ZX-Calculus: Cyclotomic Supplementarity and Incompleteness for Clifford+T Quantum Mechanics

Emmanuel Jeandel[1], Simon Perdrix[2], Renaud Vilmart[3], and Quanlong Wang[4]

1   Université de Lorraine, CNRS, Inria, LORIA, Nancy, France
    `emmanuel.jeandel@loria.fr`
2   Université de Lorraine, CNRS, Inria, LORIA, Nancy, France
    `simon.perdrix@loria.fr`
3   Université de Lorraine, CNRS, Inria, LORIA, Nancy, France
    `renaud.vilmart@loria.fr`
4   Dept. of Computer Science, University of Oxford, UK
    `quanlong.wang@cs.ox.ac.uk`

Abstract

The ZX-Calculus is a powerful graphical language for quantum mechanics and quantum information processing. The completeness of the language – i.e. the ability to derive any true equation – is a crucial question. In the quest of a complete ZX-calculus, supplementarity has been recently proved to be necessary for quantum diagram reasoning (MFCS 2016). Roughly speaking, supplementarity consists in merging two subdiagrams when they are parameterized by antipodal angles. We introduce a generalised supplementarity – called cyclotomic supplementarity – which consists in merging n subdiagrams at once, when the n angles divide the circle into equal parts. We show that when n is an odd prime number, the cyclotomic supplementarity cannot be derived, leading to a countable family of new axioms for diagrammatic quantum reasoning.

We exhibit another new simple axiom that cannot be derived from the existing rules of the ZX-Calculus, implying in particular the incompleteness of the language for the so-called Clifford+T quantum mechanics. We end up with a new axiomatisation of an extended ZX-Calculus, including an axiom schema for the cyclotomic supplementarity.

1   Introduction

The ZX-Calculus is a powerful diagrammatic language for reasoning in quantum mechanics introduced by Coecke and Duncan [8]. Every diagram is composed of three kinds of vertices: red and green dots which are parametrised by an angle, and a yellow box; and each diagram represents a matrix thanks to the so-called standard interpretation. Moreover, any quantum transformation can be expressed using ZX-diagrams, meaning they are *universal*. For instance, some particular states can be represented as evidenced by [9]. The language initially describes pure quantum state transformations, though some work has been made to adapt it to non pure evolutions [7, 10] and measurement-based quantum computing [12].

Unlike quantum circuits, the ZX-Calculus comes with a set of equalities between diagrams that preserve the matrix that is represented. Hence, using a succession of locally applied

such equalities, one can prove that two diagrams represent the same matrix, for the language is *sound* i.e. all the equalities do indeed preserve the matrix.

The converse of soundness is called *completeness*. Here, it amounts to being able to transform any diagram into another one, as long as both represent the same matrix. Hence, the concept of completeness is here totally defined by one particular interpretation, the *standard interpretation*, unlike other definitions of completeness such as in [18] in which it is related to a whole family of interpretations.

It has been proven that the ZX-Calculus is in general not complete [17]. Yet, some restrictions have been proven to be complete. The $\frac{\pi}{2}$-fragment – the language restricted to angles that are multiples of $\frac{\pi}{2}$, which represents the *stabiliser quantum mechanics* – is complete [1], its pseudo-normal form using graph states introduced in the case of the ZX-Calculus in [11]. Moreover, this proof can be adapted to show the completeness of a ZX-like calculi used for graphically representing Spekken's toy model [4, 20] or for graphically representing the real matrices [15]. The π-fragment – representing the *real stabiliser quantum mechanics* – is also complete, with a slightly adapted set of rules [13].

A fragment is *approximately universal* when any quantum transformation can be approached with arbitrarily great precision using only the angles in the fragment. Sadly, the $\frac{\pi}{2}$-fragment is not approximately universal, but the $\frac{\pi}{4}$-fragment is [19]. It is called the *Clifford+T quantum mechanics*. Completeness for this fragment was an open question, one of the main ones in the fields of categorical quantum mechanics [6] – even though a partial answer has been given for the fragment composed of path diagrams involving angles multiple of $\frac{\pi}{4}$ [2].

In this paper, we show that in the ZX-Calculus, the $\frac{\pi}{4}$-fragment is not complete, showing that a scalar equality is derivable using matrices, but not diagrammatically. We propose to replace the "inverse rule" by this equality, and show that it can prove the former one as well as a third one: the "zero rule". Notice that this axiomatisation has been recently turned into complete axiomatisation of the ZX-calculus for this fragment [14].

We also show that an infinite number of fragments are also incomplete, by proving that a generalised form of the "supplementarity rule" [16] cannot be derived in them. Supplementarity, which has been proved to be necessary, consists in merging two subdiagrams when they are parameterized by antipodal angles. The generalised supplementarity – called cyclotomic supplementarity – consists in merging n subdiagrams at once, when the n angles divide the circle into equal parts. We show that when n is an odd prime number, the cyclotomic supplementarity cannot be derived, leading to a countable family of new axioms for diagrammatic quantum reasoning.

Finally, we propose to add the new scalar equation, as well as the cyclotomic supplementarity to the set of rules, and to get rid of the now obsolete "inverse" and "zero" rules. We address the question of the incompleteness of the – new – general ZX-calculus with a modified version of the proof by Zamdzhiev and Schröder de Witt [17], for theirs does not stand any more because of the introduction of the cyclotomic supplementarity.

We present the ZX-Calculus in section 2, prove the incompleteness of the $\frac{\pi}{4}$-fragment and give a new scalar rule in section 3, and in section 4 we show how to generalise the supplementarity rule, discuss the way some are derivable from others, present the altered set of rules and give a new proof of incompleteness of the general ZX-Calculus. Some proofs are omitted in this paper, you can find them at `https://hal.archives-ouvertes.fr/hal-01445707`.

