Hybrid VCSPs with Crisp and Valued Conservative Templates

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Abstract

A constraint satisfaction problem (CSP) is a problem of computing a homomorphism $\mathbf{R} \to \mathbf{\Gamma}$ between two relational structures, e.g. between two directed graphs. Analyzing its complexity has been a very fruitful research direction, especially for *fixed template CSPs* (or, *non-uniform CSPs*), denoted $\mathrm{CSP}(\mathbf{\Gamma})$, in which the right side structure $\mathbf{\Gamma}$ is fixed and the left side structure $\mathbf{R}$ is unconstrained.

Recently, the *hybrid* setting, written $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$, where both sides are restricted simultaneously, attracted some attention. It assumes that $\mathbf{R}$ is taken from a class of relational structures $\mathcal{H}$ (called the *structural restriction*) that additionally is *closed under inverse homomorphisms*. The last property allows to exploit an algebraic machinery that has been developed for fixed template CSPs. The key concept that connects hybrid CSPs with fixed-template CSPs is the so called "lifted language". Namely, this is a constraint language $\mathbf{\Gamma}_{\mathbf{R}}$ that can be constructed from an input $\mathbf{R}$. The tractability of the language $\mathbf{\Gamma}_{\mathbf{R}}$ for any input $\mathbf{R} \in \mathcal{H}$ is a necessary condition for the tractability of the hybrid problem.

In the first part we investigate templates $\mathbf{\Gamma}$ for which the latter condition is not only necessary, but also is sufficient. We call such templates $\mathbf{\Gamma}$ *widely tractable*. For this purpose, we construct from $\mathbf{\Gamma}$ a new finite relational structure $\mathbf{\Gamma}'$ and define a "maximal" structural restriction $\mathcal{H}_0$ as a class of structures homomorphic to $\mathbf{\Gamma}'$. For the so called strongly BJK templates that probably captures all templates, we prove that wide tractability is equivalent to the tractability of $\mathrm{CSP}_{\mathcal{H}_0}(\mathbf{\Gamma})$. Our proof is based on the key observation that $\mathbf{R}$ is homomorphic to $\mathbf{\Gamma}'$ if and only if the core of $\mathbf{\Gamma}_{\mathbf{R}}$ is preserved by a Siggers polymorphism. Analogous result is shown for conservative valued CSPs.

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1 Introduction

The *constraint satisfaction problems (CSPs)* and the *valued constraint satisfaction problems (VCSPs)* provide a powerful framework for the analysis of a large set of computational problems arising in propositional logic, combinatorial optimization, graph theory, artificial intelligence, scheduling, biology (protein folding), computer vision etc. CSP can be formalized either as a problem of (a) finding an assignment of values to a given set of variables, subject to constraints on the values that can be assigned simultaneously to specified subsets of variables, or as a problem of (b) finding a homomorphism between two finite relational structures A and B (e.g., two oriented graphs). These two formulations are polynomially equivalent under the condition that input constraints in the first case or input relations in the second case are given by lists of their elements. A soft version of CSP, the Valued CSP, generalizes the

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CSP by changing crisp constraints to cost functions applied to tuples of variables. In the VCSP we are asked to find a minimum (or maximum) of a sum of cost functions applied to corresponding variables.

The CSPs have been a very active research field since 70s. One of the topics that revealed the rich logical and algebraic structure of the CSPs was the problem's computational complexity when constraint relations are restricted to a given set of relations or, alternatively, when the second relational structure is some fixed $\mathbf{\Gamma}$. Thus, this problem is parameterized by $\mathbf{\Gamma}$, denoted as CSP($\mathbf{\Gamma}$) and called a fixed template CSP with a template $\mathbf{\Gamma}$ (another name is a non-uniform CSP). E.g., if the domain set is boolean and $\mathbf{\Gamma}$ is a structure with four ternary relations $x \vee y \vee z$, $\overline{x} \vee y \vee z$, $\overline{x} \vee \overline{y} \vee z$, $\overline{x} \vee \overline{y} \vee \overline{z}$, CSP($\mathbf{\Gamma}$) models 3-SAT which is historically one of the first NP-complete problems [8]. At the same time, if we restrict that all relations in $\mathbf{\Gamma}$ are binary, then we obtain tractable 2-SAT. Schaeffer proved [25] that for any template $\mathbf{\Gamma}$ over the boolean set, CSP($\mathbf{\Gamma}$) is either in P or NP-complete. For the case when $\mathbf{\Gamma}$ is a graph (without loops) Hell and Nešetřil [14] proved an analogous statement, by showing that only for bipartite graphs the problem is tractable. Feder and Vardi [11] found that all fixed template CSPs can be expressed as problems in a fragment of SNP, called the Monotone Monadic SNP (MM SNP), and showed that for any problem in MM SNP there is a polynomial-time Turing reduction to a fixed template CSP. Thus, non-uniform CSPs' complexity classification would yield a classification for MM SNP problems. This result placed fixed-template CSPs into a broad logical context which naturally lead to a conjecture that such CSPs are either tractable or NP-hard, the so called dichotomy conjecture.

In [16] Jeavons showed that the complexity of CSP($\mathbf{\Gamma}$) is determined by the polymorphisms of $\mathbf{\Gamma}$. Research in this direction lead to a conjectured description of tractable templates through properties of their polymorphisms. The key formulation was given by Bulatov, Jeavons, and Krokhin [5], with subsequent reformulations of this conjecture by Maroti and McKenzie [23]. Later, it was shown by Siggers [26] that if the Bulatov-Jeavons-Krokhin formulation is true, then for a relational structure to be tractable it is necessary and sufficient that its core is preserved by a single 6-ary polymorphism that satisfies a certain term identity. Further, an arity of a polymorphism in the latter formulation was decreased to 4 [18]. We will use the last fact as a key ingredient for our results. Very recently, several independent proofs of the Bulatov-Jeavons-Krokhin formulation were announced [24, 6, 30]. Since the papers have not yet been thoroughly verified and widely accepted by the CSP community, in this paper we refer to the formulation as a hypothesis.

