Custom Hypergraph Categories via Generalized Relations*

Dan Marsden[1] and Fabrizio Genovese[2]

1 **Department of Computer Science, University of Oxford, UK**
2 **Department of Computer Science, University of Oxford, UK**

Abstract

Process theories combine a graphical language for compositional reasoning with an underlying categorical semantics. They have been successfully applied to fields such as quantum computation, natural language processing, linear dynamical systems and network theory. When investigating a new application, the question arises of how to identify a suitable process theoretic model.

We present a conceptually motivated parameterized framework for the construction of models for process theories. Our framework generalizes the notion of binary relation along four axes of variation, the truth values, a choice of algebraic structure, the ambient mathematical universe and the choice of proof relevance or provability. The resulting categories are preorder-enriched and provide analogues of relational converse and taking graphs of maps. Our constructions are functorial in the parameter choices, establishing mathematical connections between different application domains. We illustrate our techniques by constructing many existing models from the literature, and new models that open up ground for further development.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Process Theory, Categorical Compositional Semantics, Generalized Relations, Hypergraph Category, Compact Closed Category

Digital Object Identifier 10.4230/LIPIcs.CALCO.2017.17

1 Introduction

The term "process theory" has recently been introduced [11] to describe compositional theories of abstract processes. These process theories typically consist of a graphical language for reasoning about composite systems, and a categorical semantics tailored to the application domain. This compositional perspective has been incredibly successful in reasoning about questions in quantum computation and quantum foundations. The scope of the process theoretic perspective encompasses many other application domains, including natural language processing [12], signal flow graphs [8], control theory [3], Markov processes [5], electrical circuits [4] and even linear algebra [31].

When considering a new application of the process theoretic approach, the question arises of how to find a suitable categorical setting capturing the phenomena of interest. Dagger compact closed categories are of particular importance as they have an elegant graphical calculus, and many of the examples cited above live in compact closed categories.

We illustrate the process of constructing new dagger compact closed categories with two examples in the theory of human cognition, as developed in [18, 19]. This is an unconven-

* This work was partially funded by the AFSOR grant "Algorithmic and Logical Aspects when Composing Meanings" and the FQXi grant "Categorical Compositional Physics".

tional application area, and therefore highlights clearly the challenges faced when trying to model a new problem domain in a process theoretic manner.

As our first example, we consider how the notion of convexity can be incorporated into a compact closed setting. Convexity is important in mathematical models of cognition, where it is argued that the meaningful concepts should be closed under forming mixtures. Informally, if we have a space representing animals, then if two points describe dogs, we would expect any points "in between" should also be models of the concept of being a dog.

An algebraic model of convexity is given by the Eilenberg-Moore algebras of the finite distribution monad. These algebras, referred to as convex algebras, are sets equipped with a well behaved operation for forming convex mixtures of elements. Informally, we denote such a convex mixture as $\sum_i p_i x_i$ where the p_i are positive reals summing to one, and the x_i are elements of the algebra. This notation is not intended to imply there are independent addition and scaling operations that can be applied to the individual elements.

The Eilenberg-Moore category of any monad on **Set**[1], is itself a regular category. Therefore the category of convex algebras is regular and we can form its category of relations, denoted **ConvexRel**. It is well known that the category of relations over a regular category is a dagger compact closed category [20]. Concretely, a convex relation is an ordinary binary relation R which is closed under forming convex mixtures, in the sense that implications of the following form hold

$$R(a_1, b_1) \wedge ... \wedge R(a_n, b_n) \Rightarrow R(\sum_i p_i a_i, \sum_i p_i b_i) \tag{1}$$

A **state** of an object A in a monoidal category is a morphism of type $I \to A$ where I is the monoidal unit. The states in **ConvexRel** are the convex subsets, as we may have hoped. This model was used as the mathematical basis for a compositional model of cognition [7].

As our second example, we return to the mathematics of cognition. It is natural to think about notions of nearness and distance for models of reasoning, a wolf is nearly a dog, a squirrel is closer to being a rat than an elephant. We would therefore like to capture metrics within our model. We now consider how to introduce metrics into a compact closed setting. The construction used in the previous example is not applicable as the various natural categories of metric spaces are not regular. Therefore, we will require a new approach, which will entail a small detour. We begin by introducing the notion of a quantale.

▶ **Definition 1** (Quantale). A **quantale** is a join complete partial order Q with a monoid structure $(\otimes, k)$ satisfying the following distributivity axioms, for all $a, b \in Q$ and $A, B \subseteq Q$

$$a \otimes \left[\bigvee B\right] = \bigvee\{a \otimes b \mid b \in B\} \qquad \left[\bigvee A\right] \otimes b = \bigvee\{a \otimes b \mid a \in A\}$$

A quantale is said to be **commutative** if its monoid structure is commutative.

▶ **Example 2.** Every locale [22] is a commutative quantale, and in particular any complete chain is a commutative quantale with

$$\bigvee A = \sup A \qquad a_1 \otimes a_2 = \min(a, b) \qquad k = \top$$

The **Boolean quantale B** and the **interval quantale I** are the chains $\{0, 1\}$ and $[0, 1]$ of real numbers, with their usual orderings. The quantale **F** is given by the chain $[0, \infty]$ of

[1] In fact, any regular category in which every regular epimorphism has a section.