2 ZX-Calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \to l$ is an open graph with k inputs and l outputs – read from top to bottom – and is generated by:

$R_Z^{(n,m)}(\alpha) : n \to m$	(green spider, n inputs, m outputs, α)	$R_X^{(n,m)}(\alpha) : n \to m$	(red spider, n inputs, m outputs, α)
$H : 1 \to 1$	(yellow box)	$e : 0 \to 0$	(empty dashed box)
$\mathbb{I} : 1 \to 1$	(vertical line)	$\sigma : 2 \to 2$	(swap)
$\epsilon : 2 \to 0$	(cup)	$\eta : 0 \to 2$	(cap)

where $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$

and the two compositions:

- Spacial Composition: for any $D_1 : a \to b$ and $D_2 : c \to d$, $D_1 \otimes D_2 : a + c \to b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1.
- Sequential Composition: for any $D_1 : a \to b$ and $D_2 : b \to c$, $D_2 \circ D_1 : a \to c$ consists in placing D_1 on the top of D_2, connecting the outputs of D_1 to the inputs of D_2.

The standard interpretation of the stabiliser ZX-diagrams associates to any diagram $D : n \to m$ a linear map $[\![D]\!] : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m}$ inductively defined as follows:

$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \qquad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$$

$$\left[\!\!\left[\; \raisebox{0pt}{\scriptsize$\square$} \;\right]\!\!\right] := (1) \qquad \left[\!\!\left[\; | \;\right]\!\!\right] := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left[\!\!\left[\; \raisebox{0pt}{\scriptsize$\square$} \;\right]\!\!\right] := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\left[\!\!\left[\; \times \;\right]\!\!\right] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \left[\!\!\left[\; \cup \;\right]\!\!\right] := \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \qquad \left[\!\!\left[\; \cap \;\right]\!\!\right] := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[\![\,\text{green } \alpha\,]\!] := \left(1 + e^{i\alpha}\right) \qquad \left[\!\!\left[\; \raisebox{0pt}{\scriptsize green α, n in m out} \;\right]\!\!\right] := 2^m \overbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \quad (n + m > 0)$$

For any $n, m \geq 0$ and $\alpha \in \mathbb{R}$, $\left[\!\!\left[\; \raisebox{0pt}{\scriptsize red α, n in m out} \;\right]\!\!\right] = \left[\!\!\left[\; \raisebox{0pt}{\scriptsize$\square$} \;\right]\!\!\right]^{\otimes m} \circ \left[\!\!\left[\; \raisebox{0pt}{\scriptsize green α, n in m out} \;\right]\!\!\right] \circ \left[\!\!\left[\; \raisebox{0pt}{\scriptsize$\square$} \;\right]\!\!\right]^{\otimes n}$

(where $M^{\otimes 0} = (1)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$).

To simplify, the red and green nodes will be represented empty when holding a 0 angle:

Also in order to make the diagrams a little less heavy, when n copies of the same subdiagram occur, we will use the notation $(.)^{\otimes n}$ as defined above.

With the general calculus – with angles being in $\mathbb{R}$ – we can represent any matrix of size a power of 2 i.e. ZX-Diagrams are universal:

$$\forall A \in \mathbb{C}^{2^n} \times \mathbb{C}^{2^m}, \ \exists D, \ [\![D]\!] = A$$

This requires dealing with an uncountable set of angles, so it is generally preferred to work with *approximate* universality – the ability to approximate any linear map with arbitrary accuracy – in which only a finite set of angles is involved. The $\frac{\pi}{4}$-fragment – ZX-diagrams where all angles are multiples of $\frac{\pi}{4}$ – is one such approximately universal fragment, whereas the $\frac{\pi}{2}$-fragment is not.

2.2   Calculus

The diagrammatic representation of a matrix is not unique in the ZX-Calculus. Hence, a set of equalities has been proposed to axiomatise the language. This set is summed up in Figure 1.

The initial set of axioms [8] included the rules (S1), (S2), (S3), (B1), (B2), (K1), (K2) and (H). The rule (EU) has been proven to be necessary in [11] and the rules (ZO) and (IV) result from [3, 5]. Finally, the rule (SUP) has been added in [16].

When we can show that a diagram D_1 is equal to another one, D_2, using a succession of equalities of this set, we write $ZX \vdash D_1 = D_2$. Given that the rules are sound, this implies that $[\![D_1]\!] = [\![D_2]\!]$. The rules can be applied to any subdiagram, meaning, for any diagram D:

$$(ZX \vdash D_1 = D_2) \quad \Rightarrow \quad \left\{ \begin{array}{ll} (ZX \vdash D_1 \circ D = D_2 \circ D) & \wedge \quad (ZX \vdash D \circ D_1 = D \circ D_2) \\ (ZX \vdash D_1 \otimes D = D_2 \otimes D) & \wedge \quad (ZX \vdash D \otimes D_1 = D \otimes D_2) \end{array} \right.$$

Scalars.   We will identify diagrams with 0 input and 0 output – hence representing a 1×1 matrix – with scalars. We will not ignore them in this paper, while in some versions of the ZX-Calculus, the global phase or even all the scalars are ignored. Ignoring them would imply taking the risk of ignoring a zero scalar, which can lead to false statements – if $ZX \vdash 0 \times D_1 = 0 \times D_2$, we can not say that $ZX \vdash D_1 = D_2$. The first rules to palliate it appeared in [3] and were simplified in [5].