Related work. A meta-problem of the VCSP topic is to establish the complexity of VCSP given that an input is restricted to an arbitrary subset of all input pairs $(\mathbf{R}, \mathbf{\Gamma})$. A natural approach to this problem is to construct a new structure for any input $(\mathbf{R}, \mathbf{\Gamma})$, $G_{\mathbf{R},\mathbf{\Gamma}}$, and shift the analysis to $G_{\mathbf{R},\mathbf{\Gamma}}$. In case of binary CSPs (i.e. when all relations of an input are binary) it is natural to define $G_{\mathbf{R},\mathbf{\Gamma}}$ as a microstructure graph [17] of an instance $(\mathbf{R}, \mathbf{\Gamma})$. Thereby, a set of inputs, in which certain local substructures in $G_{\mathbf{R},\mathbf{\Gamma}}$ are forbidden, forms a parametrized problem. Cooper and Živný [9] investigated this formulation and found examples of specific forbidden substructures that result in tractable hybrid CSPs. Microstructure graphs also naturally appear in the context of fixed template CSPs. Specifically, if a template $\mathbf{\Gamma}$ with binary relations is such that the arc and path consistency preprocessing of an instance of CSP($\mathbf{\Gamma}$) always results in a perfect microstructure graph, then additionally to satisfying all constraints (by finding a maximum clique) one can also optimize arbitrary sums of unary terms over a set of solutions (by assigning weights to vertices of the microstructure graph). The latter optimization problem is called *the minimum cost homomorphism problem* and all such templates were completely classified in [28].

Recently, a hybrid framework for VCSP has attracted some attention [21], that is when left structures are restricted to some set $\mathcal{H}$ and a right structure $\mathbf{\Gamma}$ is fixed (the corresponding CSP is denoted as $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$) and $\mathcal{H}$ is closed under inverse homomorphisms. The specific feature of this case is that for any input $\mathbf{R} \in \mathcal{H}$ one can construct a new language $\mathbf{\Gamma}_{\mathbf{R}}$, called a lifted language, so that tractability of this language is a necessary condition for the tractability of $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$.

Our results. The first question that we address is a characterization of those templates $\mathbf{\Gamma}$ for which the tractability of $\mathbf{\Gamma}_{\mathbf{R}}$ for any $\mathbf{R} \in \mathcal{H}$ is not only necessary, but also is sufficient for the tractability of $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$. We call $\mathbf{\Gamma}$ that possesses this property for any $\mathcal{H}$ (closed under inverse homomorphisms) *widely tractable*. It turns out that the statement that the core of $\mathbf{\Gamma}_{\mathbf{R}}$ is preserved by a Siggers polymorphism (i.e. satisfies the Bulatov-Jeavons-Krokhin test for non-NP-hardness) is equivalent to the statement that $\mathbf{R}$ is homomorphic to a certain structure $\mathbf{\Gamma}'$ (constructed from $\mathbf{\Gamma}$). Based on this observation we prove that, for a class of templates (that is likely to capture all templates), wide tractability is equivalent to the tractability of $\mathrm{CSP}_{\mathcal{H}_0}(\mathbf{\Gamma})$, where $\mathcal{H}_0$ is equal to a set of structures homomorphic to $\mathbf{\Gamma}'$. Moreover, we prove that $\mathrm{CSP}(\mathbf{\Gamma})$ can be in polynomial-time Turing reduced to $\mathrm{CSP}(\mathbf{\Gamma}')$ and, therefore, $\mathbf{\Gamma}'$ is at least as hard as $\mathbf{\Gamma}$. We develop an analogous theory for conservative valued CSPs.

2 Preliminaries

Throughout the paper it is assumed that $P \neq NP$. A problem is called *tractable* if it can be solved in polynomial time. Let $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ denote the set of rational numbers with (positive) infinity and $[k] = \{1, ..., k\}$. Also, D and V are finite sets, D^V is a set of mappings from V to D. We denote the tuples in lowercase boldface such as $\mathbf{a} = (a_1, \dots, a_k)$. Also for mappings $h\colon A \to B$ and tuples $\mathbf{a} = (a_1, \dots, a_k)$, where $a_j \in A$ for $j = 1, \dots, k$, we will write $\mathbf{b} = (h(a_1), \dots, h(a_k))$ simply as $\mathbf{b} = h(\mathbf{a})$. Relational structures are denoted in uppercase boldface as $\mathbf{R} = (R, r_1, \dots, r_k)$. Finally let $\mathrm{ar}(\varrho)$, $\mathrm{ar}(\mathbf{a})$, and $\mathrm{ar}(f)$ stand for the arity of a relation ϱ, the size of a tuple $\mathbf{a}$, and the arity of a function f, respectively.

2.1 Fixed template VCSPs

Let us formulate the general CSP as a homomorphism problem.

▶ **Definition 1.** Let $\mathbf{R} = (R, r_1, \dots, r_k)$ and $\mathbf{R}' = (R', r'_1, \dots, r'_k)$ be relational structures with a common signature (that is $\mathrm{ar}(r_i) = \mathrm{ar}(r'_i)$ for every $i = 1, \dots, k$). A mapping $h\colon R \to R'$ is called a *homomorphism* from $\mathbf{R}$ to $\mathbf{R}'$ if for every $i = 1, \dots, k$ and for any $(x_1, \dots, x_{\mathrm{ar}(r_i)}) \in r_i$ we have that $\big((h(x_1), \dots, h(x_{\mathrm{ar}(r'_i)})\big) \in r'_i$. In that case, we write $\mathbf{R} \xrightarrow{h} \mathbf{R}'$ or sometimes just $\mathbf{R} \to \mathbf{R}'$.

▶ **Definition 2. The general CSP** is the following problem. Given a pair of relational structures with a common signature $\mathbf{R} = (V, r_1, \dots, r_k)$ and $\mathbf{\Gamma} = (D, \varrho_1, \dots, \varrho_k)$, the question is whether there is a homomorphism $h : \mathbf{R} \to \mathbf{\Gamma}$. The second structure $\mathbf{\Gamma}$ is called *a template*.

▶ **Definition 3.** Let D be a finite set and $\mathbf{\Gamma}$ be a finite relational structure over D. Then the **fixed template CSP** for template $\mathbf{\Gamma}$, denoted $\mathrm{CSP}(\mathbf{\Gamma})$, is defined as follows: given a relational structure $\mathbf{R} = (V, r_1, \dots, r_k)$ of the same signature as $\mathbf{\Gamma}$, the question is whether there is a homorphism $h : \mathbf{R} \to \mathbf{\Gamma}$.

A more general framework operates with *cost functions* $f : D^n \to \overline{\mathbb{Q}}$ instead of relations $\varrho \subseteq D^n$.

