Finite Behaviours and Finitary Corecursion[*]

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─── **Abstract** ───

In the coalgebraic approach to state-based systems, semantics is captured up to behavioural equivalence by special coalgebras such as the final coalgebra, the final locally finitely presentable coalgebra (Adámek, Milius, and Velebil), or the final locally finitely generated coalgebra (Milius, Pattinson, and Wißmann). The choice of the proper semantic domain is determined by finiteness restrictions imposed on the systems of interest. We propose a unifying perspective by introducing the concept of a final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra, where the two parameters $\mathbb{I}$ and $\mathcal{M}$ determine what a "finite" system is. Under suitable conditions on the categories and type functors, we show that the final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra exists and coincides with the initial $(\mathbb{I}, \mathcal{M})$-iterative algebra, thereby putting a common roof over several results on iterative, fg-iterative and completely iterative algebras that were given a separate treatment before.

1 Introduction

Coalgebras model a wide variety of state-based systems, including deterministic and non-deterministic automata, weighted automata, labelled transition systems and probabilistic systems. One striking advantage of the coalgebraic approach is its beautiful account of semantics: the possible behaviours of states are captured by the final coalgebra, and the behaviour of a system is given by its final homomorphism. As a prominent example, the set functor $\{0, 1\} \times \mathsf{Id}^\Sigma$ (modelling deterministic automata) has a final coalgebra carried by the set of all languages over Σ, and the final homomorphism maps a state of an automaton to its accepted language [21].

In computer science, the focus is typically on systems satisfying some finiteness restrictions. For example, classical automata theory investigates finite automata (i.e. automata with finitely many states and input symbols) and their behaviours, the regular languages. And in the recently developed theory of nominal automata [9], the objects of interest are automata with possibly infinite, but orbit-finite nominal sets of states and inputs. From a "finite" point of view, the semantics given by the final coalgebra is sometimes unsatisfactory because it may identify states of finite coalgebras even though there is no finite witness for their behavioural equivalence. Hence, to obtain the semantics of finite systems, the final coalgebra needs to be replaced by another semantic domain that properly captures finite behaviours.

One prime challenge in systematically developing a theory of finite behaviours is that category theory offers several natural ways of modelling finite objects in categories, e.g. as finitely presentable, finitely generated, or perfectly presentable objects. These concepts are

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equivalent in the category of sets (and give precisely the finite sets) but generally differ, e.g. in categories of algebraic structures. The first categorical treatment of finite coalgebras and their final semantics was given in the work of Adámek, Milius, and Velebil [2] and later elaborated by Milius [16]. These authors modelled finite systems as coalgebras in a locally finitely presentable category carried by a *finitely presentable* object of states and introduced the *rational fixpoint* of a functor. The latter is essentially a "finitely presentable" version of the final coalgebra: it forms the final locally finitely presentable coalgebra, i.e. every coalgebra with a finitely presentable carrier has a unique homomorphism into it, and two states of such a coalgebra are identified by the unique homomorphism if and only if there is a finitely presentable witness for their behavioural equivalence. In the recent work of Milius, Pattinson, and Wißmann [19], a similar and largely parallel theory is developed for coalgebras based on *finitely generated* (instead of finitely presentable) objects of states: the authors introduce the *locally finite fixpoint* of a functor and prove it to be the final locally finitely generated coalgebra. The locally finite fixpoint has some technical advantages over the rational fixpoint, e.g. it is always a subcoalgebra of the final coalgebra, and it captures many instances of finite behaviours such as regular languages, context-free languages, and algebraic trees.

The goal of the present paper is to propose a uniform account that captures both the classical final semantics and its finite versions in the literature as special cases. For this purpose, we introduce the concept of an $(\mathbb{I}, \mathcal{M})$-*accessible* category, a generalisation of accessible and locally finitely presentable categories [6]. The two parameters $\mathbb{I}$ (a class of diagram schemes) and $\mathcal{M}$ (a class of morphisms) give rise to the notion of an $(\mathbb{I}, \mathcal{M})$-*presentable* object in a category that we take as our general definition of "finite object". By taking different choices of $\mathbb{I}$ and $\mathcal{M}$, the $(\mathbb{I}, \mathcal{M})$-presentable objects of a category instantiate e.g. to finitely presentable, finitely generated, perfectly presentable, or arbitrary objects. We then develop the theory of coalgebras based on $(\mathbb{I}, \mathcal{M})$-presentable objects: under suitable conditions on the categories and coalgebraic type functors, we show the existence of a final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra, providing a general finite version of final semantics and putting a common roof over the earlier work in [2, 16, 19]. As a new instance of our setting, we investigate coalgebras in algebraic categories carried by finitely generated free algebras of states. Such coalgebras arise naturally as determinisations of coalgebras with side effects, using the generalised powerset construction of Silva, Bonchi, Bonsangue, and Rutten [22].

Besides providing semantics of state-based systems, final coalgebras are also known to allow for an abstract and elegant category-theoretic approach to the semantics of guarded recursive specifications. Generalising classical work of Elgot [11, 12], Nelson [20] and Tiuryn [23] on algebraic properties of infinite trees, Milius [15] observed that the final coalgebra for a functor is also its initial completely iterative algebra. Analogous characterisations are known for the final locally finitely presentable coalgebra [2] and the final locally finitely generated coalgebra [19], where guarded recursive specifications restricted to a finitely presentable (resp. finitely generated) object of variables are considered. In Section 5, we will establish these results uniformly at our level of generality.

2   Preliminaries

We start by reviewing some concepts from category theory that we use in the paper.

2.1 Filtered colimits and locally finitely presentable categories

A category I is *filtered* if I-colimits commute with finite limits in **Set**. Equivalently, the following two properties hold: (i) for any two objects $X, Y \in I$, there exist morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ with a common codomain Z, and (ii) for any two parallel morphisms $f, g\colon X \to Y$ in I, there exists a morphism $h\colon Y \to Z$ with $h \cdot f = h \cdot g$. A *filtered colimit* in a category $\mathcal{A}$ is a colimit over a diagram $D\colon I \to \mathcal{A}$ with I filtered. If all colimit injections (and thus also all connecting maps of the diagram) are monomorphisms, a filtered colimit is called a *directed union*. An object A of $\mathcal{A}$ is called *finitely presentable* if the hom-functor $\mathcal{A}(A, -)\colon \mathcal{A} \to$ **Set** preserves filtered colimits, and *finitely generated* if it preserves directed unions. The full subcategories $\mathcal{A}_{fp}$ and $\mathcal{A}_{fg}$ of all finitely presentable (resp. finitely generated) objects of $\mathcal{A}$ are closed under finite colimits. Moreover, $\mathcal{A}_{fg}$ is closed under strong quotients, i.e. quotients carried by strong epimorphisms. The category $\mathcal{A}$ is *locally finitely presentable* if it is cocomplete, $\mathcal{A}_{fp}$ is essentially small (that is, its objects taken up to isomorphism form a set), and every object can be expressed as a filtered colimit of finitely presentable objects. This implies that also every object is a directed union of finitely generated objects. Examples of locally finitely presentable categories include the category of sets, the category of posets, and every variety of (many-sorted) algebras, e.g. groups, rings, vector spaces, or graphs. The finitely generated objects of a variety are the algebras with finitely many generators, and the finitely presentable objects are the algebras presentable with finitely many generators and relations. See [6] for more on locally finitely presentable categories.