Only Topology Matters   is a paradigm – to be taken as a rule – stating that any wire of a ZX-diagram can be bent at will, without changing its semantics:
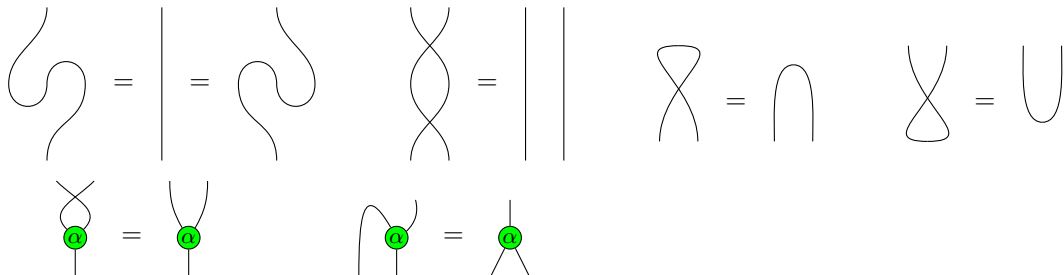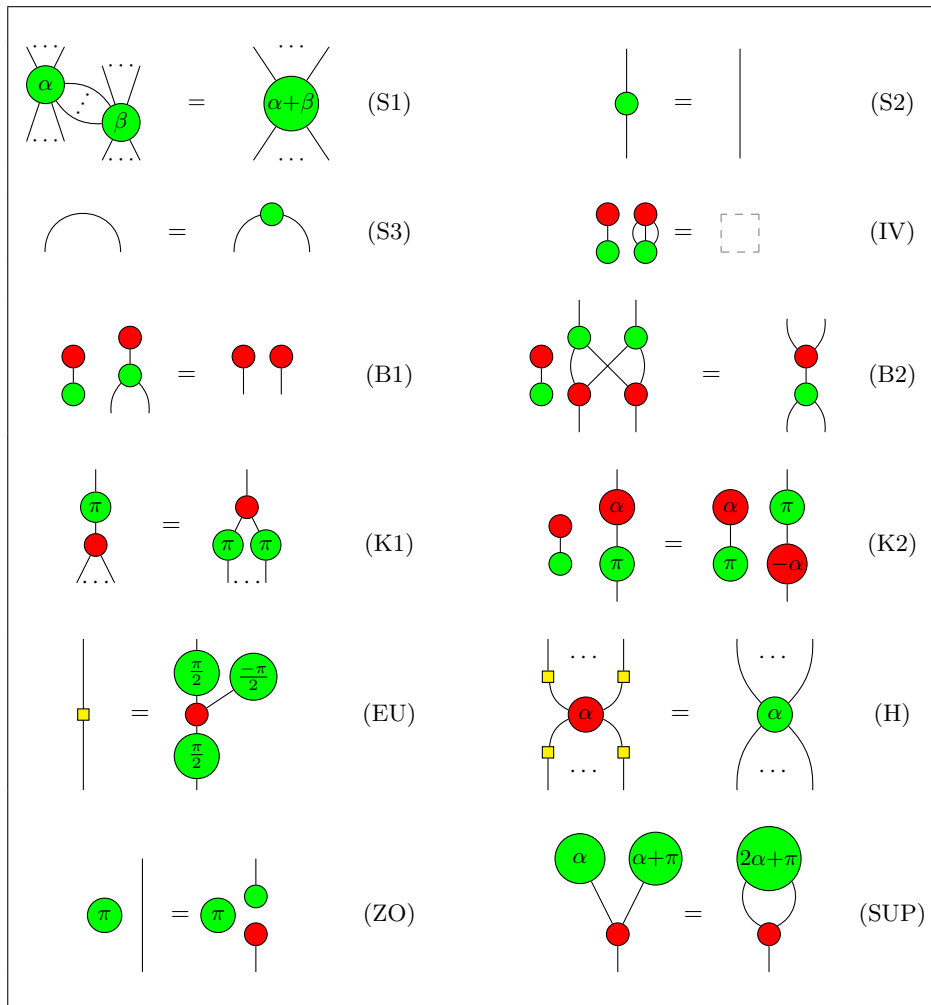
Figure 1 Rules for the ZX-calculus with scalars, augmented with the supplementarity rule [16]. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (IV) is an empty diagram. $(\cdots)$ denote zero or more wires, while $(.\cdot^{\cdot})$ denote one or more wires.

3 The $\frac{\pi}{4}$-Fragment is not Complete

In this section, we identify the following simple equation (E), which is sound – both sides of the equation represent the scalar 1 – but which cannot be derived from the rules of the ZX-Calculus expressed in Figure 1.

                                                                    (E)

Since equation (E) only involves angles multiple of $\frac{\pi}{4}$, it implies the incompleteness of the $\frac{\pi}{4}$-fragment of the ZX-Calculus. In the following, we exhibit a simple invariant of the ZX-Calculus to prove that (E) is not derivable, and then we show that (E) subsumes two existing rules of the ZX-Calculus – namely (IV) and (ZO) –, leading to a simpler – (IV) and (ZO) rules are replaced by (E) – but more expressive ZX-Calculus that we call ZX_E.

3.1   A Graphical Invariant for the ZX-Calculus

We introduce a simple graphical quantity for ZX-diagrams, the parity of the number of odd-degree red dots plus the number of H-dots (yellow squares), formally defined as follows:

▶ **Definition 1.** Given a ZX-diagram $D : n \to m$, let $[\![D]\!]^{\bullet} \in \{0, 1\}$ be inductively defined as

$$[\![D_1 \otimes D_2]\!]^{\bullet} = [\![D_1 \circ D_2]\!]^{\bullet} = [\![D_1]\!]^{\bullet} + [\![D_2]\!]^{\bullet} \bmod 2, \quad \left[\!\!\left[\overset{\overset{n}{\cdots}}{\underset{\underset{m}{\cdots}}{\alpha}} \right]\!\!\right]^{\bullet} = n + m \bmod 2, \quad \left[\!\!\left[\begin{array}{c} \downarrow \\ \square \\ \vert \end{array} \right]\!\!\right]^{\bullet} = 1,$$

and $[\![.]\!]^{\bullet} = 0$ for all the other generators.

One can define similarly $[\![.]\!]^{\circ}$ as the parity of the number of odd-degree green dots plus the number of H-dots. Notice that for any scalar $D : 0 \to 0$, $[\![D]\!]^{\circ} + [\![D]\!]^{\bullet} = 0 \bmod 2$, thanks to the well known degree sum formula which implies that the sum of the degree of the vertices of a graph is even. More generally, for any $D : n \to m$, $[\![D]\!]^{\circ} + [\![D]\!]^{\bullet} = n + m \bmod 2$, which is clearly an invariant of the ZX-calculus since all the rules preserve the number of inputs/outputs. As a consequence, a rule preserves $[\![.]\!]^{\circ}$ if and only if it preserves $[\![.]\!]^{\bullet}$.
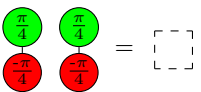
▶ **Lemma 2** (Invariant). *For any ZX-diagram D_1, if $[\![D_1]\!] \neq 0$ and $ZX \vdash D_1 = D_2$, then $[\![D_1]\!]^{\bullet} = [\![D_2]\!]^{\bullet}$.*