extended positive reals with the *reverse* ordering. An important example of a commutative quantale that does not correspond to a locale is the **Lawvere quantale C** with underlying set the extended positive reals with reverse order and algebraic structure

$$\bigvee A = \inf A \qquad a_1 \otimes a_2 = a_1 + a_2 \qquad k = 0$$

A binary relation between two sets A and B can be described by its characteristic function $A \times B \to 2$ where 2 is the two element set of Boolean truth values. We can generalize the notion of binary relation by allowing the truth values to be taken in a suitable choice of quantale Q, as a function of the form $A \times B \to Q$. We can see this as a potentially infinite matrix of truth values. These binary relations form a category $\mathbf{Rel}(Q)$, with identities and composition given by suitable generalizations of their matrix theoretic analogues. We will prove that if the quantale of truth values is commutative, $\mathbf{Rel}(Q)$ is in fact dagger compact closed. So we have found another dagger compact closed category, but what has this got to do with metrics? In order to establish the required connection, we note that we can order relations pointwise in the quantale order. This order structure makes $\mathbf{Rel}(Q)$ into a poset-enriched symmetric monoidal category. This means we can consider internal monads, in the sense of formal category theory [32]. These identify important "structured objects" within our categories. For example, internal monads in $\mathbf{Rel}(\mathbf{B})$ are preorders on their underlying set, and the internal monads of $\mathbf{Rel}(\mathbf{I})$ can be seen as a fuzzy generalization of the notion of a preorder. The key example is the internal monads of $\mathbf{Rel}(\mathbf{C})$. These are endo-relations such that $R(a, a) = 0$ and $R(a, b) + R(b, c) \geq R(a, c)$, that is, they are *generalized metric spaces* [25]. The internal monads of $\mathbf{Rel}(\mathbf{F})$ are similar, satisfying $R(a, a) = 0$ and $\max(R(a, b), R(b, c)) \geq R(a, c)$, and can be thought of as *generalized ultrametric spaces*. So in particular, $\mathbf{Rel}(\mathbf{C})$ gives us a partial order enriched dagger compact closed category in which the internal monads are generalized metric spaces. Such categories of relations have been proposed as a unifying categorical setting for investigating various topological notions, see [10, 21]. Multi-valued relations have also been investigated for compositional models of natural language [13].

To recap, we have constructed two compact closed categories using differing techniques that can be found in the literature. Firstly, by exploiting relations respecting algebraic structure, standard monad and regular category theory provided us with a category where the states are exactly convex subsets. Secondly, generalizing the notion of relations in a different direction, we produced a category where the internal monads are generalized metric spaces. So, using rather ad-hoc methods, we have solved two modelling problems using generalizations of binary relations. This prompts several questions:

- How do these constructions relate to each other? In particular, can we simultaneously work with convexity and metrics in an appropriate setting? Can they be seen as instances of a general construction?
- Does the notion of binary relation permit further axes of variation, producing additional examples of compact closed categories? As these parameters vary, can the resulting categories be related? Formally, this is a question of functoriality in a suitable sense.

These questions provide the starting point for our investigations. We also observe that the categories we identified in our examples are both in fact instances of Fong and Kissinger's hypergraph categories [16]. These are a particularly well behaved class of dagger compact closed categories, and this will be our technical setting for the remainder of the paper. We summarize our contributions as follows

- We provide parameterized constructions of hypergraph categories of generalized relations and spans in Theorems 13 and 19, and show they have analogues of relational converse and taking the graph of an underlying morphism. Many further aspects are shown to commute with this important structure. In Section 5 the resulting categories are shown to be appropriately order enriched.

- We address questions of functoriality. Theorem 29 shows that generalized spans can be functorially mapped to generalized relations. Section 6 shows that homomorphisms of truth values functorially induce functors between models. In Section 7 we show that our constructions are functorial in the choices of algebraic structure. We also describe how the algebraic and truth value structures interact, providing connections with notions of resource sensitivity in the sense of linear logic. Finally, in Theorem 47 we show that the functors induced by changes of parameters commute with each other.
- Our methods give explicit concrete descriptions of the mathematical objects of interest, suitable for use in applications. We provide many examples illustrating the flexibility of our techniques, particularly to the construction of new and existing models of natural language processing and cognition applications.

Related Work

Categories of relations have been studied in the form of allegories [17]. This work is somewhat removed from our approach, the heavy use of the modular law does not directly yield the graphical phenomena of interest. Of more direct relevance is the concept of cartesian bicategory of [9]. Although graphical notation is not used directly in this work, these categories can be seen as close relatives of the hypergraph categories resulting from our constructions. The emphasis in the study of cartesian bicategories was characterization rather than construction of models.

A somewhat syntactic approach to constructing categories with graphical calculi is the use of PROPs [26, 24]. They have recently been used to construct various categorical models relating to control theory [8, 33, 14]. These methods begin with syntax and equations, and freely derive a resulting category. This style is most effective when the application under consideration has well understood calculational properties. Our approach instead emphasizes the direct construction of models which can then be investigated for their suitability to a given application.

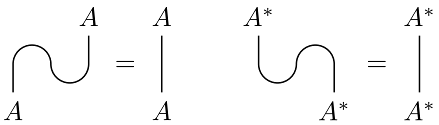
The beautiful work on decorated cospans and corelations of [15, 16], motivated by the program of network theory initiated in [2], is of most direct relevance to our approach. In a precise sense, the decorated corelation construction is completely generic, every hypergraph category is produced by that construction. Our emphasis is different, we do not aim for maximum generality. Instead, our aim is conceptually motivated parameterization. By providing four clearly motivated features that can be adjusted to application needs, we aim for a practical construction with which investigators using process theories can construct new models with desirable features.

2 Mathematical Background

In this section we briefly establish some background. We will be interested in particular types of symmetric monoidal categories, and will make use of their corresponding graphical languages [30]. Technical background on monoidal categories and general categorical notions can be found in [27]. We will also refer to toposes and their internal languages in places, a standard reference is [23]. The paper has been written with the intention that it should be readable without any detailed knowledge of topos theory. For such readers, definitions should be read as if they pertain to ordinary sets, functions and predicate logic. We will write **Set**, **Pos** and **Preord** for the categories of sets, partially ordered sets and preordered sets, with their usual homomorphisms.

▶ **Definition 3** (Compact Closed Category)**.** An object A in a symmetric monoidal category is said to have dual A^* if there exist unit $\eta : I \to A^* \otimes A$ and counit $\epsilon : A \otimes A^* \to I$ morphisms. These morphisms are depicted in the graphical calculus using special notation of Diagram (2) and are required to satisfy the **snake equations** of Diagram (3). [2]

(2)

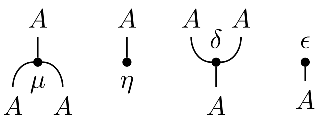


(3)

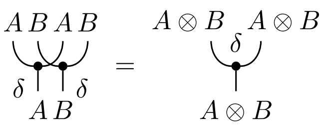
A **compact closed category** is a symmetric monoidal category in which every object has a dual. A compact closed category $\mathcal{A}$, equipped with an identity on objects involution $(-)^\dagger : \mathcal{A}^{op} \to \mathcal{A}$ coherent with the symmetric monoidal compact closed structure, is referred to as a **dagger compact closed category** [1].