2.2 Sifted colimits and algebraic categories

A category I is *sifted* if I-colimits commute with finite products in **Set**. Every filtered category is sifted, as is every category with binary coproducts. A *sifted colimit* in a category $\mathcal{A}$ is a colimit over a diagram $D\colon I \to \mathcal{A}$ with I sifted. An object A of $\mathcal{A}$ is *perfectly presentable* if the hom-functor $\mathcal{A}(A, -)\colon \mathcal{A} \to$ **Set** preserves sifted colimits. The category $\mathcal{A}$ is called *algebraic* if it is cocomplete, its full subcategory $\mathcal{A}_{pp}$ of perfectly presentable objects is essentially small, and every object can be expressed as a sifted colimit of perfect presentable objects. Algebraic categories can be characterised as the categories of models of Lawvere theories; in particular, every variety of (many-sorted) algebras is algebraic. The perfectly presentable objects of an algebraic category are exactly the split quotients of finitely generated free algebras. Sifted colimits in varieties are formed on the level of underlying sets. See [7] for more on algebraic categories.

2.3 Factorisation systems

A *factorisation system* in a category $\mathcal{A}$ is a pair $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ and $\mathcal{M}$ are classes of morphims such that (i) both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms, (ii) every morphism f of $\mathcal{A}$ has a factorisation $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and (iii) the diagonal fill-in property holds: given morphisms e, f, g, m with $e \in \mathcal{E}$, $m \in \mathcal{M}$ and $f \cdot e = m \cdot g$, there exists a unique morphism d with $d \cdot e = g$ and $m \cdot d = f$. We mention the two important properties of factorisation systems:

1. For any two morphisms m and n with $n \cdot m, n \in \mathcal{M}$ one has $m \in \mathcal{M}$.
2. If $(a_i\colon A_i \to A)_{i \in I}$ is a colimit in $\mathcal{A}$ and $A_i \xrightarrow{e_i} \overline{A}_i \xrightarrow{m_i} A$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of a_i, then also $(m_i\colon \overline{A}_i \to A)_{i \in I}$ is a colimit, provided that all morphisms in $\mathcal{E}$ are epic.

Factorisation systems are discussed in detail in [1].

2.4   Coalgebras and algebras

A *coalgebra* for an endofunctor $H\colon \mathcal{A} \to \mathcal{A}$ is a pair (C, γ) of an object $C \in \mathcal{A}$ and a morphism $\gamma\colon C \to HC$. A *homomorphism* between coalgebras (C, γ) and (D, δ) is a morphism $h\colon C \to D$ in $\mathcal{A}$ with $\delta \cdot h = Hh \cdot \gamma$. We denote the category of coalgebras and homomorphisms by $\mathbf{Coalg}\,(H)$. The forgetful functor from $\mathbf{Coalg}\,(H)$ to $\mathcal{A}$ creates colimits, that is, colimits of coalgebras are formed in the underlying category $\mathcal{A}$. If $\mathcal{A}$ has a factorisation system $(\mathcal{E}, \mathcal{M})$ and H preserves $\mathcal{M}$ (i.e. $m \in \mathcal{M}$ implies $Hm \in \mathcal{M}$), then $\mathbf{Coalg}\,(H)$ has the factorisation system of $\mathcal{E}$-carried and $\mathcal{M}$-carried homomorphisms.

Dually, an *algebra* for an endofunctor $H\colon \mathcal{A} \to \mathcal{A}$ is a pair (A, α) of an object $A \in \mathcal{A}$ and a morphism $\gamma\colon HA \to A$. A *homomorphism* between algebras (A, α) and (B, β) is a morphism $h\colon A \to B$ in $\mathcal{A}$ with $h \cdot \alpha = \beta \cdot Hh$.

2.5   Final functors

A functor $F\colon I \to J$ is *final* if (i) for every $j \in J$ there exists a morphism $f\colon j \to Fi$ for some $i \in I$, and (ii) any two morphisms $f\colon j \to Fi$ and $\overline{f}\colon j \to F\overline{i}$ with $i, \overline{i} \in I$ are connected by a zig-zag, i.e. there exist morphisms $k_0, \ldots, k_n \in I$ and $f_1, \ldots, f_n \in J$ making the diagram below commutative:

$$Fi \xleftarrow{Fk_0} Fi_1 \xrightarrow{Fk_1} Fi_2 \xleftarrow{Fk_2} \cdots \qquad \cdots \xleftarrow{Fk_{n-1}} Fi_n \xrightarrow{Fk_n} F\overline{i}$$

with f, f_1, f_2, f_n, $\overline{f}$, j.

If F is final and $D\colon I \to \mathcal{A}$ and $D'\colon J \to \mathcal{A}$ are two diagrams in a category $\mathcal{A}$ with $D' \cdot F = D$, then D and D' have the same colimits. More precisely, if $(D'_j \xrightarrow{a_j} A)_{j \in J}$ is a colimit cocone over D', then $(D_i = D'_{Fi} \xrightarrow{a_{Fi}} A)_{i \in I}$ is a colimit cocone over D. Conversely, if $(D_i \xrightarrow{a_i} A)_{i \in I}$ is a colimit cocone over D, then $(D'_j \xrightarrow{D'f_j} D'_{Fi_j} = D_{i_j} \xrightarrow{a_{i_j}} A)_{j \in J}$ is a colimit cocone over D', where $f_j\colon j \to Fi_j$ is an arbitrary morphism in J with $i_j \in I$, which exists by the finality of F.

2.6   Colimits in $\mathbf{Set}$

A cocone $(c_i\colon C_i \to C)_{i \in I}$ in $\mathbf{Set}$ forms a colimit if and only if (i) the maps c_i are jointly surjective (i.e. every element of C lies in the image of some c_i) and (ii) for any two elements $x \in C_i$ and $y \in C_j$ $(i, j \in I)$ with $c_i(x) = c_j(y)$, there exists a zig-zag of connecting morphisms and elements $x_1, \ldots, x_n$ as shown below:

$$C_i \xleftarrow{c_{i_1,i}} C_{i_1} \xrightarrow{c_{i_1,i_2}} C_{i_2} \xleftarrow{c_{i_3,i_2}} \ldots \xleftarrow{c_{i_n,i_{n-1}}} C_{i_n} \xrightarrow{c_{i_n,j}} C_j$$
$$x \xleftarrow{\quad} x_1 \longmapsto x_2 \qquad\qquad x_n \longmapsto y$$

If I is filtered, (ii) can be simplified as follows: for any $x, y \in C_i$ $(i \in I)$ with $c_i(x) = c_i(y)$, there exists a connecting morphism $c_{ij}\colon C_i \to C_j$ $(j \in I)$ with $c_{ij}(x) = c_{ij}(y)$.

3 $(\mathbb{I}, \mathcal{M})$-accessible categories

As the foundation for our approach to the semantics of finite behaviours we present in this section a mild generalisation of the concept of a locally finitely presentable category (cf. 2.1).
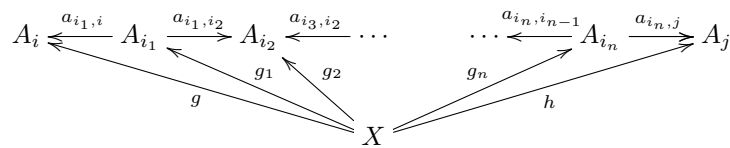
While in the latter the concept of a "finite" object is given by finitely presentable objects, we introduce two parameters that allow for some flexibility in choosing the desired concept of finiteness. Throughout this paper let us fix a category $\mathcal{A}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$, with $\mathcal{E}$ a class of epimorphisms, and a collection $\mathbb{I}$ of (not necessarily small) sifted categories.

