p \prec_k q \text{ and } q \neq q_1 & \text{otherwise.} \end{cases}$$

The schedule η is *valid* if for every visible index i in R_2 we have $p \prec_k q$ for the unique k, p, q such that $\eta(k) = \langle \varphi(i), i\rangle$, $(t^1_{\varphi(i)})^\bullet = \{p\}$ and $(t^2_i)^\bullet = \{q\}$.

▶ **Example 15.** Recall the runs R'_2 and R_1 from Example 11. The following sequence is a schedule from R_1 to R'_2.

$$\eta = (0,0);(1,0);(1,1);(2,2);(3,3);(4,4);(5;5);(6,5);(6,6);(6,7);(6,8)$$

We may then draw the two runs side by side according to this schedule:



From this schedule, we can compute the successive $\prec_k$ relations, and check that $4 \prec_3 5$, $5 \prec_4 6$, $6 \prec_5 8$, and $7 \prec_6 7$. Hence η is valid.

▶ Remark. If $\eta(k) = \langle i, j\rangle$, $\eta(k') = \langle i', j'\rangle$, $\langle p, i\rangle \approx_{R_1} \langle p', i'\rangle$ and $\langle q, j\rangle \approx_{R_2} \langle q', j'\rangle$, then $p \prec_k q$ if and only if $p' \prec_{k'} q'$.

▶ **Lemma 16.** *If $\eta(k) = \langle i, j\rangle$, then $p \prec_k q$ entails $\varphi(\mathit{vpred}\,\langle q, j\rangle) \subseteq \mathit{vpred}\,\langle p, i\rangle$.*

The algorithm we define in the next section looks for valid schedules. The following proposition establishes soundness of this strategy.

▶ **Proposition 17.** *If there exists a valid schedule from R_1 to R_2, then $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$.*

Proof. The bijection φ induced by η works. Let i, j be visible indices in R_2 such that $i \leq_{R_2} j$. We need to show that $\varphi(i) \leq_{R_1} \varphi(j)$. Take the unique k, p, q such that $\eta(k) = \langle \varphi(j), j\rangle$, $(t^1_{\varphi(j)})^\bullet = \{p\}$ and $(t^2_j)^\bullet = \{q\}$. By Lemma 12.(1) we have $i \in \mathit{vpred}\,\langle q, j\rangle$. Since η is valid, we have $p \prec_k q$, and thus $\varphi(\mathit{vpred}\,\langle q, j\rangle) \subseteq \mathit{vpred}\,\langle p, \varphi(j)\rangle$ by Lemma 16. This means that $\varphi(i) \in \mathit{vpred}\,\langle p, \varphi(j)\rangle$, and using Lemma 12.(1) again we obtain $\varphi(i) \leq_{R_1} \varphi(j)$. ◀

For completeness of the algorithm we need to exhibit valid schedules. Under appropriate assumptions—see Proposition 19 below—the following *canonical schedule* $\boldsymbol{\eta}$ from R_1 to R_2 will work. We define it recursively. Intuitively, we schedule the silent transitions of R_1 as early as possible and those of R_2 as late as possible:

- $\boldsymbol{\eta}(0) = \langle 0, 0\rangle$;
- if $\boldsymbol{\eta}(k) = \langle i, j\rangle$ then
  1. if t^1_{i+1} is silent, then $\boldsymbol{\eta}(k+1) = \langle i+1, j\rangle$;
  2. if t^1_{i+1} is visible and t^2_{j+1} is silent then $\boldsymbol{\eta}(k+1) = \langle i, j+1\rangle$;
  3. if t^1_{i+1} and t^2_{j+1} are visible, then $\boldsymbol{\eta}(k+1) = \langle i+1, j+1\rangle$.

We write $\boldsymbol{\varphi}$ for the bijection induced by $\boldsymbol{\eta}$. The schedule from Example 15 is actually the canonical schedule. The converse of Lemma 16 holds for the canonical schedule:

▶ **Lemma 18.** *If R_2 is series-parallel and economical, then, for every k with $\boldsymbol{\eta}(k) = \langle i, j\rangle$, if $\boldsymbol{\varphi}(\mathit{vpred}\,\langle q, j\rangle) \subseteq \mathit{vpred}\,\langle p, i\rangle$, then $p \prec_k q$.*

▶ **Proposition 19.** *If R_1 and R_2 are series-parallel, if $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$, and if R_2 is economical and follows R_1 then the canonical schedule $\boldsymbol{\eta}$ from R_1 to R_2 is valid.*

Proof. First note that since R_2 follows R_1, $\boldsymbol{\varphi}$ and the bijection witnessing $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$ must coincide. Let i be a visible index in R_2, and let k, p, q such that $\boldsymbol{\eta}(k) = \langle \boldsymbol{\varphi}(i), i\rangle$, $(t^1_{\boldsymbol{\varphi}(i)})^\bullet = \{p\}$ and $(t^2_i)^\bullet = \{q\}$. We have to prove $p \prec_k q$. By Lemma 18, it suffices to prove the inclusion $\boldsymbol{\varphi}(\mathit{vpred}\,\langle q, i\rangle) \subseteq \mathit{vpred}\,\langle p, \boldsymbol{\varphi}(i)\rangle$, i.e., that for every $j \in \mathit{vpred}\,\langle q, i\rangle$, we have $\boldsymbol{\varphi}(j) \in \mathit{vpred}\,\langle p, \boldsymbol{\varphi}(i)\rangle$. This is equivalent to checking that for every visible j, $j \leq_{R_2} i$ implies $\boldsymbol{\varphi}(j) \leq_{R_1} \boldsymbol{\varphi}(i)$, which is true because $\boldsymbol{\varphi}$ is an order preserving bijection from the pomsets of R_2 to that of R_1. ◀

Note that this lemma relies on the fact that R_2 is series-parallel. Indeed, there can be pairs of runs R_1 and R_2 satisfying $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$, and R_2 being economical and following R_1, but such that the canonical schedule is not valid.

5.4 Reduction to finite automata

Now that we have the notion of valid schedule, we use a technique similar to [10] to reduce the problem of comparing Petri nets modulo subsumption to the comparison of plain automata. For this end, we define the following automaton that aims at recognising those runs of $\mathcal{N}_1$ for which there exists a valid schedule to some run in $\mathcal{N}_2$.

▶ **Definition 20.** The *composite automaton* $\mathcal{N}_1 \prec \mathcal{N}_2$ is the nondeterministic finite state automaton with epsilon-transitions $\langle Q, T_1, q_0, F\rangle$ where:

- the alphabet is the set of transitions of $\mathcal{N}_1$;
- the set of states Q consists of triples $\langle C_1, C_2, \prec\rangle$ with C_1 and C_2 respectively configurations of $\mathcal{N}_1$ and $\mathcal{N}_2$ and $\prec \subseteq C_1 \times C_2$;
- the initial state q_0 is the triple $\left\langle \left\{p^1_{in}\right\}, \left\{p^2_{in}\right\}, \left\{\left\langle p^1_{in}, p^2_{in}\right\rangle\right\}\right\rangle$;
- final states are those triples of the shape $\left\langle \{p^1_{fin}\}, \{p^2_{fin}\}, \_\right\rangle$;
- transitions are split into three kinds:

1. if t is a silent transition of $\mathcal{N}_1$, $C_1 \xrightarrow{t}_{\mathcal{N}_1} C_1'$, then from every state $\langle C_1, C_2, \prec\rangle$ there is a transition labelled with t going to the state $\langle C_1', C_2, \prec'\rangle$ with

$$p \prec' q \Leftrightarrow \begin{cases} p \prec q & \text{if } p \notin t^\bullet, \\ \exists p' \in {}^\bullet t, p' \prec q & \text{otherwise.} \end{cases}$$

2. if t is a silent transition of $\mathcal{N}_2$, $C_2 \xrightarrow{t}_{\mathcal{N}_2} C_2'$, then from every state $\langle C_1, C_2, \prec\rangle$ there is an epsilon-transition going to the state $\langle C_1, C_2', \prec'\rangle$ with

$$p \prec' q \Leftrightarrow \begin{cases} p \prec q & \text{if } q \notin t^\bullet, \\ \forall q' \in {}^\bullet t, p \prec q' & \text{otherwise.} \end{cases}$$

3. if t_1 and t_2 are visible transitions of $\mathcal{N}_1$ and $\mathcal{N}_2$ with the same label, inputs p_0 and q_0 and outputs p_1 and q_1, if $C_1 \xrightarrow{t_1}_{\mathcal{N}_1} C_1'$ and $C_2 \xrightarrow{t_2}_{\mathcal{N}_2} C_2'$, then from every state $\langle C_1, C_2, \prec\rangle$ such that $p_0 \prec q_0$, there is a transition labelled with t_1 going to the state $\langle C_1', C_2', \prec'\rangle$ with

$$p \prec' q \Leftrightarrow \begin{cases} p_0 \prec q \text{ or } q = q_1 & \text{if } p = p_1, \\ p \prec q \text{ and } q \neq q_1 & \text{otherwise.} \end{cases}$$

By definition of this composite automaton, we have

▶ **Lemma 21.** *The language of the automaton $\mathcal{N}_1 \prec \mathcal{N}_2$ is the set of accepting runs R_1 in $\mathcal{N}_1$ such that there is an accepting run R_2 in $\mathcal{N}_2$ and a valid schedule from R_1 to R_2.*

Finally, we can reduce the comparison of Petri nets modulo subsumption to that of (word) automata.

▶ **Proposition 22.** *If the runs in $\mathcal{N}_1$ and those in $\mathcal{N}_2$ are all series-parallel, then $\mathcal{L}(\mathcal{A}(\mathcal{N}_1)) \subseteq \mathcal{L}(\mathcal{N}_1 \prec \mathcal{N}_2)$ if and only if $[\![\mathcal{N}_1]\!] \sqsubseteq [\![\mathcal{N}_2]\!]$.*

Proof. Suppose $\mathcal{L}(\mathcal{A}(\mathcal{N}_1)) \subseteq \mathcal{L}(\mathcal{N}_1 \prec \mathcal{N}_2)$ and let $P \in [\![\mathcal{N}_1]\!]$. There exists $R_1 \in \mathcal{L}(\mathcal{A}(\mathcal{N}_1))$ such that $P = \mathbb{P}(R_1)$. By assumption we also have $R_1 \in \mathcal{L}(\mathcal{N}_1 \prec \mathcal{N}_2)$, which means, by Lemma 21, that there is an accepting run R_2 in $\mathcal{N}_2$ and a valid schedule η from R_1 to R_2. Proposition 17 then tells us that $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$, thus proving $P \in {}^{\sqsubseteq}[\![\mathcal{N}_2]\!]$.

Conversely, assume that $[\![\mathcal{N}_1]\!] \sqsubseteq [\![\mathcal{N}_2]\!]$. Let $R_1 \in \mathcal{L}(\mathcal{A}(\mathcal{N}_1))$ be an accepting run in $\mathcal{N}_1$. By assumption, there is an accepting run R_2 in $\mathcal{N}_2$ such that $\mathbb{P}(R_1) \sqsubseteq \mathbb{P}(R_2)$. By hypothesis, both R_1 and R_2 are series-parallel. By Proposition 10, there exists an economical series-parallel run R_2' that follows R_1 and is equivalent to R_2. Hence using Proposition 19, there is a valid schedule from R_1 to R_2'. With Lemma 21 we conclude that $R_1 \in \mathcal{L}(\mathcal{N}_1 \prec \mathcal{N}_2)$. ◀

6 Decidability & Complexity of CKA

Putting together the results from Sections 4 and 5 yields the announced algorithm.

▶ **Proposition 23.** CKA *lies in the class* ExpSpace.

Proof. We build the Petri nets $\mathcal{N}(e)$ and $\mathcal{N}(f)$, and then the finite automata $\mathcal{A}(\mathcal{N}(e))$ and $\mathcal{N}(e) \prec \mathcal{N}(f)$. By Proposition 22, to answer the original question, we simply need to test these automata for language inclusion. It is well known that this requires polynomial space with respect to the size of the automata.

Let n be the size of e and m the size of f (their number of symbols). The Petri nets $\mathcal{N}(e)$ and $\mathcal{N}(f)$ are linear in the size of e and f, with at most $2n$ and $2m$ places. The automaton $\mathcal{A}(\mathcal{N}(e))$ uses at most 2^{2n} states (recall these are sets of places). The automaton $\mathcal{N}(e) \prec \mathcal{N}(f)$ uses at most $2^{2n} \times 2^{2m} \times 2^{2n \times 2m}$ states. Hence testing language equivalence of these two automata will use an amount of space polynomial in 2^{2n} and $2^{2n+2m+4nm}$, whence the announced result. ◀

For the lower bound, we reduce the problem of universality of regular expressions with shuffle [16] to the containment of downward-closed rational languages. We briefly recall the former. *Regular expressions with shuffle* over the alphabet Σ are terms over the syntax

$$e, f \in \mathrm{Rat}^{\bowtie}\langle\Sigma\rangle ::= e + f \mid e \cdot f \mid e \bowtie f \mid e^\star \mid 0 \mid 1 \mid a \qquad (a \in \Sigma).$$

Given two words u and v over Σ, the *shuffle product* of u and v, written $u \bowtie v$, is the language of all words of the form $u_1 v_1 u_2 v_2 \cdots u_k v_k$; where $u = u_1 \cdots u_k$, $v = v_1 \cdots v_k$, and the words u_i, v_i can be of arbitrary length (including the empty word).

The *language of a regular expression with shuffle* is defined recursively as follows.

$$[\![0]\!] := \emptyset \qquad [\![1]\!] := \{\varepsilon\} \qquad [\![a]\!] := \{a\} \qquad [\![e + f]\!] := [\![e]\!] \cup [\![f]\!] \qquad [\![e \cdot f]\!] := [\![e]\!] \cdot [\![f]\!]$$

$$[\![e \bowtie f]\!] := \bigcup_{u \in [\![e]\!],\ v \in [\![f]\!]} u \bowtie v \qquad [\![e^\star]\!] := \bigcup_{n \in \mathbb{N}} \{u_1 \dots u_n \mid \forall i \leqslant n, u_i \in [\![e]\!]\}\,.$$

▶ **Theorem 24** (Mayer and Stockmeyer [16])**.** *The problem of testing whether the language of a regular expression with shuffle is equal to* $\Sigma^\star$ *is* ExpSpace*-complete.*

The key observations for our reduction are due to Grabowski [6]: words are isomorphic to totally ordered pomsets, and given two words u and v, the set of totally ordered pomsets in ${}^{\sqsubseteq}[\![u \parallel v]\!]$ is isomorphic to the shuffle product of u and v.

Concretely, we associate a series-parallel expression $\lfloor e \rfloor$ to any regular expression with shuffle e by replacing every occurrence of $\bowtie$ with $\parallel$. This encoding has the following property.

▶ **Lemma 25.** *For every word* $w \in \Sigma^\star$*, we have* $w \in [\![e]\!]$ *if and only if* w *seen as a totally ordered pomset is in* ${}^{\sqsubseteq}[\![\lfloor e \rfloor]\!]$*.*

Proof. By a simple induction on e, using the above observation for the shuffle case. (Each subcase can be found in [6].) ◀

As a consequence, the language of e is $\Sigma^\star$ if and only if $[\![\Sigma^\star]\!] \sqsubseteq [\![\lfloor e \rfloor]\!]$. We thus have a linear encoding of the universality of regular expressions with shuffle into containment of downward-closed rational languages, hence our final theorem.

▶ **Theorem 26.** *The problem* CKA *is* ExpSpace*-complete.*

An implementation of the algorithm is available at `http://paul.brunet-zamansky.fr/cka.html`.

7 Related work

Several constructions in the literature are similar to those presented in Section 4. Here we list some of them, highlighting the differences between these developments and our own.

Lodaya and Weil introduced *branching automata* that recognise series-parallel rational pomset languages [15], which include the series-rational languages we use here. These automata impose a strong notion of bracketing (opening and closing τ-transitions must match exactly), which we do not know how to handle when it comes to comparing automata. This is why we used plain Petri nets instead.

Jategaonkar and Meyer presented a construction almost equivalent to ours [10], albeit for different purposes: their goal was to obtain a lower complexity bound by a reduction from the universality problem of regular languages with shuffle. The main differences in the constructions are that we use an initial *place* instead of an initial *marking*, and that we consider a unique final *place* while they have a distinguished final *transition*. These differences mainly impact the star and parallel product constructs. Jategaonkar and Meyer's construction could in fact be adapted to obtain an alternative proof of Theorem 5. However, their construction does not satisfy the structural constraints needed for the completeness of the algorithm we develop in Section 5: the runs of the automata produced by their construction are not always series-parallel.

Finally, a third construction that produces safe Petri nets from expressions was developed by Lodaya [14]. It is, however, quite different from the present approach. In particular, it requires initial and final markings, and it is not appropriate for a precise complexity analysis, as it produces nets that are exponentially large with respect to input expressions.

References

1 Maurice Boffa. Une remarque sur les systèmes complets d'identités rationnelles. *Informatique Théorique et Applications*, 24:419–428, 1990. URL: `http://archive.numdam.org/article/ITA_1990__24_4_419_0.pdf`.

2 Maurice Boffa. Une condition impliquant toutes les identités rationnelles. *Informatique Théorique et Applications*, 29(6):515–518, 1995.

3 Paul Brunet, Damien Pous, and Georg Struth. On decidability of concurrent Kleene algebra, 2017. Full version of this extended abstract, with proofs. URL: `https://hal.archives-ouvertes.fr/hal-01558108/`.

4 J. H. Conway. *Regular algebra and finite machines*. Chapman and Hall, 1971.

5 Jay L. Gischer. The equational theory of pomsets. *Theoretical Computer Science*, 61(2):199–224, 1988. `doi:10.1016/0304-3975(88)90124-7`.

6 J. Grabowski. On partial languages. *Fundamenta Informaticae*, 4:427–498, 1981.

7 Christoph Haase and Piotr Hofman. Tightening the complexity of equivalence problems for commutative grammars. In *STACS*, volume 47 of *LIPIcs*, pages 41:1–41:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

8 T. Hoare, B. Möller, G. Struth, and I. Wehrman. Concurrent Kleene algebra and its foundations. *Journal of Logic and Algebraic Programming*, 80(6):266–296, 2011.

9 A. Horn and D. Kroening. On partial order semantics for SAT/SMT-based symbolic encodings of weak memory concurrency. In *FORTE*, volume 9039 of *Lecture Notes in Computer Science*, pages 19–34. Springer Verlag, 2015.

10 Lalita Jategaonkar and Albert R. Meyer. Deciding true concurrency equivalences on safe, finite nets. *Theoretical Computer Science*, 154(1):107–143, 1996. `doi:10.1016/0304-3975(95)00132-8`.

11 Dexter Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. In *LICS*, pages 214–225. IEEE, 1991. doi:10.1109/LICS.1991.151646.

12 Daniel Krob. A Complete System of B-Rational Identities. In *ICALP*, volume 443 of *Lecture Notes in Computer Science*, pages 60–73. Springer Verlag, 1990. doi:10.1007/BFb0032022.

13 Michael R. Laurence and Georg Struth. Completeness theorems for bi-Kleene algebras and series-parallel rational pomset languages. In *RAMiCS*, volume 8428 of *Lecture Notes in Computer Science*, pages 65–82. Springer Verlag, 2014. doi:10.1007/978-3-319-06251-8_5.

14 K. Lodaya, D. Ranganayakulu, and K. Rangarajan. Hierarchical structure of 1-safe petri nets. In *ASIAN*, Progamming Languages and Distributed Computation, pages 173–187. Springer Verlag, 2003. doi:10.1007/978-3-540-40965-6_12.

15 Kamal Lodaya and Pascal Weil. Series–parallel languages and the bounded-width property. *Theoretical Computer Science*, 237(1):347–380, 2000. doi:10.1016/S0304-3975(00)00031-1.

16 Alain J. Mayer and Larry J. Stockmeyer. The complexity of word problems – this time with interleaving. *Information and Computation*, 115(2):293–311, 1994. doi:10.1006/inco.1994.1098.

17 A. R. Meyer and L. J. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential space. In *SWAT*, pages 125–129. IEEE, 1972. doi:10.1109/SWAT.1972.29.

18 K. Thompson. Regular expression search algorithm. *Communications of the ACM*, 11:419–422, 1968. URL: http://www.fing.edu.uy/inco/cursos/intropln/material/p419-thompson.pdf.

19 Jacobo Valdes, Robert E. Tarjan, and Eugene L. Lawler. The recognition of series parallel digraphs. In *STOC*, pages 1–12. ACM, 1979. doi:10.1145/800135.804393.

Model-Checking Counting Temporal Logics on Flat Structures*†

Normann Decker[1], Peter Habermehl[2], Martin Leucker[3], Arnaud Sangnier[4], and Daniel Thoma[5]

1 **ISP, University of Lübeck, Lübeck, Germany**
decker@isp.uni-luebeck.de
2 **IRIF, Univ. Paris Diderot, Paris, France**
haberm@irif.fr
3 **ISP, University of Lübeck, Lübeck, Germany**
leucker@isp.uni-luebeck.de
4 **IRIF, Univ. Paris Diderot, Paris, France**
sangnier@irif.fr
5 **ISP, University of Lübeck, Lübeck, Germany**
thoma@isp.uni-luebeck.de

Abstract

We study several extensions of linear-time and computation-tree temporal logics with quantifiers that allow for counting how often certain properties hold. For most of these extensions, the model-checking problem is undecidable, but we show that decidability can be recovered by considering flat Kripke structures where each state belongs to at most one simple loop. Most decision procedures are based on results on (flat) counter systems where counters are used to implement the evaluation of counting operators.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Counting Temporal Logic, Model checking, Flat Kripke Structure

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.29

1 Introduction

Model checking [8] is a method to verify automatically the correct behaviour of systems. It takes as input a model of the system to be verified and a logical formula encoding the specification and checks whether the behaviour of the model satisfies the formula. One key aspect of this method is to find the appropriate balance between expressiveness of models and logical formalisms and efficiency of the model-checking algorithms. If the model is too expressive, e.g. Turing machines, then the model-checking problem, even with very simple logical formalisms, becomes undecidable. On the other hand, some expressive logics have been proposed in order to reason on the temporal executions of simple models such as Kripke structures. This is the case for the linear temporal logic LTL [22] and the branching-time temporal logics CTL [7] and CTL* [14], for which the model-checking problem has been shown to be PSPACE-complete, contained in P and PSPACE-complete, respectively (see, e.g., [3]).

* Partially supported by EGIDE/DAAD-Procope (FREQS) and European Commission H2020 Project COEMS.
† An extended version is available [9] providing additional proof details.

Even though these logical formalisms allow for stating classical properties like safety or liveness over executions of Kripke structures, their expressiveness is limited. In particular they cannot describe quantitative aspects, as for instance the fact that a property has been true twice as often as another along an execution. One approach to solve this issue is to extend the logic with some ability to *count* positions of an execution satisfying some property and to check constraints over such numbers at some positions. Such a counting extension is proposed in [19] for CTL leading to a logic denoted here as ${}_{\mathtt{c}}\mathtt{CTL}$. This formalism can state properties such as an event p will eventually occur and before that, the number of events q is larger than two. The authors propose further an extension called (here) ${}_{\mathtt{c}}\mathtt{CTL}_{\pm}$ that admits diagonal comparisons (i.e., negative and positive coefficients) to state, for instance that the number of events b is greater than the number of events c. It is shown that the model-checking problem for ${}_{\mathtt{c}}\mathtt{CTL}$ is decidable in polynomial time and that the satisfiability problem for ${}_{\mathtt{c}}\mathtt{CTL}_{\pm}$ is undecidable. A similar extension for LTL is considered in [18] where it is proven that model checking of ${}_{\mathtt{c}}\mathtt{LTL}$ is ExpSpace-complete while that of ${}_{\mathtt{c}}\mathtt{LTL}_{\pm}$ is undecidable.

Following the same motivation, *regular availability expressions (RAE)* were introduced in [16] extending regular expressions by a mechanism to express that on a (sub-)word matching an expression specific letters occur with a given relative frequency. Unfortunately, emptiness of the intersection of two such expressions was shown undecidable. Even for single expressions only a non-elementary procedure is known for verification (inclusion in regular languages) and deciding emptiness [1]. The case is similar for the logic fLTL [5], a variant of LTL that features an until operator extended by a frequency constraint. The operator is intended to relax the classical semantics where $\varphi\,\mathtt{U}\,\psi$ requires φ to hold at all positions before ψ. For example, the fLTL formula $p\,\mathtt{U}^{\frac{1}{3}}\,q$ states that q holds eventually and before that the proportion of positions satisfying p should be at least one third. The concept of relative frequencies embeds naturally into the context of counting logics as it can be understood as a restricted form of counting. In fact, fLTL can be considered as a fragment of ${}_{\mathtt{c}}\mathtt{LTL}_{\pm}$ and still has an undecidable satisfiability problem [5] implying the same for model-checking Kripke structures. Moreover, most techniques employed for obtaining results on RAE as well as fLTL involve variants of counter systems.

Looking at the model-checking problem from the model point of view, recent work has shown that restrictions can be imposed on Kripke structures to obtain better complexity bounds. As a matter of fact if the structure is *flat* (or weak), which means every state belongs to at most one simple cycle in the graph underlying the structure, then the model-checking problem for LTL becomes NP-complete [17]. Such a restriction has as well been successfully applied to more complex classes of models. It is well known that the reachability problem for two-counter systems is undecidable [21] whereas for flat systems the problem is decidable for any number of counters [15], even more, model checking of LTL is NP-complete [11]. Flat structures are not only interesting because of their algorithmic properties, but also because they can be used as a way to under-approximate the behaviour of non-flat systems. For instance for counter systems one gets a semi-decision procedure for the reachability problem which consists in enumerating flat sub-systems and testing for reachability. In simple words, flat structures can be understood as an extension of paths typically used in bounded model checking and we expect that bounded model checking using flat structures rather than paths improves practical model checking approaches.

Contributions. We consider the model-checking problem for a counting logic that we call $\mathtt{CCTL}^{*}$ where we use variables to mark positions on a run from where we begin to count the

Table 1 Complexity characterisation of the model-checking problems of fragments of CCTL*. PH indicates polynomial reducibility to the (decidable) satisfiability problem of PH.

	CTL	LTL	CTL*	fLTL	fCTL	fCTL*	CLTL	CCTL	CCTL*
KS	P	PSPACE-c.	PSPACE-c.	undec. [5]	**Exp**	undec.	undec.	undec. [19]	undec.
FKS	P	NP-c. [17]	PSPACE	**NExp**	**Exp**	**ExpSpace**	**PH**	**PH**	**PH**

number of times a subformula is satisfied. Such a way of counting was also introduced in [19], see Section 2.2 for a comparison. We study as well its fragments fCTL, fLTL and fCTL* where the explicit counting mechanism is replaced by a generalized version of the until operator capable of expressing frequency constraints.

First we prove that fCTL model checking is at most exponential in the formula size and polynomial in the structure size by using an algorithm similar to the one for CTL model checking. To deal with frequency constraints a counter is employed for tracking the number of times a subformula is satisfied in a run of a Kripke structure. We then show that for flat Kripke structures the model-checking problems of fLTL and CCTL* are decidable. For the former, our method is a guess and check procedure based on the existence of a flat counter system as witness of a run of the Kripke structure satisfying the fLTL formula. For the latter, we use a technique which consists in encoding the run of a flat Kripke structure into a Presburger arithmetic formula and then we show that model checking of CCTL* can be translated into the satisfiability problem of a decidable extension of Presburger arithmetic, called PH, featuring a counting quantifier known as Härtig quantifier. We hence provide new decidability results for CCTL* which in practice could be used as an under-approximation approach to the general model-checking problem. We furthermore relate an extension of Presburger arithmetic, for which the complexity of the satisfiability problem is open, to a concrete model-checking problem. In summary, for model checking different fragments of CCTL* on Kripke structures (KS) or flat Kripke structures (FKS) we obtain the picture shown in Table 1 where bold entries are our novel results.

2 Definitions

2.1 Preliminaries

We write $\mathbb{N}$ and $\mathbb{Z}$ to denote the sets of natural numbers (including zero) and integers, respectively, and $[i, j]$ for $\{k \in \mathbb{Z} \mid i \leq k \leq j\}$. We consider integers encoded with a binary representation. For a finite alphabet Σ, Σ^* represents the set of finite words over Σ, Σ^+ the set of finite non-empty words over Σ and Σ^ω the set of infinite words over Σ. For a finite set E of elements, $|E|$ represents its cardinality. For (finite or infinite) words and general sequences $u = a_0 a_1 \dots a_k \dots$ of length at least $k+1 > 0$ we denote by $u(k) = a_k$ the $(k+1)$-th element and refer to its indices $0, 1, \dots$ as positions on u. If u is finite then $|u|$ denotes its length. For arbitrary functions $f : A \to B$ and elements $a \in A, b \in B$ we denote by $f[a \mapsto b]$ the function f' that is equal to f except that $f'(a) = b$. We write $\mathbf{0}$ and $\mathbf{1}$ for the functions $f_0 : A \to \{0\}$ and $f_1 : A \to \{1\}$, respectively, if the domain A is understood. By B^A for sets A and B we denote the set of all functions from A to B.

Kripke structures. Let AP be a finite set of *atomic propositions*. A *Kripke structure* is a tuple $\mathcal{K} = (S, s_I, E, \lambda)$ where S is a finite set of control states, $s_I \in S$ the initial control state, $E \subseteq S \times S$ the set of edges and $\lambda : S \mapsto 2^{AP}$ the labelling function. A finite *path* in $\mathcal{K}$ is a sequence $u = s_0 s_1 \dots s_k \in S^+$ with $(s_i, s_{i+1}) \in E$ for all $i \in [0, k-1]$. Infinite paths are defined analogously. A *run* ρ of $\mathcal{K}$ is an infinite path with $\rho(0) = s_I$. We denote by $\texttt{Runs}(\mathcal{K})$ the set of runs of $\mathcal{K}$. Due to the single initial state, we assume without loss of generality that the graph of $\mathcal{K}$ is connected, i.e. all states are reachable. A *simple loop* in $\mathcal{K}$ is a finite path $u = s_0 s_1 \dots s_k$ such that $i \neq j$ implies $s_i \neq s_j$ for all $i, j \in [0, k]$ and $(s_k, s_0) \in E$. A Kripke structure $\mathcal{K}$ is called *flat* if for each state $s \in S$ there is at most one simple loop u in $\mathcal{K}$ with $u(0) = s$. See Figure 1 for an example. The classes of all Kripke structures and all flat Kripke structures are denoted $\texttt{KS}$ and $\texttt{FKS}$, respectively.

Counter systems. Our proofs use systems with integer counters and simple guards. A *counter system* is a tuple $\mathcal{S} = (S, s_I, C, \Delta)$ where S is a finite set of control states, $s_I \in S$ is the initial state, C is a finite set of counter names and $\Delta \subseteq S \times \mathbb{Z}^C \times 2^{\mathfrak{G}(C)} \times S$ is the transition relation where $\mathfrak{G}(C) = \{(c < 0), (c \geq 0) \mid c \in C\}$. An infinite sequence $s_0 s_1 \dots \in S^\omega$ of states starting in $s_0 = s_I$ is called a *run* of $\mathcal{S}$ if there is a sequence $\theta_0 \theta_1 \dots \in (\mathbb{Z}^C)^\omega$ of valuation functions $\theta_i : C \to \mathbb{Z}$ with $\theta_0 = \mathbf{0}$ and a transition $(s_i, \mathbf{u}_i, G_i, s_{i+1}) \in \Delta$ for every $i \in \mathbb{N}$ such that $\theta_{i+1} = \theta_i + \mathbf{u}_i$ (defined point-wise as usual), $\theta_{i+1}(c) < 0$ if $(c < 0) \in G_i$ and $\theta_{i+1}(c) \geq 0$ if $(c \geq 0) \in G_i$ for all $c \in C$. Again, we denote by $\texttt{Runs}(\mathcal{S})$ the set of all such runs and assume the graph of control states underlying $\mathcal{S}$ is connected.

2.2 Temporal Logics with Counting

We now introduce the different formalisms we use in this work as specification language. The most general one is the branching-time logic $\texttt{CCTL}^*$ which extends the branching-time logic $\texttt{CTL}^*$ (see e.g. [3]) with the following features: it has operators that allow for counting along a run the number of times a formula is satisfied and which stores the result into a variable. The counting starts when the associated variable is "placed" on the run. These variables may be shadowed by nested quantification, similar to the semantics of the freeze quantifier in linear temporal logic [13].

Let V be a set of *variables* and AP a set of atomic propositions. The syntax of $\texttt{CCTL}^*$ formulae φ over V and AP is given by the grammar rules

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \texttt{X}\,\varphi \mid \varphi\,\texttt{U}\,\varphi \mid \texttt{E}\,\varphi \mid x.\varphi \mid \tau \leq \tau \qquad \tau ::= a \mid a \cdot \#_x(\varphi) \mid \tau + \tau$$

for $p \in AP$, $x \in V$ and $a \in \mathbb{Z}$. Common abbreviations such as $\top \equiv p \vee \neg p$, $\bot \equiv \neg\top$, $\texttt{F}\,\varphi \equiv \top\,\texttt{U}\,\varphi$, $\texttt{G}\,\varphi \equiv \neg\,\texttt{F}\,\neg\varphi$ and $\texttt{A}\,\varphi \equiv \neg\,\texttt{E}\,\neg\varphi$ may also be used. The set of all *subformulae* of a formula φ (including itself) is denoted $\texttt{sub}(\varphi)$ and $|\varphi|$ denotes the length of φ, with binary encoding of numbers.

Semantics. Intuitively, a variable x is used to mark some position on the concerned run. Within the scope of x a term $\#_x(\varphi)$ refers to the number of times the formula φ holds between the current position and that marked by x. The semantics of $\texttt{CCTL}^*$ is hence defined with respect to a Kripke structure $\mathcal{K} = (S, s_I, E, \lambda)$, a run $\rho \in \texttt{Runs}(\mathcal{K})$, a position $i \in \mathbb{N}$ on ρ and a valuation function $\theta : V \to \mathbb{N}$ assigning a position (index) on ρ to each variable. The

satisfaction relation $\models$ is defined inductively for $p \in AP$, formulae φ, ψ and terms τ_1, τ_2 by

$$\begin{aligned}
(\rho,i,\theta) &\models p &&\overset{\text{def}}{\Leftrightarrow}\ p \in \lambda(\rho(i)),\\
(\rho,i,\theta) &\models \mathtt{X}\,\varphi &&\overset{\text{def}}{\Leftrightarrow}\ (\rho,i+1,\theta) \models \varphi,\\
(\rho,i,\theta) &\models \varphi\,\mathtt{U}\,\psi &&\overset{\text{def}}{\Leftrightarrow}\ \exists k \geq i : (\rho,k,\theta) \models \psi \text{ and } \forall j \in [i,k-1] : (\rho,j,\theta) \models \varphi,\\
(\rho,i,\theta) &\models \mathtt{E}\,\varphi &&\overset{\text{def}}{\Leftrightarrow}\ \exists \rho' \in \mathtt{Runs}(\mathcal{K}) : \forall j \in [0,i] : \rho'(j) = \rho(j) \text{ and } (\rho',i,\theta) \models \varphi,\\
(\rho,i,\theta) &\models x.\varphi &&\overset{\text{def}}{\Leftrightarrow}\ (\rho,i,\theta[x \mapsto i]) \models \varphi,\\
(\rho,i,\theta) &\models \tau_1 \leq \tau_2 &&\overset{\text{def}}{\Leftrightarrow}\ [\![\tau_1]\!](\rho,i,\theta) \leq [\![\tau_2]\!](\rho,i,\theta),
\end{aligned}$$

where the Boolean cases are omitted and the semantics of terms is given, for $a \in \mathbb{Z}$, by

$$\begin{aligned}
[\![a]\!](\rho,i,\theta) &\overset{\text{def}}{=} a,\\
[\![\tau_1+\tau_2]\!](\rho,i,\theta) &\overset{\text{def}}{=} [\![\tau_1]\!](\rho,i,\theta) + [\![\tau_2]\!](\rho,i,\theta),\\
[\![a \cdot \#_x(\varphi)]\!](\rho,i,\theta) &\overset{\text{def}}{=} a \cdot |\{j \in \mathbb{N} \mid \theta(x) \leq j \leq i, (\rho,j,\theta) \models \varphi\}|.
\end{aligned}$$

We abbreviate $(\rho, i, \mathbf{0}) \models \varphi$ by $(\rho, i) \models \varphi$ and $(\rho, 0) \models \varphi$ by $\rho \models \varphi$ and say that ρ satisfies φ (at position i) in these cases. Moreover, we say a state $s \in S$ satisfies φ, denoted $s \models \varphi$ if there are $\rho_s \in \mathtt{Runs}(\mathcal{K})$ and $i \in \mathbb{N}$ such that $\rho_s(i) = s$ and $(\rho_s, i) \models \varphi$. The Kripke structure $\mathcal{K}$ satisfies φ, denoted by $\mathcal{K} \models \varphi$, if $s_I \models \varphi$. Note that we choose to define the model-checking relation existentially but since the formalism is closed under negation, this does not have major consequences on our results.

Fragments. We define the following fragments of CCTL* in analogy to the classical logics LTL and CTL. The *linear* time fragment CLTL consists of those CCTL* formulae that do not use the path quantifiers E and A. The *branching* time logic CCTL restricts the use of temporal operators X and U such that each occurrence must be preceded immediately by either E or A. Similar branching-time logics have been considered in [19].

Frequency logics. A major subject of our investigation are frequency constraints. This concept embeds naturally into the context of counting logics as it can be understood as a restricted form of counting. We therefore define in the following the frequency temporal logics fCTL*, fLTL and fCTL as fragments of CCTL*. Consider the following grammar defining the syntax of formulae φ for natural numbers $n, m \in \mathbb{N}$ with $n \leq m > 0$ and $p \in AP$.

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \alpha \qquad\qquad \beta ::= \mathtt{X}\,\varphi \mid \varphi\,\mathtt{U}^{\frac{n}{m}}\,\varphi$$

With the additional rule $\alpha ::= \mathtt{E}\,\varphi \mid \beta$ it defines precisely the set of fCTL* formulae while it defines fCTL for $\alpha ::= \mathtt{E}\,\beta \mid \mathtt{A}\,\beta$ and fLTL for $\alpha ::= \beta$. The semantics is defined by interpreting fCTL* formulae as CCTL* with the additional equivalence

$$\varphi\,\mathtt{U}^{\frac{n}{m}}\,\psi \overset{\text{def}}{\equiv} \psi \vee x\,.\,\mathtt{F}\,((\mathtt{X}\,\psi) \wedge m \cdot \#_x(\varphi) \geq n \cdot \#_x(\top)) \tag{1}$$

for fCTL* formulae φ and ψ and a variable $x \in V$ not being used in either φ or ψ.

▶ **Example 1.** *Consider the Kripke structure given by Figure 1 and the CCTL formula* $\varphi_1 = z.\,\mathtt{A}\,\mathtt{G}\,(q \to (\#_z(p) \leq \#_z(\mathtt{E}\,\mathtt{X}\,r)))$*. It basically states that on every path reaching* s_5 *there must be a position where the states* s_2 *and* s_4 *(satisfying* $\mathtt{E}\,\mathtt{X}\,r$*) together have been visited at least as often as the state* s_0*. A different, yet similar statement can be formulated using only frequency constraints:* $\varphi_1' = \mathtt{A}((\mathtt{E}\,\mathtt{X}\,r)\,\mathtt{U}^{\frac{1}{2}}\,q)$ *states that* s_5 *must always be reached while visiting*

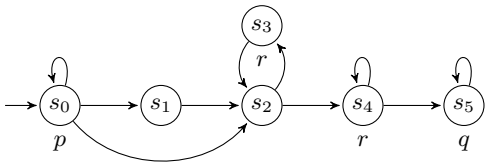


Figure 1 A flat Kripke over $AP = \{p, q, r\}$.

s_2 and s_4 together at least as often as s_0, s_1 and s_3. Both φ_1 and φ_1' are violated, e.g. by the path $s_0^3 s_1 s_2 s_4 s_5^\omega$. The Kripke structure however satisfies $\varphi_2 = z.\,\mathtt{A}\,\mathtt{G}\,(\neg q \to \mathtt{E}\,\mathtt{F}\,\#_z(p) < \#_z(r))$ because from every state except s_5 the number of positions that satisfy r can be increased arbitrary without increasing the number of those satisfying p. Notice that this would not be the case, e.g., if s_4 was labelled by p.

While the positional variables in CCTL* are a very flexible way of defining the scope of a constraint, frequency constraints in fCTL* are always bound to the scope of an until operator. The same applies to the counting constraints of ${}_{\mathtt{c}}\mathtt{LTL}$ as defined in [19]. For example, the ${}_{\mathtt{c}}\mathtt{LTL}$ formula $\varphi\,\mathtt{U}_{[a_1\#(\varphi_1)+\cdots+a_n\#(\varphi_n)\geq k]}\,\psi$ is equivalent to the CLTL formula $z.\varphi\,\mathtt{U}(\psi \wedge a_1\#_z(\varphi_1) + \cdots + a_n\#_z(\varphi_n) \geq k)$. Admitting only natural coefficients, ${}_{\mathtt{c}}\mathtt{LTL}$ can be encoded even in LTL making it thus strictly less expressive than fLTL. On the other hand, ${}_{\mathtt{c}}\mathtt{LTL}_{\pm}$ admits arbitrary integer coefficients, which is more general than the frequency until operator of fLTL. For example, $p\,\mathtt{U}^{\frac{a}{b}}\,q$ can be expressed as $\top\,\mathtt{U}_{[b\#(p)-a\#(\top)\geq 0]}\,q$ in ${}_{\mathtt{c}}\mathtt{LTL}_{\pm}$. The relation between ${}_{\mathtt{c}}\mathtt{CTL}_{\pm}$ and fCTL, as well as ${}_{\mathtt{c}}\mathtt{CTL}^*_{\pm}$ and fCTL* is analogous.

Model-checking problem. We now present the problem on which we focus our attention. The *model-checking problem* for a class $\mathfrak{K} \subseteq \mathtt{KS}$ of Kripke structures and a specification language $\mathcal{L}$ (in our case all the specification languages are fragments of CCTL*) is denoted by $\mathtt{MC}(\mathfrak{K}, \mathcal{L})$ and defined as the following decision problem.

Input: A Kripke structure $\mathcal{K} \in \mathfrak{K}$ and a formula $\varphi \in \mathcal{L}$.
Decide: Does $\mathcal{K} \models \varphi$ hold?

For temporal logics without counting variables, the model-checking problem over Kripke structure has been studied intensively and is known to be PSpace-complete for LTL and CTL* and in P for CTL (see e.g. [3]). It has recently been shown that when restricting to flat (or weak) structures the complexity of the model-checking problem for LTL is lower than in the general case [17]: it drops from PSpace to NP. As we show later, in the case of CCTL*, flatness of the structures allows us to regain decidability of the model-checking problem which is in general undecidable. In this paper, we propose various ways to solve the model-checking problem of fragments of CCTL* over flat structures. For some of them we provide a direct algorithm, for others we reduce our problem to the satisfiability problem of a decidable extension of Presburger arithmetic.

3 Model-checking Frequency CTL

Satisfiability of fLTL is undecidable [5] implying the same for model-checking fLTL, CLTL and CCTL* over Kripke structures. This applies moreover to CCTL [19]. In contrast, we show in the following that MC(KS, fCTL) is decidable using an extension of the well-known labelling algorithm for CTL (see e.g. [3]).

Let $\mathcal{K} = (S, s_I, E, \lambda)$ be a Kripke structure and Φ an fCTL formula. We compute recursively subsets $S_\varphi \subseteq S$ of the states of $\mathcal{K}$ for every subformula $\varphi \in \mathtt{sub}(\Phi)$ of Φ such that for all $s \in S$ we have $s \in S_\varphi$ iff $s \models \varphi$. Checking whether the initial state s_I is contained in S_Φ then solves the problem. Propositions ($p \in AP$), negation ($\neg\varphi$), conjunction ($\varphi \wedge \psi$) and *temporal next* ($\mathtt{E}\,\mathtt{X}\,\varphi$, $\mathtt{A}\,\mathtt{X}\,\varphi$) are handled as usual, e.g. $S_p = \{q \in S \mid p \in \lambda(q)\}$ and $S_{\mathtt{E}\,\mathtt{X}\,\varphi} = \{q \in S \mid \exists q' \in S_\varphi : (q, q') \in \delta\}$.

To compute if a state $s \in S$ satisfies a formula of the form $\mathtt{E}\,\varphi\,\mathtt{U}^r\,\psi$ or $\mathtt{A}\,\varphi\,\mathtt{U}^r\,\psi$, assume that S_φ and S_ψ are given inductively. If $s \in S_\psi$ we immediately have $s \in S_{\mathtt{E}\,\varphi\,\mathtt{U}^r\,\psi}$ and $s \in S_{\mathtt{A}\,\varphi\,\mathtt{U}^r\,\psi}$. For the remaining cases, the problem of deciding whether $s \in S_{\mathtt{E}\,\varphi\,\mathtt{U}^r\,\psi}$ or $s \in S_{\mathtt{A}\,\varphi\,\mathtt{U}^r\,\psi}$, respectively, can be reduced in linear time to the *repeated control-state reachability* problem in systems with one integer counter. The idea is to count the ratio along paths $\rho \in S^\omega$ in $\mathcal{K}$ as follows, in direct analogy to the semantics defined in Equation 1. Assume $r = \frac{n}{m}$ for $n, m \in \mathbb{N}$ and $n \leq m$. For passing any position on ρ we pay a fee of n and for those positions that satisfy φ we gain a reward of m. Thus, we obtain a non-negative balance of rewards and gains at some position on ρ if, in average, among every m positions there are at least n positions that satisfy φ, meaning the ratio constraint is satisfied. In $\mathcal{K}$, this balance along a path can be tracked using an *integer counter* that is increased by $m - n$ when leaving a state $s' \in S_\varphi$ and decreased by adding $-n$ whenever leaving a state $s' \notin S_\varphi$. Thus, let $\hat{\mathcal{K}}_s = (S, s, \{c\}, \Delta)$ be the counter system with

$$\Delta = \{(t, \mathbf{u}, \emptyset, t') \mid (t, t') \in E,\ t \notin S_\varphi \Rightarrow \mathbf{u}(c) = -n,\ t \in S_\varphi \Rightarrow \mathbf{u}(c) = m - n\}.$$

The state s satisfies the formula $\mathtt{A}\,\varphi\,\mathtt{U}^r\,\psi$ if there is no path starting in state s violating the formula $\varphi\,\mathtt{U}^r\,\psi$. The latter is the case if at every position where ψ holds, the balance computed up to this position is negative. Therefore, consider an extension $\mathcal{R}_s$ of $\hat{\mathcal{K}}_s$ where every edge leading into a state $s' \in S_\psi$ is guarded by the constraint $c < 0$. Every (infinite) run of $\mathcal{R}_s$ is now a counter example for the property holding at s. To decide whether $s \in S_{\mathtt{A}\,\varphi\,\mathtt{U}^r\,\psi}$ it suffices to check that in $\mathcal{R}_s$ no state is repeatedly reachable from s.

A formula $\mathtt{E}\,\varphi\,\mathtt{U}^r\,\psi$ is satisfied by s if there is some state $s' \in S_\psi$ reachable from s with a non-negative balance. Hence, consider the counter system $\mathcal{U}_s = (S \uplus \{\mathtt{t}\}, s, \{c\}, \Delta')$ obtained from $\hat{\mathcal{K}}_s$ featuring a new sink state $\mathtt{t} \notin S$. The transition relation

$$\Delta' = \Delta \cup \{(s', \mathbf{0}, \{c \geq 0\}, \mathtt{t}) \mid s' \in S_\psi\} \cup \{(\mathtt{t}, \mathbf{0}, \emptyset, \mathtt{t})\}$$

extends Δ such that precisely the paths starting in s and reaching a state $s' \in S_\psi$ with non-negative counter value (i.e. sufficient ratio) can be extended to reach $\mathtt{t}$. Checking if s is supposed to be contained in $S_{\mathtt{E}\,\varphi\,\mathtt{U}^r\,\psi}$ then amounts to decide whether $\mathtt{t}$ is (repeatedly) reachable from s in $\mathcal{U}_s$.

Finally, repeated reachability is easily translated to the *accepting run* problem of *Büchi pushdown systems (BPDS)* and the latter is in P [6]. A counter value $n \geq 0$ can be encoded into a stack of the form $\oplus^n$ while $\ominus^n$ encodes $-n \leq 0$ and for evaluating the guards $c \geq 0$ and $c < 0$ only the top symbol is relevant. Simulating an update of the counter by a number $a \in \mathbb{Z}$ requires to perform $|a|$ push or pop actions. The size of the system is therefore linear in the largest absolute update value and hence exponential in its binary representation. Since the updates of the constructed counter systems originate from the ratios in Φ, the corresponding BPDS are of up to exponential size in $|\Phi|$. During the labelling procedure this step must be performed at most a polynomial number of times giving an exponential-time algorithm.

▶ **Theorem 2.** *MC(KS, fCTL) is in* Exp.

It is worth noting that for a fixed formula (program complexity) or a unary encoding of numbers in frequency constraints, the size of the constructed Büchi pushdown systems and thus the runtime of the algorithm remains polynomial.

▶ **Corollary 3.** *MC(KS, fCTL) with unary number encoding is in* P.

4 Model-checking Frequency LTL over Flat Kripke Structures

We show in this section that model-checking fLTL is decidable over flat Kripke structures. As decision procedure we employ a *guess and check* approach: given a flat Kripke structure $\mathcal{K}$ and an fLTL formula Φ, we choose non-deterministically a set of satisfying runs to witness $\mathcal{K} \models \Phi$. As representation for such sets we introduce *augmented path schemas* that extend the concept of path schemas [20, 11] and provide for each of its runs a labelling by formulae. We show that if an augmented path schema features a syntactic property that we call *consistency* then the associated runs actually satisfy the formulae they are labelled with. Moreover, we show that every run of $\mathcal{K}$ is in fact represented by some consistent schema of size at most exponential in $|\mathcal{K}| + |\Phi|$. This gives rise to the following non-deterministic procedure.

1. **Read as input** an FKS $\mathcal{K}$ and an fLTL formula Φ.
2. **Guess** an augmented path schema $\mathcal{P}$ in $\mathcal{K}$ of at most exponential size.
3. **Terminate** successfully if $\mathcal{P}$ is consistent and accepts a run that is initially labelled by Φ.

We fix for this section a flat Kripke structure $\mathcal{K} = (S, s_I, E, \lambda)$ and an fLTL formula Φ. For convenience we assume that $AP \subseteq \mathtt{sub}(\Phi)$. Omitted technical details can be found in [9].

4.1 Augmented Path Schemas

The set of runs of $\mathcal{K}$ can be represented as a finite number of so-called path schemas that consist of a sequence of paths and simple loops consecutive in $\mathcal{K}$ [20, 11]. A path schema represents all runs that follow the given shape while repeating each loop arbitrarily often. For our purposes we extend this idea with additional labellings and introduce integer counters, updates and guards that can restrict the admitted runs.

▶ **Definition 4** (Augmented Path Schema)**.** An *augmented state* of $\mathcal{K}$ is a tuple $a = (s, L, G, \mathbf{u}, t) \in S \times 2^{\mathtt{sub}(\Phi)} \times 2^{\mathfrak{G}(C)} \times \mathbb{Z}^C \times \{\mathtt{L}, \mathtt{R}\}$ comprised of a state s of $\mathcal{K}$, a set of formula labels L, guards G and an update $\mathbf{u}$ over a set of counter names C, and a *type* indicating whether the state is part of a loop (L) or a not (R). We denote by $\mathtt{st}(a) = s$, $\mathtt{lab}(a) = L$, $\mathtt{g}(a) = G$, $\mathtt{u}(a) = \mathbf{u}$ and $\mathtt{t}(a) = t$ the respective components of a. An *augmented path* in $\mathcal{K}$ is a sequence $u = a_0 \dots a_n$ of augmented states a_i such that $(\mathtt{st}(a_i), \mathtt{st}(a_{i+1})) \in E$ for $i \in [0, n-1]$. If $\mathtt{t}(a_i) = \mathtt{R}$ for all $i \in [0, n-1]$ then u is called a *row*. It is called an *augmented simple loop* (or simply *loop*) if it is non-empty and $(\mathtt{st}(a_n), \mathtt{st}(a_1)) \in E$ and $\mathtt{st}(a_i) \neq \mathtt{st}(a_j)$ for $i \neq j$ and $\mathtt{t}(a_i) = \mathtt{L}$ for all $i \in [0, n-1]$.

An *augmented path schema (APS)* in $\mathcal{K}$ is a tuple $\mathcal{P} = (P_0, \dots, P_n)$ where each component P_k is a row or a loop, P_n is a loop and their concatenation $P_1 P_2 \dots P_n$ is an augmented path.

Thanks to counters we can, for example, restrict to those runs satisfying a specific frequency constraint at some positions tracking it as discussed in Section 3. Figure 2 shows an example of an APS with edges indicating the possible state progressions. It features a single counter that tracks the frequency constraint of a formula $r \,\mathtt{U}^{\frac{2}{3}}\, q$ from state 1.

We denote by $|\mathcal{P}| = |P_0 \dots P_n|$ the size of $\mathcal{P}$ and use global indices $\ell \in [0, |\mathcal{P}| - 1]$ to address the $(\ell+1)$-th augmented state in $P_0 \dots P_n$, denoted $\mathcal{P}[\ell]$. To distinguish these global indices from positions in arbitrary sequences, we refer to them as *locations* of $\mathcal{P}$. Moreover,

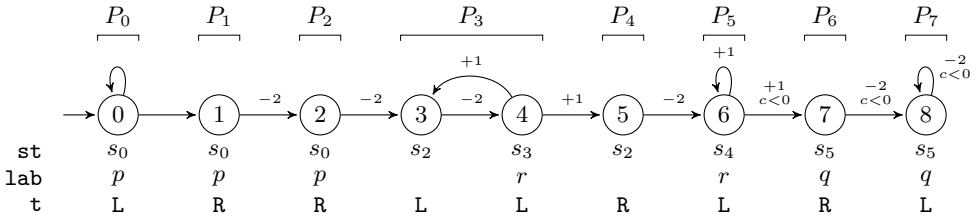


Figure 2 An APS $\mathcal{P} = (P_0, \ldots, P_7)$ of the flat Kripke structure in Figure 1.

$\mathtt{loc}_{\mathcal{P}}(k) = \{\ell \mid |P_0P_1 \ldots P_{k-1}| \leq \ell < |P_0P_1 \ldots P_k|\}$ denotes for $0 \leq k \leq n$ the set of locations belonging to component P_k and for all locations $\ell \in \mathtt{loc}_{\mathcal{P}}(k)$ we denote the corresponding component index in $\mathcal{P}$ by $\mathtt{comp}_{\mathcal{P}}(\ell) = k$. For example, in Figure 2 we have $\mathtt{loc}_{\mathcal{P}}(3) = \{3, 4\}$ and $\mathtt{comp}_{\mathcal{P}}(6) = 5$ because the seventh state of $\mathcal{P}$ belongs to P_5. We extend the component projections for augmented states to (sequences of) locations of $\mathcal{P}$ and write, e.g., $\mathtt{st}_{\mathcal{P}}(\ell_1\ell_2)$ for $\mathtt{st}(\mathcal{P}[\ell_1])\mathtt{st}(\mathcal{P}[\ell_2])$ and $\mathtt{u}_{\mathcal{P}}(\ell)$ for $\mathtt{u}(\mathcal{P}[\ell])$.

An APS $\mathcal{P}$ gives rise to a counter system $\mathtt{CS}(\mathcal{P}) = (Q, 0, C, \Delta)$ where $Q = \{0, \ldots, |P| - 1\}$, C are the counters used in the augmented states of $\mathcal{P}$ and Δ consists of those transitions $(\ell, \mathtt{u}_{\mathcal{P}}(\ell), \mathtt{g}_{\mathcal{P}}(\ell'), \ell')$ such that $0 \leq \ell' = \ell + 1 < |\mathcal{P}|$ or $\ell' < \ell$ and $\{\ell', \ell' + 1, \ldots, \ell\} = \mathtt{loc}_{\mathcal{P}}(k)$ for some loop P_k. Notice that the APS in Figure 2 is presented as its corresponding counter system. Let $\mathtt{succ}_{\mathcal{P}}(\ell)$ denote the set $\{\ell' \in Q \mid \exists \mathbf{u}, G : (\ell, \mathbf{u}, G, \ell') \in \Delta\}$ of successors of ℓ in $\mathtt{CS}(\mathcal{P})$. A *run* of $\mathcal{P}$ is a run of $\mathtt{CS}(\mathcal{P})$ that visits each location $\ell \in S$ at least once. The set of all runs of $\mathcal{P}$ is denoted $\mathtt{Runs}(\mathcal{P})$. As a consequence, a run visits the last loop infinitely often. We say that an APS $\mathcal{P}$ is non-empty iff $\mathtt{Runs}(\mathcal{P}) \neq \emptyset$. Since every run $\sigma \in \mathtt{Runs}(\mathcal{P})$ corresponds, by construction of $\mathcal{P}$, to a path $\mathtt{st}_{\mathcal{P}}(\rho) \in Q^\omega$ in $\mathcal{K}$ we define the satisfaction of an $\mathtt{fLTL}$ formula φ at position i by $(\sigma, i) \models_{\mathcal{P}} \varphi$ iff $(\mathtt{st}_{\mathcal{P}}(\sigma), i) \models \varphi$.

Finally, notice that $\mathtt{CS}(\mathcal{P})$ is in fact a *flat* counter system. It is shown in [11] that $\mathtt{LTL}$ properties can be verified over flat counter systems in non-deterministic polynomial time. Since $\mathtt{LTL}$ can express that each location of $\mathtt{CS}(\mathcal{P})$ is visited we obtain the following result.

▶ **Lemma 5** ([11]). *Deciding non-emptiness of APS is in* NP.

4.2 Labellings of Consistent APS are Correct

An APS $\mathcal{P}$ assigns to every position i on each of its runs σ the labelling $L_i = \mathtt{lab}_{\mathcal{P}}(\sigma(i))$. We are interested in this labelling being *correct* with respect to some $\mathtt{fLTL}$ formula Φ in the sense that $\Phi \in L_i$ if and only if $(\sigma, i) \models \Phi$. The notion of consistency introduced in the following provides a sufficient criterion for correctness of the labelling of all runs of an APS.

An augmented path $u = a_0 \ldots a_n$ is said to be *good*, *neutral* or *bad* for an $\mathtt{fLTL}$ formula $\Psi = \varphi \mathtt{U}^{\frac{x}{y}} \psi$ if the number $d = |\{0 \leq i < |u| \mid \varphi \in \mathtt{lab}(u(i))\}|$ of positions labelled with φ is larger than $(d > \frac{x}{y} \cdot |u|)$, equal to $(d = \frac{x}{y} \cdot |u|)$ or smaller than $(d < \frac{x}{y} \cdot |u|)$, respectively, the fraction $\frac{x}{y}$ of all positions of u. A tuple $(P_0, \ldots, P_n)$ of rows and loops (not necessarily an APS) is called *L-periodic* for a set $L \subseteq \mathtt{sub}(\Phi)$ of labels if all augmented paths P_k share the same labelling with respect to L, that is for all $0 \leq k < n - 1$ we have $|P_k| = |P_{k+1}|$ and $\mathtt{lab}(P_k(i)) \cap L = \mathtt{lab}(P_{k+1}(i)) \cap L$ for all $0 \leq i < |P_k|$.

▶ **Definition 6** (Consistency). Let $\mathcal{P} = (P_0, \ldots, P_n)$ be an APS in $\mathcal{K}$, $k \in [0, n]$ and $\ell \in \mathtt{loc}_{\mathcal{P}}(k)$ a location on component P_k. The location ℓ is *consistent* with respect to an $\mathtt{fLTL}$ formula Ψ if all locations of $\mathcal{P}$ are consistent with respect to all strict subformulae of Ψ and one of the following conditions applies.

1. $\Psi \in AP$ and $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell) \Leftrightarrow \Psi \in \lambda(\mathtt{st}_{\mathcal{P}}(\ell))$, or $\Psi = \varphi \wedge \psi$ and $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell) \Leftrightarrow \varphi, \psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$, or $\Psi = \neg\varphi$ and $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell) \Leftrightarrow \varphi \notin \mathtt{lab}_{\mathcal{P}}(\ell)$.
2. $\Psi = \mathtt{X}\,\varphi$ and $\forall \ell' \in \mathtt{succ}_{\mathcal{P}}(\ell) : \Psi \in \mathtt{lab}_{\mathcal{P}}(\ell) \Leftrightarrow \varphi \in \mathtt{lab}_{\mathcal{P}}(\ell')$.
3. $\Psi = \varphi\, \mathtt{U}^{\frac{x}{y}}\, \psi$ and one of the following holds:
   - **a.** $\Psi, \psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$
   - **b.** $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$ and P_n is good for Ψ and $\exists \ell' \in \mathtt{loc}_{\mathcal{P}}(n) : \psi \in \mathtt{lab}_{\mathcal{P}}(\ell')$
   - **c.** $\mathtt{t}_{\mathcal{P}}(\ell) = \mathtt{R}$ and there is a counter $c \in C$ such that $\forall \ell' < \ell : \mathtt{u}_{\mathcal{P}}(\ell')(c) = 0$ and $\forall \ell' \geq \ell : \varphi \in \mathtt{lab}_{\mathcal{P}}(\ell') \Rightarrow \mathtt{u}_{\mathcal{P}}(\ell')(c) = y - x$ and $\forall \ell' \geq \ell : \varphi \notin \mathtt{lab}_{\mathcal{P}}(\ell') \Rightarrow \mathtt{u}_{\mathcal{P}}(\ell')(c) = -x$ and
     - if $\Psi \notin \mathtt{lab}_{\mathcal{P}}(\ell)$ then $\psi \notin \mathtt{lab}_{\mathcal{P}}(\ell)$ and $\forall \ell' > \ell : \psi \in \mathtt{lab}_{\mathcal{P}}(\ell') \Rightarrow (c < 0) \in \mathtt{g}_{\mathcal{P}}(\ell')$ and
     - if $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$ then $\exists \ell' > \ell : \psi \in \mathtt{lab}_{\mathcal{P}}(\ell') \wedge (c \geq 0) \in \mathtt{g}_{\mathcal{P}}(\ell')$.
   - **d.** There is $k' \in [0, n]$ such that all locations $\ell' \in \mathtt{loc}_{\mathcal{P}}(k')$ are consistent wrt. Ψ and
     - if $k = n$ then $k' < k$ and $(P_{k'}, P_{k'+1}, \ldots, P_k)$ is $\{\varphi, \psi, \Psi\}$-periodic,
     - if $k < n$ and P_k is good or neutral for Ψ and $\Psi \notin \mathtt{lab}_{\mathcal{P}}(\ell)$, or P_k is bad for Ψ and $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$ then $k' < k < n$ and $(P_{k'}, P_{k'+1}, \ldots, P_{k+1})$ is $\{\varphi, \psi, \Psi\}$-periodic, and
     - if $k < n$ and P_k is good or neutral for Ψ and $\Psi \in \mathtt{lab}_{\mathcal{P}}(\ell)$, or P_k is bad for Ψ and $\Psi \notin \mathtt{lab}_{\mathcal{P}}(\ell)$ then $k < k' < n$ and $(P_k, P_{k+1}, \ldots, P_{k'+1})$ is $\{\varphi, \psi, \Psi\}$-periodic.

The APS $\mathcal{P}$ is consistent with respect to Ψ if it is the case for all its locations.

The cases **1** and **2** reflect the semantics syntactically. For instance, location 0 in Figure 2 can be labelled consistently with $\mathtt{X}\,p$ since all its sucessor (0 and 1) are labelled with p. Case **3**, concerning the (frequency) until operator, is more involved.

Assume that $\Phi = \varphi\, \mathtt{U}^{\frac{x}{y}}\, \psi$ is an until formula and that the labelling of $\mathcal{K}$ by φ and ψ is consistent. In some cases, it is obvious that Φ holds, namely at positions labelled by ψ (case **3a**) or if the final loop already guarantees that Φ always holds (case **3b**). If neither is the case we can apply the idea discussed in Section 3 and use a counter to check explicitly if at some point the formula Φ holds (case **3c**). Recall that to validate (or invalidate) the labelling of a location by the formula Φ a specific counter tracks the frequency constraint in terms of the balance between fees and rewards along a run. For the starting point to be unique this case only applies to locations that are not part of a loop. For those labelled with Φ there should exist a location in the future where ψ holds and the balance counter is non-negative. For those not labelled with Φ all locations in the future where ψ holds must be entered with negative balance. Finally, case **3d** can apply (not only) to loops and is based on the following reasoning: if a loop is good (bad) and Φ is supposed to hold at some of its locations then it suffices to verify that this is the case during any of its future (past) iterations, e.g. the last (first) and vice versa if Φ is supposed not to hold. This is the reason why this case allows for delegating consistency along a periodic pattern.

For instance, consider the formula $\Psi = r\, \mathtt{U}^{\frac{2}{3}}\, q$ and the APS shown in Figure 2. It is consistent to *not* label location 1 by Ψ because the counter c tracks the balance and locations 7 and 8 are guarded as required. If a run takes, e.g., the loop P_5 seven times, it has to take P_3 at least twice to satisfy all guards. This ensures that the ratio for the proposition r is strictly less than $\frac{2}{3}$ upon reaching the first (and thus any) occurrence of q. Note that to also make location 2 consistent, an additional counter needs to be added. Consistency with respect to Ψ is then inherited by location 0 from location 1 according to case **3d** of the definition. Intuitively, additional iterations of the bad loop P_0 can only diminish the ratio.

The definition of consistency guarantees that if an APS is consistent with respect to Φ then for every run of the APS, each time the formula Φ is encountered, it holds at the current position (see [9] for further details). Hence we obtain the following lemma that guarantees correctness of our decision procedure.

▶ **Lemma 7** (Correctness). *If there is an APS $\mathcal{P}$ in $\mathcal{K}$ such that $\mathcal{P}$ is consistent wrt. Φ and $\Phi \in \mathtt{lab}_{\mathcal{P}}(0)$ and $\mathtt{Runs}(\mathcal{P}) \neq \emptyset$ then $\mathcal{K} \models \Phi$.*

4.3 Constructing Consistent APS

Assuming that our flat Kripke structure $\mathcal{K}$ admits a run ρ such that $\rho \models \Phi$, we show how to construct a non-empty APS that is initially labelled by and consistent with respect to Φ. It will be of at most exponential size in $|\mathcal{K}| + |\Phi|$ and is built recursively over the structure of Φ.

Concerning the base case where $\Phi \in AP$, all paths in a flat structure can be represented by a path schema of linear size [20, 11]. Intuitively, since $\mathcal{K}$ is flat, every subpath $s_i s_{i+1} \dots s_{i'} \dots s_{i''}$ of ρ where a state $s_i = s_{i'} = s_{i''}$ occurs more than twice is equal to $(s_i s_{i+1} \dots s_{i'-1})^k s_{i''}$ for some $k \in \mathbb{N}$. Hence, there are simple subpaths $u_0, \dots, u_m \in S^+$ of ρ and positive numbers of iterations $n_0, \dots, n_{m-1} \in \mathbb{N}$ such that $\rho = u_0^{n_0} u_1^{n_1} \dots u_{m-1}^{n_{m-1}} u_m^{\omega}$ and $|u_0 u_1 \dots u_m| \leq 2|S|$. From this decomposition, we build an APS being consistent with respect to all propositions. Henceforth, we assume by induction an APS $\mathcal{P}$ being consistent with respect to all strict subformulae of Φ and a run $\sigma \in \mathtt{Runs}(\mathcal{P})$ with $\mathtt{st}_{\mathcal{P}}(\sigma) = \rho$. If $\Phi = \varphi \wedge \psi$ or $\Phi = \neg\varphi$, Definition 6 determines for each augmented state of $\mathcal{P}$ whether it is supposed to be labelled by Φ or not. It remains hence to deal with the next and frequency until operators.

Labelling $\mathcal{P}$ by $\mathtt{X}\,\varphi$. If $\Phi = \mathtt{X}\,\varphi$ the labelling at some location ℓ is extended according to the labelling of its successors. These may disagree upon φ (only) if ℓ has more than one successor, i.e., being the last location on a loop P_k of $\mathcal{P} = (P_0, \dots, P_m)$. In that case we consult the run σ: if it takes P_k only once, this loop can be *cut* and replaced by P'_k that we define to be an exact copy except that all augmented states have type $\mathtt{R}$ instead of $\mathtt{L}$. If otherwise σ takes P_k at least twice, the loop can be *unfolded* by inserting P'_k between P_k and P_{k+1}, i.e. letting $\mathcal{P}' = (P_0, \dots, P_k, P'_k, P_{k+1}, \dots, P_m)$. Either way, σ remains a run of the obtained APS, up to shifting the locations $\ell' > \ell$ if the extra component was inserted (recall that locations are indices). Importantly, cutting or unfolding any loop, even any number of times, in $\mathcal{P}$ preserves consistency.

Labelling $\mathcal{P}$ by $\varphi\,\mathtt{U}^r\,\psi$. The most involved case is to label a location ℓ by $\Phi = \varphi\,\mathtt{U}^r\,\psi$. First, assume that ℓ is part of a row. Whether it must be labelled by Φ is uniquely determined by σ. This is consistent if case **3a** or **3b** of Definition 6 applies. The conditions of case **3c** are also realised easily in most situations. Only, if Φ holds at ℓ but every location ℓ' witnessing this (by being reachable with sufficient frequency and labelled by ψ) is part of some loop P'. Adding the required guard directly to ℓ' may be too strict if σ traverses P' more than once. However, the first iteration (if P' is bad for Φ) or the last iteration (if P' is good) on σ contains a position (labelled with ψ) witnessing that Φ holds if any iteration does. Thus it suffices to unfold the loop once in the respective direction. For example, consider in Figure 2 location 5 and a formula $\varphi = r\,\mathtt{U}^{\frac{2}{5}}\,q$. Location 8 could witness that φ holds but a corresponding guard would be violated eventually since P_7 is bad for φ. The first iteration is thus the optimal choice. The unfolding P_6 separates it such that location 7 can be guarded instead without imposing unnecessary constraints.

Now assume that location ℓ, to be labelled or not with Φ, is part of a loop P which is *stable* in the sense that Φ holds either at all positions i with $\sigma(i) = \ell$ or at none of them. With two unfoldings of P, made consistent as above, case **3d** applies. However, σ may go through ℓ several, say $n > 1$, times where Φ holds at some but not all of the corresponding positions. If n is small we can replace P by precisely n unfoldings, thus reducing to the

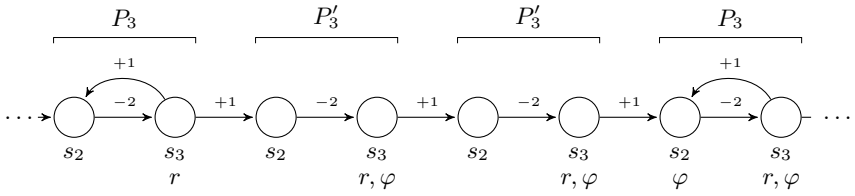


Figure 3 A decomposition of loop P_3 from Figure 2 allowing for a correct labelling wrt. $\varphi = r \,\mathtt{U}^{\frac{2}{3}}\, q$.

previous case without increasing the size of the structure too much. We can moreover show that if n is not small then it is possible to decompose such a problematic loop into a constant number of unfoldings and two stable copies based on the following observation.

▶ **Lemma 8** (Decomposition). *Let $P = \mathcal{P}[\ell_0] \dots \mathcal{P}[\ell_{|P|-1}]$ be a non-terminal loop in $\mathcal{P}$ with corresponding location sequence $v = \ell_0 \dots \ell_{|P|-1}$ and $\hat{n} = |P| \cdot y$ for some $y > 0$. For every run $\sigma = uv^n w \in \mathtt{Runs}(\mathcal{P})$ where $n \geq \hat{n} + 2$ there are n_1 and n_2 such that $\sigma = uv^{n_1} v^{\hat{n}} v^{n_2} w$ and for all positions i on σ with $|u| \leq i < |uv^{n_1 - 1}|$ or $|uv^{n_1} v^{\hat{n}}| \leq i < |uv^{n_1} v^{\hat{n}} v^{n_2 - 2}|$ we have $(\sigma, i) \models_{\mathcal{P}} \Phi$ iff $(\sigma, i + |P|) \models_{\mathcal{P}} \Phi$.*

▶ **Example 9.** *Consider again the APS $\mathcal{P}$ in Figure 2, a run $\sigma \in \mathtt{Runs}(\mathcal{P})$ and the location 3. Whether or not $\varphi = r \,\mathtt{U}^{\frac{2}{3}}\, q$ holds at some position i with $\sigma(i) = 3$ depends on how often σ traverses the good loop P_5 (the more the better) and how often it repeats P_3 after position i (the more the worse). Assume σ traverses P_5 exactly five times and P_3 sufficiently often, say 10 times. Then, during the last three iterations of P_3, φ holds when visiting location 3, and also location 4. In the two iterations before, the formula holds exclusively at location 4 and in any preceding iteration, it does not hold at all. Thus any labelling of P_3 would necessarily be incorrect. However, we can replace P_3 by four copies of it that are labelled as indicated in Figure 3 and σ can easily be mapped onto this modified structure.*

The presented procedure for constructing an APS from the run ρ in $\mathcal{K}$ performs only linearly many steps in $|\Phi|$, namely one step for each subformula. It starts with a structure of size at most $2|\mathcal{K}|$ and all modifications required to label an APS increase its size by a constant factor. Hence, we obtain an APS $\mathcal{P}_\Phi$ of size at most exponential in the length of Φ and polynomial in the number of states of $\mathcal{K}$. This consistent APS still contains a run corresponding to ρ and hence its first location must be labelled by Φ because $(\rho, 0) \models \Phi$ and we have seen that consistency implies correctness.

▶ **Lemma 10** (Completeness). *If $\mathcal{K} \models \Phi$ then there is a consistent APS $\mathcal{P}$ in $\mathcal{K}$ of at most exponential size in $\mathcal{K}$ and Φ where $\Phi \in \mathtt{lab}(\mathcal{P}(0))$ and $\mathcal{P}$ is non-empty.*

We have seen in this section that the decision procedure presented in the beginning is sound and complete due to Lemma 7 and 10, respectively. The guessed APS is of exponential size in $|\Phi|$ and of polynomial size in $|\mathcal{K}|$. Since both checking consistency and non-emptiness (cf. Lemma 5) require polynomial time (in the size of the APS) the procedure requires at most exponential time.

▶ **Theorem 11.** *MC(FKS, fLTL) is in* NExp.

This result immediately extends to $\mathtt{fCTL}^*$. For a state q of a flat Kripke structure $\mathcal{K}$ and an arbitrary $\mathtt{fLTL}$ formula φ, the procedure allows us to decide in NExp whether $q \models \mathtt{E}\,\varphi$

holds. It allows us further to decide if $q \models \mathtt{A}\,\varphi$ holds in ExpSpace by the dual formulation $q \not\models \mathtt{E}\,\neg\varphi$ and Savitch's theorem. Following otherwise the standard labeling procedure for CTL (cf. Section 3) requires to invoke the procedure a polynomial number of times in $|\mathcal{K}| + |\Phi|$.

▶ **Theorem 12.** *MC(FKS, fCTL*) is in* ExpSpace.

5 On model-checking CCTL* over flat Kripke structures

In this section, we prove decidability of MC(FKS, CCTL*). We provide a polynomial encoding into the satisfiability problem of a decidable extension of Presburger arithmetic featuring a quantifier for counting the solutions of a formula. For the reverse direction an exponential reduction provides a corresponding hardness result for CLTL, CCTL and CCTL*.

Presburger arithmetic with Härtig quantifier. First-order logic over the natural numbers with addition was shown to be decidable by M. Presburger [23]. It has been extended with the so-called *Härtig quantifier* [2, 24, 25] that allows for referring to the number of values for a specific variable that satisfy a formula. We denote this extension by PH. The syntax of PH formulae φ and PH terms τ over a set of variables V is defined by the grammar

$$\varphi ::= \tau \leq \tau \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \exists^{=x} y.\varphi \qquad\qquad \tau ::= a \mid a \cdot x \mid \tau + \tau$$

for natural constants $a \in \mathbb{N}$ and variables $x, y \in V$. Since the structure $(\mathbb{N}, +)$ is fixed, the semantics is defined over valuations $\eta : V \to \mathbb{N}$ that are extended to terms t as expected, e.g., $\eta(3 \cdot x + 1) = 3 \cdot \eta(x) + 1$. We define the satisfaction relation $\models_{\mathtt{PH}}$ as usual for first-order logics and by $\eta \models_{\mathtt{PH}} \exists^{=x} y.\varphi \overset{\text{def}}{\Leftrightarrow} \mathbb{N} \ni |\{b \in \mathbb{N} \mid \eta[y \mapsto b] \models_{\mathtt{PH}} \varphi\}| = \eta(x)$ for the Härtig quantifier. Notice that the solution set has to be finite.

The *satisfiability problem* of PH consists in determining whether for a PH formula φ there exists a valuation η such that $\eta \models_{\mathtt{PH}} \varphi$. It is decidable [2, 24, 25] via eliminating the Härtig quantifier, but its complexity is not known. For what concerns classic Presburger arithmetic, the complexity of its satisfiability problem lies between 2Exp and 2ExpSpace [4].

Lower bound for MC(FKS, CCTL*). Let $\mathcal{K}$ be the flat Kripke structure over $AP = \emptyset$ that consists of a single loop of length one. We can encode satisfiability of a PH formula Φ into the question whether the (unique) run ρ of $\mathcal{K}$ satisfies a CLTL formula $\hat{\Phi}$. Assume without loss of generality that Φ has no free variables. Let V_Φ be the variables used in Φ and $z_1, z_2, \ldots \notin V_\Phi$ additional variables. Recall that $\rho \models \hat{\Phi}$ if $(\rho, \theta, 0) \models \hat{\Phi}$ for some valuation θ of the positional variables in $\hat{\Phi}$.

The idea is essentially to encode the value given to a variable $x \in V_\Phi$ of Φ into the distance between the positions assigned to two variables of $\hat{\Phi}$. Technically, a mapping $Z \in \mathbb{N}^{V_\Phi}$ associates with each variable $x \in V_\Phi$ an index $j = Z(x)$ and the constraints that Φ imposes on x are translated to constraints on positional variables z_j and z_{j-1} (more precisely, the distance $\theta(z_j) - \theta(z_{j-1})$ between the assigned positions). The following transformation $\mathtt{t} : \mathtt{PH} \times \mathbb{N}^{V_\Phi} \times \mathbb{N} \to \mathtt{CLTL}$ constructs the CLTL formula from Φ. When a variable is encountered, the mapping Z is updated by assigning to it the next free index (third parameter). Let

$$\begin{array}{lcllcl}
\mathtt{t}(\varphi_1 \odot \varphi_2, Z, i) & = & \mathtt{t}(\varphi_1, Z, i) \odot \mathtt{t}(\varphi_2, Z, i) & \mathtt{t}(\neg\varphi, Z, i) & = & \neg\mathtt{t}(\varphi, Z, i) \\
\mathtt{t}(a \cdot x, Z, i) & = & a \cdot \#_{z_{Z(x)-1}}(\top) - a \cdot \#_{z_{Z(x)}}(\top) & \mathtt{t}(a, Z, i) & = & a \\
\mathtt{t}(\exists x.\varphi, Z, i) & = & \mathtt{F}\, z_i.\mathtt{t}(\varphi, Z[x \mapsto i], i+1) & & & \\
\mathtt{t}(\exists^{=x} y.\varphi, Z, i) & = & \mathtt{F}\,\mathtt{G}\left(\mathtt{t}(x, Z, i) = \#_{z_{i-1}}(z_i.\mathtt{t}(\varphi, Z[y \mapsto i], i+1))\right) & & &
\end{array}$$

for $x, y \in V_\Phi$, $a, i \in \mathbb{N}$ and $\odot \in \{\wedge, \leq, +\}$. Then, we obtain $\hat{\Phi} = z_0.\mathtt{t}(\Phi, \mathbf{1}, 1)$, initialising Z and the first free index with 1. Notice that the translation of the Härtig quantifier instantiates

the scope effectively twice when substituting the equality and thus the size of $\hat{\Phi}$ may at worst double with each nesting. Finally, we can equivalently add path quantifiers to all temporal operators in $\hat{\Phi}$ and obtain, syntactically, a CCTL formula.

▶ **Theorem 13.** *The satisfiability problem of PH is reducible in exponential time to both MC(FKS, CLTL) and MC(FKS, CCTL).*

Deciding MC(FKS, CCTL*). We provide a polynomial reduction to the satisfiability problem of PH. Given a flat Kripke structure $\mathcal{K}$ we can represent each run ρ by a fixed number of naturals. We use a predicate $Conf$ that allows for accessing the i-th state on ρ given its encoding and a predicate Run characterising all (encodings of) runs in $\mathtt{Runs}(\mathcal{K})$. Such predicates were shown to be definable by Presburger arithmetic formulae of polynomial size and used to encode MC(FKS,CTL*) [12, 10]. We adopt this idea for MC(FKS, CCTL*) and PH. Let $\mathcal{K} = (S, s_I, E, \lambda)$ and assume $S \subseteq \mathbb{N}$ without loss of generality. For $N \in \mathbb{N}$ let $V_N = \{r_1, \ldots, r_N, i, s\}$ be a set of variables that we use to encode a run, a position and a state, respectively.

▶ **Lemma 14** ([10]). *There is a number $N \in \mathbb{N}$, a mapping $enc : \mathbb{N}^N \to S^\omega$ and predicates $Conf(r_1, \ldots, r_N, i, s)$ and $Run(r_1, \ldots, r_N)$ such that for all valuations $\eta : V_N \to \mathbb{N}$ we have*

1. $\eta \models_{PH} Run(r_1, \ldots, r_N) \;\Leftrightarrow\; enc(\eta(r_1), \ldots, \eta(r_N)) \in \mathtt{Runs}(\mathcal{K})$ *and*
2. *if $\eta \models_{PH} Run(r_1, \ldots, r_N)$ then*
   $\eta \models_{PH} Conf(r_1, \ldots, r_N, i, s) \;\Leftrightarrow\; enc(\eta(r_1), \ldots, \eta(r_N))(\eta(i)) = \eta(s)$.

Both predicates are definable by PH formulae over variables $V \supseteq V_N$ of polynomial size in $|\mathcal{K}|$.

Now, let Φ be a CCTL* formula to be verified on $\mathcal{K}$. Without loss of generality we assume that all comparisons $\varphi_\leq \in \mathtt{sub}(\Phi)$ of the form $\tau_1 \leq \tau_2$ have the shape $\varphi_\leq = \sum_{\ell=1}^{k} a_\ell \cdot \#_{x_\ell}(\varphi_\ell) + b \leq \sum_{\ell=k+1}^{m} a_\ell \cdot \#_{x_\ell}(\varphi_\ell) + c$ for some $k, m, b, c \in \mathbb{N}$, coefficients $a_\ell \in \mathbb{N}$ and subformulae φ_ℓ. As it is done in [10] for CTL, using the predicates $Conf$ and Run, we construct a PH formula that is satisfiable if and only if $\mathcal{K} \models \Phi$. Given the encoding of relevant runs into natural numbers we can express path quantifiers with quantification over the variables $r_1, \ldots, r_N$. Temporal operators can be expressed by using $Conf$ to access specific positions. Storing of positions is done explicitly by assigning them as value to specific variables x. Variables z are introduced to hold the number of positions satisfying a formula and can then be used in constraints. For example, to translate a term $\#_x(\varphi)$ we specify a variable, e.g., z_1 holding this value by $\exists z_1.\exists^{=z_1} i'.x \leq i' \leq i \wedge \hat{\varphi}$ where i holds the current position and $\hat{\varphi}$ expresses that φ holds at position i' of the current run. Constraints like $\#_x(\varphi) + 1 \leq \#_x(\psi)$ can now directly be translated to, e.g., $z_1 + 1 \leq z_2$. We use a syntactic translation function chk that takes the formula φ to be translated, the names of N variables encoding the current run and the name of the variable holding the current position. Let

$$
\begin{aligned}
\mathtt{chk}(p, r_1, \ldots, r_N, i) &= \exists s.Conf(r_1, \ldots, r_N, i, s) \wedge \textstyle\bigvee_{a \mid p \in \lambda(a)} s = a \\
\mathtt{chk}(\varphi \wedge \psi, r_1, \ldots, r_N, i) &= \mathtt{chk}(\varphi, r_1, \ldots, r_N, i) \wedge \mathtt{chk}(\psi, r_1, \ldots, r_N, i) \\
\mathtt{chk}(\neg\varphi, r_1, \ldots, r_N, i) &= \neg\mathtt{chk}(\varphi, r_1, \ldots, r_N, i) \\
\mathtt{chk}(\mathtt{X}\,\varphi, r_1, \ldots, r_N, i) &= \exists i'.i' = i + 1 \wedge \mathtt{chk}(\varphi, r_1, \ldots, r_N, i') \\
\mathtt{chk}(\varphi\,\mathtt{U}\,\psi, r_1, \ldots, r_N, i) &= \exists i''.i \leq i'' \wedge \mathtt{chk}(\psi, r_1, \ldots, r_N, i'') \wedge \\
&\quad \forall i'.(i \leq i' \wedge i' < i'') \rightarrow \mathtt{chk}(\psi, r_1, \ldots, r_N, i') \\
\mathtt{chk}(\mathtt{E}\,\varphi, r_1, \ldots, r_N, i) &= \exists r'_1 \ldots \exists r'_N.Run(r'_1, \ldots, r'_N) \wedge \mathtt{chk}(\varphi, r'_1, \ldots, r'_N, i) \wedge \forall i'. \\
&\quad (i' \leq i) \rightarrow \exists s.Conf(r_1, \ldots, r_N, i', s) \wedge Conf(r'_1, \ldots, r'_N, i', s) \\
\mathtt{chk}(x.\varphi, r_1, \ldots, r_N, i) &= \exists x.x = i \wedge \mathtt{chk}(\varphi, r_1, \ldots, r_N, i) \\
\mathtt{chk}(\varphi_\leq, r_1, \ldots, r_N, i) &= \exists z_1 \ldots \exists z_m.\,(\textstyle\bigwedge_{\ell=1}^{m} \exists^{=z_\ell} i'.x_\ell \leq i' \leq i \wedge \mathtt{chk}(\varphi_\ell, r_1, \ldots, r_N, i')) \\
&\quad \wedge\; a_1 \cdot z_1 + \ldots + a_k \cdot z_k + b \leq a_{k+1} \cdot z_{k+1} + \ldots + a_m \cdot z_m + c
\end{aligned}
$$

for $\varphi_{\leq} = \sum_{\ell=1}^{k} a_\ell \cdot \#_{x_\ell}(\varphi_\ell) + b \leq \sum_{\ell=k+1}^{m} a_\ell \cdot \#_{x_\ell}(\varphi_\ell) + c$. Primed variables denote fresh copies of the corresponding input variables, e.g. i' becomes $(i')' = i''$ and i'' becomes i'''. Now, $\Phi \models \mathcal{K}$ if and only if $\exists r_1 \ldots \exists r_N . \exists i . Run(r_1, \ldots, r_N) \wedge i = 0 \wedge \mathtt{chk}(\Phi, r_1, \ldots, r_N, i)$ is satisfiable.

▶ **Theorem 15.** *$MC(FKS, CCTL^*)$ is reducible to PH satisfiability in polynomial time.*

6 Conclusion

In this paper, we have seen that model checking flat Kripke structures with some expressive counting temporal logics is possible whereas this is not the case for general, finite Kripke structures. However, our results provide an under-approximation approach to this latter problem that consists in constructing flat sub-systems of the considered Kripke structure. We furthermore believe our method works as well for flat counter systems. We left as open problem the precise complexity for model checking fCTL, fLTL and fCTL* over flat Kripke structures. It follows from [17] that the latter two problems are NP-hard while we obtain exponential upper bounds. However, we believe that if we fix the nesting depth of the frequency until operator in the logic, the complexity could be improved.

This work has shown, as one could have expected, a strong connection between CLTL and counter systems and as future work we plan to study automata-based formalisms inspired by fLTL where we will equip our automata with some counters whose role will be to evaluate the relative frequency of particular events.

References

1 Parosh Aziz Abdulla, Mohamed Faouzi Atig, Roland Meyer, and Mehdi Seyed Salehi. What's decidable about availability languages? In Prahladh Harsha and G. Ramalingam, editors, *35th IARCS Annual Conference on Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2015, December 16-18, 2015, Bangalore, India*, volume 45 of *LIPIcs*, pages 192–205. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. doi:10.4230/LIPIcs.FSTTCS.2015.192.

2 H. Apelt. Axiomatische Untersuchungen über einige mit der Presburgerschen Arithmetik verwandten Systeme. *Z. Math. Logik Grundlagen Math.*, 12:131–168, 1966.

3 Christel Baier and Joost-Pieter Katoen. *Principles of model checking*. MIT Press, 2008.

4 Leonard Berman. The complexitiy of logical theories. *Theor. Comput. Sci.*, 11:71–77, 1980. doi:10.1016/0304-3975(80)90037-7.

5 Benedikt Bollig, Normann Decker, and Martin Leucker. Frequency linear-time temporal logic. In Tiziana Margaria, Zongyan Qiu, and Hongli Yang, editors, *Sixth International Symposium on Theoretical Aspects of Software Engineering, TASE 2012, 4-6 July 2012, Beijing, China*, pages 85–92. IEEE Computer Society, 2012. doi:10.1109/TASE.2012.43.

6 Ahmed Bouajjani, Javier Esparza, and Oded Maler. Reachability analysis of pushdown automata: Application to model-checking. In Antoni W. Mazurkiewicz and Józef Winkowski, editors, *CONCUR '97: Concurrency Theory, 8th International Conference, Warsaw, Poland, July 1-4, 1997, Proceedings*, volume 1243 of *Lecture Notes in Computer Science*, pages 135–150. Springer, 1997. doi:10.1007/3-540-63141-0_10.

7 Edmund M. Clarke and E. Allen Emerson. Design and synthesis of synchronization skeletons using branching-time temporal logic. In Dexter Kozen, editor, *Logics of Programs, Workshop, Yorktown Heights, New York, May 1981*, volume 131 of *Lecture Notes in Computer Science*, pages 52–71. Springer, 1981. doi:10.1007/BFb0025774.

8 Edmund M. Clarke, E. Allen Emerson, and Joseph Sifakis. Model checking: algorithmic verification and debugging. *Commun. ACM*, 52(11):74–84, 2009. doi:10.1145/1592761.1592781.

9 Normann Decker, Peter Habermehl, Martin Leucker, Arnaud Sangnier, and Daniel Thoma. Model-checking counting temporal logics on flat structures. *CoRR*, abs/1706.08608, 2017. URL: http://arxiv.org/abs/1706.08608.

10 Stéphane Demri, Amit Kumar Dhar, and Arnaud Sangnier. Equivalence between model-checking flat counter systems and Presburger arithmetic. In Joël Ouaknine, Igor Potapov, and James Worrell, editors, *Reachability Problems - 8th International Workshop, RP 2014, Oxford, UK, September 22-24, 2014. Proceedings*, volume 8762 of *Lecture Notes in Computer Science*, pages 85–97. Springer, 2014. doi:10.1007/978-3-319-11439-2_7.

11 Stéphane Demri, Amit Kumar Dhar, and Arnaud Sangnier. Taming past LTL and flat counter systems. *Inf. Comput.*, 242:306–339, 2015. doi:10.1016/j.ic.2015.03.007.

12 Stéphane Demri, Alain Finkel, Valentin Goranko, and Govert van Drimmelen. Model-checking CTL* over flat Presburger counter systems. *Journal of Applied Non-Classical Logics*, 20(4):313–344, 2010. doi:10.3166/jancl.20.313-344.

13 Stéphane Demri and Ranko Lazic. LTL with the freeze quantifier and register automata. *ACM Trans. Comput. Log.*, 10(3):16:1–16:30, 2009. doi:10.1145/1507244.1507246.

14 E. Allen Emerson and Joseph Y. Halpern. "Sometimes" and "not never" revisited: On branching versus linear time. In John R. Wright, Larry Landweber, Alan J. Demers, and Tim Teitelbaum, editors, *Conference Record of the Tenth Annual ACM Symposium on Principles of Programming Languages, Austin, Texas, USA, January 1983*, pages 127–140. ACM Press, 1983. doi:10.1145/567067.567081.

15 Alain Finkel and Jérôme Leroux. How to compose Presburger-accelerations: Applications to broadcast protocols. In Manindra Agrawal and Anil Seth, editors, *FST TCS 2002: Foundations of Software Technology and Theoretical Computer Science, 22nd Conference Kanpur, India, December 12-14, 2002, Proceedings*, volume 2556 of *Lecture Notes in Computer Science*, pages 145–156. Springer, 2002. doi:10.1007/3-540-36206-1_14.

16 Jochen Hoenicke, Roland Meyer, and Ernst-Rüdiger Olderog. Kleene, rabin, and scott are available. In Paul Gastin and François Laroussinie, editors, *CONCUR 2010 - Concurrency Theory, 21th International Conference, CONCUR 2010, Paris, France, August 31-September 3, 2010. Proceedings*, volume 6269 of *Lecture Notes in Computer Science*, pages 462–477. Springer, 2010. doi:10.1007/978-3-642-15375-4_32.

17 Lars Kuhtz and Bernd Finkbeiner. Weak Kripke structures and LTL. In Joost-Pieter Katoen and Barbara König, editors, *CONCUR 2011 - Concurrency Theory - 22nd International Conference, CONCUR 2011, Aachen, Germany, September 6-9, 2011. Proceedings*, volume 6901 of *Lecture Notes in Computer Science*, pages 419–433. Springer, 2011. doi:10.1007/978-3-642-23217-6_28.

18 François Laroussinie, Antoine Meyer, and Eudes Petonnet. Counting LTL. In Nicolas Markey and Jef Wijsen, editors, *TIME 2010 - 17th International Symposium on Temporal Representation and Reasoning, Paris, France, 6-8 September 2010*, pages 51–58. IEEE Computer Society, 2010. doi:10.1109/TIME.2010.20.

19 François Laroussinie, Antoine Meyer, and Eudes Petonnet. Counting CTL. *Logical Methods in Computer Science*, 9(1), 2012. doi:10.2168/LMCS-9(1:3)2013.

20 Jérôme Leroux and Grégoire Sutre. Flat counter automata almost everywhere! In Doron A. Peled and Yih-Kuen Tsay, editors, *Automated Technology for Verification and Analysis, Third International Symposium, ATVA 2005, Taipei, Taiwan, October 4-7, 2005, Proceedings*, volume 3707 of *Lecture Notes in Computer Science*, pages 489–503. Springer, 2005. doi:10.1007/11562948_36.

21 M. Minsky. *Computation, Finite and Infinite Machines*. Prentice Hall, 1967.

22 Amir Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October - 1 November 1977*, pages 46–57. IEEE Computer Society, 1977. doi:10.1109/SFCS.1977.32.

23 M. Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. In *Comptes Rendus du premier congrès de mathématiciens des Pays Slaves, Warszawa*, pages 92–101, 1929.

24 William Pugh. Counting solutions to Presburger formulas: How and why. In Vivek Sarkar, Barbara G. Ryder, and Mary Lou Soffa, editors, *Proceedings of the ACM SIGPLAN'94 Conference on Programming Language Design and Implementation (PLDI), Orlando, Florida, USA, June 20-24, 1994*, pages 121–134. ACM, 1994. doi:10.1145/178243.178254.

25 Nicole Schweikardt. Arithmetic, first-order logic, and counting quantifiers. *ACM Trans. Comput. Log.*, 6(3):634–671, 2005. doi:10.1145/1071596.1071602.

Concurrent Reversible Sessions*

Ilaria Castellani[1], Mariangiola Dezani-Ciancaglini†[2], and Paola Giannini‡[3]

1 **Université Côte d'Azur, INRIA,**
2004 Route des Lucioles, 06902 Sophia Antipolis, France
ilaria.castellani@inria.fr

2 **Dipartimento di Informatica, Università di Torino,**
corso Svizzera 185, 10131 Torino, Italy
dezani@di.unito.it

3 **DiSIT, Università del Piemonte Orientale,**
Via Teresa Michel 11, 15121 Alessandria, Italy
paola.giannini@uniupo.it

Abstract

We present a calculus for concurrent reversible multiparty sessions, which improves on recent proposals in several respects: it allows for concurrent and sequential composition within processes and types, it gives a compact representation of the *past* of processes and types, which facilitates the definition of rollback, and it implements a fine-tuned strategy for backward computation. We propose a refined session type system for our calculus and show that it enforces the expected properties of session fidelity, forward and backward progress, as well as causal consistency. In conclusion, our calculus is a conservative extension of previous proposals, offering enhanced expressive power and refined analysis techniques.

1998 ACM Subject Classification F.1.2 Parallelism and Concurrency, F.3.3 Type Structure

Keywords and phrases Communication-centric Systems, Reversible Computation, Process Calculi, Multiparty Session Types

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.30

1 Introduction

Building on recent proposals, we argue that session types and reversibility can be fruitfully combined to yield a model for correct and reliable communication-centric systems. Session types are a simple but expressive type formalism that specifies the structure of interactions. Traditionally, session types have been used to ensure safety properties of interactions, such as absence of communication errors, deadlock freedom and race freedom.

Reversibility, on its side, may be viewed as a means to improve system flexibility and reliability. Reversing a computation may be defined as the act of undoing some suffix of the computation, in order to return to a previously visited state. A condition which is usually required for undoing a computational step is that all its effects have been already undone [21, 22, 30, 33, 8, 16, 15]. This property is generally referred to as *causal consistency*.

* The authors acknowledge a partial support of COST Action IC1405 on Reversible Computation – extending horizons of computing.
† Partially supported by EU H2020-644235 Rephrase project, EU H2020-644298 HyVar project, IC1402 ARVI and Ateneo/CSP project RunVar.
‡ This original research has the financial support of the Università del Piemonte Orientale.

A stronger property, called *full reversibility* in [25], enables a system to restore exactly a past state, keeping no trace of the rollback [9, 20, 24]. Sometimes neither of these properties can be achieved, particularly in distributed systems, and one has to go for weaker properties. In some cases, it is even desirable that a restored state be not exactly the same as the original one: for instance, the restored state could keep some memory of the undone computational path, so as to avoid engaging in that path again in case it led to an unsuccessful state.

In the setting of structured communications, reversibility has been first studied for contracts [2, 3] and transactions [10, 11, 19]. Only recently has this issue started to be addressed for session calculi, both binary [33, 23] and multiparty [14, 34, 28, 24] (see Section 6 for more discussion on related work).

When reversing a structured interaction, one has to face the problem of preserving consistency of the global state: if one of the partners triggers a rollback, then all its communicating partners should roll back accordingly. This is where session types come to the rescue, with their precise specification of the functionality of communications (sender, receiver and message), and of the order in which they should occur.

We present a calculus for concurrent reversible multiparty sessions, which extends previous proposals in several ways. First of all, our procotols are more general than those specified by standard multiparty session types [18]: concurrent communications are allowed both between disjoint groups of participants, and, in a controlled way that excludes auto-concurrency, also within participants themselves. More specifically, we relax the linearity constraint of standard multiparty session types, to allow protocol participants to perform communications in parallel, as long as they do not generate races. To enable the control flows to join again after a bunch of parallel interactions, besides parallel composition we also introduce full sequential composition in the syntax of both types and processes.

Moreover, our calculus gives a compact representation of the *past* of processes and types, which facilitates the definition of rollback, and it implements a fine-tuned strategy for backward computation, which is geared towards achieving compliance.

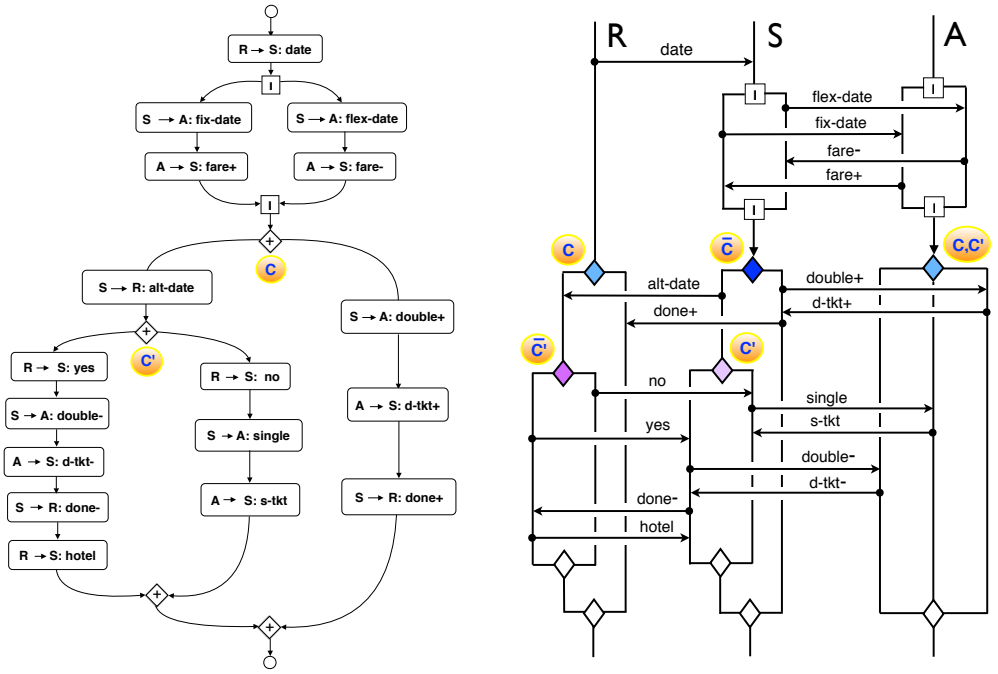
The main contributions of our paper may be summarised as follows:

- the introduction of *concurrent session types* in multiparty session frameworks;
- the proposal of a reversible session calculus which is more general than the existing ones, and which conservatively extends them, preserving the expected properties of session fidelity, forward and backward progress, and causal consistency;
- a fine-tuned strategy for rollback to checkpointed choices, which can only be triggered by choice leaders in predefined states of the computation, leading back to the choice state stripped off the unsuccessful path, so as redirect the computation towards alternative, potentially successful, paths.

The rest of the paper is organised as follows. In Section 2 we present our running example. In Section 3 we define the syntax and operational semantics – both forward and backward – of our calculus. In Section 4 we introduce our extended syntax for global types and session types and we establish well-formedness conditions for global types. Section 5 presents our type system and proves its soundness, namely that it ensures the expected semantic properties. We conclude in Section 6 with some discussion on related and future work.

2 Travel protocol example

To illustrate our approach, we present a simple travel protocol, involving three parties, a Researcher **R**, a Student **S** and an Airline company **A**. This protocol, henceforth called the *RSA protocol*, will serve as our running example throughout the paper.



(a) Global graph. **(b)** Message Sequence Graph.

Figure 1 RSA protocol.

When graphically representing protocols, we shall use two kinds of graphs: the first one, called *global graph*, is borrowed from [13] and gives a global description of the protocol; the second one, which we call *message sequence graph*, is closer to message sequence charts and emphasises the control flows of the individual partners. In both cases, diamonds will be used as delimiters for the branching construct, and squares as delimiters for the parallel construct.

In the global graph, boxes represent communications and their whole functionality: sender, receiver and message. In the message sequence graph, the control flow of the protocol is split into the individual contributions of the partners: each partner is denoted by a vertical line; communications are represented by arrows from the sender to the receiver, labelled by messages; and branching and parallel constructs are replicated for each involved partner.

The RSA protocol is depicted in Figure 1a and Figure 1b according to these conventions. Let us now informally describe the protocol. Suppose that **R** and **S** wish to travel together to some conference, and that **R** has date constraints but is flexible about prices, while **S** has funding constraints but is flexible about dates. The protocol starts by **R** sending to **S** her chosen date. Then **S** sends two parallel flight requests to **A**: one for the date chosen by **R**, and one for more flexible dates. The airline **A** replies by sending in parallel the respective fares: a higher fare for the fixed date, and a lower fare for the flexible date. At this point, if **S** can afford the higher fare, she will book two tickets at this fare, then send a message done+ to **R**, and the whole interaction will end here.

Suppose now that **S** cannot afford the higher fare. In this case, she will propose to **R** the alternative date corresponding to the cheaper fare. If **R** cannot travel at this date she replies no. Then **S** will book a single ticket for herself at the cheaper fare, and the interaction terminates. If instead this date is suitable for **R**, she replies yes to **S**. Now **S** books two tickets at the cheaper fare and sends a message done- to **R**. Then **R** informs **S** that the hotel

booking has been changed to fit the new date (supposing **R** had previously booked the hotel for her chosen date), thereby closing the interaction.

So far we have described only the forward behaviour of the session. Suppose now that after proposing the alternative date to **R** and receiving **R**'s agreement, **S** discovers that this date is not suitable anymore. Then **S** will trigger a rollback to her internal choice with checkpoint label $\overline{C}$, and **R** will have to roll back to her corresponding external choice with checkpoint label C. At this point, since the first branch of the choice has been already explored with no success, **S** will have to choose the second branch and book the more expensive flight. On her side, **R** could decide to roll back to her internal choice with label $\overline{C'}$ if she discovers that the hotel is fully booked at the alternative date, and then both **S** and **A** will have to roll back to their corresponding external choices with label C'.

In our calculus, only "the leader" of a choice, i.e., the participant who solved the choice by sending the first message, will be authorised to trigger a rollback. In the RSA protocol, the leader of the first choice is **S**, while the leader of the second choice is **R**. In the message sequence graph, the leader is distinguished by the fact that her choice has an overlined checkpoint. In the global graph, which collapses the distributed structure of the interaction, the choice leader is the sender in the first communication after the choice. In fact, the graph of Figure 1b can always be derived from that of Figure 1a through a projection operation.

3 Calculus

In this section, we introduce the syntax and semantics of our calculus. As usual in session calculi, we distinguish between user processes, which have not started to be executed yet, and runtime processes. The executed part of runtime processes will be marked with hats: this extension is instrumental to reversing computations. Processes may be decorated by checkpoint labels, marking them as possible rollback points. Our syntax generalises that of standard multiparty session calculi [18, 12], featuring the additional operators of parallel and sequential composition (the latter replacing the prefixing operator).

We assume the following base sets: *messages*, ranged over by $\lambda, \lambda', \ldots$ and forming the set Msg; *checkpoint labels*, ranged over by C, C' and forming the set ChLa; and *session participants*, ranged over by $\mathsf{p}, \mathsf{q}, \mathsf{r}$ and forming the set Part.

We use δ to range over general sets of checkpoint labels, and Δ to stand for either δ or δ extended by exactly one overlined checkpoint label:

$$\delta ::= \emptyset \quad | \quad \delta, C \qquad\qquad \Delta ::= \delta \quad | \quad \delta, \overline{C}$$

Sets of checkpoint labels are associated with choices. More precisely, an overlined checkpoint label can only be associated with an internal choice and is said to be *active*. A simple checkpoint label is *passive* and may be associated with both internal and external choices. Intuitively, an overlined checkpoint label is the handle of a backward move: a participant who crossed an internal choice (henceforth also called the *choice leader*) with an active checkpoint label, and then proceeded in the computation, may decide to return to that choice whenever she has the ability to send a message. Within a network, this backward move of the choice leader will have to be matched by backward moves of all the participants who did some action after that checkpoint label.

Let $\pi \in \{\mathsf{p}?\lambda, \mathsf{p}!\lambda \mid \mathsf{p} \in \mathsf{Part}, \lambda \in \mathsf{Msg}\}$ denote an *atomic action*, namely a directed input or output action. An atomic action can bear a hat, in which case it represents an already executed or *past* action. We use $\widetilde{\pi}$ to stand for either π or $\widehat{\pi}$. External and internal choices can also bear hats, indicating that one branch has been chosen. We use $\widetilde{\sum}$ to stand for either $\sum$ or $\widehat{\sum}$, and similarly for $\bigoplus$.