▶ **Definition 4.** Let us denote the set of all functions $f : D^n \to \overline{\mathbb{Q}}$ by $\mathbf{\Phi}_D^{(n)}$ and let $\mathbf{\Phi}_D = \bigcup_{n\geq 1} \mathbf{\Phi}_D^{(n)}$. We call the functions in $\mathbf{\Phi}_D$ *cost functions* over D. For every cost function $f \in \mathbf{\Phi}_D^{(n)}$, let $\operatorname{dom} f = \{x \mid f(x) < \infty\}$.

▶ **Definition 5.** An instance of the **valued constraint satisfaction problem** (VCSP) is a triple $(\mathbf{R}, \mathbf{\Gamma}, \{w_i(\mathbf{v})\}_{i\in[k],\mathbf{v}\in r_i})$ where $\mathbf{R} = (V, r_1, \ldots, r_k)$ is a relational structure, $\mathbf{\Gamma} = (D, f_1, \ldots, f_k)$ is a tuple where D is finite and $f_i \in \mathbf{\Phi}_D^{(\mathrm{ar}(r_i))}$, $\{w_i(\mathbf{v})\}_{i\in[k],\mathbf{v}\in r_i}$ are positive rationals, and the goal is to find an *assignment* $h \in D^V$ that minimizes a function from D^V to $\overline{\mathbb{Q}}$ given by

$$f_{\mathcal{I}}(h) = \sum_{i=1}^{k} \sum_{\mathbf{v}\in r_i} w_i(\mathbf{v}) f_i(h(\mathbf{v})), \tag{1}$$

A tuple $\mathbf{\Gamma} = (D, f_1, \ldots, f_k)$ is called a *valued template*.

▶ **Definition 6.** We will denote by VCSP($\mathbf{\Gamma}$) a class of all VCSP instances in which the valued template is $\mathbf{\Gamma}$.

For such $\mathbf{\Gamma}$ we will denote by Γ (without boldface) the set of cost functions $\{f_1, \ldots, f_k\}$. A set Γ is called a constraint language. The complexity of VCSP($\mathbf{\Gamma}$) does not depend on the order of cost functions, therefore, we will use VCSP(Γ) and VCSP($\mathbf{\Gamma}$) interchangeably.

This framework captures many specific well-known problems, including k-Sat, Graph k-Colouring, Minimum Cost Homomorphism Problem and others (see [15]).

A function $f \in \mathbf{\Phi}_D^{(n)}$ that takes values in $\{0, \infty\}$ is called *crisp*. We will often view it as a relation in D^n, and vice versa (this should be clear from the context). If a language Γ is crisp (i.e. it contains only crisp functions) then VCSP(Γ) is a search problem corresponding to CSP(Γ).

▶ Remark. Note that we formulated CSP as a decision problem, whereas VCSP as a search optimizational problem. This convention is followed throughout the text and further it becomes more important because decision and search problems are not computationally equivalent for hybrid CSPs (see after definition 20).

▶ **Definition 7.** A constraint language Γ (or, a template $\mathbf{\Gamma}$) is said to be tractable, if VCSP(Γ_0) is tractable for each finite $\Gamma_0 \subseteq \Gamma$. Also, Γ (or, $\mathbf{\Gamma}$) is NP-hard if there is a finite $\Gamma_0 \subseteq \Gamma$ such that VCSP(Γ_0) is NP-hard.

An important problem in the CSP research is to characterize all tractable languages.

2.2 Polymorphisms and fractional polymorphisms

Let $\mathcal{O}_D^{(m)}$ denote a set of all operations $g : D^m \to D$ and let $\mathcal{O}_D = \bigcup_{m\geq 1} \mathcal{O}_D^{(m)}$.

Any language Γ over a domain D can be associated with a set of operations on D, known as the polymorphisms of Γ, defined as follows.

▶ **Definition 8.** An operation $g \in \mathcal{O}_D^{(m)}$ is a *polymorphism* of a relation $\rho \subseteq D^n$ (or, g *preserves* ρ) if, for any $\mathbf{x}^1, \ldots, \mathbf{x}^m \in \rho$, we have that $g(\mathbf{x}^1, \ldots, \mathbf{x}^m) \in \rho$ where g is applied component-wise. For any crisp constraint language Γ over a set D, we denote by $\mathrm{Pol}(\Gamma)$ a set of all operations on D which are polymorphisms of every $\rho \in \Gamma$.

Polymorphisms play a key role in the algebraic approach to the CSP, but, for VCSPs, more general constructs are necessary, which we now define.

▶ **Definition 9.** An m-ary *fractional operation* ω on D is a probability distribution on $\mathcal{O}_D^{(m)}$. *The support of* ω is defined as $\mathrm{supp}(\omega) = \{g \in \mathcal{O}_D^{(m)} \mid \omega(g) > 0\}$.

▶ **Definition 10.** An m-ary fractional operation ω on D is said to be a *fractional polymorphism* of a cost function $f \in \mathbf{\Phi}_D$ if, for any $\mathbf{x}^1, \ldots, \mathbf{x}^m \in \mathrm{dom}\, f$, we have

$$\sum_{g \in \mathrm{supp}(\omega)} \omega(g) f(g(\mathbf{x}^1, \ldots, \mathbf{x}^m)) \leq \frac{1}{m}(f(\mathbf{x}^1) + \ldots + f(\mathbf{x}^m)). \tag{2}$$

For a constraint language Γ, $\mathrm{fPol}(\Gamma)$ will denote a set of all fractional operations that are fractional polymorphisms of each function in Γ.

We will also use symbols $\mathrm{Pol}(\mathbf{\Gamma})$, $\mathrm{fPol}(\mathbf{\Gamma})$ meaning $\mathrm{Pol}(\Gamma)$, $\mathrm{fPol}(\Gamma)$ respectively.

2.3 Algebraic dichotomy conjecture

An algebraic characterization for tractable templates was first conjectured by Bulatov, Krokhin and Jeavons [5], and a number of equivalent formulations were later given in [23, 1, 26, 18]. We will use the formulation from [18] that followed a discovery by M. Siggers [26]; it is crucial for our purposes that in the next definition an operation has a fixed arity (namely, 4) and, therefore, there is only a finite number of them on a finite domain D.