▸ **Definition 3.1.** A category $\mathcal{A}$ is $\mathbb{I}$-*cocomplete* if every $\mathbb{I}$-diagram (i.e. every diagram $I \to \mathcal{A}$ with $I \in \mathbb{I}$) has a colimit. A colimit cocone $(a_i : A_i \to A)_{i \in I}$ over an $\mathbb{I}$-diagram is called an $(\mathbb{I}, \mathcal{M})$-*colimit* if all injections a_i lie in $\mathcal{M}$. (The latter implies that all connecting morphisms $a_{ij} : A_i \to A_j$ lie in $\mathcal{M}$, see 2.3). A functor $F : \mathcal{A} \to \mathcal{B}$ *preserves* $(\mathbb{I}, \mathcal{M})$-*colimits* if $(Fa_i : FA_i \to FA)_{i \in I}$ is a colimit cocone in $\mathcal{B}$ for any $(\mathbb{I}, \mathcal{M})$-colimit $(a_i : A_i \to A)_{i \in I}$ in $\mathcal{A}$.

▸ **Definition 3.2.** An object X of $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-*presentable* if the hom-functor $\mathcal{A}(X, -) : \mathcal{A} \to$ **Set** preserves $(\mathbb{I}, \mathcal{M})$-colimits. We denote by $\mathcal{A}_{\mathbb{I}, \mathcal{M}}$ the full subcategory of all $(\mathbb{I}, \mathcal{M})$-presentable objects.

▸ **Remark 3.3.** Using the characterisation of colimits in **Set**, see 2.6, an object X is $(\mathbb{I}, \mathcal{M})$-presentable if and only if for any $(\mathbb{I}, \mathcal{M})$-colimit $(a_i : A_i \to A)_{i \in I}$ in $\mathcal{A}$ the following holds:

(i) For every morphism $f : X \to A$, there exists $i \in I$ and $g : X \to A_i$ with $f = a_i \cdot g$.

(ii) For any two morphisms $g : X \to A_i$ and $h : X \to A_j$ $(i, j \in I)$ with $a_i \cdot g = a_j \cdot h$, there exists a zig-zag of connecting morphisms along with morphisms $g_1, \ldots, g_n$ such that the diagram below commutes:

$$A_i \xleftarrow{a_{i_1,i}} A_{i_1} \xrightarrow{a_{i_1,i_2}} A_{i_2} \xleftarrow{a_{i_3,i_2}} \cdots \qquad \cdots \xleftarrow{a_{i_n,i_{n-1}}} A_{i_n} \xrightarrow{a_{i_n,j}} A_j$$

with arrows g, g_1, g_2, g_n, h from X

If I is filtered, (ii) simplifies as follows: for any two morphisms $g, h : X \to A_i$ $(i \in I)$ with $a_i \cdot g = a_i \cdot h$, there is a connecting morphism a_{ij} $(j \in J)$ with $a_{ij} \cdot g = a_{ij} \cdot h$.

▸ **Definition 3.4.** The category $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-*accessible* if (i) $\mathcal{A}$ is $\mathbb{I}$-cocomplete, and (ii) every object of $\mathcal{A}$ is an $\mathbb{I}$-colimit of $(\mathbb{I}, \mathcal{M})$-presentable objects.

▸ **Example 3.5.**

1. Every category $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-accessible w.r.t. the parameters

$\mathbb{I} =$ categories with a terminal object and $(\mathcal{E}, \mathcal{M}) = ($ isomorphisms, all morphisms $)$.

Note that $(\mathbb{I}, \mathcal{M})$-colimits are trivial, and thus every object of $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-presentable.

2. Every locally finitely presentable category $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-accessible w.r.t.

$\mathbb{I} =$ small filtered categories and $(\mathcal{E}, \mathcal{M}) = ($ isomorphisms, all morphisms $)$.

Then $(\mathbb{I}, \mathcal{M})$-colimits are filtered colimits, and thus $(\mathbb{I}, \mathcal{M})$-presentable objects are precisely the finitely presentable ones.

3. Every locally finitely presentable category $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-accessible w.r.t.

$\mathbb{I} =$ small filtered categories and $(\mathcal{E}, \mathcal{M}) = ($ strong epimorphisms, monomorphisms $)$.

Then $(\mathbb{I}, \mathcal{M})$-colimits are directed unions, and thus $(\mathbb{I}, \mathcal{M})$-presentable objects coincide with the finitely generated ones.

4. Every algebraic category $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-accessible w.r.t.

$$\mathbb{I} = \text{small sifted categories} \quad \text{and} \quad (\mathcal{E}, \mathcal{M}) = (\text{isomorphisms, all morphisms}).$$

Then $(\mathbb{I}, \mathcal{M})$-colimits are sifted colimits, and thus $(\mathbb{I}, \mathcal{M})$-presentable objects are precisely the perfectly presentable ones.

In the following we generalise some important technical properties of locally finitely presentable categories to $(\mathbb{I}, \mathcal{M})$-accessible categories.

▸ **Lemma 3.6.** $\mathcal{A}_{\mathbb{I}, \mathcal{M}}$ *is closed under binary coproducts and $\mathcal{E}$-carried quotients.*

Proof. *Closure under finite coproducts.* Let $X, Y \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$, i.e. the functors $\mathcal{A}(X, -)$ and $\mathcal{A}(Y, -)$ preserve $(\mathbb{I}, \mathcal{M})$-colimits, and suppose that X and Y have a coproduct $X + Y$ in $\mathcal{A}$. Since all categories in $\mathbb{I}$ are sifted by assumption, $\mathbb{I}$-colimits commute with finite products in **Set** (see 2.2). Therefore the functor $\mathcal{A}(X + Y, -) \simeq \mathcal{A}(X, -) \times \mathcal{A}(Y, -)$ preserves $(\mathbb{I}, \mathcal{M})$-colimits, which proves $X + Y \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$.

Closure under $\mathcal{E}$-quotients. Let $e : X \to Y$ be a morphism in $\mathcal{E}$ with $X \in \mathcal{A}_{\mathbb{I}, M}$, and suppose that $(a_i : A_i \to A)_{i \in I}$ is an $(\mathbb{I}, \mathcal{M})$-colimit. To show that $Y \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$, we verify the criterion of Remark 3.3.

(i) Let $f : Y \to A$. Since $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$, the morphism $f \cdot e$ factorises through the cocone, i.e. $f \cdot e = a_i \cdot g$ for some $i \in I$ and $g : X \to A_i$. Diagonal fill-in gives a morphism $d : Y \to A_i$ with $a_i \cdot d = f$.
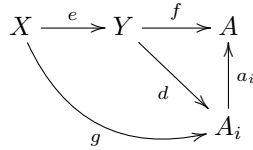
**(ii)** Let $g : Y \to A_i$ and $h : Y \to A_j$ be two morphisms with $i, j \in I$ and $a_i \cdot g = a_j \cdot h$. Since $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$, there exists a zig-zag connecting $g \cdot e$ and $h \cdot e$ as in the diagram below:
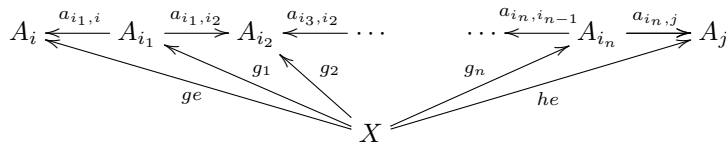
Since the connecting morphism $a_{i_1, i}$ lies in $\mathcal{M}$, diagonal fill-in yields a morphism $d_1 : Y \to A_{i_1}$ with $d_1 \cdot e = g_1$ and $a_{i_1, 1} \cdot d_1 = g$.
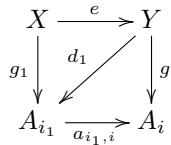
Put $d_2 := a_{i_1, i_2} \cdot d_1$. Again by diagonal fill-in, there exists $d_3 : X \to A_{i_3}$ with $a_{i_3, i_2} \cdot d_3 = d_2$. Proceeding in this fashion, we obtain morphisms $d_1, \ldots, d_n$ making the following diagram commute:
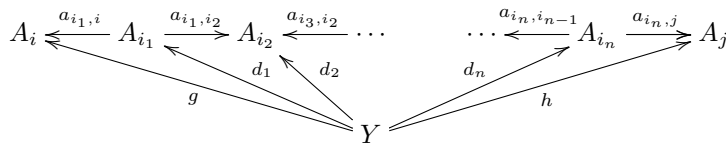
Thus g and h are connected by a zig-zag, as required. ◂