Proof. Notice that all the rules in Figure 1, but (ZO), preserve $[\![.]\!]^{\bullet}$. Since $[\![D_1]\!] \neq 0$, the scalar ⊚ cannot appear in any derivation transforming D_1 into D_2, thus (ZO) is not applied, as a consequence $[\![D_1]\!]^{\bullet} = [\![D_2]\!]^{\bullet}$. ◀

▶ **Proposition 3.** *Equation (E) is not derivable using the rules in Figure 1: $ZX \nvdash$ (E).*

Proof. The two diagrams of equation (E) are non zero, and they differ for $[\![.]\!]^{\bullet}$, so according to lemma 2, $ZX \nvdash$ (E). ◀

Since the diagrams of equation (E) are in the $\frac{\pi}{4}$-fragment, it implies that the $\frac{\pi}{4}$-fragment of the ZX-calculus is not complete.

▶ Remark. $\begin{array}{c} \text{\large ⬤⬤} \\ \text{\large ⬤⬤} \end{array} = \boxed{}$ the "doubled" version of (E), contrary to it, is derivable in the ZX-Calculus.

By completeness of the $\frac{\pi}{2}$-fragment, for any ZX-diagrams D_1 and D_2 in this particular fragment, if $[\![D_1]\!] = [\![D_2]\!] \neq 0$, then $[\![D_1]\!]^{\bullet} = [\![D_2]\!]^{\bullet}$. This property is obviously not true in the $\frac{\pi}{4}$-fragment, equation (E) being a counter example. However, this property is also satisfied by other, a priori not complete, fragments:

▶ **Proposition 4.** *For any $k \neq 0 \bmod 4$ and any two diagrams D_1, D_2 with angles multiple of $\frac{\pi}{k}$, if $[\![D_1]\!] = [\![D_2]\!] \neq 0$ then $[\![D_1]\!]^{\bullet} = [\![D_2]\!]^{\bullet}$.*

Proof. Given $k > 0$ and D a ZX-diagram with angles multiple of $\frac{\pi}{k}$, one can show, by induction on D, that $(\sqrt{2}^{[\![D]\!]^{\bullet}}) [\![D]\!]$ is a matrix whose entries are in $\mathbb{Q}[e^{\frac{i\pi}{k}}]$, the smallest subfield of $\mathbb{C}$ which contains $e^{\frac{i\pi}{k}}$. Since there is a non-zero entry in $[\![D_1]\!]$, there exist $q_1, q_2 \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$ such that $\sqrt{2}^{[\![D_1]\!]^{\bullet}} q_1 = \sqrt{2}^{[\![D_2]\!]^{\bullet}} q_2 \neq 0$, so $\sqrt{2}^{([\![D_1]\!]^{\bullet} - [\![D_2]\!]^{\bullet})} \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$. Suppose $\sqrt{2} \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$:

(i) If $k = 2 \bmod 4$, then $i = e^{\frac{i\pi}{2}} \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$, so $e^{\frac{-i\pi}{4}} = \sqrt{2}\frac{1-i}{2} \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$. Therefore $e^{\frac{i\pi}{2k}} = (e^{\frac{i\pi}{k}})^{\frac{k+2}{4}} \times e^{\frac{-i\pi}{4}} \in \mathbb{Q}[e^{\frac{i\pi}{k}}]$. This implies $\mathbb{Q}[e^{\frac{i\pi}{2k}}] = \mathbb{Q}[e^{\frac{i\pi}{k}}]$ which is not possible as they are vector spaces over $\mathbb{Q}$ of dimension respectively $\varphi(4k) = 2\varphi(2k)$ and $\varphi(2k)$ where φ is Euler's totient function, and $\varphi(2k) \neq 0$.

(ii) If k is odd then $2k = 2 \bmod 4$. Moreover, since $\mathbb{Q}[e^{i\frac{\pi}{k}}] \subseteq \mathbb{Q}[e^{i\frac{\pi}{2k}}]$, $\sqrt{2} \in \mathbb{Q}[e^{i\frac{\pi}{2k}}]$ which is impossible according to the previous case (i).

Thus $\sqrt{2} \notin \mathbb{Q}[e^{i\frac{\pi}{k}}]$ when $k \neq 0 \bmod 4$, so $[\![D_1]\!]^{\bullet} = [\![D_2]\!]^{\bullet}$. ◀

3.2 A Simpler and More Expressive ZX-calculus

Equation (E) cannot be derived in the ZX-calculus (proposition 3), as a consequence we propose to add this equation (E) as a rule of the language to make it more expressive. We show in the following that the introduction of this new rule makes the two scalar rules (ZO) and (IV) obsolete, leading to a language with less rules than the one define in Figure 1.

Let us define $ZX_E = \{(E)\} \cup ZX \setminus \{(IV),(ZO)\}$. First, notice that, thanks to [5], the so-called Hopf law is derivable from $ZX \setminus \{(IV),(ZO)\}$, and hence from ZX_E:

▶ **Lemma 5.**

$$ZX \setminus \{(IV),(ZO)\} \vdash \qquad\qquad = \qquad\qquad \tag{HL}$$

▶ **Proposition 6.** (IV) *is derivable from* ZX_E.

Proof. Using (E), (B1), (HL), (S2) and (S1):

◀

▶ **Proposition 7.** (ZO) *is derivable from* ZX_E.

Hence $ZX_E \vdash (IV),(ZO)$.

▶ Remark. The other rules of the language remain, a priori, necessary in the presence of equation (E). In particular the supplementarity which has been recently proved to be necessary in ZX [16] is necessary in ZX_E: one can prove using the interpretation $[\![.]\!]^{\sharp}_{k,l}$ – defined in [16] – with $k = 3$ and $l = 8$ that $ZX_E \setminus \{(SUP)\} \nvdash (SUP)$.

▶ Remark. One may want to generalise equation (E), replacing the particular angles $\pm\frac{\pi}{4}$ by some generic angle α. Proposition 4 is a strong evidence that such a generalisation is not possible and that the language requires at least one rule which is specific to the $\frac{\pi}{4}$ angle.