▶ **Definition 1.** *Processes* are defined by:

$$P \quad ::= \quad \widetilde{\sum}_{i\in I}\widetilde{\pi}_i; P_i \quad | \quad \widetilde{\bigoplus}_{i\in I}\widetilde{\pi}_i; P_i \quad | \quad {}_{\Delta}P \quad | \quad P \mid P \quad | \quad P; P \quad | \quad \mu X.P \quad | \quad X \quad | \quad \mathsf{skip}$$

Processes without and with hats are called respectively *user processes* and *runtime processes*.

We will omit empty sets of checkpoint labels, choice symbols in one-branch choices, and trailing skip processes. External and internal choices are assumed to be associative, commutative, idempotent, and non-empty (except when combined with binary choices in evaluation contexts or finished processes, see below). A choice with a single branch may be either an *input process* or an *output process*. Parallel composition is associative and commutative, with neutral element skip. Sequential composition is associative, with neutral element skip. The operators have the following precedence: ';', '+', '$\oplus$','|'.

Following [27], we require recursion to be:

- *guarded*, i.e. a recursion variable X can only appear free in the second argument of a sequential composition whose first argument is different from skip;
- *sequential*, i.e. a recursion variable X cannot occur free in any branch of a parallel composition.

Processes are treated equi-recursively, i.e. they are identified with their generated tree [31].

The typing rules of Section 5 will ensure the following well-formedness conditions for processes:

1. External choices are between input processes;
2. Internal choices are between output processes;
3. A choice without a hat has branches without hats;
4. A choice with a hat has exactly one branch with hats;
5. Only choices are decorated with sets of checkpoint labels;
6. Only internal choices are decorated with an active checkpoint label.

Conditions 1 and 2 reflect the active role of outputs and the passive role of inputs. Conditions 3 and 4 express the fact that executing a choice amounts to executing one of its branches. Conditions 5 and 6 specify that choices are the only return points for backward reductions. In the following we will take advantage of these restrictions to simplify definitions.

In a full-fledged calculus, messages would carry values, namely they would be of the form $\lambda(v)$. Here, for simplicity we consider only pure messages.

Networks are parallel compositions of pairs $\mathsf{p}[\![\, P \,]\!]$, where participant p has behaviour P.

▶ **Definition 2.** *Networks* are defined by: $\quad \mathbb{N} \quad ::= \quad \mathsf{p}[\![\, P \,]\!] \quad | \quad \mathbb{N} \parallel \mathbb{N}$

The operator $\parallel$ is associative and commutative, with neutral element $\mathsf{p}[\![\, \mathsf{skip} \,]\!]$ for each p.

The operational semantics is given by two LTSs, one for processes and one for networks. In the LTS for processes, *forward transitions* have the form $P \xrightarrow{\pi} P'$ and *backward transitions* have the form $P \overset{\overline{C}}{\circlearrowleft} P'$ or $P \overset{C}{\circlearrowleft} P'$. We define $P \downarrow_{out}$ if $P \xrightarrow{\mathsf{p}!\lambda} P'$ for some p, λ, P'.

In the LTS for networks, forward and backward transitions have respectively the form $N \xrightarrow{\mathsf{p}\lambda\mathsf{q}} N'$ and $N \overset{C}{\circlearrowleft} N'$.

▶ **Definition 3.** *Evaluation contexts* $\mathcal{E}$ are defined by:

$$\mathcal{E} \quad ::= \quad {}_{\delta}(\widehat{\sum}_{i\in I}\pi_i; P_i \mathbin{\widehat{+}} \widehat{\pi}; \mathcal{E}) \quad | \quad {}_{\Delta}(\widehat{\bigoplus}_{i\in I}\pi_i; P_i \mathbin{\widehat{\oplus}} \widehat{\pi}; \mathcal{E}) \quad | \quad \mathcal{E} \mid P \quad | \quad \mathcal{E}; P \quad | \quad F; \mathcal{E} \quad | \quad [\,]$$

The definition of evaluation context ensures that in a sequential composition the evaluation of the second component can only start when the first is a *finished process* F, defined by:

$$F \quad ::= \quad {}_{\delta}(\widehat{\sum}_{i\in I}\pi_i; P_i \mathbin{\widehat{+}} \widehat{\pi}; F) \quad | \quad {}_{\Delta}(\widehat{\bigoplus}_{i\in I}\pi_i; P_i \mathbin{\widehat{\oplus}} \widehat{\pi}; F) \quad | \quad F \mid F \quad | \quad F; F \quad | \quad \mathsf{skip}$$

$$ {}_\delta\textstyle\sum_{i\in I}\pi_i;P_i \xrightarrow{\pi_j} {}_\delta(\widehat{\sum}_{i\in I\setminus\{j\}}\pi_i;P_i \mathbin{\widehat{+}} \widehat{\pi}_j;P_j) \quad j\in I \qquad [\textsc{ExtCh}] $$

$$ {}_\Delta\textstyle\bigoplus_{i\in I}\pi_i;P_i \xrightarrow{\pi_j} {}_\Delta(\widehat{\bigoplus}_{i\in I\setminus\{j\}}\pi_i;P_i \mathbin{\widehat{\oplus}} \widehat{\pi}_j;P_j) \quad j\in I \qquad [\textsc{IntCh}] $$

$$ \frac{P\downarrow_{out} \quad \overline{C}\in\Delta \quad I\neq\emptyset}{{}_\Delta(\widehat{\bigoplus}_{i\in I}P_i \mathbin{\widehat{\oplus}} P) \overset{\overline{C}}{\circlearrowright} {}_\Delta\textstyle\bigoplus_{i\in I}P_i} [\textsc{BackCkt}] \qquad \frac{C\in\Delta}{{}_\Delta S \overset{C}{\circlearrowright} {}_\Delta\wr S\int} [\textsc{BackP}] $$

$$ \frac{P\xrightarrow{\pi}P'}{\mathcal{E}[P]\xrightarrow{\pi}\mathcal{E}[P']} [\textsc{CtFAt}] \qquad \frac{P\overset{\overline{C}}{\circlearrowright}P' \quad \mathcal{E}\ \mathsf{ok}\text{ for }\overline{C}}{\mathcal{E}[P]\overset{\overline{C}}{\circlearrowright}\mathcal{E}[P']} [\textsc{CtBA}] \qquad \frac{P\overset{C}{\circlearrowright}P' \quad \mathcal{E}\ \mathsf{ok}\text{ for }C}{\mathcal{E}[P]\overset{C}{\circlearrowright}\mathcal{E}[P']} [\textsc{CtBP}] $$

$$ \frac{P_1\xrightarrow{\mathsf{p}?\lambda}P_1' \quad P_2\xrightarrow{\mathsf{q}!\lambda}P_2'}{\mathsf{q}[\![\,P_1\,]\!]\parallel\mathsf{p}[\![\,P_2\,]\!]\parallel\mathbb{N}\xrightarrow{\mathsf{p}\lambda\mathsf{q}}\mathsf{q}[\![\,P_1'\,]\!]\parallel\mathsf{p}[\![\,P_2'\,]\!]\parallel\mathbb{N}} [\textsc{Com}] $$

$$ \frac{P\overset{\overline{C}}{\circlearrowright}P' \quad P_i\overset{C}{\circlearrowright}P_i' \ \ i\in I \quad P_j\overset{C}{\not\circlearrowright} \ \ j\in J}{\mathsf{p}[\![\,P\,]\!]\parallel\Pi_{i\in I}\mathsf{p}_i[\![\,P_i\,]\!]\parallel\Pi_{j\in J}\mathsf{p}_j[\![\,P_j\,]\!]\overset{C}{\circlearrowright}\mathsf{p}[\![\,P'\,]\!]\parallel\Pi_{i\in I}\mathsf{p}_i[\![\,P_i'\,]\!]\parallel\Pi_{j\in J}\mathsf{p}_j[\![\,P_j\,]\!]} [\textsc{Back}] $$

Figure 2 LTS for processes and networks.

The LTSs for processes and networks are given in Figure 2. Rules [ExtCh] and [IntCh] allow an action to be extracted from one of the summands, as usual, but instead of discarding the other summands they record the fact that the choice has been crossed by marking the choice operator with a hat. With this technique, inspired by [6] and already used for reversible computations in [24], all the dynamic operators are turned into static operators, and nothing is lost of the original user process. Notice that when $I=\{j\}$ these rules become $\pi_j;P_j \xrightarrow{\pi_j} \widehat{\pi}_j;P_j$. Rule [BackCkt] is the main backward rule: it applies to a past internal choice, where one branch has been partially executed, and it allows the process to roll back to the original choice where the executed branch is removed. For this to be possible, the choice must have at least one non executed branch P_i and a set of checkpoint labels containing an overlined label $\overline{C}$, which will label the back transition. This is essential to ensure (by means of typing) that the choice leader will be the only one who can decide to roll back to this choice. The condition $P\downarrow_{out}$ means that in order to trigger a rollback, P should be "in lead" again, namely able to do an output.

Rule [BackP], where S denotes either $\widehat{\sum}_{i\in I}P_i$, or $\widehat{\bigoplus}_{i\in I}P_i$, is needed to allow the remaining participants to roll back. The mapping $\wr\ \int$ erases hats from processes, yielding user processes, i.e. $\wr\widehat{\pi};P\int = \pi;\wr P\int$ and $\wr\ \int$ acts homomorphically otherwise. Note that the rollback rules can only be applied to processes that are not user processes: in particular, one branch can be erased only if at least one of its actions has been executed. An evaluation context $\mathcal{E}$ is ok for $\overline{C}$ (C) if $\overline{C}\notin\Delta$ ($C\notin\Delta$) whenever $\mathcal{E}$ is a sub-context of a context of the shape ${}_\Delta(\widehat{\sum}_{i\in I}P_i \mathbin{\widehat{+}} \widehat{\pi};\mathcal{E}')$ or ${}_\Delta(\widehat{\bigoplus}_{i\in I}P_i \mathbin{\widehat{\oplus}} \widehat{\pi};\mathcal{E}')$. We use this condition in rules [CtBA] and [CtBP] to assure that all participants involved in a recursion go back to the same checkpoint, namely to the first one, as in [28]. This is needed to assure subject reduction, see Example 8.

Rule [Com] is standard. We write $P\overset{C}{\not\circlearrowright}$ if neither Rule [BackP] nor Rule [CtBP] (with label C) can be applied to P. This means that C can only occur in user processes within P.

In a well-typed network, Rule [Back] will make participant p roll back to an internal choice and moreover, all participants that can roll back to this choice will do so in the same step. This will be the basis for our soundness result in Section 5. A direct implementation of this rule is clearly unrealistic. To that purpose, asynchronous communications including rollback messages should be used, as in [24]. As expected, sequences of network transitions generate *traces*, which are words over the infinite alphabet $\{\mathsf{p}\lambda\mathsf{q} \mid \mathsf{p},\mathsf{q} \in \mathsf{Part}, \lambda \in \mathsf{Msg}\} \cup \mathsf{ChLa}$.

When the labels of transitions are not relevant, we write them simply as $\longrightarrow$ and $\curvearrowright$. In this case, we use $\longrightarrow^*$ to denote the reflexive and transitive closure of $\longrightarrow$ and $\overset{\curvearrowright}{\longrightarrow}^*$ to denote the reflexive and transitive closure of $\longrightarrow \cup \curvearrowright$.

Note that our semantics does not enforce any scheduling policy. Hence, a network may generate an infinite trace that does not involve all participants. For instance, the network:

$$\mathsf{p}[\![\, \mu X_1.\mathsf{q}!\lambda_1;X_1 \mid \mu X_2.\mathsf{r}!\lambda_2;X_2 \,]\!] \parallel \mathsf{q}[\![\, \mu X_3.\mathsf{p}?\lambda_1;X_3 \,]\!] \parallel \mathsf{r}[\![\, \mu X_4.\mathsf{p}?\lambda_2;X_4 \,]\!]$$

may generate the infinite trace $\mathsf{p}\lambda_1\mathsf{q}\,\mathsf{p}\lambda_1\mathsf{q}\cdots$. In a more elaborate calculus, we could impose a *fair scheduling policy*, forcing p to communicate alternately with q and r, and in general, no communication to be iterated until each participant has communicated at least once.

A network for our RSA travel protocol in Section 2 is $\mathbb{N}_{in} = \mathbf{R}[\![\, P^{\mathbf{R}} \,]\!] \parallel \mathbf{S}[\![\, P^{\mathbf{S}} \,]\!] \parallel \mathbf{A}[\![\, P^{\mathbf{A}} \,]\!]$, defined as follows (where messages are abbreviated in a programming language style):

- The process $P^{\mathbf{S}}$ is $\mathbf{R}?\mathtt{dt};(P_1^{\mathbf{S}} \mid P_2^{\mathbf{S}})\,;\,(P_3^{\mathbf{S}} \,{}_{\{\overline{C}\}}\!\oplus P_4^{\mathbf{S}})$ where

$$\begin{array}{ll} P_1^{\mathbf{S}} = \mathbf{A}!\mathtt{fxDt};\mathbf{A}?\mathtt{frPl} & P_2^{\mathbf{S}} = \mathbf{A}!\mathtt{flDt};\mathbf{A}?\mathtt{frMn} \\ P_3^{\mathbf{S}} = \mathbf{R}!\mathtt{alDt};(P_5^{\mathbf{S}} \,{}_{\{C'\}}\!+ P_6^{\mathbf{S}}) & P_4^{\mathbf{S}} = \mathbf{A}!\mathtt{dbPl};\mathbf{A}?\mathtt{dTkPl};\mathbf{R}!\mathtt{dnPl} \\ P_5^{\mathbf{S}} = \mathbf{R}?\mathtt{yes};\mathbf{A}!\mathtt{dbMn};P_7^{\mathbf{S}} \quad P_7^{\mathbf{S}} = \mathbf{A}?\mathtt{dTkMn};\mathbf{R}!\mathtt{dnMn};\mathbf{R}?\mathtt{htl} & P_6^{\mathbf{S}} = \mathbf{R}?\mathtt{no};\mathbf{A}!\mathtt{sn};\mathbf{A}?\mathtt{snTk} \end{array}$$

- The process $P^{\mathbf{R}}$ is $\mathbf{S}!\mathtt{dt};(\; P_1^{\mathbf{R}} \,{}_{\{C\}}\!+ \mathbf{S}?\mathtt{dnPl}\;)$ where

$$P_1^{\mathbf{R}} = \mathbf{S}?\mathtt{alDt};(\; P_2^{\mathbf{R}} \,{}_{\{\overline{C'}\}}\!\oplus \mathbf{S}!\mathtt{no}\;) \qquad P_2^{\mathbf{R}} = \mathbf{S}!\mathtt{yes};\mathbf{S}?\mathtt{dnMn};\mathbf{S}!\mathtt{htl}$$

- The process $P^{\mathbf{A}}$ is $(P_1^{\mathbf{A}} \mid P_2^{\mathbf{A}});\, {}_{\{C,C'\}}\!\sum_{i\in\{3,4,5\}} P_i^{\mathbf{A}}$ where

$$\begin{array}{c} P_1^{\mathbf{A}} = \mathbf{S}?\mathtt{fxDt};\mathbf{S}!\mathtt{frPl} \qquad P_2^{\mathbf{A}} = \mathbf{S}?\mathtt{flDt};\mathbf{S}!\mathtt{frMn} \\ P_3^{\mathbf{A}} = \mathbf{S}?\mathtt{dbMn};\mathbf{S}!\mathtt{dTkMn} \qquad P_4^{\mathbf{A}} = \mathbf{S}?\mathtt{sn};\mathbf{S}!\mathtt{snTk} \qquad P_5^{\mathbf{A}} = \mathbf{S}?\mathtt{dbPl};\mathbf{S}!\mathtt{dTkPl} \end{array}$$

In the computation below, we denote with $\widehat{P}$ the process P entirely marked with hats.

$$\begin{array}{lll} \mathbb{N}_{in} & \xrightarrow{\mathbf{R}\,\mathtt{dt}\,\mathbf{S}} & \mathbf{R}[\![\, \widehat{\mathbf{S}!\mathtt{dt}};(P_1^{\mathbf{R}} \,{}_{\{C\}}\!+ \mathbf{S}?\mathtt{dnPl}) \,]\!] \;\parallel\; \mathbf{S}[\![\, \widehat{\mathbf{R}?\mathtt{dt}};(P_1^{\mathbf{S}} \mid P_2^{\mathbf{S}})\,;\,(P_3^{\mathbf{S}} \,{}_{\{\overline{C}\}}\!\oplus P_4^{\mathbf{S}}) \,]\!] \;\parallel\; \mathbf{A}[\![\, P^{\mathbf{A}} \,]\!] \\ & \longrightarrow^* & \mathbf{R}[\![\, \widehat{\mathbf{S}!\mathtt{dt}};(P_1^{\mathbf{R}} \,{}_{\{C\}}\!+ \mathbf{S}?\mathtt{dnPl}) \,]\!] \;\parallel\; \mathbf{S}[\![\, F^{\mathbf{S}}\,;\,(P_3^{\mathbf{S}} \,{}_{\{\overline{C}\}}\!\oplus P_4^{\mathbf{S}}) \,]\!] \;\parallel\; \mathbf{A}[\![\, Q^{\mathbf{A}} \,]\!] \\ & & \text{where } F^{\mathbf{S}} = \widehat{\mathbf{R}?\mathtt{dt}};(\widehat{P_1^{\mathbf{S}}} \mid \widehat{P_2^{\mathbf{S}}}),\; Q^{\mathbf{A}} = (\widehat{P_1^{\mathbf{A}}} \mid \widehat{P_2^{\mathbf{A}}});\, {}_{\{C,C'\}}\!\sum_{i\in\{3,4,5\}} P_i^{\mathbf{A}} \\ & \xrightarrow{\mathbf{S}\,\mathtt{alDt}\,\mathbf{R}} & \mathbf{R}[\![\, \widehat{\mathbf{S}!\mathtt{dt}};(\; \widehat{\mathbf{S}?\mathtt{alDt}};(\; P_2^{\mathbf{R}} \,{}_{\{\overline{C'}\}}\!\oplus \mathbf{S}!\mathtt{no}\;) \,{}_{\{C\}}\!\widehat{+}\; \mathbf{S}?\mathtt{dnPl}\;) \,]\!] \;\parallel \\ & & \mathbf{S}[\![\, F^{\mathbf{S}}\,;\,(\widehat{\mathbf{R}!\mathtt{alDt}};(P_5^{\mathbf{S}} \,{}_{\{C'\}}\!+ P_6^{\mathbf{S}}) \,{}_{\{\overline{C}\}}\!\widehat{\oplus}\, P_4^{\mathbf{S}}) \,]\!] \;\parallel\; \mathbf{A}[\![\, Q^{\mathbf{A}} \,]\!] \\ & \longrightarrow^* & \mathbf{R}[\![\, \widehat{\mathbf{S}!\mathtt{dt}};(\; \widehat{\mathbf{S}?\mathtt{alDt}};(\; \widehat{\mathbf{S}!\mathtt{yes}};\mathbf{S}?\mathtt{dnMn};\mathbf{S}!\mathtt{htl} \,{}_{\{\overline{C'}\}}\!\widehat{\oplus}\, \mathbf{S}!\mathtt{no}\;) \,{}_{\{C\}}\!\widehat{+}\; \mathbf{S}?\mathtt{dnPl}\;) \,]\!] \;\parallel \\ & & \mathbf{S}[\![\, F^{\mathbf{S}}\,;\,(\widehat{\mathbf{R}!\mathtt{alDt}};(\widehat{\mathbf{R}?\mathtt{yes}};\mathbf{A}!\mathtt{dbMn};P_7^{\mathbf{S}} \,{}_{\{C'\}}\!\widehat{+}\; P_6^{\mathbf{S}}) \,{}_{\{\overline{C}\}}\!\widehat{\oplus}\, P_4^{\mathbf{S}}) \,]\!] \;\parallel\; \mathbf{A}[\![\, Q^{\mathbf{A}} \,]\!] \\ & \overset{C}{\curvearrowright} & \mathbf{R}[\![\, \widehat{\mathbf{S}!\mathtt{dt}};(P_1^{\mathbf{R}} \,{}_{\{C\}}\!+ \mathbf{S}?\mathtt{dnPl}\;) \,]\!] \;\parallel\; \mathbf{S}[\![\, F^{\mathbf{S}}\,;\,P_4^{\mathbf{S}} \,]\!] \;\parallel\; \mathbf{A}[\![\, Q^{\mathbf{A}} \,]\!] \end{array}$$

Note that the last network differs from that in Line 2 only by the absence of process $P_3^{\mathbf{S}}$.

4 Global Types and Session Types

A *multiparty session* is a series of communications among a fixed number of participants [18], which follows a predefined protocol specified by a *global type*.

We use γ to denote either the empty set or a singleton made of a checkpoint label:

$$\gamma ::= \emptyset \quad | \quad \{C\}$$

Sets γ will be associated with choices in global types.

Let $\alpha^{\mathsf{p}} \in \{\mathsf{p} \xrightarrow{\lambda} \mathsf{q} \mid \mathsf{q} \in \mathsf{Part}, \lambda \in \mathsf{Msg}\}$ denote an *atomic communication* with sender p. An atomic communication can bear a hat, in notation $\widehat{\alpha^{\mathsf{p}}}$, in which case it represents an executed or *past* communication. The symbol $\widetilde{\alpha^{\mathsf{p}}}$ stands for either α^{p} or $\widehat{\alpha^{\mathsf{p}}}$.

Global types G are built from choices among atomic communications with the same sender, with the constructs of parallel composition, sequential composition, and recursion. A global type K specifies an interaction that is still to start. On the opposite, a global type H specifies a completely executed interaction, in which there is one path entirely marked with hats. A general global type G specifies a partially executed interaction, whose executed part (history) is specified by subterms H, while the parts that have been discarded in choices or remain to be executed are specified by subterms K.

▶ **Definition 4.** *Global types* G are defined by:

$$\begin{array}{lcl} \mathsf{K} & ::= & {}_{\gamma}\boxplus_{i\in I}\ \alpha_i^{\mathsf{p}};\mathsf{K}_i \ \mid\ \mathsf{K} \parallel \mathsf{K} \ \mid\ \mathsf{K};\mathsf{K} \ \mid\ \mu\mathbf{t}.\mathsf{K} \ \mid\ \mathbf{t} \ \mid\ \mathsf{Skip} \\ \mathsf{H} & ::= & {}_{\gamma}(\widehat{\boxplus}_{i\in I}\ \alpha_i^{\mathsf{p}};\mathsf{K}_i \mathbin{\widehat{\boxplus}} \widehat{\alpha^{\mathsf{p}}}\ ;\mathsf{H}) \ \mid\ \mathsf{H} \parallel \mathsf{H} \ \mid\ \mathsf{H};\mathsf{H} \ \mid\ \mathsf{Skip} \\ \mathsf{G} & ::= & \mathsf{K} \ \mid\ {}_{\gamma}(\widehat{\boxplus}_{i\in I}\ \alpha_i^{\mathsf{p}};\mathsf{K}_i \mathbin{\widehat{\boxplus}} \widehat{\alpha^{\mathsf{p}}}\ ;\mathsf{G}) \ \mid\ \mathsf{G} \parallel \mathsf{G} \ \mid\ \mathsf{G};\mathsf{K} \ \mid\ \mathsf{H};\mathsf{G} \end{array}$$

The choice operator $\boxplus$ is n-ary, commutative and idempotent. We use $\widetilde{\boxplus}$ for either $\boxplus$ or $\widehat{\boxplus}$. In the binary case we write $\widetilde{\alpha_1^{\mathsf{p}}};\mathsf{G}\ {}_{\gamma}\widetilde{\boxplus}\ \widetilde{\alpha_2^{\mathsf{p}}};\mathsf{G}'$. The notation ${}_{\gamma}(\widehat{\boxplus}_{i\in I}\ \alpha_i^{\mathsf{p}};\mathsf{K}_i \mathbin{\widehat{\boxplus}} \widehat{\alpha^{\mathsf{p}}}\ ;\mathsf{G})$ stands for a choice where the branch initiating with $\widehat{\alpha^{\mathsf{p}}}$ has started to be executed. By abuse of notation, we will write ${}_{\gamma}\widetilde{\boxplus}_{i\in I}\ \widetilde{\alpha_i^{\mathsf{p}}};\mathsf{G}_i$ for either ${}_{\gamma}\boxplus_{i\in I}\ \alpha_i^{\mathsf{p}};\mathsf{G}_i$ or ${}_{\gamma}(\widehat{\boxplus}_{i\in I\setminus\{j\}}\ \alpha_i^{\mathsf{p}};\mathsf{G}_i \mathbin{\widehat{\boxplus}} \widehat{\alpha_j^{\mathsf{p}}}\ ;\mathsf{G}_j)$. When the checkpoint set γ associated with a choice is empty, we omit it. We call *rooted interaction* a subterm $\widetilde{\alpha^{\mathsf{p}}};\mathsf{G}$. Hence a type ${}_{\gamma}\widetilde{\boxplus}_{i\in I}\ \widetilde{\alpha_i^{\mathsf{p}}};\mathsf{G}_i$ is a choice among a number of rooted interactions with the same sender, at most one of which bears a hat. If I is a singleton we write $\widetilde{\alpha^{\mathsf{p}}};\mathsf{G}$. So $\widetilde{\alpha^{\mathsf{p}}};\mathsf{G}$ can denote either a rooted interaction or a choice with a single branch (the context will disambiguate if needed). We use $\mathsf{pa}(\mathsf{G})$ to denote *the set of participants of* G, namely all p,q such that $\mathsf{p} \xrightarrow{\lambda} \mathsf{q}$ occurs in G.

As for processes, parallel and sequential composition are associative, with neutral element Skip, parallel composition is commutative, and recursion is guarded, sequential, and treated equi-recursively. The operators have the following precedence: ';', '$\boxplus$', '$\parallel$'.

Session types are projections of global types onto participants. They represent the contributions of individual participants to the session. The projection of a choice yields a union for the choice leader and intersections for the receivers. Checkpoint labels of global types are preserved by the projection onto session types, and the checkpoint label of the choice leader is distinguished by overlining it.

We now define *session pre-types*, which are a superset of session types. In this paper, session types will only be defined as projections of well-formed global types, see Definition 7. (There is no circularity since global type well-formedness does not depend on session types.) Session pre-types are obtained from processes by replacing external and internal choices with intersections and unions, X with $\mathbf{t}$ and skip with Skip, with similar conventions.

▶ **Definition 5.** *Session pre-types* are defined by:

$$\mathsf{T} \quad ::= \quad \widetilde{\bigwedge}_{i\in I}\widetilde{\pi}_i;\mathsf{T}_i \ \mid\ \widetilde{\bigvee}_{i\in I}\widetilde{\pi}_i;\mathsf{T}_i \ \mid\ {}_{\Delta}\mathsf{T} \ \mid\ \mathsf{T}\,|\,\mathsf{T} \ \mid\ \mathsf{T};\mathsf{T} \ \mid\ \mu\mathbf{t}.\mathsf{T} \ \mid\ \mathbf{t} \ \mid\ \mathsf{Skip}$$

We call an intersection $\widetilde{\bigwedge}_{i\in I}\widetilde{\mathsf{p}_i?\lambda_i};\mathsf{T}_i$ *non-ambiguous* if its initial inputs are all distinct, namely if $i \neq j$ implies either $\mathsf{p}_i \neq \mathsf{p}_j$ or $\lambda_i \neq \lambda_j$. Projectability will require non-ambiguity of intersections. Our definition of projection (see Figure 3) is more liberal than in other session calculi, since we have an extended syntax for global types and we want to maximise the set of global types that are projectable. In particular, we allow equal messages between the same pair of participants in choices (this is usually forbidden [12, 28]). In any case, we want projection to ensure that in a global choice, the choice leader makes the decision and all the other participants act accordingly. We do so by requiring that, for any participant except the choice leader, the set of projections of the choice branches on that participant, say $\{\mathsf{T}_i \mid i \in I\}$, be consistent, i.e., the *join of the types in the set,* $\bigsqcup_{i\in I}\mathsf{T}_i$, be defined.

The join $\bigsqcup_{i\in I}\mathsf{T}_i$ is a partial operator, which checks that the T_i's are compatible behaviours and then combines them into a single session type. (In the definition of join we may assume that all the T_i's are projections of global types.) Intuitively, $\bigsqcup_{i\in I}\mathsf{T}_i$ is defined if the concerned participant either has the same behaviour in all T_i, or, if this is not the case, if she receives a message that "notifies" her about the chosen T_i before she starts differentiating her behaviour.

If one of the T_i's is an intersection of input behaviours, then so must be all the other T_i's.

To join intersection types, we define an auxiliary operator $\biguplus$, which takes a non-ambiguous intersection $\widetilde{\bigwedge}_{i\in I}\widetilde{\mathsf{p}_i?\lambda_i};\mathsf{T}_i$ and a session type $\widetilde{\mathsf{p}?\lambda};\mathsf{T}$, and combines them, if possible:

$$(\widetilde{\bigwedge}_{i\in I}\widetilde{\mathsf{p}_i?\lambda_i};\mathsf{T}_i)\biguplus\widetilde{\mathsf{p}?\lambda};\mathsf{T} = \begin{cases} \widetilde{\bigwedge}_{i\in I}\widetilde{\mathsf{p}_i?\lambda_i};\mathsf{T}_i \mathbin{\widetilde{\wedge}} \widetilde{\mathsf{p}?\lambda};\mathsf{T} & \text{if } \mathsf{p}_i \neq \mathsf{p} \text{ or } \lambda_i \neq \lambda \text{ for all } i \in I \\ \widetilde{\bigwedge}_{i\in I\setminus\{j\}}\widetilde{\mathsf{p}_i?\lambda_i};\mathsf{T}_i \mathbin{\widetilde{\wedge}} (\widetilde{\mathsf{p}_j?\lambda_j} \mathbin{\widetilde{\cup}} \widetilde{\mathsf{p}?\lambda});\mathsf{T}\bigsqcup\mathsf{T}_j & \text{if } \mathsf{p}_j = \mathsf{p} \\ & \quad \text{and } \lambda_j = \lambda \ (j \in I) \end{cases}$$

where $\widehat{\mathsf{p}?\lambda} \mathbin{\widetilde{\cup}} \mathsf{p}?\lambda = \mathsf{p}?\lambda \mathbin{\widetilde{\cup}} \widehat{\mathsf{p}?\lambda} = \widehat{\mathsf{p}?\lambda}$ and $\mathsf{p}?\lambda \mathbin{\widetilde{\cup}} \mathsf{p}?\lambda = \mathsf{p}?\lambda$ and the obtained intersections have hats if one of the input behaviours has a hat. If the prefix $\mathsf{p}?\lambda$ is different from all the $\mathsf{p}_i?\lambda_i$, then $\biguplus$ produces the intersection of the two types. In this case, receiving message λ from p amounts to being notified of the choice of branch T. If instead $\mathsf{p}?\lambda = \mathsf{p}_j?\lambda_j$ for some $j \in I$, then we try to combine the types $\widetilde{\mathsf{p}?\lambda};\mathsf{T}$ and $\widetilde{\mathsf{p}_j?\lambda_j};\mathsf{T}_j$ by factoring out their common prefix and producing $\widetilde{\mathsf{p}?\lambda};\mathsf{T}\bigsqcup\mathsf{T}_j$ (where $\mathsf{p}?\lambda$ has a hat if one of the two building prefixes has a hat), so that the resulting intersection is again non-ambiguous. If $\mathsf{T}\bigsqcup\mathsf{T}_j$ is defined, then T and T_j will be discriminated later if they are different, see G_1 in Example 6.

To define $\biguplus$ on two intersection types, we just iterate the above definition on the members of one of the intersections. In a similar way, we can extend $\biguplus$ to a set of intersection types.

We may now formally define the join of a set of session types.

$$\bigsqcup_{i\in I}\mathsf{T}_i = \begin{cases} {}_\Delta\biguplus_{i\in I, j\in J_i}\widetilde{\mathsf{p}_{i,j}?\lambda_{i,j}};\mathsf{T}_{i,j} & \text{if } \mathsf{T}_i = {}_{\Delta_i}\widetilde{\bigwedge}_{j\in J_i}\widetilde{\mathsf{p}_{i,j}?\lambda_{i,j}};\mathsf{T}_{i,j} \text{ for all } i \in I \\ & \quad \text{and } \Delta = \bigcup_{i\in I}\Delta_i \\ \mathsf{T};\bigsqcup_{i\in I}\mathsf{T}'_i & \text{if } \mathsf{T}_i = \mathsf{T};\mathsf{T}'_i \text{ for all } i \in I \\ \mathsf{Skip} & \text{if } \mathsf{T}_i = \mathsf{Skip} \text{ for all } i \in I \end{cases}$$

The definitions of $\bigsqcup_{i\in I}\mathsf{T}_i$ and $\biguplus$ are mutually recursive. Let us examine the three clauses. If all T_i's are intersections of session types, then we combine them with $\biguplus$ as explained above. The checkpoint set of the resulting intersection is the union of the checkpoint sets of the T_i's.

If all T_i's start with the same interaction T, and their continuations T'_i's are consistent, then we factor out T and we produce the session type T followed by the join of the T'_i's. If some T_i is Skip, it means that the participant terminates in the branch T_i. Then it must terminate in all branches, so all T_i's must be Skip.

The join is not defined between intersections and unions, nor between unions with different prefixes. Our join extends that of [12], to deal with parallel and sequential composition.

$$(\widetilde{\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{G}}) \upharpoonright \mathsf{p} = \widetilde{\mathsf{q}!\lambda; \mathsf{G} \upharpoonright \mathsf{p}} \quad (\widetilde{\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{G}}) \upharpoonright \mathsf{q} = \widetilde{\mathsf{p}?\lambda; \mathsf{G} \upharpoonright \mathsf{q}} \quad (\widetilde{\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{G}}) \upharpoonright \mathsf{r} = \mathsf{G} \upharpoonright \mathsf{r} \text{ , if } \mathsf{r} \neq \mathsf{p}, \mathsf{q}$$

$$(\ {}_{\gamma}\widetilde{\boxplus}_{i \in I}\, \widetilde{\alpha_i^{\mathsf{p}}; \mathsf{G}_i}\ ) \upharpoonright \mathsf{r} = \begin{cases} \overline{\gamma}\widetilde{\bigvee}_{i \in I}(\ \widetilde{\alpha_i^{\mathsf{p}}; \mathsf{G}_i}\ ) \upharpoonright \mathsf{r} & \text{if } \mathsf{r} = \mathsf{p} \\ \gamma \lfloor \bigsqcup_{i \in I}(\ \widetilde{\alpha_i^{\mathsf{p}}; \mathsf{G}_i}\ ) \upharpoonright \mathsf{r} \rfloor & \text{otherwise} \end{cases}$$

Figure 3 Projection of global types onto participants.

To define projection we need one last auxiliary operator, the prefixing of a session pre-type T by a set γ of at most one checkpoint label (notation $\gamma\lfloor \mathsf{T} \rfloor$), defined by:

$$\gamma\lfloor \widetilde{\bigwedge}_{i \in I} \widetilde{\pi_i}; \mathsf{T}_i \rfloor = {}_{\gamma}\widetilde{\bigwedge}_{i \in I} \widetilde{\pi_i}; \mathsf{T}_i \qquad \gamma\lfloor \widetilde{\bigvee}_{i \in I} \widetilde{\pi_i}; \mathsf{T}_i \rfloor = {}_{\gamma}\widetilde{\bigvee}_{i \in I} \widetilde{\pi_i}; \mathsf{T}_i \qquad \gamma\lfloor {}_{\Delta}\mathsf{T} \rfloor = {}_{\gamma \cup \Delta}\mathsf{T} \text{ if } \gamma \cap \Delta = \emptyset$$
$$\gamma\lfloor \mathsf{T}_0 \mid \mathsf{T}_1 \rfloor = \gamma\lfloor \mathsf{T}_0 \rfloor \mid \gamma\lfloor \mathsf{T}_1 \rfloor \quad \gamma\lfloor \mathsf{T}_0; \mathsf{T}_1 \rfloor = \gamma\lfloor \mathsf{T}_0 \rfloor; \mathsf{T}_1 \quad \gamma\lfloor \mu \mathbf{t}.\mathsf{T}' \rfloor = \mu \mathbf{t}.\gamma\lfloor \mathsf{T}' \rfloor \quad \gamma\lfloor \mathsf{Skip} \rfloor = \mathsf{Skip}$$

Prefixing adds γ to the first (possibly empty) set Δ found in T. Intuitively, the checkpoint labels that are spread along successive choices in the global type may get grouped together on a single local choice when projected on participants. We do not need to define $\gamma\lfloor \mathbf{t} \rfloor$ since recursion is guarded. The condition in the clause for $\gamma\lfloor {}_{\Delta}\mathsf{T} \rfloor$ is needed to avoid checkpoint labels which can never be used. This condition rules out for instance the global type:

$$(\mathsf{p} \xrightarrow{\lambda_1} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_1'} \mathsf{r}; (\mathsf{p} \xrightarrow{\lambda_2} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_2'} \mathsf{r}\ {}_{\{C\}}\boxplus\ \mathsf{p} \xrightarrow{\lambda_3} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_3'} \mathsf{r}))\ {}_{\{C\}}\boxplus\ \mathsf{p} \xrightarrow{\lambda_4} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_4'} \mathsf{r}$$

in which no backward reduction could return to the innermost occurrence of C. This is due to the condition on evaluation contexts in rules [CTBA] and [CTBP].

The projection of global types uses the projections of rooted interactions, see the first line of Figure 3. The projection of a choice is a union for the sending participant, and it is otherwise computed as the join of the projections on the branches. With $\overline{\gamma}$ we denote $\{\overline{C}\}$ if $\gamma = \{C\}$ and $\emptyset$ if $\gamma = \emptyset$. The join operation can be undefined, and therefore also the projection of global types can be undefined. Since $\mathsf{Skip} \bigsqcup \mathsf{T}$ is defined only if $\mathsf{T} = \mathsf{Skip}$, the definition of projection ensures that all the branches of a choice ${}_{\gamma}\widetilde{\boxplus}_{i \in I}\, \widetilde{\alpha_i^{\mathsf{p}}; \mathsf{G}_i}$ have the same participants, i.e., $\mathsf{pa}(\widetilde{\alpha_i^{\mathsf{p}}; \mathsf{G}_i}) = \mathsf{pa}(\widetilde{\alpha_j^{\mathsf{p}}; \mathsf{G}_j})$ for all $i, j \in I$. The omitted cases in Figure 3 are either standard (recursion and Skip) or homomorphic projections (parallel and sequential composition). Notice that projection respects hats and checkpoint labels.

▶ **Example 6.** Let $\mathsf{G}_1 = (\ \widehat{\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_1} \mathsf{q}}; \mathsf{q} \xrightarrow{\lambda_2} \mathsf{r}\ ) \mathbin{\widehat{\boxplus}} (\ \mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_3} \mathsf{q}; \mathsf{q} \xrightarrow{\lambda_4} \mathsf{r}\ )$. The projections of G_1 onto its participants are: $\mathsf{G}_1 \upharpoonright \mathsf{p} = (\ \widehat{\mathsf{q}!\lambda}; \widehat{\mathsf{q}!\lambda_1}\ ) \mathbin{\widehat{\vee}} (\ \mathsf{q}!\lambda; \mathsf{q}!\lambda_3\ )$, $\mathsf{G}_1 \upharpoonright \mathsf{q} = \widehat{\mathsf{p}?\lambda}; (\ \widehat{\mathsf{p}?\lambda_1}; \mathsf{r}!\lambda_2\ ) \mathbin{\widehat{\wedge}} (\ \mathsf{p}?\lambda_3; \mathsf{r}!\lambda_4\ )$ and $\mathsf{G}_1 \upharpoonright \mathsf{r} = \mathsf{q}?\lambda_2 \wedge \mathsf{q}?\lambda_4$. Even though q receives the same message λ from p in the two branches of the choice, before sending two different messages to r it receives a second message from p, which identifies the chosen branch. Also the global type $\mathsf{G}_2 = (\ \mathsf{p} \xrightarrow{\lambda_1} \mathsf{q}; \mathsf{r} \xrightarrow{\lambda_2} \mathsf{q}\ ) \boxplus (\ \mathsf{p} \xrightarrow{\lambda_3} \mathsf{q}; \mathsf{r} \xrightarrow{\lambda_2} \mathsf{q}\ )$ is projectable, since r sends the same message λ_2 in both branches.

Instead, the global type $\mathsf{G}_3 = (\ \widehat{\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{q} \xrightarrow{\lambda_1} \mathsf{r}}\ ) \mathbin{\widehat{\boxplus}} (\ \mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{q} \xrightarrow{\lambda_2} \mathsf{r}\ )$ is not projectable on the participant q: in fact $(\ \widehat{\mathsf{p}?\lambda}; \mathsf{r}!\lambda_1\ ) \bigsqcup (\ \mathsf{p}?\lambda; \mathsf{r}!\lambda_2\ )$ is not defined, since it yields $\widehat{\mathsf{p}?\lambda}; (\mathsf{r}!\lambda_1 \bigsqcup \mathsf{r}!\lambda_2)$ and $\mathsf{r}!\lambda_1 \bigsqcup \mathsf{r}!\lambda_2$ is not defined. This is justified since q should send two different messages to r in the two branches, while no previous communication identifies the chosen branch.

The next example features a checkpoint label set containing both a simple label and an overlined one. Let $\mathsf{G}_4 = \mathsf{p} \xrightarrow{\lambda_1} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_1'} \mathsf{r}; \mathsf{G}_5\ {}_{\{C'\}}\boxplus\ \mathsf{p} \xrightarrow{\lambda_2} \mathsf{q}; \mathsf{p} \xrightarrow{\lambda_2'} \mathsf{r}; \mathsf{G}_5$ where $\mathsf{G}_5 =$

$\mathsf{s} \xrightarrow{\lambda_3} \mathsf{p};\mathsf{s} \xrightarrow{\lambda'_3} \mathsf{r}\ {}_{\{C\}}\boxplus\ \mathsf{s} \xrightarrow{\lambda_4} \mathsf{p};\mathsf{s} \xrightarrow{\lambda'_4} \mathsf{r}$. Then the projection of G_4 onto s is $\mathsf{G}_4 \upharpoonright \mathsf{s} = \mathsf{p}!\lambda_3\ \mathsf{r}!\lambda'_3\ {}_{\{C',\overline{C}\}}\curlyvee\ \mathsf{p}!\lambda_4\ \mathsf{r}!\lambda'_4$.

Let us now look back at the RSA protocol. Its global type G, which may be seen as a syntactic description of the graph in Figure 1a, is the following:

$$\mathsf{G} = \mathbf{R} \xrightarrow{\mathtt{dt}} \mathbf{S}\,;(\mathsf{G}_1 \parallel \mathsf{G}_2);(\mathsf{G}_3\ {}_{\{C\}}\boxplus\ \mathsf{G}_4) \text{ where}$$
$$\mathsf{G}_1 = \mathbf{S} \xrightarrow{\mathtt{fxDt}} \mathbf{A};\mathbf{A} \xrightarrow{\mathtt{frPl}} \mathbf{S} \qquad \mathsf{G}_2 = \mathbf{S} \xrightarrow{\mathtt{flDt}} \mathbf{A};\mathbf{A} \xrightarrow{\mathtt{frMn}} \mathbf{S}$$
$$\mathsf{G}_3 = \mathbf{S} \xrightarrow{\mathtt{alDt}} \mathbf{R};(\mathsf{G}_5\ {}_{\{C'\}}\boxplus\ \mathsf{G}_6) \qquad \mathsf{G}_4 = \mathbf{S} \xrightarrow{\mathtt{dbPl}} \mathbf{A}\,;\mathbf{A} \xrightarrow{\mathtt{dTkPl}} \mathbf{S};\mathbf{S} \xrightarrow{\mathtt{dnPl}} \mathbf{R}$$
$$\mathsf{G}_5 = \mathbf{R} \xrightarrow{\mathtt{yes}} \mathbf{S}\,;\mathbf{S} \xrightarrow{\mathtt{dbMn}} \mathbf{A};\mathbf{A} \xrightarrow{\mathtt{dTkMn}} \mathbf{S};\mathbf{S} \xrightarrow{\mathtt{dnMn}} \mathbf{R};\mathbf{R} \xrightarrow{\mathtt{htl}} \mathbf{S} \quad \mathsf{G}_6 = \mathbf{R} \xrightarrow{\mathtt{no}} \mathbf{S}\,;\mathbf{S} \xrightarrow{\mathtt{sn}} \mathbf{A};\mathbf{A} \xrightarrow{\mathtt{snTk}} \mathbf{S}$$

The projection of G on the participant $\mathbf{S}$ ($\mathsf{T}^{\mathbf{S}}$) has the same structure as G, since $\mathbf{S}$ is involved in all communications of the protocol:

$$\mathsf{T}^{\mathbf{S}} = \mathbf{R}?\mathtt{dt};(\mathsf{T}^{\mathbf{S}}_1 \mid \mathsf{T}^{\mathbf{S}}_2)\,;\ (\mathsf{T}^{\mathbf{S}}_3\ {}_{\{\overline{C}\}}\curlyvee\ \mathsf{T}^{\mathbf{S}}_4) \quad \text{where}$$
$$\mathsf{T}^{\mathbf{S}}_1 = \mathbf{A}!\mathtt{fxDt};\mathbf{A}?\mathtt{frPl} \qquad \mathsf{T}^{\mathbf{S}}_2 = \mathbf{A}!\mathtt{flDt};\mathbf{A}?\mathtt{frMn}$$
$$\mathsf{T}^{\mathbf{S}}_3 = \mathbf{R}!\mathtt{alDt};(\mathsf{T}^{\mathbf{S}}_5\ {}_{\{C'\}}\curlyvee\ \mathsf{T}^{\mathbf{S}}_6) \qquad \mathsf{T}^{\mathbf{S}}_4 = \mathbf{A}!\mathtt{dbPl};\mathbf{A}?\mathtt{dTkPl};\mathbf{R}!\mathtt{dnPl}$$
$$\mathsf{T}^{\mathbf{S}}_5 = \mathbf{R}?\mathtt{yes};\mathbf{A}!\mathtt{dbMn};\mathbf{A}?\mathtt{dTkMn};\mathbf{R}!\mathtt{dnMn};\mathbf{R}?\mathtt{htl} \qquad \mathsf{T}^{\mathbf{S}}_6 = \mathbf{R}?\mathtt{no};\mathbf{A}!\mathtt{sn};\mathbf{A}?\mathtt{snTk}$$

On the other hand, the projections of G on $\mathbf{A}$ and $\mathbf{R}$ are simpler, and are not given here.

In the presence of sequentialisation and recursion, projectability of global types is not enough to ensure *starvation freedom* in communication protocols. For example, the global type $\mu\mathbf{t}.\mathsf{p} \xrightarrow{\lambda} \mathsf{q};\mathbf{t};\mathsf{p} \xrightarrow{\lambda'} \mathsf{r}$ exhibits starvation, since the participant r will indefinitely wait for message λ' from participant p. This problem can be avoided by requiring that the communications which follow a loop involve only participants who are also communicating in the body of the loop. We formalise this by the predicate *sequentially well-formed* on global types. This predicate applied to $\mathsf{G};\mathsf{G}'$ requires that the participants of G' be contained in the set of *allowed followers* of G, denoted by $\mathsf{af}(\mathsf{G})$ and defined by

$$\mathsf{af}(\gamma\widetilde{\boxplus}_{i\in I}\,\widetilde{\alpha^{\mathsf{p}}_i};\mathsf{G}_i) = \textstyle\bigcap_{i\in I}\mathsf{af}(G_i) \qquad \mathsf{af}(\mathsf{G} \parallel \mathsf{G}') = \mathsf{af}(\mathsf{G}) \cup \mathsf{af}(\mathsf{G}') \qquad \mathsf{af}(\mathsf{G};\mathsf{G}') = \mathsf{af}(\mathsf{G}) \cap \mathsf{af}(\mathsf{G}')$$
$$\mathsf{af}(\mu\mathbf{t}.\mathsf{G}) = \begin{cases} \mathsf{pa}(\mathsf{G}) & \text{if } \mathbf{t} \in \mathsf{G} \\ \mathsf{af}(\mathsf{G}) & \text{otherwise} \end{cases} \qquad \mathsf{af}(\mathsf{Skip}) = \mathsf{Part}$$

In the definition of af for parallel composition of global types, the union is justified by the example $(\mu\mathbf{t}.\mathsf{p} \xrightarrow{\lambda_1} \mathsf{q};\mathbf{t} \parallel \mu\mathbf{t}.\mathsf{r} \xrightarrow{\lambda_2} \mathsf{s};\mathbf{t});\mathsf{p} \xrightarrow{\lambda_3} \mathsf{r}$.

In order to avoid deadlocks, we need to impose some restrictions on the use of checkpoint labels and of messages in parallel communications. The same checkpoint label should not occur in two parallel global types. For suppose we allowed the global type:

$$(\mathsf{p} \xrightarrow{\lambda_1} \mathsf{q}\,;\ \mathsf{p} \xrightarrow{\lambda'_1} \mathsf{r}\ {}_{\{C\}}\boxplus\ \mathsf{p} \xrightarrow{\lambda_2} \mathsf{q}\,;\ \mathsf{p} \xrightarrow{\lambda'_2} \mathsf{r}) \parallel (\mathsf{p} \xrightarrow{\lambda_3} \mathsf{q};\mathsf{p} \xrightarrow{\lambda'_3} \mathsf{r}\ {}_{\{C\}}\boxplus\ \mathsf{p} \xrightarrow{\lambda_4} \mathsf{q};\mathsf{p} \xrightarrow{\lambda'_4} \mathsf{r})$$

Here there are two parallel interactions, both involving participants p,q and r. Assume that message λ_1 has been exchanged in the first interaction, and message λ_3 has been exchanged in the second. Now participant p may want to roll back to the choice labelled by $\{C\}$ in the first interaction. Then participant q should roll back to the same choice, while the above type would allow her to roll back to the choice labelled by $\{C\}$ in the second interaction.

As regards messages, we already observed that we allow different branches of a choice to have equal senders, equal messages and equal receivers. Our definition of projection ensures that the behaviour of a participant is either independent of the chosen branch or identifiable through some communications. For parallel global types, instead, we require that different branches do not have equal senders, equal messages and equal receivers. Indeed, if we allowed

$$\frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I)}{\Gamma \vdash {}_{\delta}\textstyle\sum_{i\in I} \pi_i; P_i : {}_{\delta}\bigwedge_{i\in I} \pi_i; \mathsf{T}_i}\ [\text{T-ExtCh}] \qquad \frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I)}{\Gamma \vdash {}_{\Delta}\textstyle\bigoplus_{i\in I} \pi_i; P_i : {}_{\Delta}\bigvee_{i\in I} \pi_i; \mathsf{T}_i}\ [\text{T-IntCh}]$$

$$\frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I) \quad \Gamma \vdash P : \mathsf{T}}{\Gamma \vdash {}_{\delta}\widehat{\textstyle\sum}_{i\in I} \pi_i; P_i \mathbin{\widehat{+}} \widehat{\pi}; P : {}_{\delta}\widehat{\textstyle\bigwedge}_{i\in I} \pi_i; \mathsf{T}_i \mathbin{\widehat{\wedge}} \widehat{\pi}; \mathsf{T}}\ [\text{T-ExtCh-RT}]$$

$$\frac{\Gamma \vdash P_i : \mathsf{T}_i \ (i \in I) \quad \Gamma \vdash P : \mathsf{T}}{\Gamma \vdash {}_{\Delta}\widehat{\textstyle\bigoplus}_{i\in I} \pi_i; P_i \mathbin{\widehat{\oplus}} \widehat{\pi}; P : {}_{\Delta}\widehat{\textstyle\bigvee}_{i\in I} \pi_i; \mathsf{T}_i \mathbin{\widehat{\vee}} \widehat{\pi}; \mathsf{T}}\ [\text{T-IntCh-RT}]$$

$$\frac{\vdash P_i : \mathsf{G}\upharpoonright \mathsf{p}_i \quad (1 \le i \le n) \quad \mathsf{pa}(\mathsf{G}) \subseteq \{\mathsf{p}_1, \ldots, \mathsf{p}_n\}}{\vdash \mathsf{p}_1[\![\, P_1 \,]\!] \parallel \cdots \parallel \mathsf{p}_n[\![\, P_n \,]\!] : \mathsf{G}}\ [\text{T-Net}]$$

Figure 4 Main typing rules for processes and networks.

two parallel communications with equal messages between the same pair of participants, then we could have a global type $\mathsf{p} \xrightarrow{\lambda} \mathsf{q}\,;\ \mathsf{G}_1 \parallel \mathsf{p} \xrightarrow{\lambda} \mathsf{q}\,;\ \mathsf{G}_2$, and processes implementing participants p and q could "cross" their communications.

A global type meeting the above two requirements on messages and checkpoint labels is called respectively *message well formed and checkpoint label well formed*.

Last but not least, following [7, 26] we require that the order of communications prescribed by a global type be witnessed by at least one of the session participants. This excludes, for instance, the global type $\mathsf{p} \xrightarrow{\lambda} \mathsf{q}; \mathsf{r} \xrightarrow{\lambda'} \mathsf{s}$, since the two communications have disjoint sets of participants, and therefore they should be independent. The correct global type representing this situation is $\mathsf{p} \xrightarrow{\lambda} \mathsf{q} \parallel \mathsf{r} \xrightarrow{\lambda'} \mathsf{s}$. This condition is necessary for a global type to be *implementable* by a collection of processes, and we refer to [7] for further justifications of this choice. It is easy to formalise this requirement as a well-formedness condition on the set of traces generated by global types, defined in the standard way.

To sum up, we define well-formedness of global types as follows.

▶ **Definition 7** (Well-formed global types). *A global type is well formed* when:

1. Its projections are defined on all participants;
2. It is sequentially well formed, message well formed and checkpoint label well formed;
3. Its set of traces is well formed.

In the following we will consider only well-formed global types.

5 Type System and Soundness

The shape of *typing judgements* is $\Gamma \vdash P : \mathsf{T}$, where the environment Γ associates process variables with session types: $\Gamma ::= \emptyset \mid \Gamma, X : \mathsf{T}$. Process typing exploits the correspondence between external choices and intersections, internal choices and unions. Figure 4 shows the interesting rules. Typing respects hats, in rules [T-ExtCh], [T-IntCh], [T-ExtCh-RT], [T-IntCh-RT] intersections and unions have hats if choices have hats. Subtyping $\leqslant$ on session types takes into account the standard rules for intersection and union and preserves hats, while checkpoint sets can decrease with deletion of types in intersections and increase with addition of types in unions. Subtyping is used in a standard subsumption rule (omitted). The other omitted rules for processes are just homomorphisms as expected from the syntax of processes and session types. Rule [T-Net] is the only rule for typing networks: it

requires that the types of all processes be projections of a unique global type. The condition $\mathsf{pa}(\mathsf{G}) \subseteq \{\mathsf{p}_1, \ldots, \mathsf{p}_n\}$ ensures the presence of all session participants and allows the typing of sessions containing $\mathsf{p}[\![\, \mathsf{skip} \,]\!]$ for any p, a property needed to guarantee invariance of types under structural equivalence of networks. Clearly, typing imposes constraints on the way hats and checkpoint labels are placed within processes.

▶ **Example 8.** Let $P = \mu X.\mathsf{q}!\lambda_1; \mathsf{r}!\lambda_2; X \;{}_{\{C\}}\!\oplus\; \mathsf{q}!\lambda_3; \mathsf{r}!\lambda_4; X$ and $Q = \mu Y.\mathsf{p}?\lambda_1; Y \;{}_{\{C\}}\!+\; \mathsf{p}?\lambda_3; Y$ and $R = \mu Z.\mathsf{p}?\lambda_2; Z \;{}_{\{C\}}\!+\; \mathsf{p}?\lambda_4; Z$. The network $\mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathsf{r}[\![\, R \,]\!]$ reduces by forward reductions to $\mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathsf{r}[\![\, R' \,]\!]$ where $P' = \widehat{\mathsf{q}!\lambda_1}; \widehat{\mathsf{r}!\lambda_2}; (\mathsf{q}!\lambda_1; \mathsf{r}!\lambda_2; P \;{}_{\{C\}}\!\oplus\; \mathsf{q}!\lambda_3; \mathsf{r}!\lambda_4; P) \;{}_{\{C\}}\!\oplus\; \mathsf{q}!\lambda_3; \mathsf{r}!\lambda_4; P$ and $Q' = \widehat{\mathsf{p}?\lambda_1}; (\mathsf{p}?\lambda_1; Q \;{}_{\{C\}}\!+\; \widehat{\mathsf{p}?\lambda_3}; Q) {}_{\{C\}}\!+\; \mathsf{p}?\lambda_3; Q$ and $R' = \widehat{\mathsf{p}?\lambda_2}; R \;{}_{\{C\}}\!+\; \mathsf{p}?\lambda_4; R$. By Rule [Back], $\mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathsf{r}[\![\, R' \,]\!] \overset{C}{\circlearrowleft} \mathsf{p}[\![\, P'' \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathsf{r}[\![\, R \,]\!]$ where $P'' = \mathsf{q}!\lambda_3; \mathsf{r}!\lambda_4; P$. Without the condition $\mathcal{E}\;\mathsf{ok}$ for $\overline{C}$ on rule [CTBA], we could have also the backward move: $\mathsf{p}[\![\, P' \,]\!] \parallel \mathsf{q}[\![\, Q' \,]\!] \parallel \mathsf{r}[\![\, R' \,]\!] \overset{C}{\circlearrowleft} \mathsf{p}[\![\, P''' \,]\!] \parallel \mathsf{q}[\![\, Q'' \,]\!] \parallel \mathsf{r}[\![\, R \,]\!]$, where $P''' = (\widehat{\mathsf{q}!\lambda_1}; \widehat{\mathsf{r}!\lambda_2}; \mathsf{q}!\lambda_1; \mathsf{r}!\lambda_2; P) \;{}_{\{C\}}\!\oplus\; \mathsf{q}!\lambda_3; \mathsf{r}!\lambda_4; P$ and $Q'' = \widehat{\mathsf{p}?\lambda_1}; Q \;{}_{\{C\}}\!+\; \mathsf{p}?\lambda_3; Q$. Then Subject Reduction would fail, since the network $\mathsf{p}[\![\, P \,]\!] \parallel \mathsf{q}[\![\, Q \,]\!] \parallel \mathsf{r}[\![\, R \,]\!]$ is typable, while $\mathsf{p}[\![\, P''' \,]\!] \parallel \mathsf{q}[\![\, Q'' \,]\!] \parallel \mathsf{r}[\![\, R \,]\!]$ is not typable. In fact the output $\mathsf{r}!\lambda_2$ has a hat in the session type of P''', while the corresponding input $\mathsf{p}?\lambda_2$ does not have a hat in the session type of R. This example shows also why it would not be possible to roll back to the last checkpoints in recursive processes, since they could be different for different participants.

It is easy to verify that the type $\mathsf{T}^{\mathbf{S}}$ given at page 11 can be derived for the process $P^{\mathbf{S}}$ given at the end of Section 3.

The present type system is not informative enough: $\vdash \mathbb{N} : \mathsf{G}$ does not imply that a communication in G can be done in a forward computation of $\mathbb{N}$. For example, consider $\vdash \mathbb{N}_0 : \mathsf{G}_0$ where $\mathbb{N}_0 = \mathsf{p}[\![\, \mathsf{q}!\lambda \,]\!] \parallel \mathsf{q}[\![\, \mathsf{p}?\lambda + \mathsf{p}?\lambda' \,]\!]$ and $\mathsf{G}_0 = \mathsf{p} \xrightarrow{\lambda} \mathsf{q} \boxplus \mathsf{p} \xrightarrow{\lambda'} \mathsf{q}$. The communication $\mathsf{p} \xrightarrow{\lambda'} \mathsf{q}$ cannot occur in $\mathbb{N}_0$. To discuss properties, we introduce a type system which represents more closely the evolution of networks.

We write $\Gamma \vdash^{\star} P : \mathsf{T}$ if this judgement can be derived using the typing rules of Figure 4 without the help of the subtyping rule. Let $\leqslant^{\star}$ be the subtyping relation between session types obtained from $\leqslant$ by requiring that unions have the same number of disjuncts.

We write $\vdash^{\star} \mathbb{N} : \mathsf{G}$ if this judgement can be derived using the typing rule:

$$\frac{\vdash^{\star} P_i : \mathsf{T}_i \quad \mathsf{T}_i \leqslant^{\star} \mathsf{G} \upharpoonright \mathsf{p}_i \quad (1 \leq i \leq n) \quad \mathsf{pa}(\mathsf{G}) \subseteq \{\mathsf{p}_1, \ldots, \mathsf{p}_n\}}{\vdash \mathsf{p}_1[\![\, P_1 \,]\!] \parallel \cdots \parallel \mathsf{p}_n[\![\, P_n \,]\!] : \mathsf{G}} \;[\text{T-Net}^{\star}]$$

The relation between the type systems $\vdash^{\star}$ and $\vdash$ is expressed by Theorem 9, whose proof uses standard Inversion Lemmas. The property of Subject Reduction for $\vdash^{\star}$ is then established by Theorem 10. Its proof relies on the close correspondence between the evolution of well-typed nets and the evolution of their types along forward and backward computations.

▶ **Theorem 9.** *If* $\vdash^{\star} \mathbb{N} : \mathsf{G}$, *then* $\vdash \mathbb{N} : \mathsf{G}$. *If* $\vdash \mathbb{N} : \mathsf{G}$, *then* $\vdash^{\star} \mathbb{N} : \mathsf{G}'$ *for some* G'.

▶ **Theorem 10** (Subject Reduction). $\vdash^{\star} \mathbb{N} : \mathsf{G}$ *and* $\mathbb{N} \overset{\circlearrowleft}{\longrightarrow}^{*} \mathbb{N}'$ *imply* $\vdash^{\star} \mathbb{N}' : \mathsf{G}'$ *for some* G'.

Notice that Theorems 9 and 10 imply Subject Reduction for $\vdash$.

A standard property enforced by session types is Session Fidelity: all communications occur as specified by global types. To deal with backward reductions we introduce a mapping $\llparenthesis \;\; \rrparenthesis$ on global types, which erases all communications with hats and all discarded branches of choices. A checkpoint label C is *alive* in G if $\llparenthesis\, \mathsf{G} \,\rrparenthesis$ contains a choice ${}_{\{C\}}\!\boxplus_{i \in I}\; \mathsf{p} \xrightarrow{\lambda_i} \mathsf{q}_i; \mathsf{G}_i$ with more than one branch, such that for some $i \in I, \lambda, \mathsf{q}$, the global type G_i contains $\mathsf{p} \xrightarrow{\lambda} \mathsf{q}$.

▶ **Theorem 11** (Session Fidelity). *If* $\vdash \mathbb{N} : \mathsf{G}$*, then all traces in forward computations of* $\mathbb{N}$ *are suffixes of traces of* G*. Moreover if* $\mathbb{N} \longrightarrow^* \mathbb{N}' \overset{C}{\backsim} \mathbb{N}''$*, then* C *is alive in* G*.*

We discuss now forward and backward progress.

▶ **Theorem 12** (Forward Progress). *If* $\vdash \mathsf{p}[\![\, \mathcal{E}[P] \,]\!] \parallel \mathbb{N} : \mathsf{G}$ *with* P *user process,* $P \neq \mathsf{skip}$*, then there are* P'*,* $\mathbb{N}'$ *such that* $\mathsf{p}[\![\, \mathcal{E}[P] \,]\!] \parallel \mathbb{N} \longrightarrow^* \mathsf{p}[\![\, \mathcal{E}[P'] \,]\!] \parallel \mathbb{N}'$ *and* P' *has one hat.*

Notice that the standard formulation of progress [12], which requires that each input/output that is persistently offered be eventually consumed, is an easy consequence of this theorem.

▶ **Theorem 13** (Backward Progress). *If* $\vdash^\star \mathbb{N} : \mathsf{G}$ *and* C *is alive in* G*, then there is* $\mathbb{N}'$ *such that* $\mathbb{N} \longrightarrow^* \mathbb{N}'$ *and* $\mathbb{N}' \overset{C}{\backsim} \mathbb{N}''$*.*

We end this section with a remark on causal consistency and full reversibility. Clearly, our calculus enjoys causal consistency since a rollback to a checkpointed choice cancels all communications done after that checkpoint. Instead, full reversibility does not hold, since rollbacks erase explored branches in internal choices.

6 Related Work and Conclusion

Since the seminal work by Danos and Krivine on reversible CCS [9], reversible computation has been widely studied in process calculi. In [29], Phillips and Ulidowski proposed a method for reversing process operators defined in a general SOS format, and noted that thread identifiers and histories were needed to record the past of computations. In [21], Lanese et al. defined a reversible variant of the higher-order π-calculus, using tags to identify threads, and explicit memory processes. This calculus was enriched with a fine-grained rollback primitive in [20]. In [8], Cristescu et al. proposed a causal semantic model for the reversible π-calculus.

Reversibility for structured communications was first studied in [10, 11, 19], where *transactions* with rollback and coordinated checkpoints were modelled in an extended CCS. More recently, reversibility has been incorporated into *contracts* [1, 4] and *session calculi* [17, 18]. In [2] and [3], Barbanera et al. investigated the notions of compliance and sub-behaviour for contracts with checkpoints.While rollbacks are forgetful in [2], in [3] they are used as a strategy to achieve compliance: in this case, after a rollback a process cannot engage again in the previously explored branch, presumably unsuccessful.

Our work is most closely related to the recent proposals [33, 34, 23, 22, 24, 28, 26, 14]. Tiezzi and Yoshida [33] use tags and memories to allow full reversibility of binary sessions with delegation. In [34], two forms of reversibility are considered: either a session is completely reversed in a single backward step, or any intermediate state is restored. Mezzina and Pérez [23, 22] use monitors as memories for reversing binary sessions. A key novelty of this work is the use of session types with present and past. In [24], this approach is generalised to multiparty sessions, asynchronous higher-order communications, and decoupled rollbacks. In [28], Neykova and Yoshida provide an algorithm to analyse and extract causal dependencies from a given multiparty global type, and use it to ensure that communicating processes are safely recovered from consistent states in the presence of a failure. In [26], Mezzina and Tuosto propose a semantic control of reversibility: a computation along a branch is reversed according to the guards on the current configuration. A feature of [26] is that inputs are potentially irreversible actions, unless they appear within a loop. Finally, our paper builds on [14] and improves it in several respects: it has a more liberal syntax for types and processes, gives a more compact representation of past communications and implements a fine-tuned strategy for backward moves, geared towards achieving compliance.

It should be noted that in all the above-mentioned proposals, the syntax of global types (if any) does not include recursion, parallel composition nor sequential composition. Thus, some of our examples cannot be expressed in these models. On the other hand, the context-free session types of [32], recently proposed to capture the type-safe serialisation of recursive data types, include sequential composition and recursion. Our session types generalise them by offering in addition parallel composition, and by handling multiparty sessions.

The properties of our calculus are standard for session types, but their proofs require some ingenuity due to the generality of our syntax and to the specificity of our rollback mechanism. A limitation of our work is that the starting points of rollbacks are statically determined. By contrast, these points are determined dynamically in [26], offering a more realistic solution. We plan to introduce similar runtime conditions for rollback in our calculus.

Since our session types and global types are "truly concurrent", we would also like to study their interpretation into some non-interleaving semantic model. A natural candidate that comes to mind is the model of Event Structures, for which reversible variants have already been proposed [30, 16]. We plan to explore a reversible variant of *Flow Event Structures* [5], a model that has already been used to interpret CCS processes with past in [6].

Acknowledgements. We are grateful to the anonymous reviewers for their useful suggestions, which led to substantial improvements. We also thank Claudio Antares Mezzina for pointing out a technical problem in an earlier version of the paper.

References

1 Franco Barbanera and Ugo de' Liguoro. Sub-behaviour relations for session-based client/server systems. *Mathematical Structures in Computer Science*, 25(6):1339–1381, 2015. doi:10.1017/S096012951400005X.

2 Franco Barbanera, Mariangiola Dezani-Ciancaglini, and Ugo de'Liguoro. Reversible client/server interactions. *Formal Aspects of Computing*, 28(4):697–722, 2016. doi:10.1007/s00165-016-0358-2.

3 Franco Barbanera, Mariangiola Dezani-Ciancaglini, Ivan Lanese, and Ugo de' Liguoro. Retractable contracts. In *PLACES*, volume 203 of *EPTCS*, pages 61–72, 2016. doi:10.4204/EPTCS.203.

4 Giovanni Bernardi and Matthew Hennessy. Modelling session types using contracts. *Mathematical Structures in Computer Science*, 26(3):510–560, 2016. doi:10.1017/S0960129514000243.

5 Gérard Boudol and Ilaria Castellani. Permutation of transitions: an event structure semantics for CCS and SCCS. In *REX*, volume 354 of *LNCS*, pages 411–427. Springer, 1988. doi:10.1007/BFb0013028.

6 Gérard Boudol and Ilaria Castellani. Flow models of distributed computations: three equivalent semantics for CCS. *Information and Computation*, 114(2):247–314, 1994. doi:10.1006/inco.1994.1088.

7 Giuseppe Castagna, Mariangiola Dezani-Ciancaglini, and Luca Padovani. On Global Types and Multi-Party Sessions. *Logical Methods in Computer Science*, 8:1–45, 2012. doi:10.2168/LMCS-8(1:24)2012.

8 Ioana Cristescu, Jean Krivine, and Daniele Varacca. Rigid families for the reversible π-calculus. In *RC*, volume 9720 of *LNCS*, pages 3–19. Springer, 2016. doi:10.1007/978-3-319-40578-0_1.

9 Vincent Danos and Jean Krivine. Reversible communicating systems. In *CONCUR*, volume 3170 of *LNCS*, pages 292–307. Springer, 2004. doi:10.1007/978-3-540-28644-8_19.

10 Edsko de Vries, Vasileios Koutavas, and Matthew Hennessy. Communicating transactions - (extended abstract). In *CONCUR*, volume 6269 of *LNCS*, pages 569–583. Springer, 2010. doi:10.1007/978-3-642-15375-4_39.

11 Edsko de Vries, Vasileios Koutavas, and Matthew Hennessy. Liveness of communicating transactions - (extended abstract). In *APLAS*, volume 6461 of *LNCS*, pages 392–407. Springer, 2010. doi:10.1007/978-3-642-17164-2_27.

12 Pierre-Malo Deniélou and Nobuko Yoshida. Dynamic multirole session types. In *POPL*, pages 435–446. ACM Press, 2011. doi:10.1145/1926385.1926435.

13 Pierre-Malo Deniélou and Nobuko Yoshida. Multiparty session types meet communicating automata. In *ESOP*, volume 7211 of *LNCS*, pages 194–213. Springer, 2012. doi:10.1007/978-3-642-28869-2_10.

14 Mariangiola Dezani-Ciancaglini and Paola Giannini. Reversible multiparty sessions with checkpoints. In *EXPRESS/SOS*, volume 222 of *EPTCS*, pages 60–74, 2016. doi:10.4204/EPTCS.222.5.

15 Elena Giachino, Ivan Lanese, Claudio Antares Mezzina, and Francesco Tiezzi. Causal-consistent rollback in a tuple-based language. *Journal of Logic and Algebraic Methods in Programming*, 88:99–120, 2017. URL: https://doi.org/10.1016/j.jlamp.2016.09.003, doi:10.1016/j.jlamp.2016.09.003.

16 Eva Graversen, Iain Phillips, and Nobuko Yoshida. Towards a categorical representation of reversible event structures. In *PLACES*, volume 246 of *EPTCS*, pages 49–60, 2017. doi:10.4204/EPTCS.246.9.

17 Kohei Honda, Vasco T. Vasconcelos, and Makoto Kubo. Language primitives and type disciplines for structured communication-based programming. In *ESOP*, volume 1381 of *LNCS*, pages 22–138. Springer, 1998. doi:10.1007/BFb0053567.

18 Kohei Honda, Nobuko Yoshida, and Marco Carbone. Multiparty asynchronous session types. In *POPL*, pages 273–284. ACM Press, 2008. doi:10.1145/1328897.1328472.

19 Vasileios Koutavas, Carlo Spaccasassi, and Matthew Hennessy. Bisimulations for communicating transactions - (extended abstract). In *FOSSACS*, volume 8412 of *LNCS*, pages 320–334. Springer, 2014. doi:10.1007/978-3-642-54830-7_21.

20 Ivan Lanese, Claudio Antares Mezzina, Alan Schmitt, and Jean-Bernard Stefani. Controlling reversibility in higher-order pi. In *CONCUR*, volume 6901 of *LNCS*, pages 297–311. Springer, 2011. doi:10.1007/978-3-642-23217-6_20.

21 Ivan Lanese, Claudio Antares Mezzina, and Jean-Bernard Stefani. Reversing higher-order pi. In *CONCUR*, volume 6269 of *LNCS*, pages 478–493. Springer, 2010. doi:10.1007/978-3-642-15375-4_33.

22 Claudio Antares Mezzina and Jorge A. Pérez. Reversible semantics in session-based concurrency. In *ICTCS*, volume 1720 of *CEUR Workshop Proceedings*, pages 221–226. CEUR-WS.org, 2016.

23 Claudio Antares Mezzina and Jorge A. Pérez. Reversible sessions using monitors. In *PLACES*, volume 211 of *EPTCS*, pages 56–64, 2016. doi:10.4204/EPTCS.211.6.

24 Claudio Antares Mezzina and Jorge A. Pérez. Causally consistent reversible choreographies. *CoRR*, abs/1703.06021, 2017.

25 Claudio Antares Mezzina, Rudolf Schlatte, and al. COST Action on Reversible Computation, State of the Art Report, Working Group 2 on Software and Systems, 2012. URL: http://www.informatik.uni-bremen.de/ictcost/wg2_soar.pdf.

26 Claudio Antares Mezzina and Emilio Tuosto. Choreographies for automatic recovery. *CoRR*, abs/1705.09525, 2017.

27 Robin Milner. *A Calculus of Communicating Systems*, volume 92 of *LNCS*. Springer, 1980. doi:10.1007/3-540-10235-3.

28 Rumyana Neykova and Nobuko Yoshida. Let it recover: multiparty protocol-induced recovery. In *CC*, pages 98–108. ACM Press, 2017. doi:10.1145/3033019.

29 Iain Phillips and Irek Ulidowski. Reversing algebraic process calculi. *Journal of Logic and Algebraic Methods in Programming*, 73(1-2):70–96, 2007. doi:10.1016/j.jlap.2006.11.002.

30 Iain Phillips and Irek Ulidowski. Reversibility and asymmetric conflict in event structures. *Journal of Logic and Algebraic Methods in Programming*, 84(6):781–805, 2015. doi:10.1016/j.jlamp.2015.07.004.

31 Benjamin C. Pierce. *Types and Programming Languages*. MIT Press, 2002.

32 Peter Thiemann and Vasco T. Vasconcelos. Context-free session types. In *ICFP*, pages 462–475. ACM Press, 2016. doi:10.1145/2951913.2951926.

33 Francesco Tiezzi and Nobuko Yoshida. Reversible session-based pi-calculus. *Journal of Logical and Algebraic Methods in Programming*, 84(5):684–707, 2015. doi:10.1016/j.jlamp.2015.03.004.

34 Francesco Tiezzi and Nobuko Yoshida. Reversing single sessions. In *RC*, volume 9720 of *LNCS*, pages 52–69. Springer, 2016. doi:10.1007/978-3-319-40578-0_4.

Unbounded Product-Form Petri Nets*†

Patricia Bouyer[1], Serge Haddad[2], and Vincent Jugé[3]

1 **LSV, CNRS, ENS Paris-Saclay, Université Paris-Saclay, France**
2 **LSV, CNRS, ENS Paris-Saclay, Inria, Université Paris-Saclay, France**
3 **LSV, CNRS, ENS Paris-Saclay, Université Paris-Saclay, France**

Abstract

Computing steady-state distributions in infinite-state stochastic systems is in general a very difficult task. Product-form Petri nets are those Petri nets for which the steady-state distribution can be described as a natural product corresponding, up to a normalising constant, to an exponentiation of the markings. However, even though some classes of nets are known to have a product-form distribution, computing the normalising constant can be hard. The class of (closed) Π^3-nets has been proposed in an earlier work, for which it is shown that one can compute the steady-state distribution efficiently. However these nets are bounded. In this paper, we generalise queuing Markovian networks and closed Π^3-nets to obtain the class of open Π^3-nets, that generate infinite-state systems. We show interesting properties of these nets: (1) we prove that liveness can be decided in polynomial time, and that reachability in live Π^3-nets can be decided in polynomial time; (2) we show that we can decide ergodicity of such nets in polynomial time as well; (3) we provide a pseudo-polynomial time algorithm to compute the normalising constant.

1998 ACM Subject Classification D.2.2 Petri Nets

Keywords and phrases Performance evaluation, infinite-state systems, Petri nets, steady-state distribution

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.31

1 Introduction

Quantitative analysis of stochastic infinite-state systems. Performance measures of stochastic systems can be roughly classified in two categories: those related to the transient behaviour as expressed by a temporal logic formula and those related to the long-run behaviour whose main measure is the steady-state distribution (when it exists).

There are different relevant questions concerning the steady-state distribution of infinite-state systems: (1) given a state and a threshold, one can ask whether the steady-state probability of this state is above (or below) the threshold; (2) given a state, one can compute the steady-state probability of this state, either in an exact way, or in an accurate approximate way; and (3) one can give a symbolic representation of both the set of reachable states and its associated distribution (or an accurate approximation thereof).

Clearly the last question is the most difficult one, and the first breakthrough in that direction has been obtained in the framework of open queuing Markovian networks: in those systems, the measure of a state is obtained as the product of terms, where each term is related to the parameters of a queue (service and visit rate) and the number of clients in the queue [13]. In order to get a probability distribution over the set of reachable states,

* A full version of the paper is available at `https://arxiv.org/abs/1708.05847`.
† This work has been supported by ERC project EQualIS (FP7-308087).

this product is normalised by a constant, whose computation is easy when the service rates of the queues do not depend on the number of clients. This work has been adapted to closed networks, and the main contribution in [9] consists in computing the normalising constant without enumerating the (finite) reachability set, leading to an algorithm which runs in polynomial-time w.r.t. the size of the network and the number of clients. Later, Markov chains generated by a stochastic Petri net with a single unbounded place (that is, quasi-birth death processes) have been investigated [8], and an algorithm which approximates up to arbitrary precision the steady-state distribution has been proposed; however the complexity of the algorithm is very high, since it requires the computation of the finite reachability sets of some subnets, whose size may be non primitive recursive. More recently, the abstract framework of (infinite-state) Markov chains with a finite eager attractor (e.g. probabilistic lossy channel systems) has been used to develop an algorithm which approximates up to arbitrary precision the steady-state distribution as well [1], but there is no complexity bound for the algorithm.

Product-form Petri nets. While queuing networks are very interesting since they allow for an explicit representation of the steady-state distribution, they lack two important features available in Petri nets [19, 20], which are very relevant for modelling concurrent systems: resource competition and process synchronisation. So very soon researchers have tried to get the best of the two formalisms and they have defined subclasses of Petri nets for which one can establish product-form steady-state distributions. Historically, solutions have been based on purely behavioural properties (i.e. by an analysis of the reachability graph) like in [15], and then progressively have moved to more and more structural characterisations [16, 6]. Building on the work of [6], [11] has established the first purely structural condition for which a product-form steady-state distribution exists, and designed a polynomial-time algorithm to check for the condition (see also [17] for an alternative characterisation). These nets are called Π^2-nets. However the computation of the normalising constant remains a difficult open issue since a naive approach in the case of finite-state Π^2-nets would require to enumerate the potentially huge reachability state-space. Furthermore, the lower bounds shown in [10] for behavioural properties of Π^2-nets strongly suggest that the computation of the normalising constant can probably not be done in an efficient way. In [6, 23], the authors introduce semantical classes of product-form Petri nets for which this constant is computed in pseudo-polynomial time. However their approach suffers two important drawbacks: (1) checking whether a net fufills this condition as least as hard as the reachability problem, and (2) the only syntactical class for which this condition is fulfilled boils down to queuing networks.

To overcome this problem, the model of Π^3-nets is defined in [10] as a subclass of Π^2-nets obtained by structuring the synchronisation between concurrent activity flows in layers. This model strictly generalises closed product-form queuing networks (in which there is a single activity flow). Two interesting properties of those nets is that liveness for Π^3-nets and reachability for live Π^3-nets can both be checked in polynomial time. Furthermore, from a quantitative point-of-view, the normalising constant of the steady-state distribution can be efficiently computed using an elaborated dynamic programming algorithm.

Product-form Petri nets have been applied for the specification and analysis of complex systems. From a modelling point-of-view, compositional approaches have been proposed [18, 2] as well as hierarchical ones [12]. Application fields have also been identified, for instance, hardware design and more particularly RAID storage [12], or software architectures [3].

Our contributions. Unfortunately Π^3-nets generate finite-state systems. Here we address this problem by introducing and studying open Π^3-nets. Informally, an open Π^3-net has a main activity flow, which roughly corresponds to an open queuing network, and has other activity flows which are structured as in (standard, or closed) Π^3-nets. More precisely, in the case of a single activity flow this model is exactly equivalent to an open queuing network, but the general model is enriched with other activity flows, raising difficult computation issues. In particular, several places may in general be unbounded. Open Π^3-nets are particularly appropriate when modelling, in an open environment, protocols and softwares designed in layers. In adddition, they allow to specify dynamical management of resources where processes may produce and consume them with no *a priori* upper bound on their number. Our results on open Π^3-nets can be summarised as follows:

- We first establish that the liveness problem can be solved in polynomial time, and that the boundedness as well as the reachability problem in live nets can also be solved in polynomial time. On the other side, we show that the unboundedness, the reachability and even the covering problem become NP-hard without the liveness assumption.
- Contrary to the case of closed Π^3-nets, open Π^3-nets may not be ergodic (that is, there may not exist a steady-state distribution). We design a polynomial-time algorithm to decide ergodicity of an open Π^3-net.
- Our main contribution is the computation of the normalising constant for ergodic live Π^3-nets. Our procedure combines symbolic computations and dynamic programming. The time complexity of the algorithm is polynomial w.r.t. the size of the structure of the net and the maximal value of integers occuring in the description of the net (thus pseudo-polynomial). As a side result, we improve the complexity for computing the normalising constant of closed Π^3-nets that was given in [10] (the complexity was the same, but was assuming that the number of activity flows is a constant).

In Section 2, we introduce and illustrate product-form nets, and recall previous results. In Section 3, we focus on qualitative behavioural properties, while quantitative analysis is developed in Section 4. All proofs are postponed to the full version of this article.

2 Product-form Petri nets

Notations. Let A be a matrix over $I \times J$, one denotes $A(i,j)$ the item whose row index is i and column index is j. When I and J are disjoint, $W(k)$ denotes the row (resp. column) vector indexed by k when $k \in I$ (resp. $k \in J$). Given a real vector $\mathbf{v}$ indexed by I its norm, denoted $\|\mathbf{v}\|$, is defined by $\|\mathbf{v}\| = \sum_{i\in I} |\mathbf{v}(i)|$. Sometimes, one writes $\mathbf{v}_i$ for $\mathbf{v}(i)$. Given two vectors $\mathbf{v}$, $\mathbf{w}$ indexed by I their scalar product denoted $\mathbf{v} \cdot \mathbf{w}$ is defined by $\sum_{i\in I} \mathbf{v}_i\mathbf{w}_i$. Finally, if $\mathbf{v}$ is a vector over I, we define its support as $\mathsf{Supp}(\mathbf{v}) = \{i \in I \mid \mathbf{v}(i) \neq 0\}$.

We briefly recall Petri nets and stochastic Petri nets. The state of a Petri net, called a *marking* is defined by the number of *tokens* contained in every *place*. A Petri net models concurrent activities by *transitions* whose enabling requires tokens to be consumed in some places and then tokens to be produced in some places.

▶ **Definition 1** (Petri net)**.** A *Petri net* is a tuple $\mathcal{N} = (P, T, W^-, W^+)$ where:

- P is a finite set of *places*;
- T is a finite set of *transitions*, disjoint from P;
- W^- and W^+ are $P \times T$ matrices with coefficients in $\mathbb{N}$.

W^- (resp. W^+) is called the backward (resp. forward) incidence matrix, $W^-(p,t)$ (resp. $W^+(p,t)$) specifies the number of tokens consumed (resp. produced) in place p by the *firing* of transition t, and $W^-(t)$ (resp. $W^+(t)$) is the t-th column of W^- (resp. W^+). One

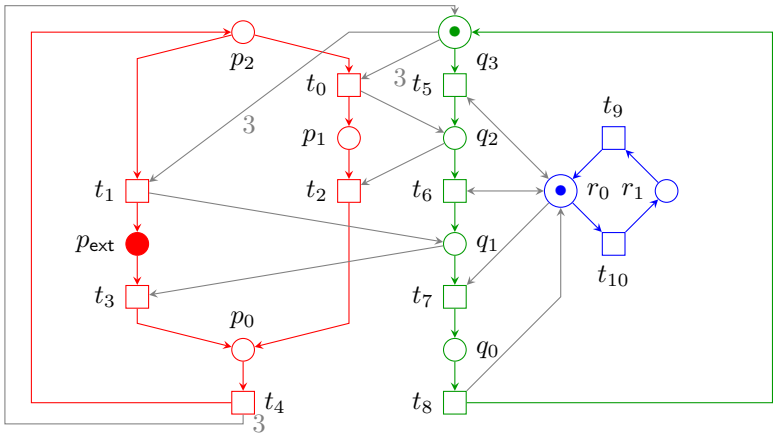


Figure 1 A marked Petri net (initial marking: $q_3 + r_0$).

assumes that for all $t \in T$, $W^-(t) \neq W^+(t)$ (i.e. no useless transition) and for all $t' \neq t$, either $W^-(t) \neq W^-(t')$ or $W^+(t) \neq W^+(t')$ (i.e. no duplicated transition); this will not affect our results.

A *marking* of $\mathcal{N}$ is a vector of $\mathbb{N}^P$; in the sequel we will often see m as a multiset ($m(p)$ is then the number of occurrences of p), or as a symbolic sum $\sum_{p\in P|m(p)>0} m(p)\, p$. The symbolic sum $\sum_{p\in P'} p$ will be more concisely written P'. Transition t is *enabled* by marking $m \in \mathbb{N}^P$ if for all $p \in P$, $m(p) \geqslant W^-(p,t)$. When enabled, its firing leads to the marking m' defined by: for all $p \in P$, $m'(p) = m(p) - W^-(p,t) + W^+(p,t)$. This firing is denoted by $m \xrightarrow{t} m'$. The *incidence matrix* $W = W^+ - W^-$ allows one to rewrite the marking evolution as $m' = m + W(t)$ if $m \geqslant W^-(t)$. Given an initial marking $m_0 \in \mathbb{N}^P$, the *reachability set* $\mathcal{R}_{\mathcal{N}}(m_0)$ is the smallest set containing m_0 and closed under the firing relation. When no confusion is possible one denotes it more concisely by $\mathcal{R}(m_0)$. Later, if $m \in \mathcal{R}(m_0)$, we may also write $m_0 \to^* m$. We will call $(\mathcal{N}, m_0)$ a *marked* Petri net

From a qualitative point of view, one is interested by several standard relevant properties including reachability. *Liveness* means that the modelled system never loses its capacity: for all $t \in T$ and $m \in \mathcal{R}(m_0)$, there exists $m' \in \mathcal{R}(m)$ such that t is enabled in m'. *Boundedness* means that the modelled system is a finite-state system: there exists $B \in \mathbb{N}$ such that for all $m \in \mathcal{R}(m_0)$, $\|m\| \leqslant B$. While decidable, these properties are costly to check: (1) Reachability is EXPSPACE-hard in general and PSPACE-complete for 1-bounded nets [7], (2) using results of [21] liveness has the same complexity, and (3) boundedness is EXPSPACE-complete [22]. Furthermore there is a family of bounded nets $\{\mathcal{N}_n\}_{n\in\mathbb{N}}$ whose size is polynomial in n such that the size of their reachability set is lower bounded by some Ackermann function [14].

▶ **Example 2.** An example of marked Petri net is given on Figure 1. Petri nets are represented as bipartite graphs where places are circles containing their initial number of tokens and transitions are rectangles. When $W^-(p,t) > 0$ (resp. $W^+(p,t) > 0$) there is an edge from p (resp. t) to t (resp. p) labelled by $W^-(p,t)$ (resp. $W^+(p,t)$). This label is called the *weight* of this edge, and is omitted when its value is equal to one. For sake of readability, one merges edges $p \xrightarrow{W^-(p,t)} t$ and $t \xrightarrow{W^+(p,t)} p$ when $W^-(p,t) = W^+(p,t)$ leading to a pseudo-edge with two arrows, as in the case of (r_0, t_5).

The net of Figure 1 is not live. Indeed t_0, t_1, t_2, t_3 and t_4 will never be enabled due to the absence of tokens in p_0, p_1, p_2, p_{ext}. Suppose that one deletes the place p_{ext} and its input

and output edges. Consider the firing sequence $q_3 + r_0 \xrightarrow{t_5} q_2 + r_0 \xrightarrow{t_6} q_1 + r_0 \xrightarrow{t_3} p_0 + r_0 \xrightarrow{t_4} p_2 + 3q_3 + r_0$. Since the marking $p_2 + 3q_3 + r_0$ is (componentwise) larger than the initial marking $q_3 + r_0$, we can iterate this sequence and generate markings with an arbitrarily large number of tokens in p_2 and q_3; this new net is unbounded. Applying technics that we will develop in this paper (Section 3), we will realize that this new net is actually live.

▶ **Definition 3** (Stochastic Petri net)**.** A *stochastic Petri net* (SPN) is a pair $(\mathcal{N}, \lambda)$ where:
- $\mathcal{N} = (P, T, W^-, W^+)$ is a Petri net;
- λ is a mapping from T to $\mathbb{R}_{>0}$.

A marked stochastic Petri net $(\mathcal{N}, \lambda, m_0)$ is a stochastic Petri net equipped with an initial marking. In a marked stochastic Petri net, when becoming enabled a transition triggers a random delay according to an exponential distribution with (firing) rate $\lambda(t)$. When several transitions are enabled, a *race* between them occurs. Accordingly, given some initial marking m_0, the stochastic process underlying a SPN is a continuous time Markov chain (CTMC) whose (possibly infinite) set of states is $\mathcal{R}(m_0)$ and such that the rate $\mathbf{Q}(m, m')$ of a transition from some m to some $m' \neq m$ is equal to $\sum_{t : m \xrightarrow{t} m'} \lambda(t)$ (as usual $\mathbf{Q}(m, m) = -\sum_{m' \neq m} \mathbf{Q}(m, m')$). Matrix $\mathbf{Q}$ is called the *infinitesimal generator* of the CTMC (see [5] for more details).

From a quantitative point of view, one may be interested in studying the long-run behaviour of the net and in particular in deciding whether there exists a steady-state distribution and in computing it in the positive case. When the underlying graph of the CTMC is strongly connected (i.e. an *irreducible* Markov chain) it amounts to deciding whether there exists a non-zero distribution π over $\mathcal{R}(m_0)$ such that $\pi \cdot \mathbf{Q} = 0$.

It is in general non-trivial to decide whether there exists a steady-state distribution, and even when such a distribution exists, given some state it is hard to compute its steady-state probability (see the introduction). Furthermore, even when the net is bounded, the size of the reachability set may prevent any feasible computation of π.

Thus one looks for subclasses of nets where the steady-state distribution π can be computed more easily and in particular when π has a *product-form*, that is: there exist a constant vector $\mu \in \mathbb{R}_{\geq 0}^P$ only depending on $\mathcal{N}$ such that for all $m \in \mathcal{R}(m_0)$, $\pi(m) = G \cdot \prod_{p \in P} \mu_p^{m(p)}$ where $G = \left(\sum_{m \in \mathcal{R}(m_0)} \prod_{p \in P} \mu_p^{m(p)} \right)^{-1}$ is the so-called *normalising constant* [9].

The most general known class of nets admitting a structural product-form distribution is the class of Π^2-nets [11]. It is based on two key ingredients: *bags* and *witnesses*. A bag is a multiset of tokens that is consumed or produced by some transition. Considering a bag as a whole, one defines the bag graph whose vertices are bags and, given a transition t, an edge goes from the bag consumed by t to the bag produced by t. Observe that there are at most $2|T|$ vertices and exactly $|T|$ edges. This alternative representation of a net via the bag graph does not lose any information: from the bag graph, one can recover the original net. Formally:

▶ **Definition 4** (Bag graph of a Petri net)**.** Let $\mathcal{N} = (P, T, W^-, W^+)$ be a Petri net. Then its bag graph is a labelled graph $G_{\mathcal{N}} = (V_{\mathcal{N}}, E_{\mathcal{N}})$ defined by:
- $V_{\mathcal{N}} = \{W^-(t), W^+(t) \mid t \in T\}$ the finite set of *bags*;
- $E_{\mathcal{N}} = \{W^-(t) \xrightarrow{t} W^+(t) \mid t \in T\}$.

▶ **Example 5.** The bag graph of the net of Figure 1 is described in Figure 2. The bag is written inside the vertex (the external label of the vertices will be explained later). Observe that this graph has three connected components, both of them being strongly connected.

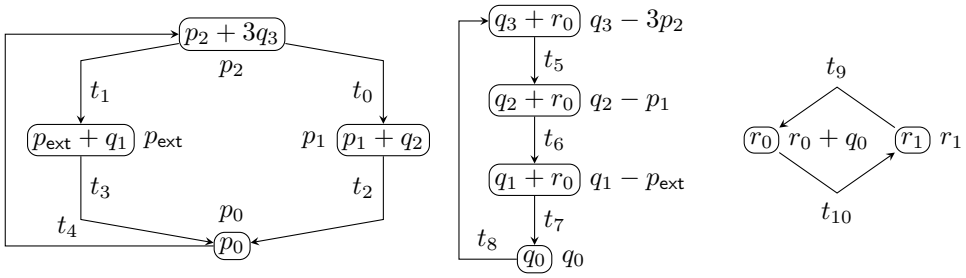


Figure 2 Bags and witnesses.

We now turn to the notion of witness. A queuing network models a single activity flow where activities are modelled by queues and clients leave their current queue when served and enter a new one depending on a routing probability. In Π^2-nets there are several activity flows, one per component of the bag graph. So one wants to witness production and consumption of every bag b by the transition firings. In order to witness it, one looks for a linear combination of the places *wit* such that for every firing of a transition t that produces (resp. consumes) the bag b, for every marking m, $m \cdot wit$ is increased (resp. decreased) by one unit, and such that all other transition firings let $m \cdot wit$ invariant.

▶ **Definition 6** (Witness of a bag)**.** Let $\mathcal{N} = (P, T, W^-, W^+)$ be a a Petri net, $b \in V_{\mathcal{N}}$ and $wit \in \mathbb{Q}^P$. Then wit is a witness of b if:

$$\begin{cases} wit \cdot W(t) = -1 & \text{if } W^-(t) = b; \\ wit \cdot W(t) = 1 & \text{if } W^+(t) = b; \\ wit \cdot W(t) = 0 & \text{otherwise.} \end{cases}$$

▶ **Example 7.** All bags of the net of Figure 1 have (non unique) witnesses. We have depicted them close to their vertices in Figure 2. For instance, consider the bag $b = q_1 + r_0$: transition t_6 produces b while transition t_7 consumes it. Let us check that $w = q_1 - p_{\text{ext}}$ is a witness of b. t_6 produces a token in q_1 and the marking of p_{ext} is unchanged. t_7 consumes a token in q_1 and the marking of p_{ext} is unchanged. The other transitions that change the marking of q_1 and p_{ext} are t_1 and t_3. However since they simultaneously produce or consume a token in both places, $m \cdot w$ is unchanged (m is the current marking).

The definition of Π^2-nets relies on structural properties of the net and on the existence of witnesses. Every connected component of the graph bag will represent an activity flow of some set of processes where every activity (i.e. a bag) has a witness.

▶ **Definition 8** (Π^2-net)**.** Let $\mathcal{N}$ be a Petri net. Then $\mathcal{N}$ is a Π^2-net if:
- every connected component of $G_{\mathcal{N}}$ is strongly connected;
- every bag b of $V_{\mathcal{N}}$ admits a witness (denoted wit_b).

Observe that the first condition called *weak reversibility* ensures that the reachability graph is strongly connected since the firing of any transition t can be "withdrawn" by the firing of transitions occurring along a path from $W^+(t)$ to $W^-(t)$ in the bag graph. The complexity of reachability in weakly reversible nets is still high: EXPSPACE-complete [4].

The next theorem shows the interest of Π^2-nets. Let us define $\lambda(b)$ the *firing rate* of a bag b by $\lambda(b) = \sum_{t|W^-(t)=b} \lambda(t)$ and the choice probability pr_t of transition t by $pr_t = \frac{\lambda(t)}{\lambda(W^-(t))}$.

The routing matrix $\mathbf{P}$ of bags is the stochastic matrix indexed by bags such that for all t, $\mathbf{P}(W^-(t), W^+(t)) = pr_t$ and $\mathbf{P}(b, b') = 0$ otherwise. Consider $\mathbf{vis}$ some positive solution of $\mathbf{vis} \cdot \mathbf{P} = \mathbf{vis}$. Since $\mathbf{P}$ is a stochastic matrix such a vector always exists but is not unique in general; however given b, b' two bags of the same connected component, $\frac{\mathbf{vis}(b')}{\mathbf{vis}(b)}$ measures the ratio between visits of b' and b in the discrete time Markov chain induced by $\mathbf{P}$.

▶ **Theorem 9** ([11]). *Let $(\mathcal{N}, \lambda, m_0)$ be a marked stochastic Π^2-net. Then defining for all $m \in \mathcal{R}(m_0)$, $\mathbf{v}(m) = \prod_{b \in V_\mathcal{N}} \left(\frac{\mathbf{vis}(b)}{\lambda(b)}\right)^{m \cdot wit_b}$, the next assertions hold:*

- $\mathbf{v} \cdot \mathbf{Q} = 0$;
- *$(\mathcal{N}, \lambda, m_0)$ is ergodic iff $\|\mathbf{v}\| < \infty$*
- *when ergodic, the associated Markov chain admits $\|\mathbf{v}\|^{-1}\mathbf{v}$ as steady-state distribution.*

Let us discuss the computational complexity of the product-form of the previous theorem. First deciding whether a net is a Π^2-net is straightforwardly performed in polynomial time. The computation of the visit ratios, the witnesses and the rate of bags can also be done in polynomial time. So computing an item of vector $\mathbf{v}$ is easy. However without additional restriction on the nets the normalising constant $\|\mathbf{v}\|^{-1}$ requires to enumerate all items of $\mathcal{R}_\mathcal{N}(m_0)$, which can be prohibitive.

So in [10], the authors introduce Π^3-net, a subclass of Π^2-net which still strictly generalises closed queuing networks, obtained by structuring the activity flows of the net represented by the components of the bag graph. First there is a bijection between places and bags such that the input (resp. output) transitions of the bag produce (consume) one token of this place. The other places occurring in the bag may be viewed as resources associated with the bag and thus the *potential* of the bag is its total number of resources. Second the components of the graph may be ordered as N layers such that the resources of a bag occurring in layer i correspond to places associated with bags of layer $i-1$ (for $i > 1$) and more precisely to those with maximal potential. Informally a token in such a place means that it is a resource available for the upper layer.

▶ **Definition 10** (Π^3-net). Let $\mathcal{N}$ be a net. Then $\mathcal{N}$ is an N-closed Π^3-net if:

- There is a bijection between P and $V_\mathcal{N}$. Denoting b_p the bag associated with place p (and p_b the place associated with bag b), we have $b_p(p) = 1$.
  The potential of a place $\mathsf{pot}(p)$ is equal to $\|b_p\| - 1$.
- $V_\mathcal{N}$ is partitioned into N strongly connected components $V_1, \ldots, V_N$. One denotes:
  $P_i = \{p_b \mid b \in V_i\}$ and $P_i^{\max} = \operatorname{argmax}(\mathsf{pot}(p) \mid p \in P_i)$. By convention, $P_0^{\max} = P_0 = \emptyset$.
- For all $b \in V_i$ and $p \in P \setminus \{p_b\}$, $b(p) > 0$ implies $p \in P_{i-1}^{\max}$.

A net is an N-open Π^3-net if it is obtained by deleting some place $p_{\mathsf{ext}} \in P_N$ (and its input/output edges) from an N-closed Π^3-net.

Given an open Π^3-net $\mathcal{N}$, $\overline{\mathcal{N}}$ denotes the closed net based on which $\mathcal{N}$ has been defined. For every $1 \leq i \leq N$, we will later write $P_i^{\neg \max}$ for the set $P_i \setminus P_i^{\max}$, and T_i for the set of transitions t such that $W^-(t) \in V_i$. The next proposition establishes that Π^3-nets are product-form Petri nets.

▶ **Proposition 11.** *Let $\mathcal{N}$ be a (closed or open) Π^3-net. Then $\mathcal{N}$ is a Π^2-net.*

▶ **Example 12.** The net of Figure 1 is a 3-closed Π^3-net. We have used different colors for the layers: $P_3 = \{p_0, p_1, p_2, p_{\mathsf{ext}}\}$ (in red), $P_2 = \{q_0, q_1, q_2, q_3\}$ (in green) and $P_1 = \{r_0, r_1\}$ (in blue). The place q_0 (resp. q_1, q_2, q_3) is associated with the bag q_0 (resp. $q_1+r_0, q_2+r_0, q_3+r_0$) and has potential 0 (resp. $1, 1, 1$). Thus $P_2^{\max} = \{q_1, q_2, q_3\}$ and indeed q_0 does not occur in bags of layer 3. In order to highlight the use of resources, edges between places of P_i and transitions of T_{i+1} are depicted in gray.

The next theorem shows the interest of closed Π^3-nets.

▶ **Theorem 13** ([10]). *Let $(\mathcal{N}, \lambda, m_0)$ be a N-closed Π^3-net. Then:*

- *$(\mathcal{N}, m_0)$ is bounded.*
- *One can decide whether $(\mathcal{N}, m_0)$ is live in polynomial time.*
- *When $(\mathcal{N}, m_0)$ is live, one can decide in polynomial time, given m, whether $m \in \mathcal{R}_{\mathcal{N}}(m_0)$.*
- *For any fixed N, when $(\mathcal{N}, m_0)$ is live, one can compute the normalising constant of the steady-state distribution (i.e. $\|\mathbf{v}\|^{-1}$ in Theorem 9) in polynomial time with respect to $|P|$, $|T|$, the maximal weight of the net's edges, and $\|m_0\|$ (thus in pseudo-polynomial time w.r.t. the size of m_0).*

The above results do not apply to infinite-state systems and in particular to the systems generated by open Π^3-nets. In addition, the polynomial-time complexity for computing the normalising constant requires to fix N. We address these issues in the next sections.

3 Qualitative analysis

In this section we first give a simple characterisation of the liveness property in a marked Π^3-net. We then fully characterise the set of reachable markings in a live marked Π^3-net. These characterisations give polynomial-time algorithms for deciding liveness of a marked Π^3-net, and the boundedness property of a live marked Π^3-net. We end the section with a coNP-hardness result for the boundedness property of a marked Π^3-net, when it is not live. For the rest of this section, we assume $\mathcal{N} = (P, T, W^-, W^+)$ is an open or closed Π^3-net with N layers. We further use the notations of Definition 10. In particular, if $\mathcal{N}$ is open, then we write p_{ext} for the place which has been removed (and we call it virtual). We therefore set $P_N^\star = P_N \cup \{p_{\mathsf{ext}}\}$ if the net is open and $P_N^\star = P_N$ otherwise; For every $1 \leq i \leq N-1$ we define $P_i^\star = P_i$; And we set $P^\star = \bigcup_{i=1}^N P_i^\star$.

3.1 Liveness analysis

We give a simple characterisation of the liveness property through a dependence between the number of tokens at some layer and potentials of places activated on the next layer. More precisely, for every $1 \leq i \leq N-1$, Live_i is defined as the set of markings m such that:

$$m \cdot P_i \geq \min\{\mathsf{pot}(p) \mid p \in P_{i+1}^\star \text{ and } (m(p) > 0 \text{ or } p = p_{\mathsf{ext}})\}.$$

Note that $p = p_{\mathsf{ext}}$ can only happen when $\mathcal{N}$ is open and $i = N-1$. We additionally define Live_N as the set of markings m such that $m \cdot P_N > 0$ if $\mathcal{N}$ is closed, and as the set of all markings if $\mathcal{N}$ is open.

For every $1 \leq i \leq N$ (resp. $1 \leq i \leq N-1$), we define $\mathsf{POT}_i = \max\{\mathsf{pot}(p) \mid p \in P_i\}$ when $\mathcal{N}$ is closed (resp. open). When $\mathcal{N}$ is open, we write $\mathsf{POT}_N = \mathsf{pot}(p_{\mathsf{ext}})$. Given a marking m, when no place p fulfills $p \in P_{i+1}^\star$ and $(m(p) > 0$ or $p = p_{\mathsf{ext}})$, the minimum is equal to POT_{i+1}. Thus given a marking m, the condition $m \in \mathsf{Live}_i$ for $i < n$ only depends on the values of $m(p)$ for $p \in P_i \cup P_{i+1}^{\neg \max}$.

The intuition behind condition Live_i is the following: transitions in $\bigcup_{j \leq i} T_j$ cannot create new tokens on layer i (layer i behaves like a state machine, and smaller layers do not change the number of tokens in that layer); therefore, to activate a transition of T_{i+1} out of some marked place $p \in P_{i+1}$, it must be the case that enough tokens are already present on layer i; hence there should be at least as many tokens in layer i as the minimal potential of a marked place in layer $i+1$. When $\mathcal{N}$ is open, the virtual place p_{ext} behaves like a source

of tokens, hence it is somehow always "marked"; this is why it is taken into account in the right part of Live_i. The following characterisation was already stated in [10] in the restricted case of closed nets.

▶ **Theorem 14.** *A marking m is live if and only if for every $1 \leq i \leq N$, $m \in \mathsf{Live}_i$.*

▶ **Example 15.** Building on the marked Petri net of Figure 1, the marking $q_3 + r_0$ is live when the net is open but not live if the net is closed. Indeed, transitions of the two first layers can trivially be activated from q_3+r_0 (hence by weak-reversibility from every reachable marking). We see that in the case of the closed net, transitions of layer 3 cannot be activated (no fresh token can be produced on that layer). On the contrary, in the case of the open net, the token in q_3 can be moved to q_1, which will activate transition t_3; from there, all transitions of layer 3 will be eventually activated.

As a consequence of the characterisation of Theorem 14, we get:

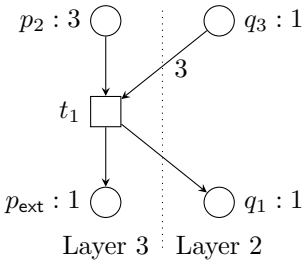
▶ **Corollary 16.** *We can decide the liveness of a marked Π^3-net in polynomial time.*

3.2 Reachable markings

We will now give a characterisation of the set of reachable markings $\mathcal{R}_{\mathcal{N}}(m_0)$ when m_0 is live. We will first give linear invariants of the net: those are vectors in the left kernel of W (or P-flows). The name "invariants" comes from the fact that they will allow to infer real invariants satisfied by the reachable markings. Furthermore, for every $1 \leq i \leq N$, for every $p \in P_i$, we define $\mathsf{cin}(p) = \mathsf{POT}_i - \mathsf{pot}(p)$. Except when $\mathcal{N}$ is open and $i = N$, the cin value of a place is nonnegative.