▸ **Corollary 3.7.** *In an* $(\mathbb{I}, \mathcal{M})$*-accessible category* $\mathcal{A}$*, every object is an* $(\mathbb{I}, \mathcal{M})$*-colimit of* $(\mathbb{I}, \mathcal{M})$*-presentable objects.*

Proof. Since $\mathcal{A}$ is $(\mathbb{I}, \mathcal{M})$-accessible, every object is an $\mathbb{I}$-colimit of $(\mathbb{I}, \mathcal{M})$-presentable objects. Therefore the statement follows from 2.3(2) and the closure of $\mathcal{A}_{\mathbb{I}, \mathcal{M}}$ under $\mathcal{E}$-quotients.   ◂

▸ **Notation 3.8.** For any object $A \in \mathcal{A}$, denote by $\mathcal{A}_{\mathbb{I}, \mathcal{M}} \downarrow A$ the comma category whose objects are the morphisms $f \colon X \to A$ with $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$; a morphism from $(f \colon X \to A)$ to $(g \colon Y \to A)$ is a morphism $h \colon X \to Y$ in $\mathcal{A}$ with $f = g \cdot h$. The *canonical diagram* of A is the diagram $\pi_A \colon (\mathcal{A}_{\mathbb{I}, \mathcal{M}} \downarrow A) \to \mathcal{A}$ mapping $(f \colon X \to A)$ to X and $h \colon f \to g$ to h.

▸ **Lemma 3.9.** *If* $\mathcal{A}$ *is an* $(\mathbb{I}, \mathcal{M})$*-accessible category, then every object* $A \in \mathcal{A}$ *is the colimit of its canonical diagram, with colimit injections* $f \colon \pi_A(f) \to A$ $(f \in \mathcal{A}_{\mathbb{I}, \mathcal{M}} \downarrow A)$.

Proof. Express A as an $(\mathbb{I}, \mathcal{M})$-colimit $(a_i \colon D_i \to A)_{i \in I}$ of $(\mathbb{I}, \mathcal{M})$-presentable objects D_i, see Corollary 3.7. Then the functor $F \colon I \to \mathcal{A}_{\mathbb{I}, \mathcal{M}}$ mapping $i \in I$ to the colimit injection $(a_i \colon D_i \to A) \in \mathcal{A}_{\mathbb{I}, \mathcal{M}} \downarrow A$ is final; indeed, the two criteria (i) and (ii) in 2.5 state precisely that every object of $\mathcal{A}_{\mathbb{I}, \mathcal{M}}$ is $(\mathbb{I}, \mathcal{M})$-presentable. Since $D = \pi_A \cdot F$, it follows from 2.5 that the morphisms $f \colon \pi_A(f) \to A$ form a colimit cocone.   ◂

4 Locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebras

In this section, we consider finite systems modelled as coalgebras based on $(\mathbb{I}, \mathcal{M})$-presentable objects and show the existence of a final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra, which serves as the semantic domain of finite behaviours.

▸ **Assumptions 4.1.** From now on, fix an endofunctor $H \colon \mathcal{A} \to \mathcal{A}$ on an $(\mathbb{I}, \mathcal{M})$-accessible category $\mathcal{A}$. Let $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$ denote the full subcategory of $\mathbf{Coalg}(H)$ of all coalgebras (A, α) with $A \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$. We assume that (i) $\mathcal{A}$ has binary coproducts, (ii) H preserves $\mathbb{I}$-colimits, (iii) H preserves $\mathcal{M}$ (i.e. $m \in \mathcal{M}$ implies $Hm \in \mathcal{M}$), and (iv) $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H) \in \mathbb{I}$.

▸ **Notation 4.2.** Let $T_H \xrightarrow{\tau} H(T_H)$ be the colimit of the inclusion $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H) \rightarrowtail \mathbf{Coalg}(H)$. (The colimit exists by Assumption 4.1(iv) and because colimits in $\mathbf{Coalg}(H)$ are formed in the underlying category $\mathcal{A}$). We denote the colimit injections by

$$\alpha^{\#} \colon (A, \alpha) \to (T_H, \tau) \qquad ((A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)).$$

▸ **Example 4.3.** The following settings of categories and functors satisfy our assumptions.

1. Let $\mathcal{A}$ be a category with binary coproducts and $H \colon \mathcal{A} \to \mathcal{A}$ a functor with a final coalgebra. Choose $\mathbb{I} =$ categories with a terminal object and $(\mathcal{E}, \mathcal{M})$ trivial as in Example 3.5.1. Then the above assumptions (i)-(iv) are clearly satisfied. The coalgebra T_H is the colimit of all H-coalgebras, i.e. the final coalgebra νH of H.

2. Let $\mathcal{A}$ be a locally finitely presentable category and $H \colon \mathcal{A} \to \mathcal{A}$ a functor preserving filtered colimits (a *finitary* functor for short). Choose $\mathbb{I} =$ small filtered categories and $(\mathcal{E}, \mathcal{M})$ trivial as in Example 3.5.2. Then (i), (ii) and (iii) are clearly true, and (iv) holds because finitely presentable objects are stable under finite colimits and colimits of H-coalgebras are formed in $\mathcal{A}$. The coalgebra T_H is the colimit of all H-coalgebras with finitely presentable carrier. This coalgebra is the *rational fixpoint* of H introduced in the work of Adámek, Milius, and Velebil [2], and is denoted by ϱH. The term "fixpoint" will be justified in Lemma 4.5 below.

3. Let $\mathcal{A}$ be locally finitely presentable and $H\colon \mathcal{A} \to \mathcal{A}$ a finitary functor preserving monomorphisms. Choose $\mathbb{I}$ = small filtered categories and $(\mathcal{E}, \mathcal{M})$ = (strong epimorphisms, mononomorphisms) as in Example 3.5.3. Then (i), (ii) and (iii) are clear, and (iv) holds because finitely generated objects are stable under finite colimits. The coalgebra T_H is the colimit of all H-coalgebras with finitely generated carrier; this is the *locally finite fixpoint* of H investigated by Milius, Pattinson, and Wißmann [19]. We denote it by ϑH.

4. Let $\mathcal{A}$ be an algebraic category and $H\colon \mathcal{A} \to \mathcal{A}$ a functor preserving sifted colimits. Choose $\mathbb{I}$ = small sifted categories and $(\mathcal{E}, \mathcal{M})$ trivial as in Example 3.5.4. Then (i), (ii) and (iii) are clear, and (iv) holds because perfectly presentable objects are stable under finite coproducts. The coalgebra T_H is formed as the colimit of all coalgebras with perfectly presentable carrier, and is in the following denoted by φH.