▶ **Example 4.** The canonical example of a dagger compact closed category of relevance to the current work is the category **Rel** of sets and binary relations between them. The symmetric monoidal structure is given by cartesian products of sets, and the dagger by the usual converse of relations. Objects are self-dual, with the unit on a set A given by the relation $\{(*, (a, a)) \mid a \in A\}$ and the counit as its converse.

▶ **Definition 5** (Hypergraph Category)**.** A **hypergraph category** is a symmetric monoidal category such that every object A carries both a commutative monoid (η, μ) and a cocommutative comonoid (ϵ, δ) structure, as depicted in Diagram (4), satisfying the coherence conditions of Diagram (5), and their obvious dual [16].

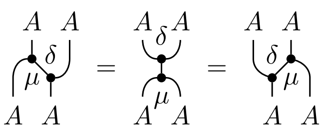


(4)



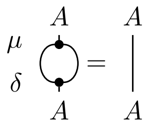
(5)

Here, we overload the use of the symbols μ, η, δ and ϵ to avoid cluttering our diagrams with indices or subscripts. We will exploit similar overloading of names in many places in what follows. The morphisms μ and δ must also satisfy the Frobenius (6) and special (7) axioms:



(6)

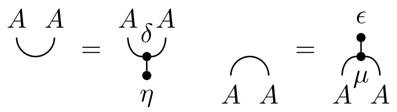
[2] Our string diagrams are oriented bottom to top.



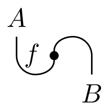
(7)

Our interest in hypergraph categories is that they are a particularly pleasant form of dagger compact closed category, as established by the following well known observation.

▶ **Proposition 6.** *Every hypergraph category is a dagger compact closed category, with the cup and cap given by Equation* (8) *and the dagger of a morphism* $f : A \to B$ *given by its* **transpose***, shown in Diagram* (9)*.*



(8)



(9)

▶ **Example 7.** The category **Rel** is also an example of a hypergraph category. The cocommutative comonoid is given by the relations

$$\epsilon = \{(a, *) \mid a \in A\} \qquad \delta = \{(a, (a, a)) \mid a \in A\}$$

The monoid is the relational converse of the comonoid structure. The induced dagger compact closed structure of Proposition 6 is exactly that described in Example 4.

As a final technical point, we will be working with various categories with finite products. Throughout, we will implicitly assume a choice of terminal object and binary products has been given. To reduce clutter, we therefore resist repeating this assumption in the statements of our subsequent theorems.

3 Relations

The aim in this section is to broadly generalize the notion of binary relation between sets, in order to support our motivating examples, and to provide scope for many other variations. We observed, for sets A and B, and quantale Q, that we can consider a function $A \times B \to Q$ as a relation, with truth values taken in the quantale. For such generalized relations, we define identities[3] and composition of relations $R : A \to B$ and $S : B \to C$ by analogy with the usual notions:

$$1_A(a_1, a_2) = \bigvee \{k \mid a_1 = a_2\}$$

$$(S \circ R)(a, c) = \bigvee \{R(a, b) \otimes S(b, c) \mid b \in B\}$$

We then observe that all of these definitions actually make sense in the internal language of an arbitrary topos. This leads us to the following definition.

[3] Our definition of identities is suitable for interpretation in the internal logic of a topos, unlike the more natural definition by cases.

▶ **Definition 8** (Q-relation)**.** Let $\mathcal{E}$ be a topos, and Q an internal quantale. A Q**-relation** between $\mathcal{E}$ objects A and B is an $\mathcal{E}$-morphism of type $A \times B \to Q$. $\mathcal{E}$-objects and Q-relations between them form a category $\mathbf{Rel}(Q)$, with identities and composition as described above.

Definition 8 is a first step in the right direction, but in order to capture convexity, as discussed in the introduction, we must find a way of incorporating algebraic structure. If we consider an algebraic signature (Σ, E) with set of operations Σ and equations E, the general form of Equation (1), for n-ary operation $\sigma \in \Sigma$, is

$$R(a_1, b_1) \wedge ... \wedge R(a_n, b_n) \Rightarrow R(\sigma(a_1, ..., a_n), \sigma(b_1, ..., b_n))$$

We will require throughout that all operation symbols have finite arity, as is conventional in universal algebra.

It is then natural to consider replacing the logical components of this definition with the structure of our chosen quantale. This leads to the definition we require.

▶ **Definition 9** (Algebraic Q-relation)**.** Let $\mathcal{E}$ be a topos, and Q an internal quantale. Let (Σ, E) be an algebraic variety in $\mathcal{E}$. An **algebraic** Q-relation between (Σ, E)-algebras A and B is a Q-relation between their underlying $\mathcal{E}$-objects such that for each $\sigma \in \Sigma$ the following axiom holds

$$R(a_1, b_1) \otimes ... \otimes R(a_n, b_n) \leq R(\sigma(a_1, ..., a_n), \sigma(b_1, ..., b_n)) \tag{10}$$

(Σ, E)-algebras and algebraic Q-relations form a category $\mathbf{Rel}_{(\Sigma,E)}(Q)$, with identities and composition as for their underlying Q-relations.

There is some subtlety to the interaction between truth values and algebraic structure, we will return to this topic in Section 7. We now continue studying the categorical structure of algebraic Q-relations.

▶ **Proposition 10.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma,E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite products in $\mathcal{E}$.*

The notions of taking the converse of a relation, and taking the graph of an underlying function generalize smoothly to algebraic Q-relations, in a manner that respects all the relevant categorical structure.