4 Cyclotomic Supplementarity

4.1 Generalisation of (SUP)

The concept of supplementarity in quantum diagram reasoning has been first introduced by Coecke and Edwards [9], turned into a simple but necessary rule (SUP in Figure 1) in [16]. Roughly speaking, supplementarity consists in merging two dots sharing the same neighbour when the difference of their angles is π, i.e. when the two angles are antipodal.

We generalize this concept to cyclotomic supplementarity as follows: for any $n \in \mathbb{N}^*$, n dots sharing the same neighbour can be merged when their angles divide the circle into equal parts (cyclotomy), i.e. when their angles are of the form $\alpha + \frac{2k\pi}{n}$ for $k \in [\![0; n-1]\!]$:
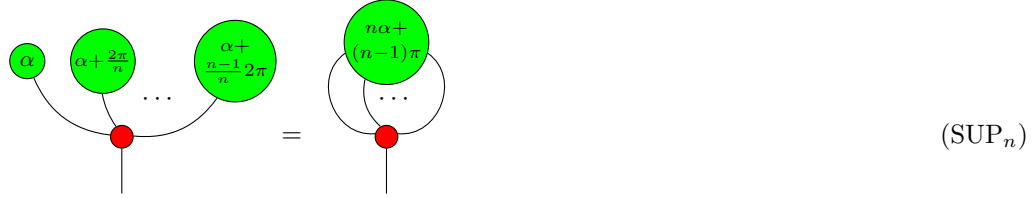
$$(\mathrm{SUP}_n)$$

Notice that there are n green dots in the left diagram, and n parallel wires in the right diagram.

Any of these equations is valid for the standard interpretation of ZX-diagrams:

▶ **Proposition 8.** (SUP_n) *is sound.*

Cyclotomic supplementarity has a generalisation: the green dots can be merged not only when they share the same neighbour, but also when they share the same neighbourhood. It leads to the notion of cyclotomic twins, which generalise the notion of antiphase twins [13]:

▶ **Definition 9** (Cyclotomic Twins). n dots in a ZX-diagram are cyclotomic twins if:

- they have the same colour
- their angles divide the circle into equal parts $\left(\alpha + \frac{2k\pi}{n} \text{ for } k \in [\![0; n-1]\!]\right)$
- they have the same neighbourhood: for any vertex, the number of wires connecting it to any of the twins is the same

▶ **Proposition 10** (Cyclotomic Twins and Supplementarity). *With $ZX \cup \{(\mathrm{SUP}_n)\}_{n \in \mathbb{N}}$, cyclotomic twins can be merged.*

The rest of the section is dedicated to the structures of this family of equations: we show that (SUP_n) is necessary when n is an odd prime number and that (SUP_n) can be derived when n is not prime. As a consequence, we exhibit a countable family of equations that cannot be derived in the ZX-calculus.

4.2 The Set of Supplementarity Rules for Prime Numbers

It is not necessary to define the supplementarity rules for all numbers $n \in \mathbb{N}$ as axioms. For instance, we will prove that their restriction to the set of prime numbers is enough to show all the others.

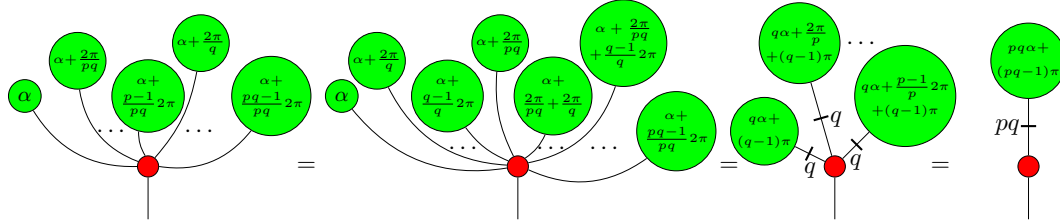Let $\mathbb{P}$ be the set of prime numbers.

▶ **Theorem 11.**

- $\forall p, q \in \mathbb{N}^*, \quad ZX_E \cup \{(\mathrm{SUP}_p), (\mathrm{SUP}_q)\} \vdash (\mathrm{SUP}_{pq})$
- $\forall n \in \mathbb{N}^*, \quad ZX_E \cup \{(\mathrm{SUP}_p)\}_{p \in \mathbb{P}} \vdash (\mathrm{SUP}_n)$
- $\forall p \in \mathbb{P}, p \geq 3, \quad ZX_E \cup \{(\mathrm{SUP}_q)\}_{q \in \mathbb{P} \setminus \{p\}} \nvdash (\mathrm{SUP}_p)$

Proof.

First statement: If n is not prime, its supplementarity can be derived. Indeed, suppose n can be decomposed in two numbers p and q ($n = pq$), for which we know the supplementarity rule.


with p-ticked edge representing p parallel wires. The first equality is just a rearranging of the branches, the second uses (SUP$_q$) p times and the last one exploits Proposition 10 with $p(q\alpha + (q-1)\pi) + (p-1)\pi = pq\alpha + (pq-1)\pi$.

Second Statement: As a direct consequence of the previous statement, since (SUP$_1$) is trivial, the supplementarity rules for prime numbers are enough to derive all the others.

Third Statement: Let $p \in \mathbb{P}$ and $p \geq 3$. Let us consider the interpretation $[\![.]\!]_{p^2}$ which amounts to multiplying all the angles of a diagram by p^2.