▸ **Remark 4.4.** Since H preserves $\mathcal{M}$, the factorisation system of $\mathcal{A}$ lifts to $\mathbf{Coalg}\,(H)$, see 2.4. Consequently, we can express the coalgebra (T_H, τ) as an $(\mathbb{I}, \mathcal{M})$-colimit of coalgebras in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$. Indeed, for each $(A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$ factorise

$$\alpha^{\#} \equiv (A, \alpha) \xrightarrow{e_\alpha} (\overline{A}, \overline{\alpha}) \xrightarrow{m_\alpha} (T_H, \tau)$$

in $\mathbf{Coalg}\,(H)$, where $e_\alpha \in \mathcal{E}$ and $m_\alpha \in \mathcal{M}$. Then $m_\alpha = \overline{\alpha}^{\#}$ because $m_\alpha \cdot e_\alpha = \alpha^{\#} = \overline{\alpha}^{\#} \cdot e_\alpha$ (using that $(-)^{\#}$ forms a cocone) and e_α is an epimorphism. Therefore, by 2.3(2), the homomorphisms

$$\overline{\alpha}^{\#} \colon (\overline{A}, \overline{\alpha}) \to (T_H, \tau) \qquad ((A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H))$$

form an $(\mathbb{I}, \mathcal{M})$-colimit cocone in $\mathbf{Coalg}\,(H)$. Given a homomorphism $h\colon (A, \alpha) \to (B, \beta)$ in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$, we denote by $\overline{h}\colon (\overline{A}, \overline{\alpha}) \to (\overline{B}, \overline{\beta})$ the unique homomorphism (obtained via diagonal fill-in) with $\overline{h} \cdot e_\alpha = e_\beta \cdot h$.

▸ **Lemma 4.5** (Lambek Lemma for T_H)**.** *The coalgebra structure $T_H \xrightarrow{\tau} H(T_H)$ is an isomorphism in $\mathcal{A}$.*

Proof sketch. By Remark 4.4 and since colimits in $\mathbf{Coalg}\,(H)$ are formed in $\mathcal{A}$, we know that τ is the unique mediating morphism with $\tau \cdot \overline{\alpha}^{\#} = H\overline{\alpha}^{\#} \cdot \overline{\alpha}$ for all $(A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$. One can show that the morphisms

$$\overline{A} \xrightarrow{\overline{\alpha}} H\overline{A} \xrightarrow{H\overline{\alpha}^{\#}} HT_H \qquad ((A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H))$$

form a colimit cocone over the diagram $D\colon \mathbf{Coalg}_{\mathbb{I}, M}(H) \to \mathcal{A}$ mapping (A, α) to $\overline{A}$ and $h\colon (A, \alpha) \to (B, \beta)$ to $\overline{h}$. Then the uniqueness of colimits implies that τ is an isomorphism. The details of the proof are given in the Appendix. ◂

▸ **Example 4.6.**

1. Consider the setting of Example 4.3.1. Then the above lemma is precisely the classical Lambek lemma [14]: the final coalgebra νH is a fixpoint of H.
2. In the setting of Example 4.3.2, the lemma shows that ϱH forms a fixpoint of H. This was shown in [2] with a conceptually different proof method.
3. In the setting of Example 4.3.3, the lemma shows that ϑH is a fixpoint of H. This result is known from [19] where again a different proof method was used.
4. In the setting of Example 4.3.4, we obtain a new fixpoint φH for any sifted colimit preserving endofunctor H on an algebraic category $\mathcal{A}$. The fixpoint φH models the behaviours of *pp-coalgebras*, i.e. coalgebras with perfectly presentable carrier. Given

two pp-coalgebras (A, α) and (B, β), two states $a \in A$ and $b \in B$ are merged by the colimit injections $\alpha^{\#}: A \to \varphi H$ and $\beta^{\#}: B \to \varphi H$ if and only if there exists a "perfectly presentable reason" for it, in the sense that a and b are connected by a zig-zag of pp-coalgebras as in the diagram below:

$$(A, \alpha) \xleftarrow{g} (A_1, \alpha_1) \xrightarrow{h_1} (A_2, \alpha_2) \xleftarrow{h_2} \ldots \xleftarrow{h_{n-1}} (A_n, \alpha_n) \xrightarrow{h} (B, \beta)$$
$$a \longleftarrow\!\!\mid a_1 \longmapsto a_2 \qquad\qquad a_n \longmapsto b$$

Indeed, this follows from the fact that the sifted colimit defining φH is formed on the level of **Set**, see 2.2 and 2.6.

One natural occurrence of pp-coalgebras, and in particular coalgebras carried by finitely generated free algebras, arises in the generalised powerset construction of Silva, Bonchi, Bonsangue, and Rutten [22]: given a monad $\mathbf{T} = (T, \eta, \mu)$ on **Set** and an endofunctor $H: \mathbf{Set} \to \mathbf{Set}$ that admits a lifting $\overline{H}: \mathcal{A}^{\mathbf{T}} \to \mathcal{A}^{\mathbf{T}}$ to the category of $\mathbf{T}$-algebras, a coalgebra $X \to HTX$ for the functor HT can be transformed into a coalgebra $\mathbf{T}X \to \overline{H}\mathbf{T}X$ for the functor $\overline{H}$ whose carrier is the free $\mathbf{T}$-algebra $\mathbf{T}X = (TX, \mu_X)$ on X. For example, the classical powerset construction for nondeterministic automata is an instance of the generalised one by taking $H = \{0, 1\} \times \mathsf{Id}^{\Sigma}$ and the finite powerset monad $\mathbf{T} = \mathcal{P}_f$.

▸ **Remark 4.7.** In $\mathcal{A} = \mathbf{Set}$ perfectly presentable, finitely presentable and finitely generated objects coincide with the finite sets, and moreover every finitary set functor H preserves sifted colimits, see [7, Corollary 6.30]. Therefore, for any such H we have $\varphi H = \varrho H = \vartheta H$. For example, if $H = H_{\Sigma} = \coprod_{\sigma \in \Sigma} \mathsf{Id}^{\mathsf{ar}(\sigma)}$ is the polynomial set functor associated to a finitary signature Σ, then the final coalgebra νH_{Σ} is carried by the set of finite or infinite Σ-trees, and $\varrho H_{\Sigma} = \vartheta H_{\Sigma} = \varphi H_{\Sigma}$ is carried by the set of *rational trees* [13], i.e. finite or infinite trees that up to isomorphism have only finitely many subtrees. We shall see in Example 4.12 below that in general algebraic categories, the fixpoint φH may differ from ϱH and ϑH.

Our next goal is to characterise the coalgebra T_H by a universal property.

▸ **Definition 4.8.** An H-coalgebra is called *locally $(\mathbb{I}, \mathcal{M})$-presentable* if it is an $\mathbb{I}$-colimit of coalgebras in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$.

▸ **Remark 4.9.** By factorising as in Remark 4.4, it follows that a locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra can also be expressed as an $(\mathbb{I}, \mathcal{M})$-colimit of coalgebras in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$.

▸ **Lemma 4.10.** *If all categories in $\mathbb{I}$ are filtered, then every coalgebra in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$ is an $(\mathbb{I}, \mathcal{M})$-presentable object of $\mathbf{Coalg}(H)$.*

For the case where H is a finitary functor on a locally finitely presentable category $\mathcal{A}$ (the setting of Example 4.3.2), this is shown in [8, Lemma III.2]. The following proof is a straightforward generalisation.