▶ **Proposition 11.** *[Converse and Graph] Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal commutative quantale. There are identity on objects strict symmetric monoidal* **converse** *and* **graph** *functors with actions on morphisms:*

$$(-)^\circ : \mathbf{Rel}_{(\Sigma,E)}(Q)^{op} \to \mathbf{Rel}_{(\Sigma,E)}(Q)$$
$$R^\circ(b, a) = R(a, b)$$

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \to \mathbf{Rel}_{(\Sigma,E)}(Q)$$
$$f_\circ(a, b) = \bigvee\{k \mid f(a) = b\}$$

The symmetric monoidal structure on $\mathbf{Alg}(\Sigma, E)$ is the finite product structure.

The graph functor allows us to lift structures from the underlying category of algebras. The following canonical comonoids are of particular conceptual importance.

▶ **Proposition 12.** *Let $\mathcal{E}$ be a category with finite products. Each object A carries a cocommutative comonoid structure satisfying the coherence equations* (5)*, via the canonical morphisms* $! : A \to 1$ *and* $\langle 1_A, 1_A \rangle : A \to A \times A$.

Finally, we are in a position to establish that our categories of algebraic Q-relations are hypergraph categories.

▶ **Theorem 13.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma,E)}(Q)$ is a hypergraph category. The cocommutative comonoid structure is given by the graphs of the canonical comonoids described in Proposition 12, and the monoid structure is given by their converses.*

We quickly return to one of the examples discussed in the introduction.

▶ **Example 14.** The convex algebras discussed in the introduction can be presented by a family of binary operations $+^p$, where $p \in (0,1)$, for forming pairwise convex combinations satisfying suitable equations. Writing Convex for this signature, we can construct **ConvexRel** as $\mathbf{Rel}_{\mathrm{Convex}}(\mathbf{B})$, where $\mathbf{B}$ is the Boolean quantale.

4 Spans

Generalizing the truth values, algebraic structure and ambient category has provided three degrees of freedom for describing custom hypergraph categories. Currently we can vary the underlying topos, quantale and choice of algebraic structure. We now investigate a fourth, final direction of variation.

If we consider a span of sets $A \xleftarrow{f} X \xrightarrow{g} B$, we can consider an element $x \in X$ as a proof witness relating $f(x)$ and $g(x)$. Spans $A \xleftarrow{f} X \xrightarrow{g} B$ and $B \xleftarrow{h} Y \xrightarrow{k} C$ are composed by pulling back g along h. Recall that in **Set** this pullback is given explicitly by $\{(x, y) \mid g(x) = h(y)\}$, with the obvious projections. Therefore, a pair (x, y) relates a and c exactly if x relates a to some b and this b is related to c by y. So, at least for the category **Set**, we can think of spans as proof relevant relations. This is the intuition we now pursue, starting by adjusting the notion of Q-relation in Definition 8 to the setting of spans.

▶ **Definition 15** **(Q-span).** Let $\mathcal{E}$ be a finitely complete category, and Q an internal monoid. A **Q-span** of type $A \to B$ is a quadruple (X, f, g, χ) where $(X, f : X \to A, g : X \to B)$ is a span in $\mathcal{E}$ and $\chi : X \to Q$ is a $\mathcal{E}$-morphism, referred to as the **characteristic morphism**. Two Q-spans $(X, f, g, \chi), (Y, h, k, \xi)$ are composed by composing their underlying spans by pullback, and taking the resulting characteristic morphism to be

$$X \times_C Y \xrightarrow{\langle p_1, p_2 \rangle} X \times Y \xrightarrow{\chi \times \xi} Q \times Q \xrightarrow{\otimes} Q$$

where p_1 and p_2 are the pullback projections.

A **morphism of Q-spans** $\alpha : (X_1, f_1, g_1, \chi_1) \to (X_2, f_2, g_2, \chi_2)$ between two Q-spans of type $A \to B$ is a $\mathcal{E}$-morphism $\alpha : X_1 \to X_2$ such that

$$f_1 = f_2 \circ \alpha \qquad g_1 = g_2 \circ \alpha \qquad \chi_1 = \chi_2 \circ \alpha$$

▶ Remark. When discussing Q-spans in the remainder of this paper, we actually intend isomorphism classes of spans with respect to the homomorphisms of Definition 15. This convention is common when considering categories of ordinary spans, where composition of spans via pullback is only defined up to isomorphism. All definitions and calculations using representatives will respect this isomorphism structure. These isomorphism classes of Q-spans form a category $\mathbf{Span}(Q)$. If we write χ_k for the constant morphism $\chi_k = A \xrightarrow{!} 1 \xrightarrow{k} Q$ then the identity at A is given by the Q-span $(A, 1, 1, \chi_k)$.

The key step now is to incorporate algebraic structure into the picture, paralleling the ideas of Definition 9. In this case, things are slightly more complicated as we have to explicitly administer the proof witnesses in the spans. We also must introduce an ordering on our truth values in order to specify the necessary axiom.

▶ **Definition 16.** Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. For (Σ, E)-algebras A and B, an **algebraic** Q**-span** is a quadruple (X, f, g, χ) which is a Q-span between the underlying $\mathcal{E}$-objects such that for every $\sigma \in \Sigma$ if

$$\bigwedge_i (f(x_i) = a_i \wedge g(x_i) = b_i)$$

then there exists x such that $f(x) = \sigma(a_1, ..., a_n)$ and $g(x) = \sigma(b_1, ..., b_n)$ and

$$\bigotimes_i \chi(x_i) \leq \chi(x)$$

(Σ, E)-algebras and algebraic Q-spans form a category $\mathbf{Span}_{(\Sigma,E)}(Q)$ with identities and composition given as for the underlying Q-spans.

As with the algebraic Q-relations in Section 3, we obtain a symmetric monoidal category with analogues of relational converse and taking graphs.