- The interpretation $[\![.]\!]_{p^2}$ coincides with the interpretation $[\![.]\!]_{1,p^2-1}^\sharp$ defined in [16]. As stated in this article, since the first parameter is odd and the second one is even, all the rules of ZX$\setminus\{$(SUP$_2$)$\}$ hold.
- The rule (E) also holds. Indeed, p is odd, and whether $p \bmod 8$ is $1, 3, 5$ or 7, $p^2 \bmod 8 = 1$, so $p^2 \frac{\pi}{4} = \frac{\pi}{4} \bmod 2\pi$.
- The rule (SUP$_q$) when $q \in \mathbb{P}$, $q \neq p$ holds, since $gcd(p^2, q) = 1$:
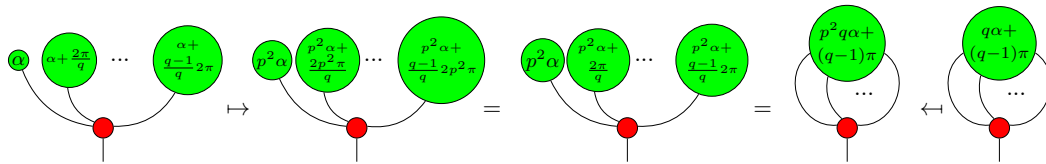
- The rule (SUP$_p$) does not hold:
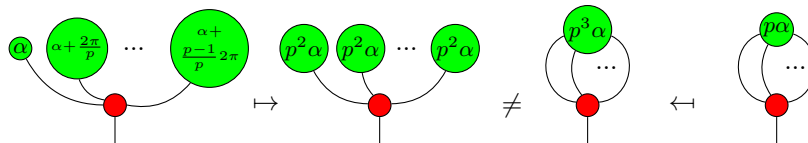
Indeed, when $\alpha = 0$, for instance, using on the left side $p-1$ times (B1) and (IV), and on the right side (IV) and $\frac{p-1}{2}$ times the Hopf law (HL), since $p \geq 3$:
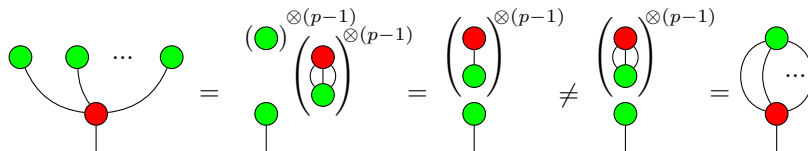
Every rule but the p-supplementarity (with $p \in \mathbb{P}$ and $p \geq 3$) holds with this interpretation, so it cannot be derived from the others:

$$\forall p \in \mathbb{P}, p \geq 3, \quad ZX_E \cup \{(\text{SUP}_n)\}_{n \in \mathbb{P}} \setminus \{(\text{SUP}_p)\} \nvdash (\text{SUP}_p) \qquad \blacktriangleleft$$

▶ **Corollary 12.** *For any $n \geq 3$ odd, the $\frac{\pi}{2n}$-fragment of the ZX_E-Calculus is incomplete.*

Proof. Let p be an odd prime factor of n. Theorem 11 proves that $ZX_E \nvdash (SUP_p)$, and notice that all the angles involved in the rule are multiples of $\frac{\pi}{2p}$, hence in the $\frac{\pi}{2n}$-fragment. ◀

▶ **Remark.** We can also notice that all the rules (SUP_n) respect the quantity $\llbracket . \rrbracket^{\bullet}$, so that the rule (E) remains necessary.

4.3   Discussion on the Supplementarity's Derivability Structure

Let p and q be two natural numbers. We have previously shown $ZX_E \cup \{(SUP_p), (SUP_q)\} \vdash (SUP_{pq})$. In other words, (SUP_p) can be deduced from the supplementarity of the dividers of p. Now, can we deduce this same equality from the supplementarity of some of its multiples?

The first result comes when p is odd:

▶ **Proposition 13.**

$$\forall p, q \in \mathbb{N}^*, \qquad (p = 1 \bmod 2) \quad \Rightarrow \quad \{(HL), (IV), (SUP_p), (SUP_{pq})\} \vdash (SUP_q)$$

There exists another – weaker – derivation when p is even:

▶ **Proposition 14.**

$$\forall p, q \in \mathbb{N}^*, \qquad \{(HL), (IV), (SUP_p), (SUP_{p^2q})\} \vdash (SUP_{pq})$$

▶ **Remark.** In the last two propositions, we require that the ZX be "general" i.e. with angles either real or a rational multiple of π because we need α/p to be in the fragment in both cases. Though, the result can be expanded to any fragment for some α provided α/p be in the fragment.

To sum up:

$$ZX_E \cup \{(SUP_p), (SUP_q)\} \vdash (SUP_{pq})$$

$$ZX_E \cup \{(SUP_p), (SUP_{p^2q})\} \vdash (SUP_{pq})$$

$$(p = 1 \bmod 2) \quad \Rightarrow \quad ZX_E \cup \{(SUP_p), (SUP_{pq})\} \vdash (SUP_q)$$

4.4   Updated Set of Rules

We propose to add the generalisation of the supplementarity rule to the set of rules of the ZX-Calculus, and to restrict to the set necessary when dealing with particular fragments. We notice that the rule (K1) is derivable from the others [5] so we can get rid of it, and the new set of rules of the ZX-Calculus is shown in figure 2.

▶ **Remark.** We can prove that (SUP_2) is not derivable from $ZX_E \setminus \{(SUP_2)\}$, using the interpretation $\llbracket . \rrbracket^{\sharp}_{k,l}$ defined in [16] with $k = 3$ and $l = 8$.

However, it is important to notice that we have not proven that (SUP_2) can not be derived from the rest once the cyclotomic supplementarity is added. Indeed, the family of interpretations used in the proof of Theorem 11 only works when p is odd, and the one used previously, $\llbracket . \rrbracket^{\sharp}_{k,l}$, does not hold for many supplementarity rules.