Proof. Let $(b_i: (B_i, \beta_i) \to (B, \beta))_{i \in I}$ be an $(\mathbb{I}, \mathcal{M})$-colimit in $\mathbf{Coalg}(H)$, and suppose that $h: (A, \alpha) \to (B, \beta)$ is a coalgebra homomorphism with $(A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$. We need to show that h factorises through the cocone essentially uniquely. Since the colimit is formed in $\mathcal{A}$ and the object A is $(\mathbb{I}, \mathcal{M})$-presentable, there exists $i \in I$ and a morphism $g: A \to B_i$ in $\mathcal{A}$ with $h = b_i \cdot g$. Moreover, since H preserves $\mathbb{I}$-colimits we have that also $(Hb_i)_{i \in I}$ is a colimit cocone. The two morphisms $\beta_i \cdot g, Hg \cdot \alpha: A \to HB_i$ are merged by Hb_i because

$$Hb_i \cdot \beta_i \cdot g = \beta \cdot b_i \cdot g = \beta \cdot h = Hh \cdot \alpha = Hb_i \cdot Hg \cdot \alpha,$$

using that h and b_i are coalgebra homomorphisms. Therefore, since I is filtered, there exists a connecting morphism $b_{ij} \colon (B_i, \beta_i) \to (B_j, \beta_j)$ with $Hb_{ij} \cdot \beta_i \cdot g = Hb_{ij} \cdot Hg \cdot \alpha$. An easy computation now shows that the morphism $f := b_{ij} \cdot g$ is a coalgebra homomorphism from (A, α) to (B_j, β_j) with $h = b_j \cdot f$. Thus h factorises through b_j in **Coalg** (H).

The uniqueness of factorisations is clear because this holds in the underlying category. ◄

▸ **Theorem 4.11.** *If all categories in* $\mathbb{I}$ *are filtered, then* (T_H, τ) *is the final locally* $(\mathbb{I}, \mathcal{M})$-*presentable H-coalgebra.*

Proof. By definition, the coalgebra (T_H, τ) is locally $(\mathbb{I}, \mathcal{M})$-presentable. To show the finality, it suffices to prove that every coalgebra (A, α) in **Coalg** $_{\mathbb{I}, \mathcal{M}}(H)$ has a unique homomorphism into (T_H, τ). Clearly the colimit injection $\alpha^{\#} \colon (A, \alpha) \to (T_H, \tau)$ is a homomorphism. For the uniqueness, suppose that $h \colon (A, \alpha) \to (T_H, \tau)$ is any homomorphism. Since (A, α) is $(\mathbb{I}, \mathcal{M})$-presentable in **Coalg** (H) by Lemma 4.10 and the homomorphisms $\overline{\beta}^{\#} \colon (\overline{B}, \overline{\beta}) \to (T_H, \tau)$ $((B, \beta) \in \textbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H))$ form an $(\mathbb{I}, \mathcal{M})$-colimit cocone by Remark 4.4, there exists a coalgebra (B, β) in **Coalg** $_{\mathbb{I}, \mathcal{M}}(H)$ and a homomorphism $g \colon (A, \alpha) \to (\overline{B}, \overline{\beta})$ with $\overline{\beta}^{\#} \cdot g = h$. Since the morphisms $(-)^{\#}$ form a cocone, this implies $h = \overline{\beta}^{\#} \cdot g = \alpha^{\#}$. ◄

▸ **Example 4.12.** In the setting of Example 4.3.4 where the categories in $\mathbb{I}$ are not filtered, the universal property in the above theorem generally fails, that is, a pp-coalgebra can admit more than one homomorphism into φH. To see this, consider the category $\mathcal{A}$ of algebras with a single unary operation u and the identity functor $H = \mathsf{Id}$ on $\mathcal{A}$. Thus H-coalgebras are input-free deterministic transition systems endowed with an additional unary operation that commutes with the transitions. Let FX denote the free algebra of $\mathcal{A}$ over the set X, carried by the set of all terms $u^n(x)$ with $n \geqslant 0$ and variables $x \in X$. Note that split quotients of a term algebra are again term algebras (arising by identifiying variables). Therefore the perfectly presentable objects of $\mathcal{A}$ are exactly the finitely generated free algebras, i.e. the algebras FX with X finite (cf. 2.2). We write **Coalg** $_{\text{free}}(H)$ for the category of all H-coalgebras with finitely generated free carrier.

Consider the two H-coalgebras

$$F\{x\} \xrightarrow{\alpha} F\{x\} \quad \text{with} \quad \alpha(x) = x$$

and

$$F\{y\} \xrightarrow{\beta} F\{y\} \quad \text{with} \quad \beta(y) = u(y),$$

and let $g \colon F\{x\} \to \varphi H$ be the unique morphism in $\mathcal{A}$ with $g(x) = \beta^{\#}(y)$. We will prove that (i) $g \colon (F\{x\}, \alpha) \to (\varphi H, \tau)$ is a coalgebra homomorphism and (ii) $g \neq \alpha^{\#}$, which shows that there are two distinct homomorphisms from $(F\{x\}, \alpha)$ into $(\varphi H, \tau)$.

To prove (i), observe first that clearly $\beta \colon (F\{y\}, \beta) \to (F\{y\}, \beta)$ is a coalgebra homomorphism, and thus $\beta^{\#} = \beta^{\#} \cdot \beta$ because $(-)^{\#}$ forms a cocone. This implies

$$
\begin{aligned}
g(\alpha(x)) &= g(x) && (\text{def. } \alpha) \\
&= \beta^{\#}(y) && (\text{def. } g) \\
&= \beta^{\#}(\beta(y)) && (\text{see above}) \\
&= \tau(\beta^{\#}(y)) && (\beta^{\#} \text{ coalg. hom.}) \\
&= \tau(g(x)) && (\text{def. } g)
\end{aligned}
$$

and thus $g \cdot \alpha = \tau \cdot g$ because x generates $F\{x\}$. Hence g is a coalgebra homomorphism.

To prove (ii), it suffices to show that $\alpha^{\#}(x) \neq \beta^{\#}(y)$. Since the sifted colimit defining φH is formed in **Set** (see 2.2), this requires us to show that there is no zig-zag of coalgebra homomorphisms in $\mathbf{Coalg}_{\mathrm{free}}(H)$ connecting x and y. In the following, let us call an element a of an H-coalgebra (A, γ) *finite* if the set $\{\gamma^n(a) : n \geqslant 0\}$ of all states that are reachable from a by transitions is finite.

$(*)$ *Claim.* Let X and Y be finite sets and let $h : (FX, \gamma) \to (FY, \delta)$ be a coalgebra homomorphism. Then a state $t \in FX$ is finite if and only if the state $h(t) \in FY$ is finite.

Proof. Since h is a coalgebra homomorphism we have $h(\gamma^n(t)) = \delta^n(h(t))$ for every $n \geqslant 0$. This immediately implies that $h(t)$ is finite whenever t is finite. Conversely, suppose that t is not finite. Since the set X of variables is finite, there are only finitely many terms of any given height in FX. Thus, for every $k \geqslant 0$, there exists a term $u^n(x) \in TX$ with $x \in X$ and $n \geqslant k$ that is reachable from t. Then $h(u^n(x)) = u^n(h(x))$ is a term of height at least $n \geqslant k$ in FY, and this term is reachable from $h(t)$ because h is a coalgebra homomorphism. Thus the state $h(t) \in FY$ is not finite. ◄

Since the state x of the coalgebra $(F\{x\}, \alpha)$ is finite and the state y of $(F\{y\}, \beta)$ is infinite, $(*)$ shows that no zig-zag in $\mathbf{Coalg}_{\mathrm{free}}(H)$ connecting x and y exists.