▶ **Proposition 17.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma,E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite product structure in $\mathcal{E}$.*

▶ **Proposition 18.** *[Converse and Graph] Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. There are identity on objects strict symmetric monoidal* **converse** *and* **graph** *functors with actions on morphisms:*

$$(-)^\circ : \mathbf{Span}_{(\Sigma,E)}(Q)^{op} \to \mathbf{Span}_{(\Sigma,E)}(Q)$$
$$(X, f, g, \chi)^\circ = (X, g, f, \chi)$$

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \to \mathbf{Span}_{(\Sigma,E)}(Q)$$
$$f_\circ = (A, 1, f, \chi_k)$$

As before, we can exploit the graph construction and the canonical comonoids of Proposition 12 to establish the existence of a hypergraph structure.

▶ **Theorem 19.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma,E)}(Q)$ is a hypergraph category. The cocommutative comonoid structure is given by the graphs of the canonical comonoids described in Proposition 12, and the monoid structure is given by their converses.*

This construction presents new possibilities, that can be combined with other features, opening fresh directions for investigation that may not have been immediately apparent.

▶ **Example 20.** The span construction allows us to build variations on the models we are already interested in. For example, we can now consider a proof relevant version of the model in Example 14. From a practical perspective, this presents the possibility of models in which we can describe the interaction of cognitive phenomena, and provide quantitative evidence for any relationships that we conclude hold.

▶ **Example 21.** Instead of using **Set** as our base topos in our models, we could consider using a presheaf topos $[\mathcal{C}^{op}, \mathbf{Set}]$ for a small category $\mathcal{C}$. This allows us to construct models using "sets varying with context", incorporating all the features discussed in the previous examples. In linguistic or cognitive examples, contexts could describe time, the agents involved or the broader setting in which meaning should be interpreted. These context sensitive models present a lot of new expressive potential, and will be investigated in detail in future work.

5 Order Enrichment

In order to meaningfully discuss internal monads, we require some 2-categorical structure on our relational constructions. Specifically, we introduce an appropriate ordering on our morphisms. Order enrichment is also important from a practical perspective when modelling real world applications. For example, in natural language applications, we are often interested in phenomena such as ambiguity [29, 28] and lexical entailment [6], and these are best studied from an order theoretic perspective.

Generalizing the situation for ordinary set theoretic binary relations, we introduce an ordering on Q-relations.

▶ **Definition 22.** Let $\mathcal{E}$ be a topos and Q an internal quantale. We define a partial order on Q-relations as follows

$$R \subseteq R' \quad \text{iff} \quad \forall a, b. R(a, b) \leq R'(a, b)$$

Algebraic Q-relations are ordered similarly, by comparing their underlying Q-relations.

▶ **Theorem 23.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma,E)}(Q)$ is a partially ordered symmetric monoidal category.*

Q-spans can also be ordered, in a manner analogous to that for relations, but explicitly taking into account the proof witnesses.

▶ **Definition 24.** For topos $\mathcal{E}$ and internal partially ordered monoid Q, we define a preorder on Q-spans by saying $(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$ if there is a $\mathcal{E}$-monomorphism $m : X_1 \to X_2$ such that $f_1 = f_2 \circ m$, $g_1 = g_2 \circ m$ and $\forall x. \chi_1(x) \leq \chi_2(m(x))$. Algebraic Q-spans are ordered similarly, by comparing their underlying Q-spans.

▶ **Theorem 25.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma,E)}(Q)$ is a preorder enriched symmetric monoidal category.*

The orders are respected by the important converse operation

▶ **Proposition 26.** *Let $\mathcal{E}$ be a topos, and (Σ, E) a variety in $\mathcal{E}$. If Q is an internal quantale, the converse functor of Proposition 11 is a partially ordered functor. If Q is an internal partially order monoid, the converse functor of Proposition 18 is a preordered functor.*

The order enrichment of Q-relations and Q-spans is crucial for us to be able to consider the internal monads central to the second example of the introduction.

▶ **Example 27.** The model incorporating metric spaces as internal monads, as discussed in the introduction, can be constructed with base topos **Set**, the empty algebraic signature and using the Lawvere quantale $\mathbf{C}$ as the choice of truth values.

Now that we have both algebraic and order structure available to us within the same construction, we can consider combining the features we are interested in, by making appropriate choices for the parameters used in the construction.

▶ **Example 28** (Convexity and Metrics)**.** We now see that we can combine *both* the convex and metric features in a single model. With underlying topos **Set**, we take our algebraic structure as in Example 14 and our quantale $\mathbf{C}$ as in Example 27. In this case we find the internal monads are distance measures $d : A \times A \to [0, \infty]$ satisfying the conditions

$$d(a, a) = 0 \qquad d(a, b) + d(b, c) \geq d(a, c) \qquad d(a_1, a_2) + d(b_1, b_2) \geq d(a_1 +^p b_1, a_2 +^p b_2)$$

These are generalized metric spaces that respect convex structure. The usual metric on $\mathbb{R}^n$ is an example of such a metric.

We can now consider the simplest aspect of functors induced by changes of parameters, the binary choice between proof relevance and provability. The next theorem shows that the orders on relations and spans are compatible, in the sense that we can collapse spans to relations using the join of the quantale to choose optimal truth values, and this mapping is functorial and respects the order structure.

▶ **Theorem 29.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$ and Q an internal commutative quantale. There is an identity on objects, strict symmetric monoidal* **Preord***-functor V, with action on morphisms:*

$$V : \mathbf{Span}_{(\Sigma,E)}(Q) \to \mathbf{Rel}_{(\Sigma,E)}(Q)$$
$$V(X, f, g, \chi)(a, b) = \bigvee\{\chi(x) \mid f(x) = a \wedge g(x) = b\}$$

V commutes with graphs and converses in that $V \circ (-)_\circ = (-)_\circ$ and $(-)^\circ \circ V^{op} = V \circ (-)^\circ$.

6 Changing Truth Values

We would expect that homomorphisms between our structures of truth values lead to functorial relationships between models. This all goes through very smoothly, as we now elaborate. Firstly, for algebraic Q-relations, it is natural to consider internal quantale homomorphisms.