The rule (SUP_2) is all the more peculiar as, due to the Hopf law (HL), it is the only supplementarity rule that creates a non-trivial scalar – except for the supplementarity rules for even numbers, which anyway derive from (SUP_2). For instance it may create a scalar worth 0 – when applied with $\alpha = 0$ – which is the first step towards proving the rule (ZO) in the $\frac{\pi}{4}$-fragment. Moreover, it is the only supplementarity that can not be proven to be necessary by simply multiplying the angles by a constant.
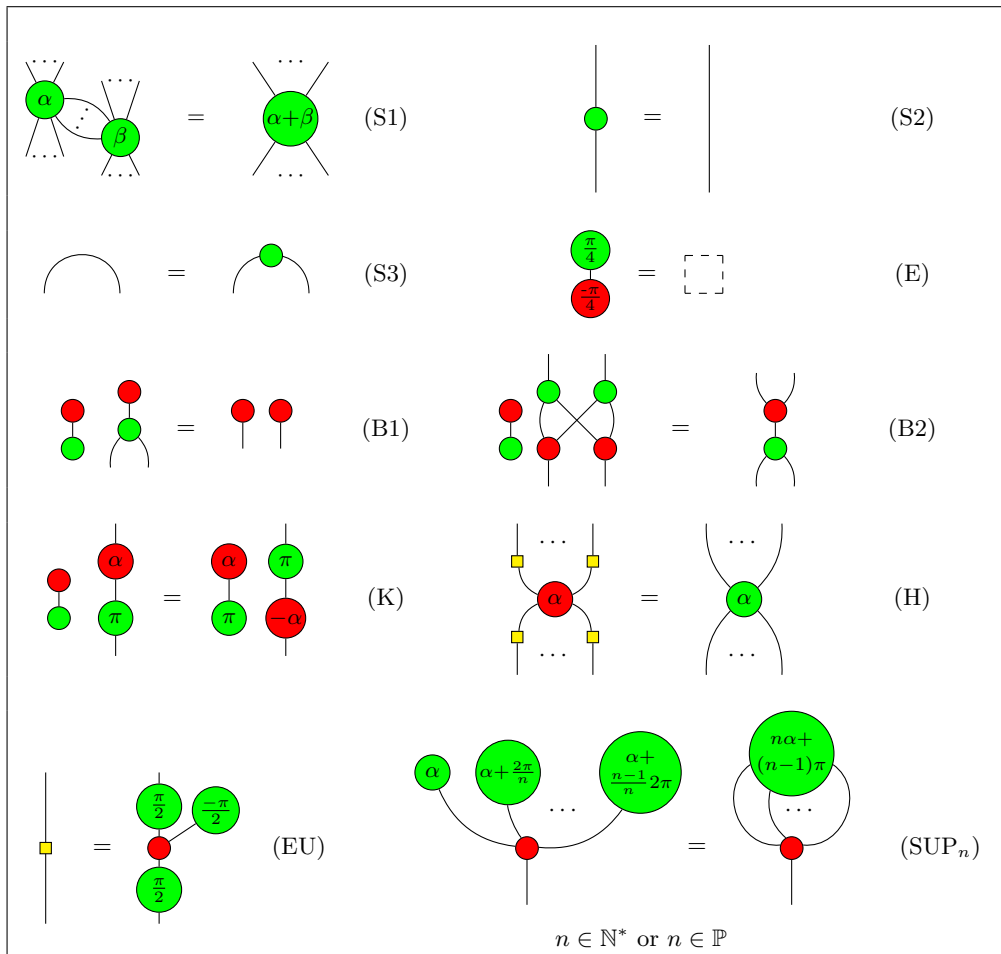
Figure 2 New set of rules for the ZX-calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (E) is an empty diagram. $(\cdots)$ in (S1) and (H) denote 0 or more wires, while $(.\,\dot{}\,)$ denote 1 or more wires.

4.5 The General ZX-Calculus is still Not Complete

The argument given by Schröder de Witt and Zamdzhiev [17] to show the incompleteness of the general ZX-Calculus is not valid anymore – when multiplying the angles by any integer, there is at least one supplementarity that does not hold. But we can patch the demonstration to make it valid again.

▶ **Theorem 15.** *The general ZX-Calculus is incomplete with the set of rules in figure 2.*

Proof. We will make the proof using a combination of ZX-rules and matrix calculus on the interpretations of the diagrams. Consider the following diagrams:
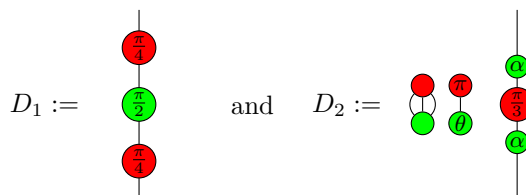
We will try to express D_1 in the form D_2. One can notice that $[\![D_1]\!] = [\![D_2]\!]$ when $\alpha_0 = \frac{\pi}{2} - \arccos\left(\sqrt{\frac{2}{3}}\right)$ and $\theta_0 = \arccos\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{6}\right)$. We can even show:

No more than four values for α are possible when decomposing D_1 in the form D_2:
When applying the π green state at the top and the 0 green state at the bottom of both D_1 and D_2, we end up with:
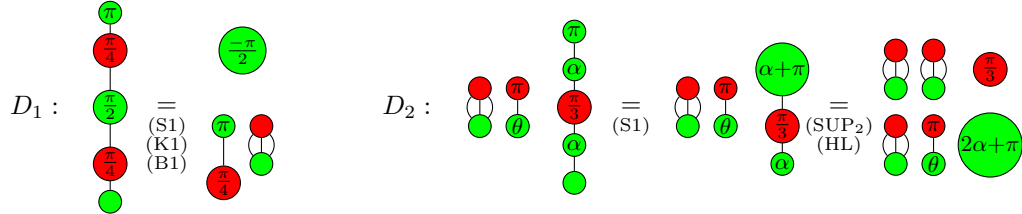
In order for their interpretations to be equal, we need:

$$e^{i\frac{\pi}{4}}\sqrt{2}e^{-i\frac{\pi}{4}} = \frac{1}{2}e^{i\theta}(1 + e^{i\frac{\pi}{3}})(1 + e^{i(2\alpha+\pi)}) \quad \text{i.e.} \quad \frac{\sqrt{2}}{2} = e^{i(\theta + \frac{\pi}{6} + \alpha + \frac{\pi}{2})}\cos\left(\frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{2}\right)$$

So using the modulus, $\left| \cos\left(\alpha + \frac{\pi}{2}\right) \right| = \sqrt{\frac{2}{3}}$, thus $\alpha = \pm\frac{\pi}{2} \pm \arccos\left(\sqrt{\frac{2}{3}}\right) \bmod 2\pi$.