▶ **Definition 11.** An operation $s\colon D^4 \to D$ is called **a Siggers operation on** $D' \subseteq D$ if $s(x, y, z, t) \in D'$ whenever $x, y, z, t \in D'$ and for each $x, y, z \in D'$ we have:

$$s(x, y, x, z) = s(y, x, z, y)$$
$$s(x, x, x, x) = x$$

▶ **Definition 12.** Let g be a unary and s be a 4-ary operations on D and $g(D) = \{g(x) | x \in D\}$. A pair (g, s) is called *a Siggers pair on* D if s is a Siggers operation on $g(D)$. A crisp constraint language Γ is said to admit a Siggers pair (g, s) if g and s are polymorphisms of Γ.

▶ **Theorem 13** ([18])**.** *A crisp constraint language* Γ *that does not admit a Siggers pair is NP-Hard.*

▶ **Definition 14.** A crisp language Γ is called a *BJK* language if it satisfies one of the following:
- $CSP(\Gamma)$ is tractable
- Γ does not admit a Siggers pair.

Algebraic dichotomy conjecture: Every crisp language Γ is a BJK language.

This theorem first has been verified for domains of size 2 [25], 3 [3], or for languages containing all unary relations on D [4]. It has also been shown that it is equivalent to its restriction for directed graphs (that is when Γ contains a single binary relation ϱ) [7]. Just recently, a number of authors [24, 6, 30] independently claimed the proof of the conjecture.

3 Hybrid VCSP setting

▶ **Definition 15.** Let us call a family $\mathcal{H}$ of relational structures with a common signature a **structural restriction**.

▶ **Definition 16** (Hybrid CSP). Let D be a finite domain, $\mathbf{\Gamma}$ a template over D, and $\mathcal{H}$ a structural restriction of the same signature as $\mathbf{\Gamma}$. We define $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ as the following problem: given a relational structure $\mathbf{R} \in \mathcal{H}$ as input, decide whether there is a homomorphism $h : \mathbf{R} \to \mathbf{\Gamma}$.

▶ **Definition 17** (Hybrid VCSP). Let D be a finite domain, $\mathbf{\Gamma} = (D, f_1, \dots, f_k)$ a valued template over D, and $\mathcal{H}$ a structural restriction of the same signature as $\mathbf{\Gamma}$. We define $\mathrm{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ as a class of instances of the following form.

An instance is a function from D^V to $\overline{\mathbb{Q}}$ given by

$$f_{\mathcal{I}}(h) = \sum_{i=1}^{k} \sum_{\mathbf{v} \in r_i} w_i(\mathbf{v}) f_i(h(\mathbf{v})), \tag{3}$$

where $\mathbf{R} = (V, r_1, \dots, r_k) \in \mathcal{H}$ is a relational structure, $\{w_i(\mathbf{v})\}_{i \in [k], \mathbf{v} \in r_i}$ are positive rationals. The goal is to find an *assignment* $h \in D^V$ that minimizes $f_{\mathcal{I}}$.

For certain classes of structural restrictions the tractability/intractability can be explained by algebraic means, and of special interest is the case when $\mathcal{H}$ is *up-closed.*

▶ **Definition 18.** A family of relational structures $\mathcal{H}$ is called **closed under inverse homomorphisms** (or **up-closed** for short) if whenever $\mathbf{R}' \to \mathbf{R}$ and $\mathbf{R} \in \mathcal{H}$, then also $\mathbf{R}' \in \mathcal{H}$.

Examples of hybrid CSPs with up-closed structural restrictions include such studied problems as a digraph H-coloring for an acyclic input digraph [27] or for an input digraph with odd girth at least k [21], renamable Horn Boolean CSPs [12] etc. The key tool in their analysis is a construction of the so called lifted language that appeared first in [21]. In this construction, given arbitrary $\mathbf{R} \in \mathcal{H}$ one constructs a language $\Gamma_{\mathbf{R}}$ over a finite domain, such that for tractability of $\mathrm{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$, the tractability of $\mathrm{VCSP}(\Gamma_{\mathbf{R}})$ is *necessary.*

Let us give a detailed description of $\Gamma_{\mathbf{R}}$. Given $\mathbf{R} = (V, r_1, \dots, r_k)$ and $\mathbf{\Gamma} = (D, f_1, ..., f_k)$ we define $D_{\mathbf{R}} = V \times D$ and $D_v = \{(v, a) | a \in D\}, v \in V$.

For tuples $\mathbf{a} = (a_1, \dots, a_p) \in D^p$ and $\mathbf{v} = (v_1, \dots, v_p) \in V^p$ denote $d(\mathbf{v}, \mathbf{a}) = ((v_1, a_1), ..., (v_p, a_p))$.

Now for a cost function $f \in \mathbf{\Phi}_D$ and $\mathbf{v} \in V^{\mathrm{ar}(f)}$ we will define a cost function on $D_{\mathbf{R}}$ of the same arity as f via

$$f^{\mathbf{v}}(\mathbf{x}) = \begin{cases} f(\mathbf{y}) & \text{if } \mathbf{x} = d(\mathbf{v}, \mathbf{y}) \text{ for some } \mathbf{y} \in D^{\mathrm{ar}(f)} \\ \infty & \text{otherwise} \end{cases} \qquad \forall \mathbf{x} \in D_{\mathbf{R}}^{\mathrm{ar}(f)} \tag{4}$$

Finally, we construct the sought language $\Gamma_{\mathbf{R}}$ on domain $D_{\mathbf{R}}$ as follows:

$$\Gamma_{\mathbf{R}} = \{f_i^{\mathbf{v}} \,:\, i \in [k], \mathbf{v} \in r_i\} \cup \{D_v \,:\, v \in V\}$$

where relation $D_v \subseteq D_{\mathbf{R}}$ is treated as a unary function $D_v : D_{\mathbf{R}} \to \{0, \infty\}$.