▶ **Theorem 30.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and $h : Q_1 \to Q_2$ a morphism of internal commutative quantales. There is an identity on objects, strict symmetric monoidal* **Pos***-functor $h^* : \mathbf{Rel}_{(\Sigma,E)}(Q_1) \to \mathbf{Rel}_{(\Sigma,E)}(Q_2)$, with action on morphisms $R \mapsto h \circ R$. The assignment $h \mapsto h^*$ is functorial.*

In the case of the span constructions, morphisms of partially ordered monoids are the appropriate notion of homomorphism to consider.

▶ **Theorem 31.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and $h : Q_1 \to Q_2$ a morphism of internal partially ordered commutative monoids. There is an identity on objects, strict symmetric monoidal* **Preord***-functor $h^* : \mathbf{Span}_{(\Sigma,E)}(Q_1) \to \mathbf{Span}_{(\Sigma,E)}(Q_2)$, with action on morphisms $(X, f, g, \chi) \mapsto (X, f, g, h \circ \chi)$. The assignment $h \mapsto h^*$ is functorial.*

Both these functors commute with graphs and converses.

▶ **Proposition 32.** *With the same assumptions, the induced functors of Theorems 30 and 31 commute with graphs and converses. That is, $h^* \circ (-)_\circ = (-)_\circ$ and $(-)^\circ \circ (h^*)^{op} = h^* \circ (-)^\circ$.*

▶ **Example 33.** For any commutative quantale Q there is a partially ordered monoid morphism $1 \to Q$, induced by the monoid unit. Here, 1 is the terminal quantale. Therefore there is a strict symmetric monoidal functor $\mathbf{Span}_{(\Sigma,E)}(1) \to \mathbf{Span}_{(\Sigma,E)}(Q)$. This example motivates our use of partially ordered monoids, rather than simply restricting to the quantales of interest in our primary applications, as the required morphism is not a quantale morphism.

▶ **Example 34.** There is a quantale morphism $\mathbf{B} \to \mathbf{C}$ from the Boolean to the Lawvere quantale. The induced functor identifies the ordinary binary relations as living within the category $\mathbf{Rel}(\mathbf{C})$ that we introduced to capture metric spaces as internal monads.

When using ordinary relational models of natural language, the meanings of two sentences are typically compared using the inner product of the corresponding states. This is a crude measure of similarity as it is a simple Boolean test of overlap. By embedding Boolean relations into $\mathbf{Rel}(\mathbf{C})$, more subtle comparisons can be made using a metric. For example we can measure how far apart two states are at their nearest point.

7 Algebraic Structure

We now investigate the interaction between truth values and algebraic structure. Again, this will lead to functorial relationships between models, but the subject is more delicate than in the previous sections. The essential detail is that in Equation (10) is only required to hold for the operations in our signature. It does not directly say anything about derived terms and operations. We will require several definitions in order to make the situation precise.

▶ **Definition 35.** Let (Σ, E) be an algebraic signature. We say that a term τ over a finite set of variables is **affine** if it uses each variable at most once and **relevant** if it uses each variable at least once. A term is **linear** if it is both affine and relevant. We will refer to a term as **cartesian** to emphasize that its use of variables is unrestricted. We use the same terminology for the derived operation associated to τ. An **interpretation** of signature (Σ_1, E_1) in signature (Σ_2, E_2) is a mapping assigning each $\sigma \in \Sigma_1$ to a derived term of (Σ_2, E_2) of the same arity, such that the equations E_1 can be proved in equational logic from E_2. We say that an interpretation is **linear**, **affine**, **relevant** or **cartesian** if all the derived terms used in the interpretation are suitably restricted. It is standard that every interpretation i contravariantly induces a functor $\hat{\imath}$ between the categories of algebras.

▶ **Definition 36.** Let $\mathcal{E}$ be a topos. If Q is an internal quantale, we say that a Q-relation R is **affine** if $R(a_1, b_1) \otimes R(a_2, b_2) \leq R(a_1, b_1)$ and **relevant** if $R(a, b) \leq R(a, b) \otimes R(a, b)$. R is **cartesian** if it is both affine and linear. We say that R is **linear** to emphasize that no additional axioms are assumed to hold. Similarly, if Q is an internal partially ordered monoid, we say that a Q-span (X, f, g, χ) is **affine** if $\chi(x_1) \otimes \chi(x_2) \leq \chi(x_1)$, and **relevant** if $\chi(x) \leq \chi(x) \otimes \chi(x)$. A Q-span is said to be **cartesian** if it is both affine and relevant, and **linear** if no additional axioms are assumed to hold.

Our terminology is derived from that sometimes used for variants of linear logic. If we view truth values as resources, the question is when can these resources be "copied" or "deleted". The next proposition shows that if our truth values are well behaved, so are our morphisms.

▶ **Lemma 37.** *Let Q be an internal quantale. If $p \otimes q \leq p$ holds, every relation is affine and if $p \leq p \otimes p$ then every relation is relevant. Similarly, if Q is an internal partially ordered monoid, if $p \otimes q \leq p$ holds, every span is affine and if $p \leq p \otimes p$ then every span is relevant.*

Each of our special classes of relations and spans forms a hypergraph category.

▶ **Theorem 38.** *Let $\mathcal{E}$ be a topos and (Σ, E) a variety in $\mathcal{E}$. If Q is a commutative quantale, the affine, relevant and cartesian relations each form a sub-hypergraph category of $\mathbf{Rel}_{(\Sigma,E)}(Q)$. If Q is a commutative partially ordered monoid, the affine, relevant and cartesian spans each form a sub-hypergraph category of $\mathbf{Span}_{(\Sigma,E)}(Q)$. In each case, the morphisms in the image of the graph functor are all cartesian.*

▶ **Definition 39.** We will write $\mathbf{Rel}^{\mathrm{cart}}_{(\Sigma,E)}(Q)$ and $\mathbf{Span}^{\mathrm{cart}}_{(\Sigma,E)}(Q)$ for the sub-hypergraph categories of cartesian relations and spans described in Theorem 38, and use similar notation for the other restricted classes of morphisms.

Our restricted classes of relations respect the corresponding classes of derived terms.