α is not a rational multiple of π: One can check that $e^{i\alpha_0}$ is a root of the polynomial $3X^4 + 2X^2 + 3$ which is irreducible in $\mathbb{Z}$ (since 30203 is a prime number, thanks to Cohn's irreducibility criterion, $3X^4 + 2X^2 + 3$ is irreducible in $\mathbb{Z}$). The polynomial is not cyclotomic because its coefficient of higher degree is not 1, hence $e^{i\alpha_0}$ is not a root of unity, i.e. α_0 is not a rational multiple of π. As a consequence, none of the $\pm\frac{\pi}{2} \pm \arccos\left(\sqrt{\frac{2}{3}}\right)$ are rational multiples of π.

Now, let us put back all the pieces together. Assume $ZX \vdash D_1 = D_2$ for some α and θ. Then there exists a finite sequence of rules of the ZX that transforms D_2 into D_1. We define $q \in \mathbb{N}^*$ such that for any (SUP_p) in the sequence, $p \leq q$, and $S = \{k(q+4)! + 1 \mid k \in \mathbb{N}\}$. For all $q' \in S$ and for $(.)_{q'}$ the interpretation that multiplies the angles by q', the rules of the ZX are preserved, $(D_1)_{q'} = D_1$, and $(D_2)_{q'}$ is in the form D_2. Indeed:

- $(q+4)!$ is clearly a multiple of 8 so $q'\frac{\pi}{4} = \frac{\pi}{4} \bmod 2\pi$, so all the rules but the supplementarity rules hold, and $(D_1)_{q'} = D_1$ since it is in the $\frac{\pi}{4}$-fragment.
- for any $p \in \mathbb{P}$ such that $p \leq q$, then $(q+4)! = 0 \bmod p$ so $gcd(p, q') = 1$, which implies that (SUP_p) also holds.
- $(q+4)!$ is a multiple of 6 so $q'\frac{\pi}{3} = \frac{\pi}{3} \bmod 2\pi$ hence $(D_2)_{q'}$ is in the form D_2.

Then, $ZX \vdash D_1 = (D_2)_{q'}$.

D_1 has a finite number of decompositions in the form D_2, but $\left\{(D_2)_{q'} \mid q' \in S\right\}$ is infinite – since α is an irrational multiple of π – and all these diagrams are decompositions of D_1 in the form D_2, hence we end up with a contradiction.

So $ZX \nvdash D_1 = D_2$, which proves the incompleteness. ◀

---- **References** ----

1 Miriam Backens. The zx-calculus is complete for stabilizer quantum mechanics. *New Journal of Physics*, 16(9):093021, 2014. `doi:10.1088/1367-2630/16/9/093021`.

2 Miriam Backens. The zx-calculus is complete for the single-qubit clifford+t group. *Electronic Proceedings in Theoretical Computer Science*, 2014. `doi:10.4204/EPTCS.172.21`.

3   Miriam Backens. Making the stabilizer zx-calculus complete for scalars. *Electronic Proceedings in Theoretical Computer Science*, 2015. `doi:10.4204/EPTCS.195.2`.

4   Miriam Backens and Ali Nabi Duman. A complete graphical calculus for spekkens' toy bit theory. *Foundations of Physics*, pages 1–34, 2014. `doi:10.1007/s10701-015-9957-7`.

5   Miriam Backens, Simon Perdrix, and Quanlong Wang. A simplified stabilizer zx-calculus. *Electronic Proceedings in Theoretical Computer Science*, 2016. `doi:10.4204/EPTCS.236.1`.

6   Categorical quantum mechanics: Zx-completeness. URL: `http://cqm.wikidot.com/zx-completeness`.

7   Bob Coecke. Axiomatic description of mixed states from selinger's cpm-construction. *Electron. Notes Theor. Comput. Sci.*, 210:3–13, July 2008. `doi:10.1016/j.entcs.2008.04.014`.

8   Bob Coecke and Ross Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13(4):043016, 2011. `doi:10.1088/1367-2630/13/4/043016`.

9   Bob Coecke and Bill Edwards. Three qubit entanglement within graphical z/x-calculus. *Electronic Proceedings in Theoretical Computer Science*, 52:22–33, 2011. `doi:10.4204/EPTCS.52.3`.

10   Bob Coecke and Simon Perdrix. Environment and classical channels in categorical quantum mechanics. *Logical Methods in Computer Science*, Volume 8, Issue 4, November 2012. `doi:10.2168/LMCS-8(4:14)2012`.

11   Ross Duncan and Simon Perdrix. Graphs states and the necessity of euler decomposition. *Mathematical Theory and Computational Practice*, 5635:167–177, 2009. `doi:10.1007/978-3-642-03073-4`.

12   Ross Duncan and Simon Perdrix. Rewriting measurement-based quantum computations with generalised flow. *Lecture Notes in Computer Science*, 6199:285–296, 2010. `doi:10.1007/978-3-642-14162-1_24`.

13   Ross Duncan and Simon Perdrix. Pivoting makes the zx-calculus complete for real stabilizers. *Electronic Proceedings in Theoretical Computer Science*, 2013. `doi:10.4204/EPTCS.171.5`.

14   Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A complete axiomatisation of the zx-calculus for clifford+ t quantum mechanics. *arXiv preprint arXiv:1705.11151*, 2017.

15   Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Y-calculus: A language for real matrices derived from the zx-calculus. In *Conference on Quantum Physics and Logics (QPL'17)*, 2017.

16   Simon Perdrix and Quanlong Wang. Supplementarity is necessary for quantum diagram reasoning. In *41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016)*, volume 58 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 76:1–76:14, Krakow, Poland, August 2016. `doi:10.4230/LIPIcs.MFCS.2016.76`.

17   Christian Schröder de Witt and Vladimir Zamdzhiev. The zx-calculus is incomplete for quantum mechanics. *Electronic Proceedings in Theoretical Computer Science*, 2014. `doi:10.4204/EPTCS.172.20`.

18   Peter Selinger. Finite dimensional hilbert spaces are complete for dagger compact closed categories. *Logical Methods in Computer Science*, 8(4):1–12, 2012. `doi:10.2168/LMCS-8(3:06)2012`.

19   Peter Selinger. Quantum circuits of *t*-depth one. *Phys. Rev. A*, 87:042302, Apr 2013. `doi:10.1103/PhysRevA.87.042302`.

20   Robert Spekkens. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A*, 75:032110, Mar 2007. `doi:10.1103/PhysRevA.75.032110`.