After ordering of its relations $\Gamma_{\mathbf{R}}$ becomes a template $\mathbf{\Gamma_R}$. The following is true [21]:

▶ **Theorem 19.** *Suppose that $\mathcal{H}$ is up-closed, $\mathbf{R} \in \mathcal{H}$ and $\mathbf{\Gamma}$ is a (valued) template. Then there is a polynomial-time reduction from* $\mathrm{(V)CSP}(\mathbf{\Gamma_R})$ *to* $\mathrm{(V)CSP}_{\mathcal{H}}(\mathbf{\Gamma})$*. Consequently,*
(a) if $\mathrm{(V)CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ *is tractable then so is* $\mathrm{(V)CSP}(\mathbf{\Gamma_R})$*;*
(b) if $\mathrm{(V)CSP}(\mathbf{\Gamma_R})$ *is NP-hard then so is* $\mathrm{(V)CSP}_{\mathcal{H}}(\mathbf{\Gamma})$*.*

Let us give a proof of the latter theorem that slightly differs from the original one. For this purpose we will need a special case of hybrid VCSP, called *the VCSP with input prototype.* Given a finite relational structure $\mathbf{R}$, denote $\mathbf{Up}(\mathbf{R}) = \{\mathbf{I} | \mathbf{I} \to \mathbf{R}\}$.

▶ **Definition 20.** For a given valued template $\mathbf{\Gamma}$ and a relational structure $\mathbf{R}$ a problem $\text{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ where $\mathcal{H} = \mathbf{Up}(\mathbf{R})$ is called **the VCSP with input prototype R** and is denoted as $\text{VCSP}_{\mathbf{R}}(\mathbf{\Gamma})$. If $\mathbf{\Gamma}$ is crisp, then the decision version of $\text{VCSP}_{\mathbf{R}}(\mathbf{\Gamma})$ is denoted as $\text{CSP}_{\mathbf{R}}(\mathbf{\Gamma})$.

It is easy to see that $\mathcal{H} = \mathbf{Up}(\mathbf{R})$ is up-closed. Note that an input of $(\text{V})\text{CSP}_{\mathbf{R}}(\mathbf{\Gamma})$ is a relational structure $\mathbf{I}$ that is homomorphic to $\mathbf{R}$ but this homomorphism itself is not a part of the input. If we also assume that together with a structure $\mathbf{I}$ we are given a homomorphism $h : \mathbf{I} \to \mathbf{R}$, then the latter problem is denoted as $(\text{V})\text{CSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$.

▶ Remark. Note that the complexities of $\text{VCSP}_{\mathbf{R}}(\mathbf{\Gamma})$ and $\text{VCSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$ can be sharply different. For example, consider $\mathbf{\Gamma} = ([4]; \mathbf{neq}_4)$ and $\mathbf{R} = ([3]; \mathbf{neq}_3)$ where $\mathbf{neq}_k = \{(i, j)|i, j \in [k], i \neq j\}$. While $\text{VCSP}_{\mathbf{R}}(\mathbf{\Gamma})$, a problem of 4-coloring of a 3-colorable graph, is known to be NP-hard [19], $\text{VCSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$ is a trivial one. This example also demonstrates the distinction between decision and search in the hybrid framework: the decision problem $\text{CSP}_{\mathbf{R}}(\mathbf{\Gamma})$ is also trivial, whereas its search version is NP-hard.

▶ **Lemma 21.** $(\text{V})\text{CSP}(\mathbf{\Gamma}_{\mathbf{R}})$ *is polynomially equivalent to* $(\text{V})\text{CSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$

Theorem 19 (a). Since $\mathcal{H}$ is up-closed, then for any $\mathbf{R} \in \mathcal{H}$, $\{\mathbf{I}|\mathbf{I} \to \mathbf{R}\} \subseteq \mathcal{H}$. I.e. a problem $\text{VCSP}_{\mathbf{R}}(\mathbf{\Gamma})$ is a restriction of $\text{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ to certain inputs. Therefore, $\text{VCSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$ is polynomially reducible to $\text{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$. Using the previous lemma, we conclude that for the tractability of $\text{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ it is necessary that $\text{VCSP}^{+}_{\mathbf{R}}(\mathbf{\Gamma})$ and $\text{VCSP}(\mathbf{\Gamma}_{\mathbf{R}})$ are tractable. Part (b) can be proved analogously. ◀

4 Wide tractability of a crisp language

Throughout this section we will assume that $\mathbf{\Gamma}$ is crisp.

4.1 Widely tractable languages

For up-closed structural restrictions $\mathcal{H}$, the construction of a lifted language gives us the necessary conditions for the tractability of $\text{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ (Theorem 19 (a)). Let us now define *widely tractable templates* $\mathbf{\Gamma}$ as those for which the necessary conditions for the tractability of $\text{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ are, in fact, sufficient:

▶ **Definition 22.** A template $\mathbf{\Gamma}$ is called **widely tractable** if for any up-closed $\mathcal{H}$, $\text{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ is tractable if and only if $\text{CSP}(\mathbf{\Gamma}_{\mathbf{R}})$ is tractable for any $\mathbf{R} \in \mathcal{H}$.

The concept of wide tractability is important in the hybrid CSPs setting due to the following theorem:

▶ **Theorem 23.** *If a template* $\mathbf{\Gamma}$ *is widely tractable, then there is an up-closed* $\mathcal{H}^{\mathbf{\Gamma}}$ *such that for any up-closed* $\mathcal{H}$*,* $\text{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ *is tractable if and only if* $\mathcal{H} \subseteq \mathcal{H}^{\mathbf{\Gamma}}$*.*

Proof. Let us define

$$\mathcal{H}^{\mathbf{\Gamma}} = \{\mathbf{R}|\text{CSP}(\mathbf{\Gamma}_{\mathbf{R}}) \textit{ is tractable}\} \tag{5}$$

It is easy to see that $\mathcal{H}^{\mathbf{\Gamma}}$ is up-closed itself. By definition, $\mathcal{H}^{\mathbf{\Gamma}}$ contains only such $\mathbf{R}$ for which $\text{CSP}(\mathbf{\Gamma}_{\mathbf{R}})$ is tractable, and this together with wide tractability of $\mathbf{\Gamma}$, implies that $\text{CSP}_{\mathcal{H}^{\mathbf{\Gamma}}}(\mathbf{\Gamma})$ is tractable.

Suppose that for some up-closed $\mathcal{H}$, $\text{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ is tractable. From the wide tractability of $\mathbf{\Gamma}$ we obtain that it is equivalent to stating that $\text{CSP}(\mathbf{\Gamma}_{\mathbf{R}})$ is tractable for any $\mathbf{R} \in \mathcal{H}$. But the last is equivalent to $\mathcal{H} \subseteq \mathcal{H}^{\mathbf{\Gamma}}$. ◀

4.2 Wide tractability in case of strongly BJK languages

In this section we will give necessary and sufficient conditions of wide tractability in a very important case of crisp languages, namely, *strongly BJK* languages.