▸ **Remark 4.13.** In the above example the fixpoints ϱH, ϑH and νH are carried by the terminal object 1, while φH is nontrivial. In general, all four fixpoints may be pairwise distinct. This holds, e.g., for the endofunctor $H = \mathbb{N} \times \mathsf{Id}$ on the category $\mathcal{A}$ of sets with two unary operations, with both operations on $\mathbb{N}$ given by the successor map; see Milius [17]. In addition, in *loc. cit.* the author discusses sufficient conditions on the functor H ensuring that the three fixpoints φH, ϱH and ϑH (each of which represents different flavours of finite behaviours) coincide.

5 $(\mathbb{I}, \mathcal{M})$-**iterative algebras**

In this section, we establish another universal property of T_H: it is the initial $(\mathbb{I}, \mathcal{M})$-iterative algebra and thus forms the universal domain of solutions for guarded recursive specifications. The results of this section put a common roof over results from [15, 2, 19]. Since the proofs are essentially identical to the ones of [2], we confine ourselves to describing the constructions involved.

For motivation, recall that for an algebra A over a finitary signature Σ, a *flat system of equations* is a finite system of recursive equations of the form $x_1 = t_1, \ldots, x_n = t_n$ where $x_1, \ldots x_n$ are the variables and each t_i is either an element of A or a Σ-term of height 1 in the variables $x_1, \ldots, x_n$. Thus, a flat system corresponds to a function $e : X \to H_\Sigma X + A$ with $X = \{x_1, \ldots x_n\}$. The algebra A is called *iterative* if every flat system of equations has a unique solution in A.

▸ **Example 5.1.** The Σ-algebra of finite or infinite trees is iterative, as is the Σ-algebra of rational trees (see Remark 4.7). Given the signature Σ of a binary operation symbol $*$ and a constant symbol c, the flat system $x_1 = x_2 * x_1$, $x_2 = c$, has the following unique solution in the algebra of rational trees:

Replacing H_Σ by a general endofunctor $H \colon \mathcal{A} \to \mathcal{A}$ and the finite set X of variables an arbitrary $(\mathbb{I}, \mathcal{M})$-presentable object, the concept of an iterative Σ-algebra generalises has the following categorical generalisation:

▸ **Definition 5.2.** Let $HA \xrightarrow{\alpha} A$ be an H-algebra. By a *flat equation morphism* is meant a morphism $e \colon X \to HX + A$ with $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$. A *solution* of e in (A, α) is a morphism $e^\dagger \colon X \to A$ making the following square commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad e^\dagger \quad} & A \\
{\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle [\alpha, A]} \\
HX + A & \xrightarrow[\;He^\dagger + A\;]{} & HA + A
\end{array}
$$

The algebra (A, α) is called $(\mathbb{I}, \mathcal{M})$-*iterative* if every flat equation morphism admits a unique solution.

▸ **Example 5.3.** In the settings of Example 4.3.1-3, $(\mathbb{I}, \mathcal{M})$-iterative algebras are called *completely iterative algebras* [15], *iterative algebras* [2] and *fg-iterative algebras* [19], respectively.

Since the coalgebra structure τ of T_H is an isomorphism by Lemma 4.5, we can view T_H as an H-algebra (T_H, τ^{-1}). We aim to show that this algebra is $(\mathbb{I}, \mathcal{M})$-iterative and, in fact, the initial $(\mathbb{I}, \mathcal{M})$-iterative algebra. This requires further assumptions on our setting:

▸ **Assumptions 5.4.** In addition to the Assumptions 4.1 given in the previous section, we assume that (i) all categories in $\mathbb{I}$ are filtered, (ii) $\mathcal{M}$ is closed under coproducts, i.e. $m, m' \in \mathcal{M}$ implies $m + m' \in \mathcal{M}$, and (iii) $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X) \in \mathbb{I}$ for every $X \in \mathcal{A}$.

Here $H + X \colon \mathcal{A} \to \mathcal{A}$ is the endofunctor given by $Y \mapsto HY + X$.

▸ **Example 5.5.** In the setting of Example 4.3.1, the above assumption (iii) states precisely that H is an *iteratable endofunctor*, i.e. the functor $H + X$ admits a final coalgebra for every $X \in \mathcal{A}$. In Example 4.3.2/3, (iii) is trivially satisfied.

▸ **Lemma 5.6.** (T_H, τ^{-1}) *is an* $(\mathbb{I}, \mathcal{M})$-*iterative algebra.*

Proof sketch. Let $e \colon X \to HX + T_H$ be a flat equation morphism with $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$. Express the coalgebra (T_H, τ) as an $(\mathbb{I}, \mathcal{M})$-colimit

$$\overline{\alpha}^{\#} \colon (\overline{A}, \overline{\alpha}) \to (T_H, \tau) \qquad ((A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H),$$

see Remark 4.4. Since colimits commute with coproducts and $\mathcal{M}$ is closed under coproducts, we have the $(\mathbb{I}, \mathcal{M})$-colimit

$$HX + \overline{\alpha}^{\#} \colon HX + \overline{A} \to HX + T_H \qquad ((A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H))$$

in $\mathcal{A}$. Therefore, since X is $(\mathbb{I}, \mathcal{M})$-presentable, there exists a coalgebra $(A, \alpha) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$ and a morphism e_0 making the triangle below commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\;e\;} & HX + T_H \\
 & {\scriptstyle e_0}\searrow & \big\uparrow{\scriptstyle HX + \overline{\alpha}^{\#}} \\
 & & HX + \overline{A}
\end{array}
$$

Form the coalgebra

$$s \equiv X + \overline{A} \xrightarrow{[e_0, \mathsf{inr}]} HX + \overline{A} \xrightarrow{HX + \overline{\alpha}} HX + H\overline{A} \xrightarrow{\mathsf{can}} H(X + \overline{A})$$

where inl and inr denote the left and right coproduct injections, and can is the canonical morphism determined by $\mathsf{can} \cdot \mathsf{inl} = H\mathsf{inl}$ and $\mathsf{can} \cdot \mathsf{inr} = H\mathsf{inr}$. Letting $s^{\#}$ be the unique homomorphism into (T_H, τ), the morphism

$$e^{\dagger} \equiv X \xrightarrow{\mathsf{inl}} X + \overline{A} \xrightarrow{s^{\#}} T_H$$

can be shown to be the unique solution of e. The argument is identical to the proof of [2, Lemma 3.5]. ◄

▸ **Theorem 5.7.** (T_H, τ^{-1}) *is the initial* $(\mathbb{I}, \mathcal{M})$*-iterative algebra.*

Proof sketch. Let $HA \xrightarrow{\alpha} A$ be an $(\mathbb{I}, \mathcal{M})$-iterative algebra. Any coalgebra $(X, \xi) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$ induces a flat equation morphism

$$e_\xi \equiv X \xrightarrow{\xi} HX \xrightarrow{\mathsf{inl}} HX + A$$

with the unique solution $e_\xi^{\dagger} \colon X \to A$. The morphisms $e_\xi^{\dagger}$ form a cocone in A over the diagram defining T_H. Therefore there exists a unique $h \colon T_H \to A$ in $\mathcal{A}$ with $h \cdot \xi^{\#} = e_\xi^{\dagger}$ for all $(X, \xi) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H)$, which can be shown to be the unique H-algebra homomorphism from (T_H, τ^{-1}) to (A, α). The proof is analogous to [2, Theorem 3.3] ◄

▸ **Example 5.8.** By specialising the above theorem to the settings of Example 4.3.1-3, we recover the following three results from the literature [15, 2, 19]:

1. If $\mathcal{A}$ is a category with binary coproducts and H is a functor with a final coalgebra νH, then νH is the initial completely iterative algebra for H.
2. If $\mathcal{A}$ is locally finitely presentable and H is a finitary functor, then ϱH is the initial iterative algebra for H.
3. If $\mathcal{A}$ is locally finitely presentable with coproducts stable under monomorphisms, and H is a finitary functor preserving monomorphisms, then ϑH is the initial fg-iterative algebra for H.