▶ **Proposition 40.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$ and Q an internal commutative quantale. For linear (affine, relevant, cartesian) algebraic Q-relation $R : A \to B$ the axiom*

$$R(a_1, b_1) \otimes ... \otimes R(a_n, b_n) \leq R(\tau(a_1, ..., a_n), \tau(b_1, ..., b_n))$$

holds for every linear (affine, relevant, cartesian) n-ary derived operation τ.

Spans with sufficient structure also respect the corresponding types of derived terms.

▶ **Proposition 41.** *Let $\mathcal{E}$ be a topos, (Σ, E) a variety in $\mathcal{E}$, and Q an internal partially ordered commutative monoid. For (Σ, E)-algebras A and B, and linear (affine, relevant, cartesian) algebraic Q-span (X, f, g, χ) and n-ary linear (affine, relevant, cartesian) term τ if $\bigwedge_i (f(x_i) = a_i \wedge g(x_i) = b_i)$ then there exists x such that $f(x) = \tau(a_1, ..., a_n)$, $g(x) = \tau(b_1, ..., b_n)$ and $\bigotimes_i \chi(x_i) \leq \chi(x)$.*

Finally, we can establish a contravariant functorial relationship between interpretations and functors between relational models.

▶ **Theorem 42.** *Let $\mathcal{E}$ be a topos and Q an internal commutative quantale. Let $i : (\Sigma_1, E_1) \to (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is an identity on morphisms strict symmetric monoidal functor $i^* : \mathbf{Rel}^{\mathrm{lin}}_{(\Sigma_2,E_2)}(Q) \to \mathbf{Rel}^{\mathrm{lin}}_{(\Sigma_1,E_1)}(Q)$, sending each (Σ_2, E_2)-algebra to the corresponding (Σ_1, E_1)-algebra under the interpretation. The assignment $i \mapsto i^*$ extends to a contravariant functor. Similar results hold for affine, relevant and cartesian interpretations and relations. In each case, the induced functors commute with graphs and converses. That is, $(-)_\circ \circ \hat{i} = i^* \circ (-)_\circ$ and $(-)^\circ \circ (i^*)^{op} = i^* \circ (-)^\circ$, where $\hat{i}$ is the interpretation induced functor of Definition 35.*

A similar contravariant functorial relationship holds between interpretations and functors between span based models.

▶ **Theorem 43.** *Let $\mathcal{E}$ be a topos and Q an internal partially ordered commutative monoid. Let $i : (\Sigma_1, E_1) \to (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is an identity* *on morphisms strict monoidal functor* $i^* : \mathbf{Span}^{\mathrm{lin}}_{(\Sigma_2,E_2)}(Q) \to \mathbf{Span}^{\mathrm{lin}}_{(\Sigma_1,E_1)}(Q)$*, sending each* (Σ_2, E_2)*-algebra to the corresponding* (Σ_1, E_1)*-algebra under the interpretation. The assignment* $i \mapsto i^*$ *extends to a contravariant functor. Similar results hold for affine, relevant and cartesian interpretations and spans. In each case, the induced functors commute with graphs and converses. That is,* $(-)_\circ \circ \hat{i} = i^* \circ (-)_\circ$ *and* $(-)_\circ \circ (i^*)^{op} = i^*(-)_\circ$ *where* $\hat{i}$ *is the interpretation induced functor of Definition 35.*

We also note that the extensional collapse functor of Theorem 29 also respects our different classes of spans and relations.

▶ **Proposition 44.** *Let* $\mathcal{E}$ *be a topos,* (Σ, E) *a variety in* $\mathcal{E}$ *and* Q *an internal commutative quantale. The functor* V *of Theorem 29 maps cartesian, affine and linear algebraic* Q*-spans to the corresponding class of algebraic* Q*-relations.*

▶ **Example 45.** Let $(\emptyset, \emptyset)$ denote the signature with no operations or equations. For any signature (Σ, E) there is a trivial *linear* interpretation $(\emptyset, \emptyset) \to (\Sigma, E)$. We therefore have, for every choice of internal quantale Q, strict symmetric monoidal forgetful functors $\mathbf{Rel}_{(\Sigma,E)}(Q) \to \mathbf{Rel}_{(\emptyset,\emptyset)}(Q)$ and $\mathbf{Span}_{(\Sigma,E)}(Q) \to \mathbf{Span}_{(\emptyset,\emptyset)}(Q)$.

▶ **Example 46.** The signature for convex algebras has linear interpretations in both real vector spaces and affine semilattices. Therefore for any commutative quantale Q, we find relations between real vector spaces and relations between affine join semilattices as sub-hypergraph categories of $\mathbf{Rel}_{Convex}(Q)$.

Finally, we establish that our various induced functors between models are independent, in that they all commute with each other.

▶ **Theorem 47.** *Let* $\mathcal{E}$ *be a topos,* $h : Q_1 \to Q_2$ *a morphism of internal commutative quantales and* $i : (\Sigma_1, E_1) \to (\Sigma_2, E_2)$ *a cartesian interpretation. For the induced functors of Theorems 30, 31, 42 and 43, the following diagram commutes:*

$$\begin{tikzcd}
& \mathbf{Span}^{\mathrm{cart}}_{(\Sigma_2,E_2)}(Q_1) \arrow[rr, "i^*"] \arrow[dl, "h^*"'] \arrow[dd] & & \mathbf{Span}^{\mathrm{cart}}_{(\Sigma_1,E_1)}(Q_1) \arrow[dl, "h^*"'] \arrow[dd] \\
\mathbf{Span}^{\mathrm{cart}}_{(\Sigma_2,E_2)}(Q_2) \arrow[rr, "i^*"] \arrow[dd] & & \mathbf{Span}^{\mathrm{cart}}_{(\Sigma_1,E_1)}(Q_2) \arrow[dd] & \\
& \mathbf{Rel}^{\mathrm{cart}}_{(\Sigma_2,E_2)}(Q_1) \arrow[rr, "i^*"'] \arrow[dl, "h^*"'] & & \mathbf{Rel}^{\mathrm{cart}}_{(\Sigma_1,E_1)}(Q_1) \arrow[dl, "h^*"] \\
\mathbf{Rel}^{\mathrm{cart}}_{(\Sigma_2,E_2)}(Q_2) \arrow[rr, "i^*"] & & \mathbf{Rel}^{\mathrm{cart}}_{(\Sigma_1,E_1)}(Q_2) &
\end{tikzcd}$$

where the vertical arrows are the V *functors of Theorem 29. Similar diagrams commute for affine, relevant and linear interpretations, relations and spans.*