▶ **Proposition 17.** *The following vectors are linear invariants of $\mathcal{N}$:*

- *for every $1 \leqslant i \leqslant N-1$, $v^{(i)} = \sum_{p\in P_i} p + \sum_{p\in P_{i+1}} \mathsf{cin}(p)\ p$;*
- *if $\mathcal{N}$ is closed, $v^{(N)} = \sum_{p\in P_N} p$.*



First observe that for $i < N-1$, $\mathsf{Supp}(v^{(i)}) = P_i \cup P_{i+1}^{\neg\max}$. Thus given a marking m, only firing of transitions $t \in T_i \cup T_{i+1}$ could change $m \cdot v^{(i)}$. This is not the case of a transition $t \in T_i$ since it moves a token from a place of P_i to another one. To give an intuition why transitions in T_{i+1} do not change $m \cdot v^{(i)}$, we consider part of the closed net (that is, p_{ext} is a real place) of Figure 1 depicted on the right, where numbers close to place names are potential values. We focus on transition t_1 and explain why $m \cdot v^{(2)}$ is unchanged by its firing. The impact of transition t_1 is to decrease the sum $\sum_{p\in P_2} p$ by 2; due to the weights of places of P_3 in $v^{(2)}$, place p_2 counts as 0 and place p_{ext} counts as +2. This intuition extends into a formal proof.

For every $1 \leq i \leq N-1$, we set $\mathbf{C}_i^{m_0} = m_0 \cdot v^{(i)}$, and if $\mathcal{N}$ is closed, we set $\mathbf{C}_N^{m_0} = m_0 \cdot v^{(N)}$. For every $1 \leq i \leq N-1$, we define $\mathsf{Inv}_i(m_0)$ as the set of markings m such that $m \cdot v^{(i)} = \mathbf{C}_i^{m_0}$, and if $\mathcal{N}$ is closed, we define $\mathsf{Inv}_N(m_0)$ as the set of markings m such that $m \cdot v^{(N)} = \mathbf{C}_N^{m_0}$. For uniformity, if $\mathcal{N}$ is open, we define $\mathsf{Inv}_N(m_0)$ as the set of all markings. As a consequence of Proposition 17, we get:

▶ **Corollary 18.** $\mathcal{R}_{\mathcal{N}}(m_0) \subseteq \bigcap_{i=1}^{N} \mathsf{Inv}_i(m_0)$.

More complex invariants were given in [10] for closed Π^3-nets. The advantage of the above invariants is that each of them only involves two neighbouring layers. This will have a huge impact on various complexities, and will allow the development of our methods for quantitative analysis.

▶ **Example 19.** Going back to the Petri net of Figure 1, with initial marking $m_0 = q_3 + r_0$. We first consider the closed net. Then: $\mathsf{POT}_3 = 2$ and $\mathsf{POT}_2 = 1$. Therefore:

$$\begin{cases} \mathsf{Inv}_3(m_0) = \{m \mid \sum_{i=0}^{2} m(p_i) + m(p_{\mathsf{ext}}) = 0\} \\ \mathsf{Inv}_2(m_0) = \{m \mid \sum_{i=0}^{3} m(q_i) + 3m(p_0) + 2m(p_1) + 2m(p_{\mathsf{ext}}) = m_0(q_3) = 1\} \\ \mathsf{Inv}_1(m_0) = \{m \mid m(r_0) + m(r_1) + m(q_0) = m_0(r_0) = 1\} \end{cases}$$

We now turn to the open net, obtained by deleting p_{ext}. Definition of POT_3 differs from the previous case: $\mathsf{POT}_3 = \mathsf{pot}(p_{\mathsf{ext}}) = 1$ and $\mathsf{POT}_2 = 1$. Therefore:

$$\begin{cases} \mathsf{Inv}_3(m_0) = \mathbb{N}^P \\ \mathsf{Inv}_2(m_0) = \{m \mid \sum_{i=0}^{3} m(q_i) + m(p_0) - 2m(p_2) = m_0(q_3) = 1\} \\ \mathsf{Inv}_1(m_0) = \{m \mid m(r_0) + m(r_1) + m(q_0) = m_0(r_0) = 1\} \end{cases}$$

The invariants of Corollary 18 do not fully characterise the set of reachable markings, since they do not take into account the enabling conditions of the transitions. However, they will be very helpful for characterising the reachable markings when m_0 is live.

▶ **Theorem 20.** *Suppose that $(\mathcal{N}, m_0)$ is a live Π^3-net. Then:*

$$\mathcal{R}_{\mathcal{N}}(m_0) = \bigcap_{i=1}^{N} \mathsf{Inv}_i(m_0) \ \cap \ \bigcap_{i=1}^{N} \mathsf{Live}_i$$

Thus reachability in live Π^3-nets can be checked in polynomial time.

▶ **Example 21.** In the open Petri net of Figure 1, with initial marking $m_0 = q_3 + r_0$, the sets Live_i are $\mathsf{Live}_1 = \{m \mid m(r_0) + m(r_1) + m(q_0) \geqslant 1\}$, $\mathsf{Live}_2 = \{m \mid \sum_{i=0}^{3} m(q_i) + m(p_0) \geqslant 1\}$ and $\mathsf{Live}_3 = \mathbb{N}^P$. Observing that $\mathsf{Inv}_i(m_0) \subseteq \mathsf{Live}_i$ for $1 \leqslant i \leqslant 3$, the net has reachability set

$$\mathcal{R}_{\mathcal{N}}(m_0) = \{m \mid m(r_0) + m(r_1) + m(q_0) = 1 \text{ and } \textstyle\sum_{i=0}^{3} m(q_i) + m(p_0) - 2m(p_2) = 1\}.$$

The idea of the proof when the net is closed is to show that, from every marking m satisfying the right handside condition in the theorem, one can reach a specific marking m_0^* (where, for every $1 \leq i \leq N$, $\mathbf{C}_i^{m_0}$ tokens are in one arbitrary place of $P_i^{\max}$). Hence, given two markings m and m' satisfying the conditions, $m \to^* m_0^*$ and $m' \to^* m_0^*$, which implies by weak reversibility of the net: $m \to^* m_0^* \to^* m'$. This is in particular the case from m_0: every marking satisfying the conditions is reachable from m_0. In the case of an open net, this is a bit more tricky, and a joint marking to every pair (m, m') of markings satisfying the conditions has to be chosen.

3.3 Boundedness analysis

As a consequence of the characterisation given in Theorem 20, we get:

▶ **Corollary 22.** *We can decide the boundedness of a live marked Π^3-net in polynomial time.*

Indeed, it can be shown that, if $\mathcal{N}$ is closed, then $\mathcal{N}$ is bounded, and that if $\mathcal{N}$ is open, then it is bounded if and only if $\mathsf{cin}(q) > 0$ for all $q \in P_N$ (that is, p_{ext} has maximal potential, and no other place of P_N has maximal potential). Furthermore, if $\mathcal{N}$ is bounded, then the overall number of tokens in the net is bounded by $\sum_{i=1}^{N-1} \mathbf{C}_i^{m_0}$ (resp. $\sum_{i=1}^{N} \mathbf{C}_i^{m_0}$) if the net is open (resp. closed).

The polynomial-time complexity of Corollary 22 is in contrast with the following hardness result, which can be obtained by a reduction from the independent-set problem.

▶ **Proposition 23.** *Deciding the boundedness of a marked Π^3-net which is not live is coNP-hard. The reachability (and even the coverability) problem is NP-hard.*

4 Quantitative analysis

Contrary to closed Π^3-nets, open Π^3-nets may not be ergodic. In this section, we first give a simple characterisation of the ergodicity property for open Π^3-nets, which gives us a polynomial-time algorithm for deciding ergodicity. We then provide a polynomial-time algorithm for computing the steady-state distribution of ergodic (open and closed) Π^3-nets.

For the rest of this section, we assume that $(\mathcal{N}, \lambda, m_0)$ is a stochastic Π^3-net with N layers, and that m_0 is live. Let $\mathbf{W}$ be the maximal weight of the edges of $\mathcal{N}$. Then we assume that the constants $\mu_p = \prod_{b \in \mathcal{V}_\mathcal{N}} (\mathbf{vis}(b)/\lambda_b)^{wit_b \cdot p}$ have already been precomputed (in polynomial time with respect to $|P|$, $|T|$ and $\log(1 + \mathbf{W})$). [1]

In what follows, and for every vector $\delta \in \mathbb{N}^P$, we denote by $\hat{v}(\delta)$ the product $\prod_{p \in P} \mu_p^{\delta(p)}$. Consequently, the vector $\mathbf{v}$ mentioned in Theorem 9 is then defined by $\mathbf{v}(m) = \hat{v}(m)$ for all markings $m \in \mathcal{R}(m_0)$, and its norm is $\|\mathbf{v}\| = \sum_{m \in \mathcal{R}(m_0)} \hat{v}(m)$. Note that $\mathbf{v}$ and $\hat{v}$ only differ by their domain. In addition, in what follows, and for every set $Z \subseteq P$, we simply denote by $\mathsf{cin}(Z)$ the formal sum $\sum_{p \in Z} \mathsf{cin}(p)p$.

4.1 Ergodicity analysis

We assume here that $\mathcal{N}$ is open. We give a simple characterisation of the ergodicity property through a comparison of the constants μ_p for a limited number of places p. Those constraints express congestion situations that may arise; we show that they are sufficient. These places are the elements of the subset Y of places that is defined by $Y = P_N \cup P_{N-1}^{\max}$. In particular, as soon as the initial marking m_0 is live, then the ergodicity of the stochastic net $(\mathcal{N}, \lambda, m_0)$ does *not* depend on m_0.

According to Theorem 9, the net is ergodic if and only if the norm $\|\mathbf{v}\| = \sum_{m \in \mathcal{R}(m_0)} \hat{v}(m)$ is finite. Hence, deciding ergodicity amounts to deciding the convergence of a sum. The following characterisation holds.

▶ **Theorem 24.** *Let $(\mathcal{N}, \lambda, m_0)$ be a live open stochastic Π^3-net with N layers. This net is ergodic if and only if all of the following inequalities hold:*

- *for all places $p \in P_N$, if $\mathsf{cin}(p) = 0$, then $\mu_p < 1$;*
- *for all places $p, q \in P_N$, if $\mathsf{cin}(p) > 0 > \mathsf{cin}(q)$, then $\mu_p^{|\mathsf{cin}(q)|} \mu_q^{\mathsf{cin}(p)} < 1$;*
- *for all places $p \in P_{N-1}^{\max}$ and $q \in P_N$, if $0 > \mathsf{cin}(q)$, then $\mu_p^{|\mathsf{cin}(q)|} \mu_q < 1$.*

Proof (sketch). Let $\mathcal{F}$ be the family formed of the vectors p (for $p \in P_N$ such that $\mathsf{cin}(p) = 0$), $\mathsf{cin}(p)q - \mathsf{cin}(q)p$ (for $p, q \in P_N$ such that $\mathsf{cin}(p) > 0 > \mathsf{cin}(q)$) and $q - \mathsf{cin}(q)p$ (for

[1] In the rest of this section, we will design polynomial-time procedures w.r.t. $\mathbf{W}$, hence polynomial if it is encoded in unary and pseudo-polynomial if it is encoded in binary.

$p \in P_{N-1}^{\max}$ and $q \in P_N$ such that $0 > \mathsf{cin}(q)$). Let also $\mathcal{L}$ be the sublattice of $\mathbb{N}^P$ generated by the vectors in $\mathcal{F}$, and let $\mathcal{G}$ be the finite subset of $\mathbb{N}^P$ formed of those vectors whose entries are not greater than some adequately chosen constant $\mathbf{G}$.

Since m_0 is live, Theorem 20 applies, which allows us to prove the inclusions $\{m_0\}+\mathcal{L} \subseteq \mathcal{R}(m_0) \subseteq \mathcal{G}+\mathcal{L}$. Hence, the sum $\sum_{m\in\mathcal{R}(m_0)} \hat{v}(m)$ is finite iff the sum $\sum_{m\in\mathcal{L}} \hat{v}(m)$ is finite, i.e. iff each constant $\hat{v}(\delta)$ is (strictly) smaller than 1, for $\delta \in \mathcal{F}$. ◀

▶ **Example 25.** Going back to the open Π^3-net of Figure 1, with any live initial marking m_0, we obtain the following necessary and sufficient conditions for being ergodic:

$$\mu_{p_1} < 1,\ \mu_{p_0}^2\mu_{p_2} < 1,\ \mu_{q_1}^2\mu_{p_2} < 1,\ \mu_{q_2}^2\mu_{p_2} < 1 \text{ and } \mu_{q_3}^2\mu_{p_2} < 1.$$

As a consequence of the characterisation of Theorem 24, we get:

▶ **Corollary 26.** *We can decide the ergodicity of a marked, live and open stochastic Π^3-net in polynomial time.*

4.2 Computing the steady-state distribution

In case the Π^3-net is ergodic, it remains to compute its steady-state distribution, given by $\pi(m) = \|\mathbf{v}\|^{-1}\mathbf{v}(m)$ for all $m \in \mathcal{R}(m_0)$. Since we already computed $\mathcal{R}(m_0)$ and $\mathbf{v}(m)$ for all markings $m \in \mathcal{R}(m_0)$, it remains to compute the normalising constant $\|\mathbf{v}\|$.

This section is devoted to proving the following result.

▶ **Theorem 27.** *Let $(\mathcal{N}, \lambda, m_0)$ be a live ergodic stochastic Π^3-net. There exists an algorithm for computing the normalising constant $\|\mathbf{v}\|$ in polynomial time with respect to $|P|$, $|T|$, $\mathbf{W}$, and $\|m_0\|$ (thus in pseudo-polynomial time).*

This theorem applies to both closed and open Π^3-nets. For closed nets, it provides a similar yet stronger result than Theorem 13, where the polynomial-time complexity was obtained only for a fixed value of N (the number of layers of the net).

We prove below Theorem 27 in the case of open nets. The case of closed nets is arguably easier: one can transform a closed net into an equivalent open net by adding a layer $N+1$ with one place (and one virtual place p_{ext}), and set a firing rate $\lambda_t = 0$ for all transitions t of the layer $N+1$. We therefore assume for the rest of this section that $\mathcal{N}$ is open.

We first describe a naive approach. The normalisation constant $\|\mathbf{v}\|$ can be computed as follows. Recall the family $\mathcal{F}$ introduced in the proof of Theorem 24. We may prove that the set $\mathcal{R}(m_0)$ is a union of (exponentially many) translated copies of the lattice $\mathcal{L}$ generated by $\mathcal{F}$. These copies may intersect each other, yet their intersections themselves are translated copies of $\mathcal{L}$. Hence, using an inclusion-exclusion formula and a doubly exponential computation step, computing the sum $\|\mathbf{v}\| = \sum_{\mathfrak{m}\in\mathcal{R}(m_0)} \hat{v}(\mathfrak{m})$ reduces to computing the sum $\sum_{\ell\in\mathcal{L}} \hat{v}(\ell)$.

The family $\mathcal{F}$ is not free a priori, hence computing this latter sum is not itself immediate. Using again inclusion-exclusion formulæ, we may write $\mathcal{L}$ as a finite, disjoint union of exponentially many lattices generated by free subfamilies of $\mathcal{F}$. This last step allows us to compute $\sum_{\ell\in\mathcal{L}} \hat{v}(\ell)$, and therefore $\|\mathbf{v}\|$.

Such an approach suffers from a prohibitive computational cost. Yet it is conceptually simple, and it allows proving rather easily that $\|\mathbf{v}\|$ is a rational fraction in the constants μ_p, whose denominator is the product $\prod_{\ell\in\mathcal{F}}(1-\hat{v}(\ell))$.

▶ **Example 28.** Going back to the open Π^3-net of Figure 1, with initial marking $m_0 = q_3 + r_0$, this algorithm allows us to compute

$$\|\mathbf{v}\| = \frac{\mu_{q_0} a + (\mu_{r_0} + \mu_{r_1}) b}{c}, \text{ with}$$
$$a = \mu_{p_2}^2 \mu_{p_0} \mu_{q_1} \mu_{q_2} \mu_{q_3} + \mu_{p_2}(\mu_{p_0}\mu_{q_1} + \mu_{p_0}\mu_{q_2} + \mu_{p_0}\mu_{q_3} + \mu_{q_1}\mu_{q_2} + \mu_{q_1}\mu_{q_3} + \mu_{q_2}\mu_{q_3}) + 1,$$
$$b = \mu_{p_2}(\mu_{p_0}\mu_{q_1}\mu_{q_2} + \mu_{p_0}\mu_{q_1}\mu_{q_3} + \mu_{p_0}\mu_{q_2}\mu_{q_3} + \mu_{p_1}\mu_{q_2}\mu_{q_3}) + \mu_{p_0} + \mu_{q_1} + \mu_{q_2} + \mu_{q_3} \text{ and}$$
$$c = (1 - \mu_{p_0}^2 \mu_{p_2})(1 - \mu_{p_1})(1 - \mu_{q_1}^2 \mu_{p_2})(1 - \mu_{q_2}^2 \mu_{p_2})(1 - \mu_{q_3}^2 \mu_{p_2}).$$

Recall Example 25, which states that $\|\mathbf{v}\|$ is finite if and only if all of $1 - \mu_{p_1}$, $1 - \mu_{p_0}^2 \mu_{p_2}$, $1 - \mu_{q_1}^2 \mu_{p_2}$, $1 - \mu_{q_2}^2 \mu_{p_2}$ and $1 - \mu_{q_3}^2 \mu_{p_2}$ are positive. We observe that, as suggested above, the denominator c is precisely the product of these five factors.

Our approach for computing $\|\mathbf{v}\|$ will involve first computing variants of $\|\mathbf{v}\|$. More precisely, for all subsets Z of P, we consider a congruence relation $\sim_Z$ on markings, such that $m \sim_Z m'$ iff m and m' coincide on all places $p \in Z$. Then, we denote by $\mathfrak{M}_Z$ the quotient set $\mathbb{N}^P / \sim_Z$ and, for every element $\mathfrak{m}$ of $\mathfrak{M}_Z$, we denote by $\hat{v}(\mathfrak{m})$ the product $\prod_{p \in Z} \mu_p^{\mathfrak{m}(p)}$. Two remarkable subsets Z are the set $X = \bigcup_{i \leqslant N-2} P_i \cup P_{N-1}^{\neg \max}$ and its complement $Y = P_{N-1}^{\max} \cup P_N = P \setminus X$, which was already mentioned in Section 4.1. Indeed, places in X are necessarily bounded while, if the net is unbounded, then so are the places in Y.

Based on these objects, and for all integers $c \geqslant 0$, we define two sets $\mathcal{C}_{m_0}(c)$ and $\mathcal{D}_{m_0}(c)$, which are respective subsets of $\mathfrak{M}_X$ and of $\mathfrak{M}_Y$. In some sense, the sets $\mathcal{C}_{m_0}(c)$ are meant to describe the "bounded" part of the markings in $\mathcal{R}(m_0)$, whose "unbounded" part is described by the sets $\mathcal{D}_{m_0}(c)$.

▶ **Definition 29.** The set $\mathcal{C}_{m_0}(c)$ is the set of those classes $\mathfrak{m}_X$ in $\mathfrak{M}_X$ such that $\mathfrak{m}_X \cdot P_{N-1}^{\neg \max} = c$, and such that $\mathfrak{m}_X$ contains some marking in $\mathcal{R}(m_0)$. The set $\mathcal{D}_{m_0}(c)$ is the set of those classes $\mathfrak{m}_Y$ in $\mathfrak{M}_Y$ such that $c + \mathfrak{m}_Y \cdot P_{N-1}^{\max} + \mathfrak{m}_Y \cdot \mathsf{cin}(P_N) = \mathbf{C}_{N-1}^{m_0}$, and such that $\mathfrak{m}_Y$ contains some marking in $\mathcal{R}(m_0)$.

These two sets of classes allow us to split nicely the huge sum $\|\mathbf{v}\|$ into smaller independent sums, as stated in the result below. This decomposition result has the flavour of convolution algorithms, yet it requires a specific treatment since its terms are infinite sums.

▶ **Lemma 30.** *The normalisation constant $\|\mathbf{v}\|$ is equal to the following finite sum:*

$$\|\mathbf{v}\| = \sum_{c=0}^{|P|\|\mathbf{W}\|m_0\|} \left(\sum_{\mathfrak{m}_X \in \mathcal{C}_{m_0}(c)} \hat{v}(\mathfrak{m}_X) \right) \left(\sum_{\mathfrak{m}_Y \in \mathcal{D}_{m_0}(c)} \hat{v}(\mathfrak{m}_Y) \right).$$

It remains to compute sums $\sum_{\mathfrak{m}_X \in \mathcal{C}_{m_0}(c)} \hat{v}(\mathfrak{m}_X)$ and $\sum_{\mathfrak{m}_Y \in \mathcal{D}_{m_0}(c)} \hat{v}(\mathfrak{m}_Y)$ for polynomially many values of c. It turns out that such computations can be performed in polynomial time, as mentioned in Propositions 31 and 32.

▶ **Proposition 31.** *Let $(\mathcal{N}, \lambda, m_0)$ be a live ergodic stochastic open Π^3-net, and let $c \geqslant 0$ be an integer. There exists an algorithm for computing the sum $\sum_{\mathfrak{m}_X \in \mathcal{C}_{m_0}(c)} \hat{v}(\mathfrak{m}_X)$ in polynomial time with respect to $|P|$, $|T|$, $\mathbf{W}$, c and $\|m_0\|$.*

Proposition 31 is similar to the results of [10], and the algorithm can be adapted to closed Π^3-nets. The only major improvement here is that, instead of obtaining an algorithm that is polynomial-time for fixed values of N only, our choice of invariants leads to a polynomial-time algorithm independently of N.

▶ **Proposition 32.** *Let $(\mathcal{N}, \lambda, m_0)$ be a live ergodic stochastic open Π^3-net, and let $c \geqslant 0$ be an integer. There exists an algorithm for computing the sum $\sum_{\mathfrak{m}_Y \in \mathcal{D}_{m_0}(c)} \hat{v}(\mathfrak{m}_Y)$ in polynomial time with respect to $|P|$, $|T|$, $\mathbf{W}$, c and $\|m_0\|$.*

Proof (sketch). We compute the sum $\sum_{\mathfrak{m}_Y \in \mathcal{D}_{m_0}(c)} \hat{v}(\mathfrak{m}_Y)$ by using a dynamic-programming approach. Let $a = |P_{N-1}^{\max}|$, $b = |P_N|$, and define t and u to be integers such that (i) $\mathsf{cin}(p_i^N) > 0$ iff $1 \leqslant i \leqslant t$, (ii) $\mathsf{cin}(p_i^N) = 0$ iff $t < i < u$ and (iii) $\mathsf{cin}(p_i^N) < 0$ iff $u \leqslant i \leqslant b$. In addition, for all $i \leqslant b$, let $\Delta_i = \max\{1, |\mathsf{cin}(p_i^N)|\}$. We consider below the lattice $\mathbb{L} = \mathbb{Z}^{a-1} \times \prod_{i=1}^{b} (\Delta_i \mathbb{Z})$.

Now, consider integers $A, B \in \mathbb{Z}$ and $1 \leqslant \alpha \leqslant a$, $1 \leqslant \gamma \leqslant \beta \leqslant b$, as well as a vector $\mathbf{w}$ in the quotient set $\mathbb{Z}^{a+b-1}/\mathbb{L}$. We define auxiliary sets of vectors of the form

$$\overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma) = \left\{ \mathbf{xy} \in \mathbb{N}^{a+b-1} \;\middle|\; \begin{array}{l} A + \sum_{j=u}^{\beta} y_j \geqslant \sum_{i=1}^{\alpha} x_i + \sum_{j=\gamma}^{t} y_j \\ B + \sum_{j=u}^{\beta} y_j \geqslant \sum_{j=\gamma}^{t} y_j \\ \mathbf{xy} - \mathbf{w} \in \mathbb{L} \\ x_i = 0 \text{ for all } i > \alpha \\ y_j = 0 \text{ for all } j < \gamma \text{ and all } j > \beta \end{array} \right\},$$

where $\mathbf{xy}$ denotes the vector $(x_1, \ldots, x_{a-1}, y_1, \ldots, y_b)$.

First, we exhibit a bijection $\mathfrak{m} \mapsto \overline{\mathfrak{m}}$, from the set $\mathcal{D}_{m_0}(c)$ to a finite union of sets $\overline{\mathcal{D}}(\mathbf{A}, \mathbf{B}, \mathbf{0}, a-1, b, \mathbf{s})$, where the integers $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{s}$ can be computed efficiently. We can also prove a relation of the form $\hat{v}(\mathfrak{m}) = \hat{\hat{v}}(\overline{\mathfrak{m}})$ for all markings $\mathfrak{m} \in \mathcal{D}_{m_0}(c)$, where $\hat{\hat{v}}$ is a product-form function $\hat{\hat{v}} : \mathbf{xy} \mapsto \mathbf{P} \prod_{i=1}^{a-1} \nu_{x,i}^{x_i} \prod_{i=1}^{b} \nu_{y,i}^{y_i}$ such that the constants $\mathbf{P}$, $\nu_{x,i}$ and $\nu_{y,j}$ can be computed efficiently. It remains to compute sums of the form

$$\mathcal{U}(A, B, \mathbf{w}, \alpha, \beta, \gamma) = \sum_{\mathbf{xy} \in \overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma)} \hat{\hat{v}}(\mathbf{xy}).$$

We do it by using recursive decompositions of the sets $\overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$, where the integers A and B belong to the finite set $\{-|\mathbf{A}| - |\mathbf{B}|, \ldots, |\mathbf{A}| + |\mathbf{B}|\}$, and where the vector $\mathbf{w}$ is constrained to have at most one non-zero coordinate, chosen from some finite domain. These decompositions involve computing polynomially many auxiliary sums, which will prove Proposition 32. ◀

We illustrate the last part of this sketch of proof in the case where $\gamma \leqslant t$, $u \leqslant \beta$, and $\mathbf{w}$ is of the form $\lambda\, \mathbf{1}_{y,\beta}$, for some $\lambda \in \mathbb{Z}/\Delta_\beta \mathbb{Z}$ (where we denote by $\mathbf{1}_{y,j}$ the vector of the canonical basis whose unique non-zero entry is y_j). In this case, we show how the sum $\mathcal{U}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$ can be expressed in terms of "smaller" sums $\mathcal{U}(A', B', \mathbf{w}', \alpha', \beta', \gamma')$.

Let us split the set $\overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$ into two subsets:

$\overline{\mathcal{D}}_{\oplus}(A, B, \mathbf{w}, \alpha, \beta, \gamma) = \{\mathbf{xy} \in \overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma) \mid y_\beta \leqslant y_\gamma\}$ and
$\overline{\mathcal{D}}_{\ominus}(A, B, \mathbf{w}, \alpha, \beta, \gamma) = \{\mathbf{xy} \in \overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma) \mid y_\beta \geqslant y_\gamma\}$, whose intersection is
$\overline{\mathcal{D}}_{\odot}(A, B, \mathbf{w}, \alpha, \beta, \gamma) = \{\mathbf{xy} \in \overline{\mathcal{D}}(A, B, \mathbf{w}, \alpha, \beta, \gamma) \mid y_\beta = y_\gamma\}$.

Computing $\mathcal{U}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$ amounts to computing the three sums $\sum_{\mathbf{xy}} \hat{\hat{v}}(\mathbf{xy})$, where $\mathbf{xy}$ ranges respectively on $\overline{\mathcal{D}}_{\oplus}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$, $\overline{\mathcal{D}}_{\ominus}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$ and $\overline{\mathcal{D}}_{\odot}(A, B, \mathbf{w}, \alpha, \beta, \gamma)$. We show here how to compute the first sum, which we simply denote by $\mathcal{U}_{\oplus}$. The two latter sums are computed similarly.

Splitting every vector $\mathbf{xy}$ into one vector $y_\beta(\mathbf{1}_{y,\gamma} + \mathbf{1}_{y,\beta})$ and one vector $\mathbf{xy}' = \mathbf{xy} - y_\beta(\mathbf{1}_{y,\gamma} + \mathbf{1}_{y,\beta})$, we observe that $\mathbf{xy} - \lambda\, \mathbf{1}_{y,\beta} \in \mathbb{L}$ if and only if (i) $y_\beta \equiv \lambda \mod \Delta_\beta$, and (ii)

$\mathbf{xy}' - y_\beta \mathbf{1}_{y,\gamma} \in \mathbb{L}$. It follows that

$$\begin{aligned}\overline{\mathcal{D}}_\oplus(A,B,\lambda\ \mathbf{1}_{y,\beta},\alpha,\beta,\gamma) &= \bigsqcup_{j\geqslant 0}\{\mathbf{xy}\in\overline{\mathcal{D}}_\oplus(A,B,\lambda\ \mathbf{1}_{y,\beta},\alpha,\beta,\gamma)\mid y_\beta = j\}\\ &= \bigsqcup_{j\geqslant 0, j\equiv\lambda \bmod \Delta_\beta} \left(\overline{\mathcal{D}}(A,B,j\ \mathbf{1}_{y,\gamma},\alpha,\beta-1,\gamma) + j(\mathbf{1}_{y,\gamma}+\mathbf{1}_{y,\beta})\right).\end{aligned}$$

Then, using the change of variable $j = k + \Delta_\gamma\Delta_\beta\ell$ (with $0\leqslant k<\Delta_\gamma\Delta_\beta$), observe that $j\equiv k \bmod \Delta_\beta$ and that $j\ \mathbf{1}_{y,\gamma}\equiv k\ \mathbf{1}_{y,\gamma} \mod \mathbb{L}$, whence

$$\begin{aligned}\mathcal{U}_\oplus &= \sum_{j\geqslant 0}\mathbf{1}_{j\equiv\lambda \bmod \Delta_\beta}\ \nu_{y,\gamma}^j\ \nu_{y,\beta}^j\ \mathcal{U}(A,B,j\ \mathbf{1}_{y,\gamma},\alpha,\beta-1,\gamma)\\ &= \sum_{\ell\geqslant 0}\sum_{k=0}^{\Delta_\gamma\Delta_\beta-1}\mathbf{1}_{k\equiv\lambda \bmod \Delta_\beta}\ (\nu_{y,\gamma}\nu_{y,\beta})^{k+\Delta_\gamma\Delta_\beta\ell}\mathcal{U}(A,B,k\ \mathbf{1}_{y,\gamma},\alpha,\beta-1,\gamma)\\ &= \frac{1}{1-(\nu_{y,\gamma}\nu_{y,\beta})^{\Delta_\gamma\Delta_\beta}}\sum_{k=0}^{\Delta_\gamma\Delta_\beta-1}\mathbf{1}_{k\equiv\lambda \bmod \Delta_\beta}\ (\nu_{y,\gamma}\nu_{y,\beta})^{k}\mathcal{U}(A,B,k\ \mathbf{1}_{y,\gamma},\alpha,\beta-1,\gamma).\end{aligned}$$

This shows, in this specific case, that computing $\mathcal{U}_\oplus$ reduces to computing finitely many sums of the form $\mathcal{U}(A,B,\mathbf{w}',\alpha,\beta-1,\gamma)$. Similar constructions are successfully used in all other cases.

5 Conclusion

Performance analysis of infinite-state stochastic systems is a very difficult task. Already checking the ergodicity is difficult in general, and even for systems which are known to be ergodic and which have product-form steady-state distributions, computing the normalising constant can be hard. In this work, we have proposed the model of open Π^3-nets; this model generalises queuing networks and closed produc-form Petri nets and generates a potentially infinite state-space. We have shown that we can efficiently decide (in polynomial time!) many behavioural properties, like the boundedness, the reachability in live nets, and the most important quantitative property: ergodicity. Furthermore, using dynamic programming algorithms managing infinite sums, we have shown that we can compute the normalising constant of the steady-state distribution in pseudo-polynomial time.

We believe our approach can be extended to Π^3-nets in which one place is removed in every layer, without affecting too much the complexity. This setting would allow to model production of resources by the environment while in the current version resources may grow in an unbounded way but only when the number of processes of the main layer also grows. We leave this as future work.

References

1 Parosh Aziz Abdulla, Noomene Ben Henda, Richard Mayr, and Sven Sandberg. Limiting behavior of markov chains with eager attractors. In *Proc. 3rd International Conference on the Quantitative Evaluation of Systems (QEST'06)*, pages 253–264, 2006.

2 Simonetta Balsamo and Andrea Marin. Determining product-form steady-state solutions of generalized stochastic Petri nets by the analysis of the reversed process. In *Proc. 7th IEEE/ACS International Conference on Computer Systems and Applications (AICCSA'09)*, pages 808–815. IEEE Computer Society, 2009.

3 Simonetta Balsamo and Andrea Marin. Performance engineering with product-form models: efficient solutions and applications. In *Proc. 2nd Joint WOSP/SIPEW International Conference on Performance Engineering (ICPE'11)*, pages 437–448. ACM Press, 2011.

4 E. Cardoza, Richard J. Lipton, and Albert R. Meyer. Exponential space complete problems for Petri nets and commutative semigroups: Preliminary report. In *Proc. 8th Annual ACM Symposium on Theory of Computing (STOC'76)*, pages 50–54. ACM Press, 1976.

5 E. Cinlar. *Introduction to Stochastic Processes*. Prentice Hall, 1975.

6 J. L. Coleman, William Henderson, and Peter G. Taylor. Product form equilibrium distributions and a convolution algorithm for stochastic Petri nets. *Performance Evaluation*, 26(3):159–180, 1996.

7 Javier Esparza and Mogens Nielsen. Decidability issues for Petri nets – A survey. *Bulletin of the EATCS*, 52:244–262, 1994.

8 Gerard Florin and Stéphane Natkin. One-place unbounded stochastic Petri nets: Ergodic criteria and steady-state solutions. *Journal of Systems and Software*, 6(1):103 – 115, 1986.

9 William J. Gordon and Gordon F. Newell. Closed queuing systems with exponential servers. *Operations Research*, 15(2):254–265, 1967.

10 Serge Haddad, Jean Mairesse, and Hoang-Thach Nguyen. Synthesis and analysis of product-form Petri nets. *Fundamenta Informaticae*, 122(1-2):147–172, 2013.

11 Serge Haddad, Patrice Moreaux, Matteo Sereno, and Manuel Silva. Product-form and stochastic Petri nets: A structural approach. *Performance Evaluation*, 59(4):313–336, 2005.

12 Peter G. Harrison and Catalina M. Lladó. Hierarchically constructed Petri-nets and product-forms. In *Proc. 5th International ICST Conference on Performance Evaluation Methodologies and Tools Communications (VALUETOOLS'11)*, pages 101–110. ICST/ACM Press, 2011.

13 James R. Jackson. Jobshop-like queueing systems. *Management Science*, 10(1):131–142, 1963.

14 Matthias Jantzen. Complexity of place/transition nets. In *Proc. Advances in Petri Nets (1986)*, volume 254 of *Lecture Notes in Computer Science*, pages 413–434. Springer, 1987.

15 Aurel A. Lazar and Thomas G. Robertazzi. Markovian Petri net protocols with product form solution. *Performance Evaluation*, 12(1):67 – 77, 1991.

16 Man Li and Nicolas D.Georganas. Exact parametric analysis of stochastic Petri nets. *IEEE Transactions on Computers*, 41:1176–1180, 1992.

17 Jean Mairesse and Hoang-Thach Nguyen. Deficiency zero Petri nets and product form. *Fundamenta Informaticae*, 105(3):237–261, 2010.

18 Andrea Marin, Simonetta Balsamo, and Peter G. Harrison. Analysis of stochastic Petri nets with signals. *Performance Evaluation*, 69(11):551–572, 2012.

19 Marco Ajmone Marsan, Gianfranco Balbo, Gianni Conte, Susanna Donatelli, and Giuliana Franceschinis. *Modelling with Generalized Stochastic Petri Nets*. John Wiley & Sons, Inc., 1995.

20 Michael K. Molloy. Performance analysis using stochastic Petri nets. *IEEE Transactions on Computers*, 31(9):913–927, 1982.

21 J.L. Peterson. *Petri Net Theory and the Modeling of Systems*. Prentice Hall, 1981.

22 Charles Rackoff. The covering and boundedness problems for vector addition systems. *Theoretical Computer Science*, 6:223–231, 1978.

23 Matteo Sereno and Gianfranco Balbo. Mean value analysis of stochastic Petri nets. *Performance Evaluation*, 29(1):35–62, 1997.

Efficient Coalgebraic Partition Refinement*†

Ulrich Dorsch[1], Stefan Milius[2], Lutz Schröder[3], and Thorsten Wißmann[4]

1 **Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany**
`Ulrich.Dorsch@fau.de`
2 **Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany**
`mail@stefan-milius.eu`, `http://orcid.org/0000-0002-2021-1644`
3 **Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany**
`Lutz.Schroeder@fau.de`
4 **Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany**
`Thorsten.Wissmann@fau.de`

Abstract

We present a generic partition refinement algorithm that quotients coalgebraic systems by behavioural equivalence, an important task in reactive verification; coalgebraic generality implies in particular that we cover not only classical relational systems but also various forms of weighted systems. Under assumptions on the type functor that allow representing its finite coalgebras in terms of nodes and edges, our algorithm runs in time $\mathcal{O}(m \cdot \log n)$ where n and m are the numbers of nodes and edges, respectively. Instances of our generic algorithm thus match the runtime of the best known algorithms for unlabelled transition systems, Markov chains, and deterministic automata (with fixed alphabets), and improve the best known algorithms for Segala systems.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.2 Modes of Computation

Keywords and phrases coalgebra, markov chains, partition refinement, transition systems

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.32

1 Introduction

Minimization under bisimilarity is the task of identifying all states in a reactive system that exhibit the same behaviour. Minimization appears as a subtask in state space reduction (e.g. [5]) or non-interference checking [34]. The notion of bisimulation was first defined for relational systems [33, 24, 26]; it was later extended to other system types including probabilistic systems [23, 9] and weighted automata [6]. In fact, the importance of minimization under bisimilarity appears to increase with the complexity of the underlying system type. E.g., while in LTL model checking, minimization drastically reduces the state space but, depending on the application, does not necessarily lead to a speedup in the overall balance [11], in probabilistic model checking, minimization under strong bisimilarity does lead to substantial efficiency gains [19].

The algorithmics of minimization, often referred to as *partition refinement* or *lumping*, has received a fair amount of attention. Since bisimilarity is a greatest fixpoint, it is more or less immediate that it can be calculated in polynomial time by approximating this fixpoint from above following Kleene's fixpoint theorem. In the relational setting, Kanellakis and

* Full version with all proof details available at `http://arxiv.org/abs/1705.08362`.
† This work forms part of the DFG-funded project COAX (MI 717/5-1 and SCHR 1118/12-1).

Smolka [18] introduced an algorithm that in fact runs in time $\mathcal{O}(nm)$ where n is the number of nodes and m is the number of transitions. An even more efficient algorithm running in time $\mathcal{O}(m \log n)$ was later described by Paige and Tarjan [25]; this bound holds even if the number of action labels is not fixed [31]. Current algorithms typically apply further optimizations to the Paige-Tarjan algorithm, thus achieving better average-case behaviour but the same worst-case behaviour [10]. Probabilistic minimization has undergone a similarly dynamic development [3, 7, 36], and the best algorithms for minimization of Markov chains now have the same $\mathcal{O}(m \log n)$ run time as the relational Paige-Tarjan algorithm [15, 8, 32]. Using ideas from abstract interpretation, Ranzato and Tapparo [27] have developed a relational partition refinement algorithm that is generic over *notions of process equivalence*. As instances, they recover the classical Paige-Tarjan algorithm for strong bisimilarity and an algorithm for stuttering equivalence, and obtain new algorithms for simulation equivalence and for a new process equivalence.

In this paper we follow an orthogonal approach and provide a generic partition refinement algorithm that can be instantiated for many different *types* of systems (e.g. nondeterministic, probabilistic, weighted). We achieve this by methods of *universal coalgebra* [28]. That is, we encapsulate transition types of systems as endofunctors on sets (or a more general category), and model systems as coalgebras for a given type functor.

Our work proceeds on several levels of abstraction. On the most abstract level (Section 3) we work with coalgebras for a monomorphism-preserving endofunctor on a category with image factorizations. Here we present a quite general category-theoretic partition refinement algorithm, and we prove its correctness. The algorithm is parametrized over a select routine that determines which observations are used to split blocks of states; the corner case where all available observations are used yields known coalgebraic final chain algorithms, e.g. [22].

Next, we present an optimized version of our algorithm (Section 4) that needs more restrictive conditions to ensure correctness; specifically, we need to assume that the type endofunctor satisfies a condition we call *zippability* in order to allow for incremental computation of partitions. This property holds, e.g., for all polynomial endofunctors on sets and for the type functors of labelled and weighted transition systems, but not for all endofunctors of interest. In particular, zippable functors fail to be closed under composition, as exemplified by the double covariant powerset functor $\mathcal{PP}$ on sets, for which the optimized algorithm is in fact incorrect. However, it turns out that obstacles of this type can be removed by moving to multi-sorted coalgebras [29], so we do eventually obtain an efficient partition refinement algorithm for coalgebras of composite functors, including $\mathcal{PP}$-coalgebras as well as (probabilistic) Segala systems [30].

Finally, we analyse the run time of our algorithm (Section 5). To this end, we make our algorithm parametric in an abstract *refinement interface* to the type functor, which encapsulates the incremental calculation of partitions in the optimized version of the algorithm. We show that if the interface operations can be implemented in linear time, then the algorithm runs in time $\mathcal{O}(m \log n)$, where n is the number of states and m the number of 'edges' in a syntactic encoding of the input coalgebra. We thus recover the most efficient known algorithms for transition systems (Paige and Tarjan [25]) and for weighted systems (Valmari and Franceschinis [32]). Using the mentioned modularity results, we also obtain an $\mathcal{O}((m + n) \log(m + n))$ algorithm for Segala systems, to our knowledge a new result (more precisely, we improve an earlier bound established by Baier, Engelen, and Majster-Cederbaum [3], roughly speaking by letting only non-zero probabilistic edges enter into the time bound). The algorithm and its analysis apply also to generalized polynomial functors on sets; in particular, for the functor $2 \times (-)^A$, which models deterministic finite automata, we obtain the same complexity $\mathcal{O}(n \log n)$ as for Hopcroft's classical minimization algorithm for a fixed alphabet A [14, 21, 12].

2 Preliminaries

We assume that readers are familiar with basic category theory [2]. For the convenience of the reader we recall some concepts that are central for the categorical version of the algorithm.

▶ **Notation 2.1.** The terminal object is denoted by 1, with unique arrows $! : A \to 1$, and the product of objects A, B by $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$. Given $f : D \to A$ and $g : D \to B$, the morphism induced by the universal property of the product $A \times B$ is denoted by $\langle f, g\rangle : D \to A \times B$. The *kernel* $\mathsf{ker}\, f$ of a morphism f is the pullback of f along itself. We write $\twoheadrightarrow$ for regular epimorphisms (i.e. coequalizers), and $\rightarrowtail$ for monomorphisms.

Kernels allow us to talk about equivalence relations in a category. In particular in Set, there is a bijection between kernels and equivalence relations in the usual sense: For a map $f : D \to A$, $\mathsf{ker}\, f = \{(x, y) \mid fx = fy\}$ is the equivalence relation induced by f. Generally, relations (i.e. jointly monic spans of morphisms) in a category are ordered by inclusion in the obvious way. We say that a kernel K is *finer* than a kernel K' if K is included in K'. We use intersection $\cap$ and union $\cup$ of kernels for meets and joins in the inclusion ordering on *relations* (not equivalence relations or kernels); in this notation, $\mathsf{ker}\langle f, g\rangle = \mathsf{ker}\, f \cap \mathsf{ker}\, g$. In Set, a map $f : D \to A$ factors through the partition $D/\mathsf{ker}\, f$ induced by its kernel, via the map $[-]_f : D \twoheadrightarrow D/\mathsf{ker}\, f$ taking equivalence classes

$$[x]_f := \{x' \in D \mid fx = fx'\} = \{x' \in D \mid (x, x') \in \mathsf{ker}\, f\}.$$

Well-definedness of functions on $D/\mathsf{ker}\, f$ is determined precisely by the universal property of $[-]_f$ as a coequalizer of $\mathsf{ker}\, f \rightrightarrows D$. In particular, f induces an injection $D/\mathsf{ker}\, f \rightarrowtail A$; together with $[-]_f$, this is the factorization of f into a regular epimorphism and a monomorphism. Categorically, this is captured by the following assumptions.

▶ **Assumption 2.2.** We assume throughout that $\mathcal{C}$ is a finitely complete category that has coequalizers and *image factorizations*, i.e. every morphism f has a factorization $f = m \cdot e$ as a regular epimorphism e followed by a monomorphism m. We call the codomain of e the *image* of f, and denote it by $D/\mathsf{ker}\, f$. Regular epis in $\mathcal{C}$ are closed under composition and right cancellation [2, Prop. 14.14].

▶ **Examples 2.3.** Examples of categories satisfying Assumption 2.2 abound. In particular, every regular category with coequalizers satisfies our assumptions. The category Set of sets and maps is, of course, regular. Every topos is regular, and so is every finitary variety, i.e. a category of algebras for a finitary signature satisfying given equational axioms (e.g. monoids, groups, vector spaces etc.). Posets and topological spaces fail to be regular but still satisfy our assumptions. If $\mathcal{C}$ is regular, so is the functor category $\mathcal{C}^{\mathcal{E}}$ for any category $\mathcal{E}$.

For a set $\mathcal{S}$ of sorts, the category $\mathsf{Set}^{\mathcal{S}}$ of $\mathcal{S}$-sorted sets has $\mathcal{S}$-tuples of sets as objects. We write $\chi_S : X \to 2$ for the characteristic function of a subset $S \subseteq X$, i.e. for $x \in X$ we have $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. We will also use a three-valued version:

▶ **Definition 2.4.** For $S \subseteq C \subseteq X$, define $\chi_S^C : X \to 3$ by $C \not\ni x \mapsto 0, C \setminus S \ni x \mapsto 1$, and $S \ni x \mapsto 2$. (This is essentially $\langle \chi_S, \chi_C\rangle : X \to 4$ without the impossible case $x \in S \setminus C$.)

Coalgebras. We briefly recall basic notions from coalgebra. For introductory texts, see [28, 17, 1, 16]. Given an endofunctor $H : \mathcal{C} \to \mathcal{C}$, a *coalgebra* is pair (C, c) where C is an object of $\mathcal{C}$ called the *carrier* and thought of as an object of *states*, and $c : C \to HC$ is a morphism called the *structure* of the coalgebra. Our leading examples are the following.

▶ **Example 2.5.**

1. Labelled transition systems with labels from a set A are coalgebras for the functor $HX = \mathcal{P}(A \times X)$ (and unlabelled transition systems are simply coalgebras for $\mathcal{P}$). Explicitly, a coalgebra $c : C \to HC$ assigns to each state x a set $c(x) \in \mathcal{P}(A \times X)$, and this represents the transition structure at x: x has an a-transition to y iff $(a, y) \in c(x)$.
2. Weighted transition systems with weights drawn from a commutative monoid are modelled as coalgebras as follows. For the given commutative monoid $(M, +, 0)$, we consider the *monoid-valued* functor $M^{(-)}$ on Set given for any map $h : X \to Y$ by

$$M^{(X)} = \{f : X \to M \mid f(x) \neq 0 \text{ for finitely many } x\}, \qquad M^{(h)}(f)(y) = \textstyle\sum_{hx=y} f(x).$$

   M-weighted transition systems are in bijective correspondence with coalgebras for $M^{(-)}$ [13] (and for M-weighted labelled transition systems one takes $(M^{(-)})^A$).
3. Probabilistic transition systems are modelled coalgebraically using the distribution functor $\mathcal{D}$. This is the subfunctor $\mathcal{D}X \subseteq \mathbb{R}_{\geq 0}^{(X)}$, where $\mathbb{R}_{\geq 0}$ is the monoid of addition on the non-negative reals, given by $\mathcal{D}X = \{f \in \mathbb{R}_{\geq 0}^{(X)} \mid \sum_{x \in X} f(x) = 1\}$.
4. The finite powerset functor $\mathcal{P}_\mathrm{f}$ is a monoid-valued functor for the Boolean monoid $\mathbb{B} = (2, \vee, 0)$. The *bag functor* $\mathcal{B}_\mathrm{f}$, which assigns to a set X the set of bags (i.e. finite multisets) on X, is the monoid-valued functor for the additive monoid of natural numbers.
5. Simple (resp. general) Segala systems [30] strictly alternate between non-deterministic and probabilistic transitions; they can be modeled as coalgebras for the set functor $\mathcal{P}_\mathrm{f}(A \times \mathcal{D}(-))$ (resp. $\mathcal{P}_\mathrm{f}\mathcal{D}(A \times -)$).

A *coalgebra morphism* from a coalgebra (C, c) to a coalgebra (D, d) is a morphism $h : C \to D$ such that $d \cdot h = Hh \cdot c$; intuitively, coalgebra morphisms preserve observable behaviour. Coalgebras and their morphisms form a category $\mathsf{Coalg}(H)$. The forgetful functor $\mathsf{Coalg}(H) \to \mathcal{C}$ creates all colimits, so $\mathsf{Coalg}(H)$ has all colimits that $\mathcal{C}$ has.

A *subcoalgebra* of a coalgebra (C, c) is represented by a coalgebra morphism $m : (C, c) \to (D, d)$ such that m is a monomorphism in $\mathcal{C}$. Likewise, a *quotient* of a coalgebra (C, c) is represented by a coalgebra morphism $q : (C, c) \to (D, d)$ carried by a regular epimorphism q of $\mathcal{C}$. If H preserves monomorphisms, then the image factorization structure on $\mathcal{C}$ lifts to coalgebras.

▶ **Definition 2.6.** A coalgebra is *simple* if it does not have any non-trivial quotients.

Equivalently, a coalgebra (C, c) is simple if every coalgebra morphism with domain (C, c) is carried by a monomorphism. Intuitively, in a simple coalgebra all states exhibiting the same observable behaviour are already identified. This paper is concerned with the design of algorithms for computing *the* simple quotient of a given coalgebra:

▶ **Lemma 2.7.** *The simple quotient of a coalgebra is unique (up to isomorphism).*

Intuitively speaking, two elements (possibly in different coalgebras) are called behaviourally equivalent if they can be identified by coalgebra morphisms. Hence, the simple quotient of a coalgebra is its quotient modulo behavioural equivalence. In our main examples, this means that we minimize w.r.t. standard bisimilarity-type equivalences.

▶ **Example 2.8.** Behavioural equivalence instantiates to various notions of bisimilarity:

1. Park-Milner bisimilarity on labelled transition systems;
2. weighted bisimilarity on weighted transition systems [20, Proposition 2];
3. stochastic bisimilarity on probabilistic transition systems [20];
4. Segala bisimilarity on simple and general Segala systems [4, Theorem 4.2].

3 A Categorical Algorithm for Behavioural Equivalence

We proceed to describe a categorical partition refinement algorithm that computes the simple quotient of a given coalgebra under fairly general assumptions.

▶ **Assumption 3.1.** Assume that H is an endofunctor on $\mathcal{C}$ that preserves monomorphisms.

Note that mono preservation is w.l.o.g. for $\mathcal{C} = \mathsf{Set}$. Roughly, for a given coalgebra $\xi : X \to HX$ in Set, a *partition refinement algorithm* maintains a quotient $q : X \twoheadrightarrow X/Q$ that distinguishes some (but possibly not all) states with different behaviour, and in fact, initially q typically identifies everything. The algorithm repeats the following steps:

1. Gather new information on which states should become separated by using $X \xrightarrow{\xi} HX \xrightarrow{Hq} HX/Q$, i.e., by identifying equivalence classes under q that contain states whose behaviour is observed to differ under one more step of the transition structure ξ.
2. Use parts of this information to refine q and repeat until q does not change any more.

One of the core ideas of the Paige-Tarjan partition refinement algorithm [25] is to not use all information immediately in the second step. Recall that the algorithm maintains two partitions Y and Z of the state set X of the given transition system; the elements of Y are called *subblocks* and the elements of Z are called *compound blocks*. The partition Y is a refinement of the partition Z. The key to the time efficiency of the algorithm is to select in each iteration a subblock that is at most half of the size of the compound block it belongs to. At the present high level of generality (which in particular does not know about sizes of objects), we encapsulate the subblock selection in a routine select, assumed as a parameter to our algorithm:

▶ **Definition 3.2.** A select routine is an operation that receives a chain of two regular epis $X \xrightarrow{y} Y \xrightarrow{z} Z$ and returns some morphism $k : Y \to K$ into some object K. We call Y the *subblocks* and Z the *compound blocks*.

The idea is that the morphism k throws away some of the information provided by the refinement Y. For example, in the Paige-Tarjan algorithm it models the selection of one compound block to be split in two parts, which then induce the further refinement of Y.

▶ **Example 3.3.**

1. In the classical Paige-Tarjan algorithm [25], i.e., for $\mathcal{C} = \mathsf{Set}$, one wants to find a proper subblock that is at most half of the size of the compound block it sits in. So let $S \in Y$ such that $2 \cdot |y^{-1}[\{S\}]| \leq |(zy)^{-1}[\{z(S)\}]|$. Here, $z(S)$ is the compound block containing S. Then we let $\mathsf{select}(z, y)$ be $k : Y \to 3$ given by $k(x) = 2$ if $x = S$, else $k(x) = 1$ if $z(x) = z(S)$, and $k(x) = 0$ otherwise; i.e. $k = \chi^{[S]_z}_{\{S\}}$ (Theorem 2.4). If Y and Z are encoded as partitions of X, then S and $C := z(S)$ are subsets of X and $k \cdot y = \chi^{C}_{S}$. If there is no such $S \in Y$, then z is bijective, i.e., there is no compound block from Z that needs to be refined. In this case, k does not matter and we simply put $k = ! : Y \to 1$.
2. One obvious choice for k is to take the identity on Y, so that *all* of the information present in Y is used for further refinement. We will discuss this in Theorem 3.
3. Two other, trivial, choices are $k = ! : Y \to 1$ and $k = z$. Since both of these choices provide no extra information, this will leave the partitions unchanged, see Theorem 3.12.

Given a select routine, the most general form of our partition refinement works as follows.

▶ **Algorithm 3.4.** Given a coalgebra $\xi : X \to HX$, we successively refine equivalence relations Q and P on X, maintaining the invariant that P is finer than Q. In each step, we take into account new information on the behaviour of states, represented by a map

$q : X \to K$, and accumulate this information in a map $\bar{q} : X \to \bar{K}$. To facilitate the analysis, these variables are indexed over loop iterations in the description. Initial values are

$$Q_0 = X \times X \qquad q_0 = ! : X \to 1 = K_0 \qquad P_0 = \ker(X \xrightarrow{\xi} HX \xrightarrow{H!} H1).$$

We then iterate the following steps while $P_i \neq Q_i$, for $i \geq 0$:

1. $X/P_i \overset{k_{i+1}}{\twoheadrightarrow} K_{i+1} := \mathsf{select}\big(X \twoheadrightarrow X/P_i \twoheadrightarrow X/Q_i\big)$, using that X/P_i is finer than X/Q_i
2. $q_{i+1} := X \longrightarrow\!\!\!\!\!\rightarrow X/P_i \xrightarrow{k_{i+1}} K_{i+1}$, $\bar{q}_{i+1} := \langle \bar{q}_i, q_{i+1} \rangle : X \longrightarrow \bar{K}_i \times K_{i+1}$
3. $Q_{i+1} := \ker \bar{q}_{i+1} \quad (= \ker\langle \bar{q}_i, q_{i+1}\rangle = \ker \bar{q}_i \cap \ker q_{i+1})$
4. $P_{i+1} := \ker\big(X \xrightarrow{\xi} HX \xrightarrow{H\bar{q}_{i+1}} H \prod_{j \leq i+1} K_j \big)$

Upon termination, the algorithm returns $X/P_i = X/Q_i$ as the simple quotient of (X, ξ).

▶ **Notation 3.5.** For spans $R \rightrightarrows X$, we will denote the canonical quotient by $\kappa_R : X \twoheadrightarrow X/R$.

We proceed to prove correctness, i.e. that the algorithm really does return the simple quotient of (X, ξ). We fix the notation in Algorithm 3.4 throughout. Since $\bar{q}$ accumulates more information in every step, it is clear that P and Q are really being successively refined:

▶ **Lemma 3.6.** *For every i, P_{i+1} is finer than P_i, Q_{i+1} is finer than Q_i, and P_i is finer than Q_{i+1}.*

$$\begin{array}{ccccccccccc} Q_0 & \longleftarrow & Q_1 & \longleftarrow & Q_2 & \leftarrow\!\cdots\!\prec & Q_{i+1} & \longleftarrow & Q_{i+2} & \leftarrow\cdots \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ P_0 & \longleftarrow & P_1 & \leftarrow\!\cdots\!\prec & P_i & \longleftarrow & P_{i+1} & \leftarrow\cdots & & \end{array} \tag{3.1}$$

If we suppress the termination on $P_i = Q_i$ for a moment, then the algorithm thus computes equivalence relations refining each other. At each step, select decides which part of the information present in P_i but not in Q_i should be used to refine Q_i to Q_{i+1}.

▶ **Proposition 3.7.** *There exist morphisms $\xi/Q_i : X/P_i \to H(X/Q_i)$ for $i \geq 0$ (necessarily unique) such that* (3.2) *commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\kappa_{P_i}} & X/P_i \\ {\scriptstyle\xi}\downarrow & & \vdots\,{\scriptstyle \xi/Q_i} \\ HX & \xrightarrow{H\kappa_{Q_i}} & H(X/Q_i) \end{array} \tag{3.2}$$

Upon termination the morphism ξ/Q_i yields the structure of a quotient coalgebra of ξ:

▶ **Corollary 3.8.** *If $P_i = Q_i$ then X/Q_i carries a unique coalgebra structure forming a quotient of $\xi : X \to HX$.*

This means intuitively that all states that are merged by the algorithm are actually behaviourally equivalent. The following property captures the converse:

▶ **Lemma 3.9.** *Let $h : (X, \xi) \to (D, d)$ be a quotient of (X, ξ). Then $\ker h$ is finer than both P_i and Q_i, for all $i \geq 0$.*

▶ **Theorem 3.10** (Correctness)**.** *If $P_i = Q_i$, then $\xi/Q_i : X/Q_i \to H\, X/Q_i$ is a simple coalgebra.*

▶ Remark. Most classical partition refinement algorithms are parametrized by an initial partition $\kappa_{\mathcal{I}} : X \twoheadrightarrow X/\mathcal{I}$. We start with the trivial partition $! : X \to 1$ because a non-trivial initial partition might split equivalent behaviours and then would invalidate Theorem 3.9. To accomodate an initial partition $X/\mathcal{I}$ coalgebraically, replace (X, ξ) with the coalgebra $\langle \xi, \kappa_{\mathcal{I}} \rangle$ for the functor $H(-) \times X/\mathcal{I}$ – indeed, already P_0 will then be finer than $\mathcal{I}$.

We look in more detail at two corner cases of the algorithm, where the select routine retains all available information, respectively none:

▶ Remark. Recall that H induces the *final sequence*:

$$1 \xleftarrow{!} H1 \xleftarrow{H!} H^2 1 \xleftarrow{H^2!} \cdots \xleftarrow{H^{i-1}!} H^i 1 \xleftarrow{H^i!} H^{i+1}1 \xleftarrow{H^{i+1}!} \cdots$$

Every coalgebra $\xi : X \to HX$ then induces a *canonical cone* $\xi^{(i)} : X \to H^i 1$ on the final sequence, defined inductively by $\xi^{(0)} = !$, $\xi^{(i+1)} = H\xi^{(i)} \cdot \xi$. The objects $H^n 1$ may be thought of as domains of n-step behaviour for H-coalgebras. If $\mathcal{C} = \mathsf{Set}$ and X is finite, then states x and y are behaviourally equivalent iff $\xi^{(i)}(x) = \xi^{(i)}(y)$ for all $i < \omega$ [35].

The vertical inclusions in (3.1) reflect that only some and not necessarily all of the information present in the relation P_i (resp. the quotient X/P_i) is used for further refinement. If indeed everything is used, i.e., we have $k_{i+1} := \mathsf{id}_{X/P_i}$, then these inclusions become isomorphisms and then our algorithm simply computes the kernels of the morphisms in the canonical cone, i.e. $Q_i = \ker \xi^{(i)}$.

That is, when select retains all available information, then Algorithm 3.4 just becomes a standard final chain algorithm (e.g. [22]). The other extreme is the following:

▶ **Definition 3.11.** We say that select *discards all new information at* $i+1$ if k_{i+1} factors through the morphism $X/P_i \twoheadrightarrow X/Q_i$ witnessing that P_i is finer than Q_i, see Theorem 3.6.

▶ **Lemma 3.12.** *The algorithm fails to progress in the* $i+1$*-th iteration, i.e.* $Q_{i+1} = Q_i$*, iff* select *discards all new information at* $i+1$.

▶ **Corollary 3.13.** *If* $\mathcal{C}$ *is (concrete over)* $\mathsf{Set}^{\mathcal{S}}$ *and* select *never discards all new information, then Algorithm 3.4 terminates and computes the simple quotient of a given finite coalgebra.*

Indeed, Proposition 3.7 shows that we obtain a chain of successively finer quotients of X, and by Theorem 3.12 this chain must finally converge (i.e. $P_i = Q_i$ will hold).

4 Incremental Partition Refinement

In the most generic version of the partition refinement algorithm (Algorithm 3.4), the partitions are recomputed from scratch in every step: In Step 4 of the algorithm, $P_{i+1} = \ker(H\langle \bar{q}_i, q_{i+1}\rangle \cdot \xi)$ is computed from the information $\bar{q}_i$ accumulated so far and the new information q_{i+1}, but in general one cannot exploit that the kernel of $\bar{q}_i$ has already been computed. We now present a refinement of the algorithm in which the partitions are computed incrementally, i.e. P_{i+1} is computed from P_i and q_{i+1}. This requires the type functor H to be *zippable* (Theorem 4.1). The algorithm will be further refined in the next section.

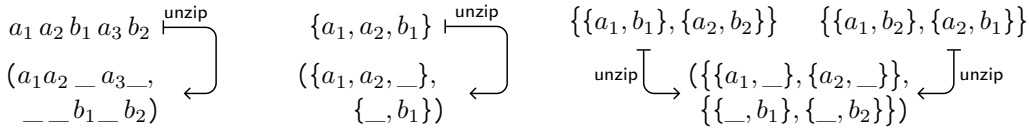
Note that in Step 3, Algorithm 3.4 computes a kernel $Q_{i+1} = \ker \bar{q}_{i+1} = \ker\langle \bar{q}_i, q_{i+1}\rangle$. In general, the kernel of a pair $\langle a, b\rangle : D \to A \times B$ is an intersection $\ker a \cap \ker b$. Hence, the partition for such a kernel can be computed in two steps:

1. Compute $D/\ker a$.
2. Refine every block in $D/\ker a$ with respect to $b : D \to B$.

Algorithm 3.4 can thus be implemented to keep track of the partition X/Q_i and then refine this partition by q_{i+1} in each iteration.

However, the same trick cannot be applied immediately to the computation of X/P_i, because of the functor H inside the computation of the kernel: $P_{i+1} = \ker(H\langle \bar{q}_i, q_{i+1}\rangle \cdot \xi)$. In the following, we will provide sufficient conditions for H, $a : D \to A$, $b : D \to B$ to satisfy

$$\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle.$$



(a) $(-)^*$ is zippable. **(b)** $\mathcal{P}_\mathrm{f}$ is zippable. **(c)** $\mathcal{P}_\mathrm{f}\mathcal{P}_\mathrm{f}$ is not zippable.

Figure 1 Zippability of Set-Functors for sets $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$.

As soon as this holds for $a = \bar{q}_i, b = q_{i+1}$, we can optimize the algorithm by changing Step 4 to

$$P'_{i+1} := \mathsf{ker}\langle H\bar{q}_i \cdot \xi, Hq_{i+1} \cdot \xi\rangle. \tag{4.1}$$

▶ **Definition 4.1.** A functor H is *zippable* if the following morphism is a monomorphism:

$$\mathsf{unzip}_{H,A,B} : H(A + B) \xrightarrow{\langle H(A+!), H(!+B)\rangle} H(A + 1) \times H(1 + B)$$

Intuitively, if H is a functor on Set, we think of elements t of $H(A + B)$ as shallow terms with variables from $A + B$. Then zippability means that each t is uniquely determined by the two terms obtained by replacing A- and B-variables, respectively, by some placeholder $\_$, viz. the element of 1, as in the examples in Figure 1.

In the following, we work in the category $\mathcal{C} = \mathsf{Set}^{\mathcal{S}}$ of $\mathcal{S}$-sorted sets. However, most proofs are category-theoretic to clarify where sets are really needed and where the arguments are more generic.

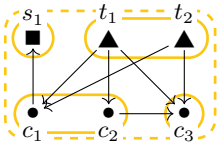
▶ **Example 4.2.** **1.** Constant functors $X \mapsto A$ are zippable: unzip is the diagonal $A \to A \times A$.
2. The identity functor is zippable since $A + B \xrightarrow{\langle A+!, !+B\rangle} (A + 1) \times (1 + B)$ is monic in $\mathsf{Set}^{\mathcal{S}}$.
3. From Lemma 4.3 it follows that every polynomial endofunctor is zippable.

▶ **Lemma 4.3.** *Zippable endofunctors are closed under products, coproducts and subfunctors.*

▶ **Lemma 4.4.** *If H has a componentwise monic natural transformation $H(X + Y) \rightarrowtail HX \times HY$, then H is zippable.*

▶ **Example 4.5.**
1. For every commutative monoid, the monoid-valued functor $M^{(-)}$ admits a natural isomorphism $M^{(X+Y)} \cong M^{(X)} \times M^{(Y)}$, and hence is zippable by Lemma 4.4.
2. As special cases of monoid-valued functors we obtain that the finite powerset functor $\mathcal{P}_\mathrm{f}$ and the bag functor $\mathcal{B}_\mathrm{f}$ are zippable.
3. The distribution functor $\mathcal{D}$ (see Example 2.5) is a subfunctor of the monoid-valued functor $M^{(-)}$ for M the additive monoid of real numbers, and hence is zippable by Item 1 and Lemma 4.3.
4. The previous examples together with the closure properties in Lemma 4.3 show that a number of functors of interest are zippable, e.g. $2 \times (-)^A$, $2 \times \mathcal{P}(-)^A$, $\mathcal{P}(A \times (-))$, $2 \times \big((-) + 1\big)^A$, and variants where $\mathcal{P}$ is replaced with $\mathcal{B}_\mathrm{f}$, $M^{(-)}$, or $\mathcal{D}$.

▶ **Example 4.6.** The finitary functor $\mathcal{P}_\mathrm{f}\mathcal{P}_\mathrm{f}$ fails to be zippable, as shown in Figure 1. First, this shows that zippable functors are not closed under quotients, since any finitary functor is a quotient of a polynomial, hence zippable, functor (recall that a Set-functor F is finitary if $FX = \bigcup\{Fi[Y] \mid i : Y \rightarrowtail X \text{ and } Y \text{ finite}\}$). Secondly, this shows that zippable functors are not closed under composition. One can extend the counterexample to a coalgebra to show that the optimization is incorrect for $\mathcal{P}_\mathrm{f}\mathcal{P}_\mathrm{f}$ and $\mathsf{select} = \chi^C_S$. We will remedy this later by making use of a second sort, i.e. by working in Set^2 (Theorem 4).



(a) Q_0, P_0 for $\bar{q}_0 = !$ **(b)** Q_1, P_1 for $\bar{q}_1 = \kappa_{P_0}$

Figure 2 Partitions of a coalgebra ξ for $H = \{\blacktriangle, \blacksquare, \bullet\} \times \mathcal{P}_\mathrm{f}(-)$. X/Q_i is indicated by dashed, X/P_i by solid lines.

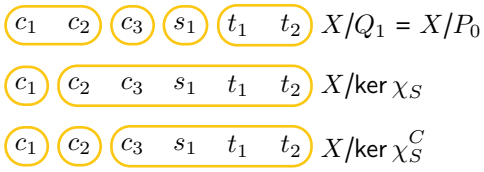


Figure 3 Grouping of elements when $S := \{c_1\}$ is chosen as the next subblock and $C := \{c_1, c_2\}$ as the compound block.

Additionally, we will need to enforce constraints on the select routine to arrive at the desired optimization (4.1). This is because in general, $\ker H\langle a, b\rangle$ differs from $\ker\langle Ha, Hb\rangle$ even for H zippable; e.g. for $H = \mathcal{P}$ and for π_1, π_2 denoting binary product projections, $\langle \mathcal{P}\pi_1, \mathcal{P}\pi_2\rangle$ in general fails to be injective although $\mathcal{P}\langle \pi_1, \pi_2\rangle = \mathcal{P}\mathsf{id} = \mathsf{id}$.

The next example illustrates this issue, and a related one: One might be tempted to implement splitting by a subblock S by $q_i = \chi_S$. While this approach is sufficient for systems with real-valued weights [32], it may in general let $\ker(H\langle \bar{q}_i, q_{i+1}\rangle \cdot \xi)$ and $\ker\langle H\bar{q}_i \cdot \xi, Hq_{i+1} \cdot \xi\rangle$ differ even for zippable H, thus rendering the algorithm incomplete:

▶ **Example 4.7.** Consider the coalgebra $\xi : X \to HX$ for the zippable functor $H = \{\blacktriangle, \blacksquare, \bullet\} \times \mathcal{P}_\mathrm{f}(-)$ illustrated in Figure 2 (essentially a Kripke model). The initial partition X/P_0 splits by shape and by $\mathcal{P}_\mathrm{f}!$, i.e. states with and without successors are split (Figure 2a). Now, suppose that select returns $k_1 := \mathsf{id}_{X/P_0}$, i.e. retains all information (cf. Remark 3), so that $Q_1 = P_0$ and P_1 puts c_1 and c_2 into different blocks (Figure 2b). We now analyse the next partition that arises when we split w.r.t. the subblock $S = \{c_1\}$ but not w.r.t. the rest $C \setminus S$ of the compound block $C = \{c_1, c_2\}$; in other words, we take $k_2 := \chi_{\{\{c_1\}\}} : X/P_1 \to 2$, making $q_2 = \chi_{\{c_1\}} : X \to 2$. Then, $H\langle \bar{q}_1, q_2\rangle \cdot \xi$ splits t_1 from t_2, because t_1 has a successor c_2 with $\bar{q}_1(c_2) = \{c_1, c_2\}$ and $q_2(c_2) = 0$ whereas t_2 has no such successor. However, t_1, t_2 fail to be split by $\langle H\bar{q}_1, Hq_2\rangle \cdot \xi$ because their successors do not differ when looking at successor blocks in X/Q_1 and $X/\ker \chi_S$ separately: both have $\{c_1, c_2\}$ and $\{c_3\}$ as successor blocks in X/Q_1 and $\{c_1\}$ and $X \setminus \{c_1\}$ as successors in $X/\ker \chi_S$. Formally:

$$H\bar{q}_1 \cdot \xi(t_1) = (\mathsf{id} \times \mathcal{P}_\mathrm{f}\kappa_{P_0}) \cdot \xi(t_1) = (\blacktriangle, \{\{c_1, c_2\}, \{c_3\}\}) = H\bar{q}_1 \cdot \xi(t_2)$$
$$Hq_2 \cdot \xi(t_1) = (\mathsf{id} \times \mathcal{P}_\mathrm{f}\chi_{\{c_1\}}) \cdot \xi(t_1) = (\blacktriangle, \{0, 1\}) = Hq_2 \cdot \xi(t_2)$$

So if we computed P_2 iteratively as in (4.1) for $q_2 = \chi_S$, then t_1 and t_2 would not be split, and we would reach the termination condition $P_2 = Q_2$ before all behaviourally inequivalent states have been separated.

Already Paige and Tarjan [25, Step 6 of the Algorithm] note that one additionally needs to split by $C \setminus S = \{c_3\}$, which is accomplished by splitting by $q_i = \chi_S^C$. This is formally captured by the condition we introduce next.

▶ **Definition 4.8.** A select routine *respects compound blocks* if whenever $k = \mathsf{select}(X \overset{y}{\twoheadrightarrow} Y \overset{z}{\twoheadrightarrow} Z)$ then the union $\ker k \cup \ker z$ is a kernel.

In $\mathsf{Set}^{\mathcal{S}}$, $\cup$ denotes the usual union of multi-sorted relations; and since reflexive and symmetric relations are closed under unions, the definition boils down to $\ker k \cup \ker z$ being transitive. We can rephrase the condition more explicitly, restricting to the single-sorted case for readability:

▶ **Lemma 4.9.** *For $a : D \to A$, $b : D \to B$ in* Set*, the following are equivalent:*

1. $\ker a \cup \ker b \rightrightarrows D$ *is a kernel.*
2. $\ker a \cup \ker b \rightrightarrows D$ *is the kernel of the pushout of a and b.*
3. *For all $x, y, z \in D$, $ax = ay$ and $by = bz$ implies $ax = ay = az$ or $bx = by = bz$.*
4. *For all $x \in D$, $[x]_a \subseteq [x]_b$ or $[x]_b \subseteq [x]_a$.*

The last item states that when going from a-equivalence classes to b-equivalence classes, the classes either merge or split, but do not merge with other classes and split at the same time. Note that in Figure 3, $Q_1 \cup \ker \chi_S$ fails to be transitive, while $Q_1 \cup \ker \chi_S^C$ is transitive.

▶ **Example 4.10.** All $\mathsf{select}\big(X \overset{y}{\twoheadrightarrow} Y \overset{z}{\twoheadrightarrow} Z\big)$ routines in Theorem 3.3 respect compound blocks.

▶ **Proposition 4.11.** *Let $a : D \to A$, $b : D \to B$ be a span such that $\ker a \cup \ker b$ is a kernel, and let $H :$ Set $\to \mathcal{D}$ be a zippable functor preserving monos. Then we have*

$$\ker\langle Ha, Hb\rangle = \ker H\langle a, b\rangle. \tag{4.2}$$

We thus obtain soundness of optimization (4.1); summing up:

▶ **Corollary 4.12.** *Suppose that H is a zippable endofunctor on* Set *and that* select *respects compound blocks and never discards all new information. Then Algorithm 3.4 with optimization* (4.1) *terminates and computes the simple quotient of a given finite H-coalgebra.*

▶ Remark. Like most results on set coalgebras, the above extends to multisorted sets by componentwise arguments, and this allows dealing with complex composite functors [29]. We restrict to a case with lightweight notation: Let F and G be zippable Set-functors, recalling from Theorem 4.6 that the composite FG need not itself be zippable, and let F be finitary. Then in lieu of FG-coalgebras, we can equivalently consider coalgebras for the endofunctor $H : (X, Y) \mapsto (FY, GX)$ on Set^2. In particular, a coalgebra $\xi : X \to FGX$ with finite carrier X induces, since F is finitary, a finite set $Y \subseteq_{\mathrm{f}} GX$, with inclusion y, such that $\xi = (X \xrightarrow{x} FY \xrightarrow{Fy} FGX)$, so we obtain a finite H-coalgebra $(x, y) : (X, Y) \to (FY, GX)$. Mutatis mutandis, Proposition 4.11 holds also for H, since kernels and pairs in Set^2 are computed componentwise, so we obtain a version of Theorem 4.12 for H. Explicitly, when computing the kernel of $H\bar{q}_{i+1} \cdot (x, y)$, we can use the optimization (4.1) in both sorts. The first component of the simple quotient of $((X, Y), (x, y))$ computed by the algorithm then yields the simple quotient of the original (X, ξ). Composites of more than two functors are treated similarly.

▶ **Example 4.13.** Applying this to the functors $F = \mathcal{P}_{\mathrm{f}}$, $G = A \times (-)$, and $H = \mathcal{D}$, we obtain simple (resp. general) Segala systems as coalgebras for FGH (resp. FHG). For simple Segala systems, the Set^3 functor is defined by $(X, Y, Z) \mapsto (FY, GZ, HX)$.

5 Efficient Calculation of Kernels

In Algorithm 3.4, it is left unspecified how the kernels are computed concretely. We proceed to define a more concrete algorithm based on a *refinement interface* of the functor. This interface is aimed at efficient implementation of the *refinement step* in the algorithm. Specifically from now on, we split along $\xi : X \to HY$ w.r.t. a subblock $S \subseteq C \in Y/Q$, and need to compute how the splitting of C into S and $C \setminus S$ within Y/Q affects the partition X/P.

The low complexity of Paige-Tarjan-style algorithms hinges on this refinement step running in time $\mathcal{O}(|\mathsf{pred}[S]|)$, where $\mathsf{pred}(y)$ denotes the set of predecessors of some $y \in Y$

in the given transition system. In order to speak about "predecessors" w.r.t. more general $\xi : X \to HY$, the refinement interface will provide an encoding of H-coalgebras as sets of states with successor states encoded as bags (implemented as lists up to ordering; recall that $\mathcal{B}_\mathrm{f}Z$ denotes the set of bags over Z) of A-labelled edges, where A is an appropriate label alphabet. Moreover, the interface will allow us to talk about the behaviour of elements of X w.r.t. the splitting of C into S and $C \setminus S$, looking only at points in S.

▶ **Definition 5.1.** A *refinement interface* for a Set-functor H is formed by a set A of *labels*, a set W of *weights* and functions

$$\begin{aligned} &\mathsf{b} : HY \to \mathcal{B}_\mathrm{f}(A \times Y), && \mathsf{init} : H1 \times \mathcal{B}_\mathrm{f}A \to W, \\ &w : \mathcal{P}_\mathrm{f}Y \to HY \to W, && \mathsf{update} : \mathcal{B}_\mathrm{f}A \times W \to W \times H3 \times W \end{aligned}$$

such that for every $S \subseteq C \subseteq Y$, the diagrams

$$\begin{array}{ccccc} HY & & \xrightarrow{w(Y)} & & \\ {\scriptstyle\langle H!,\, \mathcal{B}_\mathrm{f}\pi_1 \cdot \mathsf{b}\rangle}\downarrow & & & & \\ H1 \times \mathcal{B}_\mathrm{f}A & & \xrightarrow{\mathsf{init}} & & W \end{array} \qquad \begin{array}{ccccccc} HY & & & & \xrightarrow{\langle w(S), H\chi_S^C, w(C\setminus S)\rangle} & & \\ {\scriptstyle\langle \mathsf{b}, w(C)\rangle}\downarrow & & & & & & \\ \mathcal{B}_\mathrm{f}(A \times Y) \times W & & \xrightarrow{\mathsf{fil}_S \times W} & & \mathcal{B}_\mathrm{f}A \times W & \xrightarrow{\mathsf{update}} & W \times H3 \times W \end{array} \tag{5.1}$$

commute, where $\mathsf{fil}_S : \mathcal{B}_\mathrm{f}(A \times Y) \to \mathcal{B}_\mathrm{f}(A)$ is the filter function $\mathsf{fil}_S(f)(a) = \sum_{y \in S} f(a, y)$ for $S \subseteq Y$. The significance of the set $H3$ is that when using a set $S \subseteq C \subseteq X$ as a splitter, we want to split every block B in such a way that it becomes *compatible* with S and $C \setminus S$, i.e. we group the elements $s \in B$ by the value of $H\chi_S^C \cdot \xi(s) \in H3$. The set W depends on the functor. But in most cases $W = H2$ and $w(C) = H\chi_C : HY \to H2$ are sufficient.

In an implementation, we do not require a refinement interface to provide w explicitly, because the algorithm will compute the values of w incrementally using (5.1), and b need not be implemented because we assume the input coalgebra to be already *encoded* via b:

▶ **Definition 5.2.** Given an interface of H (Theorem 5.1), an *encoding* of a morphism $\xi : X \to HY$ is given by a set E and maps

$$\mathsf{graph} : E \to X \times A \times Y \qquad\qquad \mathsf{type} : X \to H1$$

such that $(\mathsf{b} \cdot \xi(x))(a, y) = |\{e \in E \mid \mathsf{graph}(e) = (x, a, y)\}|$, and with $\mathsf{type} = H! \cdot \xi$.

Intuitively, an encoding presents the morphism ξ as a graph with edge labels from A.

▶ **Lemma 5.3.** *Every morphism $\xi : X \to HY$ has a canonical encoding where E is the obvious set of edges of $\mathsf{b} \cdot \xi : X \to \mathcal{B}_\mathrm{f}(A \times Y)$. If X is finite, then so is E.*

▶ **Example 5.4.** In the following examples, we take $W = H2$ and $w(C) = H\chi_C : HY \to H2$. We use the helper function $\mathsf{val} := \langle H(=2), \mathsf{id}, H(=1)\rangle : H3 \to H2 \times H3 \times H2$, where $(=x) : 3 \to 2$ is the equality check for $x \in \{1, 2\}$, and in each case define $\mathsf{update} = \mathsf{val} \cdot \mathsf{up}$ for some function $\mathsf{up} : \mathcal{B}_\mathrm{f}A \times H2 \to H3$. We implicitly convert sets into bags.

1. For the monoid-valued functor $G^{(-)}$, for an Abelian group $(G, +, 0)$, we take labels $A = G$ and define $\mathsf{b}(f) = \{(f(y), y) \mid y \in Y, f(y) \neq 0\}$ (which is is finite because f is finitely supported). With $W = H2 = G \times G$, the weight $w(C) = H\chi_C : HY \to G \times G$ is the accumulated weight of $Y \setminus C$ and C. Then the remaining functions are

   $$\mathsf{init}(h_1, e) = (0, \Sigma e) \quad \text{and} \quad \mathsf{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e),$$

   where $\Sigma : \mathcal{B}_\mathrm{f}G \to G$ is the obvious summation map.

2. Similarly to the case $\mathbb{R}^{(-)}$, one has the following init and up functions for the distribution functor $\mathcal{D}$: put $\mathsf{init}(h_1, e) = (0, 1) \in \mathcal{D}2 \subset [0,1]^2$ and $\mathsf{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e)$ if the latter lies in $\mathcal{D}3$, and $(0, 0, 1)$ otherwise.
3. Similarly, one obtains a refinement interface for $\mathcal{B}_\mathrm{f} = \mathbb{N}^{(-)}$ adjusting the one for $\mathbb{Z}^{(-)}$; in fact, init remains unchanged and $\mathsf{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e)$ if the middle component is a natural number and $(0, 0, 0)$ otherwise.
4. Given a polynomial functor H_Σ for a signature Σ with bounded arity (i.e. there exists k such that every arity is at most k), the labels $A = \mathbb{N}$ encode the indices of the parameters:

$$\mathfrak{b}(\sigma(y_1, \dots, y_n)) = \{(1, y_1), \dots, (n, y_n)\} \quad \mathsf{init}(\sigma(0, \dots, 0), f) = \sigma(1, \dots, 1)$$
$$\mathsf{up}(I, \sigma(b_1, \dots, b_n)) = \sigma(b_1 + (1 \in I), \dots, b_i + (i \in I), \dots, b_n + (n \in I))$$

   Here $b_i + (i \in I)$ means $b_i + 1$ if $i \in I$ and b_i otherwise. Since I are the indices of the parameters in the subblock, $i \in I$ happens only if $b_i = 1$.

One example where $W = H2$ does not suffice is the powerset functor $\mathcal{P}$: Even if we know for a $t \in \mathcal{P}Y$ that it contains elements in $C \subseteq Y$, in $S \subseteq C$, and outside C (i.e. we know $\mathcal{P}\chi_S(t), \mathcal{P}\chi_C \in \mathcal{P}2$), we cannot determine whether there are any elements in $C \setminus S$ – but as seen in Theorem 4.7, we need to include this information.

▶ **Example 5.5.** The interface for the powerset functor needs to count the edges into blocks $C \supseteq S$ in order to know whether there are edges into $C \setminus S$, as described by Paige and Tarjan [25]. What happens formally is that first the interface for $\mathbb{N}^{(-)}$ is implemented for edge weights at most 1, and then the middle component of the result of update is adjusted. So $W = \mathbb{N}^{(2)}$, and the encoding $\mathfrak{b} : \mathcal{P}_\mathrm{f}Y \hookrightarrow \mathcal{B}_\mathrm{f}(1 \times Y)$ is the obvious inclusion. Then

$$\mathsf{init}(h_1, e) = (0, |e|) \qquad w(C)(M) = \mathcal{B}_\mathrm{f}\chi_C(M) = (|M \setminus C|, |C \cap M|)$$
$$\begin{aligned}\mathsf{update}(n, (c, r)) &= \langle \mathcal{B}_\mathrm{f}(=2),\ (\overset{?}{>}0)^3,\ \mathcal{B}_\mathrm{f}(=1)\rangle(r, c - |n|, |n|) \\ &= \big((r + c - |n|, |n|), (r \overset{?}{>} 0, c - |n| \overset{?}{>} 0, |n| \overset{?}{>} 0), (r + |n|, c - |n|)\big),\end{aligned}$$

where $x \overset{?}{>} 0$ is 0 if $x = 0$ and 1 otherwise.

▶ **Assumption 5.6.** From now on, assume a Set-functor H with a refinement interface such that init and update run in linear time and elements of $H3$ can be compared in constant time.

▶ **Example 5.7.** The refinement interfaces in Examples 5.4 and 5.5 satisfy Assumption 5.6.

▶ Remark. In the implementation, we encode the partitions X/P, Y/Q as doubly linked lists of the blocks they contain, and each block is in turn encoded as a doubly linked list of its elements. The elements $x \in X$ and $y \in Y$ each hold a pointer to the corresponding list entry in the blocks containing them. This allows removing elements from a block in $\mathcal{O}(1)$.

The algorithm maintains the following mutable data structures:

- An array $\mathsf{toSub} : X \to \mathcal{B}_\mathrm{f}E$, mapping $x \in X$ to its outgoing edges ending in the currently processed subblock.
- A pointer mapping edges to memory addresses: $\mathsf{lastW} : E \to \mathbb{N}$.
- A store of last values $\mathsf{deref} : \mathbb{N} \to W$.
- For each block B a list of markings $\mathsf{mark}_B \subseteq B \times \mathbb{N}$.

▶ **Notation 5.8.** In the following we write $e = x \xrightarrow{a} y$ in lieu of $\mathsf{graph}(e) = (x, a, y)$.

1: **for** $e \in E$, $e = x \xrightarrow{a} y$ **do**
2: add e to $\mathsf{toSub}[x]$ and $\mathsf{pred}[x]$.
3: **for** $x \in X$ **do**
4: $p_X :=$ new cell in deref containing $\mathsf{init}(\mathsf{type}[x], \mathcal{B}_\mathrm{f}(\pi_2 \cdot \mathsf{graph})(\mathsf{toSub}[x]))$
5: **for** $e \in \mathsf{toSub}[x]$ **do** $\mathsf{lastW}[e] = p_X$
6: $\mathsf{toSub}[x] := \varnothing$
7: $X/P :=$ group X by $\mathsf{type} : X \to H1$, $Y/Q := \{Y\}$.

Figure 4 The initialization procedure.

▶ **Definition 5.9** (Invariants). Our correctness proof below establishes the following properties that we call *the invariants*:

1. For all $x \in X, \mathsf{toSub}(x) = \varnothing$, i.e. toSub is empty everywhere.
2. For $e_i = x_i \xrightarrow{a_i} y_i, i \in \{1,2\}$, $\mathsf{lastW}(e_1) = \mathsf{lastW}(e_2) \iff x_1 = x_2$ and $[y_1]_{\kappa_Q} = [y_2]_{\kappa_Q}$.
3. For each $e = x \xrightarrow{a} y$, $C := [y]_{\kappa_Q} \in Y/Q$, $w(C, \xi(x)) = \mathsf{deref} \cdot \mathsf{lastW}(e)$.
4. For $x_1, x_2 \in B \in X/P$, $C \in Y/Q$, $(x_1, x_2) \in \ker(H\chi_C \cdot \xi)$.

In the following code listings, we use square brackets for array lookups and updates in order to emphasize they run in constant time. We assume that the functions $\mathsf{graph} : E \to X \times A \times Y$ and $\mathsf{type} : X \to H1$ are implemented as arrays. In the initialization step the predecessor array $\mathsf{pred} : Y \to \mathcal{P}_\mathrm{f} E$, $\mathsf{pred}(y) = \{e \in E \mid e = x \xrightarrow{a} y\}$ is computed. Sets and bags are implemented as lists. We only insert elements into sets not yet containing them.

We say that we *group* a finite set Z by $f : Z \to Z'$ to indicate that we compute $[-]_f$. This is done by sorting the elements of $z \in Z$ by a binary encoding of $f(z)$ using any $\mathcal{O}(|Z| \cdot \log |Z|)$ sorting algorithm, and then grouping elements with the same $f(z)$ into blocks. In order to keep the overall complexity for the grouping operations low enough, one needs to use a possible majority candidate during sorting, following Valmari and Franceschinis [32].

The algorithm computing the initial partition is listed in Figure 4.

▶ **Lemma 5.10.** *The initialization procedure runs in time $\mathcal{O}(|E| + |X| \cdot \log |X|)$ and makes the invariants true.*

The algorithm for one refinement step along a morphism $\xi : X \to HY$ is listed in Figure 5.

In the first part, all blocks $B \in X/P$ are collected that have an edge into S, together with $v_\varnothing \in H3$ which represents $H\chi_S^C \cdot \xi(x)$ for any $x \in B$ that has no edge into S. For each $x \in X$, $\mathsf{toSub}[x]$ collects the edges from x into S. The markings mark_B list those elements $x \in B$ that have an edge into S, together with a pointer to $w(C, x)$.

In the second part, each block B with an edge into S is refined w.r.t. $H\chi_S^C \cdot \xi$. First, for any $(x, p_C) \in \mathsf{mark}_B$, we compute $w(S, x)$, $v^x = H\chi_S^C \cdot \xi(x)$, and $w(C \setminus S, x)$ using update. Then, the weight of all edges $x \to C \setminus S$ is updated to $w(C \setminus S, x)$ and the weight of all edges $x \to S$ needs to be stored in a new cell containing $w(S, x)$. For all unmarked $x \in B$, we know that $H\chi_S^C \cdot \xi(x) = v_\varnothing$; so all x with $v^x = v_\varnothing$ stay in B. All other $x \in B$ are removed and distributed to new blocks w.r.t. v^x.

▶ **Theorem 5.11.** $\textsc{Split}(X/P, Y/Q, S \subseteq C \in Y/Q)$ *refines X/P by $H\chi_S^C \cdot \xi : X \to H3$.*

▶ **Lemma 5.12.** *After running* Split, *the invariants hold.*

▶ **Lemma 5.13.** *Lines $1 - 23$ in* Split *run in time $\mathcal{O}(\sum_{y \in S} |\mathsf{pred}(y)|)$.*

SPLIT$(X/P, Y/Q, S \subseteq C \in Y/Q)$

1: $\mathsf{M} := \emptyset \subseteq X/P \times H3$
2: **for** $y \in S, e \in \mathsf{pred}[y]$ **do**
3: $\quad x \xrightarrow{a} y := e$
4: $\quad B :=$ block with $x \in B \in X/P$
5: $\quad$ **if** mark_B is empty **then**
6: $\quad\quad w_C^x := \mathsf{deref} \cdot \mathsf{lastW}[e]$
7: $\quad\quad v_\emptyset := \pi_2 \cdot \mathsf{update}(\emptyset, w_C^x)$
8: $\quad\quad$ add $(B, v_\emptyset)$ to M
9: $\quad$ **if** $\mathsf{toSub}[x] = \emptyset$ **then**
10: $\quad\quad$ add $(x, \mathsf{lastW}[e])$ to mark_B
11: $\quad$ add e to $\mathsf{toSub}[x]$

(a) Collecting predecessor blocks

12: **for** $(B, v_\emptyset) \in \mathsf{M}$ **do**
13: $\quad B_{\neq\emptyset} := \emptyset \subseteq X \times H3$
14: $\quad$ **for** (x, p_C) in mark_B **do**
15: $\quad\quad \ell := \mathcal{B}_\mathrm{f}(\pi_2 \cdot \mathsf{graph})(\mathsf{toSub}[x])$
16: $\quad\quad (w_S^x, v^x, w_{C\setminus S}^x) := \mathsf{update}(\ell, \mathsf{deref}[p_C])$
17: $\quad\quad \mathsf{deref}[p_C] := w_{C\setminus S}^x$
18: $\quad\quad p_S :=$ new cell containing w_S^x
19: $\quad\quad$ **for** $e \in \mathsf{toSub}[x]$ **do** $\mathsf{lastW}[e] := p_S$
20: $\quad\quad \mathsf{toSub}[x] := \emptyset$
21: $\quad\quad$ **if** $v^x \neq v_\emptyset$ **then**
22: $\quad\quad\quad$ remove x from B
23: $\quad\quad\quad$ insert (x, v^x) into $B_{\neq\emptyset}$
24: $\quad B_1 \times \{v_1\}, \ldots, B_\ell \times \{v_\ell\} :=$
$\quad\quad$ group $B_{\neq\emptyset}$ by $\pi_2 : X \times H3 \to H3$
25: $\quad$ insert $B_1, \ldots, B_\ell :=$ into X/P

(b) Splitting predecessor blocks

Figure 5 Refining X/P w.r.t $\chi_C^S : Y \to 3$ and Y/Q along $\xi : X \to HY$.

▶ **Lemma 5.14.** *For $S_i \subseteq C_i \in Y/Q_i, 0 \leq i < k$, with $2 \cdot |S_i| \leq |C_i|$ and $Q_{i+1} = \ker\langle \kappa_{Q_i}, \chi_{S_i}^{C_i} \rangle$:*

1. *For each $y \in Y$, $|\{i < k \mid y \in S_i\}| \leq \log_2 |Y|$.*
2. SPLIT$(S_i \subseteq C_i \in Y/Q_i)$ *for all $0 \leq i < k$ takes at most $\mathcal{O}(|E| \cdot \log |Y|)$ time in total.*

Bringing Sections 3, 4, and 5 together, take a coalgebra $\xi : X \to HX$ for a zippable Set-functor H with a given refinement interface where init and update run in linear and comparison in constant time. Instantiate Algorithm 3.4 with the select routine from Theorem 3.3.1, making $q_{i+1} = k_{i+1} \cdot \kappa_{P_i} = \chi_{S_i}^{C_i}$ with $2 \cdot |S| \leq |C|$, $S_i, C_i \subseteq X$. Replace line 4 by

$$X/P_{i+1} = \text{SPLIT}(X/P_i, X/Q_i, S_i \subseteq C_i). \tag{4.1'}$$

By Theorem 5.11 this is equivalent to (4.1), and hence, by Theorem 4.12, to the original line 4, since $\chi_{S_i}^{C_i}$ respects compound blocks. By Lemmas 5.10 and 5.14.2, we have

▶ **Theorem 5.15.** *The above instance of Algorithm 3.4 computes the quotient modulo behavioural equivalence of a given coalgebra in time $\mathcal{O}((m+n) \cdot \log n)$, for $m = |E|$, $n = |X|$.*

▶ **Example 5.16.**

1. For $H = \mathcal{I} \times \mathcal{P}_\mathrm{f}$, we obtain the classical Paige-Tarjan algorithm [25] (with initial partition $\mathcal{I}$), with the same complexity $\mathcal{O}((m+n) \cdot \log n)$.
2. For $HX = \mathcal{I} \times \mathbb{R}^{(X)}$, we solve Markov chain lumping with an initial partition $\mathcal{I}$ in time $\mathcal{O}((m+n) \cdot \log n)$, like the best known algorithm (Valmari and Franceschinis [32]).
3. For infinite A, we need to decompose the functor $\mathcal{P}_\mathrm{f}(A \times (-))$ for labelled transition systems into $\mathcal{P}_\mathrm{f}$ and $A \times (-)$, and thus obtain run time $\mathcal{O}(m \cdot \log m)$, like in [10] but slower than Valmari's $\mathcal{O}(m \cdot \log n)$ [31]. For fixed finite A, we run in time $\mathcal{O}(m \log n)$.
4. Hopcroft's classical automata minimization [14] is obtained by $HX = 2 \times X^A$, with running time $\mathcal{O}(n \cdot \log n)$ for fixed alphabet A. For non-fixed A the best known complexity is $\mathcal{O}(|A| \cdot n \cdot \log n)$ [12, 21]. By using decomposition into $2 \times \mathcal{P}_\mathrm{f}$ and $\mathbb{N} \times (-)$ we obtain $\mathcal{O}(|A| \cdot n \cdot \log n + |A| \cdot n \cdot \log |A|)$.
5. We quotient simple (resp. general) Segala systems [30] by bisimilarity after decomposition into three sorts (cf. Theorem 4.13). The time bound $\mathcal{O}((n+m)\log(n+m))$ slightly improves on the previous bound [3].

6 Conclusions and Future Work

We have presented a generic algorithm that quotients coalgebras by behavioural equivalence. We have started from a category-theoretic procedure that works for every mono-preserving functor on a category with image factorizations, and have then developed an improved algorithm for *zippable* endofunctors on Set. Provided the given type functor can be equipped with an efficient implementation of a *refinement interface*, we have finally arrived at a concrete procedure that runs in time $\mathcal{O}((m+n)\log n)$ where m is the number of edges and n the number of nodes in a graph-based representation of the input coalgebra. We have shown that this instantiates to (minor variants of) several known efficient partition refinement algorithms: the classical Hopcroft algorithm [14] for minimization of DFAs, the Paige-Tarjan algorithm for unlabelled transition systems [25], and Valmari and Franceschinis's lumping algorithm for weighted transition systems [32]. Moreover, we obtain a new algorithm for simple Segala systems that is asymptotically faster than previous algorithms [3]. Coverage of Segala systems is based on modularity results in multi-sorted coalgebra [29].

It remains open whether our approach can be extended to, e.g., the monotone neighbourhood functor, which is not itself zippable and also does not have an obvious factorization into zippable functors. We do expect that our algorithm applies beyond weighted systems.

References

1 Jiří Adámek. Introduction to coalgebra. *Theory Appl. Categ.*, 14:157–199, 2005.
2 Jiří Adámek, Horst Herrlich, and George Strecker. *Abstract and Concrete Categories*. Wiley Interscience, 1990.
3 Christel Baier, Bettina Engelen, and Mila Majster-Cederbaum. Deciding bisimilarity and similarity for probabilistic processes. *J. Comput. Syst. Sci.*, 60:187–231, 2000.
4 Falk Bartels, Ana Sokolova, and Erik de Vink. A hierarchy of probabilistic system types. In *Coagebraic Methods in Computer Science, CMCS 2003*, volume 82 of *ENTCS*, pages 57 – 75. Elsevier, 2003.
5 Stefan Blom and Simona Orzan. A distributed algorithm for strong bisimulation reduction of state spaces. *STTT*, 7(1):74–86, 2005.
6 Peter Buchholz. Bisimulation relations for weighted automata. *Theoret. Comput. Sci.*, 393:109–123, 2008.
7 Stefano Cattani and Roberto Segala. Decision algorithms for probabilistic bisimulation. In *Concurrency Theory, CONCUR 2002*, volume 2421 of *LNCS*, pages 371–385. Springer, 2002.
8 Salem Derisavi, Holger Hermanns, and William Sanders. Optimal state-space lumping in markov chains. *Inf. Process. Lett.*, 87(6):309–315, 2003.
9 Josee Desharnais, Abbas Edalat, and Prakash Panangaden. Bisimulation for labelled markov processes. *Inf. Comput.*, 179(2):163–193, 2002.
10 Agostino Dovier, Carla Piazza, and Alberto Policriti. An efficient algorithm for computing bisimulation equivalence. *Theor. Comput. Sci.*, 311(1-3):221–256, 2004.
11 Kathi Fisler and Moshe Vardi. Bisimulation minimization and symbolic model checking. *Formal Methods in System Design*, 21(1):39–78, 2002.
12 David Gries. Describing an algorithm by Hopcroft. *Acta Informatica*, 2:97–109, 1973.
13 Heinz-Peter Gumm and Tobias Schröder. Monoid-labelled transition systems. In *Coalgebraic Methods in Computer Science, CMCS 2001*, volume 44 of *ENTCS*, pages 185–204, 2001.
14 John Hopcroft. An $n \log n$ algorithm for minimizing states in a finite automaton. In *Theory of Machines and Computations*, pages 189–196. Academic Press, 1971.

15 Dung Huynh and Lu Tian. On some equivalence relations for probabilistic processes. *Fund. Inform.*, 17:211–234, 1992.

16 Bart Jacobs. *Introduction to Coalgebras: Towards Mathematics of States and Observations.* Cambridge University Press, 2017.

17 Bart Jacobs and Jan Rutten. A tutorial on (co)algebras and (co)induction. *Bull. EATCS*, 62:222–259, 1997.

18 Paris C. Kanellakis and Scott A. Smolka. CCS expressions, finite state processes, and three problems of equivalence. *Inf. Comput.*, 86(1):43–68, 1990.

19 Joost-Pieter Katoen, Tim Kemna, Ivan Zapreev, and David Jansen. Bisimulation minimisation mostly speeds up probabilistic model checking. In *Tools and Algorithms for the Construction and Analysis of Systems, TACAS 2007*, volume 4424 of *LNCS*, pages 87–101. Springer, 2007.

20 Bartek Klin. Structural operational semantics for weighted transition systems. In Jens Palsberg, editor, *Semantics and Algebraic Specification: Essays Dedicated to Peter D. Mosses on the Occasion of His 60th Birthday*, volume 5700 of *LNCS*, pages 121–139. Springer, 2009.

21 Timo Knuutila. Re-describing an algorithm by Hopcroft. *Theor. Comput. Sci.*, 250:333 – 363, 2001.

22 Barbara König and Sebastian Küpper. Generic partition refinement algorithms for coalgebras and an instantiation to weighted automata. In *Theoretical Computer Science, IFIP TCS 2014*, volume 8705 of *LNCS*, pages 311–325. Springer, 2014.

23 Kim Guldstrand Larsen and Arne Skou. Bisimulation through probabilistic testing. *Inf. Comput.*, 94:1–28, 1991.

24 Robin Milner. *A Calculus of Communicating Systems*, volume 92 of *LNCS*. Springer, 1980.

25 Robert Paige and Pobert E. Tarjan. Three partition refinement algorithms. *SIAM J. Comput.*, 16(6):973–989, 1987.

26 David Park. Concurrency and automata on infinite sequences. In *Theoretical Computer Science, 5th GI-Conference*, volume 104 of *LNCS*, pages 167–183. Springer, 1981.

27 Francesco Ranzato and Francesco Tapparo. Generalizing the Paige-Tarjan algorithm by abstract interpretation. *Inf. Comput.*, 206:620–651, 2008.

28 Jan Rutten. Universal coalgebra: a theory of systems. *Theor. Comput. Sci.*, 249:3–80, 2000.

29 Lutz Schröder and Dirk Pattinson. Modular algorithms for heterogeneous modal logics via multi-sorted coalgebra. *Math. Struct. Comput. Sci.*, 21(2):235–266, 2011.

30 Roberto Segala. *Modelling and Verification of Randomized Distributed Real-Time Systems.* PhD thesis, Massachusetts Institute of Technology, 1995.

31 Antti Valmari. Bisimilarity minimization in $\mathcal{O}(m \log n)$ time. In *Applications and Theory of Petri Nets, PETRI NETS 2009*, volume 5606 of *LNCS*, pages 123–142. Springer, 2009.

32 Antti Valmari and Giuliana Franceschinis. Simple $\mathcal{O}(m \log n)$ time Markov chain lumping. In *Tools and Algorithms for the Construction and Analysis of Systems, TACAS 2010*, volume 6015 of *LNCS*, pages 38–52. Springer, 2010.

33 Johann van Benthem. *Modal Correspondence Theory.* PhD thesis, Universiteit van Amsterdam, 1977.

34 Ron van der Meyden and Chenyi Zhang. Algorithmic verification of noninterference properties. In *Views on Designing Complex Architectures, VODCA 2006*, volume 168 of *ENTCS*, pages 61–75. Elsevier, 2007.

35 James Worrell. On the final sequence of a finitary set functor. *Theor. Comput. Sci.*, 338:184–199, 2005.

36 Lijun Zhang, Holger Hermanns, Friedrich Eisenbrand, and David Jansen. Flow Faster: Efficient decision algorithms for probabilistic simulations. *Log. Meth. Comput. Sci.*, 4(4), 2008.

The Complexity of Flat Freeze LTL*

Benedikt Bollig[1], Karin Quaas[2], and Arnaud Sangnier[3]

1 **CNRS, LSV, ENS Paris-Saclay, Université Paris-Saclay, France**
2 **Universität Leipzig, Germany**
3 **IRIF, Université Paris Diderot, France**

Abstract

We consider the model-checking problem for freeze LTL on one-counter automata (OCAs). Freeze LTL extends LTL with the freeze quantifier, which allows one to store different counter values of a run in registers so that they can be compared with one another. As the model-checking problem is undecidable in general, we focus on the flat fragment of freeze LTL, in which the usage of the freeze quantifier is restricted. Recently, Lechner et al. showed that model checking for flat freeze LTL on OCAs with binary encoding of counter updates is decidable and in 2NEXPTIME. In this paper, we prove that the problem is, in fact, NEXPTIME-complete no matter whether counter updates are encoded in unary or binary. Like Lechner et al., we rely on a reduction to the reachability problem in OCAs with parameterized tests (OCAPs). The new aspect is that we simulate OCAPs by alternating two-way automata over words. This implies an exponential upper bound on the parameter values that we exploit towards an NP algorithm for reachability in OCAPs with unary updates. We obtain our main result as a corollary.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases one-counter automata, freeze LTL, model checking

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.33

1 Introduction

One-counter automata (OCAs) are a simple yet fundamental computational model operating on a single counter that ranges over the non-negative integers. Albeit being a classical model, OCAs are in the focus of ongoing research in the verification community (cf., for example, [22, 19, 26, 9, 18, 20, 23]). A large body of work is devoted to model checking OCAs, i.e., the question whether all runs of a given OCA satisfy a temporal-logic specification. A natural way to model runs of OCAs is in terms of *data words*, i.e., words where every position carries two pieces of information: a set of propositions from a finite alphabet, and a datum from an infinite alphabet. In our case, the datum represents the current counter value. Now, reasoning about the sequence of propositions that is produced by a run amounts to model checking OCAs against classical temporal logics like linear-time temporal logic (LTL) and computation-tree logic (CTL) [20, 19, 18, 21]. But it is natural to also reason about the unboundedly many counter values that may occur. Several formalisms have been introduced that can handle infinite alphabets, including variants or extensions of monadic second-order logic, LTL, and CTL [10, 12, 3, 15]. Freeze LTL is an extension of LTL that allows one to remember, in terms of the *freeze quantifier*, certain counter values for later comparison in a run of the OCA under consideration (cf. [16, 27, 11, 10]). Unfortunately, satisfiability and model checking OCAs against freeze LTL are undecidable [10, 12].

* This work has been partly supported by the ANR research program PACS (ANR-14-CE28-0002) and by DFG, project QU 316/1-2.

In this paper, we study model checking of OCAs against formulas in *flat freeze LTL*, a fragment of freeze LTL that restricts the freeze quantifier, but allows for unlimited usage of comparisons. A typical property definable in flat freeze LTL is "there exists a counter value that occurs infinitely often". Moreover, the negation of many natural freeze LTL specifications can be expressed in flat freeze LTL. The approach of restricting the syntax of a temporal logic to its flat fragment has also been pursued in [7] for Constraint LTL, and in [4] for MTL.

Demri and Sangnier [13] reduced model checking OCAs against flat freeze LTL to reachability in OCAs *with parameterized tests* (OCAPs). In an OCAP, counter values may be compared with parameters whose values are arbitrary but fixed. Reachability asks whether a given state can be reached under *some* parameter instantiation. Essentially, the translation of a flat freeze LTL formula into an OCAP interprets every freeze quantifier as a parameter, whose value can be compared with counter values arbitrarily often. Decidability of the reachability problem for OCAPs, however, was left open. Recently, Lechner et al. proved decidability by a reduction to satisfiability in Presburger arithmetic [26]. As a corollary, they obtain a 2NEXPTIME upper bound for model checking OCAs against formulas in flat freeze LTL, assuming that counter updates in OCAs are encoded in binary.

Our main technical contribution is an improvement of the result by Lechner et al. We proceed in two main steps. First, we show that reachability for OCAPs (with unary counter updates) can be reduced to non-emptiness of alternating two-way (finite) automata. Interestingly, alternating two-way automata have already been used as an algorithmic framework for game-based versions of pushdown processes [24, 6], one-counter processes [28], and timed systems [1]. Our link already implies decidability of both reachability for OCAPs (in PSPACE when counter updates are unary) and model checking OCAs against flat freeze LTL (in EXPSPACE). However, we can go further. First, we deduce an exponential upper bound on the largest parameter value needed for an accepting run in the given OCAP. Exploiting this bound and a technique by Galil [17], we show that, actually, reachability for OCAPs is in NP. As a corollary, we obtain a NEXPTIME upper bound for model checking OCAs against flat freeze LTL. Using a result from [18], we can also show NEXPTIME-hardness, which establishes the precise complexity of the model-checking problem. Our result applies no matter whether counter updates in the given OCA are encoded in unary or binary.