▶ **Definition 24.** A crisp language $\mathbf{\Gamma}$ is called strongly BJK language if for any $\mathbf{R}$ the lifted $\mathbf{\Gamma_R}$ is BJK.

▶ Remark. As we have already noted it is likely that this class includes all crisp languages [24, 6, 30].

Before introducing the main theorem of this section, let us describe one construction. Let ρ be some m-ary relation over a domain D. It induces a new relation ρ' over a set of Siggers pairs on a set D, denoted D', by the following rule: a tuple of Siggers pairs $\big((g_1, s_1), \cdots, (g_m, s_m)\big) \in \rho'$ if and only if for any $(x_1, ..., x_m) \in \rho$ we have that $(g_1(x_1), ..., g_m(x_m)) \in \rho$ and for any tuples $(a_1, ..., a_m)$, $(b_1, ..., b_m)$, $(c_1, ..., c_m)$, $(d_1, ..., d_m)$ from ρ we have that $\big(s_1(a_1, b_1, c_1, d_1), \ \dots \ , s_m(a_m, b_m, c_m, d_m)\big) \in \rho$. Note that elements of D' are Siggers pairs, but not necessarily polymorphisms of ρ.

Given a relational structure $\mathbf{\Gamma} = (D, \rho_1, ..., \rho_s)$, we define $\mathbf{\Gamma}' = (D', \rho'_1, ..., \rho'_s)$.

▶ **Theorem 25.** *Let $\mathbf{\Gamma}$ be a strongly BJK language. Then $\mathbf{\Gamma}$ is widely tractable if and only if* $\mathrm{CSP}_{\mathbf{\Gamma}'}(\mathbf{\Gamma})$ *is tractable.*

A proof of theorem 25 is mainly based on the following lemma:

▶ **Lemma 26.** *For an arbitrary $\mathbf{R}$, $\mathbf{\Gamma_R}$ admits a Siggers pair if and only if there is a homomorphism $h : \mathbf{R} \to \mathbf{\Gamma}'$.*

▶ Remark. If $\mathbf{\Gamma}' \to \mathbf{\Gamma}$ then $\mathrm{CSP}_{\mathbf{\Gamma}'}(\mathbf{\Gamma})$ is a trivial problem and theorem 25 gives us that $\mathbf{\Gamma}$ is a widely tractable template. Such templates are quite common. E.g. our computational experiment showed (see section 6) that if $D = \{0, 1\}$ and $\rho \subseteq \{0, 1\}^3$ is such that $\Gamma = \{\rho\}$ is NP-hard, then $\mathbf{\Gamma}' \to \mathbf{\Gamma}$. Example of a widely tractable and NP-hard $\mathbf{\Gamma}$ for which $\mathbf{\Gamma}' \not\to \mathbf{\Gamma}$ will be given in the next section (example 29).

4.3 Relationship between $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$

The binary relation $\to$ is transitive, reflexive, but not antisymmetric. It also induces the equivalence relation $\sim$ on a set of all finite structures:

$$\mathbf{R}_1 \sim \mathbf{R}_2 \Leftrightarrow \mathbf{R}_1 \to \mathbf{R}_2, \mathbf{R}_2 \to \mathbf{R}_1$$

▶ **Theorem 27.** *For any $\mathbf{\Gamma}$, $\mathbf{\Gamma} \to \mathbf{\Gamma}'$.*

Thus, we can view $\mathrm{CSP}(\mathbf{\Gamma}')$ as a relaxation of $\mathrm{CSP}(\mathbf{\Gamma})$. Moreover, theorem 27 has the following interesting consequence.

▶ **Theorem 28.** *If $\mathbf{\Gamma}$ is strongly BJK, then there is a polynomial-time Turing reduction from* $\mathrm{CSP}(\mathbf{\Gamma})$ *to* $\mathrm{CSP}(\mathbf{\Gamma}')$

If $\mathbf{\Gamma}$ is tractable, then $\mathbf{\Gamma}'$ is preserved by a nullary constant operation $o = (g, s)$, where $(g, s) \in D'$ is a Siggers pair that is admitted by $\mathbf{\Gamma}$. I.e., $\mathbf{Up}(\mathbf{\Gamma}')$ is a set of all finite structures with the same vocabulary as $\mathbf{\Gamma}$. We can take any tractable $\mathbf{\Gamma}$ that is not constant-preserving (e.g. $\mathbf{\Gamma} = ([3]; \mathbf{neq}_3)$) as an example of a template for which $\mathbf{\Gamma} \not\sim \mathbf{\Gamma}'$, i.e. $\mathbf{\Gamma}' \not\to \mathbf{\Gamma}$.

The following example demonstrates an NP-hard $\mathbf{\Gamma}$ for which $\mathbf{\Gamma} \not\sim \mathbf{\Gamma}'$.

▶ **Example 29.** Define $\mathbf{\Gamma} = (\{0,1\}; \{0\}, \{1\}, \rho)$, where $\rho = \{0,1\}^3 \setminus \{(0,1,0),(1,0,1)\}$. A fixed-template CSP with this $\mathbf{\Gamma}$ is called *the boolean betweennes*, and it is NP-hard because $\mathbf{\Gamma}$ does not fall into any of Schaefer's classes [25].

The boolean betweennes can be popularly reformulated in the following way. Suppose that we have a number of n towns $v_1, ..., v_n$ and a system of roads (each consisting of 3 consecutive towns) $(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}), ..., (v_{\omega_1}, v_{\omega_2}, v_{\omega_3})$. Our goal is to divide those towns between 2 states (assign 0 or 1 to n variables) in such a way that unary constraints are satisfied, i.e. certain towns should be given to prespecified states, and every road should not cross administrative barriers twice.

Let $\mathbf{\Gamma}_\alpha = (\{0,1,\alpha\}; \{0,\alpha\}, \{1,\alpha\}, \rho_\alpha)$, where $\rho_\alpha = \rho \cup \{(1,1,\alpha),(\alpha,1,1),(0,0,\alpha),(\alpha,0,0),(0,\alpha,1),(1,\alpha,0)\}$. A symbol α can be interpreted as a "dual attachment" status that can be given to towns, for which we can freely change α-status to both 0 and 1 without violating ternary constraints.