8 Conclusion

We have developed a parameterized scheme for constructing order enriched hypergraph categories, by generalizing the notion of binary relation along four axes of variation: the ambient category, the truth values, the algebraic structure and the choice between proof relevance and provability. This construction provides a concrete, conceptually motivated approach for producing models of process theories when investigating new applications. As well as recovering many existing models, by allowing us to combine features, the framework points to new settings in which features such as convexity, distances, contextual meaning and proof witnesses can be incorporated into a single model. Detailed exploration of these new models in linguistic and cognition applications is left to later work.

Acknowledgments. The authors would like to thank Bob Coecke, Ignacio Funke, Kohei Kishida and Martha Lewis for feedback and discussions. We would also like to thank the anonymous reviewers for their comments and suggestions.

References

1. Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Logic in Computer Science, 2004. Proceedings of the 19th Annual IEEE Symposium on*, pages 415–425. IEEE, 2004.
2. John C Baez. Network theory (part 1). mathematical blog entry. URL: `https://johncarlosbaez.wordpress.com/2011/03/04/network-theory-part-1/`.
3. John C Baez and Jason Erbele. Categories in control. *Theory and Applications of Categories*, 30(24):836–881, 2015.
4. John C Baez and Brendan Fong. A compositional framework for passive linear networks. *arXiv preprint arXiv:1504.05625*, 2015.
5. John C Baez, Brendan Fong, and Blake S Pollard. A compositional framework for Markov processes. *Journal of Mathematical Physics*, 57(3):033301, 2016.
6. Dea Bankova. Comparing meaning in language and cognition - p-hypononymy, concept combination, asymmetric similarity. Master's thesis, University of Oxford, 2015.
7. Josef Bolt, Bob Coecke, Fabrizio Genovese, Martha Lewis, Daniel Marsden, and Robin Piedeleu. Interacting conceptual spaces. In Dimitrios Kartsaklis, Martha Lewis, and Laura Rimell, editors, *Proceedings of the 2016 Workshop on Semantic Spaces at the Intersection of NLP, Physics and Cognitive Science, SLPCS@QPL 2016, Glasgow, Scotland, 11th June 2016.*, volume 221 of *EPTCS*, pages 11–19, 2016. `doi:10.4204/EPTCS.221.2`.
8. Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. Full abstraction for signal flow graphs. *ACM SIGPLAN Notices*, 50(1):515–526, 2015.
9. Aurelio Carboni and Robert F C Walters. Cartesian bicategories I. *Journal of pure and applied algebra*, 49(1-2):11–32, 1987.
10. Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory–a common approach. *Journal of Pure and Applied Algebra*, 179(1):13–47, 2003.
11. Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes. A First Course in Quantum Theory and Diagrammatic Reasoning.* Cambridge University Press, 2017.
12. Bob Coecke, Mehrnoosh Sadrzadeh, and Stephen Clark. Mathematical foundations for distributed compositional model of meaning. Lambek festschrift. *Linguistic Analysis*, 36:345–384, 2010.
13. Matej Dostal and Mehrnoosh Sadrzadeh. Many valued generalised quantifiers for natural language in the DisCoCat model. Technical report, QMUL, 2016.
14. Jason Erbele. *Categories in Control: Applied PROPs.* PhD thesis, University of California Riverside, 2016.
15. Brendan Fong. Decorated cospans. *Theory and Applications of Categories*, 30(33):1096–1120, 2015.
16. Brendan Fong. *The Algebra of Open and Interconnected Systems.* PhD thesis, University of Oxford, 2016.
17. Peter J Freyd and Andre Scedrov. *Categories, allegories*, volume 39. Elsevier, 1990.
18. Peter Gärdenfors. *Conceptual spaces: The geometry of thought.* MIT press, 2004.

19 Peter Gärdenfors. *The geometry of meaning: Semantics based on conceptual spaces.* MIT Press, 2014.

20 Chris Heunen and Sean Tull. Categories of relations as models of quantum theory. In Chris Heunen, Peter Selinger, and Jamie Vicary, editors, Proceedings of the 12th International Workshop on *Quantum Physics and Logic,* Oxford, U.K., July 15-17, 2015, volume 195 of *EPTCS*, pages 247–261. Open Publishing Association, 2015. `doi:10.4204/EPTCS.195.18`.

21 Dirk Hofmann, Gavin J Seal, and Walter Tholen. *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, volume 153. Cambridge University Press, 2014.

22 Peter T Johnstone. *Stone spaces*, volume 3. Cambridge University Press, 1986.

23 Peter T Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume II. Oxford University Press, 2002.

24 Stephen Lack. Composing PROPs. *Theory and Applications of Categories*, 13(9):147–163, 2004.

25 F William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del seminario matématico e fisico di Milano*, 43(1):135–166, 1973.

26 Saunders MacLane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40–106, 1965.

27 Saunders MacLane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, 1998.

28 R. Piedeleu, D. Kartsaklis, B. Coecke, and M Sadrzadeh. Open system categorical quantum semantics in natural language processing. In Lawrence S. Moss and Pawel Sobocinski, editors, *6th Conference on Algebra and Coalgebra in Computer Science, CALCO 2015*, volume 35 of *LIPIcs*, pages 270–289. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. `doi:10.4230/LIPIcs.CALCO.2015.270`.

29 Robin Piedeleu. Ambiguity in categorical models of meaning. Master's thesis, University of Oxford, 2014.

30 Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

31 Pawel Sobocinski. Graphical linear algebra. mathematical blog. URL: `https://graphicallinearalgebra.net/`.

32 Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

33 Fabio Zanasi. *Interacting Hopf Algebras: The Theory of Linear Systems.* PhD thesis, l'École Normale Supérieure de Lyon, 2015.