Outline. In Section 2, we define OCAs, (flat) freeze LTL, and the corresponding model-checking problems. Section 3 is devoted to reachability in OCAPs, which is at the heart of our model-checking procedures. The reduction of model checking to reachability in OCAPs is given in Section 4, where we also present lower bounds. We conclude in Section 5.

2 Preliminaries

2.1 One-Counter Automata with Parameterized Tests

We start by defining one-counter automata *with all extras* such as parameters and succinct encodings of updates. Well-known subclasses are then identified as special cases.

For the rest of this paper, we fix a countably infinite set $\mathbb{P}$ of *propositions*, which will label states of a one-counter automaton. Transitions of an automaton may perform tests to compare the current counter value with zero or with a parameter. Thus, for a finite set $\mathcal{X}$ of parameters, we let $\mathrm{Tests}(\mathcal{X}) = \{\mathsf{zero?}\} \cup \{<x, =x, >x \mid x \in \mathcal{X}\}$.

▶ **Definition 1.** A *succinct one-counter automaton with parameterized tests (SOCAP)* is a tuple $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta, \mu)$ where Q is a finite set of *states*, $\mathcal{X}$ is a finite set of *parameters*

ranging over $\mathbb{N}$, $q_{\mathsf{in}} \in Q$ is the *initial state*, $\Delta \subseteq Q \times (\mathbb{Z} \cup \mathrm{Tests}(\mathcal{X})) \times Q$ is a finite set of *transitions*, and $\mu : Q \to 2^{\mathbb{P}}$ maps each state to a *finite* set of propositions. We define the size of $\mathcal{A}$ as $|\mathcal{A}| = |Q| + |\mathcal{X}| + \sum_{q \in Q} |\mu(q)| + |\Delta| \cdot \max\big(\{1\} \cup \big\{\log(|z|) \mid (q, z, q') \in \Delta \cap (Q \times (\mathbb{Z} \setminus \{0\}) \times Q)\big\}\big)$.

Thus, transitions of a SOCAP either compare the counter value with zero or a parameter value, or increment/decrement the counter by a value $z \in \mathbb{Z}$, which is encoded in binary.

Let $\mathcal{C}_{\mathcal{A}} := Q \times \mathbb{N}$ be the set of *configurations* of $\mathcal{A}$. In a configuration $(q, v) \in \mathcal{C}_{\mathcal{A}}$, the first component q is the current state and v is the current counter value which is always non-negative. The semantics of $\mathcal{A}$ is given w.r.t. a *parameter instantiation* $\gamma : \mathcal{X} \to \mathbb{N}$ in terms of a global transition relation $\longrightarrow_\gamma \subseteq \mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{A}}$. For two configurations (q, v) and (q', v'), we have $(q, v) \longrightarrow_\gamma (q', v')$ if there is $(q, \mathsf{op}, q') \in \Delta$ such that one of the following holds:

- $\mathsf{op} \in \mathbb{Z}$ and $v' = v + \mathsf{op}$,
- $\mathsf{op} = \mathsf{zero?}$ and $v = v' = 0$, or
- $\mathsf{op} = \bowtie x$ and $v = v'$ and $v \bowtie \gamma(x)$, for some $\bowtie \in \{<, =, >\}$.

A γ-*run* of $\mathcal{A}$ is a finite or infinite sequence $\rho = (q_0, v_0) \longrightarrow_\gamma (q_1, v_1) \longrightarrow_\gamma \cdots$ of global transitions (a run may consist of one single configuration (q_0, v_0)). We say that ρ is *initialized* if $q_0 = q_{\mathsf{in}}$ and $v_0 = 0$.

We identify some well-known special cases of the general model. Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta, \mu)$ be a SOCAP. We say that

- $\mathcal{A}$ is a **one-counter automaton with parameterized tests (OCAP)** if all transition labels are among $\{+1, 0, -1\} \cup \mathrm{Tests}(\mathcal{X})$ (i.e., counter updates are *unary*);
- $\mathcal{A}$ is a **succinct one-counter automaton (SOCA)** if $\mathcal{X} = \emptyset$;
- $\mathcal{A}$ is a **one-counter automaton (OCA)** if $\mathcal{X} = \emptyset$ and, moreover, all transition labels are among $\{+1, 0, -1\} \cup \{\mathsf{zero?}\}$.

Note that, when $\mathcal{X} = \emptyset$ (i.e., in the case of a SOCA or OCA), we may omit $\mathcal{X}$ and simply refer to $(Q, q_{\mathsf{in}}, \Delta, \mu)$. Also, the global transition relation does not depend on a parameter instantiation anymore so that we may just write $\longrightarrow$ instead of $\longrightarrow_\gamma$. Similarly, for reachability problems (see below), μ is irrelevant so that it can be omitted, too.

One of the most fundamental decision problems we can consider is whether a given state $q_f \in Q$ is reachable. Given some parameter instantiation γ, we say that q_f is γ-*reachable* if there is an initialized run $(q_0, v_0) \longrightarrow_\gamma \cdots \longrightarrow_\gamma (q_n, v_n)$ such that $q_n = q_f$. Moreover, q_f is *reachable*, written $\mathcal{A} \models \mathit{Reach}(q_f)$, if it is γ-reachable for some γ. Now, for a class $\mathcal{C} \in \{\mathrm{SOCAP}, \mathrm{OCAP}, \mathrm{SOCA}, \mathrm{OCA}\}$, the *reachability problem* is defined as follows:

$\mathcal{C}$-Reachability	
Input:	$\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta) \in \mathcal{C}$ and $q_f \in Q$
Question:	Do we have $\mathcal{A} \models \mathit{Reach}(q_f)$?

Towards the *Büchi problem* (or *repeated reachability*), we write $\mathcal{A} \models \mathit{Reach}^\omega(q_f)$ if there are a parameter instantiation γ and an infinite initialized γ-run $(q_0, v_0) \longrightarrow_\gamma (q_1, v_1) \longrightarrow_\gamma \cdots$ such that $q_i = q_f$ for infinitely many $i \in \mathbb{N}$. The Büchi problem asks whether some state

from a given set $F \subseteq Q$ can be visited infinitely often:

$\mathcal{C}$-Büchi	
Input:	$\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta) \in \mathcal{C}$ and $F \subseteq Q$
Question:	Do we have $\mathcal{A} \models \mathit{Reach}^{\omega}(q_f)$ for some $q_f \in F$?

It is known that OCA-Reachability and OCA-Büchi are NLOGSPACE-complete [29, 9], and that SOCA-Reachability and SOCA-Büchi are NP-complete [22, 21] (cf. also Table 1).

2.2 Freeze LTL and Its Flat Fragment

We now define freeze LTL. To do so, we fix a countably infinite supply of registers $\mathcal{R}$.

▶ **Definition 2** (Freeze LTL). The logic *freeze LTL*, denoted by $\mathrm{LTL}^{\downarrow}$, is given by the grammar

$$\varphi \quad ::= \quad p \quad | \quad \bowtie r \quad | \quad \neg\varphi \quad | \quad \varphi \vee \varphi \quad | \quad \varphi \wedge \varphi \quad | \quad \mathsf{X}\varphi \quad | \quad \varphi \,\mathsf{U}\, \varphi \quad | \quad \varphi \,\mathsf{R}\, \varphi \quad | \quad \downarrow_r \varphi$$

where $p \in \mathbb{P}$, $r \in \mathcal{R}$, and $\bowtie \in \{<, =, >\}$.

Note that, apart from the until operator U, we also include the dual release operator R. This is convenient in proofs, as it allows us to assume formulas to be in negation normal form. We also use common abbreviations such as $\varphi \to \psi$ for $\neg\varphi \vee \psi$, $\mathsf{F}\varphi$ for $\mathsf{true}\,\mathsf{U}\,\varphi$ (with $\mathsf{true} = p \vee \neg p$ for some proposition p), and $\mathsf{G}\varphi$ for $\neg\mathsf{F}\neg\varphi$.

We call $\downarrow_r$ the *freeze quantifier*. It stores the current counter value in register r. Atomic formulas of the form $\bowtie r$, called *register tests*, perform a comparison of the current counter value with the contents of r, provided that they are in the scope of a freeze quantifier. Actually, inequalities of the form $<r$ and $>r$ were not present in the original definition of $\mathrm{LTL}^{\downarrow}$, but our techniques will take care of them without extra cost or additional technical complication. Classical LTL is obtained as the fragment of $\mathrm{LTL}^{\downarrow}$ that uses neither the freeze quantifier nor register tests.

A formula $\varphi \in \mathrm{LTL}^{\downarrow}$ is interpreted over an infinite sequence $w = (P_0, v_0)(P_1, v_1)\ldots \in (2^{\mathbb{P}} \times \mathbb{N})^{\omega}$ with respect to a position $i \in \mathbb{N}$ and a register assignment $\nu : \mathcal{R} \to \mathbb{N}$. The satisfaction relation $w, i \models_{\nu} \varphi$ is inductively defined as follows:

- $w, i \models_{\nu} p$ if $p \in P_i$,
- $w, i \models_{\nu} \bowtie r$ if $v_i \bowtie \nu(r)$,
- $w, i \models_{\nu} \mathsf{X}\varphi$ if $w, i+1 \models_{\nu} \varphi$,
- $w, i \models_{\nu} \varphi_1 \,\mathsf{U}\, \varphi_2$ if there is $j \geq i$ such that $w, j \models_{\nu} \varphi_2$ and $w, k \models_{\nu} \varphi_1$ for all $k \in \{i, \ldots, j-1\}$,
- $w, i \models_{\nu} \varphi_1 \,\mathsf{R}\, \varphi_2$ if one of the following holds: (i) $w, k \models_{\nu} \varphi_2$ for all $k \geq i$, or (ii) there is $j \geq i$ such that $w, j \models_{\nu} \varphi_1$ and $w, k \models_{\nu} \varphi_2$ for all $k \in \{i, \ldots, j\}$,
- $w, i \models_{\nu} \downarrow_r \varphi$ if $w, i \models_{\nu[r \mapsto v_i]} \varphi$, where the register assignment $\nu[r \mapsto v_i]$ maps r to v_i and coincides with ν on all other registers.

Negation, disjunction, and conjunction are interpreted as usual.

Let φ be a *sentence*, i.e., every subformula of φ of the form $\bowtie r$ is in the scope of a freeze quantifier $\downarrow_r$. Then, the initial register assignment is irrelevant, and we simply write $w \models \varphi$ if $w, 0 \models_{\nu} \varphi$ (with ν arbitrary). Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta, \mu)$ be a SOCAP and let $\rho = (q_0, v_0) \longrightarrow_{\gamma} (q_1, v_1) \longrightarrow_{\gamma} \cdots$ be an infinite run of $\mathcal{A}$. We say that ρ satisfies φ, written $\rho \models \varphi$, if $(\mu(q_0), v_0)(\mu(q_1), v_1)\ldots \models \varphi$. Moreover, we write $\mathcal{A} \models_{\exists} \varphi$ if there *exist* γ and an infinite initialized γ-run ρ of $\mathcal{A}$ such that $\rho \models \varphi$.

Table 1 Old and new results.

	Reachability/ Büchi	MC(LTL)	MC(flatLTL$^{\downarrow}$)
OCA	NLOGSPACE-complete [29] /[9]	PSPACE-complete (e.g., [21])	NEXPTIME-complete Theorem 21
SOCA	NP-complete [22] /[21]	PSPACE-complete [18]	NEXPTIME-complete Theorem 21

Model checking for a class $\mathcal{C}$ of SOCAPs and a logic $\mathcal{L} \subseteq \text{LTL}^{\downarrow}$ is defined as follows:

$\mathcal{C}$-MC($\mathcal{L}$)	
Input:	$\mathcal{A} \in \mathcal{C}$ and a sentence $\varphi \in \mathcal{L}$
Question:	Do we have $\mathcal{A} \models_{\exists} \varphi$?

Note that, following [12, 26], we study the existential version of the model-checking problem ("Is there some run satisfying the formula?"). The reason is that the flat fragment that we consider next is not closed under negation, and the negation of many useful freeze LTL formulas are actually *flat* (cf. Example 4 below).

Unfortunately, OCA-MC(LTL$^{\downarrow}$) is undecidable [12], even if formulas use only one register. This motivates the study of the flat fragment of LTL$^{\downarrow}$, restricting the usage of the freeze quantifier [12]. Essentially, it is no longer possible to overwrite a register unboundedly often.

▶ **Definition 3** (Flat Freeze LTL). The fragment flatLTL$^{\downarrow}$ of LTL$^{\downarrow}$ contains a formula φ if, for every occurrence of a subformula $\psi = \varphi_1 \,\mathsf{U}\, \varphi_2$ (respectively, $\psi = \varphi_2 \,\mathsf{R}\, \varphi_1$) in φ, the following hold: (i) If the occurrence of ψ is in the scope of an even number of negations, then the freeze quantifier does not occur in φ_1, and (ii) if it is in the scope of an odd number of negations, then the freeze quantifier does not occur in φ_2.

▶ **Example 4.** An example flatLTL$^{\downarrow}$ formula is $\mathsf{F}{\downarrow_r}\mathsf{G}\big((<r) \vee (=r)\big)$, saying that a run takes only finitely many different counter values. The formula $\varphi = \mathsf{G}{\downarrow_r}\big(\mathit{req} \rightarrow \mathsf{F}(\mathit{serve} \wedge (=r))\big)$ (from [26]) says that every request is eventually served with the same ticket. Note that φ is *not* in flatLTL$^{\downarrow}$, but its negation $\neg\neg\big(\mathsf{true}\,\mathsf{U}\,\neg{\downarrow_r}\big(\mathit{req} \rightarrow \mathsf{F}(\mathit{serve} \wedge (=r))\big)\big)$ is: The until lies in the scope of an even number of negations, and the freeze quantifier is in the second argument of the until.

In [13], it has been shown that OCA-MC(flatLTL$^{\downarrow}$) can be reduced to OCAP-Büchi, but decidability of the latter was left open (positive results were only obtained for a restricted fragment of flatLTL$^{\downarrow}$). Recently, Lechner et al. showed that OCAP-Büchi is decidable [26]. In fact, they establish that SOCA-MC(flatLTL$^{\downarrow}$) is in 2NEXPTIME.

Our main result states that the problems OCA-MC(flatLTL$^{\downarrow}$) and SOCA-MC(flatLTL$^{\downarrow}$) are both NEXPTIME-complete. A comparison with known results can be found in Table 1. The proof outline is depicted in Figure 1. Essentially, we also rely on a reduction of SOCA-MC(flatLTL$^{\downarrow}$) to OCAP-Reachability, and the main challenge is to establish the precise complexity of the latter. In Section 3, we reduce OCAP-Reachability to non-emptiness of alternating two-way automata over infinite words, which is then exploited to prove

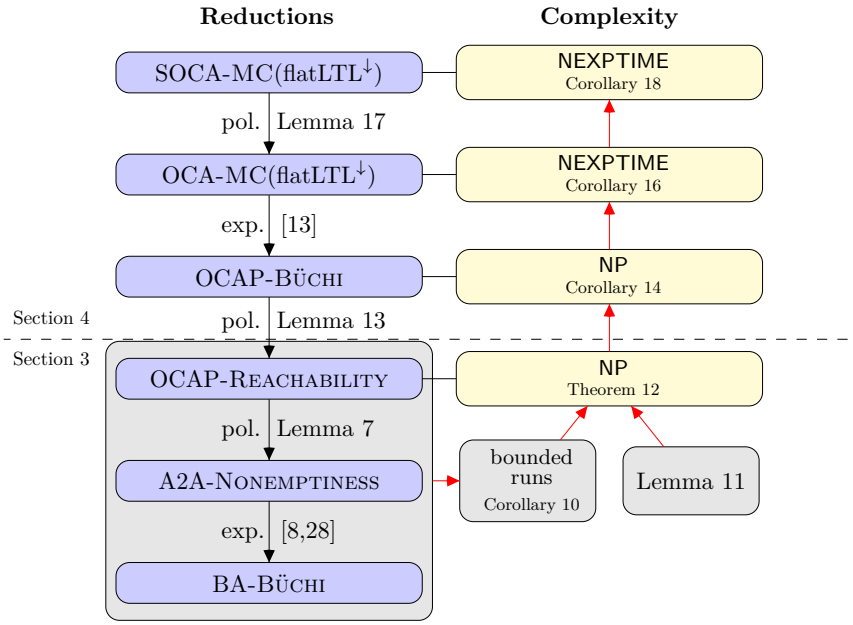


Figure 1 Proof structure for upper bounds.

an NP upper bound. The reductions from SOCA-MC(flatLTL$^{\downarrow}$) to OCAP-Reachability themselves are given in Section 4, which also contains our main results.

3 OCAP-Reachability is NP-complete

In this section, we reduce OCAP-Reachability to non-emptiness of alternating two-way automata (A2As) over infinite words. While this already implies that OCAP-Reachability is in PSPACE, we then go a step further and show that the problem is in NP.

3.1 From OCAPs to Alternating Two-Way Automata

We fix an OCAP $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta)$. The main idea is to encode a parameter instantiation $\gamma : \mathcal{X} \to \mathbb{N}$ as a word over the alphabet $\Sigma = \mathcal{X} \cup \{\square\}$, where $\square$ is a fresh symbol. A word $w = a_0 a_1 a_2 \cdots \in \Sigma^\omega$, with $a_i \in \Sigma$, is called a *parameter word* (over $\mathcal{X}$) if $a_0 = \square$ and, for all $x \in \mathcal{X}$, there is exactly one position $i \in \mathbb{N}$ such that $a_i = x$. In other words, w starts with $\square$ and every parameter occurs exactly once. Then, w determines a parameter instantiation $\gamma_w : \mathcal{X} \to \mathbb{N}$ as follows: If $x = a_i$, then $\gamma_w(x) = |a_1 \cdots a_{i-1}|_\square$ where $|a_1 \cdots a_{i-1}|_\square$ denotes the number of occurrences of $\square$ in $a_1 \cdots a_{i-1}$ (note that we start at the second position of w). For example, given $\mathcal{X} = \{x_1, x_2, x_3\}$, both $w = \square x_2 \square\square x_1 x_3 \square^\omega$ and $w' = \square x_2 \square\square x_3 x_1 \square^\omega$ are parameter words with $\gamma_w = \gamma_{w'} = \{x_1 \mapsto 2, x_2 \mapsto 0, x_3 \mapsto 2\}$. Note that, for every parameter instantiation γ, there is at least one parameter word w such that $\gamma_w = \gamma$. Let $\mathcal{W}_\mathcal{X}$ denote the set of all parameter words over $\mathcal{X}$.

From $\mathcal{A}$ and a state $q_f \in Q$, we will build an A2A that accepts the set of parameter words w such that q_f is γ_w-reachable. Like a Turing machine, an A2A can read a letter, change its state, and move its head to the left or to the right (or stay at the current position). In addition, it can spawn several independent copies, for example one that goes to the left and one that goes to the right. However, unlike a Turing machine, an A2A is not allowed

to modify the input word so that its expressive power does not go beyond finite (Büchi) automata. The simulation proceeds as follows. When the OCAP increments its counter, the A2A moves to the right to the next occurrence of $\square$. To simulate a decrement, it moves to the left until it encounters the previous $\square$. To mimic the zero test, it verifies that it is currently on the first position of the word. Moreover, it will make use of the letters in $\mathcal{X}$ to simulate parameter tests. At the beginning of an execution, the A2A spawns independent copies that check whether the input word is a valid parameter word.

Let us define A2As and formalize the simulation of an OCAP. Given a finite set Y, we denote by $\mathbb{B}^+(Y)$ the set of positive Boolean formulas over Y, including `true` and `false`. A subset $Y' \subseteq Y$ *satisfies* $\beta \in \mathbb{B}^+(Y)$, written $Y' \models \beta$, if β is evaluated to true when assigning `true` to every variable in Y', and `false` to every variable in $Y \setminus Y'$. In particular, we have $\emptyset \models$ `true`. For $\beta \in \mathbb{B}^+(Y)$, let $|\beta|$ denote the size of β, defined inductively by $|\beta_1 \vee \beta_2| = |\beta_1 \wedge \beta_2| = |\beta_1| + |\beta_2| + 1$ and $|\beta| = 1$ for atomic formulas β.

▶ **Definition 5.** An *alternating two-way automaton* (A2A) is a tuple $\mathcal{T} = (S, \Sigma, s_{\mathsf{in}}, \delta, S_f)$, where S is a finite set of states, Σ is a finite alphabet, $s_{\mathsf{in}} \in S$ is the initial state, $S_f \subseteq S$ is the set of accepting states, and $\delta \subseteq S \times (\Sigma \cup \{\mathsf{first?}\}) \times \mathbb{B}^+(S \times \{+1, 0, -1\})$ is the finite transition relation[1]. A transition $(s, \mathit{test}, \beta) \in \delta$ will also be written $s \xrightarrow{\mathit{test}} \beta$. The size of $\mathcal{T}$ is defined as $|\mathcal{T}| = |S| + |\Sigma| + \sum_{(s,\mathit{test},\beta)\in\delta} |\beta|$.

While, in an OCAP, $+1$ and -1 are interpreted as *increment* and *decrement* the counter, respectively, their interpretation in an A2A is *go to the right* and *go to the left* in the input word. Moreover, 0 means *stay*. Actually, when $\delta \subseteq S \times \Sigma \times (S \times \{+1\})$, then we deal with a classical *Büchi automaton*.

A *run* of $\mathcal{T}$ on an infinite word $w = a_0 a_1 a_2 \cdots \in \Sigma^\omega$ is a rooted tree (possibly infinite, but finitely branching) whose vertices are labeled with elements in $S \times \mathbb{N}$. A node with label (s, n) represents a proof obligation that has to be fulfilled starting from state s and position n in the input word. The root of a run is labeled by $(s_{\mathsf{in}}, 0)$. Moreover, we require that, for every vertex labeled by (s, n) with $k \in \mathbb{N}$ children labeled by $(s_1, n_1), \ldots, (s_k, n_k)$, there is a transition $(s, \mathit{test}, \beta) \in \delta$ such that (i) the set $\{(s_1, n_1 - n), \ldots, (s_k, n_k - n)\} \subseteq S \times \{+1, 0, -1\}$ satisfies β, (ii) $\mathit{test} = \mathsf{first?}$ implies $n = 0$, and (iii) $\mathit{test} \in \Sigma$ implies $a_n = \mathit{test}$. Note that, similarly to an OCAP, a transition with move -1 is blocked if $n = 0$, i.e., if $\mathcal{T}$ is at the first position of the input word. A run is *accepting* if every infinite branch visits some accepting state from S_f infinitely often. The language of $\mathcal{T}$ is defined as $L(\mathcal{T}) = \{w \in \Sigma^\omega \mid \text{there exists an accepting run of } \mathcal{T} \text{ on } w\}$. The *non-emptiness problem* for A2As is to decide, given an A2A $\mathcal{T}$, whether $L(\mathcal{T}) \neq \emptyset$.

▶ **Theorem 6** ([28]). *The non-emptiness problem for A2As is in* PSPACE.

It is worth noting that [28] also uses two-wayness to simulate one-counter automata, but in a game-based setting (the latter is reflected by alternation).

Let us show how to build an A2A from an OCAP and a target state.

▶ **Lemma 7.** *Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta)$ be an OCAP and $q_f \in Q$. There is an A2A $\mathcal{T} = (S, \Sigma, s_{\mathsf{in}}, \delta, S_f)$, with $\Sigma = \mathcal{X} \cup \{\square\}$, such that $L(\mathcal{T}) = \{w \in \mathcal{W}_{\mathcal{X}} \mid q_f$ is γ_w-reachable$\}$. Moreover, $|\mathcal{T}| = \mathcal{O}(|\mathcal{A}|^2)$.*

[1] One often considers a transition *function* $\delta : S \times \Sigma \times \{\mathsf{first?}, \neg\mathsf{first?}\} \to \mathbb{B}^+(S \times \{+1, 0, -1\})$, but a relation is more convenient for us.

Proof. The states of $\mathcal{T}$ include Q (to simulate $\mathcal{A}$), a new initial state s_{in}, and some extra states, which will be introduced below.

Starting in s_{in} and at the first letter of the input word, the A2A $\mathcal{T}$ spawns several copies: $s_{\mathsf{in}} \xrightarrow{\square} (q_{\mathsf{in}}, 0) \wedge \bigwedge_{x \in \mathcal{X}} (\mathsf{search}(x), +1)$. The copy starting in q_{in} will henceforth simulate $\mathcal{A}$. Moreover, from the new state $\mathsf{search}(x)$, we will check that x occurs exactly once in the input word. This is accomplished using the transitions $\mathsf{search}(x) \xrightarrow{x} (\checkmark_x, +1)$, as well as $\mathsf{search}(x) \xrightarrow{y} (\mathsf{search}(x), +1)$ and $\checkmark_x \xrightarrow{y} (\checkmark_x, +1)$ for all $y \in \Sigma \setminus \{x\}$. Thus, $\checkmark_x$ signifies that x has been seen and must not be encountered again. Since the whole word has to be scanned, $\checkmark_x$ is visited infinitely often so that we set $\checkmark_x \in S_f$.

It remains to specify how $\mathcal{T}$ simulates $\mathcal{A}$. The underlying idea is very simple. A configuration $(q, v) \in \mathcal{C}_{\mathcal{A}}$ of $\mathcal{A}$ corresponds to the configuration/proof obligation (q, i) of $\mathcal{T}$ where position i is the $(v+1)$-th occurrence of $\square$ in the input parameter word. The simulation then proceeds as follows.

Increment/decrement: To mimic a transition $(q, +1, q') \in \Delta$, the A2A $\mathcal{T}$ simply goes to the next occurrence of $\square$ on the right hand side and enters q'. This is accomplished by several A2A-transitions: $q \xrightarrow{\square} (\mathsf{right}(q'), +1)$, $\mathsf{right}(q') \xrightarrow{x} (\mathsf{right}(q'), +1)$ for every $x \in \mathcal{X}$, and $\mathsf{right}(q') \xrightarrow{\square} (q', 0)$. Decrements are handled similarly, using states of the form $\mathsf{left}(q')$. Finally, $(q, 0, q') \in \Delta$ is translated to $q \xrightarrow{\square} (q', 0)$.

Equality test: A zero test $(q, \mathsf{zero}?, q') \in \Delta$ corresponds to $q \xrightarrow{\mathsf{first}?} (q', 0)$. To simulate a transition $(q, =x, q') \in \Delta$, the A2A spawns two copies. One goes from q to q' and stays at the current position. The other goes to the right and accepts if it sees x before hitting another $\square$-symbol. Thus, we introduce a new state $\mathsf{present}(x)$ and transitions $q \xrightarrow{\square} (q', 0) \wedge (\mathsf{present}(x), +1)$, $\mathsf{present}(x) \xrightarrow{x} \mathtt{true}$, and $\mathsf{present}(x) \xrightarrow{y} (\mathsf{present}(x), +1)$ for all $y \in \mathcal{X} \setminus \{x\}$. Note that there is *no* transition that allows $\mathcal{T}$ to read $\square$ in state $\mathsf{present}(x)$.

Inequality tests: To simulate a test of the form $>x$, we proceed similarly to the previous case. The A2A generates a branch in charge of verifying that the parameter x lies strictly to the left of the current position. Note that transitions with label $<x$ are slightly more subtle, as x has to be retrieved strictly *beyond* the next delimiter $\square$ to the right of the current position. More precisely, $(q, <x, q') \in \Delta$ translates to $q \xrightarrow{\square} (q', 0) \wedge (\mathsf{search}^+(x), +1)$. We stay in $\mathsf{search}^+(x)$ until we encounter the next occurrence of $\square$. We then go into $\mathsf{search}(x)$ (introduced at the beginning of the proof), which will be looking for an occurrence of x. Formally, we introduce $\mathsf{search}^+(x) \xrightarrow{y} (\mathsf{search}^+(x), +1)$ for all $y \in \mathcal{X} \setminus \{x\}$, and $\mathsf{search}^+(x) \xrightarrow{\square} (\mathsf{search}(x), +1)$.

In state q_f, we accept: $q_f \xrightarrow{\square} \mathtt{true}$. The only infinite branches in an accepting run are those that eventually stay in a state of the form $\checkmark_x$. Thus, we set $S_f = \{\checkmark_x \mid x \in \mathcal{X}\}$. It is not hard to see that $L(\mathcal{T}) = \{w \in \mathcal{W}_{\mathcal{X}} \mid q_f \text{ is } \gamma_w\text{-reachable}\}$. Note that $\mathcal{T}$ has a linear number of states and a quadratic number of transitions (some transitions of $\mathcal{A}$ are simulated by $\mathcal{O}(|\mathcal{X}|)$-many transitions of $\mathcal{T}$). ◀

A similar reduction takes care of Büchi reachability. Together with Theorem 6, we thus obtain the following:

▶ **Corollary 8.** OCAP-Reachability *and* OCAP-Büchi *are both in* PSPACE.

Note that we are using alternation only to a limited extent. We could also reduce reachability in OCAPs to non-emptiness of the intersection of several two-way automata. However, this would require a more complicated word encoding of parameter instantiations, which then have to include letters of the form $<x$ and $>x$.

3.2 OCAP-Reachability is in NP

We will now show that we can improve the upper bounds given in Corollary 8 to NP, which is then optimal: NP-hardness of both problems, even in the presence of a single parameter, can be proved using a straightforward reduction from the non-emptiness problem for nondeterministic two-way automata over finite words over a *unary* alphabet, which is NP-complete [17].

In a first step, we exploit our reduction from OCAPs to A2As to establish a bound on the parameter values. To solve the reachability problems, it will then be sufficient to consider parameter instantiations up to that bound. For the rest of this section, unless stated otherwise, we fix an OCAP $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta)$ and a state $q_f \in Q$.

▶ **Lemma 9.** *There is $d \in 2^{\mathcal{O}(|\mathcal{A}|^4)}$ such that the following holds: If $\mathcal{A} \models \mathit{Reach}(q_f)$, then there is a parameter instantiation $\gamma : \mathcal{X} \to \mathbb{N}$ such that $\gamma(x) \leq d$ for all $x \in \mathcal{X}$ and q_f is γ-reachable.*

Proof. According to Lemma 7, there is an A2A $\mathcal{T} = (S, \mathcal{X} \cup \{\Box\}, s_{\mathsf{in}}, \delta, S_f)$ such that $L(\mathcal{T}) = \{w \in \mathcal{W}_{\mathcal{X}} \mid q_f \text{ is } \gamma_w\text{-reachable}\}$ and $|\mathcal{T}| = \mathcal{O}(|\mathcal{A}|^2)$. By [30], there is a (nondeterministic) Büchi automaton $\mathcal{B}$ such that $L(\mathcal{B}) = L(\mathcal{T})$ and $|\mathcal{B}| \in 2^{\mathcal{O}(|\mathcal{T}|^2)}$ (cf. also [8]).

Let $d = |\mathcal{B}|$. Suppose $\mathcal{A} \models \mathit{Reach}(q_f)$. This implies that $L(\mathcal{B}) \neq \emptyset$. But then, there must be a word $u \in \Sigma^*$ such that $|u| \leq |\mathcal{B}|$ ($|u|$ denoting the length of u) and $w = u\Box^\omega \in L(\mathcal{B})$. We have that q_f is γ_w-reachable and $\gamma_w(x) \leq |\mathcal{B}| = d$ for all $x \in \mathcal{X}$. ◀

As a corollary, we obtain that it is sufficient to consider bounded runs only. For a parameter instantiation $\gamma : \mathcal{X} \to \mathbb{N}$ and a bound $d \in \mathbb{N}$, we say that a *γ-run* $\rho = (q_0, v_0) \longrightarrow_\gamma (q_1, v_1) \longrightarrow_\gamma (q_2, v_2) \longrightarrow_\gamma \cdots$ is *d-bounded* if all its counter values $v_0, v_1, \ldots$ are at most d.

▶ **Corollary 10.** *There is $d \in 2^{\mathcal{O}(|\mathcal{A}|^4)}$ such that the following holds: If $\mathcal{A} \models \mathit{Reach}(q_f)$, then there is a parameter instantiation $\gamma : \mathcal{X} \to \mathbb{N}$ such that $\gamma(x) \leq d$ for all $x \in \mathcal{X}$ and q_f is reachable within a d-bounded γ-run.*

Proof. Consider $d \in 2^{\mathcal{O}(|\mathcal{A}|^4)}$ due to Lemma 9. A standard argument in OCAs with n states is that, for reachability, it is actually sufficient to consider runs up to some counter value in $\mathcal{O}(n^3)$ [25]. We can apply the same argument here to deduce, together with Lemma 9, that parameter and counter values can be bounded by $d + \mathcal{O}(|Q|^3)$. ◀

The algorithm that solves OCAP-Reachability in NP will guess a parameter instantiation γ and a maximal counter value in binary representation (thus, of polynomial size). It then remains to determine, in polynomial time, whether the target state is reachable under this guess. To this end, we will exploit the following lemma due to Galil [17].

Let $\mathcal{A} = (Q, q_{\mathsf{in}}, \Delta)$ be an OCA, $q, q' \in Q$, and $v, v' \in \mathbb{N}$. A run $(q_0, v_0) \longrightarrow (q_1, v_1) \longrightarrow \cdots \longrightarrow (q_{n-1}, v_{n-1}) \longrightarrow (q_n, v_n)$ of $\mathcal{A}$, $n \in \mathbb{N}$, is called a *(v, v')-run* if $\min\{v, v'\} < v_i < \max\{v, v'\}$ for all $i \in \{1, \ldots, n-1\}$. In other words, all intermediate configurations have counter values strictly between v and v'.

▶ **Lemma 11** ([17]). *The following problems can be solved in polynomial time:*

Input:	An OCA $\mathcal{A} = (Q, q_{\mathsf{in}}, \Delta)$, $q, q' \in Q$, and $v, v' \in \mathbb{N}$ given in binary.
Question 1:	Is there a (v, v')-run from (q, v) to (q', v') ?
Question 2:	Is there a (v, v')-run from (q, v) to (q', v) ?

Actually, [17] uses the polynomial-time algorithms stated in Lemma 11 to show that non-emptiness of two-way automata on finite words over a unary alphabet is in NP (cf. the

proof of the corresponding lemma on page 84 of [17], where the endmarkers of a given word correspond to v and v').

We can now deduce the following result, which is an important step towards model checking, but also of independent interest.

▶ **Theorem 12.** OCAP-Reachability *is* NP*-complete.*

Sketch of proof. Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta)$ be an OCAP with $\mathcal{X} = \{x_1, \ldots, x_n\}$, and let $q_f \in Q$ be a state of $\mathcal{A}$. Without loss of generality, we consider reachability of the *configuration* $(q_f, 0)$ (since one can add a new target state and a looping decrement transition).

Our nondeterministic polynomial-time algorithm proceeds as follows. First, it guesses $d \in 2^{\mathcal{O}(|\mathcal{A}|^4)}$ due to Corollary 10 (note that the binary representation of d is of polynomial size with respect to $|\mathcal{A}|$), as well as some parameter instantiation γ satisfying $\gamma(x_i) \leq d$ for all $1 \leq i \leq n$ where each $d_i := \gamma(x_i)$ is represented in binary, too. We may assume $n \geq 1$ and that $d_0 < d_1 < d_2 < \cdots < d_n < d_{n+1}$, where $d_0 = 0$ and $d_{n+1} = d$ (otherwise, we can rename the parameters accordingly). Second, the algorithm checks that there exists a d-bounded γ-run of the form $(q_0, v_0) \longrightarrow_\gamma^* (q'_0, v_0) \longrightarrow_\gamma^* (q_1, v_1) \longrightarrow_\gamma^* (q'_1, v_1) \longrightarrow_\gamma^* \cdots \longrightarrow_\gamma^* (q_k, v_k) \longrightarrow_\gamma^* (q'_k, v_k)$ such that (letting $D = \{d_0, d_1, \ldots, d_n, d_{n+1}\}$)

1. $(q_0, v_0) = (q_{\mathsf{in}}, 0)$ and $(q'_k, v_k) = (q_f, 0)$,
2. $v_j \in D$ for all $j \in \{0, \ldots, k\}$,
3. between (q_j, v_j) and (q'_j, v_j), the counter values always equal v_j, for all $j \in \{0, \ldots, k\}$,
4. (strictly) between (q'_j, v_j) and (q_{j+1}, v_{j+1}), the counter values are always different from the values in D, for all $j \in \{0, \ldots, k-1\}$.

Note that we can assume $k \leq |Q| \cdot (|\mathcal{X}| + 2)$, since in every longer run, the exact same configuration is encountered at least twice.

Hence, to check whether there exists a finite initialized d-bounded γ-run $(q_{\mathsf{in}}, 0) \longrightarrow_\gamma^* (q_f, 0)$, it is sufficient to guess $2k$ configurations, where $k \leq |Q| \cdot (|\mathcal{X}| + 2)$, and to verify that these configurations contribute to constructing a run of the form described above. This can be checked in polynomial time: The case (3) is obvious, and (4) is due to Lemma 11. ◀

4 From OCAP-Reachability to SOCA-Model-Checking

This section is dedicated to model checking OCAs against flatLTL$^{\downarrow}$. In Section 4.1, we establish a couple of reductions from SOCA-MC(flatLTL$^{\downarrow}$) down to OCAP-Reachability, which allow us to conclude that the former problem is in NEXPTIME. Then, in Section 4.2, we provide a matching lower bound.

4.1 Upper Bounds

First, we use the fact that OCAP-Reachability is in NP (Theorem 12) to show that OCAP-Büchi is in NP, too:

▶ **Lemma 13.** OCAP-Büchi *is polynomial-time reducible to* OCAP-Reachability.

Proof. Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta)$ be an OCAP with $\mathcal{X} = \{x_1, \ldots, x_n\}$ (without loss of generality, we suppose $n \geq 1$), and let $F \subseteq Q$. It is sufficient to give an algorithm that determines whether a particular state $q_f \in F$ can be repeated infinitely often. Thus, we construct an OCAP $\mathcal{A}'$, with a distinguished state $\hat{s}$, such that $\mathcal{A} \models \mathit{Reach}^\omega(q_f)$ iff $\mathcal{A}' \models \mathit{Reach}(\hat{s})$.

To this end, we first define an OCA $\mathcal{M}$, which is obtained from $\mathcal{A}$ by (i) removing all transitions whose labels are of the form $\mathsf{zero}?$, $=x$, or $<x$, for some $x \in \mathcal{X}$, and (ii) replacing all transition labels $>x$ (for some $x \in \mathcal{X}$) by 0. Then, we compute the set $T \subseteq Q$ of states

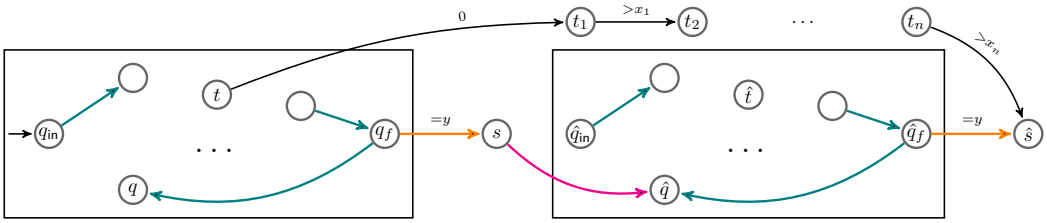


Figure 2 The OCAP $\mathcal{A}'$ constructed from $\mathcal{A}$ in the proof of Lemma 13.

t of $\mathcal{M}$ from which there is an infinite run starting in $(t, 0)$ and visiting q_f infinitely often. This can be done in NLOGSPACE [9].

Now, we obtain the desired OCAP $\mathcal{A}' = (Q', \mathcal{X}', q_{\text{in}}, \Delta')$ as follows. The set of states is $Q' = Q \uplus \{\hat{q} \mid q \in Q\} \uplus \{s, \hat{s}, t_1, \ldots, t_n\}$. That is, we introduce a copy $\hat{q}$ for every $q \in Q$ as well as fresh states $s, \hat{s}, t_1, \ldots t_n$. Recall that $\hat{s}$ will be the target state of the reachability problem we reduce to. Moreover, $\mathcal{X}' = \mathcal{X} \uplus \{y\}$ where y is a fresh parameter. It remains to define the transition relation Δ', which includes Δ as well as some new transitions (cf. Figure 2). Essentially, there are two cases to consider:

(1) First, q_f may be visited infinitely often in $\mathcal{A}$, and infinitely often along with the same counter value. To take this case into account, we use the new parameter y and a new transition $(q_f, {=}y, s)$, which allows $\mathcal{A}'$ to "store" the current counter value in y. From s, we then enter and simulate the copy of $\mathcal{A}$, i.e., we introduce transitions $(s, \mathsf{op}, \hat{q})$ for all $(q_f, \mathsf{op}, q) \in \Delta$, as well as $(\hat{q}_1, \mathsf{op}, \hat{q}_2)$ for all $(q_1, \mathsf{op}, q_2) \in \Delta$. If, in the copy, $\hat{q}_f$ is visited along with the value stored in y, we may thus enter $\hat{s}$.

(2) Suppose, on the other hand, that a γ-run ρ of $\mathcal{A}$ visits q_f infinitely often, but not infinitely often with the same counter value. If ρ visits some counter value $v \leq \max\{\gamma(x) \mid x \in \mathcal{X}\}$ infinitely often, then it necessarily contains a subrun of the form $(q, v) \longrightarrow_\gamma^* (q_f, v') \longrightarrow_\gamma^* (q, v)$ for some $q \in Q$ and $v' \in \mathbb{N}$. But then, there is an infinite γ-run that visits (q_f, v') infinitely often so that case (1) above applies. Otherwise, ρ will eventually stay strictly above $\max\{\gamma(x) \mid x \in \mathcal{X}\}$. Thus, we add the following transitions to Δ', which will allow $\mathcal{A}'$ to move from $t \in T$ to $\hat{s}$ provided the counter value is greater than the maximal parameter value: $(t, 0, t_1)$ for all $t \in T$, as well as $(t_1, {>}x_1, t_2), (t_2, {>}x_2, t_3), \ldots, (t_n, {>}x_n, \hat{s})$. ◀

From Theorem 12 and Lemma 13, we obtain the exact complexity of OCAP-Büchi (the lower bound is by a straightforward reduction from OCAP-Reachability).

▶ **Corollary 14.** OCAP-Büchi *is* NP*-complete.*

The link between OCA-MC(flatLTL$^{\downarrow}$) and OCAP-Büchi is due to [13]:

▶ **Lemma 15** ([13])**.** *Let $\mathcal{A}$ be an OCA and $\varphi \in$ flatLTL$^{\downarrow}$ be a sentence. One can compute, in exponential time, an OCAP $\mathcal{A}' = (Q, \mathcal{X}, q_{\text{in}}, \Delta)$ (of exponential size) and a set $F \subseteq Q$ such that $\mathcal{A} \models_\exists \varphi$ iff $\mathcal{A}' \models \mathit{Reach}^\omega(q_f)$ for some $q_f \in F$.*

Actually, the construction from [13] can be easily extended to handle register tests of the form $<r$ and $>r$. However, the restriction to flatLTL$^{\downarrow}$ is crucial. It allows one to assume that a register is written at most once so that, in the OCAP, it can be represented as a parameter. In particular, $\mathcal{X}$ is the set of registers that occur in φ. From Corollary 14 and Lemma 15, we now deduce our first model-checking result:

▶ **Corollary 16.** OCA-MC(flatLTL$^{\downarrow}$) *is in* NEXPTIME.

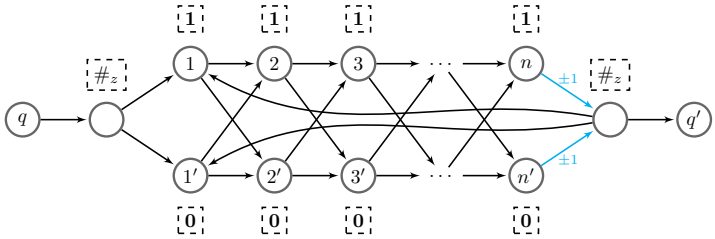


Figure 3 The OCA-gadget for simulating a SOCA-transition (q, z, q') with binary update z. The new propositions are depicted in dashed boxes. States $1, \ldots, n, 1', \ldots, n'$, where $n = \mathit{bits}(z)$, represent the bits needed to encode z in binary. At the transitions originating from n and n', the counter is updated by $+1$ or -1, depending on whether z is positive or negative, respectively.

$$\underset{\uparrow}{\mu(q)}\,\#_6\,\mathbf{100}\,\#_6\,\underset{\uparrow}{\mathbf{010}}\,\#_6\,\mathbf{110}\,\#_6\,\mathbf{001}\,\#_6\,\mathbf{101}\,\#_6\,\underset{\uparrow}{\mathbf{011}}\,\#_6\,\underset{\uparrow}{\mu(q')}$$
$$\mathit{first}(\#_6) \qquad \mathit{suff}^{=}_{\#_6} \qquad \mathit{last}^{-}(\#_6) \qquad \mathit{last}(\#_6)$$

Figure 4 A counting sequence.

However, it turns out that SOCA model checking is no harder than OCA model checking. The reason is that succinct updates can be encoded in small LTL formulas.

▶ **Lemma 17.** SOCA-MC(flatLTL$^{\downarrow}$) *is polynomial-time reducible to* OCA-MC(flatLTL$^{\downarrow}$).

Proof. Let $\mathcal{A} = (Q, q_{\mathsf{in}}, \Delta, \mu)$ be a SOCA and $\varphi \in \text{flatLTL}^{\downarrow}$ be a sentence. Without loss of generality, we assume that φ is in negation normal form, i.e., negation is applied only to atomic propositions. This is thanks to the logical equivalences $\neg(\varphi_1 \,\mathsf{U}\, \varphi_2) \equiv \neg\varphi_1 \,\mathsf{R}\, \neg\varphi_2$ and $\neg(\varphi_1 \,\mathsf{R}\, \varphi_2) \equiv \neg\varphi_1 \,\mathsf{U}\, \neg\varphi_2$. We construct an OCA $\mathcal{A}' = (Q', q'_{\mathsf{in}}, \Delta', \mu')$ and a sentence $\varphi' \in \text{flatLTL}^{\downarrow}$ of polynomial size such that $\mathcal{A} \models_{\exists} \varphi$ iff $\mathcal{A}' \models_{\exists} \varphi'$.

Let $Z = \{z \mid (q, z, q') \in \Delta \cap (Q \times \mathbb{Z} \times Q) \text{ with } |z| \geq 2\}$ and let $\Lambda = \{\#_z \mid z \in Z\} \cup \{\mathbf{0}, \mathbf{1}\}$ be a set of fresh propositions. To obtain $\mathcal{A}'$ from $\mathcal{A}$, we replace every transition $(q, z, q') \in \Delta \cap (Q \times Z \times Q)$ by the gadget depicted in Figure 3. The gadget implements a binary counter generating sequences (of sets of propositions) of the form depicted in Figure 4, for $z = 6$. We assume that the least significant bit is on the left. To make sure that the binary counter works as requested, we define an LTL formula *Counter*.

Towards its definition, let us first introduce some notation and abbreviations. For $z \in Z$, let $\mathit{bits}(z)$ denote the number of bits needed to represent the binary encoding of $|z|$. For example, $\mathit{bits}(6) = 3$. For $i \in \{1, \ldots, \mathit{bits}(z)\}$, we let $\mathit{bit}_i(z)$ denote the i-th bit in that encoding. For example, $(\mathit{bit}_1(6), \mathit{bit}_2(6), \mathit{bit}_3(6)) = (\mathbf{0}, \mathbf{1}, \mathbf{1})$. In the following, we write Λ for $\bigvee_{\gamma \in \Lambda} \gamma$ and $\neg\Lambda$ for $\bigwedge_{\gamma \in \Lambda} \neg\gamma$. For an illustration of the following LTL formulas, we refer to Figure 4. Formula $\mathit{first}(\#_z) = \neg\Lambda \wedge \mathsf{X}\#_z$ holds right before a first delimiter symbol. Similarly, $\mathit{last}(\#_z) = \#_z \wedge \mathsf{X}\neg\Lambda$ identifies all last delimiters and $\mathit{last}^{-}(\#_z) = \#_z \wedge \mathsf{X}\big((\mathbf{0} \vee \mathbf{1}) \,\mathsf{U}\, \mathit{last}(\#_z)\big)$ all second-last delimiters. Finally, we write $\curvearrowright_z \psi$ for $\mathsf{X}^{\mathit{bits}(z)+1}\psi$.

Now, $\mathit{Counter} = \mathit{Init} \wedge \mathit{Fin} \wedge \mathit{Inc} \wedge \mathit{Exit}$ is the conjunction of the following formulas:

- *Init* says that the first binary number is always one:

$$\mathit{Init} = \bigwedge_{z \in Z} \mathsf{G}\big(\mathit{first}(\#_z) \;\rightarrow\; \mathsf{X}^2(\mathbf{1} \;\wedge\; \mathsf{X}(\mathbf{0} \,\mathsf{U}\, \#_z))\big)$$

- *Fin* says that the last value represents $|z|$:

$$Fin = \bigwedge_{z \in Z} \mathsf{G}\big(last^{-}(\#_z) \;\rightarrow\; \bigwedge_{i=1}^{bits(z)} \mathsf{X}^i\, bit_i(z)\big)$$

- *Inc* implements the increments:

$$Inc = \bigwedge_{z \in Z} \mathsf{G}\left(\begin{array}{ll} & \big(\#_z \wedge \neg last^{-}(\#_z) \wedge \neg last(\#_z)\big) \\ \rightarrow & \mathsf{X}\big((\mathbf{1} \wedge \curvearrowright_z \mathbf{0})\,\mathsf{U}\,(\mathbf{0} \wedge \curvearrowright_z \mathbf{1} \wedge \mathsf{X}\, suff_z^{=})\big)\end{array}\right)$$

  where $suff_z^{=} = \big((\mathbf{0} \wedge \curvearrowright_z \mathbf{0}) \vee (\mathbf{1} \wedge \curvearrowright_z \mathbf{1})\big)\,\mathsf{U}\,\#_z$ verifies that the suffixes (w.r.t. the current position) of the current and the following binary number coincide.
- *Exit* states that we do not loop forever in the gadget from Figure 3:

$$Exit = \bigwedge_{z \in Z} \mathsf{G}\big(first(\#_z) \;\rightarrow\; \mathsf{F}\; last(\#_z)\big)$$

Note that the gadget makes sure that, in between two delimiters $\#_z$, there are exactly $bits(z)$-many bits.

Now, we set $\varphi' = [\varphi] \wedge Counter$. Here, $[\varphi]$ simulates φ on all positions that do not contain propositions from Λ. It is inductively defined as follows:

$$[p] = p \qquad [\neg p] = \neg p \qquad [\bowtie r] = \bowtie r \qquad [\downarrow_r \psi] = \downarrow_r [\psi]$$

$$[\psi_1 \vee \psi_2] = [\psi_1] \vee [\psi_2] \qquad [\psi_1 \wedge \psi_2] = [\psi_1] \wedge [\psi_2] \qquad [\mathsf{X}\varphi] = \mathsf{X}\big(\Lambda\,\mathsf{U}\,(\neg\Lambda \wedge [\varphi])\big)$$

$$[\psi_1\,\mathsf{U}\,\psi_2] = (\neg\Lambda \rightarrow [\psi_1])\,\mathsf{U}\,(\neg\Lambda \wedge [\psi_2]) \qquad [\psi_1\,\mathsf{R}\,\psi_2] = (\neg\Lambda \wedge [\psi_1])\,\mathsf{R}\,(\neg\Lambda \rightarrow [\psi_2])$$

Note that, since φ was required to be in negation normal form, φ' is indeed in flatLTL$^{\downarrow}$. ◀

Corollary 16 and Lemma 17 imply the NEXPTIME upper bound for SOCA model checking:

▶ **Corollary 18.** SOCA-MC(flatLTL$^{\downarrow}$) *is in* NEXPTIME.

4.2 Lower Bounds

This subsection presents a matching lower bound for model checking flatLTL$^{\downarrow}$. We first state NEXPTIME-hardness of OCAP-MC(LTL), which we reduce, in a second step, to OCA-MC(flatLTL$^{\downarrow}$).

▶ **Theorem 19.** OCAP-MC(LTL) *and* SOCAP-MC(LTL) *are* NEXPTIME*-complete.*

Proof. NEXPTIME-hardness of SOCAP-MC(LTL) is due to [18], where Göller et al. show NEXPTIME-hardness of LTL model checking OCAs with parametric and succinct *updates* [18]. Their proof can be adapted in a straightforward way: In Figure 3 of [18], we replace the parametric update $+x$ by an equality test $=x$ and add a self-loop to the start state with label $+1$. We remark that this lower bound already holds for SOCAPs that use only one parameter x and all parameterized tests are of the form $=x$.

Moreover, SOCAP-MC(LTL) can be reduced to OCAP-MC(LTL) in polynomial time, using the construction from Lemma 17.

Membership of OCAP-MC(LTL) in NEXPTIME is by a standard argument. The given LTL formula can be translated into a Büchi automaton of exponential size. Then, we check non-emptiness of its product with the given OCAP using Corollary 14. ◀

▶ **Lemma 20.** OCA-MC(flatLTL$^{\downarrow}$) *and* SOCA-MC(flatLTL$^{\downarrow}$) *are* NEXPTIME*-hard.*

Proof. We present a reduction from OCAP-MC(LTL), which is NEXPTIME-complete by Theorem 19. Let $\mathcal{A} = (Q, \mathcal{X}, q_{\mathsf{in}}, \Delta, \mu)$ be an OCAP and let φ be an LTL formula. We define an OCA $\mathcal{A}' = (Q', q'_{\mathsf{in}}, \Delta', \mu')$ and a sentence $\varphi' \in \text{flatLTL}^{\downarrow}$ such that $\mathcal{A} \models_{\exists} \varphi$ iff $\mathcal{A}' \models_{\exists} \varphi'$.

Without loss of generality, by [18] (cf. proof of Theorem 19), we may assume that $\mathcal{A}$ uses only one parameter x and that all parameterized tests are of the form $=x$.

The idea is as follows. A new initial state q'_{in}, in which only a fresh proposition *freeze* holds, will allow $\mathcal{A}'$ to go to an arbitrary counter value, say, v. The flatLTL$^{\downarrow}$ formula of the form $\mathsf{F}\big(\mathit{freeze} \wedge \downarrow_r \psi\big)$ stores v, which will henceforth be interpreted as parameter x. Hereby, formula ψ makes sure that, whenever $\mathcal{A}$ performs an equality check $=x$, the current counter value coincides with v. To do so, we create two copies of the original state space: $Q \times \{0,1\}$. A transition $(q, =x, q')$ of $\mathcal{A}$ is then simulated, in $\mathcal{A}'$, by a 0-labeled transition to $(q', 1)$. States of the latter form are equipped with a fresh proposition $p_{=x}$ indicating that the current counter value has to coincide with the contents of r. Thus, we can set $\psi = \mathsf{G}(p_{=x} \to {=r})$.

Formally, $\mathcal{A}'$ is given as follows. We let $Q' = (Q \times \{0,1\}) \uplus \{q'_{\mathsf{in}}\}$ and

$$\begin{aligned}\Delta' = \quad & \{(q'_{\mathsf{in}}, \mathsf{op}, q'_{\mathsf{in}}) \mid \mathsf{op} \in \{+1, -1\}\} \;\cup\; \{(q'_{\mathsf{in}}, \mathsf{zero?}, (q_{\mathsf{in}}, 0))\} \\ \cup \quad & \{((q,b), 0, (q,1)) \mid (q, =x, q') \in \Delta \text{ and } b \in \{0,1\}\} \\ \cup \quad & \{((q,b), \mathsf{op}, (q,0)) \mid (q, \mathsf{op}, q') \in \Delta \setminus (Q \times \{=x\} \times Q) \text{ and } b \in \{0,1\}\}\,.\end{aligned}$$

Moreover, we set $\mu'(q'_{\mathsf{in}}) = \{\mathit{freeze}\}$ as well as $\mu'((q,0)) = \mu(q)$ and $\mu'((q,1)) = \mu(q) \cup \{p_{=x}\}$ for all $q \in Q$. Finally, $\varphi' = \mathsf{F}\big(\mathit{freeze} \wedge \downarrow_r \mathsf{G}(p_{=x} \to {=r})\big) \wedge \big(\mathit{freeze}\, \mathsf{U}\, (\neg \mathit{freeze} \wedge \varphi)\big)$. Note that the subformula $\mathit{freeze}\, \mathsf{U}\, (\neg \mathit{freeze} \wedge \varphi)$ makes sure that φ is satisfied as soon as we enter the original initial state q_{in} (actually, $(q_{\mathsf{in}}, 0)$). ◀

We conclude stating the exact complexity of model checking (S)OCAs against flatLTL$^{\downarrow}$:

▶ **Theorem 21.** OCA-MC(flatLTL$^{\downarrow}$) *and* SOCA-MC(flatLTL$^{\downarrow}$) *are* NEXPTIME*-complete.*

5 Conclusion

In this paper, we established the precise complexity of model checking OCAs and SOCAs against flat freeze LTL. To do so, we established a tight link between OCAPs and alternating two-way automata over words. Exploiting alternation further, it is likely that we can obtain positive results for one-counter games with parameterized tests. Another interesting issue for future work concerns model checking OCAPs, which, in this paper, is defined as the following question: Are there a parameter instantiation γ and a γ-run that satisfies the given formula? In fact, it also makes perfect sense to ask whether there is such a run for *all* parameter instantiations, in particular when requiring that all system runs satisfy a given property. We believe that our techniques can be used also in that case.

Although OCAPs have mainly been used for deciding a model-checking problem for OCAs, they constitute a versatile tool as well as a natural stand-alone model. In fact, parameterized extensions of a variety of system models have been explored recently, among them OCAs with parameterized *updates* [22, 19], and parametric timed automata [2, 5] where clocks can be compared with parameters (similarly to parameterized tests in OCAPs). Our results may have an impact on those models, too, thus beyond their application to model checking flat freeze LTL of OCAs: The (existential) non-emptiness problem for parametric timed automata with a single parametric clock (1PTAs) was first studied and solved by Alur, Henzinger, and Vardi, who provided a nonelementary decision procedure [2]. Recently, Bundala and

Ouaknine improved this significantly to 2NEXPTIME [5]. They provide a polynomial-time reduction to the reachability problem for OCAPs where, in addition to parameters, the counter values can be compared with constants $c \in \mathbb{N}$ encoded in binary. Altogether, they obtain a 2NEXPTIME algorithm for the non-emptiness problem for 1PTAs. Therefore, it will be worthwhile to study whether our NP upper bound for reachability in OCAPs applies to extended OCAPs, too (however, with *binary updates*, the problem is already PSPACE-hard [14]). This would yield an optimal NEXPTIME algorithm for non-emptiness in 1PTAs.

References

1 S. Akshay, B. Bollig, P. Gastin, M. Mukund, and K. Narayan Kumar. Distributed timed automata with independently evolving clocks. *Fundamenta Informaticae*, 130(4):377–407, 2014.

2 R. Alur, T. A. Henzinger, and M. Y. Vardi. Parametric real-time reasoning. In *Proceedings of STOC'93*, pages 592–601. ACM, 1993.

3 M. Bojańczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin. Two-variable logic on data words. *ACM Trans. Comput. Log.*, 12(4):27:1–27:26, 2011.

4 P. Bouyer, N. Markey, J. Ouaknine, and J. Worrell. On expressiveness and complexity in real-time model checking. In *Proceedings of ICALP'08, Part II*, volume 5126 of *LNCS*, pages 124–135. Springer, 2008.

5 D. Bundala and J. Ouaknine. On parametric timed automata and one-counter machines. *Inf. Comput.*, 253:272–303, 2017.

6 T. Cachat. Two-way tree automata solving pushdown games. In *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *LNCS*, pages 303–317. Springer, 2002.

7 H. Comon and V. Cortier. Flatness is not a weakness. In *Proceedings of CSL'00*, volume 1862 of *LNCS*, pages 262–276. Springer, 2000.

8 C. Dax and F. Klaedtke. Alternation elimination by complementation. In *Proceedings of LPAR'08*, volume 5330 of *LNCS*, pages 214–229. Springer, 2008.

9 S. Demri and R. Gascon. The effects of bounding syntactic resources on presburger LTL. *J. Log. Comput.*, 19(6):1541–1575, 2009.

10 S. Demri and R. Lazic. LTL with the freeze quantifier and register automata. *ACM Trans. Comput. Log.*, 10(3), 2009.

11 S. Demri, R. Lazic, and D. Nowak. On the freeze quantifier in constraint LTL: decidability and complexity. *Inf. Comput.*, 205(1):2–24, 2007.

12 S. Demri, R. Lazic, and A. Sangnier. Model checking memoryful linear-time logics over one-counter automata. *Theor. Comput. Sci.*, 411(22-24):2298–2316, 2010.

13 S. Demri and A Sangnier. When model-checking freeze LTL over counter machines becomes decidable. In *Proceedings of FoSSaCS'10*, volume 6014 of *LNCS*, pages 176–190. Springer, 2010.

14 J. Fearnley and M. Jurdzinski. Reachability in two-clock timed automata is pspace-complete. In *Proceedings of ICALP'13, Part II*, volume 7966 of *Lecture Notes in Computer Science*, pages 212–223. Springer, 2013.

15 S. Feng, M. Lohrey, and K. Quaas. Path checking for MTL and TPTL over data words. In *Proceedings of DLT'15*, volume 9168 of *LNCS*, pages 326–339. Springer, 2015.

16 T. French. Quantified propositional temporal logic with repeating states. In *Proceedings of TIME-ICTL'03*, pages 155–165. IEEE Computer Society, 2003.

17 Z. Galil. Hierarchies of complete problems. *Acta Inf.*, 6:77–88, 1976.

18 S. Göller, C. Haase, J. Ouaknine, and J. Worrell. Model checking succinct and parametric one-counter automata. In *Proceedings of ICALP'10, Part II*, volume 6199 of *LNCS*, pages 575–586. Springer, 2010.

19 S. Göller, C. Haase, J. Ouaknine, and J. Worrell. Branching-time model checking of parametric one-counter automata. In *Proceedings of FOSSACS'12*, volume 7213 of *LNCS*, pages 406–420. Springer, 2012.

20 S. Göller and M. Lohrey. Branching-time model checking of one-counter processes and timed automata. *SIAM J. Comput.*, 42(3):884–923, 2013.

21 C. Haase. *On the complexity of model checking counter automata.* PhD thesis, University of Oxford, 2012.

22 C. Haase, S. Kreutzer, J. Ouaknine, and J. Worrell. Reachability in succinct and parametric one-counter automata. In *Proceedings of CONCUR'09*, volume 5710 of *LNCS*, pages 369–383. Springer, 2009.

23 P. Hofman, S. Lasota, R. Mayr, and P. Totzke. Simulation problems over one-counter nets. *Logical Methods in Computer Science*, 12(1), 2016.

24 O. Kupferman and M. Y. Vardi. An automata-theoretic approach to reasoning about infinite-state systems. In *Proceedings of CAV'00*, volume 1855 of *LNCS*, pages 36–52. Springer, 2000.

25 P. Lafourcade, D. Lugiez, and R. Treinen. Intruder deduction for AC-like equational theories with homomorphisms. In *Proceedings of RTA'05*, volume 3467 of *LNCS*, pages 308–322. Springer, 2005.

26 A. Lechner, R. Mayr, J. Ouaknine, A. Pouly, and J. Worrell. Model checking flat freeze LTL on one-counter automata. In *Proceedings of CONCUR'16*, volume 59 of *LIPIcs*, pages 29:1–29:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

27 A. Lisitsa and I. Potapov. Temporal logic with predicate lambda-abstraction. In *Proceedings of TIME'05*, pages 147–155. IEEE Computer Society, 2005.

28 O. Serre. Parity games played on transition graphs of one-counter processes. In *Proceedings of FoSSaCS'06*, volume 3921 of *LNCS*, pages 337–351. Springer, 2006.

29 L. G. Valiant and M. S. Paterson. Deterministic one-counter automata. *Journal of Computer and System Sciences*, 10(3):340–350, 1975.

30 M. Y. Vardi. Reasoning about the past with two-way automata. In *Proceedings of ICALP'98*, volume 1443 of *LNCS*, pages 628–641. Springer, 1998.

Higher-Order Linearisability*

Andrzej S. Murawski[1] and Nikos Tzevelekos[2]

1 **University of Warwick, Coventry, UK**
2 **Queen Mary University of London, London, UK**

Abstract

Linearisability is a central notion for verifying concurrent libraries: a library is proven correct if its operational history can be rearranged into a sequential one that satisfies a given specification. Until now, linearisability has been examined for libraries in which method arguments and method results were of ground type. In this paper we extend linearisability to the general higher-order setting, where methods of arbitrary type can be passed as arguments and returned as values, and establish its soundness.

1998 ACM Subject Classification F.1.2 Models of Computation

Keywords and phrases Linearisability, Concurrency, Higher-Order Computation

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.34

1 Introduction

Software libraries provide implementations of routines, often of specialised nature, to facilitate code reuse and modularity. To support the latter, they should follow specifications that describe the range of acceptable behaviours for correct and safe deployment. Adherence to specifications can be formalised using the classic notion of contextual approximation (refinement), which scrutinises the behaviour of code in any possible context. Unfortunately, the quantification makes it difficult to prove contextual approximations directly, which motivates research into sound techniques for establishing it.

In the concurrent setting, a notion that has been particularly influential is that of *linearisability* [12]. Linearisability requires that, for each history generated by a library, one should be able to find another history from the specification (a *linearisation*), which matches the former up to certain rearrangements of events. In the original formulation by Herlihy and Wing [12], these permutations were not allowed to disturb the order between library returns and client calls. Moreover, linearisations were required to be *sequential* traces, that is, sequences of method calls immediately followed by their returns.

In this paper we shall work with *open higher-order* libraries, which provide implementations of *public* methods and may themselves depend on *abstract* ones, to be supplied by parameter libraries. The classic notion of linearisability only applies to closed libraries (without abstract methods). Additionally, both method arguments and results had to be of *ground* type. The closedness limitation was recently lifted in [13, 3], which distinguished between public (or *implemented*) and abstract methods (*callable*). Although [13] did not in principle exclude higher-order functions, those works focussed on linearisability for the case where the allowable methods were restricted to first-order functions ($\mathsf{int} \to \mathsf{int}$). Herein, we give a systematic exposition of linearisability for general higher-order concurrent libraries,

* Research was partially supported by the EPSRC (EP/P004172/1).

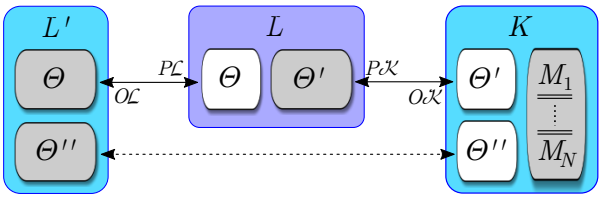


Figure 1 A library $L : \Theta \to \Theta'$ in environment comprising a parameter library $L' : \varnothing \to \Theta, \Theta''$ and a client K of the form $\Theta', \Theta'' \vdash M_1 \| \cdots \| M_N$.

where methods can be of arbitrary higher-order types. In doing so, we also propose a corresponding notion of sequential history for higher-order library interactions.

We examine libraries L that can interact with their environments by means of public and abstract methods: a library L with abstract methods of types $\Theta = \theta_1, \cdots, \theta_n$ and public methods $\Theta' = \theta'_1, \cdots, \theta'_{n'}$ is written as $L : \Theta \to \Theta'$. We shall work with arbitrary higher-order types generated from the ground types unit and int. Types in Θ, Θ' must always be function types, i.e. their order is at least 1.

A library L may be used in computations by placing it in a context that will keep on calling its public methods (via a client K) as well as providing implementations for the abstract ones (via a parameter library L'). The setting is depicted in Figure 1. Note that, as the library L interacts with K and L', they exchange functions between each other. Consequently, in addition to K making calls to public methods of L and L making calls to its abstract methods, K and L' may also issue calls to functions that were passed to them as arguments during higher-order interactions. Analogously, L may call functions that were communicated to it via library calls.

Our framework is operational in flavour and draws upon concurrent [15, 7] and operational game semantics [14, 16, 8]. We shall model library use as a game between two participants: *Player (P)*, corresponding to the library L, and *Opponent (O)*, representing the environment (L', K) in which the library was deployed. Each call will be of the form $\mathsf{call}\, m(v)$ with the corresponding return of the shape $\mathsf{ret}\, m(v)$, where v is a value. As we work in a higher-order framework, v may contain functions, which can participate in subsequent calls and returns. Histories will be sequences of *moves*, which are calls and returns paired with thread identifiers. A history is sequential just if every move produced by O is immediately followed by a move by P in the same thread. In other words, the library immediately responds to each call or return delivered by the environment. In contrast to classic linearisability, the move by O and its response by P need not be a call/return pair, as the higher-order setting provides more possibilities (in particular, the P response may well be a call). Accordingly, linearisable higher-order histories can be seen as sequences of atomic segments (linearisation points), starting at environment moves and ending with corresponding library moves.

In the spirit of [3], we are going to consider two scenarios: one in which K and L' share an explicit communication channel (the general case) as well as a situation in which they can only communicate through the library (the encapsulated case). Further, we also handle the case in which extra closure assumptions can be made about the parameter library (the relational case), which can be useful for dealing with a variety of assumptions on the use of parameter libraries that may arise in practice. In each case, we present a candidate definition of linearisability and illustrate it with tailored examples. The suitability of each kind of linearisability is demonstrated by showing that it implies the relevant form of contextual approximation (refinement). We also examine compositionality of the proposed concepts.

One of our examples will discuss the implementation of the flat-combining approach [11, 3], adapted to higher-order types.

1.1 Example: a higher-order multiset library

Higher-order libraries are common in languages like ML, Java, Python, etc. As an illustrative example, we consider a library written in ML-like syntax which implements a multiset data structure with integer elements. For simplicity, we assume that its signature contains just two methods:

$$\mathit{count} : \mathsf{int} \to \mathsf{int}, \quad \mathit{update} : (\mathsf{int} \times (\mathsf{int} \to \mathsf{int})) \to \mathsf{int}\,.$$

The former method returns for each integer its multiplicity in the multiset – this is 0 if the integer is not a member of the multiset. On the other hand, *update* takes as an argument an integer i and a function f, and updates the multiplicity of i in the multiset to $|f(i)|$ (we use the absolute value of $f(i)$ in order to meet the multiset requirement that element multiplicities not be negative; alternatively, we could have used exceptions to quarantine such client method behaviour). Methods with the same functionalities can be found e.g. in the multiset module of the ocaml-containers library [1]. While our example is simple, the same kind of analysis as below can be applied to more intricate examples such as *map* methods for integer-valued arrays, maps or multisets.

▶ **Example 1** (Multiset)**.** Consider the concurrent multiset library L_{mset} in Figure 2. It uses a private reference for storing the multiset's characteristic function and reads *optimistically*, without locking (cf. [10, 19]). The *update* method in particular reads the current multiplicity of the given element i (via *count*) and computes its new multiplicity without acquiring a lock on the characteristic function. It only acquires a lock when it is ready to write the new value (line 10) in the hope that the value at i will still be the same and the update can proceed; if not, another attempt to update the value is made.

Let us look at some example executions of the library via their resulting histories, i.e. sequences of method calls and returns between the library and a client. In the topmost block (a) of history diagrams of Figure 2, we see three such executions. Note that we do not record internal calls to *count* or *aux*, and use m and variants for method identifiers (names). We use the abbreviation *cnt* for *count*, and *upd* for *update*, and initially ignore the circled events for *cnt*. Each execution involves 2 threads.

In the first execution, the client calls $\mathit{update}(i, m)$ in the second thread, and subsequently calls $\mathit{count}(i)$ in the first thread. The code for *update* stipulates that first $\mathit{count}(i)$ be called internally, returning some multiplicity j for i, and then $m(j)$ should be called. As soon m returns a value j', *update* sets the multiplicity of i to j' and itself returns j'. The last event in this history is a return of *count* in the first thread with the old value j. According to our proposed definition, this history will be linearisable to another, intuitively correct one: the last return can be moved to the circled position. At this point the notion of linearisability is used informally, but it will be made precise in the following sections. In the second execution, the last return of *count* in the first thread returns the updated value. In this case, we will be able to move call $\mathit{cnt}(i)$ to the circled position to obtain a linearisation, which is obviously correct. Finally, in the third execution we have a history that will turn out non-linearisable to an intuitively correct history. Indeed, we should not be able to return the updated value in the first thread before m has returned it in the second one.

The two histories in block (b) in the same figure demonstrate the mechanism for updates. The first history will be linearisable to the second one. In the second history we see that both

```
1  public count, update;
2  Lock lock;
3  F := λx.0;
4
5  count = λi. (!F)i
6  update = λ(i, g). aux(i, g, count i)
7
8  aux = λ(i, g, j).
9    let y = |g j| in
10     lock.acquire();
11     let f = !F in
12       if (j == (f i)) then {
13         F := λx. if (x == i) then y
14                  else (f x) ;
15         lock.release();
16         y }
17       else {
18         lock.release();
19         aux(i, g, f i) }
```

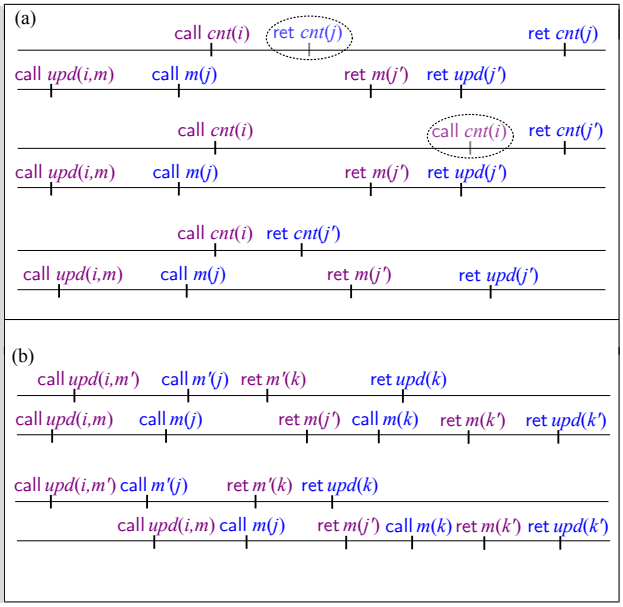


Figure 2 Multiset library L_{mset} with public methods $count : \mathsf{int} \to \mathsf{int}$ and $update : \mathsf{int} \times (\mathsf{int} \to \mathsf{int}) \to \mathsf{int}$.

threads try to update the same element i, but the first one succeeds in it first and returns k on *update*. Then, the second thread realises that the value of i has been updated to k and calls m again, this time with argument k. An important feature of the second history is that it is *sequential*: each client event (call or return) is immediately followed by a library event.

Observe that the rearrangements discussed above involve either advancing a library action or postponing an environment action and that each action could be a call or a return. Definition 6 will capture this formally. For now, we note that this generalises the classic setting [12], where library method returns could be advanced and environment method calls deferred.

2 Higher-order linearisability

We examine higher-order libraries interacting with their context by means of abstract and public methods. In particular, we shall rely on types given by the grammar on the left below. We let Meths stand for the set of *method names* and assume $\mathsf{Meths} = \biguplus_{\theta,\theta'} \mathsf{Meths}_{\theta,\theta'}$, where each set $\mathsf{Meths}_{\theta,\theta'}$ contains names for methods of type $\theta \to \theta'$. Methods are ranged over by m (and variants). We let v range over computational *values*, which include a unit value, integers, methods and pairs of values.

$$\theta ::= \mathsf{unit} \mid \mathsf{int} \mid \theta \times \theta \mid \theta \to \theta \qquad v ::= () \mid i \mid m \mid (v, v)$$

The framework of a higher-order library and its environment is depicted in Figure 1. Given $\Theta, \Theta' \subseteq \mathsf{Meths}$, a library L is said to have type $\Theta \to \Theta'$ if it defines public methods with names (and types) as in Θ', using abstract methods Θ. The environment of L consists of a *client* K (which invokes public methods of Θ'), and a *parameter library* L' (which provides code for the abstract methods Θ). In general, K and L' may interact via a disjoint set of methods $\Theta'' \subseteq \mathsf{Meths}$, to which L has no access.

In the rest of this paper, we shall implicitly assume that we work with a library L operating in an environment presented in Figure 1. The client K will consist of a fixed number N of concurrent threads. Next we introduce a notion of history tailored to the setting and define how histories can be linearised. In Section 3 we present the syntax for libraries and clients, and in Section 4 we define their semantics in terms of histories and co-histories respectively.

2.1 Higher-order histories

The operational semantics of libraries will be given in terms of *histories*, which are sequences of method calls and returns, each decorated with a thread identifier t and a *polarity index* XY, where $X \in \{O, P\}$ and $Y \in \{\mathcal{L}, \mathcal{K}\}$, as shown below.

$$(t, \mathsf{call}\, m(v))_{XY} \qquad (t, \mathsf{ret}\, m(v))_{XY}$$

We shall refer such decorated calls and returns as ***moves***. Here, m is a method name and v is a value of a matching type. The index XY specifies which of the three entities (L, L', K) produces the move, and towards whom it is addressed.

- If $XY = P\mathcal{L}$ then the move is issued by L, and is addressed to L'.
- If $XY = P\mathcal{K}$ then the move is issued by L, and is addressed to K.
- If $XY = O\mathcal{L}$ then the move is issued by L', and is addressed to L.
- If $XY = O\mathcal{K}$ then the move is issued by K, and is addressed to L.

The choice of indices is motivated by the fact that the moves can be seen as defining a 2-player game between the library (L), which represents the *Proponent* player in the game (P), and its environment (L', K) that represents the *Opponent* (O). Moves played between L and L' are moreover decorated with $\mathcal{L}$, whereas those between L and K have $\mathcal{K}$ instead. Note that any potential interaction between L' and K is invisible to L and is therefore not accounted for in the game (but we will later see how it can affect it). We use O to refer to either $O\mathcal{K}$ or $O\mathcal{L}$, and P to refer to either $P\mathcal{K}$ or $P\mathcal{L}$. Finally, we let the ***dual*** polarity of XY to be $X'Y$, where $X \neq X'$. For example, the dual of $P\mathcal{L}$ is $O\mathcal{L}$.

Next we proceed to define histories. Their definition will rely on a more primitive concept of *prehistories*, which are sequences of method calls and returns that respect a stack discipline.

▶ **Definition 2.** *Prehistories* are sequences generated by one of the grammars:

$$\begin{array}{lcl} \mathsf{PreH}_O & ::= & \epsilon \mid \mathsf{call}\, m(v)_{OY}\ \mathsf{PreH}_P\ \mathsf{ret}\, m(v')_{PY}\ \mathsf{PreH}_O \\ \mathsf{PreH}_P & ::= & \epsilon \mid \mathsf{call}\, m(v)_{PY}\ \mathsf{PreH}_O\ \mathsf{ret}\, m(v')_{OY}\ \mathsf{PreH}_P \end{array}$$

where, in each line, the two occurrences of $Y \in \{\mathcal{K}, \mathcal{L}\}$ and $m \in \mathsf{Meths}$ must each match. Moreover, if $m \in \mathsf{Meths}_{\theta,\theta'}$, the types of v, v' must match θ, θ' respectively. We let $\mathsf{PreH} = \mathsf{PreH}_O \cup \mathsf{PreH}_P$.

Thus, prehistories from PreH_O start with an O-move, while those in PreH_P start with a P-move. In each case, the polarities inside a prehistory alternate between O and P, and the polarities of calls and matching returns are always dual (*returns dual to calls*). For example, a call made by L to L' (tagged $P\mathcal{L}$) must be matched by a return from L' to L (tagged $O\mathcal{L}$).

Histories will be interleavings of prehistories tagged with thread identifiers (natural numbers), subject to a set of well-formedness constrains. In particular, a history h for library $L : \Theta \to \Theta'$ will have to begin with an O-move and satisfy the following conditions, to be formalised in Definition 3.

1. The name of any method called in h must come from Θ or Θ', or be introduced earlier in h as a higher-order argument or result (*no methods out of thin air*). In addition:
   - if the method is from Θ', the call must be tagged with $O\mathcal{K}$ (i.e. issued by K);
   - if the method is from Θ, the call must be tagged with $P\mathcal{L}$ (i.e. issued by L towards L');
   - for a call of method $m \notin \Theta \cup \Theta'$ to be valid, m must be introduced in an earlier move of dual polarity (*calls dual to introductions*).
2. Any method name appearing inside a call or return argument in h must be *fresh*, i.e. not used earlier. This reflects the assumption that methods can be called and returned, but not compared for identity (*introductions always fresh*).

Given $h \in \mathsf{PreH}$ and $t \in \mathbb{N}$, we write $t \times h$ for h in which each call or return is decorated with t. We refer to such moves with $(t, \mathsf{call}\, m(v))_{XY}$ or $(t, \mathsf{ret}\, m(v))_{XY}$ respectively. If we only want to stress the X or Y membership, we shall drop Y or X respectively. Moreover, when no confusion arises, we may sometimes drop a move's polarity altogether. We say that a move x ***introduces*** a name $m \in \mathsf{Meths}$ when $x \in \{\mathsf{call}\, m'(v), \mathsf{ret}\, m'(v)\}$ for some m', v such that v contains m.

▶ **Definition 3.** Given Θ, Θ', the set of ***histories*** over $\Theta \to \Theta'$, written $\mathcal{H}_{\Theta,\Theta'}$, is defined by

$$\mathcal{H}_{\Theta,\Theta'} = \bigcup_{N>0} \bigcup_{h_1,\cdots,h_N \in \mathsf{PreH}_O} (1 \times h_1) \mid \cdots \mid (N \times h_N)$$

where $(1 \times h_1) \mid \cdots \mid (N \times h_N)$ is the set of all interleavings of $(1 \times h_1), \cdots, (N \times h_N)$ satisfying:

1. For any $s_1(t, \mathsf{call}\, m(v))_{XY} s_2 \in \mathcal{H}_{\Theta,\Theta'}$, either $m \in \Theta'$ and $XY = O\mathcal{K}$, or $m \in \Theta$ and $XY = P\mathcal{L}$, or there is a move $(t', x)_{X'Y}$ in s_1 such that $X \neq X'$ and x introduces m.
2. For any $s_1(t, x)_{XY} s_2 \in \mathcal{H}_{\Theta,\Theta'}$ and any m, if m is introduced by x then m must not occur in s_1.

A history $h \in \mathcal{H}_{\Theta,\Theta'}$ is called ***sequential*** if it is of the form

$$h = (t_1, x_1)_{OY_1} (t_1, x_1')_{PY_1'} \cdots (t_k, x_k)_{OY_k} (t_k, x_k')_{PY_k'}$$

for some $t_i, x_i, x_i', Y_i, Y_i'$. We write $\mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$ for the set of all sequential histories from $\mathcal{H}_{\Theta,\Theta'}$.

We shall range over $\mathcal{H}_{\Theta,\Theta'}$ using h, s (and variants). The subscripts Θ, Θ' will often be omitted. Given a history h, we shall write $\overline{h}$ for the sequence of moves obtained from h by dualising all move polarities inside it. The set of ***co-histories*** over $\Theta \to \Theta'$ will be $\mathcal{H}^{co}_{\Theta,\Theta'} = \{\overline{h} \mid h \in \mathcal{H}_{\Theta,\Theta'}\}$.

While in this section histories will be extracted from example libraries informally, in Section 4 we give the formal semantics $[\![L]\!]$ of libraries. For each $L : \Theta \to \Theta'$, we shall have $[\![L]\!] \subseteq \mathcal{H}_{\Theta,\Theta'}$.

▶ Remark 4. The notion of history introduced above extends the classic notion from [12] to higher-order types. It also extends the notion presented in [3]. The intuition behind the definition is that a history is a sequence of (well-bracketed) method calls and returns, called *moves*, each tagged with a thread identifier and a polarity, where polarities track the originators and recipients of moves. Moves may be calls or returns related to methods given in the library interface ($\Theta \to \Theta'$), or dynamically created methods that appear earlier inside the histories – recall that, in a higher-order setting, methods can be passed around as arguments to calls or be returned as results by other methods. On the other hand, a sequential history is one in which the operations performed by the library can be perceived as **atomic**, that is, each move produced by O is to be immediately followed by the library's response, which is a P move in the same thread.

▶ **Example 5** (Multiset spec)**.** We now revisit our first example and provide a specification for it. Recall the multiset library L_{mset} from Figure 2. Our verification goal will be to prove linearisability of L_{mset} to a specification $A_{\mathsf{mset}} \subseteq \mathcal{H}^{\mathsf{seq}}_{\varnothing,\Theta}$, where $\Theta = \{\mathit{count}, \mathit{update}\}$, which we define below. A_{mset} will certify that L_{mset} correctly implements some integer multiset I whose elements change over time according to the moves in h. For a multiset I and natural numbers i, j, we write $I(i)$ for the multiplicity of i in I, and $I[i \mapsto j]$ for I with its multiplicity of i set to j. We shall stipulate that moves inside histories $h \in A_{\mathsf{mset}}$ be annotatable with multisets I in such a way that the multiset is empty at the start of h (i.e. $I(i) = 0$ for all i) and:

- If I is changed between two consecutive moves in h then the second move is a P-move. In other words, the client cannot directly update the elements of I.
- Each call to *count* on argument i must be immediately followed by a return with value $I(i)$, and with I remaining unchanged.
- Each call to *update* on (i, m) must be followed by a call to m on $I(i)$, with I unchanged. Moreover, m must later return with some value j. Assuming at that point the multiset will have value J, if $I(i) = J(i)$ then the next move is a return of the original *update* call, with value j; otherwise, a new call to m on $J(i)$ is produced, and so on.

We formally define the specification next.

Let $\mathcal{H}^{\circ}_{\varnothing,\Theta}$ contain sequences of moves from $\varnothing \to \Theta$ accompanied by a multiset (i.e. the sequences consist of elements of the form $(t, x, I)_{XY}$). For each $s \in \mathcal{H}^{\circ}_{\varnothing,\Theta}$, we let $\pi_1(s)$ be the history extracted by projection, i.e. $\pi_1(s) \in \mathcal{H}_{\varnothing,\Theta}$. For each t, we let $s \upharpoonright t$ be the subsequence of s of elements with first component t. Writing $\sqsubseteq_{\mathrm{pre}}$ for the prefix relation, and dropping the Y index from moves (Y is always $\mathcal{K}$ here), we define $A_{\mathsf{mset}} = \{\pi_1(s) \mid s \in A^{\circ}_{\mathsf{mset}}\}$ where:

$$A^{\circ}_{\mathsf{mset}} = \{\ s \in \mathcal{H}^{\circ}_{\varnothing,\Theta} \ \mid\ \pi_1(s) \in \mathcal{H}^{\mathsf{seq}}_{\varnothing,\Theta} \ \wedge\ \forall t.\, s \upharpoonright t \in \mathcal{S} \ \wedge\ \forall s'(\_, I)_P(\_, J)_O \sqsubseteq_{\mathrm{pre}} s.\ I = J\ \}$$

and, for each t, the set of t-indexed annotated histories $\mathcal{S}$ is given by the following grammar:

$$\begin{aligned}
\mathcal{S} \ &\to \epsilon \ \mid\ (t, \mathsf{call}\ \mathit{cnt}(i), I)_O\, (t, \mathsf{ret}\ \mathit{cnt}(I(i)), I)_P\, \mathcal{S} \\
&\quad \mid\ (t, \mathsf{call}\ \mathit{upd}(i, m), I)_O\, \mathcal{M}^{i,j}_{I,J}\, (t, \mathsf{ret}\ \mathit{upd}(|j|), J[i \mapsto |j|])_P\, \mathcal{S} \\
\mathcal{M}^{i,j}_{I,J} &\to (t, \mathsf{call}\, m(I(i)), I)_P\, \mathcal{S}\, (t, \mathsf{ret}\, m(j), J)_O \qquad \text{provided } J(i) = I(i) \\
\mathcal{M}^{i,j}_{I,J} &\to (t, \mathsf{call}\, m(I(i)), I)_P\, \mathcal{S}\, (t, \mathsf{ret}\, m(j'), J')_O\, \mathcal{M}^{i,j}_{J',J} \qquad \text{provided } J'(i) \neq I(i)
\end{aligned}$$

By definition, all histories in A_{mset} are sequential. The elements of $A^{\circ}_{\mathsf{mset}}$ carry along the multiset I that is being represented. The conditions on $A^{\circ}_{\mathsf{mset}}$ stipulate that O cannot change the value of I, while the rest of the conditions above are imposed by the grammar for $\mathcal{S}$. With the notion of linearisability to be introduced next, we will be able to show that $[\![L_{\mathsf{mset}}]\!]$ is indeed linearisable to A_{mset}.

2.2 Three notions of linearisability

We present three notions of linearisability. First introduce a general notion that generalises classic linearisability [12] and parameterised linearisability [3]. We then develop two more specialised variants: a notion of encapsulated linearisability, following [3], that captures scenarios where the parameter library and the client cannot directly interact; and a relational notion whereby context behaviour (client and parameter library) is known to be relationally invariant.

We begin by introducing a class of reorderings on histories. Suppose $X, X' \in \{O, P\}$ and $X \neq X'$. We let $\lhd_{XX'} \subseteq \mathcal{H}_{\Theta,\Theta'} \times \mathcal{H}_{\Theta,\Theta'}$ be the smallest binary relation over $\mathcal{H}_{\Theta,\Theta'}$ satisfying, for any $t \neq t'$:

$$s_1(t', x')_{Z'}(t, x)_Z\, s_2 \ \lhd_{XX'}\ s_1(t, x)_Z(t', x')_{Z'} s_2$$

whenever $Z = X$ or $Z' = X'$. Intuitively, two histories h_1, h_2 are related by $\lhd_{XX'}$ if the latter can be obtained from the former by swapping two adjacent moves from different threads in such a way that, after the swap, an X-move will occur earlier or an X'-move will occur later. Note that, because of $X \neq X'$, the relation always applies to adjacent moves of the same polarity. On the other hand, we cannot have $s_1(t,x)_X(t',x')_{X'}s_2 \lhd_{XX'} s_1(t',x')_{X'}(t,x)_X s_2$.

▶ **Definition 6** (General Linearisability)**.** Given $h_1, h_2 \in \mathcal{H}_{\Theta,\Theta'}$, we say that h_1 ***is linearised by*** h_2, written $h_1 \sqsubseteq h_2$, if $h_1 \lhd_{PO}^* h_2$. Given libraries $L, L' : \Theta \to \Theta'$ and a set of sequential histories $A \subseteq \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$ we write $L \sqsubseteq A$, and say that L *can be linearised to* A, if for any $h \in [\![L]\!]$ there exists $h' \in A$ such that $h \sqsubseteq h'$. Moreover, we write $L \sqsubseteq L'$ if $L \sqsubseteq [\![L']\!] \cap \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$ (i.e. for all $h \in [\![L]\!]$ there is sequential $h' \in [\![L']\!]$ such that $h \sqsubseteq h'$).

▶ Remark 7. The classic notion of linearisability from [12] states that h linearises to h' just if the return/call order of h is preserved in h' (and h' is sequential), i.e. if a return move precedes a call move in h then so is the case in h'. Observing that, in [12], return and call moves coincide with P- and O-moves respectively, we can see that our higher-order notion of linearisability is a generalisation of the classic notion.

We next show that a more permissive notion of linearisability applies if the parameter library L' of Figure 1 is encapsulated, that is, the client K can have no direct access to it (i.e. $\Theta'' = \varnothing$). To capture the more restrictive nature of interaction, we introduce a more constrained notion of a history. Specifically, in addition to sequentiality in every thread, we shall insist that a move made by the library in the $\mathcal{L}$ or $\mathcal{K}$ component must be followed by an O move from the *same* component.

▶ **Definition 8.** We call a history $h \in \mathcal{H}_{\Theta,\Theta'}$ ***encapsulated*** if, for each thread t, we have that if $h = s_1\,(t,x)_{PY}\,s_2\,(t,x')_{OY'}\,s_3$ and moves from t are absent from s_2 then $Y = Y'$. Moreover, we set $\mathcal{H}^{\mathsf{enc}}_{\Theta,\Theta'} = \{h \in \mathcal{H}_{\Theta,\Theta'} \mid h \text{ encapsulated}\}$ and $[\![L]\!]_{\mathrm{enc}} = [\![L]\!] \cap \mathcal{H}^{\mathsf{enc}}_{\Theta,\Theta'}$ (if $L : \Theta \to \Theta'$).

We define the corresponding linearisability notion as follows. First, let $\diamond \subseteq \mathcal{H}_{\Theta,\Theta'} \times \mathcal{H}_{\Theta,\Theta'}$ be the smallest binary relation on $\mathcal{H}_{\Theta,\Theta'}$ such that, for any $Y, Y' \in \{\mathcal{K}, \mathcal{L}\}$ with $Y \neq Y'$ and $t \neq t'$:

$$s_1(t,m)_Y(t',m')_{Y'}s_2 \;\diamond\; s_1(t',m')_{Y'}(t,m)_Y s_2$$

▶ **Definition 9** (Encapsulated linearisability)**.** Given $h_1, h_2 \in \mathcal{H}^{\mathsf{enc}}_{\Theta,\Theta'}$, we say that h_1 is ***enc-linearised*** by h_2, and write $h_1 \sqsubseteq_{\mathrm{enc}} h_2$, if $h_1(\lhd_{PO} \cup \diamond)^* h_2$ and h_2 is sequential. A library $L : \Theta \to \Theta'$ can be *enc-linearised to* A, written $L \sqsubseteq_{\mathrm{enc}} A$, if $A \subseteq \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'} \cap \mathcal{H}^{\mathsf{enc}}_{\Theta,\Theta'}$ and for any $h \in [\![L]\!]_{\mathrm{enc}}$ there exists $h' \in A$ such that $h \sqsubseteq_{\mathrm{enc}} h'$. We write $L \sqsubseteq_{\mathrm{enc}} L'$ if $L \sqsubseteq_{\mathrm{enc}} [\![L']\!]_{\mathrm{enc}} \cap \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$.

▶ Remark 10. Suppose $\Theta = \{m : \mathsf{int} \to \mathsf{int}\}$ and $\Theta' = \{m' : \mathsf{int} \to \mathsf{int}\}$. Histories from $\mathcal{H}_{\Theta,\Theta'}$ may contain the following actions only: $\mathsf{call}\, m'(i)_{O\mathcal{K}}$, $\mathsf{ret}\, m(i)_{O\mathcal{L}}$, $\mathsf{call}\, m(i)_{P\mathcal{L}}$, $\mathsf{ret}\, m'(i)_{P\mathcal{K}}$. Then $(\lhd_{PO} \cup \diamond)^*$ preserves the order between $\mathsf{call}\, m(i)_{P\mathcal{L}}$ and $\mathsf{ret}\, m(i)_{O\mathcal{L}}$ as well as that between $\mathsf{ret}\, m'(i)_{P\mathcal{K}}$ and $\mathsf{call}\, m'(i)_{O\mathcal{K}}$, i.e. it coincides with Definition 3 of [3].

▶ **Example 11** (Parameterised multiset)**.** We revisit the multiset library of Example 1 and extend it with a public method *reset*, which performs multiplicity resets to default values using an abstract method *default* as the default-value function (again, we use absolute values to avoid negative multiplicities). The extended library is shown in Figure 3 and written $L_{\mathsf{mset2}} : \{\mathit{default}\} \to \Theta'$, with $\Theta' = \{\mathit{count},\ \mathit{update},\ \mathit{reset}\}$. In contrast to the *update* method of L_{mset}, *reset* is not optimistic: it retrieves the lock upon its call, and only releases it before return. In particular, the method calls *default* while it retains the lock.

```
1  public count, update, reset;
2  abstract default;
3  Lock lock;
4  F := λx.0;
5  ...
20 reset = λi.
21   lock.acquire();
22   let y = |default i| in
23     let f = !F in
24       F := λx. if (x == i) then y
25                else (f x);
26       lock.release();
27       y
```

```
1  public run; ...;
2  Lock lock;
3  struct {fun, arg, wait, retv} requests[N];
4
5  run = λ (f,x).
6    requests[t_id].fun := f;
7    requests[t_id].arg := x;
8    requests[t_id].wait := 1;
9    while (requests[t_id].wait)
10     if (lock.tryacquire()) {
11       for (t=0; t<N; t++)
12         if (requests[t].wait) {
13           requests[t].retv :=
14                 requests[t].fun (requests[t].arg);
15           requests[t].wait := 0;
16         }; lock.release() };
17   requests[t_id].retv;
```

Figure 3 Left: Parameterised multiset library L_{mset2} (lines 5-19 as in Fig. 2) with public methods *count*, *reset*: $\mathsf{int}\to\mathsf{int}$, *update*: $\mathsf{int}\times(\mathsf{int}\to\mathsf{int})\to\mathsf{int}$; abstract method *default*: $\mathsf{int}\to\mathsf{int}$. Right: Flat combination library L_{fc}.

Observe that, were *default* able to externally call *update*, we would reach a deadlock: *default* would be keeping the lock while waiting for the return of a method that requires the lock. On the other hand, if the library is encapsulated then the latter scenario is not possible. In such a case, L_{mset2} linearises to the specification A_{mset2}, defined next. Let $A_{\mathsf{mset2}} = \{\pi_1(s) \mid s \in A^{\circ}_{\mathsf{mset2}}\}$ where:

$$A^{\circ}_{\mathsf{mset2}} = \{\ s \in \mathcal{H}^{\circ}_{\emptyset,\Theta'} \ \mid\ \pi_1(s) \in \mathcal{H}^{\mathsf{seq}}_{\emptyset,\Theta'} \ \wedge\ \forall t.\, s \upharpoonright t \in \mathcal{S} \ \wedge\ \forall s'(\_, I)_P(\_, J)_O \sqsubseteq_{\mathrm{pre}} s.\, I = J\ \}$$

and the set $\mathcal{S}$ is now given by the grammar of Example 5 extended with the rule:

$$\mathcal{S} \to\ (t, \mathsf{call}\ \mathit{reset}(i), I)_{OK}\,(t, \mathsf{call}\ \mathit{default}(i), I)_{P\mathcal{L}}\,(t, \mathsf{ret}\ \mathit{default}(j), I)_{O\mathcal{L}}\,(t, \mathsf{ret}\ \mathit{reset}(|j|), I')_{PK}\,\mathcal{S}$$

with $I' = I[i \mapsto |j|]$. Our framework makes it possible to confirm that L_{mset2} enc-linearises to A_{mset2}.

We finally extend general linearisability to cater for situations where the client and the parameter library adhere to closure constraints expressed by relations $\mathcal{R}$ on histories. Let Θ, Θ' be sets of abstract and public methods respectively. The closure relations we consider are closed under permutations of methods outside $\Theta \cup \Theta'$: if $h\,\mathcal{R}\,h'$ and π is a (type-preserving) permutation on $\mathsf{Meths} \setminus (\Theta \cup \Theta')$ then $\pi(h)\,\mathcal{R}\,\pi(h')$. The requirement represents the fact that, apart from the method names from a library interface, the other method names are arbitrary and can be freely permuted without any observable effect. Thus, $\mathcal{R}$ should not be distinguishing between such names.

▶ **Definition 12** (Relational linearisability). Let $\mathcal{R} \subseteq \mathcal{H}_{\Theta,\Theta'} \times \mathcal{H}_{\Theta,\Theta'}$ be closed under permutations of names in $\mathsf{Meths} \setminus (\Theta \cup \Theta')$. Given $h_1, h_2 \in \mathcal{H}_{\Theta,\Theta'}$, we say that h_1 is ***$\mathcal{R}$-linearised*** by h_2, and write $h_1 \sqsubseteq_{\mathcal{R}} h_2$, if $h_1(\lhd_{PO} \cup \mathcal{R})^* h_2$ and h_2 is sequential. A library $L : \Theta \to \Theta'$ can be *$\mathcal{R}$-linearised to* A, written $L \sqsubseteq_{\mathcal{R}} A$, if $A \subseteq \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$ and for any $h \in [\![L]\!]$ there exists $h' \in A$ such that $h \sqsubseteq_{\mathcal{R}} h'$. We write $L \sqsubseteq_{\mathcal{R}} L'$ if $L \sqsubseteq_{\mathcal{R}} [\![L']\!] \cap \mathcal{H}^{\mathsf{seq}}_{\Theta,\Theta'}$.

▶ **Example 13.** We consider a higher-order variant of an example from [3] that motivates relational linearisability. Flat combining [11] is a synchronisation paradigm that advocates

the use of a single thread holding a global lock to process requests of all other threads. To facilitate this, threads share an array to which they write the details of their requests and wait either until they acquire a lock or their request has been processed by another thread. Once a thread acquires a lock, it executes all requests stored in the array and the outcomes are written to the array for access by the requesting threads.

Let $\Theta' = \{run \in \mathsf{Meths}_{(\theta\to\theta')\times\theta,\theta'}\}$. The library $L_{\mathsf{fc}} : \varnothing \to \Theta'$ (Figure 3, right) is built following the flat combining approach and, on acquisition of the global lock, the winning thread acts as a combiner of all registered requests. Note that the requests will be attended to one after another (thus guaranteeing mutual exclusion) and only one lock acquisition will suffice to process one array of requests. Using our framework, one can show that L_{fc} can be $\mathcal{R}$-linearised to the specification given by the library L_{spec} defined by

$$run = \lambda\,(f,x).\ (lock.\,acquire\,();\ \ \mathsf{let}\ \ result\ = f(x)\ \ \mathsf{in}\ \ lock.\,release\,();\ \ result\,)$$

where each function call in L_{spec} is protected by a lock. Observe that we cannot hope for $L_{\mathsf{fc}} \sqsubseteq L_{\mathsf{spec}}$, because clients may call library methods with functional arguments that recognise thread identity. Consequently, we can relate the two libraries only if context behaviour is guaranteed to be independent of thread identifiers. This can be expressed through $\sqsubseteq_{\mathcal{R}}$, where $\mathcal{R} \subseteq \mathcal{H}_{\varnothing,\Theta'} \times \mathcal{H}_{\varnothing,\Theta'}$ is a relation capturing thread-blind client behaviour.

3 Library syntax

We now look at the concrete syntax of libraries and clients. Libraries comprise collections of typed methods whose argument and result types adhere to the grammar: $\theta ::= \mathsf{unit} \mid \mathsf{int} \mid \theta \to \theta \mid \theta \times \theta$.

We shall use three disjoint enumerable sets of names, referred to as Vars, Meths and Refs, to name respectively variables, methods and references. x, f (and their decorated variants) will be used to range over Vars; m will range over Meths; and r over Refs. Methods and references are implicitly typed, i.e. $\mathsf{Meths} = \biguplus_{\theta,\theta'} \mathsf{Meths}_{\theta,\theta'}$ and $\mathsf{Refs} = \mathsf{Refs}_{\mathsf{int}} \uplus \biguplus_{\theta,\theta'} \mathsf{Refs}_{\theta,\theta'}$, where $\mathsf{Meths}_{\theta,\theta'}$ contains names for methods of type $\theta \to \theta'$, $\mathsf{Refs}_{\mathsf{int}}$ contains names of integer references and $\mathsf{Refs}_{\theta,\theta'}$ contains names for references to methods of type $\theta \to \theta'$. We write $\uplus$ for disjoint set union.

The syntax for libraries and clients is given in Figure 4. Each library L begins with a series of method declarations (public or abstract) followed by a block B containing method implementations ($m = \lambda x.M$) and reference initialisations ($r := i$ or $r := \lambda x.M$). The typing rules ensure that each public method is implemented within the block, in contrast to abstract methods. Clients are parallel compositions of closed terms.

Terms M specify the shape of allowable method bodies. $()$ is the skip command, i ranges over integers, t_{id} is the current thread identifier and $\oplus$ represents standard arithmetic operations. Thanks to higher-order references, we can simulate divergence by $(!r)()$, where $r \in \mathsf{Refs}_{\mathsf{unit},\mathsf{unit}}$ is initialised with $\lambda x^{\mathsf{unit}}.(!r)()$. Similarly, $\mathsf{while}\ M\ N$ can be simulated by $(!r)()$ after $r := \lambda x^{\mathsf{unit}}.\mathsf{let}\ y = M\ \mathsf{in}\ (\mathsf{if}\ y\ \mathsf{then}\ (N; (!r)())\ \mathsf{else}\ ())$. We also use the standard derived syntax for sequential composition, i.e. $M; N$ stands for $\mathsf{let}\ x = M\ \mathsf{in}\ N$, where x does not occur in N. For each term M, we write $\mathsf{Meths}(M)$ for the set of method names occurring in M. We use the same notation for method names in blocks and libraries.

▶ Remark 14. In Section 2 we used lock-related operations in our example libraries (*acquire*, *tryacquire*, *release*), on the understanding that they can be coded using shared memory. Similarly, the array of Example 13 in the sequel can be constructed using references.

$$\begin{array}{llll} \textit{Libraries} & L ::= B \mid \mathsf{abstract}\ m;\ L \mid \mathsf{public}\ m;\ L & \textit{Clients} & K ::= M \,\|\cdots\|\, M \\ \textit{Blocks} & B ::= \epsilon \mid m = \lambda x.M;\ B \mid r := \lambda x.M;\ B \mid r := i;\ B & \textit{Values} & v ::= () \mid i \mid m \mid \langle v, v\rangle \\ \textit{Terms} & M ::= () \mid i \mid t_{\mathsf{id}} \mid x \mid m \mid M \oplus M \mid \langle M, M\rangle \mid \pi_1\, M \mid \pi_2\, M \mid \mathsf{if}\ M\ \mathsf{then}\ M\ \mathsf{else}\ M & & \\ & \quad \mid \lambda x^{\theta}.M \mid xM \mid mM \mid \mathsf{let}\ x = M\ \mathsf{in}\ M \mid r := M \mid\, !r & & \end{array}$$

$$\frac{}{\Gamma \vdash () : \mathsf{unit}} \quad \frac{}{\Gamma \vdash i : \mathsf{int}} \quad \frac{}{\Gamma \vdash t_{\mathsf{id}} : \mathsf{int}} \quad \frac{\Gamma(x) = \theta}{\Gamma \vdash x : \theta} \quad \frac{m \in \mathsf{Meths}_{\theta,\theta'}}{\Gamma \vdash m : \theta \to \theta'} \quad \frac{\Gamma \vdash M : \mathsf{int} \quad \Gamma \vdash M_0, M_1 : \theta}{\Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ M_1\ \mathsf{else}\ M_0 : \theta}$$

$$\frac{\Gamma \vdash M : \theta_1 \times \theta_2}{\Gamma \vdash \pi_i\, M : \theta_i \ \ (i = 1,2)} \quad \frac{\Gamma \vdash M_i : \theta_i \ \ (i = 1,2)}{\Gamma \vdash \langle M_1, M_2\rangle : \theta_1 \times \theta_2} \quad \frac{\Gamma \vdash M_1, M_2 : \mathsf{int}}{\Gamma \vdash M_1 \oplus M_2 : \mathsf{int}} \quad \frac{\Gamma, x : \theta \vdash M : \theta'}{\Gamma \vdash \lambda x^{\theta}.M : \theta \to \theta'}$$

$$\frac{\Gamma(x) = \theta \to \theta' \quad \Gamma \vdash M : \theta}{\Gamma \vdash xM : \theta'} \quad \frac{m \in \mathsf{Meths}_{\theta,\theta'} \quad \Gamma \vdash M : \theta}{\Gamma \vdash mM : \theta'} \quad \frac{\Gamma \vdash M : \theta \quad \Gamma, x : \theta \vdash N : \theta'}{\Gamma \vdash \mathsf{let}\ x = M\ \mathsf{in}\ N : \theta'}$$

$$\frac{r \in \mathsf{Refs}_{\mathsf{int}} \quad \Gamma \vdash M : \mathsf{int}}{\Gamma \vdash r := M : \mathsf{unit}} \quad \frac{r \in \mathsf{Refs}_{\theta,\theta'} \quad \Gamma \vdash M : \theta \to \theta'}{\Gamma \vdash r := M : \mathsf{unit}} \quad \frac{r \in \mathsf{Refs}_{\mathsf{int}}}{\Gamma \vdash\, !r : \mathsf{int}} \quad \frac{r \in \mathsf{Refs}_{\theta,\theta'}}{\Gamma \vdash\, !r : \theta \to \theta'}$$

$$\frac{}{\vdash_{\mathsf{B}} \epsilon : \varnothing} \quad \frac{m \in \mathsf{Meths}_{\theta,\theta'} \quad x : \theta \vdash M : \theta' \quad \vdash_{\mathsf{B}} B : \Theta}{\vdash_{\mathsf{B}} m = \lambda x.M;\ B : \Theta \uplus \{m\}} \quad \frac{r \in \mathsf{Refs}_{\theta,\theta'} \quad x : \theta \vdash M : \theta' \quad \vdash_{\mathsf{B}} B : \Theta}{\vdash_{\mathsf{B}} r := \lambda x.M;\ B : \Theta}$$

$$\frac{r \in \mathsf{Refs}_{\mathsf{int}} \quad \vdash_{\mathsf{B}} B : \Theta}{\vdash_{\mathsf{B}} r := i;\ B : \Theta} \quad \frac{\vdash_{\mathsf{B}} B : \Theta}{\mathsf{Meths}(B) \vdash_{\mathsf{L}} B : \varnothing \to \Theta} \quad \frac{\Theta \uplus \{m\} \vdash_{\mathsf{L}} L : \Theta' \to \Theta'' \quad m \in \Theta''}{\Theta \vdash_{\mathsf{L}} \mathsf{public}\ m; L : \Theta' \to \Theta''}$$

$$\frac{\Theta \uplus \{m\} \vdash_{\mathsf{L}} L : \Theta' \to \Theta'' \quad m \notin \Theta''}{\Theta \vdash_{\mathsf{L}} \mathsf{abstract}\ m; L : \Theta' \uplus \{m\} \to \Theta''} \quad \frac{\vdash M_j : \mathsf{unit} \ \ (j = 1, \cdots, N)}{\Theta \vdash_{\mathsf{K}} M_1 \,\|\cdots\|\, M_N : \mathsf{unit}} \ \forall j.\, \mathsf{Meths}(M_j) \subseteq \Theta$$

Figure 4 Library syntax, and typing rules for terms ($\vdash$), blocks ($\vdash_{\mathsf{B}}$), libraries ($\vdash_{\mathsf{L}}$), clients ($\vdash_{\mathsf{K}}$).