It is easy to see that $\mathbf{\Gamma}_\alpha \not\to \mathbf{\Gamma}$ (image of α cannot be both 0 and 1). If we prove that $\mathrm{CSP}(\mathbf{\Gamma}_{\mathbf{\Gamma}_\alpha})$ is tractable (and, therefore, $\mathbf{\Gamma}_{\mathbf{\Gamma}_\alpha}$ admits a Siggers pair), this will lead to a conclusion that $\mathbf{\Gamma}_\alpha \to \mathbf{\Gamma}'$ by lemma 26, and consequently, $\mathbf{\Gamma}' \not\to \mathbf{\Gamma}$.

According to lemma 21, $\mathrm{CSP}(\mathbf{\Gamma}_{\mathbf{\Gamma}_\alpha})$ is equivalent to a problem of deciding whether there is a homomorphism $h : \mathbf{R} \to \mathbf{\Gamma}$ for a relational structure $\mathbf{R} = (V, \Omega_0, \Omega_1, \Omega)$ and a homomorphism $g : \mathbf{R} \to \mathbf{\Gamma}_\alpha$ given as inputs. If $\Omega_0 \cap \Omega_1 \neq \emptyset$ we claim the nonexistence of h. Otherwise, h is defined in the following way: $h(x) = g(x)$, if $g(x) \neq \alpha$; $h(x) = 0$, if $x \in \Omega_0$ and $g(x) = \alpha$; $h(x) = 1$, if $x \in \Omega_1$ and $g(x) = \alpha$; and $h(x) = 0$, if otherwise. It can be checked that this algorithm solves $\mathrm{CSP}^+_{\mathbf{\Gamma}_\alpha}(\mathbf{\Gamma})$.

Our computational experiment showed (see section 6) showed that, in fact, $\mathbf{\Gamma}' \sim \mathbf{\Gamma}_\alpha$. It is easy to see that in the latter algorithm for $\mathrm{CSP}(\mathbf{\Gamma}_{\mathbf{\Gamma}_\alpha})$ we used a homomorphism $g : \mathbf{R} \to \mathbf{\Gamma}_\alpha$ only at the stage of the construction of h, i.e. we did not need it at the decision stage. The latter means that $\mathrm{CSP}_{\mathbf{\Gamma}_\alpha}(\mathbf{\Gamma})$ as a decision problem is also tractable and from theorem 25 we obtain that $\mathbf{\Gamma}$ is widely tractable (under condition that it is strongly BJK).

Theorem 28 gives us the idea that we can reduce $\mathrm{CSP}(\mathbf{\Gamma})$ to $\mathrm{CSP}(\mathbf{\Gamma}')$, $\mathrm{CSP}(\mathbf{\Gamma}')$ to $\mathrm{CSP}(\mathbf{\Gamma}'')$ etc. It turns out that this sequence of reductions collapses very soon:

▶ **Theorem 30.** *If $\mathbf{\Gamma}, \mathbf{\Gamma}'$ are both strongly BJK, then $\mathbf{\Gamma}' \sim \mathbf{\Gamma}''$.*

5 Valued templates: conservative case

So far, the most applicable class of fixed-template valued VCSPs was *the submodular function minimization* problems [22]. Also, *minimum cost homomorphism problems (MinHom)* appeared in such different contexts as Defense Logistics [13] and Computer Vision [10]. These two examples make the framework of *conservative valued CSPs* of special interest, since it includes both MinHom and submodular function minimization. The structure of tractable conservative languages is very clearly understood both in crisp [4] and valued cases [29]. Let us now give the definition.

▶ **Definition 31.** A valued constraint language Γ is called *conservative* if it contains $\mathbf{Un}_D$, where $\mathbf{Un}_D$ is a set of all unary $\{0,1\}$-valued cost functions over D.

In the hybrid VCSPs setting, if the right structure $\mathbf{\Gamma}$ is conservative, we have to make a certain supplementary assumption on structural restrictions, so that we do not loose the desirable property that optimized function can have an arbitrary unary part.

▶ **Definition 32.** We say that a relational structure $\mathcal{H}$ **does not restrict unaries** if for each $\mathbf{R} \in \mathcal{H}$ of the form $\mathbf{R} = (V, r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_k)$ with $\mathrm{ar}(r_i) = 1$ and for each unary relation $r_i' \subseteq V$, we have $\mathbf{R}' \in \mathcal{H}$, where $\mathbf{R}' = (V, r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_k)$.

A generalization of the wide tractability for conservative languages will be the following definition.

▶ **Definition 33.** A valued conservative language $\mathbf{\Gamma}$ is called **widely c-tractable** if for any up-closed $\mathcal{H}$ that does not restrict unaries, $\mathrm{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ is tractable if and only if $\mathrm{VCSP}(\mathbf{\Gamma}_{\mathbf{R}})$ is tractable for any $\mathbf{R} \in \mathcal{H}$.

▶ **Theorem 34.** *Any conservative valued language is widely c-tractable.*

An analog of theorem 23 is the following statement.

▶ **Theorem 35.** *For any conservative valued language $\mathbf{\Gamma}$ there is an up-closed $\mathcal{H}_c^{\mathbf{\Gamma}}$ that does not restrict unaries and such that for any up-closed $\mathcal{H}$ that does not restrict unaries,* $\mathrm{VCSP}_{\mathcal{H}}(\mathbf{\Gamma})$ *is tractable if and only if $\mathcal{H} \subseteq \mathcal{H}_c^{\mathbf{\Gamma}}$.*

Our next goal will be to prove that $\mathcal{H}_c^{\mathbf{\Gamma}} = \mathbf{Up}(\mathbf{\Gamma}_c')$ for a certain template $\mathbf{\Gamma}_c'$. If in a case of $\mathrm{CSP}_{\mathcal{H}}(\mathbf{\Gamma})$ we used a description of tractable templates in terms of polymorphisms, in the current case we will need a description via fractional polymorphisms.

▶ **Definition 36.** Let $(\sqcup, \sqcap)$ be a pair of binary operations and (Mj_1, Mj_2, Mn_3) be a triple of ternary operations defined on a domain D, and $M \subseteq \{\{a, b\} \,|a, b \in D, a \neq b\}$.

The pair $(\sqcup, \sqcap)$, is a **symmetric tournament polymorphism (STP)** on M if $\forall x, y$, $\{x \sqcup y, x \sqcap y\} = \{x, y\}$ and for any $\{a, b\} \in M$, $a \sqcup b = b \sqcup a$, $a \sqcap b = b \sqcap a$.