▸ **Remark 5.9** (Free $(\mathbb{I}, \mathcal{M})$-iterative algebras)**.** The forgetful functor from the category of all $(\mathbb{I}, \mathcal{M})$-iterative algebras and homomorphisms into $\mathcal{A}$ has a left adjoint. The free $(\mathbb{I}, \mathcal{M})$-iterative algebra over an object $X \in \mathcal{A}$ is constructed as follows.

Observe first that the functor $H + X$ satisfies the Assumptions 4.1: it preserves $\mathbb{I}$-colimits because H does and colimits commute with coproducts; it preserves $\mathcal{M}$ because H does and $\mathcal{M}$ is stable under coproducts by Assumption 5.4(ii); and one has $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X) \in \mathbb{I}$ by Assumption 5.4(iii). Therefore Theorem 4.11 (applied to the functor $H + X$ in lieu of H) shows that there exists a final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra

$$T_H X := T_{H+X}$$

for $H + X$, constructed as the colimit of all coalgebras in $\mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X)$. We denote the coalgebra structure of $T_H X$ and its inverse (see Lemma 4.5) by

$$T_H X \xrightarrow{\tau_X} H(T_H X) + X \quad \text{and} \quad H(T_H X) + X \xrightarrow{[\varrho_X, \eta_X]} T_H X,$$

respectively. The latter is the initial $(\mathbb{I}, \mathcal{M})$-iterative algebra for $H + X$ by Theorem 5.7. Then a standard argument identical to [15, Theorem 2.10] shows that $H(T_H X) \xrightarrow{\varrho_X} T_H X$ is the free $(\mathbb{I}, \mathcal{M})$-iterative H-algebra over X, with unit $\eta_X \colon X \to T_H X$.

By definition, $(\mathbb{I}, \mathcal{M})$-iterative algebras have unique solutions for every flat equation morphism. This property implies a much stronger one: every *guarded* equation morphism has a unique solution. Recall that for a Σ-algebra A, a *guarded system of equations* consists of equations $x_1 = t_1, \ldots, x_n = t_n$ where each t_i is either an element of A or a rational Σ-tree over $X + A$ of height at least 1. This concept can be generalised to our present setting as follows:

▸ **Definition 5.10.** Let (A, α) be an $(\mathbb{I}, \mathcal{M})$-iterative algebra. By a *guarded equation morphism* is meant a morphism $e \colon X \to T_H(X + A)$ with $X \in \mathcal{A}_{\mathbb{I}, \mathcal{M}}$ for which there exists a morphism e_0 making the left-hand triangle below commute. A *solution* of e is a morphism $e^\dagger \colon X \to A$ making the right-hand diagram commute. Here $\widetilde{\alpha}$ is the unique homomorphism with $\widetilde{\alpha} \cdot \eta_A = \mathrm{id}_A$, using the freeness of $T_H A$.

$$
\begin{array}{ccc}
X \xrightarrow{\;\;e\;\;} T_H(X + A) & \qquad & X \xrightarrow{\;\;e^\dagger\;\;} A \\
\searrow_{e_0} \quad \uparrow_{[\varrho, \eta \cdot \mathrm{inr}]} & & \;\;\downarrow_{e} \qquad\qquad \uparrow_{\widetilde{\alpha}} \\
H(T_H(X + A)) + A & & T_H(X + A) \xrightarrow[T_H[e^\dagger, A]]{} T_H A
\end{array}
$$

▸ **Theorem 5.11.** *Every $(\mathbb{I}, \mathcal{M})$-iterative algebra admits unique solutions of guarded equation morphisms.*

Proof sketch. Let (A, α) be an $(\mathbb{I}, \mathcal{M})$-iterative algebra, and suppose that $e \colon X \to T_H(X+A)$ is a guarded equation morphism with associated $e_0 \colon X \to HT_H(X + A) + A$. Express the coalgebra $T_H(X + A)$ as an $(\mathbb{I}, \mathcal{M})$-colimit

$$\overline{w}^\# \colon W \to T_H(X + A) \qquad ((W, w) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X + A)),$$

see Remark 4.4. Since H preserves $(\mathbb{I}, \mathcal{M})$-colimits, $\mathcal{M}$ is stable under coproducts and colimits commute with coproducts, it follows that

$$H\overline{w}^\# + A \colon H\overline{W} + A \to H(T_H(X + A)) + A \qquad ((W, w) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X + A))$$

forms an $(\mathbb{I}, \mathcal{M})$-colimit cocone in $\mathcal{A}$. Therefore, since X is $(\mathbb{I}, \mathcal{M})$-presentable, the morphism e_0 factorises through $H\overline{w}^\# + A$ for some $(W, w) \in \mathbf{Coalg}_{\mathbb{I}, \mathcal{M}}(H + X + A)$:

$$
\begin{array}{c}
X \xrightarrow{\;e_0\;} H(T_H(X + A)) + A \\
\searrow_{f_0} \qquad \uparrow_{H\overline{w}^\# + A} \\
H\overline{W} + A
\end{array}
$$

Form the following flat equation morphism, where inm is the middle coproduct injection:

$$s \equiv \overline{W} + X \xrightarrow{[\overline{w}, \mathrm{inm}]} H\overline{W} + X + A \xrightarrow{[\mathrm{inl}, f_0, \mathrm{inr}]} H\overline{W} + A \xrightarrow{H\mathrm{inl} + A} H(\overline{W} + X) + A$$

Since the algebra (A, α) is $(\mathbb{I}, \mathcal{M})$-iterative, s has the unique solution $s^\dagger \colon X \to A$, and one can verify that

$$e^\dagger \equiv X \xrightarrow{\mathrm{inr}} \overline{W} + \overline{X} \xrightarrow{s^\dagger} A$$

is the unique solution of e. The argument is identical to the proof of [2, Theorem 4.6]   ◂

6   Conclusion and Future Work

Our paper has provided the first steps towards a uniform categorical treatment of finite systems and finite recursive specifications, where the meaning of "finite" becomes a parameter that can be chosen according to the applications in mind. As our main technical result we showed that, under suitable assumptions on the categories and functors, there exists a final locally $(\mathbb{I}, \mathcal{M})$-presentable coalgebra that forms a fixpoint of the type functor and captures precisely the behaviours of finite systems. The uniformity of our setting does away with the previous need of developing coalgebraic semantics for each of the competing notions of finiteness independently, often with structurally very similar results and proofs.

In the case of finitary endofunctors on locally finitely presentable categories, the rational fixpoint and its characterisation as the initial iterative algebra formed the starting point for extensive research on iterative monads [5, 4], iteration theories [3], recursive program schemes [18], and proof systems for language equivalence [16, 10] from a (co-)algebraic perspective. We expect that many of the results in *loc. cit.* generalise to our present setting, and thus could be extended to finiteness conditions that so far have not been investigated, e.g. to finitely generated objects of variables.

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