For simplicity, we do not include private methods, yet the same effect could be achieved by storing them in higher-order references. As we explain in the next section, references present in library definitions are de facto private to the library. Note also that, according to our definition, sets of abstract and public methods are disjoint. However, given $m, m' \in \mathsf{Refs}_{\theta,\theta'}$, one can define a "public abstract" method with: $\mathsf{public}\ m;\ \mathsf{abstract}\ m';\ m = \lambda x^{\theta}.m'x$.

Terms are typed in environments $\Gamma = \{x_1 : \theta_1, \cdots, x_n : \theta_n\}$. Method blocks are typed through judgements $\vdash_{\mathsf{B}} B : \Theta$, where $\Theta \subseteq \mathsf{Meths}$. The judgments collect the names of methods defined in a block as well as making sure that the definitions respect types and are not duplicated. Also, the initialisation statements must comply with types.

Finally, we type libraries using statements of the form $\Theta \vdash_{\mathsf{L}} L : \Theta' \to \Theta''$, where $\Theta, \Theta', \Theta'' \subseteq \mathsf{Meths}$ and $\Theta' \cap \Theta'' = \varnothing$. The judgment $\varnothing \vdash_{\mathsf{L}} L : \Theta' \to \Theta''$ guarantees that any method occurring in L is present either in Θ' or Θ'', that all methods in Θ' are declared as abstract and unimplemented, while all methods in Θ'' are declared as public and defined. Thus, $\varnothing \vdash_{\mathsf{L}} L : \Theta \to \Theta'$ is a library in which Θ, Θ' are the abstract and public methods respectively. In this case, we also write $L : \Theta \to \Theta'$.

4 Semantics and soundness

The semantics of our system is given in several stages. First, we define an operational semantics for sequential and concurrent terms that may draw methods from a repository. We then adapt it to capture interactions of concurrent clients with closed libraries (no abstract methods). This notion is then used to define contextual approximation for arbitrary libraries. Finally, we introduce a trace semantics of arbitrary libraries, which generates the histories on which our notions of linearisability are based.

$$(L) \longrightarrow_{\text{lib}} (L, \varnothing, S_{\text{init}}) \qquad (r := i; B, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R}, S[r \mapsto i])$$
$$(\mathsf{abstract}\ m; L, \mathcal{R}, S) \longrightarrow_{\text{lib}} (L, \mathcal{R}, S) \qquad (m = \lambda x.M; B\,, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R}_{**}, S)$$
$$(\mathsf{public}\ m; L, \mathcal{R}, S) \longrightarrow_{\text{lib}} (L, \mathcal{R}, S) \qquad (r := \lambda x.M; B, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R}_{**}, S[r \mapsto m])$$

$$(E[t_{\mathsf{id}}], \mathcal{R}, S) \to_t (E[t], \mathcal{R}, S) \qquad (E[\mathsf{if}\ i_*\ \mathsf{then}\ M_1\ \mathsf{else}\ M_0], \mathcal{R}, S) \to_t (E[M_{j_*}], \mathcal{R}, S)$$
$$(E[i_1 \oplus i_2], \mathcal{R}, S) \to_t (E[i_{**}], \mathcal{R}, S) \qquad (E[\pi_j\langle v_1, v_2\rangle], \mathcal{R}, S) \to_t (E[v_j], \mathcal{R}, S)$$
$$(E[!r], \mathcal{R}, S) \to_t (E[S(r)], \mathcal{R}, S) \qquad (E[\mathsf{let}\ x = v\ \mathsf{in}\ M], \mathcal{R}, S) \to_t (E[M\{v/x\}], \mathcal{R}, S)$$
$$(E[\lambda x.M], \mathcal{R}, S) \to_t (E[m], \mathcal{R}_{**}, S) \qquad (E[mv], \mathcal{R}_*, S) \to_t (E[M\{v/x\}], \mathcal{R}_*, S)$$

$$E ::= \bullet \mid E \oplus M \mid i \oplus E \mid \mathsf{if}\ E\ \mathsf{then}\ M\ \mathsf{else}\ M \mid \pi_j\, E \mid \langle E, M\rangle \mid \langle v, E\rangle \mid mE \mid \mathsf{let}\ x = E\ \mathsf{in}\ M \mid r := E$$

$$\frac{(M, \mathcal{R}, S) \to_t (M', \mathcal{R}', S')}{(M_1 \| \cdots \| M_{t-1} \| M \| M_{t+1} \| \cdots \| M_N, \mathcal{R}, S) \Longrightarrow (M_1 \| \cdots \| M_{t-1} \| M' \| M_{t+1} \| \cdots \| M_N, \mathcal{R}', S')}\ (K_N)$$

Figure 5 Evaluation rules for libraries ($\longrightarrow_{\text{lib}}$), terms ($\to_t$) and clients ($\Longrightarrow$). In the rules above we use the conditions/notation: $\mathcal{R}_{**} = \mathcal{R} \uplus (m \mapsto \lambda x.M)$, $i_{**} = i_1 \oplus i_2$, $\mathcal{R}_*(m) = \lambda x.M$, and $j_* = 0$ iff $i_* = 0$.

4.1 Library-client evaluation

Libraries, terms and clients are evaluated in environments comprising:

- A method environment $\mathcal{R}$, called *own-method repository*, which is a finite partial map on Meths assigning to each m in its domain, with $m \in \mathsf{Meths}_{\theta,\theta'}$, a term of the form $\lambda y.M$ (we omit type-superscripts from bound variables for economy).
- A finite partial map $S : \mathsf{Refs} \rightharpoonup (\mathbb{Z} \cup \mathsf{Meths})$, called *store*, which assigns to each r in its domain an integer (if $r \in \mathsf{Refs}_{\mathsf{int}}$) or name from $\mathsf{Meths}_{\theta,\theta'}$ (if $r \in \mathsf{Refs}_{\theta,\theta'}$).

The evaluation rules are presented in Figure 5, where we also define *evaluation contexts* E.

▶ Remark 15. We shall assume that reference names used in libraries are library-private, i.e. sets of reference names used in different libraries are assumed to be disjoint. Similarly, when libraries are being used by client code, this is done on the understanding that the references available to that code do not overlap with those used by libraries. Still, for simplicity, we shall rely on a single set Refs of references in our operational rules.

First we evaluate the library to create an initial repository and store. This is achieved by the first set of rules in Figure 5, where we assume that S_{init} is empty. Thus, library evaluation produces a tuple $(\epsilon, \mathcal{R}_0, S_0)$ including a method repository and a store, which can be used as the initial repository and store for evaluating $M_1 \| \cdots \| M_N$ using the (K_N) rule. We shall call the latter evaluation semantics for clients (denoted by $\Longrightarrow$) the *multi-threaded operational semantics*. The latter relies on closed-term reduction ($\to_t$), whose rules are given in the middle group, where t is the current thread index. Note that the rules for $E[\lambda x.M]$ in the middle group, along with those for $m = \lambda x.M$ and $r := \lambda x.M$ in the first group, involve the creation of a fresh method name m, which is used to put the function in the repository $\mathcal{R}$. Name creation is non-deterministic: any fresh m of the appropriate type can be chosen.

We define termination for clients linked with libraries that have no abstract methods. Recall our convention (Remark 15) that L and $M_1, \cdots, M_N$ must access disjoint parts of the store. Terms $M_1, \cdots, M_N$ can share reference names, though.

▶ **Definition 16.** Let $L : \varnothing \to \Theta'$ and $\Theta' \vdash_{\mathsf{K}} M_1 \| \cdots \| M_N : \mathsf{unit}$. We say that $M_1 \| \cdots \| M_N$ *terminates with linked library* L if $(M_1 \| \cdots \| M_N, \mathcal{R}_0, S_0) \Longrightarrow^* ((\) \| \cdots \| (\), \mathcal{R}, S)$, for some $\mathcal{R}, S$, where $(L) \longrightarrow^*_{\mathrm{lib}} (\epsilon, \mathcal{R}_0, S_0)$. We then write $\mathsf{link}\ L\ \mathsf{in}\ (M_1 \| \cdots \| M_N) \Downarrow$.

We shall build a notion of contextual approximation of libraries on top of termination: one library will be said to approximate another if, whenever the former terminates when composed with any parameter library and client, so does the latter.

We will be considering the following notions for composing libraries. Let us denote a library L as $L = D; B$, where D contains all the (public/abstract) method declarations of L, and B is its method block. We write $\mathsf{Refs}(L)$ for the set of references in L. Let $L_1 : \Theta_1 \to \Theta_2$ be of the form $D_1; B_1$. Given $L_2 : \Theta_1' \to \Theta_2'$ $(= D_2; B_2)$ such that $\Theta_2 \cap \Theta_2' = \mathsf{Refs}(L_1) \cap \mathsf{Refs}(L_2) = \varnothing$, $\Theta = \{m_1, \cdots, m_n\} \subseteq \Theta_2$ and $L' : \varnothing \to \Theta_1, \Theta'$, we define the *union* of L_1 and L_2, the Θ-*hiding* of L_1, and the *sequencing* of L' with L_1 respectively as:

$$\begin{array}{ll} L_1 \cup L_2 : (\Theta_1 \cup \Theta_1') \setminus (\Theta_2 \cup \Theta_2') \to \Theta_2 \cup \Theta_2' & = (D_1; B_1) \cup (D_2; B_2) = D_1'; D_2'; B_1; B_2 \\ L_1 \setminus \Theta : \Theta_1 \to (\Theta_2 \setminus \Theta) & = (D_1; B_1) \setminus \Theta = D_1''; B_1'\{!r_1/m_1\} \cdots \{!r_n/m_n\} \\ L'; L_1 : \varnothing \to \Theta_2, \Theta' & = (L' \cup L_1) \setminus \Theta_1 \end{array}$$

where D_1' is D_1 with any $\mathsf{abstract}\, m$ declaration removed for $m \in \Theta_2'$, dually for D_2'; and where D_1'' is D_1 without $\mathsf{public}\ m$ declarations for $m \in \Theta$ and each r_i is a fresh reference matching the type of m_i, and B_1' is obtained from B_1 by replacing each $m_i = \lambda x.M$ by $r_i := \lambda x.M$. Thus, the union of L_1 and L_2 corresponds to merging their code and removing any $\mathsf{abstract}$ declarations for methods that become defined. The hiding of a public method simply renders it private via the use of references.

▶ **Definition 17.** Given $L_1, L_2 : \Theta \to \Theta'$, we say that L_1 ***contextually approximates*** L_2, written $L_1 \sqsubseteq L_2$, if for all $L' : \varnothing \to \Theta, \Theta''$ and $\Theta', \Theta'' \vdash_{\mathsf{K}} M_1 \| \cdots \| M_N : \mathsf{unit}$, if $\mathsf{link}\ L'; L_1\ \mathsf{in}\ (M_1 \| \cdots \| M_N) \Downarrow$ then $\mathsf{link}\ L'; L_2\ \mathsf{in}\ (M_1 \| \cdots \| M_N) \Downarrow$. In this case, we also say that L_2 ***contextually refines*** L_1.

Note that, according to this definition, the parameter library L' may communicate directly with the client terms through a common interface Θ''. We shall refer to this case as the *general* case. Later on, we shall also consider more restrictive testing scenarios in which this possibility of explicit communication is removed. Moreover, from the disjointness conditions in the definitions of sequencing and linking we have that L_i, L' and $M_1 \| \cdots \| M_N$ access pairwise disjoint parts of the store.

4.2 Trace semantics

Building on the earlier semantics, we next introduce a trace semantics of libraries in the spirit of game semantics [2]. As mentioned in Section 2, the behaviour of a library will be represented as an exchange of moves between two players called P and O, representing the library and its corresponding context respectively. The context consists of the client of the library as well as the parameter library, with an index on each move ($\mathcal{K}/\mathcal{L}$) specifying which of them is involved in the move.

In contrast to the previous section, we handle scenarios in which called methods need not be present in the repository $\mathcal{R}$. Calls to such undefined methods are represented by labelled transitions – calls to the context made on behalf of the library (P). The calls can later be responded to with labelled transitions corresponding to returns, made by the context (O). On the other hand, O is able to invoke methods in $\mathcal{R}$, which will also be represented through suitable labels. Because we work in a higher-order setting, calls and returns made by both players may involve methods as arguments or results. Such methods also become available

(Int) $(\mathcal{E}, M, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \rightarrow_t (\mathcal{E}, M', \mathcal{R}', \mathcal{P}, \mathcal{A}, S')$, given that $(M, \mathcal{R}, S) \rightarrow_t (M', \mathcal{R}', S')$ and $\mathsf{dom}(\mathcal{R}' \setminus \mathcal{R})$ consists of names that do not occur in $\mathcal{E}, \mathcal{A}$.

(PQy) $(\mathcal{E}, E[mv], \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\mathsf{call}\ m(v')_{PY}}_t (m :: E :: \mathcal{E}, -, \mathcal{R}', \mathcal{P}', \mathcal{A}, S)$, given $m \in \mathcal{A}_Y$ and (**PC**).

(OQy) $(\mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\mathsf{call}\ m(v)_{OY}}_t (m :: \mathcal{E}, M\{v/x\}, \mathcal{R}, \mathcal{P}, \mathcal{A}', S)$, given $m \in \mathcal{P}_Y$, $\mathcal{R}(m) = \lambda x.M$ and (**OC**).

(PAy) $(m :: \mathcal{E}, v, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\mathsf{ret}\ m(v')_{PY}}_t (\mathcal{E}, -, \mathcal{R}', \mathcal{P}', \mathcal{A}, S)$, given $m \in \mathcal{P}_Y$ and (**PC**).

(OAy) $(m :: E :: \mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\mathsf{ret}\ m(v)_{OY}}_t (\mathcal{E}, E[v], \mathcal{R}, \mathcal{P}, \mathcal{A}', S)$, given $m \in \mathcal{A}_Y$ and (**OC**).

(PC) If v contains the names $m_1, \cdots, m_k$ then $v' = v\{m_i'/m_i \mid 1 \leq i \leq k\}$ with each m_i' being a fresh name. Moreover, $\mathcal{R}' = \mathcal{R} \uplus \{m_i' \mapsto \lambda x.m_i x \mid 1 \leq i \leq k\}$ and $\mathcal{P}' = \mathcal{P} \cup_Y \{m_1', \cdots, m_k'\}$.

(OC) If v contains names $m_1, \cdots, m_k$ then $m_i \in \phi(\mathcal{P}, \mathcal{A})$, for each i, and $\mathcal{A}' = \mathcal{A} \cup_Y \{m_1, \cdots, m_k\}$.

Figure 6 Trace semantics rules. The rule (Int) is for embedding internal rules. In the rule (PQy), the library (P) calls one of its abstract methods (either the original ones or those acquired via interaction), while in (PAy) it returns from such a call. The rules (OQy) and (OAy) are dual and represent actions of the context. In all of the rules, whenever we write $m(v)$ or $m(v')$, we assume that the type of v matches the argument type of m.

for future calls: function arguments/results supplied by P are added to the repository and can later be invoked by O, while function arguments/results provided by O can be queried in the same way as abstract methods.

The trace semantics utilises configurations that carry more components than the previous semantics. We define two kinds of configurations:

$$\textit{O-configurations}\ (\mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \quad \text{and} \quad \textit{P-configurations}\ (\mathcal{E}, M, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$$

where the component $\mathcal{E}$ is an *evaluation stack*, that is, a stack of the form $[X_1, X_2, \cdots, X_n]$ with each X_i being either an evaluation context or a method name. On the other hand, $\mathcal{P} = (\mathcal{P}_\mathcal{L}, \mathcal{P}_\mathcal{K})$ with $\mathcal{P}_\mathcal{L}, \mathcal{P}_\mathcal{K} \subseteq \mathsf{dom}(\mathcal{R})$ being sets of *public* method names, and $\mathcal{A} = (\mathcal{A}_\mathcal{L}, \mathcal{A}_\mathcal{K})$ is a pair of sets of *abstract* method names. $\mathcal{P}$ will be used to record all the method names produced by P and passed to O: those passed to $O\mathcal{K}$ are stored in $\mathcal{P}_\mathcal{K}$, while those leaked to $O\mathcal{L}$ are kept in $\mathcal{P}_\mathcal{L}$. Inside $\mathcal{A}$, the story is the opposite one: $\mathcal{A}_\mathcal{K}$ ($\mathcal{A}_\mathcal{L}$) stores the method names produced by $O\mathcal{K}$ (resp. $O\mathcal{L}$) and passed to P. Consequently, the sets of names stored in $\mathcal{P}_\mathcal{L}, \mathcal{P}_k, \mathcal{A}_\mathcal{L}, \mathcal{A}_k$ will always be disjoint.

Given a pair $\mathcal{P}$ as above and a set $Z \subseteq \mathsf{Meths}$, we write $\mathcal{P} \cup_\mathcal{K} Z$ for the pair $(\mathcal{P}_\mathcal{L}, \mathcal{P}_\mathcal{K} \cup Z)$. We define $\cup_\mathcal{L}$ in a similar manner, and extend it to pairs $\mathcal{A}$ as well. Moreover, given $\mathcal{P}$ and $\mathcal{A}$, we let $\phi(\mathcal{P}, \mathcal{A})$ be the set of *fresh* method names for $\mathcal{P}, \mathcal{A}$: $\phi(\mathcal{P}, \mathcal{A}) = \mathsf{Meths} \setminus (\mathcal{P}_\mathcal{L} \cup \mathcal{P}_\mathcal{K} \cup \mathcal{A}_\mathcal{L} \cup \mathcal{A}_\mathcal{K})$.

We give the rules generating the trace semantics in Figure 6. Note that the rules are parameterised by: P/O and Y, which together determine the polarity of the next move; Q/A, which stands for the move being a call (*Question*) or a return (*Answer*) respectively. The rules depict the intuition presented above. When in an O-configuration, the context may issue a call to a public method $m \in \mathcal{P}_Y$ and pass control to the library (rule (OQy)). Note that, when this occurs, the name m is added to the evaluation stack $\mathcal{E}$ and a P-configuration is obtained. From there on, the library will compute internally using rule (Int), until: it either needs to evaluate an abstract method (i.e. some $m' \in \mathcal{A}_Y$), and hence issues a call via rule (PQy); or it completes its computation and returns the call (rule (PAy)). Calls to abstract methods, on the other hand, are met either by further calls to public methods (via (OQy)), or by returns (via (OAy)).

Finally, we extend the trace semantics to a concurrent setting where a fixed number of N-many threads run in parallel. Each thread has separate evaluation stack and term

components, which we write as $\mathcal{C} = (\mathcal{E}, X)$ (where X is a term or "$-$"). Thus, a configuration now is of the following form:

$$N\text{-}\textit{configuration}\ \ (\mathcal{C}_1 \| \cdots \| \mathcal{C}_N, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$$

where, for each i, $\mathcal{C}_i = (\mathcal{E}_i, X_i)$ and $(\mathcal{E}_i, X_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ is a sequential configuration. We shall abuse notation a little and write $(\mathcal{C}_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ for $(\mathcal{E}_i, X_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$. Also, below we write $\vec{\mathcal{C}}$ for $\mathcal{C}_1 \| \cdots \| \mathcal{C}_N$ and $\vec{\mathcal{C}}[i \mapsto \mathcal{C}'] = \mathcal{C}_1 \| \cdots \| \mathcal{C}_{i-1} \| \mathcal{C}' \| \mathcal{C}_{i+1} \| \cdots \| \mathcal{C}_N$ and, for economy, we use $\mathcal{RPAS}$ to range over tuples $(\mathcal{R}, \mathcal{P}, \mathcal{A}, S)$. The concurrent traces are produced by the following two rules

$$\frac{(\mathcal{C}_i, \mathcal{RPAS}) \to_i (\mathcal{C}', \mathcal{RPAS}')}{(\vec{\mathcal{C}}, \mathcal{RPAS}) \Longrightarrow (\vec{\mathcal{C}}[i \mapsto \mathcal{C}'], \mathcal{RPAS}')}\ (\textsc{PInt}) \qquad \frac{(\mathcal{C}_i, \mathcal{RPAS}) \xrightarrow{x_{XY}}_i (\mathcal{C}', \mathcal{RPAS}')}{(\vec{\mathcal{C}}, \mathcal{RPAS}) \xRightarrow{(i,x)_{XY}} (\vec{\mathcal{C}}[i \mapsto \mathcal{C}'], \mathcal{RPAS}')}\ (\textsc{PExt})$$

with the proviso that the names freshly produced internally in (PInt) are fresh for the whole of $\vec{\mathcal{C}}$.

We can now define the trace semantics of a library L. We call a configuration component $\mathcal{C}_i$ ***final*** if it is in one of the following forms, for O- and P-configurations respectively: $\mathcal{C}_i = ([\,], -)$ or $\mathcal{C}_i = ([\,], ())$. We call $(\vec{\mathcal{C}}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ final just if $\vec{\mathcal{C}} = \mathcal{C}_1 \| \cdots \| \mathcal{C}_N$ and each $\mathcal{C}_i$ is final.

▶ **Definition 18.** For each $L : \Theta \to \Theta'$, we define the N-trace semantics of L to be:

$$[\![L]\!]_N = \{\, s \mid (\vec{\mathcal{C}}_0, \mathcal{R}_0, (\varnothing, \Theta'), (\Theta, \varnothing), S_0) \xRightarrow{s}{}^{*} \rho \ \wedge\ \rho \text{ final} \,\}$$

where $\vec{\mathcal{C}}_0 = ([\,], -) \| \cdots \| ([\,], -)$ and $(L) \longrightarrow^{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$. We may write $[\![L]\!]_N$ simply as $[\![L]\!]$.

We are now able to revisit the linearisability claims anticipated in Examples 1, 11 and 13.

▶ **Lemma 19** (Linearisability examples)**.**

1. $L_{\mathsf{mset}} \sqsubseteq A_{\mathsf{mset}}$,
2. $L_{\mathsf{mset2}} \sqsubseteq_{\text{enc}} A_{\mathsf{mset2}}$,
3. $L_{\mathsf{fc}} \sqsubseteq_{\mathcal{R}} L_{\mathsf{spec}}$.

We conclude the presentation of the trace semantics by providing a semantics for library contexts. Recall that in our setting (Figure 1) a library $L : \Theta \to \Theta'$ is deployed in a context consisting of a parameter library $L' : \varnothing \to \Theta, \Theta''$ and a concurrent composition of client threads $\Theta', \Theta'' \vdash M_i : \mathsf{unit}$ $(i = 1, \cdots, N)$. We shall write $\mathsf{link}\ L'\mathsf{;}-\ \mathsf{in}\ (M_1 \| \cdots \| M_N)$, or simply C, to refer to such contexts.

▶ **Definition 20.** Let $\Theta', \Theta'' \vdash_{\mathsf{K}} M_1 \| \cdots \| M_N : \mathsf{unit}$ and $L' : \varnothing \to \Theta, \Theta''$. We define:

$$[\![\mathsf{link}\ L'\mathsf{;}-\ \mathsf{in}\ (M_1 \| \cdots \| M_N)]\!] = \{\, s \mid (\vec{\mathcal{C}}_0, \mathcal{R}_0, (\Theta, \varnothing), (\varnothing, \Theta'), S_0) \xRightarrow{s}{}^{*} \rho \wedge \rho \text{ final} \,\}$$

where $(L') \longrightarrow^{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$ and $\vec{\mathcal{C}}_0 = ([\,], M_1) \| \cdots \| ([\,], M_N)$.

▶ **Lemma 21.** *For any $L : \Theta \to \Theta'$, $L' : \varnothing \to \Theta, \Theta''$ and $\Theta', \Theta'' \vdash_{\mathsf{K}} M_1 \| \cdots \| M_N : \mathsf{unit}$ we have $[\![L]\!]_N \subseteq \mathcal{H}_{\Theta,\Theta'}$ and $[\![\mathsf{link}\ L'\mathsf{;}-\ \mathsf{in}\ (M_1 \| \cdots \| M_N)]\!] \subseteq \mathcal{H}^{co}_{\Theta,\Theta'}$.*

4.3 Soundness

To conclude, we clarify in what sense all the notions of linearisability are sound. Recall the general notion of contextual approximation (refinement) from Definition 17. In the encapsulated case libraries are being tested by clients that do not communicate with the parameter library explicitly. The corresponding definition of contextual approximation is defined below.

▶ **Definition 22 (Encapsulated $\sqsubseteq$).** Given libraries $L_1, L_2 : \Theta \to \Theta'$, we write $L_1 \sqsubseteq_{\text{enc}} L_2$ when, for all $L' : \varnothing \to \Theta$ and $\Theta' \vdash_{\mathsf{K}} M_1 \| \cdots \| M_N : \mathsf{unit}$, if $\mathsf{link}\ L'\,;L_1\ \mathsf{in}\ (M_1\|\cdots\|M_N)\Downarrow$ then $\mathsf{link}\ L'\,;L_2\ \mathsf{in}\ (M_1\|\cdots\|M_N)\Downarrow$.

For relational linearisability, we need yet another notion that will link $\mathcal{R}$ to contextual testing.

▶ **Definition 23.** Let $\mathcal{R} \subseteq \mathcal{H}_{\Theta,\Theta'} \times \mathcal{H}_{\Theta,\Theta'}$ be a set closed under permutation of names in $\mathsf{Meths} \setminus (\Theta \cup \Theta')$. We say that a context formed by L' and $M_1, \cdots, M_N$ is $\mathcal{R}$-closed if, for any $h \in [\![\mathsf{link}\ L'\,;-\ \mathsf{in}\ (M_1\|\cdots\|M_N)]\!]$, $\overline{h}\,\mathcal{R}\,\overline{h'}$ implies $h' \in [\![\mathsf{link}\ L'\,;-\ \mathsf{in}\ (M_1\|\cdots\|M_N)]\!]$. Given $L_1, L_2 : \Theta \to \Theta'$, we write $L_1 \sqsubseteq_{\mathcal{R}} L_2$ if, for all $\mathcal{R}$-closed contexts formed from $L', M_1, \cdots, M_N$, whenever $\mathsf{link}\ L'\,;L_1\ \mathsf{in}\ (M_1\|\cdots\|M_N)\Downarrow$ then we also have $\mathsf{link}\ L'\,;L_2\ \mathsf{in}\ (M_1\|\cdots\|M_N)\Downarrow$.

▶ **Theorem 24** (Correctness)**.**

1. *$L_1 \sqsubseteq L_2$ implies $L_1 \precsim L_2$.*
2. *$L_1 \sqsubseteq_{\text{enc}} L_2$ implies $L_1 \precsim_{enc} L_2$.*
3. *$L_1 \sqsubseteq_{\mathcal{R}} L_2$ implies $L_1 \precsim_{\mathcal{R}} L_2$.*

Finally, linearisability is compatible with library composition. $\sqsubseteq$ is closed under union with libraries that use disjoint stores, while $\sqsubseteq_{\text{enc}}$ is closed under a form of sequencing that respects encapsulations.

5 Related and future work

Linearisability has been consistently used as a correctness criterion for concurrent algorithms on a variety of data structures [18], and has inspired a variety of proof methods [5]. An explicit connection between linearisability and refinement was made in [6], where it was shown that, in base-type settings, linearisability and refinement coincide. Similar results have been proved in [4, 9, 17, 3]. Our contributions are notions of linearisability that serve as correctness criteria for libraries with methods of arbitrary order and have a similar relationship to refinement. The next natural target is to investigate proof methods for establishing linearisability of higher-order concurrent libraries. The examples proved herein are only an initial step in that direction.

At the conceptual level, [6] proposed that the verification goal behind linearisability is observational refinement. In this vein, [24] utilised logical relations as a direct method for proving refinement in a higher-order concurrent setting, while [23] introduced a program logic that builds on logical relations. On the other hand, proving conformance to a history specification has been addressed in [20] by supplying history-aware interpretations to off-the-shelf Hoare logics for concurrency. Other logic-based approaches for concurrent higher-order libraries, which do not use linearisability, include Higher-Order and Impredicative Concurrent Abstract Predicates [21, 22].

Acknowledgements. We thank the authors of [3] for bringing the higher-order linearisability problem to our attention, Radha Jagadeesan and Kasper Svendsen for constructive comments, and C. Tzevelekou for help with Figure 1.

References

1. http://c-cube.github.io/ocaml-containers/0.21/CCMultiSet.S.html.
2. S. Abramsky and G. McCusker. Game semantics. In H. Schwichtenberg and U. Berger, editors, *Logic and Computation*. Springer-Verlag, 1998. Proceedings of the 1997 Marktoberdorf Summer School.
3. A. Cerone, A. Gotsman, and H. Yang. Parameterised linearisability. In *Proceedings of ICALP'14*, volume 8573 of *Lecture Notes in Computer Science*, pages 98–109. Springer, 2014.
4. J. Derrick, G. Schellhorn, and H. Wehrheim. Mechanically verified proof obligations for linearizability. *ACM Trans. Program. Lang. Syst.*, 33(1):4, 2011.
5. B. Dongol and J. Derrick. Verifying linearisability: A comparative survey. *ACM Comput. Surv.*, 48(2):19, 2015.
6. I. Filipovic, P. W. O'Hearn, N. Rinetzky, and H. Yang. Abstraction for concurrent objects. *Theor. Comput. Sci.*, 411(51-52):4379–4398, 2010.
7. D. R. Ghica and A. S. Murawski. Angelic semantics of fine-grained concurrency. In *Proceedings of FOSSACS*, volume 2987 of *Lecture Notes in Computer Science*, pages 211–225. Springer-Verlag, 2004.
8. D. R. Ghica and N. Tzevelekos. A system-level game semantics. *Electr. Notes Theor. Comput. Sci.*, 286:191–211, 2012.
9. A. Gotsman and H. Yang. Liveness-preserving atomicity abstraction. In *Proceedings of ICALP*, volume 6756 of *Lecture Notes in Computer Science*, pages 453–465. Springer-Verlag, 2011.
10. S. Heller, M. Herlihy, V. Luchangco, M. Moir, W. N. Scherer III, and N. Shavit. A lazy concurrent list-based set algorithm. In *Proceedings of OPODIS*, volume 3974 of *Lecture Notes in Computer Science*, pages 3–16, Springer-Verlag, 2006.
11. D. Hendler, I. Incze, N. Shavit, and M. Tzafrir. Flat combining and the synchronization-parallelism tradeoff. In *Proceedings of SPAA*, pages 355–364, ACM, 2010.
12. M. Herlihy and J. M. Wing. Linearizability: A correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.
13. R. Jagadeesan, G. Petri, C. Pitcher, and J. Riely. Quarantining weakness - compositional reasoning under relaxed memory models (extended abstract). In *Proceedings of ESOP*, volume 7792 of *Lecture Notes in Computer Science*, pages 492–511, Springer-Verlag, 2013.
14. A. Jeffrey and J. Rathke. A fully abstract may testing semantics for concurrent objects. *Theor. Comput. Sci.*, 338(1-3):17–63, 2005.
15. J. Laird. A game semantics of Idealized CSP. In *Proceedings of MFPS'01*, pages 1–26. Elsevier, 2001. ENTCS, Vol. 45.
16. J. Laird. A fully abstract trace semantics for general references. In *Proceedings of ICALP*, volume 4596 of *Lecture Notes in Computer Science*, pages 667–679. Springer, 2007.
17. H. Liang and X. Feng. Modular verification of linearizability with non-fixed linearization points. In Proceedings of PLDI, pages 459–470, ACM, 2013.
18. M. Moir and N. Shavit. Concurrent data structures. In *Handbook of Data Structures and Applications*. Chapman and Hall/CRC, 2004.
19. P. W. O'Hearn, N. Rinetzky, M. T. Vechev, E. Yahav, and G. Yorsh. Verifying linearizability with hindsight. In *Proceedings of PODC*, pages 85–94, ACM, 2010.
20. I. Sergey, A. Nanevski, and A. Banerjee. Specifying and verifying concurrent algorithms with histories and subjectivity. In *Proceedings of ESOP*, volume 9032 of *Lecture Notes in Computer Science*, pages 333–358, Springer, 2015.
21. K. Svendsen and L. Birkedal. Impredicative concurrent abstract predicates. In *Proceedings of ESOP*, volume 8410 of *Lecture Notes in Computer Science*, pages 149–168, Springer, 2014.

22 K. Svendsen, L. Birkedal, and M. J. Parkinson. Joins: A case study in modular specification of a concurrent reentrant higher-order library. In *Proceedings of ECOOP*, volume 7920 of *Lecture Notes in Computer Science*, pages 327–351, Springer, 2013.

23 A. Turon, D. Dreyer, and L. Birkedal. Unifying refinement and Hoare-style reasoning in a logic for higher-order concurrency. In *Proceedings of ICFP*, pages 377–390, ACM, 2013.

24 A. J. Turon, J. Thamsborg, A. Ahmed, L. Birkedal, and D. Dreyer. Logical relations for fine-grained concurrency. In *Proceedings of POPL*, pages 343–356, ACM, 2013.

Model Checking ω-regular Properties for Quantum Markov Chains*

Yuan Feng[1], Ernst Moritz Hahn[2], Andrea Turrini[3], and Shenggang Ying[4]

1 **Centre for Quantum Software and Information, University of Technology Sydney, Sydney, Australia**
Yuan.Feng@uts.edu.au

2 **State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, P. R. China; and Saarland University, Saarland Informatics Campus, Saarbrücken, Germany**
hahn@ios.ac.cn

3 **State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, P. R. China**
turrini@ios.ac.cn

4 **Centre for Quantum Software and Information, University of Technology Sydney, Sydney, Australia**
Shenggang.Ying@uts.edu.au

Abstract

Quantum Markov chains are an extension of classical Markov chains which are labelled with super-operators rather than probabilities. They allow to faithfully represent quantum programs and quantum protocols. In this paper, we investigate model checking ω-regular properties, a very general class of properties (including, e.g., LTL properties) of interest, against this model.

For classical Markov chains, such properties are usually checked by building the product of the model with a language automaton. Subsequent analysis is then performed on this product. When doing so, one takes into account its graph structure, and for instance performs different analyses per bottom strongly connected component (BSCC). Unfortunately, for quantum Markov chains such an approach does not work directly, because super-operators behave differently from probabilities. To overcome this problem, we transform the product quantum Markov chain into a single super-operator, which induces a decomposition of the state space (the tensor product of classical state space and the quantum one) into a family of BSCC subspaces. Interestingly, we show that this BSCC decomposition provides a solution to the issue of model checking ω-regular properties for quantum Markov chains.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Quantum Markov chains, model checking, ω-regular properties, bottom strongly connected component

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.35

* This work was partially supported by the Australian Research Council (Grant Nos. DP130102764 and DP160101652), the Singapore Ministry of Education (Grant No. MOE2015-T2-1-137), the CAS/SAFEA International Partnership Program for Creative Research Teams, the National Natural Science Foundation of China (Grant Nos. 61550110506 and 61650410658), the Chinese Academy of Sciences Fellowship for International Young Scientists, and the CDZ project CAP (GZ 1023).

1 Introduction

Since its introduction, quantum computing has been considered a really promising technology for solving computationally complex tasks. Some of these tasks, such as factorisation and discrete logarithm computation, are the building blocks of cryptographic protocols developed to ensure security and privacy in communication. Quantum computing, by its own nature, allows for an easy solution to such tasks, so the future construction of a working quantum computer would compromise several important cryptography-based applications such as bank transactions and private communication. This has given rise to a large amount of research for providing a new class of communication protocols based on quantum mechanics so as to get back the desired properties. For instance, protocols such as super-dense coding [7], quantum coin-flipping protocol [6], and quantum key distribution protocols [6, 5] have been proposed as new building blocks for quantum cryptography.

However, as quantum mechanics is counter-intuitive, quantum protocol designers are more likely to make errors than their classical peers. This will become especially serious when more and more complicated quantum protocols can be implemented by future physical technology. Therefore, it is indispensable to develop methodologies and techniques for the verification of quantum systems.

This paper explores the possibility of applying model checking [11, 3], one of the dominant techniques for verification which has already a large number of successful industrial applications [8, 10, 21], to the verification of quantum protocols. In particular, we are interested in model checking ω-regular properties, a very general class of properties subsuming those expressible by LTL formulae, against quantum Markov chains (QMCs), an extension of classical Markov chains which allow to faithfully represent quantum programs and quantum protocols. Similar to the classical case, we first take the product of the QMC and a parity automaton representing the ω-regular property of interest. The model checking problem then boils down to calculating the value of the product parity quantum Markov chain (PQMC). However, we show by a counterexample that the traditional BSCC decomposition analysis used for classical model checking does not work in quantum case. To overcome this problem, we transform the product PQMC into a single super-operator on an extended Hilbert space including both the classical and quantum states. We show that due to the special structure of such an extended super-operator, the notion of BSCC *subspaces* for super-operators defined in [32] can be applied to tackle the problem.

1.1 Related works

The main obstacle of model checking quantum systems is that the set of all quantum states, traditionally regarded as the underlying state space of the model to be checked, is a continuum. Hence, the techniques of classical model checking, which normally work only for a finite state space, cannot be applied directly. Gay et al. [17] considered a special scenario where the initial state is a *stabiliser state*, and the quantum operations allowed all belong to the class of *Clifford group*, so that all the quantum states produced in the evolution are finitely describable. In this way, they proposed an efficient model checker [18] for certain quantum protocols, employing purely classical algorithms. Based on the same simplification, Ardeshir-Larijani et al. developed equivalence checkers for deterministic quantum protocols [1] as well as concurrent quantum protocols that behave *functionally* [2]. However, this approach does not work for general quantum systems. In contrast, the quantum Markov chain model adopted in this paper, which is derived from [16], is capable of describing general quantum programs and protocols, not only those in stabiliser formalism.

The state space of our quantum Markov chain is taken classical and transitions between classical states are labelled by trace-nonincreasing super-operators (thus the corresponding quantum state space is *implicitly* implied). In contrast, there is another notion of quantum Markov chains, which is a pair $(\mathcal{H}, \mathcal{E})$ with $\mathcal{H}$ being a finite dimensional Hilbert space and $\mathcal{E}$ a trace-preserving super-operator on $\mathcal{H}$, investigated in the literature. Model checking techniques for this notion of quantum Markov chains have been extensively investigated in recent years [32, 31, 22]. These two notions of quantum Markov chains turn out to be equivalent in expressive power [22]. However, they are useful in different scenarios. The model in [32] corresponds naturally to generic quantum operations and quantum communication channels, both being popular objects of study in quantum information theory. In contrast, the quantum Markov chain model considered in this paper is more suitable for analysing quantum programs and protocols where classical states such as program counters, program variables, and measurement outcomes are naturally present.

1.2 Relevance of our work

Quantum Markov chains

The notion of quantum Markov chains studied in this paper was introduced in [16] (a similar definition was given in [19] to generalise quantum walks), which has been shown to be expressive enough to describe general quantum systems. The explicit modelling of a quantum **while** program and well-known quantum protocols such as teleportation, superdense coding, quantum key distribution protocol BB84, etc., can be found in [16, 15].

One of the distinct features of this model, for verification purpose, is that it provides a way to check *once for all* in that once a property is checked to hold, it holds for all initial quantum states. This is especially important for the verification of quantum programs. For example, for the reachability problem we calculate the accumulated *super-operator*, say $\mathcal{E}$, along all valid paths. As a result, the reachability *probability* when the program is executed on the initial quantum state ρ is simply the trace $\mathrm{tr}(\mathcal{E}(\rho))$ of $\mathcal{E}(\rho)$.

ω-regular properties for QMCs

It has been shown in [16] how properties in quantum computation tree logic (QCTL), a quantum variant of the probabilistic CTL (PCTL), can be verified. We then provided a tool implementation [15] based on the probabilistic model checker IscasMC [20]. The applicability of this method so far was however hindered by the fact that the expressiveness of QCTL is rather limited. As the logic PCTL by which it was motivated, QCTL basically only allows to describe nested (single-step, bounded, and unbounded) reachability problems. To overcome this issue, in this paper we describe how ω-regular properties, and in particular linear time logic (LTL) properties, can be checked on quantum Markov chains. This allows to express and analyse a wide range of relevant properties, such as repeated reachability, reachability in a restricted order, nested Until properties, or conjunctions of such properties.

Admittedly, up to now we still do not have any quantum communication protocols that have desired properties only describable in ω-regular languages (that is also why we could not have a case study to test the effectiveness of our approach and algorithm in this paper). However, with the rapid development of quantum communication technology, especially quantum cryptographic systems, being able to check these kinds of properties for quantum Markov chains will be necessary, as they allow for instance to verify that the processes in a quantum communication protocol will repeatedly send messages, that messages are sent in

the correct order, that the key is exchanged for sure, etc., all of which cannot be expressed in QCTL.

2 Quantum Markov Chains

In this section, we recall the required notions of quantum Markov chains. For a more thorough discussion, we refer the interested reader to [24, 16].

Given a finite dimensional Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on it. Let $\mathcal{S}(\mathcal{H})$ be the set of *super-operators*, that is, completely positive linear operators from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. In particular, we denote by $\mathcal{I}_{\mathcal{H}}$ and $0_{\mathcal{H}}$ the identity and null super-operators in $\mathcal{S}(\mathcal{H})$, respectively. For simplicity, we abuse the notation slightly by denoting $\mathcal{E} = \{ E_i \mid i \in I \}$ if $\{ E_i \mid i \in I \}$ is a set of Kraus operators of $\mathcal{E}$; that is, $\mathcal{E}(A) = \sum_{i \in I} E_i A E_i^\dagger$ for all $A \in \mathcal{L}(\mathcal{H})$. For any $\mathcal{E}, \mathcal{F} \in \mathcal{S}(\mathcal{H})$, the composition of $\mathcal{E}$ and $\mathcal{F}$ is defined by $(\mathcal{E} \circ \mathcal{F})(A) = \mathcal{E}(\mathcal{F}(A))$. We sometimes omit the symbol $\circ$ and write $\mathcal{E}\mathcal{F}$ directly for $\mathcal{E} \circ \mathcal{F}$. A (pre-)order is defined in $\mathcal{S}(\mathcal{H})$ by setting $\mathcal{E} \lesssim \mathcal{F}$ if for any $\rho \in \mathcal{D}(\mathcal{H})$, $\mathrm{tr}(\mathcal{E}(\rho)) \leq \mathrm{tr}(\mathcal{F}(\rho))$. Here $\mathcal{D}(\mathcal{H})$ is the set of partial density operators in $\mathcal{L}(\mathcal{H})$, i.e., positive semidefinite operators ρ with the trace $\mathrm{tr}(\rho)$ being no larger than 1. Note that the trace of a partial density operator denotes the probability that the corresponding (normalised) quantum state is reached [28]. Intuitively, $\mathcal{E} \lesssim \mathcal{F}$ means that the success probability of performing $\mathcal{E}$ is always not greater than that of performing $\mathcal{F}$, whatever the initial state is. Let $\eqsim$ be $\lesssim \cap \gtrsim$.

We denote by $\mathcal{S}^{\mathcal{I}}(\mathcal{H})$ the set of trace-nonincreasing super-operators over $\mathcal{H}$; that is, $\mathcal{S}^{\mathcal{I}}(\mathcal{H}) = \{ \mathcal{E} \in \mathcal{S}(\mathcal{H}) \mid 0_{\mathcal{H}} \lesssim \mathcal{E} \lesssim \mathcal{I}_{\mathcal{H}} \}$. Observe that $\mathcal{E} \in \mathcal{S}^{\mathcal{I}}(\mathcal{H})$ if and only if for any $\rho \in \mathcal{D}(\mathcal{H})$, $\mathrm{tr}(\mathcal{E}(\rho)) \in [0, 1]$. Thus it is natural to regard the set $\mathcal{S}^{\mathcal{I}}(\mathcal{H})$ as the quantum counterpart of $[0, 1]$, the domain of traditional probabilities. This is exactly the key to the notion of quantum Markov chains defined in [16], that we use as our basic model.

▶ **Definition 1** (Quantum Markov Chain)**.** A *super-operator weighted Markov chain* over a Hilbert space $\mathcal{H}$, also referred to as *quantum Markov chain (QMC)* for simplicity, is a tuple $(S, \mathbf{Q})$, where

1. S is a finite set of *classical states*;
2. $\mathbf{Q} \colon S \times S \to \mathcal{S}^{\mathcal{I}}(\mathcal{H})$ is called the *transition matrix* where for each $s \in S$, the super-operator $\sum_{s' \in S} \mathbf{Q}(s, s')$ is trace-preserving, that is $\sum_{s' \in S} \mathbf{Q}(s, s') \eqsim \mathcal{I}_{\mathcal{H}}$.

Similar to classical Markov chains, the notions of paths and measures can be defined for QMCs.

▶ **Definition 2** (Paths and measures)**.** Consider a QMC $\mathcal{M} = (S, \mathbf{Q})$. A path σ of $\mathcal{M}$ is a finite or infinite sequence $s_0 s_1 \ldots$ of states in S such that for each valid index $i \geq 1$, $\mathbf{Q}(s_{i-1}, s_i) \neq 0_{\mathcal{H}}$. For a valid index i, we let $\sigma[i] \stackrel{\text{def}}{=} s_i$. We denote the set of finite paths as $\mathrm{Path}^{\mathcal{M}}_{fin}$ and the set of infinite paths as $\mathrm{Path}^{\mathcal{M}}$. We define the *cylinder set* of a finite path $\sigma = s_0 s_1 \ldots s_n$ as $Cyl(\sigma) \stackrel{\text{def}}{=} \{ \sigma' \in \mathrm{Path}^{\mathcal{M}} \mid \forall i, 0 \leq i \leq n.\ \sigma[i] = \sigma'[i] \}$. Let $(\mathrm{Path}^{\mathcal{M}}, \Sigma)$ be a measurable space where Σ is the σ-algebra generated by all the cylinder sets $Cyl(\sigma)$ where $\sigma \in \mathrm{Path}^{\mathcal{M}}_{fin}$. For any $s \in S$, we define $Q^{\mathcal{M}}_s \colon \mathrm{Path}^{\mathcal{M}}_{fin} \to \mathcal{S}(\mathcal{H})$ as

$$Q^{\mathcal{M}}_s(\sigma) \stackrel{\text{def}}{=} \begin{cases} 0_{\mathcal{H}} & s \neq s_0 \\ \mathcal{I}_{\mathcal{H}} & s = s_0 \wedge n = 0 \\ \mathbf{Q}(s_{n-1}, s_n)\mathbf{Q}(s_{n-2}, s_{n-1}) \cdots \mathbf{Q}(s_0, s_1) & s = s_0 \wedge n > 0. \end{cases}$$

Then $Q^{\mathcal{M}}_s$ induces a (super-operator valued) measure on $(\mathrm{Path}^{\mathcal{M}}, \Sigma)$, denoted by $Q^{\mathcal{M}}_s$ as well for simplicity, by setting $Q^{\mathcal{M}}_s(Cyl(\sigma)) \stackrel{\text{def}}{=} Q^{\mathcal{M}}_s(\sigma)$.

From [16, Theorem 3.2], this measure is unique up to $\eqsim$.

3 Model checking ω-regular properties for QMCs

LTL and ω-regular properties have been studied extensively for classical Markov chains [13, 12, 9, 3]. To compute the probability $\mathcal{P}(\phi)$ that a certain LTL property ϕ is satisfied in a Markov chain $\mathcal{M}$, the classical automaton-based approach works as follows. At first, ϕ is transformed into a nondeterministic Büchi automaton, which is then transformed into a deterministic automaton $\mathcal{A}$ with a more complex acceptance condition, such as Rabin or Parity acceptance. Such a determinisation step usually exploits a variant of Safra's [26] determinisation construction, such as the techniques presented in [25, 27]. Afterwards, the product $\mathcal{M} \otimes \mathcal{A}$ of $\mathcal{M}$ and $\mathcal{A}$ is constructed, which is a Markov chain equipped with an acceptance condition. Finally, using algorithms operating on the graph structure of the product Markov chain, the states of $\mathcal{M} \otimes \mathcal{A}$ are categorised into those belonging to a *bottom strongly connected component (BSCC)* and *transient states*. According to the acceptance condition, each BSCC is then marked as accepting or rejecting. The probability that ϕ holds in a transient state s can then be obtained by solving an equation system representing the probability that from s an accepting BSCC is reached.

This section is devoted to extending this approach to quantum Markov chains. However, the extension is not trivial: as will be shown by a counterexample in Section 3.2, while the product construction itself does not lead to any problem, its decomposition into BSCCs and transient states cannot be performed as in the classical case. Therefore, in Section 3.3, we provide an alternative approach which does not directly rely on the graph structure of the product. Specifically, we transform $\mathcal{M} \otimes \mathcal{A}$ into a single super-operator, and show that the BSCC decomposition of the classical-quantum Hilbert space (the tensor product of classical state space and the quantum one) induced by this super-operator, instead of the decomposition of the classical state space alone, provides a desired solution to the model checking ω-regular properties for quantum Markov chains.

3.1 Parity automata and parity quantum Markov chains

In order to define properties of QMCs, we consider an extension in which their states are decorated by a labelling function.

▶ **Definition 3** (Labelled Quantum Markov Chain)**.** A *labelled quantum Markov chain* (LQMC) is a tuple $\mathcal{M} = (S, \mathbf{Q}, AP, L)$, where $(S, \mathbf{Q})$ is a QMC and

1. AP is a finite set of *atomic propositions*; and
2. $L\colon S \to 2^{AP}$ is a *labelling function*.

The notions of paths, measures, etc. for LQMCs are as in Definitions 1 and 2. We extend the labelling functions to paths by setting

$$L(s_0 s_1 s_2 \ldots) \stackrel{\text{def}}{=} L(s_0)L(s_1)L(s_2)\ldots.$$

The properties we are interested in are the ω-regular properties (which include properties definable in LTL).

▶ **Definition 4** (ω-regular Properties)**.** An *ω-regular language* is a subset of $(2^{AP})^\omega$ which can be defined using an ω-regular expression [29]. Consider an LQMC $\mathcal{M} = (S, \mathbf{Q}, AP, L)$ and an ω-regular language $\mathcal{W} \subseteq (2^{AP})^\omega$. We define $Q_s^{\mathcal{M}}(\mathcal{W}) \stackrel{\text{def}}{=} Q_s^{\mathcal{M}}(\{\, \sigma \in \text{Path}^{\mathcal{M}} \mid L(\sigma) \in \mathcal{W} \,\})$.

We shortly restate a well-known mechanism to decide whether a word is included in a given ω-regular language. For this purpose, an additional definition is needed.

▶ **Definition 5** (Parity Automaton). A *(deterministic) parity automaton* (PA) is a tuple $\mathcal{A} = (A, \overline{a}, AP, t, \mathsf{pri})$, where

1. A is a finite set of automaton states, and $\overline{a} \in A$ is the *initial state*,
2. AP is a finite set of *atomic propositions*,
3. $t\colon A \times 2^{AP} \to A$ is a *transition function*,
4. $\mathsf{pri}\colon A \to \mathbb{N}$ is a *priority function*. Here $\mathbb{N}$ denotes the set of natural numbers.

A *path* of $\mathcal{A}$ is an infinite sequence $\sigma = a_0 L_0 a_1 L_1 \ldots \in (A \times 2^{AP})^\omega$ such that $a_0 = \overline{a}$ and for all $i \geq 0$, $t(a_i, L_i) = a_{i+1}$. We extend the priority function to paths by setting $\mathsf{pri}(\sigma) \stackrel{\text{def}}{=} \liminf_{i\to\infty} \mathsf{pri}(a_i)$. We use $\mathrm{Path}^{\mathcal{A}}$ to denote the set of all paths of $\mathcal{A}$. The *language* of $\mathcal{A}$ is defined as

$$\mathcal{L}(\mathcal{A}) \stackrel{\text{def}}{=} \{\, L_0 L_1 \ldots \in (2^{AP})^\omega \mid \exists \sigma = a_0 L_0 a_1 L_1 \ldots \in \mathrm{Path}^{\mathcal{A}}.\ \mathsf{pri}(\sigma) \text{ is even} \,\}.$$

The following result is well known from the literature; see e.g. [23, 14].

▶ **Lemma 6** (PAs represent the ω-regular languages). *A language $\mathcal{W}$ is ω-regular if and only if it is the language of a PA $\mathcal{A}$, i.e., $\mathcal{W} = \mathcal{L}(\mathcal{A})$.*

In particular this means that all properties which can be expressed in LTL can be also expressed as parity automata. Effective means to transform LTL formulas to parity automata exist in, say, [26, 25, 27].

We also need to consider QMCs with parity conditions.

▶ **Definition 7** (Parity Quantum Markov Chain). A parity quantum Markov chain (PQMC) is a tuple $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$, where $(S, \mathbf{Q})$ is a QMC and $\mathsf{pri}\colon S \to \mathbb{N}$ is a *priority function* for the classical states. We define the *value* of $\mathcal{M}$ in $s \in S$ as

$$\mathrm{val}_s^{\mathcal{M}} \stackrel{\text{def}}{=} Q_s^{\mathcal{M}}(\{\, \sigma \in \mathrm{Path}^{\mathcal{M}} \mid \mathsf{pri}(\sigma) \textit{ is even} \,\}).$$

Here again, we set $\mathsf{pri}(\sigma) \stackrel{\text{def}}{=} \liminf_{i\to\infty} \mathsf{pri}(s_i)$ provided that $\sigma = s_0 s_1 s_2 \ldots$.

3.2 Product construction

In the following, we describe how to combine an LQMC under consideration with a PA representing the property we are concerned with.

▶ **Definition 8** (LQMC-PA Product). The *product* of an LQMC $\mathcal{M} = (S, \mathbf{Q}, AP, L)$ and a PA $\mathcal{A} = (A, \overline{a}, AP, t, \mathsf{pri})$ with the same set of atomic propositions is a PQMC $\mathcal{M} \otimes \mathcal{A} \stackrel{\text{def}}{=} (S', \mathbf{Q}', \mathsf{pri}')$ where

1. $S' \stackrel{\text{def}}{=} S \times A$,
2. $\mathbf{Q}'((s,a),(s',a')) \stackrel{\text{def}}{=} \mathbf{Q}(s,s')$ if $t(a, L(s)) = a'$, and $0_{\mathcal{H}}$ otherwise,
3. $\mathsf{pri}'((s,a)) \stackrel{\text{def}}{=} \mathsf{pri}(a)$.

The following lemma shows that the value of this product is trace equivalent to the super-operator corresponding to the property under consideration in the original model.

▶ **Lemma 9.** *Consider the product $\mathcal{M}' \stackrel{\text{def}}{=} \mathcal{M} \otimes \mathcal{A} = (S', \mathbf{Q}', \mathsf{pri}')$ of an LQMC $\mathcal{M} = (S, \mathbf{Q}, AP, L)$ and a PA $\mathcal{A} = (A, \overline{a}, AP, t, \mathsf{pri})$. We have that for any $s \in S$,*

$$Q_s^{\mathcal{M}}(\mathcal{L}(\mathcal{A})) \eqsim \mathrm{val}_{(s,\overline{a})}^{\mathcal{M}'}.$$

Proof. The proof is standard. ◀

Figure 1 Example showing that BSCC decomposition for the underlying graph does not work for model checking PQMCs.

Up to now, the model checking method works as for classical Markov chains. What would fail is the subsequent part which consists of the evaluation of the PQMC.

The idea for model checking of classical parity Markov chains is quite simple: a path of a PMC is accepted if the lowest priority occurring infinitely often is even. A strongly connected component (SCC) of a classical Markov chain is a maximal set of states B such that any two states in B can reach each other with nonzero probability. A bottom SCC (BSCC) is an SCC B in which no state in B can reach any state outside B with nonzero probability. BSCCs can be computed using only the graph structure of the Markov chain. That is, concrete probabilities are irrelevant; only the information whether the probability of going from one state to another is nonzero matters. In a classical Markov chain, starting from s, the probability that s' is visited infinitely often is 1 if s and s' are in the same BSCC. The probability that a state which is not contained in any BSCC (a transient state) will be visited infinitely often is 0. Thus, model checking for PMCs can be performed as follows:

1. Identify the set of BSCCs using a graph-based algorithm, and let $ACC = \emptyset$.
2. For each BSCC B, check whether the lowest priority occurring on a state of B is even. If yes, add B to ACC, $ACC \leftarrow ACC \cup B$.
3. For any state s, if $s \in ACC$, then $\mathrm{val}_s^{\mathcal{M}} = 1$. Otherwise, $\mathrm{val}_s^{\mathcal{M}}$ is the probability that s reaches any state in ACC. That is, if s is a state of a BSCC $B \not\subseteq ACC$, then $\mathrm{val}_s^{\mathcal{M}} = 0$; and values of transient states can be computed by solving a linear equation system.

Note that a PQMC also has a set of classical states, and the transition super-operators also induce an underlying graph over these states. Thus a natural question is: can we define the notion of BSCCs in terms of the underlying graph structure for a PQMC, just as in the classical case, and employ the above technique to calculate its value? Unfortunately, this idea does not work, as the following example shows. A similar example illustrating this difficulty was also given in [22].

▶ **Example 10.** Consider the two parity Markov models in Figure 1. On the left is a classical one with $0 < p < 1$, while the right is a quantum one with $\mathcal{E}_0, \mathcal{E}_1 \neq 0_{\mathcal{H}}$ and $\mathcal{E}_0 + \mathcal{E}_1 \approx \mathcal{I}_{\mathcal{H}}$. Obviously, both models have the same classical state space, and have exactly the same underlying graph. Thus they have the same set of BSCCs, if we would define BSCCs for PQMCs according to the underlying graphs. However, we will see that this BSCC technique does not help in the evaluation of PQMCs.

In the classical model, s_0 is a transient state which will eventually reach the only BSCC $\{s_1, s_2\}$. Thus, the priority with which s_0 is labelled is irrelevant. From any state of the BSCC, the probability that both states are visited infinitely often is 1. Thus, the probability that from either state the lowest priority 0 is reached infinitely often is 1, and thus the value of the parity Markov chain is also 1.

In contrast, in the quantum model, we assume $\mathcal{E}_0 \stackrel{\text{def}}{=} \{|0\rangle\langle 0|\}$ and $\mathcal{E}_1 \stackrel{\text{def}}{=} \{|1\rangle\langle 1|\}$. Note that for $i \in \{0, 1\}$ it holds $\mathcal{E}_i\mathcal{E}_i = \mathcal{E}_i$ and $\mathcal{E}_i\mathcal{E}_{1-i} = 0_{\mathcal{H}}$. It is easy to check that if we start from s_0, the infinite path $(s_0)^\omega$, with the corresponding *nonzero* super-operator $\lim_{n\to\infty} \mathcal{E}_1^n = \{|1\rangle\langle 1|\}$,

never leaves to the set $\{s_1, s_2\}$. Thus s_0 should not be considered as a transient state at all. Furthermore, as the priority of $(s_0)^\omega$ is 0, this path also contributes to the value of PQMC. On the other hand, if we start from s_1, there are two infinite paths with nonzero super-operator, namely $(s_1)^\omega$ with the corresponding super-operator $\{|1\rangle\langle 1|\}$ and priority 0, and $(s_1 s_2)^\omega$ with the corresponding super-operator $\{|0\rangle\langle 0|\}$ and priority 0. Thus, the value of the PQMC in state s_1 is $\{|0\rangle\langle 0|\} + \{|1\rangle\langle 1|\} \eqsim \mathcal{I}_\mathcal{H}$. However, if we start from s_2 we have $(s_2)^\omega$ with the corresponding super-operator $\{|1\rangle\langle 1|\}$ and priority 1, and $(s_2 s_1)^\omega$ with the corresponding super-operator $\{|0\rangle\langle 0|\}$ and priority 0. Thus, the value in s_2 is $\{|0\rangle\langle 0|\}$, different from the one in s_1.

Thus, algorithms based on BSCC decomposition of the underlying graph do not work for PQMCs: neither are BSCCs reached with certainty, nor do all states of a BSCC have the same value. In addition, the value of a BSCC state might be equivalent to neither $0_\mathcal{H}$ nor $\mathcal{I}_\mathcal{H}$.

3.3 Computing PQMC values

We have seen from Example 10 that the notion of BSCC defined for the underlying graph over *classical states* does not help in evaluation of PQMCs. In this subsection, we show that, rather surprisingly, by encoding the behavior of $\mathcal{M}$ into a single super-operator acting on the *extended Hilbert space* which is the tensor product of the classical state space and the quantum one, the notion of BSCC *subspaces* for super-operators[1] defined in [32] can be used to compute PQMC values.

We first recall some definitions from [32]. For any $\rho \in \mathcal{D}(\mathcal{H})$, the *support* $\mathrm{supp}(\rho)$ is defined to be the space spanned by the eigenvectors of ρ with non-zero eigenvalues. Let $\{X_k\}$ be a family of subspaces of $\mathcal{H}$. The *join* of $\{X_k\}$ is defined as $\bigvee_k X_k = \mathrm{span}(\bigcup_k X_k)$. Let $\mathcal{E}$ be a super-operator acting on $\mathcal{H}$ with $\dim(\mathcal{H}) = d$. A subspace X of $\mathcal{H}$ is said to be *invariant* for $\mathcal{E}$ if $\mathcal{E}(X) \subseteq X$, and it is a BSCC of $\mathcal{E}$ if $\mathcal{R}(|\psi\rangle\langle\psi|) = X$ for any pure state $|\psi\rangle \in X$, where for any $\rho \in \mathcal{D}(\mathcal{H})$,

$$\mathcal{R}(\rho) = \bigvee_{i=0}^{\infty} \mathrm{supp}(\mathcal{E}^i(\rho))$$

is the *reachable subspace* of $\mathcal{E}$ starting in ρ. Apparently, a BSCC is also an invariant subspace. Finally, X is called *transient* if $\lim_{k\to\infty} \mathrm{tr}(P_X \mathcal{E}^k(\rho)) = 0$ for any $\rho \in \mathcal{D}(\mathcal{H})$, where P_X is the projection onto X.

Let $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ be a PQMC on $\mathcal{H}$ with $\mathbf{Q}(s,t) = \{ E_i^{s,t} \mid i \in I^{s,t} \}$. Following [22], we define a super-operator

$$\mathcal{E}_\mathcal{M} \stackrel{\mathrm{def}}{=} \{ |t\rangle\langle s| \otimes E_i^{s,t} \mid s,t \in S, i \in I^{s,t} \} \tag{1}$$

acting on the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}$, where $\mathcal{H}_c$ is a $|S|$-dimensional Hilbert space with an orthonormal basis $\{ |s\rangle \mid s \in S \}$. To see how $\mathcal{E}_\mathcal{M}$ encodes the behavior of $\mathcal{M}$, let for each

[1] We choose not to use the terminology *quantum Markov chain* as in [32], to avoid confusion with the notion of quantum Markov chain defined in this paper. Interestingly, although it has been observed that these two notions of quantum Markov chains have the same expressiveness power [22], this is the first time techniques from one model find applications in the other.

$s \in S$ that $O^s \stackrel{\text{def}}{=} \{(t,i) \mid t \in S, i \in I^{s,t}\}$, and $M^s \stackrel{\text{def}}{=} \{ E_i^{s,t} \mid (t,i) \in O^s \}$. Note that for any $\rho \in \mathcal{D}(\mathcal{H})$,

$$\mathrm{tr}\left(\sum_{(t,i)\in O^s} E_i^{s,t} \rho (E_i^{s,t})^\dagger \right) = \mathrm{tr}\left(\sum_{t\in S} \mathbf{Q}(s,t)(\rho) \right) = \mathrm{tr}(\rho)$$

where the second equality comes from the fact that $\sum_{t\in S} \mathbf{Q}(s,t) \eqsim \mathcal{I}_{\mathcal{H}}$ (see Definition 1). That is, M^s is actually a quantum measurement with the outcome set O^s. Furthermore, for any $\sigma \in \mathcal{D}(\mathcal{H}_c)$ and $\rho \in \mathcal{D}(\mathcal{H})$, we calculate that

$$\begin{aligned}\mathcal{E}_{\mathcal{M}}(\sigma \otimes \rho) &= \sum_{s,t\in S} \sum_{i\in I^{s,t}} |t\rangle\langle s|\sigma|s\rangle\langle t| \otimes E_i^{s,t} \rho (E_i^{s,t})^\dagger \\ &= \sum_{s\in S} \langle s|\sigma|s\rangle \sum_{(t,i)\in O^s} |t\rangle\langle t| \otimes E_i^{s,t} \rho (E_i^{s,t})^\dagger.\end{aligned}$$

Thus the behavior of $\mathcal{E}_{\mathcal{M}}$ can be described as the following steps, which exactly captures the intended meaning of $\mathcal{M}$:

1. a projective measurement $M \stackrel{\text{def}}{=} \{|s\rangle\langle s| : s \in S\}$ is performed on the classical system $\mathcal{H}_c$ to determine the current classical state;
2. if the measurement outcome of M is s, then the quantum measurement M^s is performed on the quantum system $\mathcal{H}$;
3. the classical state is set to be $|t\rangle\langle t|$ if (t,i), for any i, is observed in M^s.

The following lemma shows that for super-operators derived from PQMCs, the classical and quantum systems will remain separable (disentangled) during the evolution, provided that the initial state is in a product form.

▶ **Lemma 11.** *Let $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ be a PQMC on $\mathcal{H}$, $s \in S$, $k \in \mathbb{N}$, and $\rho \in \mathcal{D}(\mathcal{H})$.*

1. *For any $n \geq 0$, $\mathcal{E}_{\mathcal{M}}^n(|s\rangle\langle s| \otimes \rho)$ is block diagonal according to the classical states. Specifically,*

$$\mathcal{E}_{\mathcal{M}}^n(|s\rangle\langle s| \otimes \rho) = \sum_{t\in S} |t\rangle\langle t| \otimes \mathbf{Q}^n(s,t)(\rho).$$

2. *Let $R_s^k = Q_s^{\mathcal{M}}(\{ \sigma \in \mathrm{Path}^{\mathcal{M}} \mid \mathsf{pri}(\sigma) = k \})$. Then for any $n \geq 0$,*

$$\mathrm{tr}(R_s^k(\rho)) = \sum_{t\in S} \mathrm{tr}(R_t^k(\mathbf{Q}^n(s,t)(\rho))).$$

Proof. Statement 1 is easy by induction. For Statement 2, note that for any $n \geq 0$,

$$\begin{aligned}R_s^k &= Q_s^{\mathcal{M}}(\{ \sigma \in \mathrm{Path}^{\mathcal{M}}(s) \mid \liminf_{i\geq 0} \mathsf{pri}(\sigma[i]) = k \}) \\ &\eqsim \sum_{t\in S} Q_t^{\mathcal{M}}(\{ \sigma \in \mathrm{Path}^{\mathcal{M}}(t) \mid \liminf_{i\geq 0} \mathsf{pri}(\sigma[i]) = k \}) \circ \mathbf{Q}^n(s,t) \\ &= \sum_{t\in S} R_t^k \circ \mathbf{Q}^n(s,t).\end{aligned}$$

Then the result follows. ◀

Similar to Lemma 11, it is easy to show that for any fixed point state σ of $\mathcal{E}_{\mathcal{M}}$, i.e. $\mathcal{E}_{\mathcal{M}}(\sigma) = \sigma$, it also has the form $\sigma = \sum_{s\in S} |s\rangle\langle s| \otimes \sigma_s$. Therefore, by [32], any BSCC of $\mathcal{E}_{\mathcal{M}}$

can be spanned by pure states of the form $|s\rangle|\psi\rangle$ where $s \in S$ and $|\psi\rangle \in \mathcal{H}$. For a BSCC B of $\mathcal{E}_{\mathcal{M}}$, let

$$C(B) \stackrel{\text{def}}{=} \{\, s \in S \mid |s\rangle|\psi\rangle \in B \text{ for some } |\psi\rangle \in \mathcal{H} \,\}$$

be the set of classical states supported in B.

Exploiting the *classical-quantum separation* (Lemma 11) of super-operators derived from PQMCs, we are going to show some nice properties of their BSCC decomposition, which are key to our discussion in this paper. First, we prove that two BSCCs X and Y are orthogonal, denoted $X \perp Y$, unless they have the same set of support classical states.

▶ **Lemma 12.** *Let $\mathcal{M}$ be a PQMC. For any two BSCCs X and Y of $\mathcal{E}_{\mathcal{M}}$, if $C(X) \neq C(Y)$ then $X \perp Y$.*

Proof. Suppose $C(X) \neq C(Y)$, and, without loss of generality, let $s \in C(Y) \backslash C(X)$. Let ρ_X and ρ_Y be the fixed point states corresponding to X and Y, respectively. Since $\mathcal{E}_{\mathcal{M}}(\rho_X + \rho_Y) = \rho_X + \rho_Y$, we know that $(\rho_X + \rho_Y)/2$ is a fixed point state corresponding to $Z \stackrel{\text{def}}{=} X \vee Y$. Thus Z can be decomposed into the direct sum of some orthogonal BSCCs: $Z = X \oplus Z_1 \oplus \cdots \oplus Z_n$.

We claim that $n = 1$. Otherwise, for any i, $\dim(Z_i) < \dim(Y)$ (because $\sum_i \dim(Z_i) + \dim(X) = \dim(Z) \leq \dim(X) + \dim(Y)$), and thus $Y \perp Z_i$ by [32, Lemma 2]. This means $Y = X$, a contradiction. Now let $|s\rangle|\psi\rangle \in Y$. Since $s \notin C(X)$, we have $|s\rangle|\psi\rangle \perp X$, and thus $|s\rangle|\psi\rangle \in Z_1$. On the other hand, since both Y and Z_1 are BSCCs, $Y = Z_1 = \mathcal{R}(|s\rangle\langle s| \otimes |\psi\rangle\langle\psi|)$. Thus $X \perp Y$. ◀

Given $k \in \mathbb{N}$, let $\mathcal{BSCC}_k$ be the span of all BSCCs of $\mathcal{E}_{\mathcal{M}}$ with the minimal priority being k; that is,

$$\mathcal{BSCC}_k = \bigvee \{\, B \text{ is a BSCC of } \mathcal{E}_{\mathcal{M}} : \min\{\, \mathsf{pri}(s) \mid s \in C(B) \,\} = k \,\}.$$

Similarly, let $\mathcal{BSCC}_{k^-}$ and $\mathcal{BSCC}_{k^+}$ be the spans of all BSCCs with the minimal priority being less than and larger than k, respectively. Then by Lemma 12, $\mathcal{BSCC}_k$, $\mathcal{BSCC}_{k^-}$, and $\mathcal{BSCC}_{k^+}$ are pairwise orthogonal. From [32], the state space $\mathcal{H}_c \otimes \mathcal{H}$ can be decomposed *uniquely* into

$$\mathcal{H} = T \oplus \mathcal{BSCC}_k \oplus \mathcal{BSCC}_{k^-} \oplus \mathcal{BSCC}_{k^+},$$

where T is the maximum transient subspace of $\mathcal{E}_{\mathcal{M}}$. In the following, we denote by P_T, P_k, P_{k^-} and P_{k^+} the projections onto T, $\mathcal{BSCC}_k$, $\mathcal{BSCC}_{k^-}$ and $\mathcal{BSCC}_{k^+}$, respectively. Then $P_T + P_k + P_{k^-} + P_{k^+} = I_{\mathcal{H}_c \otimes \mathcal{H}}$, the identity operator on $\mathcal{H}_c \otimes \mathcal{H}$.

The following lemma is crucial for our purpose. Note that $\mathrm{tr}(R_t^k(\rho))$ denotes the probability that k is the lowest priority infinitely often reachable from the initial state $|t\rangle\langle t| \otimes \rho$. This lemma essentially says that such a probability will be 1 if starting from $\mathcal{BSCC}_k$ (provided that $\mathrm{tr}(\rho) = 1$; otherwise, the probability is $\mathrm{tr}(\rho)$), and it will be 0 if starting from either $\mathcal{BSCC}_{k^-}$ or $\mathcal{BSCC}_{k^+}$. Thus $\mathcal{BSCC}_k$ for each k acts like the standard BSCCs in classical Markov chains.

▶ **Lemma 13.** *For any $t \in S$ and $\rho \in \mathcal{D}(\mathcal{H})$,*

1. *if $\mathrm{supp}(|t\rangle\langle t| \otimes \rho) \subseteq \mathcal{BSCC}_k$, then $\mathrm{tr}(R_t^k(\rho)) = \mathrm{tr}(\rho)$;*
2. *if $\mathrm{supp}(|t\rangle\langle t| \otimes \rho) \subseteq \mathcal{BSCC}_{k^-}$, then $\mathrm{tr}(R_t^k(\rho)) = 0$; and*
3. *if $\mathrm{supp}(|t\rangle\langle t| \otimes \rho) \subseteq \mathcal{BSCC}_{k^+}$, then $\mathrm{tr}(R_t^k(\rho)) = 0$.*

With the above lemmas, we are now ready to prove the main theorem of this section.

▶ **Theorem 14.** *Given a PQMC* $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ *and* $k \in \mathbb{N}$*, for any* $s \in S$ *and* $\rho \in \mathcal{D}(\mathcal{H})$*,*

$$\mathrm{tr}(R_s^k(\rho)) = \mathrm{tr}(P_k \mathcal{E}_{\mathcal{M}}^{\infty}(|s\rangle\langle s| \otimes \rho))$$

where $\mathcal{E}_{\mathcal{M}}^{\infty} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{E}_{\mathcal{M}}^{n}$.

Proof. First we lift the set of super-operators R_s^k on $\mathcal{H}$ to $\mathcal{H}_c \otimes \mathcal{H}$ by defining $\widetilde{R}^k \stackrel{\text{def}}{=} \sum_{t\in S} \mathcal{E}_t \otimes R_t^k$ where $\mathcal{E}_t = \{|t\rangle\langle t|\}$. Then Lemma 11 says that for any $n \geq 0$, $\mathrm{tr}(R_s^k(\rho)) = \mathrm{tr}(\widetilde{R}^k(\mathcal{E}_{\mathcal{M}}^n(\rho_s)))$ where $\rho_s \stackrel{\text{def}}{=} |s\rangle\langle s| \otimes \rho$. Thus we have

$$\mathrm{tr}(R_s^k(\rho)) = \mathrm{tr}\left(\widetilde{R}^k\left(\frac{1}{N}\sum_{n=1}^{N} \mathcal{E}_{\mathcal{M}}^n(\rho_s)\right)\right)$$

for any $N \geq 1$, and so $\mathrm{tr}(R_s^k(\rho)) = \mathrm{tr}(\widetilde{R}^k \mathcal{E}_{\mathcal{M}}^{\infty}(\rho_s))$ by letting N tend to infinity.

On the other side, note that $\rho_s^{\infty} \stackrel{\text{def}}{=} \mathcal{E}_{\mathcal{M}}^{\infty}(\rho_s)$ is a fixed point state of $\mathcal{E}_{\mathcal{M}}$. Then by Lemma 12 and [4, Theorem 6], $P_T \rho_s^{\infty} P_T = 0$, and ρ_s^{∞} is block diagonal with respect to $\mathcal{BSCC}_k$, $\mathcal{BSCC}_{k^-}$, and $\mathcal{BSCC}_{k^+}$; that is, $\rho_s^{\infty} = P_k \rho_s^{\infty} P_k + P_{k^-} \rho_s^{\infty} P_{k^-} + P_{k^+} \rho_s^{\infty} P_{k^+}$. Thus from Lemma 13,

$$\begin{aligned}\mathrm{tr}(R_s^k(\rho)) &= \mathrm{tr}(\widetilde{R}^k(P_k \rho_s^{\infty} P_k)) + \mathrm{tr}(\widetilde{R}^k(P_{k^-} \rho_s^{\infty} P_{k^-})) + \mathrm{tr}(\widetilde{R}^k(P_{k^+} \rho_s^{\infty} P_{k^+})) \\ &= \mathrm{tr}(P_k \rho_s^{\infty}).\end{aligned}$$

◀

The following corollary, which is direct from Theorem 14, provides us a neat way to represent the value of a PQMC at a given state using certain super-operators without resorting to quantum states.

▶ **Corollary 15.** *Let* $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ *be a PQMC. Then for any* $s \in S$*,*

$$\mathrm{val}_s^{\mathcal{M}} \eqsim \mathrm{tr}_c \circ \mathcal{P}_{even} \circ \mathcal{E}_{\mathcal{M}}^{\infty} \circ \mathcal{E}_s$$

where tr_c *is the partial trace super-operator such that* $\mathrm{tr}_c(|s\rangle\langle t| \otimes \rho) = \langle t|s\rangle \cdot \rho$*,* $\mathcal{P}_{even} = \sum_{\{k \in \mathsf{pri}(S) | k \text{ is even}\}} \mathcal{P}_k$ *where* $\mathcal{P}_k = \{P_k\}$ *is the projection super-operator onto* $\mathcal{BSCC}_k$*, and* $\mathcal{E}_s(\rho) = |s\rangle\langle s| \otimes \rho$*.*

Note that in the above corollary, we do not calculate the value $\mathrm{val}_s^{\mathcal{M}}$ directly. Instead, only a super-operator which is *trace equivalent* to $\mathrm{val}_s^{\mathcal{M}}$ is obtained. Note that from [16, Theorem 3.2], the super-operator valued measure for a QMC (thus PQMC) is well-defined only up to the trace equivalence $\eqsim$. In other words, it does not make sense to talk about the *exact* super-operator associated with a measurable set of paths; only the equivalence class determined by $\eqsim$ is meaningful. Fortunately, this is sufficient for our purpose, as in practice we are interested in the *probability* of satisfying a certain ω-property, starting from an initial quantum state, which depends only on the equivalence class that the super-operator like $\mathrm{val}_s^{\mathcal{M}}$ is in.

▶ **Example 16** (Example 10 revisited)**.** Let $\mathcal{M}$ be the PQMC depicted on the right of Fig. 1 where $\mathcal{E}_0 \stackrel{\text{def}}{=} \{|0\rangle\langle 0|\}$ and $\mathcal{E}_1 \stackrel{\text{def}}{=} \{|1\rangle\langle 1|\}$. Then the super-operator encoding $\mathcal{M}$ is

$$\begin{aligned}\mathcal{E}_{\mathcal{M}} = \quad & \{|s_1\rangle\langle s_0|\} \otimes \mathcal{E}_0 + \{|s_0\rangle\langle s_0|\} \otimes \mathcal{E}_1 \\ & + \{|s_1\rangle\langle s_1|\} \otimes \mathcal{E}_1 + \{|s_2\rangle\langle s_1|\} \otimes \mathcal{E}_0 \\ & + \{|s_1\rangle\langle s_2|\} \otimes \mathcal{E}_0 + \{|s_2\rangle\langle s_2|\} \otimes \mathcal{E}_1,\end{aligned}$$

the maximal transient space of $\mathcal{E}_{\mathcal{M}}$ is $T \stackrel{\text{def}}{=} \text{span}\{|s_0\rangle|0\rangle\}$, and the BSCCs are

$$B_1 \stackrel{\text{def}}{=} \text{span}\{|s_0\rangle|1\rangle\}, \qquad B_2 \stackrel{\text{def}}{=} \text{span}\{|s_1\rangle|1\rangle\},$$
$$B_3 \stackrel{\text{def}}{=} \text{span}\{|s_1\rangle|0\rangle, |s_2\rangle|0\rangle\}, \qquad B_4 \stackrel{\text{def}}{=} \text{span}\{|s_2\rangle|1\rangle\}.$$

Thus $\mathcal{BSCC}_0 = \bigvee\{B_1, B_2, B_3\}$, and $P_0 = |s_0\rangle\langle s_0| \otimes |1\rangle\langle 1| + |s_1\rangle\langle s_1| \otimes I + |s_2\rangle\langle s_2| \otimes |0\rangle\langle 0|$. Furthermore, we calculate that for any $n \geq 1$, $\mathcal{E}_{\mathcal{M}}^{2n-1} = \mathcal{F}_0 \otimes \mathcal{E}_0 + \mathcal{F} \otimes \mathcal{E}_1$ and $\mathcal{E}_{\mathcal{M}}^{2n} = \mathcal{F}_1 \otimes \mathcal{E}_0 + \mathcal{F} \otimes \mathcal{E}_1$ where $\mathcal{F}_0 \stackrel{\text{def}}{=} \{|s_1\rangle\langle s_0|, |s_2\rangle\langle s_1|, |s_1\rangle\langle s_2|\}$, $\mathcal{F}_1 \stackrel{\text{def}}{=} \{|s_2\rangle\langle s_0|, |s_1\rangle\langle s_1|, |s_2\rangle\langle s_2|\}$, and $\mathcal{F} \stackrel{\text{def}}{=} \{|s_0\rangle\langle s_0|, |s_1\rangle\langle s_1|, |s_2\rangle\langle s_2|\}$. Thus $\mathcal{E}_{\mathcal{M}}^{\infty} = \frac{\mathcal{F}_0+\mathcal{F}_1}{2} \otimes \mathcal{E}_0 + \mathcal{F} \otimes \mathcal{E}_1$, and

$$\mathcal{P}_0 \circ \mathcal{E}_{\mathcal{M}}^{\infty} = \frac{\mathcal{F}_0 + \mathcal{F}_1}{2} \otimes \mathcal{E}_0 + (\mathcal{P}_{s_0} + \mathcal{P}_{s_1}) \otimes \mathcal{E}_1.$$

Note that $\mathcal{E}_s = \{|s\rangle \otimes I\}$ and $\text{tr}_c = \{\, \langle s_i| \otimes I \mid i = 0, 1, 2 \,\}$. It follows that

$$\text{val}_s^{\mathcal{M}} \eqsim \text{tr}_c \circ \mathcal{P}_0 \circ \mathcal{E}_{\mathcal{M}}^{\infty} \circ \mathcal{E}_s = \begin{cases} \mathcal{E}_0 + \mathcal{E}_1 \eqsim \mathcal{I}_{\mathcal{H}} & \text{if } s = s_0 \vee s = s_1 \\ \mathcal{E}_0 & \text{if } s = s_2, \end{cases}$$

coinciding with the informal discussion given in Example 10.

4 The algorithm

In this section, we propose an algorithm to compute the values of a PQMC. First, we introduce some notations. For a super-operator $\mathcal{E} = \{\, E_i \mid i \in I \,\}$ acting on Hilbert space $\hat{\mathcal{H}}$, let $M_{\mathcal{E}} = \sum_{i \in I} E_i \otimes E_i^*$ be its matrix representation which is a linear operator on $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$. Here the complex conjugate E_i^* is taken according to an orthonormal basis of $\hat{\mathcal{H}}$. It is easy to check that $M_{\mathcal{E}}$ is independent of the choices of orthonormal basis and Kraus operators E_i of $\mathcal{E}$. Let $M_{\mathcal{E}} = KJK^{-1}$ be the Jordan decomposition of $M_{\mathcal{E}}$ where $J = \bigoplus_k J_{\lambda_k}$ and J_{λ_k} is a Jordan block corresponding to the eigenvalue λ_k. Define

$$J^{\infty} = \bigoplus_{\{\, k | \lambda_k = 1 \,\}} J_{\lambda_k} \tag{2}$$

and $M_{\mathcal{E}}^{\infty} = KJ^{\infty}K^{-1}$. Then from [30, Proposition 6.3], $M_{\mathcal{E}}^{\infty}$ is the matrix representation of $\mathcal{E}_{\infty}$.

▶ **Theorem 17.** *Given a PQMC $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ on $\mathcal{H}$ and a classical state $s \in S$, Algorithm 1 computes the matrix representation of a super-operator which is trace equivalent to $\text{val}_s^{\mathcal{M}}$ in time $O(n^8 d^8)$, where $n = |S|$ and $d = \dim(\mathcal{H})$.*

Proof. Note that for any super-operators $\mathcal{E}$ and $\mathcal{F}$, the matrix representation of $\mathcal{E} \circ \mathcal{F}$ is the product of matrix representations of $\mathcal{E}$ and $\mathcal{F}$; that is, $M_{\mathcal{E}\mathcal{F}} = M_{\mathcal{E}} \times M_{\mathcal{F}}$. Then the correctness of Algorithm 1 follows from Corollary 15. The Procedure GetBSCC which, given a super-operator $\mathcal{E}$ and an invariant subspace of $\mathcal{E}$, outputs a complete set of orthogonal BSCCs in that subspace is a revised version of the procedure Decompose from [32].

Note that $\dim(\mathcal{H}_c \otimes \mathcal{H}) = nd$, and the matrix representation of $\mathcal{E}_{\mathcal{M}}$ has size $n^2d^2 \times n^2d^2$. The complexity of Algorithm 1 can be estimated as follows.

1. The time complexity of computing the matrix representation M of $\mathcal{E}_{\mathcal{M}}$ is $\sum_{s,t \in S} m_{s,t} d^4 = O(n^2 d^6)$, where $m_{s,t} \stackrel{\text{def}}{=} |I^{s,t}| \leq d^2$ is the number of Kraus operators in $\mathbf{Q}(s,t)$.
2. Note that the time complexity of Jordan decomposition is $O(m^4)$ for an $m \times m$ matrix. The computation of matrix representation M^{∞} of $\mathcal{E}_{\mathcal{M}}^{\infty}$ takes time $O(n^8 d^8)$.

Algorithm 1: Compute the values of a PQMC

input : A PQMC $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ on $\mathcal{H}$ and a classical state $s \in S$.
output : (Matrix representation of) a super-operator that is trace equivalent to $\mathrm{val}_s^{\mathcal{M}}$.
begin
  (* Compute the matrix representations of $\mathcal{E}_s$, tr_c, and $\mathcal{E}_{\mathcal{M}}$*)
  $E_s \leftarrow |s\rangle \otimes I_{\mathcal{H}}$; $M_s \leftarrow E_s \otimes E_s^*$;
  $M_c \leftarrow 0$;
  for $t \in S$ **do**
    $E \leftarrow \langle t| \otimes I_{\mathcal{H}}$; $M_c \leftarrow M_c + E \otimes E^*$;
  end
  $M \leftarrow 0$;
  for $t, t' \in S$ *and* $i \in I^{t,t'}$ **do**
    $E \leftarrow |t'\rangle\langle t| \otimes E_i^{t,t'}$; $M \leftarrow M + E \otimes E^*$; (* $\mathbf{Q}(t,t') = \{ E_i^{t,t'} \mid i \in I^{t,t'} \}$*)
  end
  (* Compute the matrix representation of $\mathcal{E}_{\mathcal{M}}^{\infty}$ *)
  $(K, J) \leftarrow$ Jordan decomposition of M; (* $M = KJK^{-1}$ *)
  $M^{\infty} \leftarrow KJ^{\infty}K^{-1}$; (* J^{∞} is defined in Eq.(2) *)
  (* Compute the matrix representation of $\mathcal{P}_{even}$ *)
  $M_{even} \leftarrow 0$; $I_c \leftarrow \sum_{t \in S} |t\rangle\langle t|$;
  $\mathcal{B} \leftarrow \mathrm{GetBSCC}(M, I_c \otimes I_{\mathcal{H}})$;
  $EP \leftarrow \{ \mathsf{pri}(t) \mid t \in S \wedge \mathsf{pri}(t) \text{ is even} \}$;
  for $k \in EP$ **do**
    $P_k \leftarrow 0$;
    for $B \in \mathcal{B}$ *with* $k = \min\{ \mathsf{pri}(t) \mid t \in C(B) \}$ **do**
      $P_k \leftarrow P_k + P_B$ where P_B is the projector onto B;
    end
    $M_{even} \leftarrow M_{even} + P_k \otimes P_k$;
  end
  return $M_c \times M_{even} \times M^{\infty} \times M_s$; (* $\times$ denotes normal matrix multiplication *)
end

3. For the Procedure $\mathrm{GetBSCC}(M, I_{\hat{\mathcal{H}}})$ where $I_{\hat{\mathcal{H}}} \stackrel{\mathrm{def}}{=} \mathcal{H}_c \otimes \mathcal{H}$, the most time-consuming step is to compute the null space of the matrix $I_{\hat{\mathcal{H}}} \otimes I_{\hat{\mathcal{H}}} - M$. This can be done by Guassian elimination with complexity being $O((n^2d^2)^3) = O(n^6d^6)$. Note that each recursive call of the procedure decreases the dimension of the subspace by at least one. The complexity of computing $\mathrm{GetBSCC}(M, I_{\hat{\mathcal{H}}})$ is $O(n^7d^7)$. ◀

At the first glance, the time complexity $O(n^8d^8)$ of Algorithm 1 looks very high. However, note that a typical super-operator on a d-dimensional Hilbert space has up to d^2 Kraus operators each of them is a $d \times d$ complex matrix. Thus the input size K of a PQMC $\mathcal{M} = (S, \mathbf{Q}, \mathsf{pri})$ is actually of order $O(n^2d^4)$ with $n = |S|$. Thus the time complexity of Algorithm 1 is indeed $O(K^4)$.

Note that the decomposition of M (the matrix representation of $\mathcal{E}_{\mathcal{M}}$) into Jordan blocks in Algorithm 1 is quite expensive. Therefore, for a practical implementation, an approximate approach might be preferable. From $\mathcal{E}_{\mathcal{M}}^{\infty} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{E}_{\mathcal{M}}^{n}$ (cf. Theorem 14) we can derive its matrix representation $M^{\infty} = \lim_{N\to\infty} M_N$ where $M_N \stackrel{\mathrm{def}}{=} \frac{1}{N} \sum_{n=1}^{N} M^n$. We then

Procedure GetBSCC(M, P)

input : The matrix representation M of a super-operator $\mathcal{E}$ acting on $\hat{\mathcal{H}} \stackrel{\text{def}}{=} \mathcal{H}_c \otimes \mathcal{H}$, and a projector P to some invariant subspace $\mathcal{H}' \subseteq \hat{\mathcal{H}}$ of $\mathcal{E}$.
output : A complete set of orthogonal BSCCs of $\mathcal{E}$ in $\mathcal{H}'$.
begin
  $\{ |\psi_i\rangle \mid i \in I \} \leftarrow$ an orthonormal basis of $\hat{\mathcal{H}}$;
  $\mathcal{X} \leftarrow$ a basis of $\{ |x\rangle \in \mathcal{H}' \otimes \mathcal{H}' \mid M|x\rangle = |x\rangle \}$;
  $F \leftarrow \emptyset$;
  for $|x\rangle \in \mathcal{X}$ **do**
    $X \leftarrow \sum_{i \in I} (I_{\hat{\mathcal{H}}} \otimes \langle\psi_i|) \cdot |x\rangle\langle\psi_i|$;
      (* The matrix X corresponds to $|x\rangle$ in that $X = \sum_{i,j \in I} x_{i,j} |\psi_i\rangle\langle\psi_j|$ iff $|x\rangle = \sum_{i,j \in I} x_{i,j} |\psi_i\rangle|\psi_j\rangle$ *)
    $X_R \leftarrow (X + X^\dagger)/2$; $X_I \leftarrow (X - X^\dagger)/2i$;
      (* $X^\dagger$ denotes the transpose and complex conjugate of X *)
    $P_R^+ \leftarrow$ the projector onto eigenspace of X_R with positive eigenvalues;
    $P_I^+ \leftarrow$ the projector onto eigenspace of X_I with positive eigenvalues;
    $X_R^+ = P_R^+ X_R P_R^+$; $X_R^- = X_R^+ - X_R$;
    $X_I^+ = P_I^+ X_I P_I^+$; $X_I^- = X_I^+ - X_I$;
    (* All of them are positive semidefinite, and $X = X_R^+ - X_R^- + i(X_I^+ - X_I^-)$ *)
    for $Y \in \{X_R^+, X_R^-, X_I^+, X_I^-\} \wedge Y \neq 0$ **do**
      $F \leftarrow F \cup \{Y/\mathrm{tr}(Y)\}$; (* Fixed point states of $\mathcal{E}$ *)
    end
  end
  if $|F| = 1$ **then**
    return $\{\mathrm{supp}(Y)\}$; (* Y is the only element of F *)
  else
    $Y_1, Y_2 \leftarrow$ two arbitrary different elements of F;
    $P^+ \leftarrow$ the projector onto eigenspace of $Y_1 - Y_2$ with positive eigenvalues;
    $P^- \leftarrow P - P^+$;
    $M^+ \leftarrow (P^+ \otimes I_{\hat{\mathcal{H}}}) M (I_{\hat{\mathcal{H}}} \otimes P^+)$;
    $M^- \leftarrow (P^- \otimes I_{\hat{\mathcal{H}}}) M (I_{\hat{\mathcal{H}}} \otimes P^-)$;
    return GetBSCC$(M^+, P^+) \cup$ GetBSCC(M^-, P^-);
  end
end

compute $M_0, M_1, M_2, \ldots$ until we have reached an N for which $\|M_N - M_{N-1}\|_{\max} < \varepsilon$ for a predefined precision ε, so as to obtain an approximation of M^∞. Note that M_N can be computed using a dynamic programming approach by means of the equality

$$M_N = \begin{cases} M & \text{if } N = 1, \\ \frac{1}{N}((N-1)M_{N-1} + M^N) & \text{if } N > 1. \end{cases}$$

In stochastic model checking, such value iteration based approaches are commonly used, and in [15], we have successfully applied a similar method for model checking QCTL Until formulas.

5 Conclusion

In this paper, we have investigated model checking ω-regular and in particular LTL properties against super-operator weighted quantum Markov chains, which can be used to faithfully model a practically relevant class of quantum processes. As future work, we would like to implement our model checking algorithm in IscasMC [20] and apply it on case studies from the area of quantum communication protocols and evaluate the actual performance of our approach.

References

1 Ebrahim Ardeshir-Larijani, Simon J. Gay, and Rajagopal Nagarajan. Equivalence checking of quantum protocols. In *TACAS*, volume 7795 of *LNCS*, pages 478–492, 2013.

2 Ebrahim Ardeshir-Larijani, Simon J. Gay, and Rajagopal Nagarajan. Verification of concurrent quantum protocols by equivalence checking. In *TACAS*, volume 8413 of *LNCS*, pages 500–514, 2014.

3 Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. The MIT Press, 2008.

4 Bernhard Baumgartner and Heide Narnhofer. The structures of state space concerning Quantum Dynamical Semigroups. *Reviews in Mathematical Physics*, 24(02):1250001, 2012.

5 Charles H. Bennett. Quantum cryptography using any two nonorthogonal states. *Physical Review Letters*, 68:3121, 1992.

6 Charles H. Bennett and Gilles Brassard. Quantum cryptography: Public-key distribution and coin tossing. In *Proceedings of the IEEE International Conference on Computer, Systems and Signal Processing*, pages 175–179, 1984.

7 Charles H. Bennett and Stephen J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. *Physical Review Letters*, 69(20):2881–2884, 1992.

8 Marco Bozzano, Alessandro Cimatti, Joost-Pieter Katoen, Panagiotis Katsaros, Konstantinos Mokos, Viet Yen Nguyen, Thomas Noll, Bart Postma, and Marco Roveri. Spacecraft early design validation using formal methods. *Reliability Engineering and System Safety*, 132:20–35, 2014.

9 Doron Bustan, Sasha Rubin, and Moshe Y Vardi. Verifying omega-regular properties of Markov chains. In *CAV*, volume 4, pages 189–201, 2004.

10 Taolue Chen, Marco Diciolla, Marta Kwiatkowska, and Alexandru Mereacre. Quantitative verification of implantable cardiac pacemakers. In *Real-Time Systems Symposium*, pages 263–272, 2012.

11 Edmund M Clarke, Orna Grumberg, and Doron Peled. *Model checking*. MIT press, 1999.

12 Costas Courcoubetis and Mihalis Yannakakis. The complexity of probabilistic verification. *JACM*, 42(4):857–907, 1995.

13 Luca de Alfaro. *Formal Verification of Probabilistic Systems*. PhD thesis, Stanford University, 1997.

14 E. Allen Emerson and Charanjit S. Jutla. Tree automata, mu-calculus and determinacy. In *FOCS*, SFCS, pages 368–377. IEEE CS, 1991.

15 Yuan Feng, Ernst Moritz Hahn, Andrea Turrini, and Lijun Zhang. QPMC: A model checker for quantum programs and protocols. In *FM'15*, volume 9109 of *Lecture Notes in Computer Science*, pages 265–272. Springer, 2015.

16 Yuan Feng, Nengkun Yu, and Mingsheng Ying. Model checking quantum Markov chains. *Journal of Computer and System Sciences*, 79(7):1181–1198, 2013.

17 Simon J. Gay, Rajagopal Nagarajan, and Nikolaos Papanikolaou. Probabilistic model-checking of quantum protocols. In *Proceedings of the 2nd International Workshop on Developments in Computational Models*, 2006.

18 Simon J. Gay, Rajagopal Nagarajan, and Nikolaos Papanikolaou. QMC: A model checker for quantum systems. In *CAV 08*, pages 543–547. Springer, 2008.

19 Stanley Gudder. Quantum Markov chains. *Journal of Mathematical Physics*, 49(7):072105, 14, 2008.

20 Ernst Moritz Hahn, Yi Li, Sven Schewe, Andrea Turrini, and Lijun Zhang. IscasMC: A web-based probabilistic model checker. In *FM'14*, volume 8442 of *Lecture Notes in Computer Science*, pages 312–317. Springer, 2014.

21 Khaza Anuarul Hoque, Otmane Aït Mohamed, and Yvon Savaria. Towards an accurate reliability, availability and maintainability analysis approach for satellite systems based on probabilistic model checking. In *Design, Automation & Test in Europe*, pages 1635–1640, 2015.

22 Lvzhou Li and Yuan Feng. Quantum markov chains: description of hybrid systems, decidability of equivalence, and model checking linear-time properties. *Information and Computation*, 244:229–244, 2015.

23 Andrzej Wlodzimierz Mostowski. Regular expressions for infinite trees and a standard form of automata. In *Computation Theory*, volume 208 of *LNCS*, pages 157–168. Springer, 1984.

24 Michael A. Nielsen and Isaac L. Chuang. *Quantum computation and quantum information*. Cambridge university press, 2000.

25 Nir Piterman. From nondeterministic Büchi and Streett automata to deterministic parity automata. *JLMCS*, 3(3:5), 2007.

26 Shmuel Safra. On the complexity of ω-automata. In *FOCS*, pages 319–327, 1988.

27 Sven Schewe. Tighter bounds for the determinisation of Büchi automata. In *FoSSaCS*, volume 5504 of *LNCS*, pages 167–181, 2009.

28 Peter Selinger. A brief survey of quantum programming languages. *Functional and Logic Programming*, 2998:1–6, 2004.

29 Wolfgang Thomas. Automata on infinite objects. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science (Vol. B)*, pages 133–191. MIT Press, Cambridge, MA, USA, 1990.

30 Michael M Wolf. Quantum channels & operations: Guided tour. *https://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf*, 2012.

31 Mingsheng Ying, Yangjia Li, Nengkun Yu, and Yuan Feng. Model-checking linear-time properties of quantum systems. *ACM Transactions on Computational Logic (TOCL)*, 15(3):22, 2014.

32 Shenggang Ying, Yuan Feng, Nengkun Yu, and Mingsheng Ying. Reachability probabilities of quantum markov chains. In *CONCUR'13*, pages 334–348. Springer, 2013.

Uniform Sampling for Networks of Automata*†

Nicolas Basset[1], Jean Mairesse[2], and Michèle Soria[3]

1 **Université libre de Bruxelles, Brussels, Belgium**
`nicolas.basset@ulb.ac.be`
2 **Sorbonne Universités, UPMC Univ Paris 06, CNRS, LIP6, 4 place Jussieu, 75252 Paris Cedex 05, France**
`jean.mairesse@lip6.fr`
3 **Sorbonne Universités, UPMC Univ Paris 06, CNRS, LIP6, 4 place Jussieu, 75252 Paris Cedex 05, France**
`michele.soria@lip6.fr`

Abstract

We call *network of automata* a family of *partially synchronised automata*, *i.e.* a family of deterministic automata which are synchronised via shared letters, and evolve independently otherwise. We address the problem of uniform random sampling of words recognised by a network of automata. To that purpose, we define the *reduced automaton* of the model, which involves only the *product of the synchronised part* of the component automata. We provide uniform sampling algorithms which are polynomial with respect to the size of the reduced automaton, greatly improving on the best known algorithms. Our sampling algorithms rely on combinatorial and probabilistic methods and are of three different types: exact, Boltzmann and Parry sampling.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.2 Modes of Computation, G.2.1 Combinatorics, G.3 Probability and Statistics

Keywords and phrases Partially synchronised automata, uniform sampling, recursive method, Boltzmann sampling, Parry measure

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.36

1 Introduction

Automata are ubiquitous in computer science in general, and in verification in particular, since they provide a good abstraction of the behaviour of *sequential* systems. *Networks of automata* are a meaningful model of *concurrent* systems where a family of component automata are synchronised via shared letters, and otherwise evolve independently. They appear under various names in the literature (see e.g. [3, 18] and references therein). An example of a network of automata is depicted in Figure 1. A challenging task when dealing with such concurrent model is to avoid the state space explosion due to an explicit construction of the product model.

In the context of either performance evaluation or model-checking of concurrent systems, it is often impossible to perform a formal or exhaustive analysis of the huge number of possible trajectories (recognised words in the context of automata). To cope with the issue, a possibility is to perform either *simulation* or *Monte Carlo model checking*, that is, to

* An extended version of this article is available in [4].

† This work was partially supported by the Fondation Simone et Cino Del Duca. N. Basset was partially supported by the ARC project "Non-Zero Sum Game Graphs: Applications to Reactive Synthesis and Beyond" funded by Fédération Wallonie-Bruxelles.

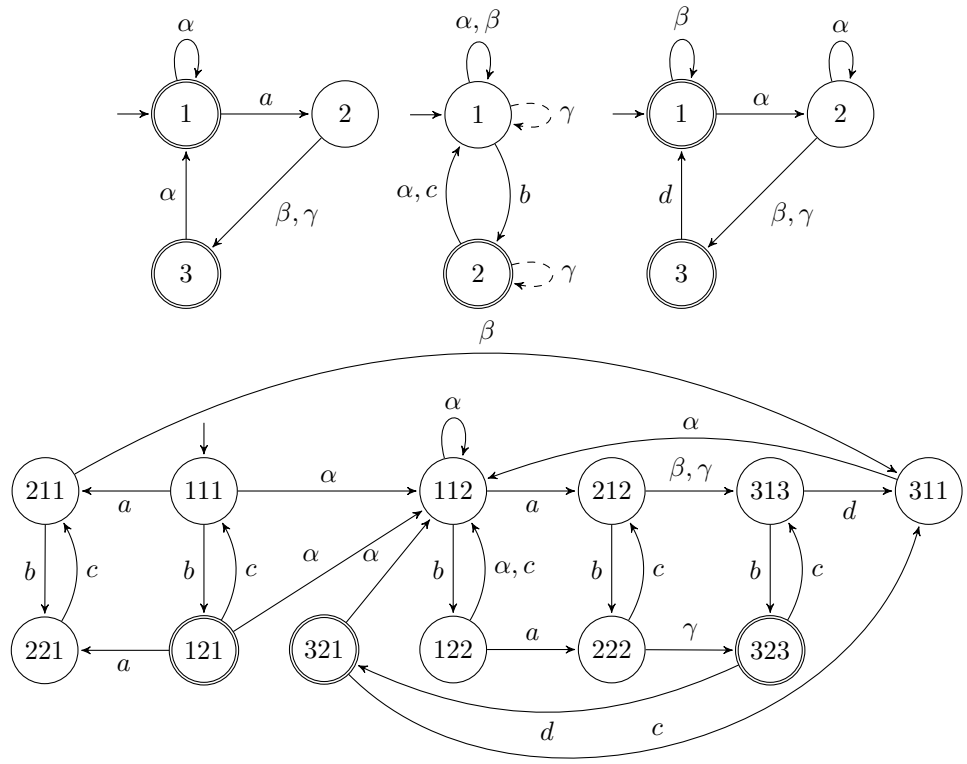


Figure 1 Top: A network of three DFAs with shared alphabet $\texttt{Synch} = \{\alpha, \beta, \gamma\}$. The dotted transitions labelled with γ (top) have been added without changing the product, see Remark 2. Bottom: the product. Unreachable states $113, 123, 213, 223, 312, 322$ are not represented. The states occurring just after a synchronisation are $112, 311, 313, 323$.

concentrate on a sample of the trajectories drawn at random (see [13, 20]). As noted in [9, 20], the classical sampling methods applied in this context use a random walk evolving on the product automaton, the most natural one being the *isotropic* random walk that chooses the next transition uniformly at random among available transitions. For instance, for the example of Figure 1, when the state 112 is visited by the random walk then each one of the transitions α, a and b is chosen with probability $1/3$. The isotropic sampling seems very natural at first sight but, unfortunately, there are simple examples (exhibited in [9, 20]) for which this sampling tends to concentrate most of the probability on a very small fraction of the trajectories. Moreover, isotropic sampling is useless to answer quantitative question such as: "With which confidence can we claim than more than 60 % of the trajectories of length 1000 visit a target state?". To answer such questions, it is necessary to use a *uniform sampling* that is to choose all equal length trajectories with the same probability. A thorough argumentation why uniform sampling should be preferred to isotropic sampling is given in [9, 20] for both Monte Carlo model checking and random model based testing. Despite these pioneering works, designing efficient uniform samplers for network of automata remains a challenge that we address in the present paper.

Before detailing our work, we need to distinguish between different notions of uniform sampling. A *fixed length uniform sampler* is a random algorithm taking as input a positive integer n and returning as output a word of length n with uniform probability. A *Boltzmann sampler* is a random algorithm returning a word of random length with the property that

two words of the same length are equiprobable. A *Parry sampler* is a random method to generate infinite words in a "uniform" way. It requires the rigorous definition of the notion of "uniform probability measure" on infinite words. Both Boltzmann and Parry samplings are relevant in Monte Carlo model checking. On the one hand, if some variability on the *length of the sampled word* is allowed, then Botzmann sampling is useful and efficient, and the expected length of the sampled word can be chosen by tuning a parameter. On the other hand, if some variability with respect to *uniformity* is allowed, then Parry sampling is of interest: with Parry, we obtain an exact uniform sampling for infinite words, but finite words can also be sampled, with approximate uniformity and an additional important feature: the sampling procedure is dynamic, that is, we can re-use a sampled word of length n as the prefix of a sampled word of length $n + k$.