The triple (Mj_1, Mj_2, Mn_3) is **an MJN** on M if $\forall x, y, z, \{Mj_1(x, y, z), Mj_2(x, y, z), Mn_3(x, y, z)\} = \{x, y, z\}$ and for each triple $(a, b, c) \in D^3$ with $\{a, b, c\} = \{x, y\} \in M$ operations $Mj_1(a, b, c)$, $Mj_2(a, b, c)$ return the unique majority element among a, b, c (that occurs twice) and $Mn_3(a, b, c)$ returns the remaining minority element.

The following theorem was established in [20].

▶ **Theorem 37.** *A conservative valued language $\mathbf{\Gamma}$ is tractable if and only if there is a symmetric tournament polymorphism $(\sqcup, \sqcap)$ on M, an MJN (Mj_1, Mj_2, Mn_3) on $\overline{M} = \{\{a, b\} \,|a, b \in D, a \neq b\} \setminus M$, such that $(\sqcup, \sqcap)$, $(Mj_1, Mj_2, Mn_3) \in \mathrm{fPol}(\Gamma)$.*

Given $\mathbf{\Gamma} = (D, f_1, ..., f_s)$, let us construct a relational structure $\mathbf{\Gamma}_c' = (D_c', f_1', ..., f_s')$. Its domain, D_c', is defined as a set of all triples $\big(M, (\sqcup, \sqcap), (Mj_1, Mj_2, Mn_3)\big)$ such that $(\sqcup, \sqcap)$ is a symmetric tournament polymorphism on M and (Mj_1, Mj_2, Mn_3) is an MJN on $\overline{M}$. All f_i' will be relations, i.e. crisp cost functions.

A tuple

$$\big(\big(M^1, (\sqcup^1, \sqcap^1), (Mj_1^1, Mj_2^1, Mn_3^1)\big), \cdots, \big(M^p, (\sqcup^p, \sqcap^p), (Mj_1^p, Mj_2^p, Mn_3^p)\big)\big)$$

is in f_i' if and only if $\big(\sqcup^1, \cdots, \sqcup^p\big)$, $\big(\sqcap^1, \cdots, \sqcap^p\big)$ and $\big(Mj_1^1, \cdots, Mj_1^p\big)$, $\big(Mj_2^1, \cdots, Mj_2^p\big)$, $\big(Mn_3^1, \cdots, Mn_3^p\big)$ are component-wise fractional polymorphisms of f_i, i.e. for any $\mathbf{x} = (x_1, \cdots, x_p)$, $\mathbf{y} = (y_1, \cdots, y_p)$, $\mathbf{z} = (z_1, \cdots, z_p)$ the following inequalities are satisfied:

$$\begin{aligned} f_i(\mathbf{x} \sqcup \mathbf{y}) + f_i(\mathbf{x} \sqcap \mathbf{y}) &\leq f_i(\mathbf{x}) + f_i(\mathbf{y}) \\ f_i(Mj_1(\mathbf{x}, \mathbf{y}, \mathbf{z})) + f_i(Mj_2(\mathbf{x}, \mathbf{y}, \mathbf{z})) + f_i(Mn_3(\mathbf{x}, \mathbf{y}, \mathbf{z})) &\leq \\ f_i(\mathbf{x}) + f_i(\mathbf{y}) + f_i(\mathbf{z}) \end{aligned}$$

where $\mathbf{x} \sqcup \mathbf{y} = \left(x_1 \sqcup^1 y_1, ..., x_p \sqcup^p y_p\right)$ and $\mathbf{x} \sqcap \mathbf{y} = \left(x_1 \sqcap^1 y_1, ..., x_p \sqcap^p y_p\right)$. Analogously, $M(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left(M^1(x_1, y_1, z_1), ..., M^p(x_p, y_p, z_p)\right)$, where instead of M we can paste Mj_1, Mj_2, or Mn_3.

The structure $\mathbf{\Gamma}'_c$ is an analog of $\mathbf{\Gamma}'$. Its domain consists of fractional polymorphisms, that play the same role for valued CSPs as polymorphisms for the crisp case.

▶ **Theorem 38.** *For conservative $\mathbf{\Gamma}$, $\mathcal{H}_c^{\mathbf{\Gamma}} = \mathbf{Up}(\mathbf{\Gamma}'_c)$.*

6 Some experiments and open problems

We list here some experimental results and open problems

- In the case when $D = \{0, 1\}$, it can be shown that in the definition of $\mathbf{\Gamma}'$ Siggers pairs can be replaced with pairs (g, w) where g is unary and w is a ternary weak near unanimity operation on $g(D)$ (the number of such pairs on $\{0, 1\}$ is moderate). This allows a practical computation of $\mathbf{\Gamma}'$s core. We experimented with random structures over the boolean domain ($\Gamma = \{\rho_1, \rho_2, \rho_3\}, \mathrm{ar}(\rho_i) \leq 3$) and found that the domain size of $\mathbf{\Gamma}'$s core is never greater than 5.
- Since $\mathrm{CSP}(\mathbf{\Gamma})$ is reducible to $\mathrm{CSP}(\mathbf{\Gamma}')$, an interesting problem is to find necessary and sufficient conditions for $\mathbf{\Gamma} \sim \mathbf{\Gamma}'$ (i.e. for the case when such reduction is trivial). Experiments showed that if $\Gamma = \{\rho\}, \rho \subseteq \{0, 1\}^3$ is NP-hard, then $\mathbf{\Gamma} \sim \mathbf{\Gamma}'$. At the same time, if $\Gamma = \{\rho, \{0\}, \{1\}\}, \rho \subseteq \{0, 1\}^3$ is NP-hard, then $\mathbf{\Gamma} \nsim \mathbf{\Gamma}'$.
- The number of Siggers pairs on D grows as $O(|D|^{|D|^4})$ which does not allow the calculation of $\mathbf{\Gamma}'$ even in the case when $|D| = 3$. Upper bounds on the domain size of $\mathbf{\Gamma}'$s core is an open problem.
- The problem of classifying all conservative $\mathbf{\Gamma}$ for which $\mathrm{CSP}(\mathbf{\Gamma}'_c)$ is tractable (modification: is solvable in Datalog [2]) is also open.
- Are all crisp templates widely tractable, or is $\mathrm{CSP}_{\mathbf{\Gamma}'}(\mathbf{\Gamma})$ always tractable?

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