The straightforward method of building explicitly the *product automaton* (the automaton defined as the direct product of the component automata), and apply to it a standard uniform sampling algorithm for automata (e.g. [5, 19]), is not efficient since the product automaton has a size which is exponential with respect to the size of the components. In fact, the whole challenge is to avoid constructing explicitly the product automaton. To be efficient, the approach has to be *compositional*: use uniform samplers for component automata and combine them in a clever way.

We now review previous literature related to the problem. Consider first the case with no shared letters, that is, no synchronisation between the component automata. In this case, the language of the network of automata is simply the *shuffle* of the languages of the component automata. The shuffle product has been widely studied from a formal language theory viewpoint (see [21] and references therein). We are not aware of any paper dealing explicitly with uniform sampling in this precise context. The most relevant article is [8] which provides a Boltzmann sampler for extended linear expressions with shuffle. Applying this algorithm in our context is not straightforward since we do not know *a priori* if there exists a polynomial method to transform a family of automata into an extended linear expression with shuffle. The direct method, which consists in shuffling the regular expressions obtained by transforming each component automaton separately, is not efficient. Indeed, the passage from automata to regular languages is known to increase the size of the description in an exponential way in the worst case (see e.g. [2]). In [9] the authors identify (amongst other contributions) the challenge of uniform sampling for network of DFAs. Our work is partly inspired by ideas of this paper like using shuffle of languages between synchronisations. However, there are several problems unsolved there that make their sampling useful only under very stringent conditions:

- **(i)** there must be at most one shared letter in the network of automata and if there is one, it must appears exactly once in every component automaton;
- **(ii)** there must be very few synchronisation in the sampled word (the algorithm having an exponential complexity wrt. the number of occurrences of the shared letter in the sampled word);
- **(iii)** the sampling algorithm for the shuffle of languages can only be used if the words that are shuffled are long and belong to strongly connected automata.

In fact, we show that the third item is not a real problem. We provide exact uniform samplers for the shuffle of languages that are built and that run in polynomial time without any restriction on the length of the words shuffled nor on the topology of the component automata. By contrast, when all the letters are synchronised, difficulties are unavoidable in the worst-case. Indeed, the problem of uniform sampling is more difficult than checking the language emptiness of the product of automata which is a PSPACE complete problem. An

option to address this issue would be to identify a subclass for which the problem would be tractable in polynomial time like in the first item above. Here we avoid such a restriction by adopting a fixed-parameter-tractable approach: our algorithms are designed for the whole class of networks of automata but run in polynomial time when a well suited parameter is fixed. This parameter controls the size of the "synchronised part" of the product automaton. We thus take advantage of the fact that the synchronised part of the product may be small, and decompose the network into pieces coming from the synchronised part and others coming from the "shuffle" of non-synchronised parts, leading to efficient sampling algorithms that use the product only when necessary. To that purpose, we introduce the notion of reduced automaton. Each component automaton of the network is transformed into a simplified automaton involving only the shared letters, and a remaining part. The *reduced automaton* is the product of the simplified automata, hence the exponential blow-up only concerns the synchronised part of the automata, and the size of the reduced automaton is a parameter that reflects the intrinsic synchronisation complexity of the problem. Our basic idea is to decompose the language recognized by the network of automata in terms of union and concatenation of shuffles of languages corresponding to the component automata. Relying on this decomposition, we then use a compositional approach to derive uniform samplers.

We now sum up our results. We propose a fixed length uniform sampler for arbitrary network of automata running in linear time in the length of the sampled word, and in polynomial time in the size of the reduced automaton. In particular we provide solutions to the problems of [9] listed above: there is no restriction on the number of shared letters; the sampling is exactly uniform; the complexity is drastically improved wrt. the number of occurrences of shared letters in the word. In the case of no shared letters, we design a fixed length uniform sampler running in linear time wrt. the length of the sampled words and in polynomial time wrt. the sum of the sizes of the component automata. This is a key ingredient used in the case where synchronisations are present. In addition to this, we provide the first Boltzmann and Parry samplers for networks of automata.

We conclude this introduction with few related works. Uniform sampling for related but different models of concurrent systems are considered in [1, 7]. Approximate uniform sampling of valuations of SAT-formulas are given in [16] (see also references therein).

2 Preliminaries: Monolithic Uniform Sampling for a DFA

In this section, we present the three types of uniform sampling for a single automaton. All the algorithms are classical and run in polynomial time. They will be used as building blocks in the compositional approach of the following sections.

Let us begin with some basics on automata. A *finite state automaton* (FSA) is a tuple $\mathcal{A} = (Q, \Sigma, \iota, F, \Delta)$ where Q is the set of *states*, Σ is the *alphabet* of *actions*, ι is the *initial* state, F is the set of *final* states and $\Delta \subseteq Q \times \Sigma \times Q$ is the set of *transitions*. A *path* of *length* $n \in \mathbb{N}$ from s to t *labelled* by $w = a_1 \cdots a_n \in \Sigma^n$ is a sequence of transitions $(s_i, a_i, t_i)_{i\in[n]}$ such that $s_1 = s$, $t_n = t$, for $i < n$, $t_i = s_{i+1}$ (where here and below, for $k \in \mathbb{N}$ we let $[k] = \{1, \ldots, k\}$). We write $s \xrightarrow{w} t$ if there is a path from s to t labelled by w. A word w is *recognised* by the automaton $\mathcal{A}$ if there is a final state $t \in F$ such that $\iota \xrightarrow{w} t$. The language *recognised* by $\mathcal{A}$, denoted by $\mathcal{L}$, is the set of words recognised by $\mathcal{A}$. We denote by $\mathcal{L}_s$ the language of words starting from s, that is recognised by the FSA $(Q, \Sigma, s, F, \Delta)$. The *size* of $\mathcal{A}$, denoted by $|\mathcal{A}|$, is the number of states and transitions. When Δ is functional, that is $\forall s \in Q$, $\forall a \in \Sigma$, $|\{t \mid (s, a, t) \in \Delta\}| \leq 1$, the FSA is called a *deterministic finite state automaton* (DFA). In automata theory, *trimming* is a standard operation which consists in

deleting useless states of a FSA (a state is *useful* if it is accessible from ι and co-accessible from a final state). We assume without loss of generality that all automata we consider are *trimmed*.

For $s \in Q$, the languages $\mathcal{L}_s$ can be characterised by the following language equations:

$$\mathcal{L}_s = \bigcup_{(s,a,t)\in\Delta} a\mathcal{L}_t \quad (\cup\{\varepsilon\} \text{ if } s \in F) \tag{1}$$

In the remainder of the section, we consider a DFA $\mathcal{A}$ of language $\mathcal{L}$. We are going to present three methods to sample a word from $\mathcal{L}$, the three being built on a common recursive scheme based on Eq. (1): randomly choose the first transition (s, a, t), output the letter a and then repeat recursively from t. These methods are *sequential*: letters are randomly chosen one after the other in the order in which they appear in the output word.

2.1 Cardinalities and fixed length uniform sampling

Recall that a *fixed length uniform sampler* is a random algorithm that takes as input a positive integer n and outputs a word of $\mathcal{L}$ of length n such that every word has the same probability to be output.

The general recursive method for uniform sampling of [12] applies in this context. The idea is to transfer recursive equations on cardinalities into recursive samplers. The equations on cardinalities are obtained directly from Eq. (1). Denoting by $l_{s,n}$ the number of words of length n in $\mathcal{L}_s$, we have

$$l_{s,n} = \sum_{(s,a,t)\in\Delta} l_{t,n-1} \quad \text{if } n > 0 \text{ and } l_{s,0} = 1_{s\in F} \tag{2}$$

A fixed length uniform sampler is then obtained as follows: choose the first transition (s, a, t) with probability $l_{t,n-1}/l_{s,n}$, output a and recursively repeat for the $n-1$ remaining letters, starting in state t.

Note that a $|Q| \times n$ table with the coefficients $(l_{s,k})_{s\in Q,k\in[n]}$ can be computed in time[1] $O(n|\mathcal{A}|)$ using Eq. (2). Cardinalities can also be expressed in terms of the power of the adjacency matrix of $\mathcal{A}$, that is, the matrix $A = (A_{st})_{s,t\in Q}$ with $A_{st} = |\{a \mid (s, a, t) \in \Delta\}|$. Indeed, for all $s \in Q, n \in \mathbb{N}$, we have: $l_{s,n} = \sum_{t\in F} A^n_{st}$.

The drawback of the above method is that the transition probabilities $(l_{t,n-1}/l_{s,n})$ depend on n so that the cardinalities should be computed and stored up to the length of the word to be generated. In the next two sampling methods on the other hand, the transition probabilities do not depend on n.

2.2 Generating functions and Boltzmann sampling

The general Boltzmann sampling of [10] applies in this context. Whereas the fixed length sampler was based on recursive equations on cardinalities, the Boltzmann sampler is based on recursive equations on generating functions.

Let us recall some basics on generating functions associated to languages. The *ordinary generating function* (OGF) of a language $\mathcal{L}$ is $L(z) = \sum_{m\in\mathbb{N}} l_m z^m$, where l_m is the number of words of length m in tne language, and the *exponential generating function* (EGF) is $\hat{L}(z) = \sum_{m\in\mathbb{N}} l_m z^m/m!$. Generating functions can be seen either as formal power series

[1] Here and in the rest of the paper, the complexity is given in terms of arithmetic complexity.

or as functions of the complex variable z. The *convergence radius* of $L(z)$ is $\mathfrak{r}(L) = \inf\{|z|\,;\, L(z) \text{ is not defined}\}$.

The equations on languages (1) transfer to equations on generating functions:

$$L_s(z) = z \sum_{(s,a,t)\in\Delta} L_t(z) + 1_{s\in F} \tag{3}$$

Define the column vector $\mathbf{L}(z) = (L_s(z))_{s\in Q}$ where $L_s(z)$ is the OGF of $\mathcal{L}_s$. Recall that A is the adjacency matrix. For $z < \mathfrak{r}(L)$, we have: $\mathbf{L}(z) = (I - zA)^{-1}e_F$, where e_F is the column vector defined by $(e_F)_s = 1$ if $s \in F$ and $(e_F)_s = 0$ otherwise. The convergence radius $\mathfrak{r}(L)$ is characterised by[2] $\mathfrak{r}(L) = \inf\{|z|\,;\, (I - zA)^{-1} \text{ is defined}\}$.

An *(ordinary) Boltzmann sampler* draws a word w with probability distribution $z^{|w|}/L(z)$ ($z < \mathfrak{r}(L)$ being a parameter to be chosen), hence the size is not fixed but the distribution is uniform when conditioned on a given size. Based on Eq. (3), a Boltzmann sampler for $\mathcal{L}_s$ of parameter z can be recursively defined: with probability $1_{s\in F}/L_s(z)$, no transition is chosen and the random generation stops; otherwise the transition (s, a, t) is chosen with probability $zL_t(z)/L_s(z)$, the letter a is output, and the sampler is called recursively from t to output the remainder of the word.

Similarly, an *exponential Boltzmann sampler* draws words with probability distribution $z^{|w|}/(|w|!\hat{L}(z))$. We give in [4] the construction of such an exponential Boltzmann sampler.

2.3 Perron-Frobenius Theorem and Parry sampling

In this section, we need to assume that the DFA under consideration is strongly connected[3].

Parry sampling is intuitively the limit case of Boltzmann sampling where $z = \mathfrak{r}(L)$ so that the probability to stop the generation is null and the output word is infinite. The formal definition is based on the Perron-Frobenius Theorem. This theorem gives fundamental properties on the spectral theory of non-negative matrices. We state it in the context of strongly connected automata. We refer the reader to [15].

▶ **Theorem 1** (Perron-Frobenius stated for automata)**.** *Consider a strongly connected DFA and its adjacency matrix A. The following holds:*

- **(i)** *The spectral radius $\rho(A)$, that is, the maximal modulus of all eigenvalues of A, is itself an eigenvalue. It satisfies $\rho(A) = 1/\mathfrak{r}(L)$.*
- **(ii)** *The eigenvalue $\rho(A)$ is simple, its unique (up to a multiplicative constant) eigenvector $\mathbf{v}$ has positive coefficients, it is called the* Perron vector*. There is no other eigenvector with only non-negative coefficients.*
- **(iii)** *If $0 \le A' \le A$ then $\rho(A') \le \rho(A)$, and $\rho(A') = \rho(A)$ only if $A' = A$.*

A *Parry sampler* for $\mathcal{L}_s$ is an algorithm that produces an infinite random word w such that for any finite word u such that $s \xrightarrow{u} t$, the probability that w begins by u is $\mathbf{v}_t/(\rho^n\mathbf{v}_s)$, where $\mathbf{v}$ is the Perron vector. A Parry sampler can be recursively defined by choosing the first transition (s, a, t) with probability $\mathbf{v}_t/(\rho\mathbf{v}_s)$ and repeating recursively from t.

A Parry sampler does not give exact uniform sampling on words of length n. However it gets closer and closer to being uniform as n gets larger. Hence it can be used as an approximate uniform sampler for large words of a given length.

---

[2] Here, we use the assumption that the DFA is trimmed, otherwise one can construct examples where a part of the DFA is useless for the language definition, but decreases the value of $\inf\{|z|\,;\, (I-zA)^{-1} \text{ is defined}\}$ which is in that case strictly smaller than $\mathfrak{r}(L)$.

[3] There is a path between each couple of states in the automaton

3 Network of automata and the reduced automaton

A *network of automata* is composed of a family of DFA that are synchronised on shared letters and evolve independently otherwise. The associated *reduced automaton* is a product automaton taking into account only the synchronised part of the components. In this section, we formally define these notions and provide equations on languages associated to states and transitions of the reduced automaton, together with a system of equations satisfied by their generating functions.

3.1 Network of automata

Consider a family of K DFAs $\mathcal{A}^{(i)} = (Q^{(i)}, \Sigma^{(i)}, \iota^{(i)}, F^{(i)}, \Delta^{(i)})$ for $i \in [K]$. The alphabets $\Sigma^{(i)}$ are *not* assumed to be disjoint. The associated product automaton is the DFA defined by $\mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)} = (Q, \Sigma, \iota, F, \Delta)$ with $Q = Q^{(1)} \times \cdots \times Q^{(K)}$; $\Sigma = \Sigma^{(1)} \cup \cdots \cup \Sigma^{(K)}$; $\iota = (\iota^{(1)}, \ldots, \iota^{(K)})$; $F = F^{(1)} \times \cdots \times F^{(K)}$ and $(s, a, t) \in \Delta$ if and only if: $\forall i \in [K]$ s.t. $a \in \Sigma^{(i)}$, $(s^{(i)}, a, t^{(i)}) \in \Delta^{(i)}$; and $\forall i \in [K]$ s.t. $a \notin \Sigma^{(i)}$, $s^{(i)} = t^{(i)}$.

An example is given in Figure 1. We call *network of automata* a family of DFAs evolving together according to the above rules of the product automaton.

Denote by $\mathtt{Synch}$ the set of letters shared by several automata of the network (we use Greek letters for the elements of $\mathtt{Synch}$).

▶ **Remark 2.** *Given $(\mathcal{A}^{(i)})_{i\in[K]}$, we define for every $i \in [K]$, the FSA $\mathcal{B}^{(i)}$ obtained from $\mathcal{A}^{(i)}$ by adding in every state a self-loop labelled by every $\alpha \in \mathtt{Synch} \setminus \Sigma^{(i)}$. Observe that we have: $\mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)} = \mathcal{B}^{(1)} \times \cdots \times \mathcal{B}^{(K)}$. See an example in Figure 1.*

Accordingly, in the following, we assume without loss of generality that every letter in $\mathtt{Synch}$ belongs to every alphabet.

The special case where $\mathtt{Synch} = \emptyset$ will be of crucial importance. It involves the shuffle of languages defined just below. The *shuffle product* of two words $w^{(1)}$ and $w^{(2)}$ on disjoint alphabets is the finite language, denoted by $w^{(1)} \sqcup\!\sqcup w^{(2)}$, containing all the possible interleavings of $w^{(1)}$ and $w^{(2)}$, e.g. $ab \sqcup\!\sqcup cd = \{abcd, acbd, acdb, cabd, cadb, cdab\}$. The shuffle product of two languages $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ on disjoint alphabets is

$$\mathcal{L}^{(1)} \sqcup\!\sqcup \mathcal{L}^{(2)} = \bigcup_{(w^{(1)}, w^{(2)}) \in \mathcal{L}^{(1)} \times \mathcal{L}^{(2)}} w^{(1)} \sqcup\!\sqcup w^{(2)}.$$

The shuffle product is associative and we denote by $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ the shuffle product of K languages $\mathcal{L}^{(1)}, \ldots, \mathcal{L}^{(K)}$. The language associated to a network of automata such that $\mathtt{Synch} = \emptyset$ is the shuffle of the component languages:

$$\mathcal{L}(\mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)}) = \mathcal{L}(\mathcal{A}^{(1)}) \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}(\mathcal{A}^{(K)}). \tag{4}$$

In the dual special case where $\mathtt{Synch} = \Sigma$, the language associated to the network of automata is the intersection of the component languages:

$$\mathcal{L}(\mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)}) = \mathcal{L}(\mathcal{A}^{(1)}) \cap \cdots \cap \mathcal{L}(\mathcal{A}^{(K)}). \tag{5}$$

3.2 Reduced automaton

We now introduce the reduced automaton of a network of automata, a product automaton which only involves the synchronised letters. All the algorithms presented in this paper will require to compute $\mathcal{A}_{\mathtt{red}}$, whereas the product automaton $\mathcal{A}$ is never explicitly computed.

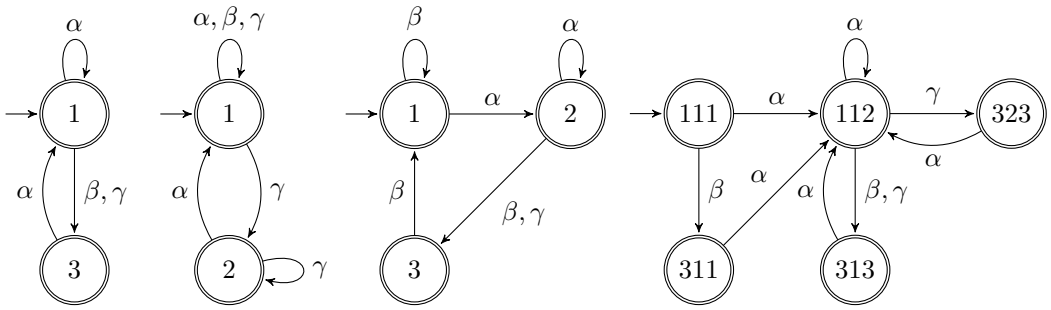


Figure 2 The reduced automata of the automata of Figure 1.

We define the *reduced automaton* of a DFA $\mathcal{A} = (Q, \Sigma, \iota, F, \Delta)$ as the FSA $\mathcal{A}_{\texttt{red}} = (Q_{\texttt{red}}, \Sigma_{\texttt{red}}, \iota_{\texttt{red}}, Q_{\texttt{red}}, \Delta_{\texttt{red}})$ such that $\Sigma_{\texttt{red}} = \texttt{Synch}$ and $Q_{\texttt{red}} \subseteq Q$ is the set of states reached from the initial state $\iota_{\texttt{red}} = \iota$ through the transition relation $\Delta_{\texttt{red}} \subseteq Q_{\texttt{red}} \times \Sigma_{\texttt{red}} \times Q_{\texttt{red}}$ defined by $\delta = (s, \alpha, t) \in \Delta_{\texttt{red}}$ if and only if there exists a word $u \in (\Sigma \setminus \texttt{Synch})^*$ such that $s \xrightarrow{u\alpha} t$. The set of final states $F_{\texttt{red}} = Q_{\texttt{red}}$ is not relevant since we are only interested in the states and transitions of $\mathcal{A}_{\texttt{red}}$, and not in its language. See Figure 2 for an example.

▶ **Proposition 3.** *Given a DFA $\mathcal{A}$, the FSA $\mathcal{A}_{red}$ can be computed by replacing every label not in* `Synch` *by ε and doing an ε-transitions removal.*

The removal of ε-transitions can be achieved in time $O(|Q|^2 + |Q|.|\Delta|)$, as shown for instance in [17].

The next proposition enables us to construct *compositionally* the reduced automaton associated to a network of automata, based on the reduced automata of each of the component automata computed monolithically as in Proposition 3.

▶ **Proposition 4.** *Consider a network of automata $(\mathcal{A}^{(i)})_{i\in[K]}$ and $\mathcal{A} = \mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)}$ its associated product. It holds that $\mathcal{A}_{red} = \mathcal{A}^{(1)}_{red} \times \cdots \times \mathcal{A}^{(K)}_{red}$.*

The problem of uniform random generation of words in the language of a product automaton requires first to check its emptiness which is PSPACE-complete in the sum of the size of the component automata [14] already for the special case with all letters synchronised (see (5)).

Our sampling algorithm described in Section 4.2 below relies on pre-computations that are polynomial in the size $|\mathcal{A}_{\texttt{red}}| = |Q_{\texttt{red}}| + |\Delta_{\texttt{red}}|$ of the reduced automaton $\mathcal{A}_{\texttt{red}}$. More precisely these problems are fixed parameter tractable when the size $|\mathcal{A}_{\texttt{red}}|$ is considered a parameter, namely their complexity is of the form $O(|\mathcal{A}_{\texttt{red}}| \cdot p(n, \sum_{i=1}^{K} |\mathcal{A}^{(i)}|))$ where p is a polynomial, n is the length of words to be generated and $\sum_{i=1}^{K} |\mathcal{A}^{(i)}|$ is the size of the network. Note that when there are only shared letters the reduced automaton has exactly the same states and transitions as the product automaton and hence can be of an exponential size. However, in the general case, when a fair proportion of letters are not shared we expect the reduced automaton to be of far smaller size than the product automaton. We give in Proposition 5 below upper-bounds on $|\mathcal{A}_{\texttt{red}}|$ that involve parameters easier to compute (in polynomial time wrt. $|\mathcal{N}|$) that we define now.

Given a letter $\alpha \in \texttt{Synch}$ and an automaton $\mathcal{A}$, we denote by $d(\alpha, \mathcal{A})$ the cardinalities of the set of states of $\mathcal{A}$ that are destinations of edges labelled by α, formally $d(\alpha, \mathcal{A}) = |\{q \mid \exists p \in Q, (p, \alpha, q) \in \Delta\}|$. Further, let $d(\mathcal{A}) = \max_\alpha d(\alpha, \mathcal{A})$. The size of the reduced automata, which is an important parameter in the complexity of our algorithms, can be bounded:

▶ **Proposition 5.** *Given a network of automata $(\mathcal{A}^{(i)})_{i\in[K]}$, the size of the reduced automaton $\mathcal{A}_{red}$ satisfies $|Q_{red}| \leq d(\mathcal{A}_{red}) \cdot |\mathtt{Synch}| + 1$ and $|\Delta_{red}| \leq |Q_{red}| \cdot d(\mathcal{A}_{red}) \cdot |\mathtt{Synch}|$ with*

$$d(\mathcal{A}_{red}) \leq \max_{\alpha\in\mathtt{Synch}} \prod_{i=1}^{K} d(\alpha, \mathcal{A}^{(i)}_{red}) \leq \left(\max_{\alpha} \max_{i\leq K} d(\alpha, \mathcal{A}^{(i)})\right)^K.$$

▶ **Remark 6.** *If every transition labelled by the same action goes to a dedicated state, then the proposition above gives $d(\mathcal{A}_{red}) \leq 1$, thus $|Q_{red}| \leq |\mathtt{Synch}| + 1$ and $|\Delta_{red}| \leq |Q_{red}| \cdot |\mathtt{Synch}|$. The particular case assumed in [9] where for each synchronised action α there is only one occurrence of α per component automaton yields further $|\Delta_{red}| = |\mathtt{Synch}|$.*

3.3 Equations on languages and generating functions

Let $\mathcal{A}$ be a DFA and $\mathcal{A}_{\mathtt{red}}$ its reduced automaton. For $s \in Q_{\mathtt{red}}$, let $\tilde{\mathcal{L}}_s$ be the language recognised by the DFA $\tilde{\mathcal{A}}_s$ obtained from $\mathcal{A}$ by removing transitions labelled by actions in $\mathtt{Synch}$ and by changing the initial state to s. For $\delta = (s, \alpha, t) \in \Delta_{\mathtt{red}}$, let $\tilde{\mathcal{L}}_\delta$ be the language recognised by the DFA $\mathcal{A}_\delta$ obtained from $\mathcal{A}$ by removing transitions labelled by actions in $\mathtt{Synch}$ and by changing the initial state to s and the final states to $\{q \mid (q, \alpha, t) \in \Delta\}$.

The following theorem constitutes the core of our study: it gives a decomposition of the language recognised by a network of automata only in terms of union and concatenation of shuffles of languages corresponding to the component automata. All our sampling algorithms for networks of automata in Section 4.2 will rely on this decomposition.

▶ **Theorem 7.** *Let $(\mathcal{A}^{(i)})_{i\in[K]}$ be a network of automata and $\mathcal{A}_{red}$ its associated reduced automaton. Using the above notations, for every $s \in Q_{red}$, it holds that:*

$$\mathcal{L}_s = \tilde{\mathcal{L}}_s \cup \bigcup_{\delta=(s,\alpha,t)\in\Delta_{red}} \tilde{\mathcal{L}}_\delta \alpha \mathcal{L}_t, \qquad \text{with} \quad \tilde{\mathcal{L}}_s = \sqcup\!\sqcup_{i=1}^{K} \tilde{\mathcal{L}}^{(i)}_{s^{(i)}} \quad \text{and} \quad \tilde{\mathcal{L}}_\delta = \sqcup\!\sqcup_{i=1}^{K} \tilde{\mathcal{L}}^{(i)}_{\delta^{(i)}}$$

The language operations (product, union, shuffle) involved in this equation are unambiguous.[4]

The language operations translate into identities on cardinalities and generating functions: union and concatenation of languages yield sum and product of OGF, and shuffle of languages yield product of EGF (see e.g. [6, 22, 11]). To do such translations, it is crucial that the language operations are unambiguous; that is why we consider only network of **deterministic** automata. Denote by $L_s(z), \tilde{L}_s(z), \tilde{L}_\delta(z)$ the OGF of languages $\mathcal{L}_s, \tilde{\mathcal{L}}_s, \tilde{\mathcal{L}}_\delta$.

▶ **Proposition 8.** *Let $(\mathcal{A}^{(i)})_{i\in[K]}$ be a network of automata and $\mathcal{A}_{red}$ its associated reduced automaton. For every $s \in Q_{red}$, the following equation holds on OGF:*

$$L_s(z) = \tilde{L}_s(z) + z \sum_{\delta=(s,\alpha,t)\in\Delta_{red}} \tilde{L}_\delta(z) L_t(z).$$

And the EGF of $\tilde{\mathcal{L}}_s$ (resp. $\tilde{\mathcal{L}}_\delta$) is the product of the EGF of $\tilde{\mathcal{L}}^{(i)}_{s^{(i)}}$ (resp. $\tilde{\mathcal{L}}^{(i)}_{\delta^{(i)}}$).

Let $M(z)$ be the $Q_{\mathtt{red}} \times Q_{\mathtt{red}}$ matrix such that $[M(z)]_{s,t} = \sum_{\delta=(s,\alpha,t)\in\Delta_{\mathtt{red}}} \tilde{L}_\delta(z)$. Then Proposition 8 can be written in terms of vectors and matrices of OGF as follows: $\mathbf{L}(z) = \tilde{\mathbf{L}}(z) + zM(z)\mathbf{L}(z)$. Based on this, we obtain the characterisation of $\mathfrak{r}(L)$ and $\mathbf{L}(z)$ in the next theorem. The important point is that it depends only on the matrix M of size $|Q_{\mathtt{red}}|$ and not on the exponentially big adjacency matrix of the product automaton.

Let $\mathfrak{r}(\tilde{\mathbf{L}}) = \min_{s\in Q_{\mathtt{red}}} \mathfrak{r}(\tilde{L}_s)$ and $\mathfrak{r}^*(M) = \inf\{|z| \,;\, (I - zM(z))^{-1}$ is defined$\}$.

[4] Recall that a language operation is said to be *unambiguous* if every word of the resulting language can be obtained in a unique way by composing different words from the operands.

▶ **Proposition 9.** *The convergence radius* $\mathfrak{r}(L)$ *and the vector of generating function* $\mathbf{L}(z)$ *for* $z < \mathfrak{r}(L)$ *are characterised as follows:*

$$\mathfrak{r}(L) = \min\left(\mathfrak{r}(\tilde{\mathbf{L}}), \mathfrak{r}^*(M)\right) \quad \text{and} \quad \mathbf{L}(z) = (I - zM(z))^{-1}\tilde{\mathbf{L}}(z).$$

4 Uniform sampling for a network of automata

Consider a network of automata. Relying on the reduced automaton, we adapt the sampling methods developed for a unique automaton, and recalled in Section 2, in order to design sampling algorithms for the network of automata. The generic method is as follows:

- choose whether a synchronisation will occur. In the case of no synchronisation, generate a word in the shuffle $\tilde{\mathcal{L}}_s = \sqcup\!\sqcup_{i=1}^{K} \tilde{\mathcal{L}}^{(i)}_{s^{(i)}}$. In the case of synchronisation:
  - choose a transition $\delta = (s, \alpha, t) \in \Delta_{\mathrm{red}}$, (this gives the next synchronisation α),
  - choose a word without synchronisation $w \in \tilde{\mathcal{L}}_\delta = \sqcup\!\sqcup_{i=1}^{K} \tilde{\mathcal{L}}^{(i)}_{\delta^{(i)}}$,
  - write $w\alpha$ on the output tape, and repeat from t to generate the rest of the word.

This method is derived from the equations on languages in Theorem 7. It will be consistently applied in all three methods of sampling. It first requires to be able to generate words in the shuffle of languages, and second to compute the right probabilities in order to make the choices. In all cases, we assume that we have algorithms dealing with a single automaton, see Section 2 (we call them *monolithic*), and we adapt and combine them.

4.1 Pure Shuffle

We first study the case of no synchronisation between the component DFAs, that is $\texttt{Synch} = \emptyset$. The language of the network of automata is the shuffle of the languages of the component automata (see (4)). We present the three sampling methods. They share the common idea of generating words for the component automata and then choosing a word in their shuffle.

4.1.1 Fixed length uniform sampler

For two languages, the cardinalities of the shuffle are easy to compute: if $\mathcal{L} = \mathcal{L}_1 \sqcup\!\sqcup \mathcal{L}_2$, then $l_n = \sum_{m=0}^{n} \binom{n}{m} l_{1,m} l_{2,n-m}$. The generalisation to K languages is more complicated:

$$l_n = \sum \frac{n!}{n^{(1)}! \times \cdots \times n^{(K)}!}\, l^{(1)}_{n^{(1)}} \cdots l^{(K)}_{n^{(K)}} \quad \text{with} \quad n^{(1)} + \cdots + n^{(K)} = n\,.$$

This is not satisfactory since it involves exponentially many terms. The difficulty was already noted in [9] where a solution was proposed under restricted assumptions (in particular the strong connectivity of the DFA). Here we propose another way to bypass the difficulty which is always valid.

The idea is to decompose the shuffle in a recursive manner. We define $\mathcal{L}^{(\le 0)} = \{\varepsilon\}$ and for $1 \le i \le K$, $\mathcal{L}^{(\le i)} = \mathcal{L}^{(\le i-1)} \sqcup\!\sqcup \mathcal{L}^{(i)}$. Then $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)} = \mathcal{L}^{(\le K)}$, and the corresponding cardinalities are recursively computed efficiently in Algorithm 1.

Algorithm 1 uses the monolithic routine $\texttt{Mono-Cardinalities}(\mathcal{A}, n)$, that outputs all the cardinalities $(l_m)_{0\le m\le n}$, in time $O(n|\mathcal{A}|)$, using Eq. (2). The cardinalities associated to $\mathcal{L}(\mathcal{A})$ corresponds to $s = \iota$, that is, $l_m = l_{\iota,m}$.

▶ **Lemma 10.** *The cardinalities* $(l^{(\le i)}_m)_{0\le m\le n, 0\le i\le K}$ *associated to the languages* $(\mathcal{L}^{(\le i)})_{0\le i\le K}$ *can be computed with Algorithm 1 in time* $O(Kn \log n + n\sum_{i=1}^{K} |\mathcal{A}^{(i)}|)$.

Algorithm 1 `Shuffle-Card`$((\mathcal{A}^{(i)})_{i\in[K]}, n)$ (precomputation for `Shuffle-Unif`).

Require: K DFAs $\mathcal{A}^{(i)}$ and a natural integer $n \in \mathbb{N}$.
Ensure: compute and store $(l_m^{(j)})_{0\le m\le n, 1\le j\le K}$ and $(l_m^{(\le j)})_{0\le m\le n, 1\le j\le K}$.
1: **for** $i = 1$ to K **do**
2: $(l_m^{(i)})_{0\le m\le n} \leftarrow$ `Mono-Cardinalities`$(\mathcal{A}^{(i)}, n)$;
3: compute $\sum_{m=0}^{n} l_m^{(\le i)} z^m/m! = \left(\sum_{m=0}^{n} l_m^{(\le i-1)} z^m/m!\right)\left(\sum_{m=0}^{n} l_m^{(i)} z^m/m!\right) \mod z^{n+1}$

Algorithm 2 Uniform sampler `Shuffle-Unif`$((\mathcal{A}^{(i)})_{i\in[K]}, n)$.

Require: K DFAs $\mathcal{A}^{(i)}$, $n \in \mathbb{N}$, and cardinalities $(l_m^{(j)})_{0\le m\le n, 1\le j\le K}$, $(l_m^{(\le j)})_{0\le m\le n, 1\le j\le K}$.
Ensure: return a word in $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ of length n uniformly at random.
1: $N \leftarrow n$;
2: **for** $i = K$ down to 1 **do**
3: choose $n^{(i)}$ with probability $n^{(i)} \mapsto \binom{N}{n^{(i)}} l_{N-n^{(i)}}^{(\le i-1)} l_{n^{(i)}}^{(i)} / l_N^{(\le i)}$;
4: $w^{(i)} \leftarrow$ `Mono-Unif`$(\mathcal{A}^{(i)}, n^{(i)})$; //(use e.g. algorithm of [5] or [19])
5: $N \leftarrow N - n^{(i)}$;
6: **return** `Shuffle-Words`$((w^{(i)})_{i\in[K]})$. //(use e.g. algorithm of [9])

Algorithm 2 repeatedly chooses, according to the precomputed cardinalities, the length of the words to be generated in each language $\mathcal{L}^{(i)}$. It generates such words w_i using a monolithic sampling algorithm on DFA `Mono-Unif` (see [5] or [19]), and then returns a random word in the shuffle language of the w_i by using a function `Shuffle-Words` that chooses uniformly at random a word in $w^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup w^{(K)}$ in linear time (see [9]).

Sampling with Algorithm 2 is no more difficult than doing independently uniform sampling for the component automata. In the theorem below, $|\texttt{Mono-Unif}(\mathcal{A}^{(i)}, n)|$ denotes the complexity of a monolithic algorithm for the uniform sampling of words of length n in the ith component automaton.

▶ **Theorem 11.** *Algorithm 2 returns a word in $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ of length n uniformly at random in time complexity $O(\sum_{i=1}^{K} |\texttt{Mono-Unif}(\mathcal{A}^{(i)}, n)|)$ after the precomputations explained in Lem. 10.*

4.1.2 Boltzmann sampler

The shuffle product of languages fits well with exponential generating functions (see Theorem 7): the EGF of the shuffle product is the product of the EGF of the components. As a consequence, the exponential Boltzmann sampler for the shuffle of languages is easy to construct from the exponential Boltzmann samplers of the component languages. Below, denote by `Mono-Boltz-Expo` a monolithic routine that realises an exponential Boltzmann sampler for a single automaton, see [4].

The situation is not as simple for ordinary generating functions. We use a method from [8] for obtaining an ordinary Boltzmann sampler by appropriately biasing an exponential Boltzmann sampler. This is the methodology followed in Algorithm 4 below. The weight functions $u \mapsto e^{-u} \prod_{i=1}^{K} \hat{L}^{(i)}(zu)$ should be computed numerically rather than symbolically.

▶ **Theorem 12.** *`Shuffle-Boltz-Expo`$((\mathcal{A}^{(i)})_{i\in[K]}, u)$ is an exponential Boltzmann sampler of parameter u for $\mathcal{L} = \mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ and `Shuffle-Boltz`$((\mathcal{A}^{(i)})_{i\in[K]}, z)$ is an ordinary Boltzmann sampler of parameter z for $\mathcal{L}$.*

Algorithm 3 Exponential Boltzmann sampler `Shuffle-Boltz-Expo`$((\mathcal{A}^{(i)})_{i\in[K]}, u)$.

Require: K DFAs $\mathcal{A}^{(i)}$ with languages $\mathcal{L}^{(i)}$.
Ensure: realise an exponential Boltzmann sampler for $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ of parameter u.
1: **for** $i = 1$ to K **do**
2: $\quad w^{(i)} \leftarrow$ `Mono-Boltz-Expo`$(\mathcal{A}^{(i)}, u)$
3: **return** `Shuffle-Words`$((w^{(i)})_{i\in[K]})$.

Algorithm 4 Ordinary Boltzmann sampler `Shuffle-Boltz`$((\mathcal{A}^{(i)})_{i\in[K]}, z)$

Require: K DFAs $\mathcal{A}^{(i)}$ with languages $\mathcal{L}^{(i)}$.
Ensure: realise an ordinary Boltzmann sampler for $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$ of parameter z.
1: Choose u according to the weight function: $u \mapsto e^{-u} \prod_{i=1}^{K} \hat{L}^{(i)}(zu)$;
2: **return** `Shuffle-Boltz-Expo`$((\mathcal{A}^{(i)})_{i\in[K]}, zu)$.

4.1.3 Parry sampler

The following theorem ensures that a Parry sampler for the shuffle language is very easy to construct given Parry samplers for the component automata.

The Parry sampler associated with a DFA is defined via the spectral attributes of its adjacency matrix (Section 2.3). For the product automaton, spectral attributes admit compact representations, thus avoiding the explicit construction of the exponentially big adjacency matrix.

▶ **Lemma 13.** *Let $\mathcal{A} = \mathcal{A}^{(1)} \times \cdots \times \mathcal{A}^{(K)}$ be the product of K strongly connected DFAs without synchronisation. Let ρ, $\mathbf{v}$, $(\rho^{(i)})_{i\in[K]}$, $(\mathbf{v}^{(i)})_{i\in[K]}$ be the spectral attributes defined as in Theorem 1. Then $\rho = \sum_{i=1}^{n} \rho^{(i)}$ and $v_s = \prod_{i=1}^{K} v^{(i)}_{s^{(i)}}$ for every $s \in Q$.*

This lemma enables us to design a Parry sampler compositionally.

▶ **Theorem 14.** *Given K strongly connected automata $(\mathcal{A}^{(i)})_{i\in[K]}$ without synchronised action, Algorithm 5 is a Parry sampler for the shuffle of their languages.*

4.2 General case with synchronisation

In the general case with synchronisation, we rely on the decomposition of Theorem 7, as stated in the beginning of Section 4.

4.2.1 Fixed length uniform sampler

The idea of our recursive method, described in Algorithm 7 is as follows: either choose to generate a word without synchronisation that leads to a final state, or choose to generate a word without synchronisation that leads to a synchronised transition, take this transition and repeat recursively from the current state. The weight we attribute to each choice is proportional to the cardinality of the language corresponding to this choice. These cardinalities are computed in Algorithm 6.

Denote by `CompInvMat`(m) the complexity of inverting a square matrix of size $m \times m$.

▶ **Lemma 15.** *Algorithm 6 runs in time $O(n \log n(\texttt{CompInvMat}(|Q_{red}|) + K|\Delta_{red}|))$.*

▶ **Theorem 16.** *Algorithm 7 is a uniform sampler that runs in linear time after precomputations made in Algorithm 6.*

Algorithm 5 Parrry sampler `Shuffle-Parry`$((\mathcal{A}^{(i)})_{i\in[K]})$

Require: K strongly connected DFAs $\mathcal{A}^{(i)}$ with languages $\mathcal{L}^{(i)}$, spectral radii $\rho^{(i)}$, and, for each $\mathcal{A}^{(i)}$, a Parry sampler $\mathcal{M}^{(i)}$.
Ensure: realise a Parry sampler for $\mathcal{L}^{(1)} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{L}^{(K)}$.
1: runs $\mathcal{M}^{(i)}$ for $i \in [K]$ in parallel to get K infinite random words $(w^{(i)})_{i\in[K]}$;
2: **while true do**
3: choose i with probability $\rho^{(i)}/(\rho^{(1)} + \cdots + \rho^{(K)})$;
4: remove from $w^{(i)}$ its first letter and write it on the output tape;

Algorithm 6 `Compo-Card`$((\mathcal{A}^{(i)})_{i\in[K]}, n)$ (precomputation of cardinalities for `Compo-Unif`).

Require: K DFAs $\mathcal{A}^{(i)}$ and a natural integer n.
Ensure: compute and store every cardinalities used in `Compo-Unif`$((\mathcal{A}^{(i)})_{i\in[K]}, s, n)$.
1: **for** $s \in Q_{\texttt{red}}$ **do**
2: `Shuffle-Card`$((\tilde{\mathcal{A}}^{(i)}_{s^{(i)}})_{i\in[K]}, n)$; (this implicitly defines $\tilde{\mathbf{L}}(z) \mod z^{n+1}$)
3: **for** $\delta \in \Delta_{\texttt{red}}$ **do**
4: `Shuffle-Card`$((\mathcal{A}^{(i)}_{\delta^{(i)}})_{i\in[K]}, n)$; (this implicitly defines $M(z) \mod z^{n+1}$)
5: Compute $\mathbf{L}(z) \mod z^{n+1}$ by solving $\mathbf{L}(z) = \tilde{\mathbf{L}}(z) + zM(z)\mathbf{L}(z)$ with all generating functions and operations modulo z^{n+1}.

4.2.2 Boltzmann sampler

Boltzmann sampling is obtained from the system of equations

$$L_s(z) = \tilde{L}_s(z) + z \sum_{\delta=(s,\alpha,t)\in\Delta_{\texttt{red}}} \tilde{L}_\delta(z) L_t(z)$$

Boltzmann sampling also applies the generic method of random sampling depicted at the beginning of this section. The procedure is described in Algorithm 8. If no synchronisation occurs (probability $\tilde{L}_s(z)/L_s(z)$), we sample a word in the shuffle of the $\tilde{\mathcal{A}}^{(i)}_{s^{(i)}}$), using Boltzmann sampling with parameter z. Otherwise we uniformly choose a transition $\delta = (s, \alpha, t) \in \Delta_{\texttt{red}}$, generate u in the shuffle of the $\mathcal{A}^{(i)}_{\delta^{(i)}}$, using Boltzmann sampling with parameter z, write $u\alpha$ and repeat from t to generate the rest of the word.

▶ **Theorem 17.** *Algorithm 8 is an ordinary Boltzmann sampler.*

4.2.3 Parry sampler

In this section, we consider a network of automata with synchronisations such that the product automaton is strongly connected (the case without synchronisations was treated in Section 4.1.3).

As before, we work with the matrix $M(z)$ associated with the reduced automaton to avoid constructing the adjacency matrix A of the product automaton. Proposition 18 is a way of stating the Perron-Frobenius theorem with $M(z)$ rather than A, enabling us to describe the Parry measure wrt. the reduced automaton in Theorem 19.

▶ **Proposition 18.** *Let $\mathfrak{r}$ and $\mathbf{v}$ be the convergence radii and Perron vector associated to the product automaton. Then $\mathfrak{r}$ and the restriction $\mathbf{v}_{red}$ of $\mathbf{v}$ to Q_{red} can be characterised wrt. $M(z)$ as follows: $\mathfrak{r} = \min\{z > 0 \mid \det(I - zM(z)) = 0\}$; $\mathbf{v}_{red}$ is the unique vector such that $\mathbf{v}_{red} \geq 0$ and $\mathfrak{r}M(\mathfrak{r})\mathbf{v}_{red} = \mathbf{v}_{red}$.*

Algorithm 7 Uniform sampler $\texttt{Compo-Unif}((\mathcal{A}^{(i)})_{i\in[K]}, s, n)$.

Require: K DFAs $\mathcal{A}^{(i)}$, a natural integer n and a state $s \in Q_{\texttt{red}}$ and cardinalities computed by $\texttt{Compo-Card}((\mathcal{A}^{(i)})_{i\in[K]}, n)$.
Ensure: return a word in $\mathcal{L}_s$ of length n uniformly at random.
1: with probability $\tilde{l}_{s,n}/l_{s,n}$ **return** $\texttt{Shuffle-Unif}((\tilde{\mathcal{A}}^{(i)}_{s^{(i)}})_{i\in[K]}, n)$;
2: choose m with weight $\sum_{\delta=(s,\alpha,t)\in\Delta_{\texttt{red}}} l_{\delta,m-1}l_{t,n-m} / \sum_{m=1}^{n}\sum_{\delta=(s,\alpha,t)\in\Delta_{\texttt{red}}} l_{\delta,m-1}l_{t,n-m}$;
3: choose $\delta = (s,\alpha,t) \in \Delta_{\texttt{red}}$ with probability $l_{\delta,m-1}l_{t,n-m} / \sum_{\delta=(s,\alpha,t)\in\Delta_{\texttt{red}}} l_{\delta,m-1}l_{t,n-m}$;
4: $w \leftarrow \texttt{Shuffle-Unif}((\mathcal{A}^{(i)}_{\delta^{(i)}})_{i\in[K]}, m)$;
5: **return** $w :: \alpha :: \texttt{Compo-Unif}((\mathcal{A}^{(i)})_{i\in[K]}, t, n-m)$.

Algorithm 8 Ordinary Boltzmann sampler $\texttt{Compo-Boltz}((\mathcal{A}^{(i)})_{i\in[K]}, s, z)$.

Require: K DFAs $\mathcal{A}^{(i)}$, a state $s \in Q_{\texttt{red}}$ and a parameter z.
Ensure: realises a ordinary Boltzmann sampler of parameter z for the language $\mathcal{L}_s$.
1: With probability $\tilde{L}_s(z)/L_s(z)$ **return** $\texttt{Shuffle-Boltz}((\tilde{\mathcal{A}}^{(i)}_{s^{(i)}})_{i\in[K]}, z)$;
2: Choose $\delta = (s,\alpha,t) \in \Delta_{\texttt{red}}$ with probability $\tilde{L}_\delta(z)L_t(z) / \sum_{\delta=(s,\alpha,t)\in\Delta_{\texttt{red}}} \tilde{L}_\delta(z)L_t(z)$;
3: **return** $\texttt{Shuffle-Boltz}((\mathcal{A}^{(i)}_{\delta^{(i)}})_{i\in[K]}, z) :: \alpha :: \texttt{Compo-Boltz}((\mathcal{A}^{(i)})_{i\in[K]}, t, z)$.

▶ **Theorem 19.** *Algorithm 9 is a Parry sampler of infinite words starting from the input state $s \in Q_{red}$.*

5 Conclusion and further work

In this paper, we propose several algorithms for uniformly sampling words recognised by networks of automata. For the purely interleaved case, with no synchronisation, the sampling algorithms are polynomial in the size of the component automata, while previous known algorithms were exponential (since they were polynomial on the exponentially big product automaton). For the general case where synchronisations occur on shared actions, our methods are efficient with respect to the reduced automaton (the product automaton where interleaved actions are removed).

We plan to implement the different algorithms presented here and apply them on benchmarks. For instance, we could use the benchmarks of [9] and [19]. We would also like to incorporate our uniform sampling into a Monte-Carlo model checker.

The recursive methods for words of a fixed length n that we presented here have a bit complexity which is essentially n times bigger than their arithmetic complexity since the cardinalities we compute grow exponentially with n and each one needs a linear amount of bits to be stored. We think that this extra factor n can be avoided using a divide-and-conquer paradigm akin to that of [5]. Further work to deal with interleaving and synchronisations has to be done though.

In the context of Monte-Carlo model checking of Büchi properties [13, 20] the meaningful objects to sample are accepting lassos (an accepting lasso is a path followed by an elementary cycle that visits accepting states). We would like to design uniform sampling of accepting lassos for networks of automata by building on top of the present work and on [20].

Algorithm 9 Parry sampler $\texttt{Parry}((\mathcal{A}^{(i)})_{i\in[K]}, s)$

Require: K DFAs $\mathcal{A}^{(i)}$ and a state $s \in Q_{\texttt{red}}$.
Ensure: realises a Parry sampler of infinite words from s.
1: choose $\delta = (s, \alpha, t) \in \Delta_{\texttt{red}}$ with probability $\mathfrak{r}\tilde{L}_\delta(\mathfrak{r})\, v_t / v_s$;
2: **return** $\texttt{Shuffle-Boltz}(\tilde{\mathcal{L}}_\delta, \mathfrak{r}) :: \alpha :: \texttt{Parry}((\mathcal{A}^{(i)})_{i\in[K]}, t)$.

References

1 Samy Abbes and Jean Mairesse. Uniform generation in trace monoids. In G. Italiano, G. Pighizzini, and D. Sannella, editors, *Math. Found. Comput. Sc. 2015 (MFCS 2015), part 1*, volume 9234 of *Lecture Notes in Comput. Sci.*, pages 63–75. Springer, 2015.

2 Alfred Aho, Rajeev Motwani, and Jeffrey Ullman. *Introduction to Automata Theory, Languages, and Computation: 2nd edition*. Addison Wesley, 2001.

3 Christel Baier and Joost-Pieter Katoen. *Principles of model checking*. MIT Press, 2008.

4 Nicolas Basset, Michèle Soria, and Jean Mairesse. Uniform sampling for networks of automata, 2017. URL: `http://hal.upmc.fr/hal-01545936`.

5 Olivier Bernardi and Omer Giménez. A linear algorithm for the random sampling from regular languages. *Algorithmica*, 62(1-2):130–145, 2012.

6 Jean Berstel and Christophe Reutenauer. *Noncommutative rational series with applications*, volume 137 of *Enc. of Math. and Appl.* Cambridge University Press, 2011.

7 Olivier Bodini, Antoine Genitrini, and Frédéric Peschanski. The combinatorics of nondeterminism. In Anil Seth and Nisheeth K. Vishnoi, editors, *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2013, December 12-14, 2013, Guwahati, India*, volume 24 of *LIPIcs*, pages 425–436. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013. `doi:10.4230/LIPIcs.FSTTCS.2013.425`.

8 Alexis Darrasse, Konstantinos Panagiotou, Olivier Roussel, and Michele Soria. Biased Boltzmann samplers and generation of extended linear languages with shuffle. *DMTCS Proceedings*, 01:125–140, 2012.

9 Alain Denise, Marie-Claude Gaudel, Sandrine-Dominique Gouraud, Richard Lassaigne, Johan Oudinet, and Sylvain Peyronnet. Coverage-biased random exploration of large models and application to testing. *STTT*, 14(1):73–93, 2012. `doi:10.1007/s10009-011-0190-1`.

10 Philippe Duchon, Philippe Flajolet, Guy Louchard, and Gilles Schaeffer. Boltzmann samplers for the random generation of combinatorial structures. *Combinatorics, Probability and Computing*, 13(4-5):577–625, 2004.

11 Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, New York, NY, USA, 1 edition, 2009.

12 Philippe Flajolet, Paul Zimmerman, and Bernard Van Cutsem. A calculus for the random generation of labelled combinatorial structures. *Theoretical Computer Science*, 132(1):1–35, 1994.

13 Radu Grosu and Scott A. Smolka. Monte carlo model checking. In *TACAS'05*, 2005.

14 Dexter Kozen. Lower bounds for natural proof systems. In *FOCS*, volume 77, pages 254–266, 1977.

15 M. Lothaire. *Applied Combinatorics on Words (Encyclopedia of Mathematics and its Applications)*. Cambridge University Press, New York, NY, USA, 2005.

16 Kuldeep S. Meel, Moshe Y. Vardi, Supratik Chakraborty, Daniel J. Fremont, Sanjit A. Seshia, Dror Fried, Alexander Ivrii, and Sharad Malik. Constrained sampling and counting: Universal hashing meets SAT solving. In Adnan Darwiche, editor, *Beyond NP, Papers from the 2016 AAAI Workshop, Phoenix, Arizona, USA, February 12, 2016.*, volume WS-16-05

of *AAAI Workshops*. AAAI Press, 2016. URL: http://www.aaai.org/ocs/index.php/WS/AAAIW16/paper/view/12618.

17 Mehryar Mohri. Generic ε-removal algorithm for weighted automata. In *International Conference on Implementation and Application of Automata*, pages 230–242. Springer, 2000.

18 Madhavan Mukund. Automata on distributed alphabets. In *Modern Applications of Automata Theory*, pages 257–288. World Scientific, 2012. doi:10.1142/9789814271059_0009.

19 Johan Oudinet, Alain Denise, and Marie-Claude Gaudel. A new dichotomic algorithm for the uniform random generation of words in regular languages. *Theor. Comput. Sci.*, 502:165–176, 2013. doi:10.1016/j.tcs.2012.07.025.

20 Johan Oudinet, Alain Denise, Marie-Claude Gaudel, Richard Lassaigne, and Sylvain Peyronnet. Uniform monte-carlo model checking. In *FASE 2011*, pages 127–140, 2011.

21 Antonio Restivo. The shuffle product: New research directions. In *International Conference on Language and Automata Theory and Applications*, pages 70–81. Springer, 2015.

22 A. Salomaa and M. Soittola. *Automata-theoretic aspects of formal power series*. Springer Verlag, 1978.

Tractability of Separation Logic with Inductive Definitions: Beyond Lists*

Taolue Chen[1], Fu Song[2], and Zhilin Wu[3]

1 **Department of Computer Science and Information Systems, Birkbeck, University of London, UK; and State Key Laboratory of Novel Software Technology, Nanjing University, China**
2 **School of Information Science and Technology, ShanghaiTech University, China**
3 **State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, China**

Abstract

In 2011, Cook et al. showed that the satisfiability and entailment can be checked in polynomial time for a fragment of separation logic that allows for reasoning about programs with pointers and linked lists. In this paper, we investigate whether the tractability results can be extended to more expressive fragments of separation logic that allow defining data structures beyond linked lists. To this end, we introduce separation logic with a *simply-nonlinear compositional* inductive predicate where source, destination, and static parameters are identified explicitly ($\mathsf{SLID}_{\mathsf{SNC}}$). We show that if the inductive predicate has more than one source (destination) parameter, the satisfiability problem for $\mathsf{SLID}_{\mathsf{SNC}}$ becomes intractable in general. This is exemplified by an inductive predicate for doubly linked list segments. By contrast, if the inductive predicate has only one source (destination) parameter, the satisfiability and entailment problems for $\mathsf{SLID}_{\mathsf{SNC}}$ are tractable. In particular, the tractability results hold for inductive predicates that define list segments with tail pointers and trees with one hole.

1998 ACM Subject Classification F.4.1 Mathematical Logic

Keywords and phrases Separation logic, Inductive definitions, Satisfiability, Entailment

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.37

1 Introduction

Separation Logic (SL, [23, 25]), an extension of Hoare logic, is a well-established formalism for the verification of heap manipulating programs. SL features a *separating conjunction operator* and *inductive predicates*, which allow to express how data structures are laid out in memory in a succinct way. Since its introduction, various verification tools based on separation logic have been developed. A notable one among them is the INFER tool [9], which was acquired by Facebook in 2013 and has been actively used in its development process [10]. To enable program verification with SL assertions in a Hoare-logic style, it is vital to be able to check satisfiability and entailment of SL formulae. Unfortunately, entailment checking in full SL is undecidable [11]. Thus one has to either resort to heuristics or consider restricted, decidable fragments. We focus on the latter approach in this paper. Earlier efforts, including the theoretical work [2] which leads to the tool SmallFoot [3], considered SL with primitives of pointers and the hardwired singly linked list segments. A remarkable result on

* This work was partially supported by the UK EPSRC grant (EP/P00430X/1), the European CHIST-ERA project SUCCESS, the NSFC grants (61472474, 61402179, 61532019, 61662035, 61572478).

this fragment was given by Cook et al. [14], who developed a polynomial-time algorithm for the satisfiability and entailment problems based on graph reasoning. This has also led to the tool SeLoger [19].

Hardwired inductive predicates have obvious limitations. More recent research has been focusing on automated reasoning about generic *user-defined* inductive predicates inside the framework of SL. This class of logic is usually referred to as *Separation Logic with Inductive Definitions* (SLID). Because of their generality, satisfiability and entailment checking are usually much more difficult. Nevertheless, notable progress has been made in this direction. For instance, Iosif et al. showed decidability of satisfiability and entailment for an expressive fragment of SLID by a reduction to Monadic Second Order Logic on graphs with bounded tree width [20]. Later on, they reduced the entailment problem to the language inclusion problem of tree automata, and implemented the tool SLIDE [21]. Brotherston et al. showed that the satisfiability problem (but not entailment) for a very expressive fragment of SLID is ExpTime-complete [6]. In the same paper, they also proved that the satisfiability problem becomes Np-complete if the arities of all predicates are bounded by a constant.

Contributions. We are mostly interested in tractable (polynomial-time) satisfiability and entailment checking for SLID. Our strategy is to generalise the graph reasoning initiated in [14], which currently is limited to singly linked lists only. To this end, we concentrate on a class of *simply-nonlinear compositional* inductive predicates, where the source, destination, and static parameters are identified explicitly. This class of inductive predicates enjoys some nice compositional properties introduced in [17], which makes, for example, the entailment problem easier to solve. The main question is *whether the satisfiability and entailment checking for the resulting SLID fragment, denoted by* $\mathsf{SLID}_{\mathsf{SNC}}$*, can be done in polynomial time*. We obtain both positive and negative results in this regard.

1. We show that, with an inductive predicate *dllseg* for *doubly linked list segments*, the satisfiability problem for $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ is Np-complete. From the perspective of data structures, doubly linked list segments are perhaps the minimal extension of singly linked list segments, and thus one would have expected the tractability result for doubly linked list segments by a straightforward adaptation of the method for singly linked list segments by Cook et al. in [14]. Our result is among the strongest intractability results for the satisfiability problem for SLID so far. Note that the Np-hardness reduction in [6] requires four predicates, while our reduction uses only one predicate *dllseg*. However, it should be noted that there are at most three parameters for predicates in [6], but *dllseg* has four parameters (i.e., two source and destination parameters respectively). The result suggests that $\mathsf{SLID}_{\mathsf{SNC}}$ becomes intractable in general if inductive predicates with more than one source (destination) parameter are allowed.
2. We show that both the satisfiability and the entailment checking can be done in polynomial time if the inductive predicate has only one source (destination) parameter. Remarkably, the fragment covers linear data structures (e.g., list segments with tail pointers), as well as non-linear data structures (e.g., trees with one hole). To our best knowledge, this is one of the first tractability results of SLID for a *class* of user-defined inductive predicates. (Independently, [8] studied a different class mostly on the model checking problem; cf. *related work* for comparison.)
   To achieve this, we generalise the graph-theoretic techniques developed in [14], in particular, the concept of graph homomorphisms, to non-linear structures. Compared to [14], the graph representations of the $\mathsf{SLID}_{\mathsf{SNC}}$ formulae have a considerably more involved structure, e.g., the unique simple path property between a pair of nodes in [14] is lost here.

An additional intricacy is that we consider the classical semantics of $\mathsf{SLID}_{\mathsf{SNC}}$ formulae rather than the (less general)*intuitionistic* semantics in [14]. The authors of [14] did briefly mention that, to tackle the classical semantics, extra conditions—although not specified—should be added to ensure that all arcs are covered in the graph homomorphism. They, however, suggested that "the details are messy and deferred to a full version." We concur with these insights. As singly linked list segments are a (very) special case of our inductive predicates, the decision procedure in this paper provides a complete account for the entailment checking under the classical semantics.

Related work. There is a vast literature on separation logic, and we will have to focus on work which is the most relevant to us. In particular, we only cover the work on SLID, and skip many results on, for instance, first-order separation logic (without inductive definitions) [4, 15]. Antonopoulos et al. established some fundamental decidability and complexity results for the entailment problem of SLID [1]. In particular, they showed that deciding the entailment $\varphi \models \psi$ for SLID is intractable where ψ may contain existentially quantified variables, even for a single predicate of list segments. This result is largely incomparable to us, since existential quantified variables are disallowed in our setting. The work [5, 13] took the cyclic-proof approach which is based on induction on the paths of proof trees, but the decision procedures therein are incomplete in general. Brotherston et al. investigated array separation logic and obtained some complexity results [7]: They showed that the satisfiability is NP-complete, entailment is decidable with high complexity, and bi-abduction is in NP. Recently, [8] gave another tractable fragment of SLID, but mostly focused on the model checking problem. The tractable fragment of [8], defined by three constraints MEM ("Memory-consuming"), CV ("Constructively valued") and DET ("Deterministic"), is incomparable to ours: "trees with one hole" is in our tractable fragment, while it does not belong to MEM+CV+DET, as the DET constraint fails; however, an MEM+CV+DET formula may contain several different inductive predicates, while an $\mathsf{SLID}_{\mathsf{SNC}}$ formula allows at most one inductive predicate. We also mention recent work considering decision procedures for SLID extended with data and size constraints [12, 18, 26, 22, 24, 27].

In the sequel, we discuss [18] and [16] which are technically more related to this paper. Strictly speaking, [18] considered SLID extended with data constraints, so here we limit the comparison to the data-free part. First of all, [18] tackled *linear* structures, while here we focus on more general non-linear structures (e.g., trees). More importantly, the decision procedures in [18] are *intractable*, even when specialised to the data-free setting. Indeed, for both satisfiability and entailment, only NP upper-bounds can be obtained. The current work improves [18] by giving polynomial-time algorithms for even non-linear structures. In particular, for satisfiability, we extend the graph-theoretic approach in [14]; for entailment, we give a new definition of graph homomorphism, which is much more involved than that in [18], but yields an efficient checking algorithm.

[16] considered a fragment of SLID where nested lists and skip lists are expressible, and provided an *incomplete* decision procedure for the entailment problem by generalising the graph homomorphism in [14]. A more technical comparison is deferred to Section 4.2.

2 Preliminaries

For $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$. We assume a set of variables Vars ranged over by $E, F, X, Y, \cdots$, and a set of *locations* $\mathbb{L}$ typically ranged over by $l, l', \cdots$. We assume a designated variable $\mathsf{nil} \in \mathsf{Vars}$ (for the "null" value of pointers in programs) and $\mathbb{L}$ contains a special element null.

Moreover, we assume a set of fields $\mathcal{F}$ ranged over by $f, f', \cdots$. We focus on separation logic with a *simply-nonlinear compositional* inductive predicate (denoted by $\mathsf{SLID}_{\mathsf{SNC}}[P]$):

$$\begin{array}{lcl} \Pi & ::= & E = F \mid E \neq F \mid \Pi \wedge \Pi, \\ \Sigma & ::= & \mathtt{emp} \mid E \mapsto \rho \mid P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B}) \mid \Sigma * \Sigma, \end{array}$$

where ρ is a tuple of elements from $\mathcal{F} \times \mathsf{Vars}$, and $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B})$ is a simply-nonlinear compositional predicate whose definition will be specified shortly.

The formulae Π and Σ are called the *pure* and *spatial* formulae respectively. Atomic formulae of the form $E \mapsto \rho$ and $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B})$ are called the *points-to* and *predicate* atoms respectively. For an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula φ, we use $\mathsf{Vars}(\varphi)$ to denote the set of variables occurring in φ, in addition, we use $\mathsf{Atoms}(\varphi)$ to denote the set of points-to or predicate atoms occurring in φ. We require that the predicate P in $\mathsf{SLID}_{\mathsf{SNC}}[P]$ is *simply-nonlinear compositional*, that is, of the following form:

1. The parameters of P are divided into three categories: the *source* parameters $E, \boldsymbol{E}'$, the *destination* parameters $F, \boldsymbol{F}'$, and the *static* parameters $\boldsymbol{B}$. (Note that $\boldsymbol{E}', \boldsymbol{F}', \boldsymbol{B}$ denote a vector of variables.) We require that the source and the destination parameters are *symmetric*, in particular, they must be of the same length. A predicate P is usually denoted by $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B})$ to highlight the parameters.
2. $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B})$ is defined by a set of rules including:
   - base rule: $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B}) ::= E = F \wedge \boldsymbol{E}' = \boldsymbol{F}' \wedge \mathtt{emp}$,
   - inductive rule:

$$\begin{array}{l} P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B}) ::= \exists \boldsymbol{X}.\ E \mapsto [(f_1, Y_1), \ldots, (f_k, Y_k)] * \\ \quad P(X_0, \boldsymbol{X_0'}; F, \boldsymbol{F}'; \boldsymbol{B}) * P(X_1, \boldsymbol{X_1'}; \mathsf{nil}, \mathbf{nil}; \boldsymbol{B}) * \cdots * P(X_l, \boldsymbol{X_l'}; \mathsf{nil}, \mathbf{nil}; \boldsymbol{B}), \end{array}$$

   such that
   - $X_0, \ldots, X_l$ are mutually distinct variables and $\boldsymbol{X} = \{X_0, \ldots, X_l\}$,
   - $Y_1, \ldots, Y_k \in \boldsymbol{E}' \cup \boldsymbol{B} \cup \boldsymbol{X} \cup \{\mathsf{nil}\}$ and each variable from $\boldsymbol{E}' \cup \boldsymbol{B} \cup \boldsymbol{X}$ occurs exactly once in $E \mapsto [(f_1, Y_1), \ldots, (f_k, Y_k)]$,
   - for each $i : 0 \leqslant i \leqslant l$, $\boldsymbol{X_i'} \subseteq \{E\} \cup \boldsymbol{X}$.

Note that above, in $Y_1, \ldots, Y_k \in \boldsymbol{E}' \cup \boldsymbol{B} \cup \boldsymbol{X} \cup \{\mathsf{nil}\}$ or $\boldsymbol{X_i'} \subseteq \{E\} \cup \boldsymbol{X}$, $\boldsymbol{E}'$ should be interpreted as the *set* of variables occurring in $\boldsymbol{E}'$, similarly for $\boldsymbol{B}, \boldsymbol{X}, \boldsymbol{X_i'}$. We assume that, if there are multiple inductive rules, the same set of fields $\mathsf{Flds}(P)$ is used. In addition, we define the set $\mathsf{PFlds}(P)$ of *principal fields*, as the set of fields $f \in \mathsf{Flds}(P)$ such that there is an inductive rule of P where (f, Z) occurs for some $Z \in \boldsymbol{X}$, and the set $\mathsf{AFlds}(P)$ of *auxiliary fields*, as the set of fields $f \in \mathsf{Flds}(P)$ such that there is an inductive rule of P where (f, Z') occurs for some $Z' \in \boldsymbol{E}' \cup \boldsymbol{B} \cup \{\mathsf{nil}\}$. We assume that $\mathsf{PFlds}(P) \cap \mathsf{AFlds}(P) = \varnothing$, which implies that $\mathsf{Flds}(P) = \mathsf{PFlds}(P) \uplus \mathsf{AFlds}(P)$.

In the base or inductive rule, the formula on the left (resp. right) of ::= is called the *head* (resp. *body*) of the rule. For each predicate $P(E, \boldsymbol{E}'; F, \boldsymbol{F}'; \boldsymbol{B})$, there is exactly one base rule, and in each inductive rule, each destination parameter from $F, \boldsymbol{F}'$ occurs exactly once in the body of the rule, namely, in $P(X_0, \boldsymbol{X_0'}; F, \boldsymbol{F}'; \boldsymbol{B})$. We use $\mathcal{P}$ to denote the set of simply-nonlinear compositional inductive predicates. Moreover, we use $\mathcal{P}_1$ to denote the set of inductive predicates $P(E; F; \boldsymbol{B}) \in \mathcal{P}$, i.e., the set of inductive predicates with a single source resp. destination parameter.

Semantics. Each $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula is interpreted on *states*. Formally, given a finite set of fields $\mathbb{F} \subseteq \mathcal{F}$, a *state* is a pair (s, h), where

- s is a *stack* which is a partial function $\mathsf{Vars} \rightharpoonup \mathbb{L}$ such that $\mathsf{dom}(s)$ is finite, $\mathsf{nil} \in \mathsf{dom}(s)$, and $s(\mathsf{nil}) = \mathsf{null}$,
- h is a *heap* which is a partial function $\mathbb{L} \times \mathbb{F} \rightharpoonup \mathbb{L}$ such that $\mathsf{dom}(h)$ is finite, $h(\mathsf{null}, f)$ is undefined for each $f \in \mathbb{F}$, and for each $l \in \mathbb{L}$, if $h(l, f)$ is defined for some $f \in \mathbb{F}$, then $h(l, f')$ is defined for each $f' \in \mathbb{F}$.

We use States to denote the set of states. For a heap h, we use $\mathsf{ldom}(h)$ to denote the set of locations $l \in \mathbb{L}$ such that $h(l, f)$ is defined for some $f \in \mathbb{F}$. Note that $\mathsf{null} \notin \mathsf{ldom}(h)$. We write $h_1 \# h_2$ if $\mathsf{ldom}(h_1) \cap \mathsf{ldom}(h_2) = \varnothing$, in which case we write $h_1 \uplus h_2$ for the union of h_1 and h_2. A heap h_1 is called a *subheap* of h_2 if $\mathsf{ldom}(h_1) \subseteq \mathsf{ldom}(h_2)$ and for each $l \in \mathsf{ldom}(h_1)$ and $f \in \mathbb{F}$, $h_1(l, f) = h_2(l, f)$.

Let $(s, h) \in \mathsf{States}$ and φ be an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula. Then the semantics of φ is defined as follows,

- $(s, h) \models E = F$ (resp. $(s, h) \models E \neq F$) if $s(E) = s(F)$ (resp. $s(E) \neq s(F)$),
- $(s, h) \models \Pi_1 \wedge \Pi_2$ if $(s, h) \models \Pi_1$ and $(s, h) \models \Pi_2$,
- $(s, h) \models \mathsf{emp}$ if $\mathsf{ldom}(h) = \varnothing$,
- $(s, h) \models E \mapsto [(f_1, X_1), \cdots, (f_k, X_k)]$ if $\mathsf{ldom}(h) = s(E)$, and for each $i : 1 \leqslant i \leqslant k$, $h(s(E), f_i) = s(X_i)$,
- $(s, h) \models P(E, \boldsymbol{E'}; F, \boldsymbol{F'}; \boldsymbol{B})$ if $(s, h) \in [\![P(E, \boldsymbol{E'}; F, \boldsymbol{F'}; \boldsymbol{B})]\!]$,
- $(s, h) \models \Sigma_1 * \Sigma_2$ if there are h_1, h_2 such that $h = h_1 \uplus h_2$, $(s, h_1) \models \Sigma_1$ and $(s, h_2) \models \Sigma_2$,

where the semantics of predicate $[\![P(E, \boldsymbol{E'}; F, \boldsymbol{F'}; \boldsymbol{B})]\!]$ is given by the least fixepoint of a monotone operator constructed from the body of rules for P. More concretely, assume that P contains m inductive rules, each of which is denoted by R_i and is of the form $R_i : P(E, \boldsymbol{E'}, F, \boldsymbol{F'}, \boldsymbol{B}) ::= \Theta_i$ (i.e., Θ_i is the body of R_i). For each R_i (where $i \in [m]$), we define a monotone operator $\tau_i : 2^{\mathsf{States}} \to 2^{\mathsf{States}}$ as follows: $\tau_i(S) := \{(s, h) \mid (s, h) \models_{P,S} \Theta_i\}$, where $\models_{P,S}$ is the satisfaction relation defined above, except that P is interpreted by S, that is, $(s, h) \models P(E, \boldsymbol{E'}; F, \boldsymbol{F'}; \boldsymbol{B})$ if $(s, h) \in S$. We finally define $[\![P]\!] := \mu S. \bigcup_i \tau_i(S)$.

▶ **Example 1.** We use predicates *lseg*, *dllseg*, *tlseg*, and *th* to exemplify our definitions, which represent list segments, doubly linked list segments, list segments with tail pointers, and binary trees with one hole, respectively.

$$
\begin{array}{lcl}
\mathit{lseg}(E; F) & ::= & E = F \wedge \mathtt{emp}, \\
\mathit{lseg}(E; F) & ::= & \exists X.\ E \mapsto (\mathtt{next}, X) * \mathit{lseg}(X, F). \\
\mathit{dllseg}(E, P; F, L) & ::= & E = F \wedge P = L \wedge \mathtt{emp}, \\
\mathit{dllseg}(E, P; F, L) & ::= & \exists X.\ E \mapsto [(\mathtt{next}, X), (\mathtt{prev}, P)] * \mathit{dllseg}(X, E; F, L). \\
\mathit{tlseg}(E; F; B) & ::= & E = F \wedge \mathtt{emp}, \\
\mathit{tlseg}(E; F; B) & ::= & \exists X.\ E \mapsto [(\mathtt{next}, X), (\mathtt{tail}, B)] * \mathit{tlseg}(X; F; B). \\
\mathit{th}(E; F) & ::= & E = F \wedge \mathtt{emp}, \\
\mathit{th}(E; F) & ::= & \exists X, Y.\ E \mapsto [(\mathtt{left}, X), (\mathtt{right}, Y)] * \mathit{th}(X; \mathsf{nil}) * \mathit{th}(Y; F), \\
\mathit{th}(E; F) & ::= & \exists X, Y.\ E \mapsto [(\mathtt{left}, X), (\mathtt{right}, Y)] * \mathit{th}(X; F) * \mathit{th}(Y; \mathsf{nil}).
\end{array}
$$

For $P \in \mathcal{P}$, we investigate the following two decision problems:

Satisfiability. Given an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula φ, decide whether there is a state $(s, h) \models \varphi$.

Entailment. Given two $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae φ and ψ with $\mathsf{Vars}(\psi) \subseteq \mathsf{Vars}(\varphi)$, decide whether $\varphi \models \psi$, i.e., for every state (s, h), $(s, h) \models \varphi$ implies $(s, h) \models \psi$.

The inductive predicates defined above fall into the categy of *compositional inductive predicates* [17] and admit the composition lemma, i.e., $P(E_1, \boldsymbol{E'_1}; E_2, \boldsymbol{E'_2}; \boldsymbol{B}) * P(E_2, \boldsymbol{E'_2}; E_3, \boldsymbol{E'_3}; \boldsymbol{B}) \models P(E_1, \boldsymbol{E'_1}; E_3, \boldsymbol{E'_3}; \boldsymbol{B})$ holds, which is crucial for the decision procedure of entailment.

In literature, there is usually a distinction of the *classical* semantics and the *intuitionistic* semantics. The aforementioned semantics for $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae are called the classical semantics. In the intuitionistic semantics, pure and spatial formulae are interpreted in the same way as the classical one. The only difference is, in the intuitionistic semantics, we have, for an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula $\varphi = \Pi \wedge \Sigma$, $(s,h) \models \varphi$ iff *there is a subheap* h_1 *of* h, $(s,h_1) \models \Pi$ and $(s,h_1) \models \Sigma$. Henceforth, when necessary, we write $\models_c$ to denote the satisfiability relation under the classical semantics, and accordingly, $\models_i$ to denote the satisfiability relation under the intuitionistic semantics. The two semantics are equivalent for the satisfiability problem, but differ in the entailment problem. For instance, $\mathit{lseg}(E_1;E_2) * \mathit{lseg}(E_2;E_3) \models_c \mathit{lseg}(E_1;E_2)$ does *not* hold, while $\mathit{lseg}(E_1;E_2) * \mathit{lseg}(E_2;E_3) \models_i \mathit{lseg}(E_1;E_2)$ *does* hold.

Graph-theoretic notions. We shall work extensively upon an arc-labeled directed graph equipped with a symmetry relation. Here we collect some notations on this matter. For simplicity, we ignore the arc-labels and symmetric relations here, since they are irrelevant right now. In general, a directed graph in $\mathcal{G}$ is a pair $(\mathcal{N},\mathcal{E})$, where $\mathcal{N}$ is a finite set of nodes, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of directed arcs. For an arc $e = (v,w) \in \mathcal{E}$, v and w are called the *source* and *destination* node of e and sometimes w is called a *successor* of v and v is called a *predecessor* of w.

A *path* in $\mathcal{G}$ is a sequence of consecutive directed arcs in $\mathcal{G}$. A node w is *reachable* from v if there is a path from v to w (note that v is reachable from v itself). In this case, we also say that v is an *ancestor* of w. A *cycle* is a path where the starting node and the ending node are equal. A path is *simple* if no nodes occur twice on the path. A cycle is *simple* if all nodes are distinct, except the ending node. For a node v and a directed arc $e = (v',w')$, e is said to be *reachable* from v if v' is reachable from v, and e is said to be *co-reachable* from v if v is reachable from w'. We use $\mathcal{R}_{\mathcal{G}}(v)$ to denote the set of *arcs* that are reachable from v.

For a graph $\mathcal{G}$, let $\mathcal{N}(\mathcal{G})$ denote the set of nodes in $\mathcal{G}$. For a set of arcs $\mathcal{E}' \subseteq \mathcal{E}$, let $\mathcal{N}(\mathcal{E}')$ denote the set of all source or destination nodes of arcs in $\mathcal{E}'$. For $\mathcal{N}' \subseteq \mathcal{N}$, *the subgraph of* $\mathcal{G}$ *induced by* $\mathcal{N}'$, denoted by $\mathcal{G}[\mathcal{N}']$, is $(\mathcal{N}', \mathcal{E} \cap (\mathcal{N}' \times \mathcal{N}'))$. On the other hand, for $\mathcal{E}' \subseteq \mathcal{E}$, *the subgraph of* $\mathcal{G}$ *induced by* $\mathcal{E}'$, denoted by $\mathcal{G}[\mathcal{E}']$, is $(\mathcal{N}(\mathcal{E}'), \mathcal{E}')$.

A connected component (CC) $\mathcal{C}$ (resp. strongly connected component, SCC, $\mathcal{S}$) of $\mathcal{G}$ is said to be *nontrivial* if $\mathcal{C}$ (resp. $\mathcal{S}$) contains at least one arc.

3 Intractability: Doubly linked list segments

Let $\varphi = \Pi \wedge \Sigma$ be an $\mathsf{SLID}_{\mathsf{SNC}}[\mathit{dllseg}]$ formula, where *dllseg* is given in Example 1. We shall show the intractability of deciding the satisfiability of $\mathsf{SLID}_{\mathsf{SNC}}[\mathit{dllseg}]$ formulae. The intractability is proved by a reduction from 3SAT. It turns out that the reduction works even for a *restricted* class of $\mathsf{SLID}_{\mathsf{SNC}}[\mathit{dllseg}]$ formulae $\Pi \wedge \Sigma$ where Π only contains inequalities and Σ only contains predicate atoms. To ease the presentation of the reduction, we will introduce a graphical representation for this restricted class of $\mathsf{SLID}_{\mathsf{SNC}}[\mathit{dllseg}]$ formulae. For an $\mathsf{SLID}_{\mathsf{SNC}}[\mathit{dllseg}]$ formula φ, recall that $\mathsf{Vars}(\varphi)$ and $\mathsf{Atoms}(\varphi)$ denote the set of variables and atoms in φ. We construct a graph $\mathcal{G}_\varphi = (\mathcal{N}, \mathcal{A}, \mathcal{E}, \mathcal{L}, R_{\neq})$ as follows.

- $\mathcal{N} = \mathsf{Vars}(\varphi)$ and $\mathcal{A} = \mathsf{Atoms}(\varphi)$.
- $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{A} \times \mathcal{N}$ is the set of arcs and $\mathcal{L}$ is the arc-labeling function, defined as: for each predicate atom $\mathfrak{a} = \mathit{dllseg}(E,P;F,L)$ in $\mathsf{Atoms}(\varphi)$, there are two arcs $e = (E,\mathfrak{a},F) \in \mathcal{E}$ and $e' = (L,\mathfrak{a},P)$ with $\mathcal{L}(e) = (\mathfrak{a},+)$ and $\mathcal{L}(e') = (\mathfrak{a},-)$ respectively.
- $R_{\neq} = \{(E,F),(F,E) \mid \Pi \models E \neq F\}$.

For convenience, we usually write $E \xrightarrow{\ell} F$ for an arc (E, F) labeled by ℓ. Clearly, $\mathcal{E}$ satisfies that, for each predicate atom $\mathfrak{a}$, there are exactly two arcs in $\mathcal{G}$, labeled by $(\mathfrak{a}, +)$ and $(\mathfrak{a}, -)$ respectively. We usually say that these two arcs are dual and have the opposite poles. All graphs for restricted $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ formulae defined as above form a class of (restricted) $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ graphs.

From each restricted $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ graph $\mathcal{G}_\varphi = (\mathcal{N}, \mathcal{A}, \mathcal{E}, \mathcal{L}, R_{\neq})$, we can recover a (restricted) $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ formula $\varphi_{\mathcal{G}} = \Pi_{\mathcal{G}} \wedge \Sigma_{\mathcal{G}}$, where $\Pi_{\mathcal{G}} = \bigwedge_{(E,F)\in R_{\neq}} E \neq F$ and $\Sigma_{\mathcal{G}} = \overset{*}{\underset{\substack{\mathfrak{a}\in\mathcal{A}, \mathcal{L}((L,\mathfrak{a},P))=(\mathfrak{a},-)}}{\underset{\mathcal{L}((E,\mathfrak{a},F))=(\mathfrak{a},+)}{}}} dllseg(E, P; F, L)$. Since $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ formulae and $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ graphs can be easily transformed to each other, we will use them interchangeably.

We are ready to present the reduction from the 3SAT Problem. Let Ψ be a CNF formula with clauses $C_1, \cdots, C_m$ over a set of variables $\{x_1, \cdots, x_n\}$. For each $j \in [m]$, C_j is a disjunction of at most three literals, that is, $C_j = l_{j,1}$, or $C_j = l_{j,1} \vee l_{j,2}$, or $C_j = l_{j,1} \vee l_{j,2} \vee l_{j,3}$, where for each $k = 1, 2, 3$, $l_{j,k} = x_i$ or $\neg x_i$ for some $i \in [n]$. Without loss of generality, we assume that for each $i \in [n]$ and $j \in [m]$, x_i occurs at most once in C_j.

We introduce a set of atoms $\mathcal{A}_\Psi = \{\mathfrak{x}_{i,j}, \overline{\mathfrak{x}_{i,j}} \mid x_i \text{ occurs in } C_j\}$. Furthermore, for each $j \in [m]$, each literal $l_{j,k}$ in C_j ($k \in \{1, 2, 3\}$) is associated with an atom $\mathfrak{l}_{j,k}$ defined as follows: if $l_{j,k} = x_i$, then $\mathfrak{l}_{j,k} = \mathfrak{x}_{i,j}$, otherwise, $\mathfrak{l}_{j,k} = \overline{\mathfrak{x}_{i,j}}$.

We then construct $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$-graphs for variables and clauses respectively.

- For each variable x_i, assume that x_i appears in $C_{j_1}, C_{j_2}, \cdots, C_{j_{k_i}}$ (where $1 \leqslant j_1 < \cdots < j_{k_i} \leqslant m$), we have the following arcs: (1) $E_i \xrightarrow{(\mathfrak{x}_{i,j_1},+)} X_i^1$, $E_i \xrightarrow{(\overline{\mathfrak{x}_{i,j_1}},+)} Y_i^1$, and (2) $X_i^s \xrightarrow{(\mathfrak{x}_{i,j_{s+1}},+)} X_i^{s+1}$, $Y_i^s \xrightarrow{(\overline{\mathfrak{x}_{i,j_{s+1}}},+)} Y_i^{s+1}$ for $s : 1 \leqslant s < k_i$. Moreover, for convenience, we write F_i for $X_i^{k_i}$ and F_i' for $Y_i^{k_i}$.Then we set $(F_i, F_i'), (F_i', F_i) \in R_{\neq}$.
- For each clause C_j, if $C_j = l_{j,1} \vee l_{j,2} \vee l_{j,3}$, then we have arcs $G_j \xrightarrow{(\mathfrak{l}_{j,1},-)} G_j' \xrightarrow{(\mathfrak{l}_{j,2},-)} G_j'' \xrightarrow{(\mathfrak{l}_{j,3},-)} H_j$. In addition, we set $(G_j, H_j), (H_j, G_j) \in R_{\neq}$. The constructions for the cases $C_j = l_{j,1}$ and $C_j = l_{j,2} \vee l_{j,3}$ are similar.

Finally, $\mathcal{G}_\Psi$ comprises the collection of $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$-graphs for all variables x_i ($i \in [n]$) and all clauses C_j ($j \in [m]$). Roughly speaking, in $\mathcal{G}_\Psi$, the semantics of separating conjunction $*$ and the variable inequalities ensure that each Boolean variable is assigned with exactly one value from $\{\texttt{true}, \texttt{false}\}$. Moreover, the synchronisation of the two arcs with opposite poles, together with variable inequalities, guarantees that the assignment of Boolean variables satisfies all clauses in Ψ.

It is easy to see that in $\mathcal{G}_\Psi$, for each pair of atoms $\mathfrak{x}_{i,j}$ or $\overline{\mathfrak{x}_{i,j}}$, one of them appears twice, but the other occurs exactly once. To make $\mathcal{G}_\Psi$ a (restricted) $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$-graph defined above, we can simply add some additional isolated arcs. For instance, if $(\mathfrak{x}_{i,j}, +)$ occurs, but no $(\mathfrak{x}_{i,j}, -)$ in $\mathcal{G}_\Psi$ (because $\neg x_i$, but not x_i, occurs in C_j), we add an arc $G_j^\dagger \xrightarrow{(\mathfrak{x}_{i,j},-)} H_j^\dagger$, where $G_j^\dagger, H_j^\dagger$ are fresh variables.

▶ **Example 2.** The following example illustrates the intuition of the reduction. Suppose $\Psi = (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$ and the graph $\mathcal{G}_\Psi$ is illustrated in Figure 1, where dashed lines represent $R_{\neq}$. In a model (s, h) of $\mathcal{G}_\Psi$, since $(G_1, H_1) \in R_{\neq}$, at least one of $\mathfrak{x}_{1,1}, \mathfrak{x}_{2,1}$, say, $\mathfrak{x}_{1,1}$, must be assigned with a nonempty subheap. By the semantics of $*$, at least one of the two paths from E_1 to F_1 and from E_1 to F_1' must be empty. Since $(\mathfrak{x}_{1,1}, +)$ appears in the path from E_1 to F_1 and $\mathfrak{x}_{1,1}$ is nonempty, we deduce that the path from E_1 to F_1' has to be assigned with an empty subheap. Accordingly, since $(\overline{\mathfrak{x}_{1,2}}, +)$ occurs in the path from E_1 to F_1', the atom $\overline{\mathfrak{x}_{1,2}}$ is assigned with an empty subheap. In addition, from $G_2 \xrightarrow{(\overline{\mathfrak{x}_{1,2}},-)} G_2' \xrightarrow{(\overline{\mathfrak{x}_{2,2}},-)} H_2$

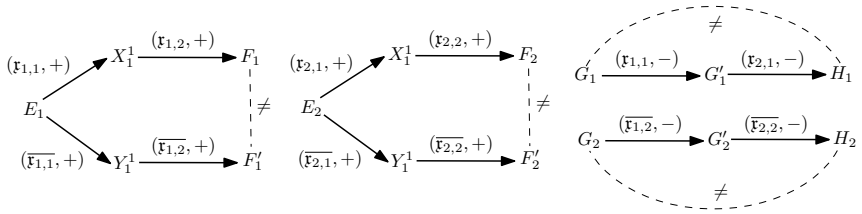


Figure 1 G_Ψ for $\Psi = (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$.

and $(G_2, H_2) \in R_{\neq}$, we have that $\overline{\mathfrak{r}_{2,2}}$ has to be assigned with a nonempty subheap, which, in turn, implies that the path from E_2 to F_2 is assigned with an empty subheap. Therefore, $\mathfrak{r}_{1,1}$ and $\overline{\mathfrak{r}_{2,2}}$ are assigned with a nonempty heap and they induce a satisfiable assignment θ of Ψ with $\theta(x_1) = \texttt{true}$ and $\theta(x_2) = \texttt{false}$.

The satisfiability of $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ can be checked easily in NP. Hence, we have:

▶ **Theorem 3.** *The satisfiability problem for* $\mathsf{SLID}_{\mathsf{SNC}}[dllseg]$ *is* NP*-complete.*

4 Tractability: Beyond list segments

In this section, we show that if $P \in \mathcal{P}_1$, then $\mathsf{SLID}_{\mathsf{SNC}}[P]$ is tractable for both satisfiability and entailment. We fix a predicate $P(E; F; \boldsymbol{B}) \in \mathcal{P}_1$.

4.1 Satisfiability

For an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula φ, we construct a graph $\mathcal{G}_\varphi$ to represent φ. Specifically, $\mathcal{G}_\varphi$ is a tuple $(\mathsf{Vars}(\varphi), \mathsf{Atoms}(\varphi), \mathcal{N}_\varphi, \mathcal{E}_\varphi, \mathcal{L}_\varphi, R_{\neq})$ defined as follows.

- $\mathcal{N}_\varphi = \{[E] \mid E \in \mathsf{Vars}(\varphi)\}$ such that $[E]$ is the equivalence class of E under (the equivalence relation) $\sim_\Pi$ where $E' \sim_\Pi F'$ iff $\Pi \models E' = F'$.
- $\mathcal{E}_\varphi$ is the set of arcs, and $\mathcal{L}_\varphi$ is the arc-labeling function, defined as follows:
  - for each points-to atom $\mathfrak{a} = E \mapsto [(f_1, F_1), \ldots, (f_k, F_k)]$ in $\mathsf{Atoms}(\varphi)$ and each $i \in [k]$ with $f_i \in \mathsf{PFlds}(P)$, $\mathcal{E}_\varphi$ contains a *field-labeled* arc $e = ([E], [F_i])$ with $\mathcal{L}_\varphi(e) = (f_i, \mathfrak{a})$;
  - for each predicate atom $\mathfrak{a} = P(E; F; \boldsymbol{B})$ in $\mathsf{Atoms}(\varphi)$, $\mathcal{E}_\varphi$ contains a *predicate-labeled* arc $e = ([E], [F])$ with $\mathcal{L}_\varphi(e) = (P, \mathfrak{a})$;

  In both cases, $\mathfrak{a}$ is the atom associated with e, denoted by $\mathsf{atom}(e)$. In some places later on, in order to emphasise that e is from $\mathcal{G}_\varphi$, we also use $\mathsf{atom}_\varphi(e)$, instead of $\mathsf{atom}(e)$.
- $R_{\neq} = \{([E], [F]), ([F], [E]) \mid \Pi \models E \neq F\}$.

As a convention, $\bot$ represents the graph for an unsatisfiable $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula.

All graphs for $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae defined as above form a class of $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graphs, whose formal definition is omitted here due to the page limit. It is easy to verify that $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae and $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graphs are equivalent, namely, they can be transformed into each other. Hence we will use them interchangeably.

▶ **Example 4.** An $\mathsf{SLID}_{\mathsf{SNC}}[th]$-graph $\mathcal{G}$ is given in Figure 2, where each node is just one variable. Labels of the arcs are abbreviated, e.g., the label $(\texttt{left}, E_1 \mapsto [(\texttt{left}, E_2), (\texttt{right}, E_3)])$ of the arc (E_1, E_2) is abbreviated as $\texttt{left}$.

Our decision procedure for satisfiability follows [14] closely. The underpinning idea is, on a high level, to reduce the $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graph by exploiting that the subheaps represented by different spatial atoms must be domain disjoint. The main difference to [14] will be

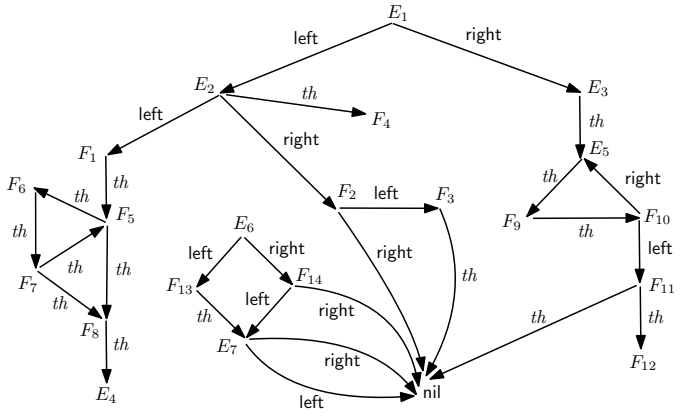


Figure 2 An example of $\mathsf{SLID}_{\mathsf{SNC}}[th]$ graphs.

discussed when they appear. We start with some notion of persistent set of arcs, largely from [14]. Intuitively, a set of arcs is persistent if in every model, a nonempty subheap has to be assigned to it.

▶ **Definition 5** (Persistent set of arcs [14]). Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{N}, \mathcal{E}, \mathcal{L}, R_{\neq})$ is an $\mathsf{SLID}_{\mathsf{SNC}}[P]$-graph and $\mathcal{E}' \subseteq \mathcal{E}$. Then $\mathcal{E}'$ is said to be *persistent* in $\mathcal{G}$ if either $\mathcal{E}'$ contains a field-labeled arc, or there are two nodes $[E], [F]$ in a *connected component* of $\mathcal{G}[\mathcal{E}']$ (the subgraph of $\mathcal{G}$ induced by $\mathcal{E}'$) such that $([E], [F]) \in R_{\neq}$.

For an $\mathsf{SLID}_{\mathsf{SNC}}[P]$-graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{N}, \mathcal{E}, \mathcal{L}, R_{\neq})$ and $e \in \mathcal{E}$, we use $\mathcal{G} - \mathsf{atom}(e)$ to denote the graph obtained from $\mathcal{G}$ by removing all the arcs $e' \in \mathcal{E}$ such that $\mathsf{atom}(e') = \mathsf{atom}(e)$.

▶ **Definition 6** (Reduced graphs). An $\mathsf{SLID}_{\mathsf{SNC}}[P]$-graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{N}, \mathcal{E}, \mathcal{L}, R_{\neq})$ is *reduced* if $\mathcal{G} = \bot$, or otherwise if it satisfies the following conditions:

(C1) For each $[E] \in \mathcal{N}$, $([E], [E]) \notin R_{\neq}$.
(C2) For each arc e out of $[E]$, $\mathcal{R}_{\mathcal{G}-\mathsf{atom}(e)}([E])$ is *not* persistent.
(C3) For each arc e out of $[E]$, $[\mathsf{nil}]$ is *not* reachable from $[E]$ in $\mathcal{R}_{\mathcal{G}-\mathsf{atom}(e)}([E])$.
(C4) For each pair of distinct nodes $[E], [F]$, there do not exist two predicate-labeled arc-disjoint simple paths from $[E]$ to $[F]$.

▶ Remark. We would like to remark that the arcs e in **(C2)** and **(C3)** may be field-labeled arcs. Moreover, we would like to mention the connection of the aforementioned conditions with those in [14, Table 2]: **(C2)** is adapted from conditions (i)—(iii) therein to deal with nonlinear structures, and **(C4)** is adapted from condition (iv) therein. On the other hand, **(C3)** is new to deal with the special variable nil ([14] did not consider nil).

▶ **Example 7.** The graph $\mathcal{G}$ in Figure 2 is not reduced, because it violates **(C2)**: For the arc $e_1 = (E_2, F_4)$, $\mathcal{R}_{\mathcal{G}-\mathsf{atom}(e_1)}(E_2)$ is persistent; **(C3)**: For the arc $e_2 = (F_{11}, F_{12})$, nil is reachable from F_{11} in $\mathcal{R}_{\mathcal{G}-\mathsf{atom}(e_2)}(F_{11})$; and **(C4)**: There are two arc-disjoint predicate-labeled simple paths from F_5 to F_8. $\mathcal{G}$ can be transformed into a reduced graph, given as $\mathcal{G}_{\varphi}$ in Figure 3.

As in [14], deciding whether a non-$\bot$ $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graph $\mathcal{G}$ is reduced can be done in polynomial time in the size of $\mathcal{G}$. To transform an arbitrary non-$\bot$ $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graph in to a reduced one, we provide a procedure which checks, for a given input non-$\bot$ $\mathsf{SLID}_{\mathsf{SNC}}[P]$ graph

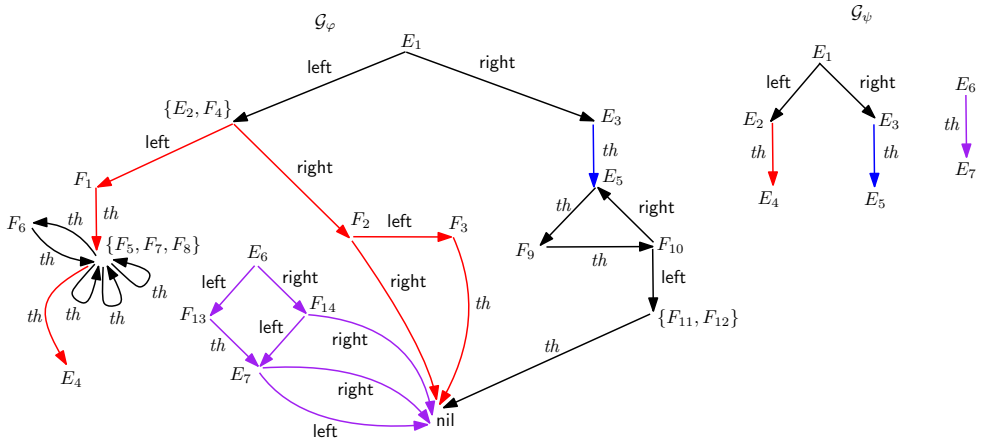


Figure 3 $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$: Running example for the entailment problem.

$\mathcal{G}$ if any condition in Definition 6 is violated. If this is the case, the algorithm performs certain operations (e.g., contract arcs, or merge nodes, or mark the graph to be $\bot$ directly) depending on which condition is violated, until $\mathcal{G}$ is reduced. Most of these are similar to [14]. However, regarding the new condition **(C3)**, if there is an arc e out of $[E]$ and $[\mathsf{nil}]$ is reachable from $[E]$ in $\mathcal{R}_{\mathcal{G}-\mathsf{atom}(e)}([E])$, then we contract the arc e in $\mathcal{G}$ if e is predicate-labeled, and mark $\mathcal{G}$ to be $\bot$ directly otherwise. The details of the procedure is omitted due to the page limit.

By Definition 6, when the graph $\mathcal{G}$ is reduced, there are only two possibilities: either $\mathcal{G} = \bot$ implying that the corresponding formula $\mathsf{SLID}_{\mathsf{SNC}}[P]$ is unsatisfiable, or $\mathcal{G} \neq \bot$ in which case one can construct a model out of $\mathcal{G}$. The similar result for linked lists is rather straightforward [14]. However, it is a bit more involved in our setting, since we need to construct a nonlinear structure to witness the satisfiability of $\mathcal{G}$.

▶ **Proposition 8.** *Given an* $\mathsf{SLID}_{\mathsf{SNC}}[P]$ *graph* $\mathcal{G}$*, one can construct, in polynomial time, a reduced* $\mathsf{SLID}_{\mathsf{SNC}}[P]$ *graph* $\mathcal{G}'$ *such that* $\mathcal{G}$ *is satisfiable iff* $\mathcal{G}' \neq \bot$*. In addition, if* $\mathcal{G}' \neq \bot$*,* $(s,h) \models \mathcal{G}$ *iff* $(s,h) \models \mathcal{G}'$ *for each state* (s,h)*.*

▶ **Theorem 9.** *Suppose* $P \in \mathcal{P}_1$*. Then the satisfiability for* $\mathsf{SLID}_{\mathsf{SNC}}[P]$ *is in polynomial time.*

4.2 Entailment

In this section, we consider the entailment problem, i.e., to decide whether $\varphi \models \psi$ for two *satisfiable* $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae φ, ψ such that $\mathsf{Vars}(\psi) \subseteq \mathsf{Vars}(\varphi)$. Our goal is to provide a polynomial time decision procedure for the entailment problem.

By Proposition 8, we assume that $\mathcal{G}_\varphi = (\mathsf{Vars}(\varphi), \mathsf{Atoms}(\varphi), \mathcal{N}_\varphi, \mathcal{E}_\varphi, \mathcal{L}_\varphi, R_{\neq,\varphi})$ and $\mathcal{G}_\psi = (\mathsf{Vars}(\psi), \mathsf{Atoms}(\psi), \mathcal{N}_\psi, \mathcal{E}_\psi, \mathcal{L}_\psi, R_{\neq,\psi})$ are *reduced*. For $E \in \mathsf{Vars}(\varphi)$ (resp. $E \in \mathsf{Vars}(\psi)$), we use $[E]_\varphi$ (resp. $[E]_\psi$) to denote the node in $\mathcal{G}_\varphi$ (resp. $\mathcal{G}_\psi$) that contains E. Moreover, we assume that for each $[E]_\varphi, [F]_\varphi \in \mathcal{N}_\varphi$ (resp. $[E]_\psi, [F]_\psi \in \mathcal{N}_\psi$), if $E = F \wedge \varphi$ (resp. $E = F \wedge \psi$) is *unsatisfiable*, then $([E],[F]),([F],[E]) \in R_{\neq,\varphi}$ (resp. $([E],[F]),([F],[E]) \in R_{\neq,\psi}$). According to Proposition 8, these (possibly implicit) inequalities can be added to $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$ in polynomial time. Figure 3 depicts two graphs $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$, which will be used as the running example. ($\mathcal{G}_\varphi$ is obtained from the graph $\mathcal{G}$ in Example 4.)

The main idea of our decision procedure is to utilise graph representations $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$, and to extend the concept of graph homomorphisms used in [14] to nonlinear structures and

the classical semantics. Let us first briefly recall the decision procedure for the entailment problem in [14]: As mentioned in the introduction, [14] focused on singly linked list segments and the intuitionistic semantics. In [14], the entailment $\varphi \models_i \psi$ is reduced to the existence of a graph homomorphism from $\mathcal{G}_\psi$ to some subgraph of $\mathcal{G}_\varphi$, that is, a mapping that assigns to each arc e from $[E]_\psi$ to $[F]_\psi$ of $\mathcal{G}_\psi$ a simple path from $[E]_\varphi$ to $[F]_\varphi$ of $\mathcal{G}_\varphi$, in addition, these simple paths are mutually arc-disjoint. The arcs that belong to these simple paths are called *covered*, and those that do not are called *uncovered*. As a result of the intuitionistic semantics, the uncovered arcs of $\mathcal{G}_\varphi$ can be ignored. The fact that the existence of a graph homomorphism from $\mathcal{G}_\psi$ to some subgraph of $\mathcal{G}_\varphi$ can be decided in polynomial time is attributed to the *unique simple path* property enjoyed by $\mathcal{G}_\varphi$, that is, for each pair of distinct nodes, say $[E]_\varphi$ and $[F]_\varphi$, there is *at most one simple path* from $[E]_\varphi$ to $[F]_\varphi$. The unique simple path property of $\mathcal{G}_\varphi$ implies that the graph homomorphism from $\mathcal{G}_\psi$ to some subgraph of $\mathcal{G}_\varphi$ is *unique* if it exists.

To deal with the entailment problem $\varphi \models_c \psi$ for $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae φ and ψ in this paper, we need to generalise the concept of graph homomorphisms in [14] to nonlinear structures and deal with the classical semantics as follows:

- Since the structures defined by the predicate atoms in $\mathsf{SLID}_{\mathsf{SNC}}[P]$ are nonlinear in general, a graph homomorphism should map each predicate-labeled arc e of $\mathcal{G}_\psi$ to a potentially nonlinear subgraph $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ of $\mathcal{G}_\varphi$, instead of just a simple path. In addition, for each predicate-labeled arc e of $\mathcal{G}_\psi$, $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ should *match* $\mathsf{atom}_\psi(e)$, that is, $\mathsf{Subgraph}_e(\mathcal{G}_\varphi) \models_c \mathsf{atom}_\psi(e)$. (Recall that we assume that an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula and its graph representation are interchangeable.) Actually, we will choose $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ to be the *minimum* subgraph of $\mathcal{G}_\varphi$ that matches $\mathsf{atom}_\psi(e)$, in order to guarantee the uniqueness of the graph homomorphism.
- Because we are considering the classical semantics in this paper, those uncovered arcs of $\mathcal{G}_\varphi$ *cannot* be simply ignored. We need to guarantee that there is a way to *dispatch* the uncovered arcs of $\mathcal{G}_\varphi$ to the predicate-labeled arcs of $\mathcal{G}_\psi$ so that for each predicate-labeled arc e of $\mathcal{G}_\psi$, $\mathsf{atom}_\psi(e)$ is matched by the graph consisting of $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ and these uncovered arcs of $\mathcal{G}_\varphi$ which are dispatched to e.
- Finally, we need to show that checking whether a subgraph of $\mathcal{G}_\varphi$ matches $\mathsf{atom}_\psi(e)$ for a predicate-labeled arc e of $\mathcal{G}_\psi$ can be done in polynomial time.

We shall use examples to illustrate the main ideas of our decision procedure. Let us first demonstrate, for each predicate-labeled arc e of $\mathcal{G}_\psi$, how we choose the graph $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ to be the *minimum* subgraph of $\mathcal{G}_\varphi$ that matches $\mathsf{atom}_\psi(e)$. The following example shows that $\mathcal{G}_\varphi$ for an $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formula φ does not enjoy the unique simple path property in general.

▶ **Example 10.** Let us consider the $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae φ, ψ whose graph representations are illustrated in Figure 3. The graph $\mathcal{G}_\varphi$ does not satisfy the unique simple path property: there are *two* arc-disjoint simple paths from E_6 to E_7 in $\mathcal{G}_\varphi$. Note that the subgraph of $\mathcal{G}_\varphi$ comprising all the arcs reachable from E_6 *does* match $th(E_6; E_7)$. This follows from the fact $E_6 \mapsto [(\mathtt{left}, F_{13}), (\mathtt{right}, F_{14})] * th(F_{13}; E_7) * th(F_{14}; \mathsf{nil}) \models th(E_6; E_7)$ and $F_{14} \mapsto ((\mathtt{left}, E_7), (\mathtt{right}, \mathsf{nil})) * E_7 \mapsto ((\mathtt{left}, \mathsf{nil}), (\mathtt{right}, \mathsf{nil})) \models th(F_{14}, \mathsf{nil})$.

Although the graph representations of $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae do not satisfy the unique simple path property in general, we can still find a way to define $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ as the minimum subgraph of $\mathcal{G}_\varphi$ that matches $\mathsf{atom}_\psi(e)$.

▶ **Definition 11** ($\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$)**.** Given a predicate-labeled arc $e = ([E]_\psi, [F]_\psi)$ in $\mathcal{G}_\psi$, $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ is defined as follows:

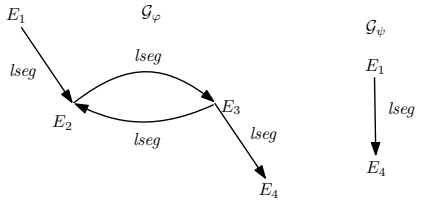


Figure 4 Graph $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$.

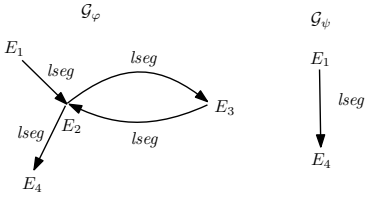


Figure 5 Graph $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$.

- If $[E]_\varphi = [F]_\varphi$, then $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ is the empty graph.
- Otherwise,
  - if $[F]_\varphi = [\mathsf{nil}]_\varphi$, then $\mathsf{Subgraph}_e(\mathcal{G}_\varphi) = \mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}([E]_\varphi)]$, that is, the subgraph of $\mathcal{G}_\varphi$ comprising all the arcs reachable from $[E]_\varphi$,
  - if $[F]_\varphi \neq [\mathsf{nil}]_\varphi$,
    - if there is a *unique* simple path from $[E]_\varphi$ to $[F]_\varphi$, denoted by π_e, then $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ is the subgraph of $\mathcal{G}_\varphi$ comprising: (1) all the arcs in π_e, (2) all field-labeled arcs $e' = ([E']_\varphi, [F']_\varphi)$ such that $[E']_\varphi \in \mathcal{N}(\pi_e)$ and $[F']_\varphi \notin \mathcal{N}(\pi_e)$, and (3) all arcs of $\mathcal{G}_\varphi$ that are reachable from $[F']_\varphi$,
    - otherwise, $\mathsf{Subgraph}_e(\mathcal{G}_\varphi) = \mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}([E]_\varphi)]$.

▶ **Example 12.** Let us consider the formulae φ, ψ illustrated in Figure 3. Let e_1, e_2, e_3 be the predicate-labeled arcs in $\mathcal{G}_\psi$, from E_2 to E_4, E_3 to E_5, and E_6 to E_7 respectively. Since $[E_2]_\varphi \neq [E_4]_\varphi$, $[E_4]_\varphi \neq [\mathsf{nil}]_\varphi$, and there is a unique path from $[E_2]_\varphi$ to $[E_4]_\varphi$, according to Definition 11, $\mathsf{Subgraph}_{e_1}(\mathcal{G}_\varphi)$ comprises the unique simple path from $[E_2]_\varphi$ to $[E_4]_\varphi$, the arc from $[E_2]_\varphi$ to $[F_2]_\varphi$, and all the arcs reachable from $[F_2]_\varphi$. Similarly, $\mathsf{Subgraph}_{e_2}(\mathcal{G}_\varphi)$ comprises the unique simple path from $[E_3]_\varphi$ to $[E_5]_\varphi$ in $\mathcal{G}_\varphi$, which is just the predicate-labeled arc from $[E_3]_\varphi$ to $[E_5]_\varphi$. Since $[E_6]_\varphi \neq [E_7]_\varphi$, $[E_7]_\varphi \neq [\mathsf{nil}]_\varphi$, and there are two distinct simple paths from $[E_6]_\varphi$ to $[E_7]_\varphi$, we have that $\mathsf{Subgraph}_{e_3}(\mathcal{G}_\varphi) = \mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}([E_6]_\varphi)]$, that is, the subgraph of $\mathcal{G}_\varphi$ comprising all the arcs reachable from $[E_6]_\varphi$.

Example 13 illustrates the difference between the intuitionistic and classical semantics.

▶ **Example 13.** Let $\varphi = \mathit{lseg}(E_1; E_2) * \mathit{lseg}(E_2; E_3) * \mathit{lseg}(E_3; E_2) * \mathit{lseg}(E_3; E_4)$, $\psi = \mathit{lseg}(E_1; E_4)$, and $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$ are depicted in Figure 4. (For simplicity, arc label (lseg, $\mathit{lseg}(E_1; E_2)$) is abbreviated as lseg, *sic passim.*) We claim $\varphi \not\models \psi$. To see this, let (s, h) be the state such that $s(E_1) = 1$, $s(E_2) = 2$, $s(E_3) = s(E_4) = 3$, $h(1, \mathtt{next}) = 2$, $h(2, \mathtt{next}) = 3$, $h(3, \mathtt{next}) = 2$. Then $(s, h) \models \varphi$, but $(s, h) \not\models \psi$ under the classical semantics, although the subheap h' of h with $\mathsf{ldom}(h') = \{1, 2\}$ satisfies ψ. This is in contrast to the intuitionistic semantics, under which $h \models \psi$ follows from $h' \models \psi$.

Regarding the homomorphism [14], although the unique simple path from E_1 to E_4 in $\mathcal{G}_\varphi$ matches $\mathit{lseg}(E_1; E_4)$, the arc (E_3, E_2) in $\mathcal{G}_\varphi$ is uncovered. This is the issue of using graph homomorphism to entailment under the classical semantics, as also pointed out in [14]. On

the other hand, let us assume another pair of graphs in Figure 5. It is not hard to see that the entailment $\varphi \models \psi$ holds. Interestingly, in this case, *all* the arcs in the cycle $E_2 \leftrightarrows E_3$ in $\mathcal{G}_\varphi$ are uncovered. ◀

As suggested in Example 13, to deal with the classical semantics, it is necessary to put some proper constraints on the uncovered arcs of nontrivial SCCs in $\mathcal{G}_\varphi$. Let us introduce some additional notations for $\mathcal{G}_\varphi$. An arc e' in $\mathcal{G}_\varphi$ is *covered* if

- either e' is an arc labeled by $(f, -)$ from $[E]_\varphi$ to $[F]_\varphi$ and there is an arc e labeled by $(f, -)$ from $[E]_\psi$ to $[F]_\psi$ in $\mathcal{G}_\psi$, where $E, F \in \mathsf{Vars}(\psi)$,
- or e' belongs to $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ for some predicate-labeled arc e in $\mathcal{G}_\psi$.

A simple cycle C of $\mathcal{G}_\varphi$ is *covered* if *all* arcs in C are covered; C is *uncovered* if *none of the arcs* in C are covered. A nontrivial SCC $\mathcal{S}$ is said to be *final w.r.t. covered arcs* if there are *no* covered arcs that are reachable from the nodes in $\mathcal{S}$, moreover, one of the following holds: 1) there is a covered arc whose destination node belongs to $\mathcal{S}$, 2) there is a predicate-labeled arc e from $[E]_\psi$ to $[F]_\psi$ in $\mathcal{G}_\psi$ such that $[E]_\varphi = [F]_\varphi$ is in $\mathcal{S}$. A collection of simple cycles $\mathcal{S} = \{C_1, \ldots, C_n\}$ is said to be *unentangled* if C_i and C_j $(i \neq j)$ do *not* share any arc, and share at most one node.

In the sequel, we first analyse the structure of nontrivial SCCs in $\mathcal{G}_\varphi$.

▶ **Proposition 14.** *Suppose φ, ψ are two $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae such that $\varphi \models_c \psi$. Then the nontrivial SCCs of $\mathcal{G}_\varphi$ satisfy the following structural constraints.*

- *If a nontrivial SCC $\mathcal{S}$ of $\mathcal{G}_\varphi$ contains only* predicate-labeled *arcs, then $\mathcal{S}$ comprises a collection of* unentangled *simple cycles $\{C_1, \ldots, C_n\}$.*
- *If a nontrivial SCC $\mathcal{S}$ of $\mathcal{G}_\varphi$ contains both an uncovered arc and a field-labeled arc, then $\mathcal{S}$ is final w.r.t. covered arcs and is exactly a simple cycle.*

A graph homomorphism from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$ is required to satisfy that *each simple cycle C of $\mathcal{G}_\varphi$ is either covered or uncovered.* In addition, the uncovered simple cycles of $\mathcal{G}_\varphi$ should satisfy some additional constraint given below.

▶ **Definition 15.** Let $\mathcal{S} = \{C_1, \cdots, C_n\}$ be a nontrivial SCC of $\mathcal{G}_\varphi$ with unentangled simple cycles, where all C_i's are either covered or uncovered, with at least one uncovered C_i.

- An *uncovered connected component (UCC)* $\mathcal{C}$ of $\mathcal{S}$ is a connected component of the graph obtained from $\mathcal{S}$ by removing all covered simple cycles.
- Suppose $\mathcal{S}$ is not final w.r.t. covered arcs. Then $\mathcal{S}$ is *indirectly covered* if for each UCC $\mathcal{C}$ of $\mathcal{S}$ and each simple cycle C in $\mathcal{C}$, and there is an arc $e = ([E]_\psi, [F]_\psi)$ in $\mathcal{G}_\psi$ with $\mathsf{atom}_\psi(e) = P(E; F; \boldsymbol{B})$, satisfying one of the following constraints:
  1. $[E]_\varphi = [F]_\varphi \in \mathcal{C}$, and for each arc e' in C with $\mathsf{atom}_\varphi(e') = P(E'; F'; \boldsymbol{B}')$, it holds that $\boldsymbol{B} \sim_\varphi \boldsymbol{B}'$ (where $\sim_\varphi$ is extended to vectors of variables in an obvious way),
  2. $[E]_\varphi \neq [F]_\varphi$, there is a node belonging to both $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ and $\mathcal{C}$, and for each arc e' in C with $\mathsf{atom}_\varphi(e') = P(E'; F'; \boldsymbol{B}')$, it holds that $\boldsymbol{B} \sim_\varphi \boldsymbol{B}'$.

  Note that if the predicate P contains no static parameters, then the condition is simplified as follows: for each UCC $\mathcal{C}$ of $\mathcal{S}$, there is an arc $e = ([E]_\psi, [F]_\psi)$ in $\mathcal{G}_\psi$ such that either $[E]_\varphi = [F]_\varphi \in \mathcal{C}$ or there is a node belonging to both $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ and $\mathcal{C}$.
- Suppose $\mathcal{S}$ is final w.r.t. covered arcs. Then $\mathcal{S}$ is *indirectly covered* if one of the following holds: 1) there is a covered arc e' in $\mathcal{G}_\varphi$ with $\mathsf{atom}_\varphi(e') = P(E; F; \boldsymbol{B})$ such that $[F]_\varphi$ is in $\mathcal{S}$ and $\mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}([F]_\varphi)]$ matches $P(F; F; \boldsymbol{B})$, 2) there is an arc e in $\mathcal{G}_\psi$ with $\mathsf{atom}_\psi(e) = P(E; F; \boldsymbol{B})$ such that $[E]_\varphi = [F]_\varphi$ is in $\mathcal{S}$ and $\mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}([F]_\varphi)]$ matches $P(F; F; \boldsymbol{B})$.

▶ **Definition 16** (Criteria of coverage of arcs)**.** A graph homomorphism from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$ is *fully covered*, if

(I) for each nontrivial SCC $\mathcal{S}$ that is *not* final w.r.t. covered arcs and contains an uncovered arc (which implies that $\mathcal{S}$ contains predicate-labeled arcs only by Proposition 14), each simple cycle C in $\mathcal{S}$ is either covered or uncovered, and each UCC of $\mathcal{S}$ is indirectly covered,

(II) for each nontrivial SCC $\mathcal{S}$ that is final w.r.t. covered arcs, $\mathcal{S}$ is indirectly covered,

(III) each uncovered arc of $\mathcal{G}_\varphi$ either belongs to some nontrivial SCC or is reachable from a node in some nontrivial SCC that is final w.r.t. covered arcs.

We now collate all the ingredients together and specify the formal definition of the graph homomorphism from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$.

▶ **Definition 17** (Graph homomorphism from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$)**.** A *homomorphism* from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$ is a function $\theta : \mathcal{N}_\psi \to \mathcal{N}_\varphi$ satisfying the following constraints.

- **Variable subsumption**: For each $E \in \mathsf{Vars}(\psi)$, $[E]_\psi \subseteq \theta([E]_\psi)$. In particular, we have $E \in \theta([E]_\psi)$. This implies that $\theta([E]_\psi)$ has to be $[E]_\varphi$, and thus, for convenience, we sometime use $[E]_\varphi$ to denote $\theta([E]_\psi)$.
- **Field-labeled arc**: For each field-labeled arc $e = ([E]_\psi, [F]_\psi)$ in $\mathcal{G}_\psi$ with $\mathcal{L}_\psi(e) = (f, E \mapsto \rho)$, there is a field-labeled arc $e' = ([E]_\varphi, [F]_\varphi)$ in $\mathcal{G}_\varphi$ with $\mathcal{L}_\varphi(e') = (f, E' \mapsto \rho')$, and for each (f', F_1) in ρ and (f', F_1') in ρ', $\theta([F_1]_\psi) = [F_1']_\varphi$.
- **Predicate-labeled arc**: For each predicate-labeled arc $e = ([E]_\psi,[F]_\psi)$ in $\mathcal{G}_\psi$ with $\mathsf{atom}_\psi(e) = P(E; F; \boldsymbol{B})$ and $[E]_\varphi \neq [F]_\varphi$, $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ matches $P(E; F; \boldsymbol{B})$.
- **Inequality**: For each $([E]_\psi, [F]_\psi) \in R_{\neq,\psi}$, $([E]_\varphi, [F]_\varphi) \in R_{\neq,\varphi}$.
- **Coverage**: θ is fully covered (cf. Definition 16).
- **Separation constraint**:
  - For each pair of distinct arcs e_1, e_2 in $\mathcal{G}_\psi$, the two subgraphs $\mathsf{Subgraph}_{e_1}(\mathcal{G}_\varphi)$ and $\mathsf{Subgraph}_{e_2}(\mathcal{G}_\varphi)$ are arc-disjoint.
  - For every two distinct nontrivial SCCs $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{G}_\varphi$ that are final w.r.t. covered arcs, the sets of arcs that are reachable from (nodes in) $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively are *disjoint*.

▶ **Proposition 18.** *$\varphi \models_c \psi$ iff there is a graph homomorphism from $\mathcal{G}_\psi$ to $\mathcal{G}_\varphi$.*

▶ **Example 19.** Consider the graphs $\mathcal{G}_\varphi, \mathcal{G}_\psi$ in Figure 3. The function $\theta : \mathcal{N}_\psi \to \mathcal{N}_\varphi$ can be defined obviously, for instance $\theta(E_2) = \{E_2, F_4\}$. For the predicate-labeled arc $e = (E_2, E_4)$ in $\mathcal{G}_\psi$, it is a routine to check that $\mathsf{Subgraph}_e(\mathcal{G}_\varphi)$ matches e. Similarly for the other predicate-labeled arcs in $\mathcal{G}_\psi$. In addition, the homomorphism θ is *fully covered*. Let $\mathcal{S}_1, \mathcal{S}_2$ be the uncovered nontrivial SCCs in $\mathcal{G}_\varphi$, which contain the node $\{F_5, F_7, F_8\}$ and E_5 respectively. $\mathcal{S}_1$ is not final w.r.t. covered arcs and contains only one uncovered connected component (i.e., itself). Since th contains no static parameters and the arc from F_1 to $\{F_5, F_7, F_8\}$ is covered, according to Definition 15, we know that $\mathcal{S}_1$ is indirectly covered. Moreover, $\mathcal{S}_2$ is final w.r.t. covered arcs. Because the arc from E_3 to E_5 is covered and $\mathcal{G}_\varphi[\mathcal{R}_{\mathcal{G}_\varphi}(E_5)]$, which comprises all the arcs reachable from E_5 in $\mathcal{G}_\varphi$, matches $th(E_5; E_5)$, we deduce that $\mathcal{S}_2$ is indirectly covered as well. Finally, it is easy to see that the separation constraint is satisfied. Therefore, θ is a graph homomorphism, and we conclude that $\varphi \models_c \psi$.

Finally, in order to show that the existence of a graph homomorphism from $\mathcal{G}_\varphi$ to $\mathcal{G}_\psi$ can be decided in polynomial time, it remains to show that checking that a subgraph of $\mathcal{G}_\varphi$ matches a predicate atom $P(E; F; \boldsymbol{B})$ can be decided in polynomial time.

▶ **Proposition 20.** *Given a subgraph $\mathcal{G}$ of $\mathcal{G}_\varphi$ and a predicate atom $P(E; F; \boldsymbol{B})$, one can decide whether $\mathcal{G}$ matches $P(E; F; \boldsymbol{B})$ in polynomial time.*

The proof of Proposition 20 relies on the fact that the problem of whether $\mathcal{G}$ matches $P(E; F; \boldsymbol{B})$ can be reduced to checking some structural properties of $\mathcal{G}$, which can be done in polynomial time.

Combining Proposition 18 and Proposition 20, we conclude:

▶ **Theorem 21.** *Let $P \in \mathcal{P}_1$. The entailment problem for $\mathsf{SLID}_{\mathsf{SNC}}[P]$ is in polynomial time.*

A detailed comparison with [16].

Our polynomial-time decision procedure for $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae with $P \in \mathcal{P}_1$ is related to that in [16], since the parameters of an inductive predicate in [16] are also divided as source, destination, and static parameters. Nevertheless, our decision procedure differs from that in [16] in the following three aspects. (1) $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae with $P \in \mathcal{P}_1$ and the fragment of SL in [16] are incomparable: The latter allows nesting inductive predicates so that nested lists and skip lists can be defined, while our fragment disallows this. On the other hand, $\mathsf{SLID}_{\mathsf{SNC}}[P]$ formulae with $P \in \mathcal{P}_1$ allows defining tree structures, which are not supported in [16]. (2) The inequality $E \neq F$ may appear in the inductive rules of predicates in [16], but is excluded here. As a result, the fragment in [16] is *precise* in the sense that when checking $\varphi \models \psi$, for each arc e in $\mathcal{G}_\psi$, there is a *unique* subgraph of $\mathcal{G}_\varphi$ that matches $\mathsf{atom}_\psi(e)$. On the other hand, $\mathsf{SLID}_{\mathsf{SNC}}[P]$ is *not* precise because of the uncovered simple cycles of $\mathcal{G}_\varphi$. Some proper constraints on these uncovered simple cycles are necessary in the graph homomorphism to achieve a *complete* decision procedure for the entailment problem. (3) In [16], checking whether a subgraph of $\mathcal{G}_\varphi$ matches a predicate atom $P(E; F; \boldsymbol{B})$ is reduced to the membership problem of a tree automaton constructed from the inductive definitions of P, which is of exponential size in the worst case, due to the nested inductive predicates. However, in our decision procedure, we simply check whether the subgraph satisfies some structural properties in polynomial time.

5 Conclusion

We have provided polynomial-time satisfiability and entailment checking algorithms for $\mathsf{SLID}_{\mathsf{SNC}}$ with one source (destination) parameter. We have also shown that the satisfiability problem is generally NP-complete if more than one source (destination) parameter is allowed, particularly for an inductive predicate of doubly linked list segments.

For the future work, we strongly believe that our polynomial-time decision procedures can be further generalised to handle inductive predicates with multiple source (destination) parameters in a constrained form. Moreover, adding data constraints certainly is one of the promising extensions of the current work.

References

1 Timos Antonopoulos, Nikos Gorogiannis, Christoph Haase, Max I. Kanovich, and Joël Ouaknine. Foundations for decision problems in separation logic with general inductive predicates. In *FoSSaCS 2014*, pages 411–425, 2014.

2 Josh Berdine, Cristiano Calcagno, and Peter W. O'Hearn. A decidable fragment of separation logic. In *FSTTCS 2004*, pages 97–109, 2004.

3 Josh Berdine, Cristiano Calcagno, and Peter W. O'Hearn. Symbolic execution with separation logic. In *APLAS 2005*, pages 52–68, 2005.

4 Rémi Brochenin, Stéphane Demri, and Étienne Lozes. On the almighty wand. *Inf. Comput.*, 211:106–137, 2012.

5 James Brotherston, Dino Distefano, and Rasmus Lerchedahl Petersen. Automated cyclic entailment proofs in separation logic. In *CADE 2011*, pages 131–146, 2011.

6 James Brotherston, Carsten Fuhs, Juan A. Navarro Perez, and Nikos Gorogiannis. A decision procedure for satisfiability in separation logic with inductive predicates. In *LICS 2014*, pages 25:1–25:10, 2014.

7 James Brotherston, Nikos Gorogiannis, and Max I. Kanovich. Biabduction (and related problems) in array separation logic. *CoRR*, abs/1607.01993, 2016. URL: `http://arxiv.org/abs/1607.01993`.

8 James Brotherston, Nikos Gorogiannis, Max I. Kanovich, and Reuben Rowe. Model checking for symbolic-heap separation logic with inductive predicates. In *POPL 2016*, pages 84–96, 2016.

9 Cristiano Calcagno and Dino Distefano. Infer: An automatic program verifier for memory safety of C programs. In *NFM 2011*, pages 459–465, 2011.

10 Cristiano Calcagno, Dino Distefano, Jérémy Dubreil, Dominik Gabi, Pieter Hooimeijer, Martino Luca, Peter W. O'Hearn, Irene Papakonstantinou, Jim Purbrick, and Dulma Rodriguez. Moving fast with software verification. In *NFM 2015*, pages 3–11, 2015.

11 Cristiano Calcagno, Hongseok Yang, and Peter W. O'Hearn. Computability and complexity results for a spatial assertion language for data structures. In *FSTTCS 2001*, pages 108–119, 2001.

12 Wei-Ngan Chin, Cristina David, Huu Hai Nguyen, and Shengchao Qin. Automated verification of shape, size and bag properties via user-defined predicates in separation logic. *Sci. Comput. Program.*, 77(9):1006–1036, 2012. `doi:10.1016/j.scico.2010.07.004`.

13 Duc-Hiep Chu, Joxan Jaffar, and Minh-Thai Trinh. Automatic induction proofs of data-structures in imperative programs. In *PLDI 2015*, pages 457–466, 2015.

14 Byron Cook, Christoph Haase, Joël Ouaknine, Matthew Parkinson, and James Worrell. Tractable reasoning in a fragment of separation logic. In *CONCUR 2011*, pages 235–249, 2011.

15 Stéphane Demri and Morgan Deters. Expressive completeness of separation logic with two variables and no separating conjunction. *ACM Trans. Comput. Log.*, 17(2):12, 2016.

16 Constantin Enea, Ondřej Lengál, Mihaela Sighireanu, and Tomáš Vojnar. Compositional entailment checking for a fragment of separation logic. In *APLAS 2014*, pages 314–333, 2014.

17 Constantin Enea, Mihaela Sighireanu, and Zhilin Wu. On automated lemma generation for separation logic with inductive definitions. In *ATVA 2015*, pages 80–96, 2015.

18 Xincai Gu, Taolue Chen, and Zhilin Wu. A complete decision procedure for linearly compositional separation logic with data constraints. In *IJCAR 2016*, pages 532–549, 2016.

19 Christoph Haase, Samin Ishtiaq, Joël Ouaknine, and Matthew J. Parkinson. Seloger: A tool for graph-based reasoning in separation logic. In *CAV 2013*, pages 790–795, 2013.

20 Radu Iosif, Adam Rogalewicz, and Jiri Simacek. The tree width of separation logic with recursive definitions. In *CADE 2013*, pages 21–38, 2013.

21 Radu Iosif, Adam Rogalewicz, and Tomás Vojnar. Deciding entailments in inductive separation logic with tree automata. In *ATVA 2014*, pages 201–218, 2014.

22 Quang Loc Le, Jun Sun, and Wei-Ngan Chin. Satisfiability modulo heap-based programs. In *CAV 2016*, pages 382–404, 2016.

23 Peter W. O'Hearn, John C. Reynolds, and Hongseok Yang. Local reasoning about programs that alter data structures. In *CSL 2001*, pages 1–19, 2001.

24 Andrew Reynolds, Radu Iosif, Cristina Serban, and Tim King. A decision procedure for separation logic in SMT. In *ATVA 2016*, pages 244–261, 2016.

25 John C. Reynolds. Separation logic: A logic for shared mutable data structures. In *LICS 2002*, pages 55–74, 2002.

26 Makoto Tatsuta, Quang Loc Le, and Wei-Ngan Chin. Decision procedure for separation logic with inductive definitions and presburger arithmetic. In *APLAS 2016*, pages 423–443, 2016.

27 Zhaowei Xu, Taolue Chen, and Zhilin Wu. Satisfiability of compositional separation logic with tree predicates and data constraints. In Leonardo de Moura, editor, *Automated Deduction - CADE 26 - 26th International Conference on Automated Deduction, Gothenburg, Sweden, August 6-11, 2017, Proceedings*, volume 10395 of *Lecture Notes in Computer Science*, pages 509–527. Springer, 2017. doi:10.1007/978-3-319-63046-5_31.

Data Multi-Pushdown Automata*

Parosh Aziz Abdulla[1], C. Aiswarya[2], and Mohamed Faouzi Atig[3]

1 **Uppsala University, Uppsala, Sweden**
parosh@it.uu.se
2 **Chennai Mathematical Institute, Chennai, India**
aiswaryanitc@gmail.com
3 **Uppsala University, Uppsala, Sweden**
mohamed_faouzi.atig@it.uu.se

Abstract

We extend the classical model of multi-pushdown systems by considering systems that operate on a finite set of variables ranging over natural numbers. The conditions on variables are defined via gap-order constraints that allow to compare variables for equality, or to check that the gap between the values of two variables exceeds a given natural number. Furthermore, each message inside a stack is equipped with a data item representing its value. When a message is pushed to the stack, its value may be defined by a variable. When a message is popped, its value may be copied to a variable. Thus, we obtain a system that is infinite in multiple dimensions, namely we have a number of stacks that may contain an unbounded number of messages each of which is equipped with a natural number. It is well-known that the verification of any non-trivial property of multi-pushdown systems is undecidable, even for two stacks and for a finite data-domain. In this paper, we show the decidability of the reachability problem for the classes of data multi-pushdown system that admit a bounded split-width (or equivalently a bounded tree-width). As an immediate consequence, we obtain decidability for several subclasses of data multi-pushdown systems. These include systems with single stacks, restricted ordering policies on stack operations, bounded scope, bounded phase, and bounded context switches.

1998 ACM Subject Classification D.2.4 Software/Program Verification

Keywords and phrases Pushdown Systems, Model-Checking, Gap-Order, Bounded Split-Width

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.38

1 Introduction

In the last few years, a lot of efforts have been devoted to the verification of *discrete* program models that have *infinite* state spaces such as Petri nets, lossy channel machines and multi-pushdown systems. In particular, multi-pushdown systems have been extensively studied as a natural model for concurrent Boolean recursive programs. Unfortunately, multi-pushdown systems are in general Turing powerful, and hence all basic decision problems are undecidable for them [31]. To overcome the undecidability barrier, several subclasses of multi-pushdown systems have been proposed (e.g., [16, 30, 10, 2, 25, 29, 26, 34, 11, 13, 12, 24, 6, 5, 33, 28]).

Bounded context-switch has been proposed in [30] as an adequate criterion for the verification of multi-pushdown systems. The idea is to restrict the analysis to executions that can be split into a given number of contexts where, in each context, pop and push operations are exclusive to one stack. The context-bounded reachability problem is NP-complete, though the state space that can be explored is still unbounded.

* This work was partially supported by DST Inspire and the Linnaeus centre of excellence UPMARC.

In [25], La Torre et al. generalize the notion of context into phase. A phase is a sequence of operations in which at most one stack can be popped, while there is no restriction on the push operations. The bounded-phase restriction considers only executions of the system that can be split into a given number of phases. In this case, the phase-bounded reachability problem is decidable in double exponential time.

Another generalization of bounded context-switch is bounded scope [26] which restricts the analysis of the multi-pushdown system to those executions in which the number of context-switches between any push operation and its corresponding pop operation is bounded by a given number. This definition extends the notion of contexts in term of coverage while being orthogonal to the notion of phase. In [26], the scope-bounded reachability problem is shown to be PSPACE-complete.

Another way to obtain decidability is to impose a linear order on stacks [16]. Stack operations are constrained in such a way that any pop operation is only allowed on the first non-empty stack. In [10], the reachability problem is shown to be 2ETIME-complete when assuming this ordering policy on stack operations. Furthermore, imposing such a restriction strictly extends the notion of phases while being orthogonal to scope-boundedness.

In [29, 21, 27], a unified technique to reason about multi-pushdown systems under such restrictions is presented. The idea is to see an execution as a graph with extra edges relating push operations and their corresponding pop operations. Then, the authors prove that the graphs generated under these restrictions have bounded split-width (or equivalently bounded tree-width). As an immediate consequence of Courcelle's theorem [20], the decidability of the reachability problem for multi-pushdown systems under these restrictions is obtained.

However, all these models assume a finite-state control, which means that the variables of the modelled programs are assumed to range over finite domains. Several extensions of (multi-)pushdown systems with data have been studied in the literature (see e.g., [14, 8, 1, 17, 22]). Most of these extensions concern the case of multi-pushdown systems with one stack except the work presented in [14] where an extension of multi-pushdown systems with data has been proposed. In order to obtain decidability of the reachability problem, the model requires the strong assumption of *data freshness*, and the restriction of the stack accesses to the bounded phase policy. Furthermore, the variable operations are restricted to checking (dis)equality.

In this paper, we consider an extension of multi-pushdown systems, which we call *Data Multi-Push-Down Automata* (Dmpda), that strengthens the classical model in two ways. First in addition to stacks, a Dmpda uses a finite set of variables ranging over the natural numbers. Moreover, each message inside the stack is equipped with a natural number which represents its value. Thus, we obtain a model that is possibly unbounded in multiple dimensions, namely we have a number of stacks such that each stack may contain an unbounded number of messages each of which is equipped with a natural number. The operations allowed on variables are defined by the *gap-order* constraint system [18, 32]. More precisely, Dmpda allow to compare the values of variables for equality, or to check that the gap between the values of two variables exceeds a given natural number. Also, a variable may be assigned a new arbitrary value, the value of another variable, or a value that is larger than at least a given natural number than the current value of another variable. Furthermore, a push operation may copy the value of a variable to the pushed message, and a pop operation may copy the value attached to the popped message to a variable. In this manner, the model of Dmpda subsumes two basic models, namely multi-pushdown systems (that we get by removing the variables and neglecting the values associated to the pushed messages) and the model of *integral relational automata* [18] (that we get by removing all the stacks).

Our main result is the decidability of the reachability problem for the classes of Dmpda

that admit a bounded split-width. To that aim, we solve a more general problem, namely we characterize the *reachability relation* on variables between each pair of control states. More precisely, we present an algorithm for computing a finite set of gap-order formulas whose denotations describe values of variables that allow to reach one state from another with empty stacks. The main ingredient of the algorithm is a symbolic representation, called *traces*, that encode certain transition sequences in the automaton. A trace represents a set of *partial runs*. A partial run does not record the contents of the stacks, but marks positions inside the run that correspond to matching push/pop operations. Furthermore, a partial run is not contiguous in the sense that it may contain a number of "holes". Our algorithm will characterize the relation between the variables at the points where the holes occur. In particular, a partial run with no holes corresponds to a concrete run that starts and ends with empty stacks. The definition of partial runs allows to extend naturally the notion of *split width* [21] that has been considered for the analysis of multi-pushdown systems (without data). Intuitively, a run has a bounded split width if it can be built from atomic runs by using a shuffle and a contraction operator without producing any intermediate runs with more holes than the given bound. An atomic run is one that consists either of a single transition, or a pair of matching push/pop transitions. We show that our algorithm is guaranteed to terminate for all classes of systems that generate runs with a bounded split width. As an immediate consequence, we obtain the decidability for several subclasses of multi-pushdown systems with data including the ones that restrict the ordering policy on stack operations, or bound the scope, the number of phases, or the number of context switches.

Related work. Several subclasses of multi-pushdown systems have been proposed in the literature including bounded-context [30], bounded-phase [25], bounded scope [26] and ordered multi-pushdown systems [16]. The reachability problem for these classes has been shown to be decidable under the assumption of finiteness of the set of control states. These classes are subclasses of our model Dmpda and our decidability result subsumes the decidability of the reachability problem for these models. In contrast, we do not provide any complexity results.

Split-width and tree-width[1] have been used for showing, in a unified way, the MSO decidability of several classes of multi-pushdown systems [29, 21, 27]. The method has been extended for message passing systems[6] and parameterized message passing systems[23]. However the considered models are restricted to the manipulation of variables over finite data domains while in our model, variables range over natural numbers. In fact the results presented in [29, 21, 27] are orthogonal to our result since we do not consider the model-checking problem against monadic second order logic.

Decidability of the reachability problem for pushdown systems (i.e., multi-pushdown systems with one stack) with data has been extensively studied in the literature (see e.g., [1, 17, 22, 3, 15, 19]). The closest work is pushdown systems with gap-order constraints [1], which is subsumed by our model. Furthermore, the techniques used to show the decidability of the reachability problem for pushdown systems with gap-order constrains are different from the ones used in this paper.

Extensions of multi-pushdown systems with data have been studied in [14]. This work uses the strong assumption of freshness of data, and bounded phase restriction on stack accesses. In contrast, we do not assume the freshness of data and our results can be applied to several subclass of multi-pushdown systems.

[1] Split-width and tree-width are not identical, but one is bounded if and only if the other is. Further the bounds are related linearly [21].

In [8] the split-width technique is lifted to analyze timed multi-pushdown systems. Timed systems give rise to an infinite data domain. However, reachability in this case can be reduced to MSO model checking of untimed systems with finite propositional labelling indicating timing constraints, since realizability of a word with timing constraints can be expressed in MSO [7]. The crux of the decidability proof in all these cases is the use of tree-automata. In contrast, the reachability problem of DMPDA under bounded split-width does not reduce to the Boolean case. Furthermore, our algorithm uses a fix-point computation which terminates, thanks to well quasi-ordering of gap-order formulas.

2 Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers. For sets A and B, we use $f : A \to B$ to denote that f is a function from A to B. We use $f[a \leftarrow a']$ to denote the function f' such that $f'(a) = a'$, and $f'(x) = f(x)$ if $x \neq a$. For $A' \subseteq A$, we use $f \odot A'$ to denote the restriction of f to A'. For sets A_1 and A_2 with $A_1 \cap A_2 = \varnothing$, and functions $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, we use $f_1 \cup f_2 : A_1 \cup A_2 \to B$ to denote the function g such that $g(a) = f_1(a)$ if $a \in A_1$ and $g(a) = f_2(a)$ if $a \in A_2$. For a finite set A, we use $|A|$ to denote the size of A.

For a set A, we use A^* to denote the set of finite words over A. We use ϵ to denote the empty word. For $w_1, w_2 \in A^*$, we use $w_1 \cdot w_2$ to denote the concatenation of w_1 and w_2.

Consider a set A and a total ordering $\preceq$ on A. We use $a_1 \prec a_2$ to denote that $a_1 \preceq a_2$ and $a_1 \neq a_2$. We use $\prec$ to denote the induced immediate successor relation, i.e., $a_1 \prec a_2$ iff $a_1 \prec a_2$ and there is no a_3 such that $a_1 \prec a_3 \prec a_2$.

3 Model

In this section, we introduce *Data Multi-Pushdown Automata* (DMPDA). A DMPDA operates on *multiple* unbounded stacks each of which allows pushing (appending) and popping (removing) messages in a last-in-first-out manner. In addition to the stacks, a DMPDA uses a finite set of variables ranging over the natural numbers.

The allowed operations on variables are defined by the *gap-order* constraint system [18, 32]. More precisely, the model allows non-deterministic value assignment, copying the value of one variable to another, and assignment of a value v to some variable such that v is larger than at least a given natural number than the current value of another variable. The transitions may be conditioned by tests that compare the values of two variables for equality, or that give the smallest allowed gap between two variables. In addition to carrying a name (taken from a finite alphabet), each message inside a stack is equipped by a natural number that represents its "value". A *push* operation may copy the value of a variable to the pushed message, and a *pop* operation may copy the value of the popped message to a variable. Notice that DMPDA extend the classical model of Push-Down Automata in three ways, namely they allow

(i) multiple stacks,
(ii) numerical variables, and
(iii) an infinite (numerical) stack alphabet.

Syntax

In the rest of the paper, we assume a finite set of variables $\mathbb{X}$, a finite set of stacks Σ, and a finite stack alphabet Γ. A DMPDA $\mathcal{A}$ is a tuple $\langle Q, \Delta \rangle$ where Q is a finite set of states, and Δ is a finite set of transitions. A transition $t \in \Delta$ is a triple $\langle q_1, op, q_2 \rangle$ where $q_1, q_2 \in Q$ are states, and op is an operation of one of the following forms:

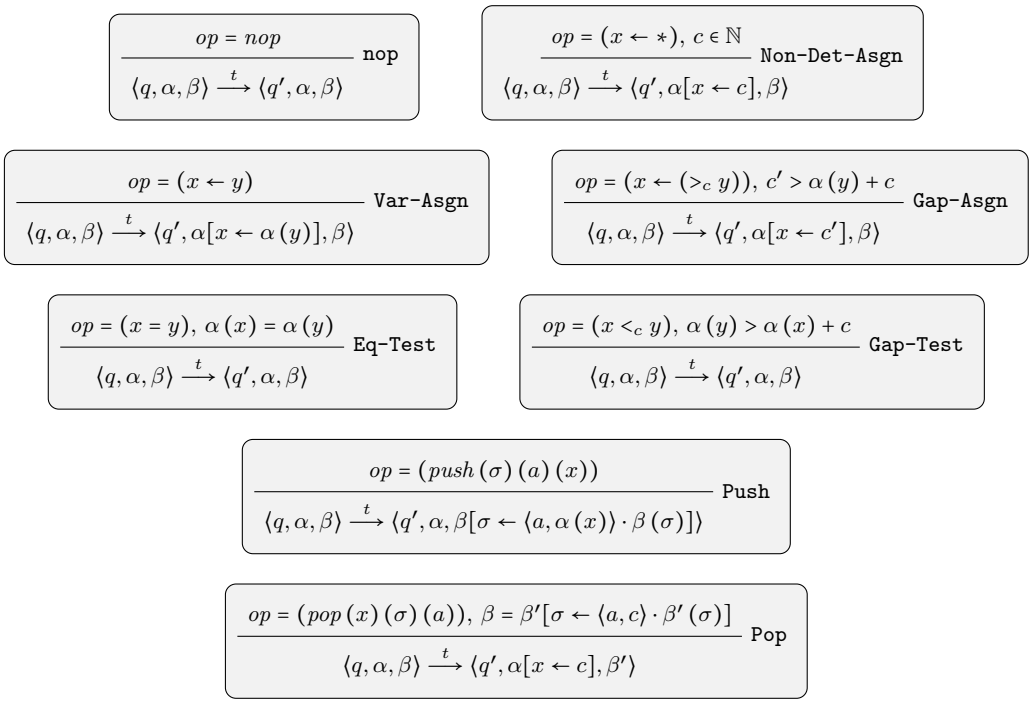


Figure 1 Inference rules defining the relation $\xrightarrow{t}$, where $t = \langle q, op, q'\rangle$.

(i) *nop* is the *empty* operation that does not change the values of the variables or the contents of the stacks.
(ii) $x \leftarrow *$ assigns non-deterministically an arbitrary value in $\mathbb{N}$ to the variable x.
(iii) $x \leftarrow y$ copies the value of variable y to x.
(iv) $x \leftarrow (>_c y)$ assigns non-deterministically to x a value that exceeds the current value of y by c (so the new value of x is $> y + c$).
(v) $x = y$ checks whether the value of x is equal to the value of y.
(vi) $x <_c y$ checks whether the gap between the values of y and x is larger than c.
(vii) $push(\sigma)(a)(x)$ pushes the symbol $a \in \Gamma$ to the stack $\sigma \in \Sigma$ and assigns to it the value of the variable x.
(viii) $pop(x)(\sigma)(a)$ pops the symbol $a \in \Gamma$ (if a is the top-most symbol at the stack $\sigma \in \Sigma$) and assigns its value to the variable x.

We define the *source* $\mathtt{src}(t) := q_1$ and the *target* $\mathtt{tgt}(t) := q_2$.

We define $\Delta^{\mathtt{intern}}$ to be the set of *internal* transitions, i.e., those that do not perform push or pop operations. We define $\Delta^{\mathtt{push}}_{\sigma,a}$ to be the set of transitions whose operations are of the form $push(\sigma)(a)(x)$ for some $x \in \mathbb{X}$. We define $\Delta^{\mathtt{push}}_{\sigma} := \cup_{a\in\Gamma}\Delta^{\mathtt{push}}_{\sigma,a}$. We define $\Delta^{\mathtt{pop}}_{\sigma,a}$ and $\Delta^{\mathtt{pop}}_{\sigma}$ analogously.

Semantics

A DMPDA induces a transition system as follows. A *configuration* $\mathbf{c}$ is a triple $\langle q,\alpha,\beta\rangle$ where $q \in Q$ is a state, $\alpha : \mathbb{X} \to \mathbb{N}$ defines the values of the variables, and $\beta : \Sigma \to (\Gamma \times \mathbb{N})^*$ defines, for each stack $\sigma \in \Sigma$, its content $\beta(\sigma)$. The content of a stack is a word whose elements are of the form $\langle a, c\rangle$ where a is a symbol and c is its value. In particular, we define β_ϵ such that $\beta_\epsilon(\sigma) = \epsilon$ for all $\sigma \in \Sigma$. We say that $\mathbf{c}$ is *plain* if $\beta = \beta_\epsilon$.

We define the transition relation $\longrightarrow := \cup_{t \in \Delta} \xrightarrow{t}$ on the set of configurations, where $\xrightarrow{t}$ describes the effect of the transition t. The semantics of the transition relation is presented through the inference rules of Fig. 1, explained below one by one.

- *nop*. The values of the variables and the stack contents are not changed.
- $x \leftarrow *$. The value of the variable x is changed non-deterministically to some natural number. The values of the other variables and the stack contents are not changed.
- $x \leftarrow y$. The value of the variable y is copied to the variable x. The values of the other variables and the stack contents are not changed.
- $x \leftarrow (>_c y)$. The variable x is assigned non-deterministically a value that exceeds the value of y by c. The values of the other variables and the stack contents are not changed.
- $x = y$ (resp. $x <_c y$). The transition is only enabled if the value of y is equal to the value of x (resp. larger than the value of x by more than c). The values of the variables and the stack contents are not changed.

- $push\,(\sigma)\,(a)\,(x)$. The symbol a is pushed onto the stack σ with a value equal to that of x.
- $pop\,(x)\,(\sigma)\,(a)$. The symbol a is popped from the stack σ (if it is the top-most symbol of σ), and its value is copied to the variable x.

We use $\xrightarrow{*}$ to denote the reflexive transitive closure of $\longrightarrow$. A *run* π is an alternating sequence $\mathbf{c}_0 t_1 \mathbf{c}_1 \cdots \mathbf{c}_{n-1} t_n \mathbf{c}_n$ of configurations and transitions such that $\mathbf{c}_{i-1} \xrightarrow{t_i} \mathbf{c}_i$ for all $i : 1 \leq i \leq n$. We say that π is *plain* if $\mathbf{c}_0$ and $\mathbf{c}_n$ are plain. For configurations $\mathbf{c}$ and $\mathbf{c}'$, we write $\mathbf{c} \xrightarrow{\pi} \mathbf{c}'$ to denote that there is a run π of the above form such that $\mathbf{c}_0 = \mathbf{c}$ and $\mathbf{c}_n = \mathbf{c}'$. Notice that $\mathbf{c} \xrightarrow{*} \mathbf{c}'$ iff $\mathbf{c} \xrightarrow{\pi} \mathbf{c}'$ for some run π.

Reachability

In the *Reachability Problem*, we are given two plain configurations $\mathbf{c}_1$ and $\mathbf{c}_2$, and are asked whether $\mathbf{c}_1 \xrightarrow{*} \mathbf{c}_2$. In order to solve the reachability problem we will study *reachability relations*. For states $q, q' \in Q$, we define $\mathbb{R}\,(q, q') := \left\{ \langle \alpha, \alpha' \rangle \mid \langle q, \alpha, \beta_\epsilon \rangle \xrightarrow{*} \langle q', \alpha', \beta_\epsilon \rangle \right\}$. Intuitively, we summarize the values of the variables that allow us to move from q to q', starting and ending with empty stacks. More precisely, $\mathbb{R}\,(q, q')$ contains all pairs $\langle \alpha, \alpha' \rangle$ such that we can start from a configuration where the state is q, the values of the variables are given by α, and the stacks are empty to another configurations where the state is q', the values of the variables are given by α', and the stacks are empty again.

4 Gap-Order Formulas

Fix a set $\mathbb{X}$ of variables ranging over $\mathbb{N}$. An *atomic gap-order formula* over $\mathbb{X}$ is either of the form $x = y$ or of the form $x <_c y$ where $x, y \in \mathbb{X}$ and $c \in \mathbb{N}$. A *gap-order formula* ϕ over $\mathbb{X}$ is a conjunction of atomic constraints over $\mathbb{X}$. Sometimes, we represent ϕ as a set (containing all its conjuncts). For a function $\mathtt{Val} : \mathbb{X} \to \mathbb{N}$, we write (as expected) $\mathtt{Val} \vDash \phi$ to denote that $\mathtt{Val}$ satisfies ϕ. We will also consider existentially quantified formulas of the form $\exists \mathbb{Y}.\ \phi$ where ϕ is a gap-order formula over $\mathbb{X}$, and $\mathbb{Y} \subseteq \mathbb{X}$. For $\mathtt{Val} : \mathbb{X} - \mathbb{Y} \to \mathbb{N}$, we write $\mathtt{Val} \vDash \exists \mathbb{Y}.\ \phi$ to denote that there is a mapping $\mathtt{Val}' : \mathbb{Y} \to \mathbb{N}$ such that $\mathtt{Val} \cup \mathtt{Val}' \vDash \phi$. For a (quantified) gap-order formula ϕ, we define its denotation $[\![\phi]\!] := \{\mathtt{Val} \mid \mathtt{Val} \vDash \phi\}$.

A gap-order formula ϕ over $\mathbb{X}$ is in *normal form* if it satisfies the following conditions:

1. If $(x <_{c_1} y) \in \phi$ and $(y <_{c_2} z) \in \phi$ then $(x <_{c_3} z) \in \phi$ for some c_3 with $c_1 + c_2 < c_3$.
2. If $(x <_c y) \in \phi$ and $(y = z) \in \phi$ (or $(z = y) \in \phi$) then $(x <_c z) \in \phi$.

3. If $(x <_c y) \in \phi$ and $(x = z) \in \phi$ (or $(z = x) \in \phi$) then $(z <_c y) \in \phi$.
4. If $(x = y) \in \phi$ and $(y = z) \in \phi$ then $(x = z) \in \phi$ or $(z = x) \in \phi$.
5. For each $x, y \in \mathbb{X}$, there is at most one conjunct in ϕ containing both x and y.

▸ **Lemma 1** ([4, 1]). *For each gap-order formula ϕ, we can effectively compute a gap-order formula ϕ' such that ϕ' is in normal form and $[\![\phi']\!] = [\![\phi]\!]$.*

We obtain ϕ' from ϕ by repeatedly adding conjuncts to maintain properties 1-4 and removing conjuncts which violate property 5 (for instance, if we have both $(x <_{c_1} y) \in \phi$ and $(x <_{c_2} y) \in \phi$, with $c_1 \leq c_2$, then we can remove the former conjunct.) Normalization can be used to check consistency: the formula is consistent iff no inequalities of the form $x <_c x$ are generated.

Furthermore, we can use normalization to perform quantifier elimination as follows. For sets of variables $\mathbb{Y} \subseteq \mathbb{X}$ and a gap-order formula ϕ, let $\phi \ominus \mathbb{Y}$ to be the gap-order formula we get from ϕ by eliminating all conjuncts in which a variable $x \in \mathbb{Y}$ occurs.

▸ **Lemma 2** ([4, 1]). *Suppose that ϕ is consistent and in normal form. Assume that $\mathtt{Val} \models \phi \ominus \mathbb{Y}$. Then there is a $\mathtt{Val}' : \mathbb{Y} \to \mathbb{N}$ such that $\mathtt{Val} \cup \mathtt{Val}' \models \phi$*

From Lemma 2 we get the following corollary.

▸ **Corollary 3.** *Suppose that ϕ is consistent and in normal form. Then $[\![\exists \mathbb{Y}.\ \phi]\!] = [\![\phi \ominus \mathbb{Y}]\!]$.*

We write $\phi_1 \sqsubseteq \phi_2$ to denote that $[\![\phi_2]\!] \subseteq [\![\phi_1]\!]$. We can check $\phi_1 \sqsubseteq \phi_2$ as follows. By Lemma 1 we can assume that ϕ_1 and ϕ_2 are in normal form. Then the following conditions should be satisfied: (1) If $(x <_{c_1} y) \in \phi_1$ then $(x <_{c_2} y) \in \phi_2$ for some $c_2 \geq c_1$, and (2) if $(x = y) \in \phi_1$ then $(x = y) \in \phi_2$ or $(y = x) \in \phi_2$.

5 Traces

We introduce a data structure, called *traces*, that encode *partial runs*. Roughly speaking, a partial run is a run with a number of "holes" inserted. Thus, a partial run consists of a sequence of *segments* each of which is a sequence of consecutive transitions. The source of one transition in a segment is identical to the target of the preceding transition. Furthermore, each push operation in the partial run is matched by a pop operation, and vice versa, such that the push/pop operations respect the stack semantics (i.e., we do not allow push/pop transitions that are "pending" in a partial run). First, we define the set of atomic traces, and describe two operations that allow to build new traces from existing ones. Then, we define an entailment relation on traces. Finally, we use traces to define the notion of *split width*. In the rest of the section, we fix a DMPDA $\mathcal{A} = \langle Q, \Delta \rangle$.

5.1 Definition

A trace τ is a tuple $\langle \mathbb{I}, \preceq, \mathtt{src}, \mathtt{tgt}, E, \phi \rangle$ defined as follows.

- $\mathbb{I}$ is a finite (index) set. Each index will be used to represent the summary of a segment. The summary is given by the starting and the end states of each segment, i.e., the source of the first transition and the target of the last transition in the segment, and by a relation on the values of the variables before and after performing the different segments. For each variable $x \in \mathbb{X}$ and index $\mathtt{i} \in \mathbb{I}$, we will introduce two new variables $x^{\mathtt{i}}_{\mathtt{s}}$ and $x^{\mathtt{i}}_{\mathtt{t}}$ representing the *source* and *target* values of x, i.e., the value of x at the start and at the end of the segment represented by the index $\mathtt{i}$. We define the set $\mathbb{X}^{\mathtt{i}} := \{x^{\mathtt{i}}_{\mathtt{s}} | (x \in \mathbb{X}) \wedge (\mathtt{i} \in \mathbb{I})\} \cup \{x^{\mathtt{i}}_{\mathtt{t}} | (x \in \mathbb{X}) \wedge (\mathtt{i} \in \mathbb{I})\}$, and define $\mathbb{X}^{\mathbb{I}} := \cup_{\mathtt{i} \in \mathbb{I}} \mathbb{X}^{\mathtt{i}}$.

- $\preceq$ is a total ordering on $\mathbb{I}$ that gives the order in which the segments represented by the indices are performed. We let $\prec$ be the induced immediate successor relation (cf. Section 2).
- $\mathtt{src} : \mathbb{I} \to Q$ maps each index to a state representing the source of the corresponding segment, i.e., the state from which the segment starts (the source of the first transition in the segment). Analogously, $\mathtt{tgt} : \mathbb{I} \to Q$ defines the target of the segment, i.e., the state at the end of the segment (the target of the last transition in the segment).
- $E : \mathbb{I} \times \mathbb{I} \to 2^{\Sigma}$ is a function representing an "edge relation" between the indices. For indices $\mathtt{i}_1, \mathtt{i}_2$ the value of $E(\mathtt{i}_1, \mathtt{i}_2)$ gives the set of stacks such that there is a push operation in the segment represented by $\mathtt{i}_1$ whose corresponding pop operation is performed in the segment represented by $\mathtt{i}_2$. We impose two conditions on E. First, we require that $E(\mathtt{i}_1, \mathtt{i}_2) \neq \varnothing$ only if $\mathtt{i}_1 \prec \mathtt{i}_2$ since a pop operation can only occur after the corresponding push operation (and furthermore, we do not record push operations whose pop operations lie in the same segment). Second, we require that, for all stacks $\sigma \in \Sigma$, there are no indices $\mathtt{i}_1 \prec \mathtt{i}_2 \prec \mathtt{i}_3 \prec \mathtt{i}_4$ such that $\sigma \in E(\mathtt{i}_1, \mathtt{i}_3) \cap E(\mathtt{i}_2, \mathtt{i}_4)$. This condition ensures that we are consistent with the stack semantics since there is no overlap between two pairs of push/pop operations on the same stack.
- ϕ is a gap-order formula over the set $\mathbb{X}^{\mathbb{I}}$, that defines the relation on values of the variables at the start and the end of the different segments.

We will equate traces that are equivalent modulo the renaming of the indices. For a trace $\tau = \langle \mathbb{I}, \preceq, \mathtt{src}, \mathtt{tgt}, E, \phi \rangle$, we define its *degree* $\#\tau = |\mathbb{I}|$, i.e., it is the size of the index set.

5.2 Atomic Traces

Atomic traces are built using the set of transitions. We will define two types of atomic traces, namely those induced by single internal transitions, and those that are induced by pairs of matching push/pop transitions.

Internal Transitions

Let $t = \langle q_1, op, q_2 \rangle \in \Delta^{\mathtt{intern}}$ be an internal transition. We will build a trace with a single index corresponding to a single segment which contains only one transition, namely t. Formally, we define $\mathtt{MkTrace}(t) := \langle \mathbb{I}, \preceq, \mathtt{src}, \mathtt{tgt}, E, \phi \rangle$ where

- $\mathbb{I} = \{\mathtt{i}\}$, i.e., the set of indices is a singleton.
- $\preceq$ is trivial since the index set contains only one node.
- $\mathtt{src}(\mathtt{i}) = q_1$ and $\mathtt{tgt}(\mathtt{i}) = q_2$, i.e., we label the single index with the source and target states of t.
- $E = \varnothing$ reflecting the fact that the operation performed by t does not affect the stacks.
- ϕ consists of all conjuncts of the following forms:
  - if $op = nop$ or $op = (x = y)$ or $op = (x <_c y)$ then $x^{\mathtt{i}}_{\mathtt{s}} = x^{\mathtt{i}}_{\mathtt{t}}$ for all $x \in \mathbb{X}$, i.e., the values of the variables are not changed during t.
  - if $op = (x = y)$ then $x^{\mathtt{i}}_{\mathtt{s}} = y^{\mathtt{i}}_{\mathtt{s}}$, and if $op = (x <_c y)$ then $y^{\mathtt{i}}_{\mathtt{s}} > x^{\mathtt{i}}_{\mathtt{s}} + c$. The values of the variables should satisfy the condition of the transition.
  - If $op = (x \leftarrow *)$ or $op = (x \leftarrow y)$ or $op = (x \leftarrow (>_c y))$ then $z^{\mathtt{i}}_{\mathtt{s}} = z^{\mathtt{i}}_{\mathtt{t}}$ for all $z \in \mathbb{X} - \{x\}$, i.e., the values of the variables different from x are not changed during t.
  - If $op = (x \leftarrow y)$ then $x^{\mathtt{i}}_{\mathtt{t}} = y^{\mathtt{i}}_{\mathtt{s}}$.
  - If $op = (x \leftarrow (>_c y))$ then $x^{\mathtt{i}}_{\mathtt{t}} > y^{\mathtt{i}}_{\mathtt{s}} + c$.

Stack Transitions

Consider two transitions $t_1 = \langle q_1, op_1, q_2\rangle \in \Delta^{\texttt{push}}_{\sigma,a}$, $t_2 = \langle q_3, op_2, q_4\rangle \in \Delta^{\texttt{pop}}_{\sigma,a}$ where $op_1 = push(\sigma)(a)(x)$ and $op_2 = pop(y)(\sigma)(a)$. Notice that the two transitions push/pop the same symbol a to/from the same stack σ. We will build a trace with two indices corresponding to two segments containing t_1 and t_2, respectively. Formally, we define $\texttt{MkTrace}(t_1, t_2) := \langle \mathbb{I}, \preceq, \texttt{src}, \texttt{tgt}, E, \phi\rangle$, where:

- $\mathbb{I} = \{\mathtt{i}_1, \mathtt{i}_2\}$ i.e., the index set contains two elements. We use the indices $\mathtt{i}_1$ and $\mathtt{i}_2$ to represent two segments each containing a single transition, namely t_1 and t_2 respectively.
- $\mathtt{i}_1 \preceq \mathtt{i}_2$. We require that $\mathtt{i}_1$ is ordered before $\mathtt{i}_2$. This reflects the fact that a pop transition occurs after the matching push transition.
- $E(\mathtt{i}_1, \mathtt{i}_2) = \{\sigma\}$, i.e., we add an edge between $\mathtt{i}_1$ to $\mathtt{i}_2$ labeled with σ corresponding to the matching push/pop operations on σ performed by t_1 resp. t_2.
- $\texttt{src}(\mathtt{i}_1) = q_1$, $\texttt{tgt}(\mathtt{i}_1) = q_2$, $\texttt{src}(\mathtt{i}_2) = q_3$, and $\texttt{tgt}(\mathtt{i}_2) = q_4$. In other words, we label the new indices with the source and target states of t_1 resp. t_2.
- ϕ consists of all conjuncts of the following forms:
  - **(i)** $z^{\mathtt{i}_1}_{\mathtt{s}} = z^{\mathtt{i}_1}_{\mathtt{t}}$ for all $z \in \mathbb{X}$, i.e., the values of the variables are not changed during t_1.
  - **(ii)** $z^{\mathtt{i}_2}_{\mathtt{s}} = z^{\mathtt{i}_2}_{\mathtt{t}}$ for all $z \in \mathbb{X} - \{y\}$, i.e., the values of the variables, except y, are not changed during t_2.
  - **(iii)** $y^{\mathtt{i}_2}_{\mathtt{t}} = x^{\mathtt{i}_1}_{\mathtt{s}}$.

  This condition corresponds to the fact that the value of a when pushed to the stack during t_1 is equal to the value of variable x. This value is identical to the value stored in y after the pop operation of transition t_2.

5.3 Operations

We define two operations for building new traces, namely shuffling and contraction.

Shuffling

Consider two traces $\tau_1 = \langle \mathbb{I}_1, \preceq_1, \texttt{src}_1, \texttt{tgt}_1, E_1, \phi_1\rangle$, and $\tau_2 = \langle \mathbb{I}_2, \preceq_2, \texttt{src}_2, \texttt{tgt}_2, E_2, \phi_2\rangle$, where $\mathbb{I}_1 \cap \mathbb{I}_2 = \varnothing$. We will build a new trace by shuffling the index sets of τ_1 and τ_2. We define $\tau_1 \otimes \tau_2$ to be the set of traces of the form $\langle \mathbb{I}, \preceq, \texttt{src}, \texttt{tgt}, E, \phi\rangle$ satisfying the following conditions:

- $\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2$, i.e., the new trace contains exactly all the segments that are in τ_1 and τ_2.
- $\preceq$ is a total ordering on $\mathbb{I}$ such that $\preceq_1 \subseteq \preceq$ and $\preceq_2 \subseteq \preceq$. We do not change the original orderings of the indices, but we do not constrain the places of the two sets of indices relative to each other.
- $\texttt{src}(\mathtt{i}) = \texttt{src}_1(\mathtt{i})$, and $\texttt{tgt}(\mathtt{i}) = \texttt{tgt}_1(\mathtt{i})$ for all $\mathtt{i} \in \mathbb{I}_1$. Furthermore, $\texttt{src}(\mathtt{i}) = \texttt{src}_2(\mathtt{i})$, and $\texttt{tgt}(\mathtt{i}) = \texttt{tgt}_2(\mathtt{i})$ for all $\mathtt{i} \in \mathbb{I}_2$. (We keep the state labelings of the old indices.)
- $E(\mathtt{i}_1, \mathtt{i}_2) = E_1(\mathtt{i}_1, \mathtt{i}_2)$ if $\mathtt{i}_1, \mathtt{i}_2 \in \mathbb{I}_1$, and $E(\mathtt{i}_1, \mathtt{i}_2) = E_2(\mathtt{i}_1, \mathtt{i}_2)$ if $\mathtt{i}_1, \mathtt{i}_2 \in \mathbb{I}_2$, i.e., all the edges in τ_1 and τ_2 are maintained in $\tau_1 \otimes \tau_2$. Also, $E(\mathtt{i}_1, \mathtt{i}_2) = \varnothing$, if $\mathtt{i}_1 \in \mathbb{I}_1$ and $\mathtt{i}_2 \in \mathbb{I}_2$ or if $\mathtt{i}_1 \in \mathbb{I}_2$ and $\mathtt{i}_2 \in \mathbb{I}_1$, i.e., we do not add any edges between the two sets of indices. Furthermore, we require that there are no $\mathtt{i}_1, \mathtt{i}_2 \in \mathbb{I}_1$ and $\mathtt{i}_3, \mathtt{i}_4 \in \mathbb{I}_2$ such that
  - **(i)** $\sigma \in E(\mathtt{i}_1, \mathtt{i}_2)$,
  - **(ii)** $\sigma \in E(\mathtt{i}_3, \mathtt{i}_4)$, and
  - **(iii)** either $\mathtt{i}_1 \prec \mathtt{i}_3 \prec \mathtt{i}_2 \prec \mathtt{i}_4$ or $\mathtt{i}_3 \prec \mathtt{i}_1 \prec \mathtt{i}_4 \prec \mathtt{i}_2$.

  This is to ensure that $\tau_1 \otimes \tau_2$ respects the stack semantics.
- $\phi = \phi_1 \wedge \phi_2$. Notice that the values of the variables indexed by elements from $\mathbb{I}_1$ are not related to the values of the variables indexed by elements from $\mathbb{I}_2$.

Contraction

We define a *contraction operation* $\downarrow$ on indices that represents merging the corresponding segments. Consider $\mathtt{i}_1, \mathtt{i}_2 \in \mathbb{I}$ such that $\mathtt{i}_1 \lessdot \mathtt{i}_2$, i.e., $\mathtt{i}_2$ is the immediate successor of $\mathtt{i}_1$. Let $\mathtt{src}(\mathtt{i}_1) = q_1$, $\mathtt{tgt}(\mathtt{i}_1) = q_2$, $\mathtt{src}(\mathtt{i}_2) = q_2$, and $\mathtt{tgt}(\mathtt{i}_1) = q_3$, i.e., the target state of $\mathtt{i}_1$ is identical to the source state of $\mathtt{i}_2$. We will merge $\mathtt{i}_1$ and $\mathtt{i}_2$ to a new (single) index $\mathtt{j}$. We define $\tau \downarrow \langle \mathtt{i}_1, \mathtt{i}_2 \rangle := \langle \mathbb{I}', \preceq', \mathtt{src}', \mathtt{tgt}', E', \phi' \rangle$ as follows:

- $\mathbb{I}' = \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\} \cup \{\mathtt{j}\}$ where $\mathtt{j} \notin \mathbb{I}$, i.e., we replace the two merged indices by a new one.
- $\mathtt{k} \preceq' \mathtt{j}$ iff $\mathtt{k} \preceq \mathtt{i}_1$, and $\mathtt{j} \preceq' \mathtt{k}$ iff $\mathtt{i}_2 \preceq \mathtt{k}$ for all $\mathtt{k} \in \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\}$, i.e., in the new ordering, the new index $\mathtt{j}$ will take the places of the two (consecutive) indices $\mathtt{i}_1$ and $\mathtt{i}_2$. Furthermore, $\mathtt{k}_1 \preceq' \mathtt{k}_2$ iff $\mathtt{k}_1 \preceq \mathtt{k}_2$ for all $\mathtt{k}_1, \mathtt{k}_2 \in \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\}$, i.e., the relative orderings of the original indices are not changed.
- $\mathtt{src}'(\mathtt{j}) = q_1$, $\mathtt{tgt}'(\mathtt{j}) = q_3$, $\mathtt{src}'(\mathtt{k}) = \mathtt{src}(\mathtt{k})$ and $\mathtt{tgt}'(\mathtt{k}) = \mathtt{tgt}(\mathtt{k})$ for all $\mathtt{k} \in \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\}$. In other words, we keep the state labelings of the old indices, while we take the source and target states of $\mathtt{j}$ to be the source state of $\mathtt{i}_1$ and the target state of $\mathtt{i}_2$ respectively.
- $E'(\mathtt{j}, \mathtt{k}) = E(\mathtt{i}_1, \mathtt{k}) \cup E(\mathtt{i}_2, \mathtt{k})$ and $E'(\mathtt{k}, \mathtt{j}) = E(\mathtt{k}, \mathtt{i}_1) \cup E(\mathtt{k}, \mathtt{i}_2)$ for all $\mathtt{k} \in \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\}$, i.e., we merge the edges originating from/to the two merged indices. Also, $E'(\mathtt{k}_1, \mathtt{k}_2) = E(\mathtt{k}_1, \mathtt{k}_2)$ for all $\mathtt{k}_1, \mathtt{k}_2 \in \mathbb{I} - \{\mathtt{i}_1, \mathtt{i}_2\}$, i.e., the edges to/from the other indices are maintained. Notice that any edges between $\mathtt{i}_1$ and $\mathtt{i}_2$ are deleted.
- $\phi' = \exists \left(\mathbb{X}^{\mathtt{i}_1} \cup \mathbb{X}^{\mathtt{i}_2}\right). \phi''$, where ϕ'' is defined as $\phi \wedge \left(\bigwedge_{x \in \mathbb{X}} \left(x_{\mathtt{t}}^{\mathtt{i}_1} = x_{\mathtt{s}}^{\mathtt{i}_2}\right)\right) \wedge \left(\bigwedge_{x \in \mathbb{X}} \left(x_{\mathtt{s}}^{\mathtt{i}_1} = x_{\mathtt{s}}^{\mathtt{j}}\right)\right) \wedge \left(\bigwedge_{x \in \mathbb{X}} \left(x_{\mathtt{t}}^{\mathtt{i}_2} = x_{\mathtt{t}}^{\mathtt{j}}\right)\right)$. We require that ϕ' is consistent. Since we merge the two segments corresponding to $\mathtt{i}_1$ and $\mathtt{i}_2$, we require the values of the variables at the end of the first segment to be consistent with the values of the variables at the start of the second segment. If these conditions are not satisfied, then the resulting formula ϕ' will not be consistent. Furthermore, the values of variables at the start of the new segment are defined to be the values of the variables at the start of the segment corresponding to $\mathtt{i}_1$. Analogously, the values of variables at the end of the new segment are defined to be the values of the variables at the end of the segment corresponding to $\mathtt{i}_2$. By Lemma 1, we know that ϕ'' can be transformed to an equivalent formula in normal form, and hence by Corollary 3, we can compute ϕ' as a gap-order formula.

We define $\tau \downarrow$ to be the set of traces τ' such that: (1) there are indices $\mathtt{i}_1, \mathtt{i}_2 \in \mathbb{I}$ with $\mathtt{i}_1 \lessdot \mathtt{i}_2$, (2) $\mathtt{tgt}(\mathtt{i}_1) = \mathtt{src}(\mathtt{i}_2)$, and (3) $\tau' = \tau \downarrow \langle \mathtt{i}_1, \mathtt{i}_2 \rangle$.

5.4 Entailment

We define an entailment relation $\sqsubseteq$ on traces. Intuitively, a trace τ_1 is *weaker* than a trace τ_2, denoted $\tau_1 \sqsubseteq \tau_2$, if their graphs are isomorphic (equivalent up to the renaming of indices), but the gap-order formula of τ_1 is weaker than the one of τ_2. Later in the paper, when we compute reachability relations, we let τ_1 "subsume" τ_2 in the sense that if we encounter both τ_1 and τ_2 then we only include τ_1 (and discard τ_2), without suffering any loss of any precision in the analysis. Formally, consider traces $\tau_1 = \langle \mathbb{I}_1, \preceq_1, \mathtt{src}_1, \mathtt{tgt}_1, E_1, \phi_1 \rangle$ and $\tau_2 = \langle \mathbb{I}_2, \preceq_2, \mathtt{src}_2, \mathtt{tgt}_2, E_2, \phi_2 \rangle$. Let $h : \mathbb{I}_1 \to \mathbb{I}_2$ be a bijection. We write $\tau_1 \sqsubseteq_h \tau_2$ to denote that the following conditions are satisfied: (1) $\mathtt{i}_1 \preceq_1 \mathtt{i}_2$ iff $h(\mathtt{i}_1) \preceq_2 h(\mathtt{i}_2)$, (2) $\mathtt{src}_1(\mathtt{i}) = \mathtt{src}_2(h(\mathtt{i}))$ and $\mathtt{tgt}_1(\mathtt{i}) = \mathtt{tgt}_2(h(\mathtt{i}))$, (3) $E_1(\mathtt{i}_1, \mathtt{i}_2) = E_2(h(\mathtt{i}_1), h(\mathtt{i}_2))$, and (4) $\phi_1^h \sqsubseteq \phi_2$. We obtain ϕ_1^h from ϕ_1 by replacing each $x_{\mathtt{s}}^{\mathtt{i}}$ by $x_{\mathtt{s}}^{h(\mathtt{i})}$ and replacing each $x_{\mathtt{t}}^{\mathtt{i}}$ by $x_{\mathtt{t}}^{h(\mathtt{i})}$. We use $\tau_1 \sqsubseteq \tau_2$ to denote that $\tau_1 \sqsubseteq_h \tau_2$ for some h.

5.5 Split Width

We say that a trace has *split width* θ if it can be derived by starting from the atomic traces and repeatedly applying the shuffling and contraction operations without letting any of the intermediately generated traces have a degree larger than θ.

A *proof tree* $\mathcal{T}$ of split width θ is a binary tree whose nodes are labeled with traces such that the following conditions are satisfied:

- The leaves are labeled with atomic traces.
- If an internal node, labeled with τ, has two children then the children are labeled with traces τ_1 and τ_2 such that $\tau \in \tau_1 \otimes \tau_2$.
- If an internal node, labeled with τ', has a single child then that child is labeled with a trace τ such that $\tau \in \tau' \downarrow$.
- For each label τ of a node in $\mathcal{T}$, we have $\#\tau \leq \theta$.

A trace τ is of *split width* θ if θ is the smallest number such that τ is the root of a proof tree of split width θ. We use $\mathtt{SW}(\tau)$ to denote the split width of τ.

We extend the notion of split width to plain runs as follows. Consider a plain run $\pi = \mathbf{c}_0 t_1 \mathbf{c}_1 \cdots \mathbf{c}_{n-1} t_n \mathbf{c}_n$ where $\mathbf{c}_i = \langle q_i, \alpha_i, \beta_i \rangle$. We define $\widehat{\pi}$ to be the set of traces of degree one, of the form $\langle \{\mathtt{i}\}, \preceq, \mathtt{src}, \mathtt{tgt}, E, \phi \rangle$ where $\preceq = \{\langle \mathtt{i}, \mathtt{i} \rangle\}$, $\mathtt{src}(\mathtt{i}) = q_0$, $\mathtt{tgt}(\mathtt{i}) = q_n$, $E = \varnothing$, and $\mathtt{Val} \vDash \phi$ with $\mathtt{Val}(x^{\mathtt{i}}_{\mathtt{s}}) = \alpha_0(x)$ and $\mathtt{Val}(x^{\mathtt{i}}_{\mathtt{t}}) = \alpha_n(x)$ for all $x \in \mathbb{X}$. We define $\mathtt{SW}(\pi)$ to be the smallest $k \in \mathbb{N}$ such that there is a trace $\tau \in \widehat{\pi}$ with $\mathtt{SW}(\tau) = k$. For states $q, q' \in Q$ and $\theta \in \mathbb{N}$, we define $\mathbb{R}^{\leq\theta}(q, q') := \left\{ \langle \alpha, \alpha' \rangle \mid \exists \pi. (\mathtt{SW}(\pi) \leq \theta) \wedge \left(\langle q, \alpha, \beta_\epsilon \rangle \xrightarrow{\pi} \langle q', \alpha', \beta_\epsilon \rangle \right) \right\}$.

For a DMPDA $\mathcal{A}$, we define the split width $\mathtt{SW}(\mathcal{A})$ to be the largest k such that there is a plain run π in $\mathcal{A}$ with $\mathtt{SW}(\pi) = k$. For a class $\mathcal{C}$ of DMPDA, we define $\mathtt{SW}(\mathcal{C})$ to be the largest k such that there is an $\mathcal{A} \in \mathcal{C}$ with $\mathtt{SW}(\mathcal{A}) = k$. We say that $\mathcal{C}$ has *bounded split width* if $\mathtt{SW}(\mathcal{C}) = k$ for some $k \in \mathbb{N}$.

6 Decidability

In this section, we present the main result of the paper:

▸ **Theorem 4.** *The reachability problem is decidable for any class of* DMPDA *with bounded split width.*

To prove Theorem 4 we first describe an algorithm computing reachability relations, and then prove its correctness in Lemma 5 (termination), Lemma 6 (soundness), and Lemma 7 (completeness).

6.1 Algorithm

The algorithm inputs a DMPDA $\mathcal{A} = \langle Q, \Delta \rangle$ together with an upper limit θ on the degrees of the traces to be considered during the analysis. The algorithm maintains a set `W` of traces that have been detected but not analyzed, and a set `V` of traces that have been both detected and analyzed. Initially, the sets `W` and `V` are empty (Line 1–2). We add all the atomic traces induced by internal transitions (Line 3), and by matching push/pop transitions (Line 5) to the set `W`. After the initialization phase, the algorithm performs a number of iterations using the repeat-loop of Line 8. In each iteration, we first select and remove an element τ from `W` (Lines 9–10). We check that τ satisfies two conditions (Line 11), namely: that (i) the degree of τ is within the allowed limit, and that (ii) τ is not subsumed by any trace already in the set `V`. If the two conditions are satisfied, we use τ to generate new traces to analyze. These new traces are added to the set `W`. First, we take the shuffle of τ with each member of the set

Algorithm 1: Computing the Reachability Relation.

Input: $\mathcal{A} = \langle Q, \Delta \rangle$: DMPDA,
θ: maximal index size
Output: characterization of the reachability relation

1 $\mathtt{V} \leftarrow \varnothing$;
2 $\mathtt{W} \leftarrow \varnothing$;
3 **for** *each* $t \in \Delta^{\mathtt{intern}}$ **do**
4 $\quad \mathtt{W} \leftarrow \mathtt{W} \cup \{\mathtt{MkTrace}(t)\}$
5 **for** *each* $t_1 \in \Delta^{\mathtt{push}}_{\sigma_1,a_1}$ *and* $t_2 \in \Delta^{\mathtt{pop}}_{\sigma_2,a_2}$ **do**
6 $\quad$ **if** $\sigma_1 = \sigma_2$ *and* $a_1 = a_2$ **then**
7 $\quad\quad \mathtt{W} \leftarrow \mathtt{W} \cup \{\mathtt{MkTrace}(t_1, t_2)\}$
8 **repeat**
9 $\quad$ select some $\tau \in \mathtt{W}$;
10 $\quad \mathtt{W} \leftarrow \mathtt{W} - \{\tau\}$;
11 $\quad$ **if** $(\#\tau \leq \theta) \wedge (\nexists\, \tau' \in \mathtt{V}.\ \tau' \sqsubseteq \tau)$ **then**
12 $\quad\quad$ **for** *each* $\tau' \in \mathtt{V}$ **do**
13 $\quad\quad\quad \mathtt{W} \leftarrow \mathtt{W} \cup (\tau \otimes \tau')$
14 $\quad\quad \mathtt{W} \leftarrow \mathtt{W} \cup \tau\downarrow$;
15 $\quad\quad \mathtt{V} \leftarrow \{\tau' \in \mathtt{V} \mid \tau \not\sqsubseteq \tau'\} \cup \{\tau\}$;
16 **until** $\mathtt{W} = \varnothing$;
17 **for** *each* $q, q' \in Q$ **do**
18 $\quad \mathcal{R}(q, q') \leftarrow \varnothing$
19 **for** *each* $\tau \in \mathtt{V}$ **do**
20 $\quad$ Let $\tau = \langle \mathbb{I}, \preceq, \mathtt{src}, \mathtt{tgt}, E, \phi \rangle$;
21 $\quad$ **if** $\#\tau = 1$ **then**
22 $\quad\quad$ let $\mathbb{I} = \{\mathtt{i}\}$, $\mathtt{src}(\mathtt{i}) = q$, $\mathtt{tgt}(\mathtt{i}) = q'$;
23 $\quad\quad \mathcal{R}(q, q') \leftarrow \mathcal{R}(q, q') \cup \{\phi\}$
24 **return** $\mathcal{R}$

$\mathtt{V}$ (Line 12). Then, we add all possible contractions of τ (Line 14). Finally, at Line 15, we add τ to $\mathtt{V}$, and at the same time remove all elements of $\mathtt{V}$ that are subsumed by τ. Notice that this means that all the traces in the set $\mathtt{V}$ will be pairwise incomparable wrt. $\sqsubseteq$. The iteration of the main loop is repeated until the set $\mathtt{W}$ becomes empty.

After the termination of the loop, we build the reachability relations successively by going through all traces that have been added to $\mathtt{V}$ (Lines 17–23). For each pair of states q and q', we maintain a set $\mathcal{R}(q, q')$ of gap-order formulas. Each time a trace τ of degree one is encountered in $\mathtt{V}$, we add the gap-order formula of τ to the set $\mathcal{R}(q, q')$ corresponding the source state q and target state q' of the (only) index of τ. At the end of the algorithm, the reachability relation between any pair of states is characterized by the union of the denotations of all the gap-order formulas in the corresponding set.

6.2 Correctness

We show correctness of the algorithm in several steps. We start by showing that the algorithm always terminates. For a set A, a pre-order $\preceq$ on A is said to be a *Well Quasi-Ordering*

(WQO) if, for each infinite sequence $a_0a_1a_2\cdots$ of elements from A, there are $i < j$ such that $a_i \preceq a_j$. For a set T of traces, we define its degree $\#T := \max_{\tau\in T}(\#\tau)$. Notice that $\#T$ need not exist in general. Gap-order formulas over $\mathbb{X}$ is WQO under the $\sqsubseteq$ relation [1, 4]. Hence, for any $k \in \mathbb{N}$ and any set of traces T with $\#T \leq k$, the entailment relation $\sqsubseteq$ is a WQO on T. This gives the following lemma.

▸ **Lemma 5.** *The algorithm is guaranteed to terminate.*

Next, we show soundness and completeness of the algorithm. To that end, we will consider the set $\mathcal{R}(q, q')$ of gap-order formulas for each pair of states q and q', returned by the algorithm. We define a denotation function for these formulas, and relate them to the reachability relation $\mathbb{R}(q, q')$. We show that each member in the denotation corresponds to a run (Lemma 6) implying the soundness of the algorithm. Conversely, we show that each run with split width θ belongs to the denotation (Lemma 7) implying the completeness of the algorithm.

A formula ϕ is said be *transitional* over $\mathbb{X}$ if ϕ is a gap-order formula over $\mathbb{X}^{\mathbb{I}}$ where $|\mathbb{I}| = 1$. Notice that, if $\mathbb{I} = \{\mathtt{i}\}$, then each variable in ϕ is either of the form $x^{\mathtt{i}}_{\mathtt{s}}$ or of the form $x^{\mathtt{i}}_{\mathtt{t}}$ where $x \in \mathbb{X}$. We define $\|\phi\|$ to be the set of pairs $\langle\alpha, \alpha'\rangle$ such that $\alpha : \mathbb{X} \mapsto \mathbb{N}$, $\alpha' : \mathbb{X} \to \mathbb{N}$, and there is a $\mathtt{Val} : \mathbb{X}^{\mathbb{I}} \to \mathbb{N}$ with $\mathtt{Val} \vDash \phi$, $\alpha(x) = \mathtt{Val}(x^{\mathtt{i}}_{\mathtt{s}})$ and $\alpha'(x) = \mathtt{Val}(x^{\mathtt{i}}_{\mathtt{t}})$ for all $x \in \mathbb{X}$. For a set of Φ of transitional gap-order formulas, we define $\|\Phi\| := \cup_{\phi\in\Phi}\|\phi\|$. Notice that all members of $\mathcal{R}(q, q')$ are transitional gap-order formulas. The following two lemmas then show the soundness and completeness of the algorithm.

▸ **Lemma 6.** $\forall q, q' \in Q.\ \|\mathcal{R}(q, q')\| \subseteq \mathbb{R}(q, q')$

▸ **Lemma 7.** $\forall q, q' \in Q.\ \|\mathcal{R}(q, q')\| \supseteq \mathbb{R}^{\leq\theta}(q, q')$

7 Applications

7.1 Pushdown automata

A data push-down automata is a DMPDA with a single stack. All plain runs of a data-push-down automaton have split-width bounded by 3 [21]. Thus, it follows from Theorem 4 that

▸ **Corollary 8.** *The reachability problem is decidable for data push-down automata.*

7.2 Multi-push-down systems

As already mentioned in the introduction, the control state reachability is undecidable even in a finite data setting for multi-pushdown systems. Several under-approximation classes (cf. Introduction) have been proposed in the literature for regaining decidability in the finite data case. We recall their definitions below.

- **Bounded context-switch** [30] A context is a sequence of operations in which at most one stack is touched. A run is k-context bounded if it is the concatenation of at most k contexts.
- **Bounded phase** [25] A phase is a sequence of operations in which at most one stack can be popped, though there are no restrictions on pushes. An run is k-phase bounded if it is the concatenation of at most k phases. This subsumes the k- context bounded runs.
- **Ordered stacks** [10, 9]. In an ordered multi-pushdown system, the stacks are ordered linearly by priority. Further a stack may be popped only if all the stacks with higher

priority are empty. An ordered-stack run respects the priority ordering in its stack operations.

- **Bounded scope** [26] A run of a multi-pushdown system is k-scope bounded if the number of context switches between a push and the corresponding pop is always bounded by k. This definition is slightly general than the original round-based definition of [26]. This subsumes the k- context bounded runs,but is orthogonal to k-phase bounded runs.

Runs falling into any of the above class have bounded split-width.

▸ Fact 1 ([21]).
- Split-width of k-context bounded runs is at most $k+2$.
- Split-width of k-phase bounded runs is at most 2^k.
- Split-width of k-scope bounded runs is at most $k+2$.
- Split-width of ordered-stack runs is at most $2^{|\Sigma|}$.

Let U be an under-approximation class above. A U-run of a DMPDA $\mathcal{A}$ is a run that is in U. For the under-approximation U, let $\mathcal{C}_U$ be the class of DMPDA such that all runs of any DMPDA $\mathcal{A} \in \mathcal{C}_U$ are U-runs. By Fact 1, $\mathcal{C}_U$ has bounded split-width. Hence by Theorem 4, we get

▸ **Corollary 9.** *The reachability problem is decidable for {bounded context-switching, bounded phase, ordered stacks, bounded scope} - DMPDA.*

Let U be an under-approximation class above. The U-reachability problem asks, given a DMPDA $\mathcal{A}$ and two plain configurations $\mathbf{c}_1$ and $\mathbf{c}_2$, whether it is possible to reach c_2 from c_1 by a U-run.

If the input DMPDA $\mathcal{A}$ belongs to $\mathcal{C}_U$, then U-reachability problem is the same as the reachability problem, which is decidable by Corollary 9. However, an arbitrary $\mathcal{A}$ may have runs outside of U in general, but still having bounded split-width. Thus running our algorithm naively on any input $\mathcal{A}$ could say "yes" to a pair of configurations $\mathbf{c}_1$ and $\mathbf{c}_2$ even when they are not U-reachable.

In order to decide the U-reachability problem, given the class U and DMPDA $\mathcal{A}$, we will construct a new DMPDA $\mathcal{A}' \in \mathcal{C}_U$. The DMPDA $\mathcal{A}'$ in effect will enforce the semantic restriction of U-runs syntactically into the automaton $\mathcal{A}$. Thus runs of $\mathcal{A}'$ are precisely U-runs of $\mathcal{A}$. We will thus reduce the U-reachability problem in $\mathcal{A}$ to the reachability problem in $\mathcal{A}' \in \mathcal{C}_U$ which is decidable by Corollary 9.

More formally, we reduce the U-reachability problem to reachability problem in $\mathcal{C}_U$. The reduction depends on the under-approximation U. On input DMPDA $\mathcal{A}$, plain configurations $\mathbf{c}_1$ and $\mathbf{c}_2$, we will construct a DMPDA $\mathcal{A}'$ belonging to $\mathcal{C}_U$. Then we will compute a finite set of pairs of plain configurations $\mathbf{c}'_1$ and $\mathbf{c}'_2$ from $\mathbf{c}_1$ and $\mathbf{c}_2$. We check whether $\mathbf{c}'_2$ is reachable from $\mathbf{c}'_1$ in $\mathcal{A}'$ for at least one pair of computed plain configurations $\mathbf{c}'_1$ and $\mathbf{c}'_2$. If this is the case, we conclude that $\mathbf{c}_2$ is U-reachable from $\mathbf{c}_1$ in $\mathcal{A}$. Otherwise, $\mathbf{c}_2$ is not U-reachable from $\mathbf{c}_1$ in $\mathcal{A}$.

Due to lack of space, we will now describe the reduction in detail for only the case of bounded-phase.

Reduction for k-phase bounded.

Given a DMPDA $\mathcal{A} = \langle Q, \Delta \rangle$ and a bound k on the number phases, we construct a new one $\mathcal{A}' = \langle Q', \Delta' \rangle$ where $Q' = Q \times \{1 \dots k\} \times \Sigma \cup Q \times \{0\}$. The state remembers, in addition to the state of $\mathcal{A}$, how many phases have been used so far, and the stack that is being popped from in the current phase. Thus a state of the form (q, i, σ) means that currently $\mathcal{A}$ would

have been in state q, and in the ith phase, which is allowed to pop only from stack σ. The state $(q,0)$ is used to start off a computation, where no stack has been popped yet. The transitions Δ' lifts Δ to smoothly extend to Q' while maintaining the intended semantics. For instance, if $\langle q_1, op, q_2\rangle \in \Delta$, then $\langle (q_1,i,\sigma), op, (q_2,i,\sigma)\rangle \in \Delta'$ and $\langle (q_1,0), op, (q_2,0)\rangle \in \Delta'$ if op is an operation of the forms (i) to (vii) (cf. Section 3). If $\langle q_1, op, q_2\rangle \in \Delta$ and if op is of the form (viii), i.e, $pop\,(x)\,(\sigma)\,(a)$, then we have i) $\langle (q_1,i,\sigma), op, (q_2,i,\sigma)\rangle \in \Delta'$, ii) $\langle (q_1,i,\sigma'), op, (q_2,i+1,\sigma)\rangle \in \Delta'$ if $\sigma' \neq \sigma$ and $i < k$, and, iii) $\langle (q_1,0), op, (q_2,1,\sigma)\rangle \in \Delta'$.

The Dmpda $\mathcal{A}'$ exhibits all and only executions of $\mathcal{A}$ in which the number of phases is bounded by k. Given the pair of plain configurations $\mathbf{c}_1 = \langle q_1, \alpha_1, \beta_\epsilon\rangle$ and $\mathbf{c}_2 = \langle q_2, \alpha_2, \beta_\epsilon\rangle$ of $\mathcal{A}$, we obtain the set of pairs of the form $(\langle q_1', \alpha_1, \beta_\epsilon\rangle, \langle q_2', \alpha_2, \beta_\epsilon\rangle)$ where $q_1' = (q_1, 0)$ and q_2' is either $(q_2, 0)$ or of the form $q_2' = (q_2, i, \sigma)$ for some i and σ.

If $\mathbf{c}_2' = \langle q_2', \alpha_2, \beta_\epsilon\rangle$ is reachable from $\mathbf{c}_1' = \langle q_1', \alpha_1, \beta_\epsilon\rangle$ in $\mathcal{A}'$ for one such computed pair, then indeed, $\mathbf{c}_2$ is k-phase reachable from $\mathbf{c}_1$ in $\mathcal{A}$. Conversely, if $\mathbf{c}_2$ is k-phase reachable from $\mathbf{c}_1$ in $\mathcal{A}$, then there exists $\mathbf{c}_2'$ and $\mathbf{c}_1'$ of the form described above such that $\mathbf{c}_2'$ is reachable from $\mathbf{c}_1'$ in $\mathcal{A}'$. This concludes our reduction.

▸ **Corollary 10.** *The k-phase reachability problem is decidable for any* Dmpda.

In a similar manner, we can have reductions for each of the under-approximations described above. Thus, we get

▸ **Corollary 11.** *{Bounded context-switching, bounded phase, ordered stacks, bounded scope}-reachability problem is decidable for any* Dmpda.

8 Conclusions

We have studied the reachability problem for multi-pushdown systems with gap-order constraints. We provide an algorithm for solving the reachability problem. The algorithm is sound and complete for the classes of automata that have a bounded split-width.

For future work, we plan to consider lifting the framework to a more general setting of auxiliary storages which include queues and multi-sets. Furthermore, it would be interesting to consider the case of distributed processes.

References

1. P. A. Abdulla, M. F. Atig, G. Delzanno, and A. Podelski. Push-down automata with gap-order constraints. In *FSEN*, volume 8161 of *LNCS*, pages 199–216. Springer, 2013.
2. P. A. Abdulla, M. F. Atig, O. Rezine, and J.Stenman. Budget-bounded model-checking pushdown systems. *Formal Methods in System Design*, 45(2):273–301, 2014.
3. P. A. Abdulla, M. F. Atig, and J. Stenman. The minimal cost reachability problem in priced timed pushdown systems. In *LATA*, volume 7183 of *LNCS*, 2012.
4. P. A. Abdulla and G. Delzanno. On the coverability problem for constrained multiset rewriting. In *Proc. AVIS'06, The fifth Int. Workshop on on Automated Verification of Infinite-State Systems*, 2006.
5. C. Aiswarya, P Gastin, and K. Narayan Kumar. Controllers for the verification of communicating multi-pushdown systems. In *CONCUR*, volume 8704 of *LNCS*, pages 297–311, 2014.
6. C. Aiswarya, P. Gastin, and K. Narayan Kumar. Verifying communicating multi-pushdown systems via split-width. In *ATVA'14*, volume 8837 of *LNCS*. Springer, 2014. To appear.
7. S. Akshay, P. Gastin, V. Juge, and S. N. Krishna. personal communication.

8 S. Akshay, P. Gastin, and S. N. Krishna. Analyzing timed systems using tree automata. In *CONCUR'16*, volume 59 of *LIPICS*, pages 27:1–27:14. Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.CONCUR.2016.27.

9 M. F. Atig. Global Model Checking of Ordered Multi-Pushdown Systems. In *FSTTCS 2010*, volume 8, pages 216–227, 2010.

10 M. F. Atig, B. Bollig, and P. Habermehl. Emptiness of multi-pushdown automata is 2ETIME-complete. In *Proceedings of DLT'08*, volume 5257 of *LNCS*, pages 121–133. Springer, 2008.

11 M. F. Atig, K. Narayan Kumar, and P. Saivasan. Adjacent ordered multi-pushdown systems. *Int. J. Found. Comput. Sci.*, 25(8):1083–1096, 2014.

12 M. F. Atig, K. Narayan Kumar, and P. Saivasan. Acceleration in multi-pushdown systems. In *TACAS 2016*, volume 9636 of *LNCS*, pages 698–714. Springer, 2016.

13 Mohamed Faouzi Atig. Model-checking of ordered multi-pushdown automata. *Logical Methods in Computer Science*, 8(3), 2012.

14 B. Bollig, A. Cyriac, P. Gastin, and K. Narayan Kumar. Model checking languages of data words. In *FoSSaCS'12*, volume 7213 of *LNCS*, pages 391–405. Springer, March 2012.

15 A. Bouajjani, R. Echahed, and R. Robbana. On the automatic verification of systems with continuous variables and unbounded discrete data structures. In *Hybrid Systems II*, volume 999 of *LNCS*, pages 64–85. Springer, 1994.

16 L. Breveglieri, A. Cherubini, C. Citrini, and S. Crespi Reghizzi. Multi-push-down languages and grammars. *International Journal of Foundations of Computer Science*, 7(3):253–292, 1996.

17 X. Cai and M. Ogawa. Well-structured pushdown systems. In *CONCUR 2013*, volume 8052 of *LNCS*, pages 121–136. Springer, 2013.

18 K. Čerāns. Deciding properties of integral relational automata. In Abiteboul and Shamir, editors, *ICALP 94*, volume 820 of *LNCS*, pages 35–46. Springer Verlag, 1994.

19 Lorenzo Clemente and Slawomir Lasota. Reachability analysis of first-order definable pushdown systems. In *CSL 2015,*, volume 41 of *LIPIcs*, pages 244–259. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.

20 B. Courcelle. The expression of graph properties and graph transformations in monadic second-order logic. In Grzegorz Rozenberg, editor, *Handbook of Graph Grammars and Computing by Graph Transformations, Volume 1: Foundations*, pages 313–400. World Scientific, 1997.

21 A. Cyriac, P. Gastin, and K. Narayan Kumar. MSO decidability of multi-pushdown systems via split-width. In *CONCUR'12*, volume 7454 of *LNCS*, pages 547–561. Springer, 2012.

22 F. S. de Boer, M. M. Bonsangue, and J. Rot. It is pointless to point in bounded heaps. *Sci. Comput. Program.*, 112:102–118, 2015.

23 M. Fortin and P. Gastin. Verification of parameterized communicating automata via split-width. In *FoSSaCS'16*, volume 9634 of *LNCS*, pages 197–213. Springer, 2016. doi:10.1007/978-3-662-49630-5_12.

24 A. Heußner, J. Leroux, A. Muscholl, and G. Sutre. Reachability analysis of communicating pushdown systems. *Logical Methods in Computer Science*, 8(3), 2012. doi:10.2168/LMCS-8(3:23)2012.

25 S. La Torre, P. Madhusudan, and G. Parlato. A robust class of context-sensitive languages. In *LICS'07*, pages 161–170. IEEE Computer Society Press, 2007.

26 S. La Torre and M. Napoli. Reachability of multistack pushdown systems with scope-bounded matching relations. In *CONCUR 2011*, volume 6901 of *LNCS*, pages 203–218. Springer, 2011.

27 S. La Torre and G. Parlato. Scope-bounded multistack pushdown systems: Fixed-point, sequentialization, and tree-width. In *FSTTCS 2012*, volume 18 of *LIPIcs*, pages 173–184. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2012.

28 A. Lal and T.W. Reps. Reducing concurrent analysis under a context bound to sequential analysis. In *CAV*, volume 5123 of *LNCS*, pages 37–51. Springer, 2008.

29 P. Madhusudan and G. Parlato. The tree width of auxiliary storage. In Thomas Ball and Mooly Sagiv, editors, *POPL*, pages 283–294. ACM, 2011.

30 S. Qadeer and J. Rehof. Context-bounded model checking of concurrent software. In N. Halbwachs and L.D. Zuck, editors, *TACAS 2005*, volume 3440 of *LNCS*, pages 93–107. Springer, 2005.

31 G. Ramalingam. Context-sensitive synchronization-sensitive analysis is undecidable. *ACM Trans. Program. Lang. Syst.*, 22(2):416–430, 2000.

32 P. Revesz. A closed form evaluation for datalog queries with integer (gap)-order constraints. *Theoretical Computer Science*, 116(1):117–149, 1993.

33 A. Seth. Global reachability in bounded phase multi-stack pushdown systems. In *CAV'10*, LNCS, 2010.

34 S. La Torre, P. Madhusudan, and G. Parlato. Reducing context-bounded concurrent reachability to sequential reachability. In *CAV*, volume 5643 of *LNCS*, pages 477–492. Springer, 2009.

Towards an Efficient Tree Automata Based Technique for Timed Systems*

S. Akshay[1], Paul Gastin[2], Shankara Narayanan Krishna[3], and Ilias Sarkar[4]

1 **Dept of CSE, IIT Bombay, India**
akshayss@cse.iitb.ac.in
2 **LSV, ENS Paris-Saclay, CNRS, France**
paul.gastin@lsv.fr
3 **Dept of CSE, IIT Bombay, India**
krishnas@cse.iitb.ac.in
4 **Dept of CSE, IIT Bombay, India**
ilias@cse.iitb.ac.in

Abstract

The focus of this paper is the analysis of real-time systems with recursion, through the development of good theoretical techniques which are implementable. Time is modeled using clock variables, and recursion using stacks. Our technique consists of modeling the behaviours of the timed system as graphs, and interpreting these graphs on tree terms by showing a bound on their tree-width. We then build a tree automaton that accepts exactly those tree terms that describe realizable runs of the timed system. The emptiness of the timed system thus boils down to emptiness of a finite tree automaton that accepts these tree terms. This approach helps us in obtaining an optimal complexity, not just in theory (as done in earlier work e.g. [4]), but also in going towards an efficient implementation of our technique. To do this, we make several improvements in the theory and exploit these to build a first prototype tool that can analyze timed systems with recursion.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Timed automata, tree automata, pushdown systems, tree-width

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.39

1 Introduction

Development of efficient techniques for the verification of real time systems is a practically relevant problem. Timed automata [6] are a prominent and well accepted abstraction of timed systems. The development of this model originally began with highly theoretical results, starting from the PSPACE-decision procedure for the emptiness of timed automata. But later, this theory has led to the development of state of the art and industrial strength tools like UPPAAL [7]. Currently, such tools are being adapted to build prototypes that handle other systems such timed games, stochastic timed systems etc. While this helps in analysis of certain systems, there are complicated real life examples that require paradigms like recursion, multi-threaded concurrency and so on.

* This work was partly supported by UMI-ReLaX, DST-CEFIPRA project AVeRTS and DST-INSPIRE faculty award [IFA12-MA-17].

For timed systems with recursion, a popular theoretical framework is the model of timed pushdown automata (TPDA). In this model, in addition to clock variables as in timed automata, a stack is used to model recursion. Depending on how clocks and stack operations are integrated, several variants [8], [1], [14], [12], [9] have been looked at. For many of these variants, the basic problem of checking emptiness has been shown decidable (and EXPTIME-complete) using different techniques. The proofs in [8], [1], [14] work by adapting the technique of region abstraction to untime the stack and obtain a usual untimed pushdown automaton, while [9] gives a proof by reasoning with sets of timed atoms. Recently, in [4], a new proof technique was introduced which modeled the behaviours of the TPDA as graphs with timing constraints and analyzed these infinite collections of time-constrained graphs using tree automata. This approach follows the template which has been explored in depth for various untimed systems in [13], [11], [3]. The basic idea can be outlined as follows: (1) describe behaviours of the underlying system as graphs, (2) show that this class of graphs has bounded width, (3) either appeal to Courcelle's theorem [10] by showing that the desired properties are MSO-defineable or explicitly construct a tree-automaton to capture the class of graphs that are the desired behaviours. The work in [4] extends this approach to timed systems, by considering their behaviors as time-constrained words. The main difficulty here is to obtain a tree automaton that accepts only those time-constrained words that are *realizable* via a valid time-stamping.

Despite the amount of theoretical work in this area [8, 13, 11, 4, 1, 9], none of these algorithms have been implemented to the best of our knowledge. Applying Courcelle's theorem is known to involve a blowup in the complexity (depending on the quantifier-alternation of the MSO formula). The algorithm for checking emptiness in [4] for the timed setting which directly constructs the tree automaton avoiding the MSO translation also turns out to be unimplementable even for small examples due to the following reasons: First, it has a pre-processing step where each transition in the underlying automaton is broken into several micro transitions, one for each constraint that is checked there, and one corresponding to each clock that gets reset on that transition. This results in a blowup in the size of the automaton. Second, the number of states of the tree automaton that is built to check realizabilty as well as the existence of a run of a system is bounded by $(M \times T)^{\mathcal{O}(K^2)} 2^{\mathcal{O}(K^2 lgK)}$, where M is one more than the maximal constant used in the given system, T is the number of transitions, and $K = 4|X| + 6$ is the so-called split-width, where $|X|$ is the number of clocks used. This implies that even for a system that has 1 clock, 5 transitions and uses a maximum constant 5, we have more than 30^{100} states.

In this paper, we take the first steps towards an efficient implementation. While we broadly follow the graph and tree-automata based approach (and in particular [4]), our main contribution is to give an efficient technique for analyzing TPDA. This requires several fundamental advances: (i) we avoid the preprocessing step, obtaining a direct bound on tree width for timed automata and TPDA. This is established by playing a *split-game* which decomposes the graph representing behaviours of the timed system into tree terms; by coloring some vertices of the graph and removing certain edges whose endpoints are colored. The minimum number of colors used in a winning strategy is 1 plus the tree-width of the graph. (ii) we develop a new algorithm for building the tree automaton for emptiness, whose complexity is in ETIME, i.e., bounded by $(M \times T)^{3|X|+3}$ with an exponent which is a linear function of the input size (improved from EXPTIME, where the exponent is a polynomial function of the input). Thus, if the system has 1 clock, 5 transitions and uses a maximum constant 5, we have only $\sim 30^6$ states. In particular, our tree-automaton is *strategy-driven*, i.e., it manipulates only those tree terms that arise out of a winning strategy of our split-game. As a result of this strategy-guided approach, the number of states of our

tree automaton is highly optimized, and an accepting run exactly corresponds to the moves in a winning strategy of our split-game. (iii) Finally, our algorithm outputs a witness for realizability (and non-emptiness). As a proof-of-concept, we implemented our algorithm and despite the worst-case complexity, we discuss optimizations, results and a modeling example where our implementation performs well.

Due to lack of space, we have not included all the proofs here; the full version of the paper, with proofs of all the results, illustrative examples, details of experimental results and benchmarks used can be found at [5].

2 Graphs for behaviors of timed systems

We fix an alphabet Σ and use Σ_ε to denote $\Sigma \cup \{\varepsilon\}$, where ε is the silent action. We also fix a finite set of intervals $\mathcal{I}$ with bounds in $\mathbb{N} \cup \{\infty\}$. For a set S, we use $\leq \subseteq S \times S$ to denote a partial or total order on S. For any $x, y \in S$, we write $x < y$ if $x \leq y$ and $x \neq y$, and $x \lessdot y$ if $x < y$ and there does not exist $z \in S$ such that $x < z < y$.

▶ **Definition 1.** A *word with timing constraints* (TCW) over $(\Sigma, \mathcal{I})$ is a structure $\mathcal{V}=(V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}})$ where V is a finite set of vertices or positions, $\lambda\colon V \to \Sigma_\varepsilon$ labels each position, the reflexive transitive closure $\leq = \to^*$ is a total order on V and $\to = \lessdot$ is the successor relation, while $\curvearrowright^I \subseteq <$ connects pairs of positions carrying a timing constraint, given by an interval in $I \in \mathcal{I}$. A TCW $\mathcal{V}=(V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}})$ is called *realizable* if there exists a timestamp map $\mathsf{ts}\colon V \to \mathbb{R}_+$ such that $\mathsf{ts}(i) \leq \mathsf{ts}(j)$ for all $i \leq j$ (time is non-decreasing) and $\mathsf{ts}(j) - \mathsf{ts}(i) \in I$ for all $i \curvearrowright^I j$ (timing constraints are satisfied).

An example of a TCW is given in Figure 1 (right), with positions $0, 1, 2, 3$ labelled by $\Sigma = \{a, b, c\}$. Curved edges decorated with intervals connect positions related by $\curvearrowright^I$, while straight edges define the successor relation $\to$. This TCW is realizable by the sequence of timestamps $0, 0.9, 2.89, 3.1$ but not by $0, 0.9, 2.99, 3.1$. We let $\mathsf{Real}(\Sigma, \mathcal{I})$ be the set of TCWs over $(\Sigma, \mathcal{I})$ which are realizable.

TPDA and their semantics as TCWs. Dense-timed pushdown automata (TPDA), introduced in [1], are an extension of timed automata, and operate on a finite set of real-valued clocks and a stack which holds symbols with their ages. The age of a symbol represents the time elapsed since it was pushed onto the stack. Formally, a TPDA $\mathcal{S}$ is a tuple $(S, s_0, \Sigma, \Lambda, \Delta, X, F)$ where S is a finite set of states, $s_0 \in S$ is the initial state, Σ, Λ, are respectively finite sets of input, stack symbols, Δ is a finite set of transitions, X is a finite set of real-valued variables called clocks, $F \subseteq S$ are final states. A transition $t \in \Delta$ is a tuple $(s, \gamma, a, \mathsf{op}, R, s')$ where $s, s' \in S$, $a \in \Sigma$, γ is a finite conjunction of atomic formulae of the kind $x \in I$ for $x \in X$ and $I \in \mathcal{I}$, $R \subseteq X$ are the clocks reset, op is one of the following stack operations:

1. nop does not change the contents of the stack,
2. $\downarrow_c$, $c \in \Lambda$ is a push operation that adds c on top of the stack, with age 0.
3. $\uparrow_c^I$, $c \in \Lambda$ is a stack symbol and $I \in \mathcal{I}$ is an interval, is a pop operation that removes the top most symbol of the stack provided it is a c with age in the interval I.

Timed automata (TA) can be seen as TPDA using nop operations only. This definition of TPDA is equivalent to the one in [1], but allows checking conjunctive constraints and stack operations together. In [9], it is shown that TPDA of [1] are expressively equivalent to timed automata with an untimed stack. As our technique is oblivious to whether the stack is timed or not, we focus on the syntactically more succinct model TPDA with a timed stack.

Next, we define the semantics of a TPDA in terms of TCWs.

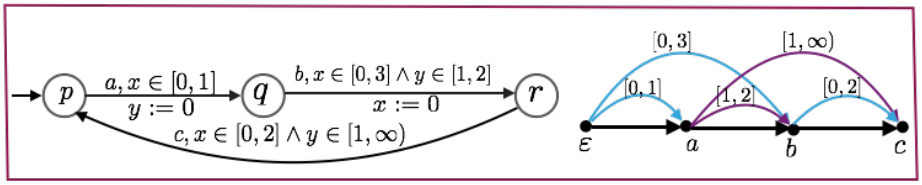


Figure 1 A timed automaton and a TCW capturing a run.

▶ **Definition 2.** A TCW $\mathcal{V} = (V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}})$ is said to be *generated or accepted* by a TPDA $\mathcal{S}$ if there is an accepting abstract run $\rho = (s_0, \gamma_1, a_1, \mathsf{op}_1, R_1, s_1)$ $(s_1, \gamma_2, a_2, \mathsf{op}_2, R_2, s_2) \cdots (s_{n-1}, \gamma_n, a_n, \mathsf{op}_n, R_n, s_n)$ of $\mathcal{S}$ such that, $s_n \in F$ and

- the sequence of push-pop operations is well-nested: in each prefix $\mathsf{op}_1 \cdots \mathsf{op}_k$ with $1 \le k \le n$, number of pops is at most number of pushes, and in the full sequence $\mathsf{op}_1 \cdots \mathsf{op}_n$, they are equal; and
- $V = \{0, 1, \ldots, n\}$ with $\lambda(0) = \varepsilon$ and $\lambda(i) = a_i$ for all $1 \le i \le n$ and $0 \to 1 \to \cdots \to n$ and, for all $I \in \mathcal{I}$, the relation $\curvearrowright^I$ is the set of pairs (i, j) with $0 \le i < j \le n$ such that
  - either for some $x \in X$ we have $x \in R_i$ (assuming $R_0 = X$) and $x \in I$ is a conjunct of γ_j and $x \notin R_k$ for all $i < k < j$,
  - or $\mathsf{op}_i = \downarrow_b$ is a push and $\mathsf{op}_j = \uparrow_b^I$ is the matching pop (same number of pushes and pops in $\mathsf{op}_{i+1} \cdots \mathsf{op}_{j-1}$).

We denote by $\mathsf{TCW}(\mathcal{S})$ the set of TCWs generated by $\mathcal{S}$. The non-emptiness problem for the TPDA $\mathcal{S}$ amounts to asking whether some TCW generated by $\mathcal{S}$ is realizable, i.e., whether $\mathsf{TCW}(\mathcal{S}) \cap \mathsf{Real}(\Sigma, \mathcal{I}) \neq \emptyset$. The TCW semantics of timed automata (TA) can be obtained from the above discussion by just ignoring the stack components (using nop operations only). Figure 1 depicts a simple example of a timed automaton and a TCW generated by it.

▶ Remark. The classical semantics of timed systems is given in terms of timed words. A *timed word* is a sequence $w = (a_1, t_1) \cdots (a_n, t_n)$ with $a_1, \ldots, a_n \in \Sigma$ and $(t_i)_{1\le i\le n}$ is a non-decreasing sequence of values in $\mathbb{R}_+$. A *realization* of a TCW $\mathcal{V} = (V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}}) \in \mathsf{TCW}(\mathcal{S})$ with $V = \{0, 1, \ldots, n\}$ is a timed word $w = (\lambda(1), \mathsf{ts}(1)) \ldots (\lambda(n), \mathsf{ts}(n))$ where the timestamp map $\mathsf{ts}\colon V \to \mathbb{R}_+$ (with $\mathsf{ts}(0) = 0$) is non decreasing and satisfies all timing constraints of $\mathcal{V}$. For example, the timed word $(a, 0.9)(b, 2.89)(c, 3.1)$ is a realization of the TCW in Figure 1 while $(a, 0.9)(b, 2.99)(c, 3.1)$ is not. It is not difficult to check that the language $\mathcal{L}(\mathcal{S})$ of timed words accepted by $\mathcal{S}$ with the classical semantics is precisely the set of realizations of TCWs in $\mathsf{TCW}(\mathcal{S})$. Therefore, $\mathcal{L}(\mathcal{S}) = \emptyset$ iff $\mathsf{TCW}(\mathcal{S}) \cap \mathsf{Real}(\Sigma, \mathcal{I}) = \emptyset$.

We now identify some important properties satisfied by TCWs generated from a TPDA. Let $\mathcal{V} = (V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}})$ be a TCW. The matching relation $(\curvearrowright^I)_{I\in\mathcal{I}}$ is used in two contexts: (i) while connecting a clock reset point (say for clock x) to a point where a guard of the form $x \in I$ is checked, and (ii) while connecting a point where a push was made to its corresponding pop, where the age of the topmost stack symbol is checked to be in interval I. We use the notations $\curvearrowright^{x\in I}$ and $\curvearrowright^{s\in I}$ to denote the matching relation $\curvearrowright^I$ corresponding to a clock-reset-check as well as push on stack-check respectively. We say that $\mathcal{V}$ is *well timed* w.r.t. a set of clocks X and a stack s if for each interval $I \in \mathcal{I}$ the matching relation $\curvearrowright^I$ can be partitioned as $\curvearrowright^I = \curvearrowright^{s\in I} \uplus \biguplus_{x\in X} \curvearrowright^{x\in I}$ where

- $(\mathbf{T_1})$ the stack relation $\curvearrowright^s = \bigcup_{I\in\mathcal{I}} \curvearrowright^{s\in I}$ corresponds to the matching push-pop events, hence it is well-nested: for all $i \curvearrowright^s j$ and $i' \curvearrowright^s j'$, if $i < i' < j$ then $j' < j$.
- $(\mathbf{T_2})$ For each $x \in X$, the clock relation $\curvearrowright^x = \bigcup_{I\in\mathcal{I}} \curvearrowright^{x\in I}$ corresponds to the timing constraints for clock x and respects the last reset condition: for all $i \curvearrowright^x j$ and $i' \curvearrowright^x j'$, if $i < i'$, then $j \le i'$. See Figure 1 for example, where $0 \curvearrowright^x 2$ and $2 \curvearrowright^x 3$.

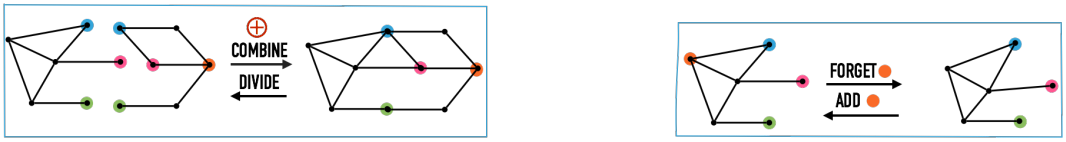


Figure 2 Operations on colored graphs.

It is then easy to check that TCWs defined by a TPDA with set of clocks X are well-timed for the set of clocks X, i.e., satisfy the properties above. We obtain the same for TA by just ignoring the stack edges, i.e., (T_1) above.

3 Tree-Width for Timed Systems

In this section, we discuss tree-algebra by introducing the basic terms, the operations on terms, their syntax and semantics. This will help us in analyzing the graphs obtained in the previous section using tree-terms, and establishing a bound on the tree-width.

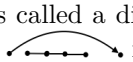
We introduce tree terms TTs from Courcelle [10] and their semantics as graphs which are both vertex-labeled and edge-labeled. Let Σ be a set of vertex labels and let Ξ be a set of edge labels. Let $K \in \mathbb{N}$. The syntax of K-tree terms K-TTs over (Σ, Ξ) is given by

$$\tau ::= (a,i) \mid (a,i)\xi(b,j) \mid \mathsf{Forget}_i\, \tau \mid \mathsf{Rename}_{i,j}\, \tau \mid \tau \oplus \tau$$

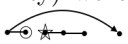
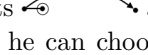
where $i, j \in \{1, 2, \ldots, K\}$ are colors $(i \neq j)$, $a, b \in \Sigma$ are vertex labels and $\xi \in \Xi$ is an edge label. The semantics of a K-TT τ is a colored graph $[\![\tau]\!] = (G_\tau, \chi_\tau)$ where $G_\tau = (V, E)$ is a graph and $\chi_\tau \colon \{1, 2, \ldots, K\} \to V$ is a partial injective function assigning a color to some vertices of G_τ. Note that any color in $\{1, 2, \ldots, K\}$ is assigned to at most one vertex of G_τ.

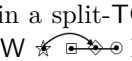
The atomic term (a, i) is a single vertex colored i and labeled a and the atomic term $(a,i)\xi(b,j)$ represents a ξ-labeled edge between two vertices colored i, j and labeled a, b respectively. Given a tree term τ, $\mathsf{Forget}_i(\tau)$ forgets the color i from a node colored i, leaving it uncolored. The operation $\mathsf{Rename}_{i,j}(\tau)$ renames the color i of a node to color j, provided no nodes are already colored j. Since any color appears at most once in G_τ, the operations $\mathsf{Forget}_i(\tau)$ and $\mathsf{Rename}_{i,j}(\tau)$ are deterministic, when colors i, j, are fixed. Finally, the operation $\tau_1 \oplus \tau_2$ (read as combine) combines two terms τ_1, τ_2 by fusing the nodes of τ_1, τ_2 which have the same color. See Figure 2.

The tree-width of a graph G is defined as the least K such that $G = G_\tau$ for some TT τ using $K + 1$ colors. Let TW_K denote the set of all graphs having tree width at most K. For TCWs, we have successor edges $\to$ and matching edges $\frown^I$ where $I \in \mathcal{I}$ is an interval. Hence, the set of edge labels is $\Xi_\mathcal{I} = \{\to\} \cup \{\frown^I \mid I \in \mathcal{I}\}$ and we use TTs over $(\Sigma, \Xi_\mathcal{I})$. An example is given in [5, Appendix A].

TCWs and Games. We find it convenient to prove that TCWs have bounded tree-width by playing a game, whose game positions are TCWs in which some successor edges may have been cut, i.e., are missing. Such TCWs, where some successor edges may be missing, are called split-TCWs. A split-TCW which is a connected graph is called a connected split-TCW, while a split-TCW which is a disconnected graph, is called a disconnected split-TCW. For example,  is a connected split-TCW, while  is a disconnected split-TCW consisting of two connected split TCWs, namely  and .

A TCW is atomic if it is denoted by an atomic term ((a, i) or $(a, i) \to (b, j)$ or $(a, i) \frown^I (b, j)$). *The split-game* is a two player turn based game $\mathcal{G} = (\mathsf{Pos}_\exists \uplus \mathsf{Pos}_\forall, \mathsf{Moves})$ where Eve's

set of game positions $\mathsf{Pos}_\exists$ consists of all connected (wrt. $\rightarrow \cup \curvearrowright$) split-TCWs and Adam's set of game positions $\mathsf{Pos}_\forall$ consists of dis-connected split-TCWs. Eve's moves consist of adding colors to the vertices of the split-TCW, and dividing the split-TCW. For example, if we have the connected split-TCW  , and Eve colors two nodes (we use shapes in place of colors for better visibility) we obtain  . This graph can be divided obtaining the disconnected graph  and  . As a result, we obtain the connected parts  and  and  . Now Adam's choices are on this disconnected split-TCW and he can choose either of the above three connected split-TCWs to continue the game. Thus, divide is the reverse of the combine operation $\oplus$. Adam's moves amount to choosing a connected component of the split-TCW. Eve has to continue coloring and dividing on the connected split-TCW chosen by Adam. Atomic split-TCWs are terminal positions in the game: neither Eve nor Adam can move from an atomic split-TCW. A play on a split-TCW $\mathcal{V}$ is a path in $\mathcal{G}$ starting from $\mathcal{V}$ and leading to an atomic split-TCW. The cost of the play is the maximum width (number of colors-1) of any split-TCW encountered in the path. In our example above,  is already an atomic split-TCW. If Adam chooses any of the other two, it is easy to see that Eve has a strategy using at most 2 colors in any of the split-TCWs that will be obtained till termination. The *cost* of a strategy σ for Eve from a split-TCW $\mathcal{V}$ is the maximal cost of the plays starting from $\mathcal{V}$ and following strategy σ. The *tree-width* of a (split-)TCW $\mathcal{V}$ is the minimal cost of Eve's (positional) strategies starting from $\mathcal{V}$. Let TCW_K denote the set of TCWs with tree-width bounded by K.

A *block* in a split-TCW is a maximal set of points of V connected by $\rightarrow$. For example, the split-TCW  has one non-trivial block  and one trivial block . Points that are not left or right endpoints of blocks of $\mathcal{V}$ are called internal.

The Bound. We show that we can find a K such that all the behaviors of the given timed system have tree-width bounded by K.

▶ **Theorem 3.** *Given a timed system $\mathcal{S}$ using a set of clocks X, all graphs in its* TCW *language have tree-width bounded by K, i.e.,* $\mathsf{TCW}(\mathcal{S}) \subseteq \mathsf{TCW}_K$*, where*
1. $K = |X| + 1$ *if $\mathcal{S}$ is a timed automaton,*
2. $K = 3|X| + 2$ *if $\mathcal{S}$ is a timed pushdown automaton.*

The following lemma completes the proof of Theorem 3 (2).

▶ **Lemma 4.** *The tree-width of a well-timed* TCW *is bounded by* $3|X| + 2$.

We prove this by playing the "split game" between *Adam* and *Eve* in which *Eve* has a strategy to disconnect the word without introducing more than $3|X| + 3$ colors. *Eve*'s strategy processes the word from right to left. Starting from any TCW, Eve colors the end points of the TCW, as well as the last reset points (from the right end) corresponding to each clock. Here she uses at most $|X| + 2$ colors. On top of this, depending on the last point, we have different cases, as sketched below (a detailed proof is in [5, Appendix B]).

If the last point is the target of a $\curvearrowright^x$ edge for some clock x, then Eve simply removes the clock edge, since both the source and target points of this edge are colored. We only discuss in some detail the case when the last point is the target of a $\curvearrowright^s$ edge, and the source of this edge is an internal point in the non-trivial block. Figure 3 illustrates this case.

To keep a bound on the number of colors needed, Eve divides the TCW as follows:
- First Eve adds a color to the source of the stack edge
- If there are any clock edges crossing this stack edge, Eve adds colors to the corresponding reset points. Note that this results in adding atmost $|X|$ colors.

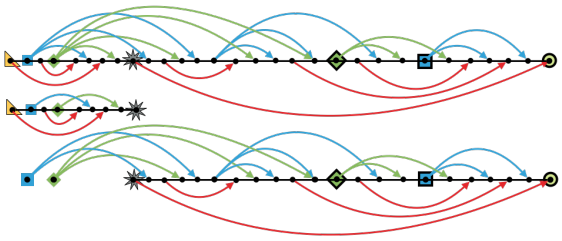


Figure 3 The last point is the target of a $\curvearrowright^s$ (top figure). After the split, we obtain the words $\mathcal{V}_1$ (the middle one) and $\mathcal{V}_2$ (the bottom one).

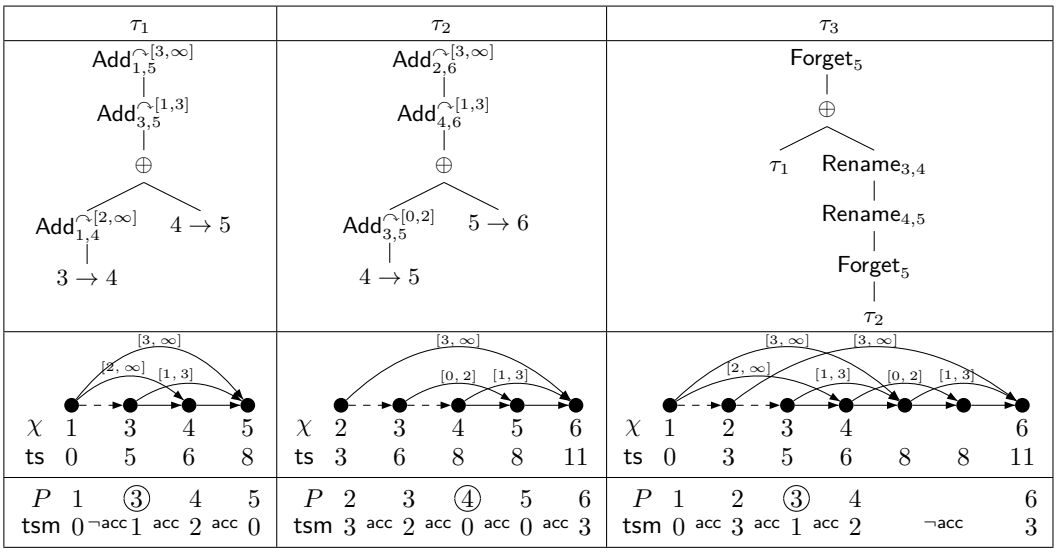
- Eve disconnects the TCW into two parts, such that the right part $\mathcal{V}_2$ consists of one non-trivial block whose end points are the source and target points of the stack edge, and also contains to the left of this block, atmost $|X|$ trivial blocks. Each of these trivial blocks are the reset points of those clock edges which cross over. The left part $\mathcal{V}_1$ is a TCW consisting of all points to the left of the source of the stack edge, and has all remaining edges other than the clock edges which have crossed over. Adam can now continue the game choosing $\mathcal{V}_1$ or $\mathcal{V}_2$. Note that in one of the words so obtained, the stack edge completely spans the non-trivial block, and can be easily removed.

Invariants and bound on tree-width. We now discuss some invariants on the structure of the split-TCWs as we play the game using the above strategy.
(I1) We have $\leq |X|$ colored trivial blocks to the left of the only non-trivial block,
(I2) The last reset node of each clock on the non-trivial block is colored,
(I3) The end points of the non-trivial block are colored.
To maintain the above invariants, we need $|X|+1$ extra colors than the at most $2|X|+2$ mentioned above. This proves that the tree-width of a TPDA with set of clocks X is bounded by $3|X|+2$. If the underlying system is a timed automaton, then we have a single non-trivial block in the game at any point of time. There are no trivial blocks, unlike the TPDA, due to the absence of stack edges. This results in using only $\leq |X|+2$ colors at any point of time, where $|X|$ colors are needed to color the last reset points of the clocks in the block, and the 2 colors are used to color the left, right end points of the block.

4 Tree automata for Validity

In this section, we give one of the most challenging constructions (Theorem 6) of the paper, namely, the tree automaton that accepts all valid and realizable K-TTs which are "good". Good K-TTs are defined below. In this section, we restrict ourselves to closed intervals; that is, those of the form $[a,b]$ and $[a,\infty)$, where $a,b \in \mathbb{N}$. Fix $K \geq 2$. Not all graphs defined by K-TTs are realizable TCWs. Indeed, if τ is such a TT, the edge relation $\rightarrow$ may have cycles or may be branching, which is not possible in a TCW. Also, the timing constraints given by $\curvearrowright^I$ need not comply with the $\rightarrow$ relation: for instance, we may have a timing constraint $e \curvearrowright^I f$ with $f \rightarrow^+ e$ ($\rightarrow^+$ is the transitive closure of $\rightarrow$, i.e., e can be reached from f after taking ≥ 1 successor edges $\rightarrow$). Moreover, some terms may define graphs denoting TCWs which are not realizable. So we use $\mathcal{A}^{K,M}_{\mathsf{valid}}$ to check for validity. Since we have only closed intervals in timing constraints, integer timestamps suffice for realizability, as can be seen from the following lemma ([5, Appendix C.1]).

Table 1 The second row gives tree representations of three good 6-TTs τ_1, τ_2, τ_3. In all these terms, we ignore vertex labels and we use $\mathsf{Add}_{i,j}^{I}\,\tau$ as a macro for $\tau \oplus i \curvearrowright^{I} j$. The third row gives their semantics $[\![\tau]\!] = (G_\tau, \chi_\tau)$ together with a realization ts, the fourth row gives possible states q of $\mathcal{A}^{K,M}_{\mathsf{valid}}$ with $M = 4$ after reading the terms. Here, L is the circled color. The boolean value $\mathsf{acc}(i)$ for each non maximal color i is written between $\mathsf{tsm}(i)$ and $\mathsf{tsm}(i^+)$.



▶ **Lemma 5.** *Let $\mathcal{V} = (V, \to, \lambda, (\curvearrowright^{I})_{I\in\mathcal{I}(M)})$ be a* TCW *using only closed intervals in its timing constraints. Then, $\mathcal{V}$ is realizable iff there exists an integer valued timestamp map satisfying all timing constraints.*

Consider a set of colors $P \subseteq \{1, \dots, K\}$. For each $i \in P$ we let $i^+ = \min\{j \in P \cup \{\infty\} \mid i < j\}$ and $i^- = \max\{j \in P \cup \{0\} \mid j < i\}$. If P is not clear from the context, then we write $\mathsf{next}_P(i)$ and $\mathsf{prev}_P(i)$. Given a K-TT τ with semantics $[\![\tau]\!] = (G, \chi)$, we denote by $\mathsf{Act} = \mathsf{dom}(\chi)$ the set of active colors in τ, we let $\mathsf{Right} = \max(\mathsf{Act})$ and $\mathsf{Left} = \min\{i \in \mathsf{Act} \mid \chi(i) \to^* \chi(\mathsf{Right})\}$. If τ is not clear from the context, then we write Act_τ, Left_τ and Right_τ. A K-TT τ is *good* if

- $\tau ::= (a, i) \to (b, j) \mid (a, i) \curvearrowright^{I} (b, j) \mid \mathsf{Forget}_i\,\tau \mid \mathsf{Rename}_{i,j}\,\tau \mid \tau \oplus \tau$,
- for every subterm of the form $(a, i) \to (b, j)$ or $(a, i) \curvearrowright^{I} (b, j)$ we have $i < j$,
- $\mathsf{Rename}_{i,j}\,\tau$ is possible only if $i^- < j < i^+$,
- $\tau_1 \oplus \tau_2$ is allowed if $\mathsf{Right}_1 = \mathsf{Left}_2$ and $\{i \in \mathsf{Act}_2 \mid \mathsf{Left}_1 \le i \le \mathsf{Right}_1\} \subseteq \mathsf{Act}_1$.

Examples of good TTs and their semantics are given in Table 1. Note that the semantics of a K-TT τ is a colored graph $[\![\tau]\!] = (G_\tau, \chi_\tau)$. Below, we provide a direct construction of a tree automaton, which gives a clear upper bound on the size of $\mathcal{A}^{K,M}_{\mathsf{valid}}$, since obtaining this bound gets very technical if we stick to MSO.

▶ **Theorem 6.** *We can build a tree automaton $\mathcal{A}^{K,M}_{\mathsf{valid}}$ with $M^{\mathcal{O}(K)}$ number of states such that $\mathcal{L}(\mathcal{A}^{K,M}_{\mathsf{valid}})$ is the set of* good K-TTs *τ such that $[\![\tau]\!]$ is a realizable* TCW *and the endpoints of $[\![\tau]\!]$ are the only colored points.*

Proof. The tree automaton $\mathcal{A}^{K,M}_{\mathsf{valid}}$ reads the TT bottom-up and stores in its state a finite abstraction of the associated graph. The finite abstraction will keep only the colored points of the graph. We will only accept *good* terms for which the natural order on the active colors

Table 2 Transitions of $\mathcal{A}^{K,M}_{\mathsf{valid}}$. See Table 1 for some intuitions. $I.up$ in row 2 represents upper bound of interval I.

$(a,i) \to (b,j)$	$\bot \xrightarrow{(a,i)\to(b,j)} q = (P, L, \mathsf{tsm}, \mathsf{acc})$ is a transition if $i < j$ and $P = \{i,j\}$, $L = i$ and $\mathsf{acc}(j) = \mathsf{ff}$. The values for $\mathsf{tsm}(i)$, $\mathsf{tsm}(j)$ and $\mathsf{acc}(i)$ are guessed.
$(a,i) \curvearrowright^I (b,j)$	$\bot \xrightarrow{(a,i)\curvearrowright^I(b,j)} q = (P, L, \mathsf{tsm}, \mathsf{acc})$ is a transition if $i < j$ and $P = \{i,j\}$, $L = j$ and $\mathsf{acc}(j) = \mathsf{ff}$. Here, i and j are trivial blocks. The values for $\mathsf{tsm}(i)$, $\mathsf{tsm}(j)$ and $\mathsf{acc}(i)$ are guessed such that ($\mathsf{acc}(i) = \mathsf{tt}$ and $d(i,j) \in I$) or ($\mathsf{acc}(i) = \mathsf{ff}$ and $I.up = \infty$).
$\mathsf{Rename}_{i,j}$	$q = (P, L, \mathsf{tsm}, \mathsf{acc}) \xrightarrow{\mathsf{Rename}_{i,j}} q' = (P', L', \mathsf{tsm}', \mathsf{acc}')$ is a transition if $i \in P$ and $i^- < j < i^+$. Then, q' is obtained from q by replacing i by j.
Forget_i	$q = (P, L, \mathsf{tsm}, \mathsf{acc}) \xrightarrow{\mathsf{Forget}_i} q' = (P', L', \mathsf{tsm}', \mathsf{acc}')$ is a transition if $L < i < \max(P)$ (endpoints should stay colored). Then, state q' is deterministically given by $P' = P \setminus \{i\}$, $L' = L$, $\mathsf{tsm}' = \mathsf{tsm}_{\mid P'}$ and $\mathsf{acc}'(i^-) = \mathsf{ACC}(i^-, i^+) \wedge (D(i^-, i^+) < M)$, the other values of acc' are inherited from acc.
$\oplus$	$q_1, q_2 \xrightarrow{\oplus} q$ where $q_1 = (P_1, L_1, \mathsf{tsm}_1, \mathsf{acc}_1)$, $q_2 = (P_2, L_2, \mathsf{tsm}_2, \mathsf{acc}_2)$ and $q = (P, L, \mathsf{tsm}, \mathsf{acc})$ is a transition if the following hold ▬ $R_1 = \max(P_1) = L_2$ and $\{i \in P_2 \mid L_1 \le i \le R_1\} \subseteq P_1$ (we cannot insert a new point from the second argument in the non-trivial block of the first argument). ▬ $P = P_1 \cup P_2$, $L = L_1$, and $\mathsf{tsm}_{\mid P_1} = \mathsf{tsm}_1$ and $\mathsf{tsm}_{\mid P_2} = \mathsf{tsm}_2$: these updates are deterministic. In particular, this implies that tsm_1 and tsm_2 coincide on $P_1 \cap P_2$. ▬ Finally, acc satisfies $\mathsf{acc}(\max(P)) = \mathsf{ff}$ and $\forall i \in P_1 \setminus \{\max(P_1)\}$ $\quad \mathsf{acc}_1(i) \iff \mathsf{ACC}_q(i, \mathsf{next}_{P_1}(i)) \wedge D_q(i, \mathsf{next}_{P_1}(i)) < M$ $\forall i \in P_2 \setminus \{\max(P_2)\}$ $\quad \mathsf{acc}_2(i) \iff \mathsf{ACC}_q(i, \mathsf{next}_{P_2}(i)) \wedge D_q(i, \mathsf{next}_{P_2}(i)) < M$. Notice that these conditions imply For all $i \in P_1$, if $\mathsf{next}_P(i) = \mathsf{next}_{P_1}(i)$ (e.g., if $L_1 \le i < R_1$) then $\mathsf{acc}(i) = \mathsf{acc}_1(i)$. For all $i \in P_2$, if $\mathsf{next}_P(i) = \mathsf{next}_{P_2}(i)$ (e.g., if $L_2 \le i$) then $\mathsf{acc}(i) = \mathsf{acc}_2(i)$.

coincides with the order of the corresponding vertices in the final TCW. The restriction to good terms ensures that the graph defined by the TT is a split-TCW.

Moreover, to ensure realizability of the TCW defined by a term, we will guess timestamps of vertices modulo M. We also guess while reading a subterm whether the time elapsed between two consecutive active colors is *big* ($\ge M$) or *small* ($< M$). We see below that the time elapsed is *small* iff it can be recovered accurately with the modulo M abstraction. Then, the automaton has to check that all these guesses are coherent and using these values it will check that every timing constraint is satisfied.

Formally, *states of* $\mathcal{A}^{K,M}_{\mathsf{valid}}$ are tuples of the form $q = (P, L, \mathsf{tsm}, \mathsf{acc})$, where $P \subseteq \{1, \ldots, K\}$, $L \in P$, $\mathsf{tsm}\colon P \to [M] = \{0, \ldots, M-1\}$ and $\mathsf{acc}\colon P \to \mathbb{B}$. acc is a flag which stands for "accurate", and is used to check if the time elapse between two points is accurate or not, based on the time stamps.

Intuitively, when reading bottom-up a K-TT τ with $[\![\tau]\!] = (V, \to, \lambda, (\curvearrowright^I)_{I\in\mathcal{I}}, \chi)$, the automaton $\mathcal{A}^{K,M}_{\mathsf{valid}}$ will reach a state $q = (P, L, \mathsf{tsm}, \mathsf{acc})$ such that

($\mathbf{A_1}$) $P = \mathsf{Act}$ is the set of *active* colors in τ, $L = \mathsf{Left}$ and $\max(P) = \mathsf{Right}$.

($\mathbf{A_2}$) For all $i \in P$, if $L \le i < \max(P)$ then $\chi(i) \to^+ \chi(i^+)$ in $[\![\tau]\!]$.

($\mathbf{A_3}$) Let $\dashrightarrow = \{(\chi(i), \chi(i^+)) \mid i \in P \wedge i < L\}$. This extra relation serves at ordering the blocks of a split-TCW. Then, $([\![\tau]\!], \dashrightarrow)$ is an *ordered* split-TCW, i.e., $< = (\to \cup \dashrightarrow)^+$ is a total order on V, timing constraints in $[\![\tau]\!]$ are $<$-compatible $\curvearrowright^I \subseteq <$ for all I, the *direct successor* relation of $<$ is $\lessdot = \to \cup \dashrightarrow$ and $\to \cap \dashrightarrow = \emptyset$. Moreover, targets of timing constraints are in the last block: for all $u \curvearrowright^I v$ in $([\![\tau]\!], \dashrightarrow)$, we have $\chi(L) \to^* v$.

(A_4) There exists a timestamp map $\mathsf{ts}\colon V \to \mathbb{N}$ such that
- all constraints are satisfied: $\mathsf{ts}(v) - \mathsf{ts}(u) \in I$ for all $u \curvearrowright^I v$ in $[\![\tau]\!]$,
- time is non-decreasing: $\mathsf{ts}(u) \leq \mathsf{ts}(v)$ for all $u \leq v$,
- $(\mathsf{tsm}, \mathsf{acc})$ is the modulo M abstraction of ts: $\forall i \in P$ we have $\mathsf{tsm}(i) = \mathsf{ts}(\chi(i))[M]$ and $\mathsf{acc}(i) = \mathsf{tt}$ iff $i^+ \neq \infty$ and $\mathsf{ts}(\chi(i^+)) - \mathsf{ts}(\chi(i)) < M$.

We say that the state q is a *realizable abstraction* of a term τ if it satisfies conditions (A_1)–(A_4).

Indeed, the finite state automaton $\mathcal{A}^{K,M}_{\mathsf{valid}}$ cannot store the timestamp map ts witnessing realizability. Instead, it stores the modulo M abstraction $(\mathsf{tsm}, \mathsf{acc})$. We will see that $\mathcal{A}^{K,M}_{\mathsf{valid}}$ can check realizability based on the abstraction $(\mathsf{tsm}, \mathsf{acc})$ of ts and can maintain this abstraction while reading the term bottom-up.

We introduce some notations. Let $q = (P, L, \mathsf{tsm}, \mathsf{acc})$ be a state and let $i, j \in P$ with $i \leq j$. We define $d(i,j) = (\mathsf{tsm}(j) - \mathsf{tsm}(i))[M]$ and $D(i,j) = \sum_{k \in P | i \leq k < j} d(k, k^+)$. We also define $\mathsf{ACC}(i,j) = \bigwedge_{k \in P | i \leq k < j} \mathsf{acc}(k)$. If the state is not clear from the context, then we write $d_q(i,j)$, $D_q(i,j)$, $\mathsf{ACC}_q(i,j)$. For instance, with the state q_3 corresponding to the term τ_3 of Table 1, we have $\mathsf{ACC}(1,4) = \mathsf{tt}$, $d(1,4) = 2$ and $D(1,4) = 6 = \mathsf{ts}(4) - \mathsf{ts}(1)$ is the accurate value of the time elapsed. Whereas, $\mathsf{ACC}(3,6) = \mathsf{ff}$ and $d(3,6) = 2 = D(3,6)$ are both *strict modulo-M under-approximations* of the time elapsed $\mathsf{ts}(6) - \mathsf{ts}(3) = 6$. The *transitions of* $\mathcal{A}^{K,M}_{\mathsf{valid}}$ are defined in Table 2.

Accepting condition. The accepting states of $\mathcal{A}^{K,M}_{\mathsf{valid}}$ should correspond to abstractions of TCWs. Hence the accepting states are of the form $(\{i,j\}, L, \mathsf{tsm}, \mathsf{acc})$ with $i, j \in \{1, \ldots, K\}$, $i < j$, $L = i$ and $\mathsf{acc}(j) = \mathsf{ff}$. The correctness of this construction is in [5, Appendix C.2], and is obtained by proving (i) the transitions of $\mathcal{A}^{K,M}_{\mathsf{valid}}$ indeed preserve the conditions (A_1)–(A_4), (ii) (A_1)–(A_4) ensure among other things, that the boolean values $\mathsf{acc}(i)$, $\mathsf{ACC}(i,j)$ for $i < j$ indeed defines when the elapse of time is accurately captured by the modulo M abstraction: that is, $\mathsf{ACC}(i,j)$ is true iff the actual time elapse between i and j is captured using the modulo M abstraction $D(i,j)$. ◀

5 Tree automata for timed systems

The goal of this section is to build a tree automaton which accepts tree terms denoting TCWs accepted by a TPDA. The existence of a tree automaton can be proved by showing the MSO definability of the runs of the TPDA $\mathcal{S}$ on a TCW. However, as seen in section 4, we directly construct a tree automaton for better complexity. Given the timed system $\mathcal{S}$, let K be the bound on tree-width given by Theorem 3 and let M be one more than the maximal constant occurring in the guards of $\mathcal{S}$. The automaton $\mathcal{A}^{K,M}_{\mathcal{S}}$ will accept *good* K-TTs with the additional restriction that a timing constraint is immediately combined with an existing term. That is, *restricted* K-TTs are *good* K-TTs restricted to the following syntax:

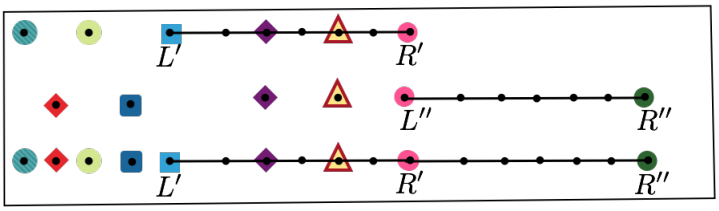
$$\tau ::= (a,i) \to (b,j) \mid \tau \oplus [(a,i) \curvearrowright^I (b,j)] \mid \mathsf{Forget}_i\, \tau \mid \mathsf{Rename}_{i,j}\, \tau \mid \tau \oplus \tau$$

▶ **Theorem 7.** *Let $\mathcal{S}$ be a TPDA of size $|\mathcal{S}|$ (constants encoded in unary) with set of clocks X and using constants less than M. Let K be the bound on tree-width given by Theorem 3. Then, we can build a tree automaton $\mathcal{A}^{K,M}_{\mathcal{S}}$ with $|\mathcal{S}|^{\mathcal{O}(K)} \cdot K^{\mathcal{O}(|X|)}$ states such that $\mathcal{A}^{K,M}_{\mathcal{S}}$ accepts the set of* restricted K-TTs τ *such that $[\![\tau]\!] \in \mathsf{TCW}(\mathcal{S})$. Further, $\mathsf{TCW}(\mathcal{S}) = [\![\mathcal{L}(\mathcal{A}^{K,M}_{\mathcal{S}})]\!] = \{[\![\tau]\!] \mid \tau \in \mathcal{L}(\mathcal{A}^{K,M}_{\mathcal{S}})\}$.*

Proof (Sketch). A state of $\mathcal{A}_{\mathcal{S}}^{K,M}$ is a tuple $q = (P, L, \delta, \mathsf{Push}, \mathsf{Pop}, G, Z)$ where,

- P is the set of active colors, and $L = \mathsf{Left} \in P$ is the left-most point that is connected to the right-end-point $R = \mathsf{Right} = \max(P)$ by successor edges on the non-trivial block.
- δ is a map that assigns to each color $k \in P$ the transition $\delta(k)$ guessed at the leaf corresponding to color k,
- Push and Pop are two boolean variables: $\mathsf{Push} = 1$ iff a push-pop edge has been added to L and $\mathsf{Pop} = 1$ iff a push-pop edge has been added to R,
- $G = (G_x)_{x \in X}$ is a boolean vector of size $|X|$: for each clock $x \in X$, $G_x = 1$ iff some constraint on x has already been checked at R,
- $Z = (Z_x)_{x \in X}$ assigns to each clock x either the color $i \in P$ with $i < L$ of the unique point on the left of the non-trivial block which is the source of a timing constraint $i \curvearrowright^I j$ for clock x, or $\perp$ if no such points exist.

For $j \in P$, let $\mathsf{Reset}(j)$ be the set of clocks that are reset in the transition $\delta(j)$. We describe here the most involved kind of transition $q' \oplus q''$ for states q', q''. The remaining transitions as well as the full proof are in [5, Appendix D]. Let $q' = (P', L', \delta', \mathsf{Push}', \mathsf{Pop}', G', Z')$, $q'' = (P'', L'', \delta'', \mathsf{Push}'', \mathsf{Pop}'', G'', Z'')$ and $q = (P, L, \delta, \mathsf{Push}, \mathsf{Pop}, G, Z)$. Then $q', q'' \xrightarrow{\oplus} q$ is a transition if the following hold:



C_1: $R' = \max(P') = L''$ and $\{i \in P'' \mid L' \le i \le R'\} \subseteq P'$ (we cannot insert a new point from the second argument in the non-trivial block of the first argument). Note that according to C_1, the points ◆, ◬ and ● in P'' lying between L', R' are already points in the non-trivial block connecting L' to R'.

C_2: $\forall i \in P' \cap P''$, $\delta'(i) = \delta''(i)$ (the guessed transitions match). By C_2, the transitions δ', δ'' of ◆, ◬ and ● must match.

C_3: if there is a Push operation in $\delta''(L'')$ then $\mathsf{Push}'' = 1$ and if there is a pop operation in $\delta'(R')$ then $\mathsf{Pop}' = 1$ (the push-pop edges corresponding to the merging point have been added, if they exist). By C_3, if $\delta(R') = \delta(L'')$ contains a pop (resp. push) operation then $R' = L''$ is the target (resp. source) of a push-pop edge.

C_4: if some guard $x \in I$ is in $\delta(R')$, then $G'_x = 1$ (before we merge, we ensure that the clock guard for x in the transition guessed at R', if any, has been checked). After the merge, $R' = L''$ becomes an internal point; hence by C_4, any guard $x \in I$ in $\delta'(R')$ must be checked already, i.e., $G'_x = 1$. After the merge, it is no more possible to add an edge $\curvearrowright^I$ leading into R'.

C_5: if $Z'_x \neq \perp$, then $\forall j \in P''$, $Z'_x < j < L'$ implies $x \notin \mathit{Reset}''(j)$ (If a matching edge starting at $Z'_x < L'$ had been seen earlier in run leading to q', then x should not have been reset in q'' between Z'_x and L', else it would violate the consistency of clocks). By C_5, if Z'_x is ◉ (resp. ●), i.e., ◉ (resp. ●) is the source of a timing constraint $\curvearrowright^I$ for clock x whose target is in the L'–R' block, then clock x cannot be reset at ◆ and ■ (resp. ■).

C_6: if $Z''_x \neq \perp$, then $\forall j \in P'$, $Z''_x < j < L''$ implies $x \notin \mathit{Reset}'(j)$ (If a matching edge starting at $Z''_x < L''$ had been seen earlier in run leading to q'', then x should not have been reset in q' between Z''_x and L''). By C_6, if Z''_x is ◆, then x cannot be reset at ●, ■, ◆, or ◬. Likewise, if Z''_x was ■, then clock x cannot be reset at ■, ◆, or ◬.

C_7: $P = P' \cup P''$, $L = L'$, $\delta = \delta' \cup \delta''$, $\mathsf{Push} = \mathsf{Push}'$, $\mathsf{Pop} = \mathsf{Pop}''$, $G = G''$ and for all $x \in X$ we have $Z_x = Z''_x$ if $Z''_x < L'$, else $Z_x = Z'_x$. C_7 says that on merging, we obtain the third split-TCW. After the merge, if Z_x is defined, it must be on the left of L', i.e., one of  ,  ,  ,  . Notice that the above three conditions ensure the well-nestedness of clocks. By C_5 and C_6 we cannot have both $Z'_x \in \{$, $\}$ and $Z''_x \in \{$, $\}$. So if $Z''_x \in \{$,$\}$ then $Z_x = Z''_x$ and otherwise $Z_x = Z'_x$ (including when $Z''_x \in \{$, $\}$ and $Z'_x = \bot$).

Accepting Condition. A state $q = (P, L, \delta, \mathsf{Push}, \mathsf{Pop}, G, Z)$ is accepting if $L = \min(P)$, $\delta(L)$ is some dummy ε-transition resetting all clocks and leading to the initial state, $\mathsf{target}(\delta(R))$ is a final state and if $\delta(R)$ has a pop operation then $\mathsf{Pop} = 1$, if it has a constraint/guard for clock x, then $G_x = 1$. Note that the above automaton only accepts restricted K-TTs; this is sufficient for emptiness checking since Eve's winning strategy in Section 3 captures all behaviours of the $\mathsf{TCW}(\mathcal{S})$ while generating only restricted K-TTs. ◀

As a corollary we obtain (see [5, Appendix D.2]),

▶ **Theorem 8.** *Let $\mathcal{S}$ be a TPDA. We have $L(\mathcal{S}) \neq \emptyset$ iff $L(\mathcal{A}^{K,M}_{\mathsf{valid}} \cap \mathcal{A}^{K,M}_{\mathcal{S}}) \neq \emptyset$.*

If the underlying system is a timed automaton, we can restrict the state space to storing just the tuple (P, δ, G) as the other components are not required and L is always $\min(P)$.

Possible Extensions. We now briefly explain how to extend our technique in the presence of diagonal guards: these are guards of the form $x - y \in I$ or $x - \mathsf{pop} \in I$ or $\mathsf{pop} - x \in I$ where x, y are clocks, and I is a time interval. The former is a guard that checks the difference between two clock values, while the latter checks the difference between the value of a clock and the age of the topmost stack symbol at the time of the pop. To handle a constraint of the form $x - y \in I$, it is enough to check the difference between the guessed time stamps at the last reset points of clocks y, x to be in I. Likewise, to check $x - \mathsf{pop} \in I$ or $\mathsf{pop} - x \in I$, we check the difference between the guessed time stamps at the points where the top symbol was pushed on the stack and the last reset of clock x. Note that the last reset points of x, y will not be forgotten until the automaton decides to accept; likewise, the push point will not be forgotten until the pop transition is encountered. Given this, our construction of the tree automaton can be extended with the above checks to handle diagonal guards as well.

6 Implementation and a case-study

We have implemented the emptiness checking procedure for TPDA using our tree-automata based approach, and describe some results here. Despite the EXPTIME-completeness of this problem for general TPDA, we present some good performance results for certain interesting subclasses of TPDA. As a concrete subclass, the complexity significantly improves when there is no extra clock other than the timing constraints associated with the stack; while popping a symbol, we simply check the time elapsed since the push. Note that this can be used to model systems where timing constraints are well-nested: clock resets correspond to push and checking guards corresponds to checking the age of the topmost stack symbol. Thus, this gives a technique for reducing the number of clocks for a timed system with nested timing constraints. For this subclass, the exact number of states of the tree automaton can be improved to $2 \times (M \times T)^2$, where M is 1 plus the maximum constant, and T is the number of transitions. This idea can be extended further to incorporate clocks whose constraints

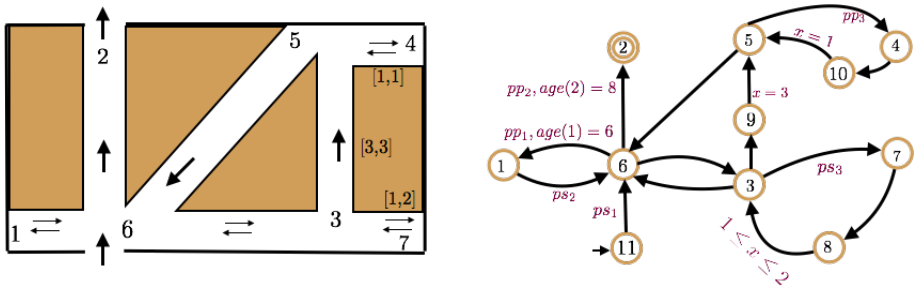


Figure 4 A simple maze. Every junction, dead end, entry point or exit point is called a place, numbered from 1 to 7. 6 is the entry, 2 the exit, 1, 7 and 4 are dead ends. Time intervals denote the time taken between adjacent places; e.g., between 1 and 2 time units must elapse between places 3 and 7. On the right, is the TPDA model of the maze.

are well-nested with respect to the stack. We can also handle clocks which are reset and checked in consecutive transitions.

For the general model (one stack + any number of clocks), we can use optimizations to reduce the number of states of the tree automaton to $(M \times T)^{2|X|+2} \times 2^{2|X|+1}$, where $|X|$ is the number of clocks, M is 1 plus the maximum constant and T is the number of transitions. To see this, consider the worst case scenario, where a state of the tree automaton has $|X|$ hanging points and $|X|$ reset points. In total there can be $2|X| + 2$ active points including the left and right end-points of the non-trivial block. After a combine operation, we can forget a point i of the new state, if it is the case that every clock x reset at the transition (guessed) at point i is also reset at some transition at a point after i. Following this strategy, if we aggressively forget as many points as we can, we will have at most $|X|$ internal (reset) active points between the left and right end-points of the non-trivial block. Thus, we reduce the number of active points from $3|X| + 2$ to $2|X| + 2$.

As a proof of concept, we have implemented our approach with these optimizations. We will now describe some examples we modelled and their experimental results. These experiments were run on a 3.5 GHz i5 PC with 8GB RAM, with number of cores=4.

A Modeling Example : Maze with Constraints

As an interesting example, we model a situation of a robot successfully traversing a maze respecting multiple constraints (see Figure 4). These constraints may include logical constraints: the robot must visit location 1 before exit, or the robot must load something at a certain place i and unload it at place j (so number of visits to i must equal visits to j). We may also have local and global time constraints which check whether adjacent places are visited within a time bound, or the total time taken in the maze is within a given duration. We show below, via an illustrative example, that certain classes of such constraints can be converted into a 1-clock TPDA.

One can go from place p to some of its adjacent place q if there is an arrow from place p to place q. In addition, the following types of constraints must be respected.

1. *Logical constraints* specify certain order between visiting places, the number of times (upper/lower bounds) to visit a place or places, and so on. The logical constraints we have in our example are (a) place 1 must be visited exactly once, (b) from the time we enter the maze, to visiting place 1, one must visit place 7 (load) and place 4 (unload) equal number of times, and at any point of time, the number of visits to place 7 is not less than number of visits to place 4. (c) from visiting place 1 to exiting the maze, one

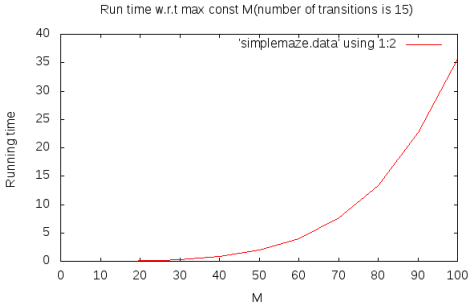
must visit place 7 and place 4 equal number of times and, at any point during time, number of visits to place 7 is not less than number of visits to place 4.

2. *Local time constraints* specify time intervals which must be respected while going from a place to its adjacent place. The time taken from some place i to an adjacent place j is given as a closed interval $[a, b]$ along with the arrow. One cannot spend any time between a pair of adjacent places other than the ones specified in the maze. For example, the time bound for going from place 7 to 3 is given, while the time taken from place 3 to place 7 and place 6 to place 1 is zero ([0,0]), since it is not mentioned. Further, one cannot stay in any place for non-zero duration.
3. *Global time constraints* specify total time that can elapse between visiting two places. From entering the maze to visiting of place 1, time taken should be exactly m units (a parameter). From visiting place 1 to exit, time should be exactly n units (another parameter).

A maze respecting multiple constraints as above is converted into a 1-clock TPDA. While the details of this conversion are given in [5, Appendix E], the main idea is to encode local time bounds with the clock which is reset on all transitions. A logical constraint specifying equal number of visits to places p_1, p_2 is modelled by pushing symbols while at p_1, and popping them at p_2. Likewise, if there is a global time constraint that requires a time elapse in $[a, b]$ between the entry and some place p, then push on the stack at entry, and check its age while at p. Note that these are *well-nested* properties.

To check the existence of a legitimate path in the maze respecting the constraints, our tool checks the existence of a run in the TPDA. By running our tool on the TPDA constructed (and fixing the parameters to be $m = 7$, $n = 8$), we obtain the following run: (described as a sequence of pairs the form : State, Entry time stamp in the state) $(6, 0.0) \to (3, 0.0) \to (7, 0.0) \to (3, 1.0) \to (7, 1.0) \to (3, 2.0) \to (5, 5.0) \to (4, 5.0) \to (5, 6.0) \to (4, 6.0) \to (5, 7.0) \to (6, 7.0) \to (1, 7.0) \to (6, 7.0) \to (3, 7.0) \to (7, 7.0) \to (3, 9.0) \to (7, 9.0) \to (3, 10.0) \to (5, 13.0) \to (4, 13.0) \to (5, 14.0) \to (4, 14.0) \to (5, 15.0) \to (6, 15.0) \to (2, 15.0)$.

The scalability is assessed by instantiating the maze for various choices of maximum constants used, as well as number of transitions. The running times with respect to various choices for the maximum constant are plotted on the right. More examples can be found in [5, Appendix E].



7 Conclusion

We have obtained a new construction for the emptiness checking of TPDA, using tree-width. The earlier approaches [1], [2] which handle dense time and discrete time push down systems respectively use an adaptation of the well-known idea of timed regions. The technique in [2] does not extend to dense time systems, and it is not clear whether the approach in [1] will work for say, multi stack push down automata even with bounded scope/phase restrictions. Unlike this, our approach is uniform : all our proofs except the tree automaton for realizability already work even if we have open guards. Our realizability proof has to be adapted for open guards and this is work under progress; in this paper, we focussed on closed guards to obtain an efficient tool based on our theory. Likewise,

our proofs can be extended to bounded phase/scope/rounds multi stack timed push down automata : we need to show a bound on the tree-width, and then adapt the tree automaton construction for the system automaton. The tree automaton checking realizability requires no change. With the theoretical improvements in this paper, we implement our approach and examine its performance on real examples. To the best of our knowledge, this is the first tool implementing timed push down systems. We plan to optimize our implementation to get a more robust and scalable tool. For the subclasses we have, it would be good to have a characterization and automatic translation (currently done by hand) that replaces well-nested clock constraints by stack edges, thus leading to better implementability.

Acknowledgments. The authors thank Vincent Jugé for insightful discussions on the MSO definability of realizability of TC words.

References

1 P. Abdulla, M. F. Atig, and J. Stenman. Dense-timed pushdown automata. In *LICS Proceedings*, pages 35–44, 2012.

2 Parosh Aziz Abdulla, Mohamed Faouzi Atig, and Jari Stenman. The minimal cost reachability problem in priced timed pushdown systems. In *Language and Automata Theory and Applications - 6th International Conference, LATA 2012, A Coruña, Spain, March 5-9, 2012. Proceedings*, pages 58–69, 2012.

3 C. Aiswarya and P. Gastin. Reasoning about distributed systems: WYSIWYG (invited talk). In *FSTTCS Proceedings*, pages 11–30, 2014.

4 S. Akshay, P. Gastin, and S. Krishna. Analyzing timed systems using tree automata. In *CONCUR Proceedings*, 2016.

5 S. Akshay, P. Gastin, S. N. Krishna, and I. Sarkar. Towards an efficient tree automata based technique for timed systems. *CoRR*, abs/1707.02297, 2017. URL: `http://arxiv.org/abs/1707.02297`.

6 R. Alur and D. Dill. A theory of timed automata. *In TCS*, 126(2):183–235, 1994.

7 G. Behrmann, A. David, and K. G. Larsen. A tutorial on Uppaal. In *International School on Formal Methods for the Design of Computer, Communication and Software Systems*, pages 200–236, 2004.

8 A. Bouajjani, R. Echahed, and R. Robbana. On the automatic verification of systems with continuous variables and unbounded discrete data structures. In *Hybrid Systems II*, pages 64–85, 1994.

9 L. Clemente and S. Lasota. Timed pushdown automata revisited. In *LICS Proceedings*, pages 738–749, 2015.

10 B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*, volume 138 of *Encyclopedia of mathematics and its applications*. CUP, 2012.

11 A. Cyriac, P. Gastin, and K. Narayan Kumar. MSO decidability of multi-pushdown systems via split-width. In *CONCUR Proceedings*, pages 547–561, 2012.

12 Guoqiang Li, Xiaojuan Cai, Mizuhito Ogawa, and Shoji Yuen. Nested timed automata. In *Formal Modeling and Analysis of Timed Systems - 11th International Conference, FORMATS 2013, Buenos Aires, Argentina, August 29-31, 2013. Proceedings*, pages 168–182, 2013.

13 P. Madhusudan and G. Parlato. The tree width of auxiliary storage. In *POPL Proceedings*, pages 283–294, 2011.

14 Ashutosh Trivedi and Dominik Wojtczak. Recursive timed automata. In *ATVA Proceedings*, pages 306–324, 2010.

On Petri Nets with Hierarchical Special Arcs

S. Akshay[1], Supratik Chakraborty[2], Ankush Das[3], Vishal Jagannath[4], and Sai Sandeep[5]

1 **Department of Computer Science and Engineering, IIT Bombay, India**
2 **Department of Computer Science and Engineering, IIT Bombay, India**
3 **Computer Science Department, Carnegie Mellon University, Pittsburgh, PA, USA**
4 **Department of Computer Science and Engineering, IIT Bombay, India**
5 **Department of Computer Science and Engineering, IIT Bombay, India**

Abstract

We investigate the decidability of termination, reachability, coverability and deadlock-freeness of Petri nets endowed with a hierarchy of places, and with inhibitor arcs, reset arcs and transfer arcs that respect this hierarchy. We also investigate what happens when we have a mix of these special arcs, some of which respect the hierarchy, while others do not. We settle the decidability status of the above four problems for all combinations of hierarchy, inhibitor, reset and transfer arcs, except the termination problem for two combinations. For both these combinations, we show that deciding termination is as hard as deciding the positivity problem for linear recurrence sequences – a long-standing open problem.

1998 ACM Subject Classification F.1.1 Models of Computation, D.2.2 Design Tools and Techniques: Petri nets

Keywords and phrases Petri Nets, Hierarchy, Reachability, Coverability, Termination, Positivity

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2017.40

1 Introduction

Petri nets are an important and versatile mathematical modeling formalism for distributed and concurrent systems. Thanks to their intuitive visual representation, precise execution semantics, well-developed mathematical theory and availability of tools for reasoning about them, Petri nets are used in varied contexts, viz. computational, chemical, biological, workflow-related etc. Several extensions to Petri nets have been proposed in the literature to augment their modeling power. From a theoretical perspective, these provide rich and interesting models of computation that warrant investigation of their expressive powers, and decidability and/or complexity of various decision problems. From a practitioner's perspective, they enable new classes of systems to be modeled and reasoned about.

In this paper, we focus on an important class of extensions proposed earlier for Petri nets, pertaining to the addition of three types of *special* arcs, namely *inhibitor*, *reset* and *transfer* arcs from places to transitions. We investigate how different combinations of these extensions affect the decidability of four key decision problems: reachability, coverability, termination, and deadlock-freeness. To start with, an inhibitor arc effectively models a zero test, and hence one can model two-counter machines with two inhibitor arcs, leading to undecidability of all of the above decision problems. However, Reinhardt [20] showed that if we impose a *hierarchy* among places with inhibitor arcs (a single inhibitor arc being a subcase), we recover decidability of reachability. Recently Bonnet [5] simplified this proof using techniques of Leroux [14] and also showed that termination and coverability are decidable

for Petri nets with hierarchical inhibitor arcs. With reset arcs (which remove all tokens from a pre-place) and transfer arcs (which transfer all tokens from a pre-place to a post-place), reachability and deadlock-freeness are known to be undecidable [9], athough termination and coverability are decidable [11].

In this paper, we are interested in what happens when *hierarchy* is introduced among all combinations of special arcs. Thus, we specify a hierarchy, or total ordering, of the places, and say that the special arcs respect the hierarchy if whenever there is a special arc from a place p to a transition t, there are also special arcs from every place lower than p in the hierarchy to t. The study of Petri nets extended with hierarchical and non-hierarchical special arcs provides a generic framework that subsumes several existing questions and raises new ones. There are only a handful of results in the literature where hierarchical special arcs have been shown to play an important role. Decidability of reachability for Petri nets with hierarchical inhibitor arcs was shown in [20] and re-visited in a special context in [4], while decidability of termination, coverability and boundedness were shown in [5]. Further, in [2] it was shown that Petri nets with hierarchical zero tests are equivalent to Petri nets with a stack encoding restricted context-free languages. Finally a specific subclass, namely Petri nets with a single inhibitor arc, has received a lot of attention, with results showing decidability of boundededness and termination [10], place-boundedness [6], and LTL model checking [7]. However, in [7], the authors remark that it would not be easy to extend their technique for the last two problems to handle hierarchical arcs. To the best of our knowledge, none of the earlier papers address mixing of reset and transfer arcs within the hierarchy of inhibitor arcs, leaving several interesting questions unanswered. Our primary goal in this paper is to comprehensively fill these gaps. Before delving into the theoretical investigations, we present two examples that illustrate why these models are interesting from a practical point of view.

Our first example is a prioritized job-shop environment in which work stations with possibly different resources are available for servicing jobs. Each job comes with a priority and with a requirement of the count of resources it needs. For simplicity, assume that all resources are identical, and that there is at most one job with any given priority. A work station can service multiple jobs simultaneously subject to availability of resources; however, a job cannot be split across multiple work stations. Additionally, we require that a job with a lower priority must not be scheduled on any work station as long as a job with higher priority is waiting to be scheduled. Once a job gets done, it can either terminate or generate additional jobs with different priorities based on some rules. An example of such a rule could be that a job with prioriy k and resource requirement m can only generate a new job with priority $\leq k$ and resource requirement $\leq m$. Given such a system, there are several interesting questions one might ask. For example, can too many jobs (above a specified threshold) of the lowest priority be left waiting for a work station? Or, can the system reach a deadlocked state from where no progress can be made? A possible approach to answering these questions is to model this prioritized job-shop environment as a Petri net with hierarchical special arcs, i.e., resets, inhibitors and transfers, and reduce the questions to decision problems (such as coverability or deadlock-freeness) for the corresponding nets.

Our second example builds on work reported in the literature on modeling integer programs with loops using Petri nets [3]. Questions pertaining to termination of such programs can be reduced to decision problems (termination or deadlock-freeness) of the corresponding Petri net model. In Section 6.1, we describe a new reduction of the termination question for integer linear loop programs to the termination problem for Petri nets with hierarchical inhibitor and transfer arcs. This underlines the importance of studying decision problems for these extensions of Petri nets.

Our main contribution is a comprehensive investigation into Petri nets extended with a mix of these special arcs, some of which respect the hierarchy, while others do not. We settle the decidability status of the four decisions problems for *all* combinations of hierarchy, inhibitor, reset and transfer arcs, except the termination problem for two combinations. For these cases, we show a reduction from the positivity problem [18, 19] – a long-standing open problem on linear recurrences. We summarize these results in Section 3, after introducing appropriate notations in Section 2. Interestingly, several of our results use completely different constructions and proof techniques, as detailed in Sections 4–6.

2 Preliminaries

We begin by recalling some key definitions and fixing notations. A Petri net, denoted PN, is defined as (P, T, F, M_0), where P is a set of *places*, T is a set of *transitions*, $M_0 : P \to \mathbb{N}$ is the *initial marking*, and $F : (P \times T) \cup (T \times P) \to \mathbb{N}$ is the *flow relation*. For every $x \in P \cup T$, we define $Pre(x) = \{y \in P \cup T \mid F(y, x) > 0\}$ and $Post(x) = \{y \in P \cup T \mid F(x, y) > 0\}$. For every $t \in T$, we use the following terminology: every $p \in Pre(t)$ is a *pre-place* of t, every $q \in Post(t)$ is a *post-place* of t, every arc (p, t) such that $F(p, t) > 0$ is a *pre-arc* of t, and every arc (t, p) such that $F(t, p) > 0$ is a *post-arc* of t.

A *marking* $M : P \to \mathbb{N}$ is a function from the set of places to non-negative integers. We say that a transition t is *firable* at marking M, denoted by $M \xrightarrow{t}$, if $\forall p \in Pre(t), M(p) \geq F(p, t)$. If t is firable at M_1, we say that firing t gives the marking M_2, where $\forall p \in P, M_2(p) = M_1(p) - F(p, t) + F(t, p)$. This is also denoted as $M_1 \xrightarrow{t} M_2$. We define the sequence of transitions $\rho = t_1 t_2 t_3 ... t_n$ to be a *run* from marking M_0, if there exist markings $M_1, M_2, ..., M_n$, such that for all i, t_i is firable at M_{i-1} and $M_{i-1} \xrightarrow{t_i} M_i$. Finally, we abuse notation and use $\leq$ to denote the component-wise ordering over markings. Thus, $M_1 \leq M_2$ iff $\forall p \in P, M_1(p) \leq M_2(p)$. A detailed account on Petri nets can be found in [17].

We now define some classical decision problems in the study of Petri nets.

▶ **Definition 2.1.** Given a Petri net $N = (P, T, F, M_0)$,

- Termination (or TERM): Does there exist an infinite run from marking M_0?
- Reachability (or REACH): Given a marking M, is there a run from M_0 which reaches M?
- Coverability (or COVER): Given a marking M, is there a marking $M' \geq M$ which is reachable from M_0?
- Deadlock-freeness (or DLFREE): Does there exist a marking M reachable from M_0, such that no transition is firable at M?

Since Petri nets are well-structured transition systems (WSTS), the decidability of coverability and termination for Petri nets follows from the corresponding results for WSTS [11]. The decidability of reachability was shown in [13]. Subsequently, there have been several alternative proofs of the same result, viz. [14]. Finally, since deadlock-freeness reduces to reachability in Petri nets [8], all the four decision problems are decidable for Petri nets. In the remainder of the paper, we concern ourselves with these decision problems for Petri nets extended with the following special arcs:

- An Inhibitor arc from place p to transition t signifies t is firable only if p has zero tokens.
- A Reset arc from place p to transition t signifies that p contains zero tokens after t fires.
- A Transfer arc from place p_1 through transition t to place p_2 signifies that on firing transition t, all tokens from p_1 get transferred to p_2.

For Petri nets with special arcs, we redefine the flow relation as $F : (P \times T) \cup (T \times P) \to \mathbb{N} \cup \{I, R\} \cup \{S_p \mid p \in P\}$, where $F(p, t) = I$ (resp. $F(p, t) = R$) signifies the presence of an

inhibitor arc (resp. reset arc) from place p to transition t. Similarly, if $F(p,t) = S_{p'}$, then there is a transfer arc from place p to place p' through transition t.

3 Problem statements and main results

We use PN to denote standard Petri nets, and I-PN, R-PN and T-PN to denote Petri nets with inhibitor, reset and transfer arcs, respectively. The following definition subsumes several additional extensions studied in this paper.

▶ **Definition 3.1.** A Petri net with *hierarchical special arcs* is defined to be a 5-tuple $(P, T, F, \sqsubseteq, M_0)$, where P is a set of places, T is a set of transitions, $\sqsubseteq$ is a total ordering over P encoding the hierarchy, $M_0 : P \to \mathbb{N}$ is the initial marking, and $F : (P \times T) \cup (T \times P) \to \mathbb{N} \cup \{I, R\} \cup \{S_p \mid p \in P\}$ is a flow relation satisfying

- $\forall (t,p) \in T \times P, \;\; F(t,p) \in \mathbb{N}$, and
- $\forall (p,t) \in P \times T, \;\; F(p,t) \notin \mathbb{N} \implies (\forall q \sqsubseteq p, \; F(q,t) \notin \mathbb{N})$

Thus, all arcs (or edges) from transitions to places are as in standard Petri nets. However, we may have special arcs from places to transitions. These can be inhibitor arcs ($F(p,t) = I$), reset arcs ($F(p,t) = R$), or transfer arcs ($F(p,t) = S_{p'}$, where p and p' are places in the Petri net). Note that all special arcs respect the hierarchy specified by $\sqsubseteq$. In other words, if there is a special arc from a place p to a transition t, there must also be special arcs from every place p' to t, where $p' \sqsubseteq p$. Depending on the subset of special arcs that are present, we can define sub-classes of Petri nets with hierarchical special arcs as follows. In the following, $\mathsf{Range}(F)$ denotes the range of the flow relation F.

▶ **Definition 3.2.** The class of Petri nets with hierarchical special arcs, where $\mathsf{Range}(F) \setminus \mathbb{N}$ is a subset of $\{I\}, \{T\}$ or $\{R\}$ is called HIPN, HTPN or HRPN respectively. Similarly, it is called HITPN, HIRPN or HTRPN if $\mathsf{Range}(F) \setminus \mathbb{N}$ is a subset of $\{I, T\}, \{I, R\}$ or $\{T, R\}$ respectively. Finally, if $\mathsf{Range}(F) \setminus \mathbb{N} \subseteq \{I, R, T\}$, we call the corresponding class HIRTPN.

We also study generalizations, in which extra inhibitor, reset and/or transfer arcs that do not respect the hierarchy specified by $\sqsubseteq$, are added to PN with hierarchical special arcs.

▶ **Definition 3.3.** Let $\mathcal{N}$ be a class of Petri nets with hierarchical special arcs as in Definition 3.2, and let $\mathcal{M}$ be a subset of $\{I, T, R\}$. We use $\mathcal{M}$-$\mathcal{N}$ to denote the class of nets obtained by adding unrestricted special arcs of type $\mathcal{M}$ to an underlying net in the class $\mathcal{N}$.

For example, R-HIPN is the class of Petri nets with hierarchical inhibitor arcs extended with reset arcs that need not respect the hierarchy. Clearly, if the special arcs in every net $N \in \mathcal{N}$ are from $\mathcal{M}$, the class $\mathcal{M}$-$\mathcal{N}$ is simply the class of Petri nets with unrestricted (no hierarchy) arcs of type $\mathcal{M}$. Hence we avoid discussing such extensions in the remainder of the paper.

As we show later, all four decision problems of interest to us are either undecidable or not known to be decidable for HIRTPN. A slightly constrained version of HIRTPN, however, turns out to be much better behaved, motivating the following definition.

▶ **Definition 3.4.** The sub-class HIRcTPN is defined to be HIRTPN with the added restriction that $\forall (p, t, p') \in P \times T \times P, \;\; F(p,t) = S_{p'} \implies F(p', t) \in \mathbb{N}$.

Thus, every place p' that has an incoming transfer arc through transition t is constrained to have only a standard PN outgoing arc (if any) to t. This restriction suffices to recover decidability for all four decision problems of interest to us. Since nets in HIRcTPN often suffice to model useful classes of systems, we present results for this class separately.

Table 1 Summary of key results; results for all other extensions are subsumed by these results.

	TERM	COVER	REACH	DLFREE
PN	✓ [11]	✓ [11]	✓ [15, 14]	✓ [8, 12]
R/T-PN	✓ [11]	✓ [11]	✗ [9]	✗ [Red. from [9]]
I-PN	✗ [16]	✗ [16]	✗ [16]	✗ [16]
HIPN	✓ [20, 5]	✓ [20, 5]	✓ [20, 5]	✓ [Thm 4.3]
HTPN	✓ [11]	✓ [11]	✗ [Thm 6.1]	✗ [Thm 6.1]
HIRPN	✓ [Thm 4.2]	✓ [Thm 4.2]	✓ [Thm 4.2]	✓ [Thm 4.3]
HITPN	Positivity-Hard [Thm 6.5]	✗ [Cor. 6.2]	✗ [Thm 6.1]	✗ [Thm 6.1]
HIRcTPN	✓ [Thm 4.2]	✓ [Thm 4.2]	✓ [Thm 4.2]	✓ [Thm 4.3]
R-HIPN	✓[Thm 5.1]	✗[Thm 5.6]	✗[[9], [1]]	✗[Red.frm [9], [1]]
T-HIPN	Positivity-Hard [Thm 6.5]	✗[Thm 5.6]	✗[[9], [1]]	✗[Red.frm [9], [1]]
R-HIRPN	✓[Thm 5.1, Thm 4.2]	✗[Thm 5.6]	✗[[9], [1]]	✗[Red.frm [9], [1]]

Status of decision problems and our contributions. Table 1 summarizes the decidability status of the four decision problems for some classes of Petri net extensions. A ✓ denotes decidability of the corresponding problem, while ✗ denotes undecidability of the problem. The shaded cells present results (and corresponding citations) already known prior to the current work, while the unshaded cells show results (and corresponding theorems) arising from this paper. Note that the table doesn't list all extensions of Petri nets that were defined above. This has been done deliberately and carefully to improve readability. Specifically, for every Petri net extension that is not represented in the table, e.g., R-HITPN, the status of all four decision problems are inferable from others shown in the table. These are explicitly listed out in a longer version of the paper [1], where we also depict the relative expressiveness of these classes. Thus, our work comprehensively addresses the four decision problems for all classes of Petri net extensions considered above.

Interestingly, several of the results use distinct constructions and proof techniques. We point out the salient features of our main results below.

- We *include reset arcs in the hierarchy* of inhibitor arcs in HIPN in Section 4. In Theorem 4.2, we show that we can model reset arcs by inhibitor arcs, while preserving hierarchy. This immediately gives decidability of all problems except DLFREE. Since the reduction may introduce deadlocks, we need a different proof for DLFREE, which we present through in Theorem 4.3. This also proves decidability of DLFREE for HIPN, which to the best of our knowledge, was not known before.
- We *add reset arcs outside the hierarchy* of inhibitor arcs in Section 5. Somewhat counter-intuitively, this class, R-HIPN, does not contain HIRPN and is incomparable to it. This is because all inhibitor arcs in R-HIPN are required to respect the hierarchy, whereas in HIRPN some inhibitor arcs can be replaced by resets, thereby violating the requirement of hierarchy among inhibitor arcs. Using a new and surprisingly simple construction of an extended finite reachability tree (FRT) which keeps track of the hierarchical inhibitor information and modifies the subsumption condition, we show in Theorem 5.1 that termination is decidable for R-HIPN. This result has many consequences. In particular, it implies an arguably simple proof for the special case of a single inhibitor arc, which was solved in [10] (using a different method of extending FRTs). In Theorem 5.6, we use a reduction from two counter machines to show that coverability is undecidable with as few as 2 reset arcs and an inhibitor arc in the absence of hierarchy.
- Finally, we *consider transfer arcs inside and outside the hierarchy* in Section 6. In Theorem 6.1, we show that unlike for reset arcs, including transfer arcs in the hierarchy

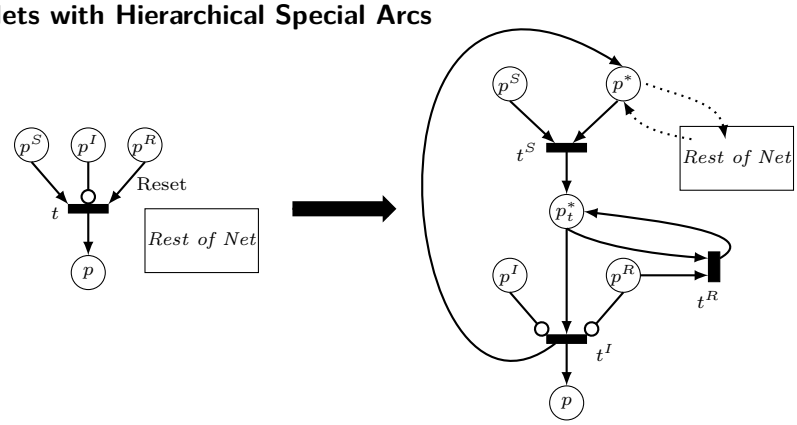


Figure 1 Transformation from $N \in \mathsf{HIRPN}_k$ (left) to $N' \in \mathsf{HIRPN}_{k-1}$ (right).

of inhibitor arcs does not necessarily preserve decidability. For both HITPN and T-HIPN, while coverability, reachability and deadlock-freeness are undecidable, we are unable to show such a result for termination. Instead, in Theorem 6.5, we show that we can reduce a long-standing open problem on linear recurrences to this problem.

4 Adding Reset Arcs with Hierarchy to HIPN

In this section, we extend hierarchical inhibitor nets [20] with reset arcs respecting hierarchy. Subsection 4.1 presents a reduction from HIRPN to HIPN that settles the decidability of termination, coverability and reachability for HIRPN. As this reduction does not work for deadlock-freeness (since it introduces new deadlocked markings), we present a separate reduction from deadlock-freeness to reachability for HIRPN in subsection 4.2.

4.1 Reduction from HIRPN to HIPN

Let HIRPN_k be the sub-class of Petri nets in HIRPN with at most k transitions having one or more reset pre-arcs. We first show that termination, reachability and coverability for HIRPN_k can be reduced to the corresponding problems for HIRPN_{k-1}, for all $k > 0$. This effectively reduces these problems for HIRPN to the corresponding problems for HIRPN_0 (or HIPN), which are known to be decidable [20, 5]. In the following, we use $\mathsf{Markings}(N)$ to denote the set of all markings of a net N.

▶ **Lemma 4.1.** *For every net N in HIRPN_k, there is a net N' in HIRPN_{k-1} and a mapping $f : \mathsf{Markings}(N) \to \mathsf{Markings}(N')$ that satisfy the following:*

- *For every $M_1, M_2 \in \mathsf{Markings}(N)$ such that M_2 is reachable from M_1 in N, the marking $f(M_2)$ is reachable from $f(M_1)$ in N'.*
- *For every $M_1', M_2' \in \mathsf{Markings}(N')$ such that $M_1' = f(M_1)$, $M_2' = f(M_2)$ and M_2' is reachable from M_1' in N', the marking M_2 is reachable from M_1 in N.*

Proof Sketch. To see how N' is constructed, consider an arbitrary transition, say t, in N with one or more reset pre-arcs. We replace t by a gadget in N' with no reset arcs, as shown in Figure 1. The gadget has two new places labeled p^* and p_t^*, with every transition in "Rest of Net" having a simple pre-arc from and a post-arc to p^*, as shown by the dotted arrows in Figure 1. The gadget also has a new transition t^S with simple pre-arcs from p^* and from every place p^S that has a simple arc to t in N. It also has a new transition labeled t^R for every reset arc from a place p^R to t in N. Thus, if there are n reset pre-arcs of t in N, the gadget has n transitions $t_1^R, \ldots t_n^R$. As shown in Figure 1, each such t_i^R has simple pre-arcs

from p_i^R and p_t^* and a post-arc to p_t^*. Finally, the gadget has a new transition labeled t^I with a simple pre-arc from p_t^* and inhibitor pre-arcs from all places p^I (resp. p^R) that have inhibitor (resp. reset) arcs to t in N.

The ordering $\sqsubseteq'$ of places in N' is obtained by extending the ordering $\sqsubseteq$ of N as follows: for each place p in N, we have $p \sqsubseteq' p_t^* \sqsubseteq' p^*$. Clearly, $N' \in \mathsf{HIRPN}_{k-1}$, since it has one less transition (i.e. t) with reset pre-arcs compared to N. It is easy to check that if the reset and inhibitor arcs in N respect $\sqsubseteq$, then the reset and inhibitor arcs in N' respect $\sqsubseteq'$ as well.

The mapping function $f : \mathsf{Markings}(N) \to \mathsf{Markings}(N')$ is defined as follows: for every place p in N', $f(M)(p) = M(p)$ if p is in N; otherwise, $f(M)(p^*) = 1$ and $f(M)(p_t^*) = 0$. The initial marking of N' is given by $f(M_0)$, where M_0 is the initial marking of N. Given a run in N, it is now easy to see that every occurrence of t in the run can be replaced by the sequence $t^S(t^R)^*t^I$ (the t^R transitions fire until the corresponding place p^R is emptied) and vice-versa. Further details of the construction are given in [1], where it is also shown that N can reach M_2 from M_1 iff N' can reach $f(M_2)$ from $f(M_1)$. ◀

In fact, the above construction can be easily adapted for HIRcTPN as well. Specifically, if we have a transfer arc from place p_x to place p_y through t, we add a new transtion $t_{x,y}^T$ with simple pre-arcs from p_t^* and p_x, and with simple post-arcs to p_t^* and p_y to the gadget shown in Figure 1. Furthermore, we add an inhibitor arc from p_x to t^I, like the arc from p^R to t^I in Figure 1. Note that the *constrained* property of transfer arcs is required here, since if we had an inhibitor arc from p_y to t in original net (hence, p_y to t^I in construction), then in the constructed net, t^I cannot be fired, since we would have added tokens in p_y through $t_{x,y}^T$. This allows us to obtain a net in HIRcTPN with at least one less transition with reset pre-arcs or transfer arcs, such that the reachability guarantees in Lemma 4.1 hold.

Finally, by repeatedly applying Lemma 4.1, we have,

▶ **Theorem 4.2.** *Termination, reachability and coverability for HIRPN and HIRcTPN are decidable.*

4.2 Reducing Deadlock-freeness to Reachability in HIRPN

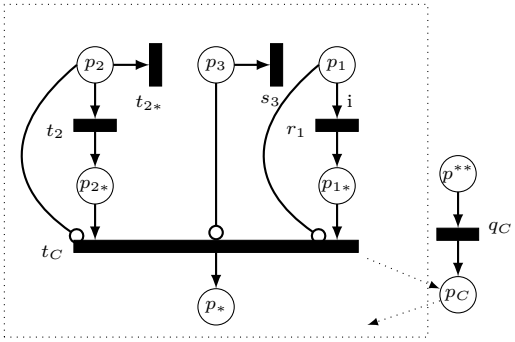
The overall idea behind our reduction is to add transitions that check whether the net is deadlocked, and to put a token in a special place, say p^*, if this is indeed the case. Note that for a net to be deadlocked, the firing of each of its transitions must be disabled. Intuitively, if M denotes a marking of a net and if T denotes the set of transitions, then $\mathsf{Deadlock}(M) = \bigwedge_{t_i \in T} \mathsf{NotFire}_i(M)$, where $\mathsf{Deadlock}(M)$ is a predicate indicating if the net is deadlocked in M, and $\mathsf{NotFire}_i(M)$ is a formula representing the enabledness of transition t_i in M. For a transition t to be disabled, atleast one of its pre-places p must fail the condition on that place for t to fire. There are three cases to consider here.

- $F(p,t) \in \mathbb{N}$: For t to be disabled, we must have $M(p) < F(p,t)$
- $F(p,t) = I$: For t to be disabled, we must have $M(p) > 0$.
- $F(p,t) = R$: Place p cannot disable t

Suppose we define $Exact_j(p) \equiv (M(p) = j)$ and $AtLeast(p) \equiv (M(p) > 0)$. Clearly, $\mathsf{NotFire}_i(M) = \bigvee_{(p,t)\in F} Check(p)$, where $Check(p) = AtLeast(p)$ if $F(p,t) = I$, and $Check(p) = \bigvee_{j<k} Exact_j(p)$ if $F(p,t) = k \in \mathbb{N}$. The formula for $\mathsf{Deadlock}(M)$ (in CNF above) is now converted into DNF by distributing conjunctions over disjunctions. Given a HIRPN, we now transform the net, preserving hierarchy, so as to reduce checking $\mathsf{Deadlock}(M)$ in DNF in the original net to a reachability problem in the transformed HIRPN.

Every conjunctive clause in the DNF of $\mathsf{Deadlock}(M)$ is a conjunction of literals of the form $AtLeast(p)$ and $Exact_j(p)$. Let S_C be the set of all literals in a conjunctive clause C,

and let P be the set of all places in the net. Define $B_i^C = \{p \in P \mid Exact_i(p) \in S_C\}$ and $A^C = \{p \in P \mid AtLeast(p) \in S_C\} \setminus \bigcup_{i \geq 1} B_i^C$. We only need to consider conjunctive clauses where the sets B_i^C are pairwise disjoint (other clauses can never be true). Similarly, we only need to consider conjunctive clauses where B_0^C and A^C are disjoint. We add a transition for each conjunctive clause that satisfies the above two properties. By definition, A^C and B_i^C are disjoint for all $i \geq 1$. Thus, the sets A^C and $B_i^C (i \geq 0)$ are pairwise disjoint for every conjunctive clause we consider.



Given the original HIRPN net, for each conjunctive clause considered, we perform the construction as shown in the above figure. For every place $p_a \in A^C$, we add a construction as for p_2 in above diagram. For every place $p_i \in B_i^C$, we add a construction as for p_1 in above diagram. For all places $p \notin A^C \cup \bigcup_i B_i^C$, we add a construction as for p_3 in above diagram. We call the transition t_C in the above diagram as the "Check Transition", and refer to the set of transitions $r_i, s_i, t_i, t_{i*}, t_C$ (excluding q_C) as *transitions for clause C*. Note that for any $p_i \in P$, exactly one of r_i, s_i, t_i exist since the sets A^C and the sets B_i^C are all pairwise disjoint.

Our construction also adds two new places, p_C and p^{**}, and one new transition q_C such that

- there is a pre-arc and a post-arc of weight 1 from p^{**} to every transition in the original net. Thus, transitions in original net can fire only if p^{**} has a token.
- there is a pre-arc of weight 1 from p_c to each transition for clause C (within dotted box).
- there is a post-arc of weight 1 to p_c from every transition for clause C (within dotted box), except from t_C to p_C.

Note that hierarchy is preserved in the transformed net, since the only new transitions which have inhibitor/reset arcs are the check transitions, which have inhibitor arcs from all places in the original net. Let N be the original net in HIRPN with P being its set of places, and let N' be the transformed net, also in HIRPN, obtained above. Define a mapping $f : \mathsf{Markings}(N) \to \mathsf{Markings}(N')$ as follows: $f(M)(p) = M(p)$ if $p \in P$; $f(M)(p^{**}) = 1$ and $f(M)(p) = 0$ in all other cases. If M_0 is the initial marking in N, define $M_0' = f(M_0)$ to be the initial marking in N'.

▶ Claim 4.1. The marking $M_\star'$ of N', defined as $M_\star'(p) = 1$ if $p = p^*$ and $M_\star'(p) = 0$ otherwise, is reachable from M_0' in N' iff there exists a deadlocked marking reachable from M_0 in N.

From, the above reduction (see [1] for proof details) and from the decidability of reachability for HIRPN we then have the following.

▶ **Theorem 4.3.** *Deadlock-freeness for HIRPN is decidable.*

5 Adding Reset Arcs without Hierarchy

The previous section dealt with extension of Petri nets where reset arcs were added within the hierarchy of the inhibitor arcs. This section discusses the decidability results when we add reset arcs outside the hierarchy of inhibitor arcs.

5.1 Termination in R-HIPN

Our main idea here is to use a modified finite reachability tree (FRT) construction to provide an algorithm for termination in R-HIPN. The usual FRT construction [11] for Petri nets does not extend to Petri nets with even a single (and hence also hierarchical) inhibitor arc.

▶ **Theorem 5.1.** *Termination is decidable for R-HIPN.*

Consider a R-HIPN net $(P, T, F, \sqsubseteq, M_0)$. We start by introducing a few definitions. For any place $p \in P$, we define the *index of the place p (Index(p))* as the number of places $q \in P$ such that $q \sqsubseteq p$. The definition of Index over places induces an Index among transitions too: For any transition $t \in T$, its *index* is defined as $Index(t) = \max_{F(p,t)=I} Index(p)$ By convention, if there is no such place, then $Index(t) = 0$. Given markings M_1 and M_2 and $i \in \mathbb{N}$, we say that M_1 and M_2 are *i-Compatible (denoted* $Compat_i(M_1, M_2)$*)* if $\forall p \in P$, $(Index(p) \leq i \implies M_1(p) = M_2(p))$.

▶ **Definition 5.2.** Consider a run $M_2 \xrightarrow{\rho} M_1$. Let $t^* = \text{argmax}_{t \in \rho} Index(t)$. We define $Subsume(M_2, M_1, \rho) = M_2 \leq M_1 \wedge \left(Compat_{Index(t^*)}(M_1, M_2) \right)$

To understand this definition note that if ρ can be fired at M_2 and reaches M_1 and if $Subsume(M_2, M_1, \rho)$ is true, then ρ can be fired at M_1 again. In classical Petri nets without inhibitor arcs, $Subsume(M_2, M_1, \rho) = M_2 \leq M_1$, and hence this is the classical monotonicity condition. However, this condition may differ in the presence of even a single inhibitor arc.

Given a net $N = (P, T, F, \sqsubseteq, M_0)$ in R-HIPN, we define the *Extended Reachability Tree* $ERT(N)$ as a directed unordered tree where the nodes are labelled by markings $M : P \to \mathbb{N}$, rooted at $n_0 : M_0$ (initial marking). If $M_1 \xrightarrow{t} M_2$ for some markings M_1 and M_2 and transition $t \in T$, then a node marked by $n' : M_2$ is a child of the node $n : M_1$. Consider any node labelled M_1. If along the path from root $n_0 : M_0$ to $n : M_1$, there is a marking $n' : M_2$ $(n \neq n')$, such that the path from $n' : M_2$ to $n : M_1$ corresponds to a run ρ and $Subsume(M_2, M_1, \rho)$ is true, then M_1 is made a leaf node. We call this a *subsumed* leaf node. Note that leaf nodes in this tree are of two types: either leaf nodes caused by subsumption as above or leaf nodes due to deadlock, where no transition is fireable.

▶ **Lemma 5.3.** *For any net* $N = (P, T, F, \sqsubseteq, M_0)$ *in R-HIPN,* $ERT(N)$ *is finite.*

Proof. Assume the contrary. By Konig's Lemma, there is an infinite path. Let the infinite path correspond to a run $\rho = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \ldots \xrightarrow{t_i} M_i \ldots$.

Let $t \in T$ be the transition which has maximum index among the transitions which are fired infinitely often in run ρ. Thus all transitions having higher index than Index(t) fire only finitely many times. Let b be chosen such that $\forall i \geq b$, $Index(t_i) \leq Index(t)$ (i.e. b is chosen after the last position where any transition with index higher than Index(t) fires). This exists by the definition of t. Since t is fired infinitely often, the sequence $\{M_i \mid i > b \wedge t_{i+1} = t\}$ is an infinite sequence. As $\leq$ over markings is a well-quasi ordering, there exist two markings M_i and M_j, such that both belong to the above sequence (i.e. $t_{i+1} = t_{j+1} = t$), $M_i \leq M_j$ and $i < j$. Since $t_{i+1} = t_{j+1} = t$, and t fires at M_i and M_j, we have $\forall p \in P$, $Index(p) \leq$

$Index(t) \implies M_i(p) = M_j(p) = 0$. Thus, $Compat_{Index(t)}(M_i, M_j)$ is true. Note that t is the maximum index transition fired in the run from M_i to M_j, since no higher index transition fires after position b and $j > i > b$. Hence, $Subsume(M_i, M_j, \rho')$ is true, where ρ' is the run from M_i to M_j. But then, the path would end at M_j, giving a contradiction. ◀

Thus, we have shown that the ERT is always finite. Next, we will show a crucial property of $Compat_i$, which will allow us to check for a non-terminating run.

▶ **Lemma 5.4.** *Consider markings M_1 and M_2 such that $M_1 \leq M_2$. Let $i \in \mathbb{N}$ be such that we have $Compat_i(M_1, M_2)$. Then for any run ρ over the set of transitions $T_i = \{t \mid t \in T \wedge Index(t) \leq i\}$, if $M_1 \xrightarrow{\rho} M_1'$, then $M_2 \xrightarrow{\rho} M_2'$, where $M_1' \leq M_2'$ and $Compat_i(M_1', M_2')$.*

Proof. We prove this by induction. We first prove that t is firable at M_2. If $F(p,t) \in \mathbb{N}$, then $M_2(p) \geq M_1(p) \geq F(p,t)$. If $F(p,t) = I$, i.e., it is an inhibitor arc, then $Index(p) \leq Index(t) \leq i$. But now, since $Compat_i(M_1, M_2)$ holds and t is firable at M_1, we obtain $M_2(p) = M_1(p) = 0$. Finally, if $F(p,t) = R$, i.e., it is a reset arc, then t can fire regardless of the value of $M_2(p)$. Hence, t is firable at M_2.

Now let $M_2 \xrightarrow{t} M_2'$. Then, for all $p \in P$, $M_2'(p) = M_2(p) - F(p,t) + F(t,p)$ and $M_1'(p) = M_1(p) - F(p,t) + F(t,p)$. Since $F(t,p)$ is constant and $F(p,t)$ can depend only on number of tokens in place p (so, if $M_1(p)$ and $M_2(p)$ were equal before firing, they remain equal now), we obtain that $Compat_i(M_1', M_2')$ and $M_1' \leq M_2'$. ◀

▶ **Lemma 5.5.** *A net N in R-HIPN has a non-terminating run iff $ERT(N)$ has a subsumed leaf node.*

Proof. In the forward direction, consider a non-terminating run. This run has a finite prefix in $ERT(N)$. This prefix ends in a leaf that is not a deadlock (as some transition is firable). Thus it is a subsumed leaf node. In the reverse direction, we now show that if $ERT(N)$ has a subsumed leaf node, then N has a non-terminating run. To see this, consider any subsumed leaf node labeled by marking M_2. Let M_1 be a marking along the path M_0 to M_2, and ρ be the run from M_1 to M_2, such that $Subsume(M_2, M_1, \rho)$ is true. Hence, we have $M_1 \xrightarrow{\rho} M_2$. Take $t^* = argmax_{t \in \rho} Index(t)$ and $i = Index(t^*)$. Since $Subsume(M_1, M_2, \rho)$ is true, we have $M_1 \leq M_2$ and $Compat_i(M_1, M_2)$ is true. We also have that ρ is a run over $T_i = \{t \mid t \in T \wedge Index(t) \leq i\}$ (by definition of i). Thus, by Lemma 5.4, we have $M_2 \xrightarrow{\rho} M_3$, where $M_2 \leq M_3$ and $Compat_i(M_2, M_3)$ is true. Hence, ρ can be fired again at M_3 and so on, resulting in a non-terminating run. ◀

Finally, we observe that checking $Subsume(M_2, M_1, \rho)$ is also easily doable. Thus, for any net in R-HIPN, one can construct its extended reachability tree and decide the termination problem using the ERT. This completes the proof of the theorem. We observe here that this construction cannot be immediately lifted to checking boundedness due to the presence of reset arcs. However, we can lift this to check for termination in HIRPN and R-HIRPN as well as to check boundedness in HIPN.

5.2 Coverability in R-HIPN

While termination turned out to be decidable, reachability is undecidable for R-HIPN in general (since it subsumes reset Petri nets). Indeed, it is shown in [9] that reachability is undecidable for Petri nets with 2 reset arcs. Using a similar strategy, in [1], we tighten the undecidability result to show that reachability in Petri nets with one inhibitor arc and one

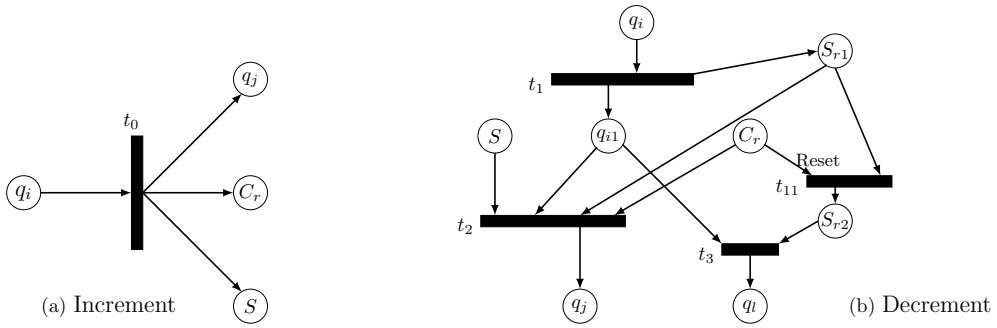


Figure 2 Modeling a Minsky Machine (unlabelled edges have weight 1).

reset arc is undecidable. Further, we can modify the construction presented to show that deadlock-freeness in Petri nets with one reset arc and one inhibitor arc is also undecidable.

Next we turn our attention to the coverability problem.

▶ **Theorem 5.6.** *Coverability is undecidable for Petri nets with two reset/transfer arcs and an inhibitor arc.*

The rest of this section proves the above theorem. To do this, we construct a Petri net with two reset arcs and one inhibitor arc that simulates a two-counter Minsky Machine. A Minsky Machine M has a finite set of instructions q_i for $0 \leq i \leq n$, where q_0 is the initial instruction and q_n is the final instruction i.e. there are no transition rules from q_n. There are two counters C_1 and C_2 and we have two types of instructions. For each counter $r \in \{1, 2\}$,

1. INC(r, j): Increase C_r by 1 and go to q_j.
2. JZDEC(r, j, l): If C_r is zero, go to q_l, else decrease C_r by 1 and go to q_j.

We construct a Petri net P such that the runs of P encode the runs of the Minsky Machine M. We use places q_i $(0 \leq i \leq n)$ to encode each instruction. The place q_i gets a token when we simulate instruction i in the Minsky Machine. We use two places C_1 and C_2 to store the number of tokens corresponding to counter values in C_1 and C_2 in the counter machine. We use a special place S which stores the sum of C_1 and C_2. Figure 2(a) shows the construction for the increment instruction "Increase C_r by 1 and go to q_j". When q_i gets a token, the transition is fired, C_r and S are incremented by 1 and q_j gets the token to proceed.

Next, we show how to simulate the decrement (along with zero check) instruction "if $C_r = 0$, then go to q_ℓ, else decrease C_r by 1 and go to q_j" by introducing non-determinism in the Petri net. The gadget for this is shown in Figure 2(b). When we reach a decrement with zero check instruction, we guess whether C_r is zero, and if so, fire t_{11} and then t_3. Else, we decrement C_r by 1 and fire t_2. We have two cases. **Case 1**: If C_r is actually zero, it runs correctly as t_2 would not fire. The transition t_3 fires and C_r remains zero. In addition, q_l gets the token. **Case 2** : If C_r has non-zero tokens, both transitions can fire. But runs in which t_3 fires are "wrong" runs. We call such transitions as *incorrect*. The crucial point is that in runs with incorrect transitions, S is not decremented whereas C_r is decremented. Hence $M(S) \neq M(C_1) + M(C_2)$ in markings reached by runs with incorrect transitions.

Note that in any run of the Petri net P, q_i and only q_i gets a token when the instruction numbered i is being simulated. The following lemma proves the correctness of the reduction.

▶ **Lemma 5.7.** *In every run of P reaching marking M, $M(S) \geq M(C_1) + M(C_2)$. Furthermore, $M(S) = M(C_1) + M(C_2)$ iff there are no incorrect transitions.*

If the Minsky Machine reaches instruction q_n, the Petri net P gets a token in the place q_n. But, if the Minsky Machine doesn't reach q_n, it is possible that the place q_n in Petri net P gets a token because of incorrect transitions. By the above lemma, to check if there had been any incorrect transitions along the run, we just check at the end (at q_n) if $M(S) = M(C_1)+M(C_2)$. This can done using an inhibitor arc. Thus q_{n+1} gets tokens iff the Minsky Machine reaches the instruction q_n. Hence reaching instruction q_n in Minsky Machine is equivalent to asking if we can cover the marking in which all places except q_{n+1} have 0 tokens and q_{n+1} has 1 token. Since checking reachability in Minksy machines is undecidable, this shows that checking coverability in Petri nets with 2 reset and an inhibitor arc is undecidable.

Note that the above proof also shows undecidability of coverability in Petri nets with 2 transfer arcs and an inhibitor arc. Additional details, the inhibitor arc construction and the extension to transfer arcs can be found in the long version of this paper [1]. Finally, the problem of coverability in Petri nets with 1 inhibitor arc and 1 reset arc is open.

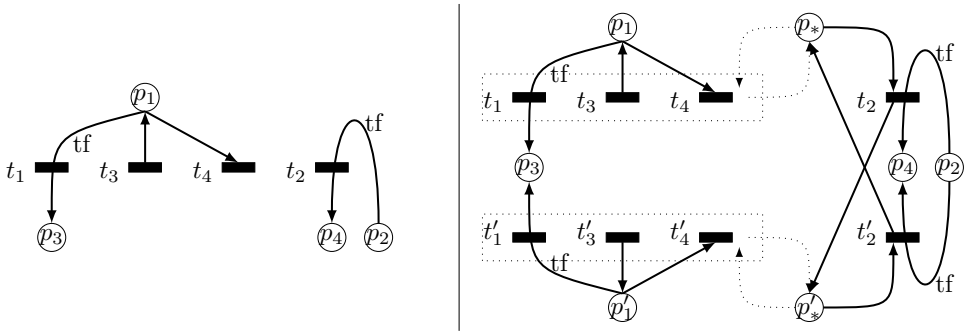
▶ Problem 1. Is coverability in Petri nets with 1 reset arc and 1 inhibitor arc decidable ?

6 Adding Transfer Arcs within and without Hierarchy

We show a reduction from Petri nets with 2 (non-hierarchical) transfer arcs to HTPN preserving reachability and deadlock-freeness. Since reachability and deadlock-freeness in Petri nets with 2 transfer arcs are undecidable [9], they are undecidable in HTPN too.

▶ **Theorem 6.1.** *Reachability and deadlock-freeness are undecidable in HTPN.*

Proof. Let N be a Petri net with 2 transfer arcs as shown in the Figure below. Here, one transfer arc is from p_1 to p_3 via t_1, while the other is from p_2 to p_4 via t_2. The transitions t_3 and t_4 are representative of other transitions to and from p_1. W.l.o.g. we assume that there is no arc from p_1 to t_2. If this is not the case, we can add a place and transition in between to create an equivalent net without adding any deadlocked reachable marking (see [1] for details). The construction of the corresponding HTPN is shown in the diagram below.



Six transfer arcs have not been shown in the construction above for clarity. These are:

- From p_1 to p_3 through t'_1.
- From p_1 to p'_1 through t_2.
- From p_1 to p'_1 through t'_2.
- From p'_1 to p_3 through t_1.
- From p'_1 to p_1 through t_2.
- From p'_1 to p_1 through t'_2.

These arcs ensure that hierarchy is respected by the transfer arcs, where $p_1 < p_1' < p_2$ in the hierarchy. The dotted arc from p_* to the upper dotted box represents a pre-arc from p_* to every transition in the box. Similarly, we have an arc from every transition in the upper dotted box to p_*. Dotted arcs between the lower dotted box and p_*' are interpreted similarly. The intuitive idea behind the construction is to represent the place p_1 in the original net by two places p_1 and p_1' in the modified net. At every marking, p_1 of the original net is represented by one of the two places p_1 or p_1' in the modified net. Places p_*' and p_* are used to keep track of which place represents p_1 in the current marking. Everytime the transition t_2 fires, the representative place swaps. Let the original net be (P, T, F) and the constructed net be (P', T', F'). The initial marking M_0' is given by $M_0'(p_*) = 1$, $M_0'(p_*') = M_0'(p_1') = 0$, and $M_0'(p) = M_0(p)$ for all other $p \in P$. Now, given marking M of original net, we define the set $S^{ext} = \{A_M, B_M\}$, where,

$$A_M(p) = \begin{cases} M(p) & p \notin \{p_1, p_1', p_*', p_*\} \\ 1 & p = p_* \\ 0 & p \in \{p_*', p_1'\} \\ M(p_1) & p = p_1 \end{cases}$$

$$B_M(p) = \begin{cases} M(p) & p \notin \{p_1, p_1', p_*', p_*\} \\ 1 & p = p_*' \\ 0 & p \in \{p_*, p_1\} \\ M(p_1) & p = p_1' \end{cases}$$

▶ Claim 6.1. Marking A_M or B_M is reachable from M_0' in the constructed net iff marking M is reachable from M_0 in the original net.

The proof of the claim is given in [1]. The proof of Theorem 6.1 follows from this claim ◀

▶ **Corollary 6.2.** *Coverability is undecidable in HITPN.*

Proof. From Theorem 5.6, it follows that coverability is undecidable in Petri nets with two transfer arcs and one inhibitor arc. Now, we can perform a construction similar to the one above to reduce the coverability of this net to the coverability problem of a net in HITPN. ◀

6.1 Hardness of Termination in HITPN

Termination in HIRcTPN was shown to be decidable in Section 4.1. Termination in HTPN is also decidable, as it is known that termination in transfer Petri nets is decidable. However, termination in HITPN, which subsumes the above two problems, is as hard as the positivity problem – a long standing open problem about linear recurrent sequences ([19],[18]). We prove this by reducing the positivity problem to termination in HITPN.

▶ **Definition 6.3** (Positivity Problem)**.** Given a matrix $M \in \mathbb{Z}^{n \times n}$ and a vector $v_0 \in \mathbb{Z}^n$, is $M^k v_0 \geq 0$ for all $k \in \mathbb{N}$?

Given matrix $M \in \mathbb{Z}^{n \times n}$ and a vector $v_0 \in \mathbb{Z}^n$, we construct a net $N \in$ HITPN such that N does not terminate the coverability problem of a net in HITPN iff $M^k v_0 \geq 0$ for all $k \in \mathbb{N}$. Consider the following while loop program `v = v0; while (v >= 0) v = Mv`. Clearly, this program is non-terminating iff $M^k v_0 \geq 0$ for all k. We construct a net N which simulates this linear program. The net N contains two phases, a forward phase that has the effect of multiplying v by M, and a backward phase that mimics assigning the new vector $M \cdot v$

$$M = \begin{bmatrix} 1 & -4 & 7 \\ 2 & -5 & -8 \\ -3 & -6 & 9 \end{bmatrix}$$

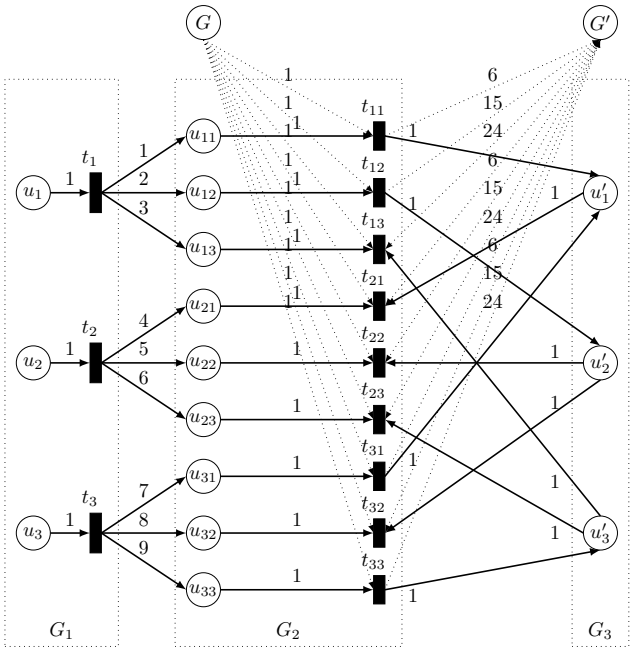


Figure 3 Forward phase (right) simulating matrix M (left).

computed in the forward phase back to v. We also check for non-negativity in the backward phase, and design the net N to terminate if any component becomes negative.

Forward Phase. The construction of the forward phase Petri net for a general matrix is explained below. An example of the construction is shown in Figure 3. We have n places, $u_1, u_2, \ldots, u_n$, corresponding to the n components of vector v. Each place u_i is connected to a transition t_i with a pre-arc of weight 1. Each t_i also has a post-arc to a new place u_{ij} for $1 \leq i, j \leq n$ with a weight $|M_{ji}|$, i.e. the absolute value of the $(j,i)^{th}$ entry of matrix M, corresponding to v_i's contribution to the new value of v_j. Finally, we have places $u'_1, u'_2, \ldots, u'_n$, corresponding to the n components of the new value of vector v. Each place u'_j is connected to place u_{ij} by a transition t_{ij}, with both the arcs being weighted 1. If $M_{ji} \geq 0$, we make use of the fact that u_{ij} has a pre-arc to t_{ij} and t_{ij} has a post-arc to u'_j. This has the effect of adding the value of u_{ij} to u'_j. On the other hand, if $M_{ji} < 0$, then we make use of the fact that both u_{ij} and u'_j have pre-arcs to t_{ij} to subtract u_{ij} from u'_j. The above run simulates the forward phase, in effect multiplying the vector v, represented by the column of u_i in Figure 3, by M, and storing the new components in the column on u'_i. To simulate the while loop program, we need to copy back each u'_i to u_i, while performing the check that each u'_i is non-negative.

Backward phase. The copy back in the backward phase (Figure 4) is effected by a transfer arc from u'_i to u_i via transition t_R. To ensure that the backward phase starts only after the forward phase completes (else, partially computed values would be copied back), we introduce a new place G. The place G stores as many tokens as the total number of times each transition t_{ij} fires, and has a pre-arc weighted 1 to each transition t_{ij}. The emptiness of G ensures that each t_{ij} has completed its firings in the current loop iteration. An inhibitor arc from G to t_R ensures that the forward phase completes before t_R fires. We introduce place G' which computes the initial value of G for the next loop iteration. G' has an arc

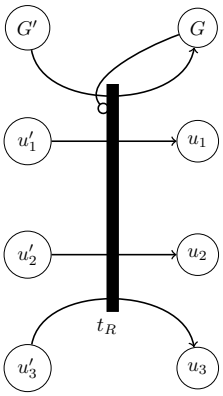


Figure 4 Backward phase: Arc from G to t_R is an inhibitor arc, rest are transfer arcs.

connected to t_{ij} with weight $\sum_{k=1}^{n} |M_{kj}|$. If u'_j has a pre-arc to t_{ij}, then G' has a pre-arc to t_{ij}, while if t_{ij} has a post-arc to u'_j, then it also has a post-arc to G'. Finally, there is a transfer arc from G' to G via t_R. Once the forward phase finishes, the place G becomes empty. Hence, the only transition that can fire is t_R, which completes the backward phase in one firing. Combining the forward and backward phases, we obtain a net N which simulates the while loop program. The initial marking assigns $(v_0)_i$, i.e. the i-th component of vector v_0, to place u_i, and $\sum_{1\leq i\leq n}(\sum_{1\leq j\leq n} |M_{ji}|)(v_0)_i$ tokens to G, while all other places are assigned 0 tokens. The following lemma (see [1] for details) relates termination of N with the Positivity problem.

▶ **Lemma 6.4.** *There exists a non-terminating run in N iff $M^k v_0 \geq 0$ for all $k \in \mathbb{N}$.*

Note that the net N constructed above has only one transition with inhibitor and transfer arcs; hence N is in T-HIPN as well as in HITPN. Thus, we have,

▶ **Theorem 6.5.** *Termination in HITPN as well as T-HIPN is as hard as positivity problem.*

7 Conclusion

In this paper, we investigated the effect of hierarchy on Petri nets extended with not only inhibitor arcs (as classically considered), but also reset and transfer arcs. For four of the standard decision problems, we settled the decidability for almost all these extensions using different reductions and proof techniques. As future work, we are interested in questions of boundedness and place-boundedness in these extended classes. We would also like to explore further links to problems on linear recurrences. We leave open one technical question of coverability for Petri nets with 1 reset and 1 inhibitor arc (without hierarchy).

Acknowledgments. We thank Alain Finkel and Mohamed Faouzi Atig for interesting and insightful discussions and pointers to earlier results and reductions.

References

1 S. Akshay, S. Chakraborty, A. Das, V. Jagannath, and S. Sandeep. On Petri Nets with Hierarchical Special Arcs. *ArXiv e-prints*, July 2017. arXiv:1707.01157.

2 Mohamed Faouzi Atig and Pierre Ganty. Approximating Petri net reachability along context-free traces. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2011, December 12-14, 2011, Mumbai, India*, pages 152–163, 2011.

3 Amir M. Ben-Amram, Samir Genaim, and Abu Naser Masud. On the termination of integer loops. *ACM Trans. Program. Lang. Syst.*, 34(4):16:1–16:24, December 2012.

4 Rémi Bonnet. The reachability problem for vector addition system with one zero-test. In *Mathematical Foundations of Computer Science 2011 - 36th International Symposium, MFCS 2011, Warsaw, Poland, August 22-26, 2011. Proceedings*, pages 145–157, 2011.

5 Rémi Bonnet. Theory of well-structured transition systems and extended vector-addition systems. *These de doctorat, ENS Cachan, France*, 2013.

6 Rémi Bonnet, Alain Finkel, Jérôme Leroux, and Marc Zeitoun. Place-boundedness for vector addition systems with one zero-test. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India*, pages 192–203, 2010.

7 Rémi Bonnet, Alain Finkel, Jérôme Leroux, and Marc Zeitoun. Model checking vector addition systems with one zero-test. *Logical Methods in Computer Science*, 8(2), 2012.

8 Allan Cheng, Javier Esparza, and Jens Palsberg. Complexity results for 1-safe nets. *Theor. Comput. Sci.*, 147(1&2):117–136, 1995. doi:10.1016/0304-3975(94)00231-7.

9 Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset nets between decidability and undecidability. In *Automata, Languages and Programming, 25th International Colloquium, ICALP'98, Aalborg, Denmark, July 13-17, 1998, Proceedings*, pages 103–115, 1998. doi:10.1007/BFb0055044.

10 Alain Finkel and Arnaud Sangnier. Mixing coverability and reachability to analyze VASS with one zero-test. In *SOFSEM 2010: Theory and Practice of Computer Science, 36th Conference on Current Trends in Theory and Practice of Computer Science, Spindleruv Mlýn, Czech Republic, January 23-29, 2010. Proceedings*, pages 394–406, 2010.

11 Alain Finkel and Philippe Schnoebelen. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1-2):63–92, 2001.

12 Michel Hack. The recursive equivalence of the reachability problem and the liveness problem for Petri nets and vector addition systems. In *15th Annual Symposium on Switching and Automata Theory, New Orleans, Louisiana, USA, October 14-16, 1974*, pages 156–164, 1974. doi:10.1109/SWAT.1974.28.

13 S. Rao Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing*, STOC '82, pages 267–281, New York, NY, USA, 1982. ACM. doi:10.1145/800070.802201.

14 Jérôme Leroux. Vector addition systems reachability problem (A simpler solution). In *Turing-100 - The Alan Turing Centenary, Manchester, UK, June 22-25, 2012*, pages 214–228, 2012. URL: http://www.easychair.org/publications/?page=1673703727.

15 Ernst W. Mayr. An algorithm for the general Petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984. doi:10.1137/0213029.

16 Marvin L Minsky. *Computation: finite and infinite machines*. Prentice-Hall, Inc., 1967.

17 Tadao Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 77(4):541–580, 1989.

18 Joël Ouaknine and James Worrell. Positivity problems for low-order linear recurrence sequences. In *Proceedings of the Twenty-fifth Annual ACM-SIAM Symposium on Discrete*

Algorithms, SODA '14, pages 366–379, Philadelphia, PA, USA, 2014. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=2634074.2634101.

19 Joël Ouaknine and James Worrell. On linear recurrence sequences and loop termination. *SIGLOG News*, 2(2):4–13, 2015. doi:10.1145/2766189.2766191.

20 Klaus Reinhardt. Reachability in Petri nets with inhibitor arcs. *Electr. Notes Theor. Comput. Sci.*, 223:239–264, 2008. doi:10.1016/j.entcs.2008.12.042.