On the Number of p4-Tilings by an n-Omino*

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Abstract

A plane tiling by the copies of a polyomino is called isohedral if every pair of copies in the tiling has a symmetry of the tiling that maps one copy to the other. We show that, for every n-omino (i.e., polyomino consisting of n cells), the number of non-equivalent isohedral tilings generated by 90 degree rotations, so called p4-tilings or quarter-turn tilings, is bounded by a constant (independent of n). The proof relies on the analysis of the factorization of the boundary word of a polyomino.

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1 Introduction

The investigation of plane tilings by polyominoes have attracted many researchers for a long time. In this paper, we focus on the following type of problem: what is the maximum number of *isohedral* tilings that a single polyomino can have? A plane tiling by a polyomino is called isohedral if every pair of copies in the tiling has a symmetry of the tiling that maps one copy to the other. Two tilings are said to be *equivalent* if they are congruent, i.e., they can be mapped onto each other by a combination of rotations, translations and reflections.

A polyomino having an isohedral tiling can be classified into seven types according to its boundary word. See the recent work by Langerman and Winslow [14, Section 3] for a clear description of the classification based on earlier works (e.g., [11]). In this paper, we focus on the isohedral tiling called *p4-tiling* (or *quarter-turn* tiling) among these seven types. A polyomino is said to have a p4-tiling if it covers the plane by only 90 degree rotations around two designated points called *rotation centers*. See Figure 1.

Some polyominoes have multiple p4-tilings. Figure 2 shows an example of a pentomino (i.e., 5-omino) having two non-equivalent p4-tilings. One can see that each pentomino is adjacent to four (five, respectively) pentominoes in the left (right, respectively) tiling.

It is known that (see e.g., [6, 7]), if an n-omino has a p4-tiling, then the relative distance (x, y) of two rotation centers satisfies

$$n = \frac{x^2 + y^2}{2}. \qquad (1)$$

This says that an n-omino can have a p4-tiling only if

$$n = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \ldots.$$

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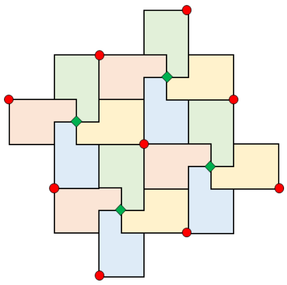

Figure 1 An example of a p4-tiling by a pentomino. The rotation centers are represented by red and green circles.

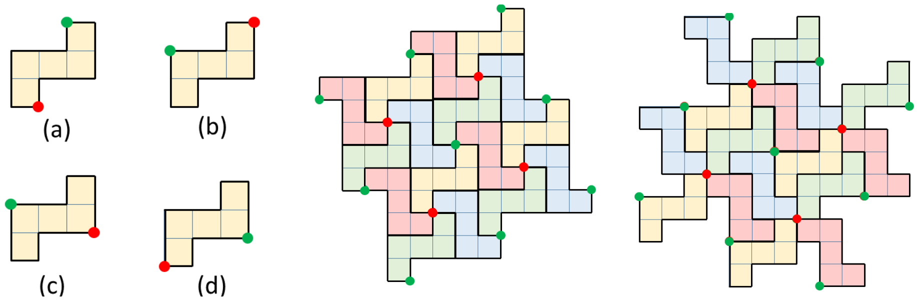

Figure 2 A pentomino having two non-equivalent p4-tilings. The rotation centers shown in (a) gives the left tiling, and (b) gives the right tiling. The rotation centers shown in (c) gives the tiling equivalent to the left tiling, and (d) is considered to be same as (b) since these are overlapped by the 180 degree rotation.

For example, every pentomino (like the one in Figure 2) has rotation centers with relative distance $(1, 3)$ (ignoring the order of x and y). We call such a tiling as a p4-tiling *with center* $(1, 3)$.

Eq. (1) not only restricts the values of n, but also arises another multiplicity of p4-tilings since Eq. (1) may have many solutions. For examples, $(x, y) = (1, 7)$ and $(5, 5)$ satisfies Eq. (1) for $n = 25$. A computer experiment shows that among 2,557,227,044,764 25-ominoes (Sequence A000105 in OEIS [15]), 3,076,890 and 1,526,416 have a p4-tiling with center $(1, 7)$ and $(5, 5)$, respectively. The size of their intersection is 10,824. See Figure 3 for one of such 25-ominoes.

The number of solutions to Eq. (1) can be unbounded as n goes to infinity (see e.g., [10, 17]). Indeed, if n is factored as $n = 2^{a_0} p_1^{2a_1} \cdots p_r^{2a_r} q_1^{b_1} \cdots q_r^{b_r}$, where the p_is are primes of the form $4k + 3$ and the q_is are primes of the form $4k + 1$, then the number of solutions $R(n)$ to Eq. (1) (allowing zeros and ignoring order and signs) is given by[1]

$$R(n) = \begin{cases} 0, & \text{if any } a_i \text{ is a half-integer,} \\ \lceil \frac{(b_1+1)(b_2+1)\cdots(b_r+1)}{2} \rceil, & \text{if all } a_i \text{ are integers.} \end{cases} \quad (2)$$

[1] In [17], the formula for the number of solutions of $n = x^2 + y^2$ not allowing zeros and ignoring order and signs is given. Eq. (2) is essentially the same to this by observing $n = x^2 + y^2$ iff $2n = (x - y)^2 + (x + y)^2$.

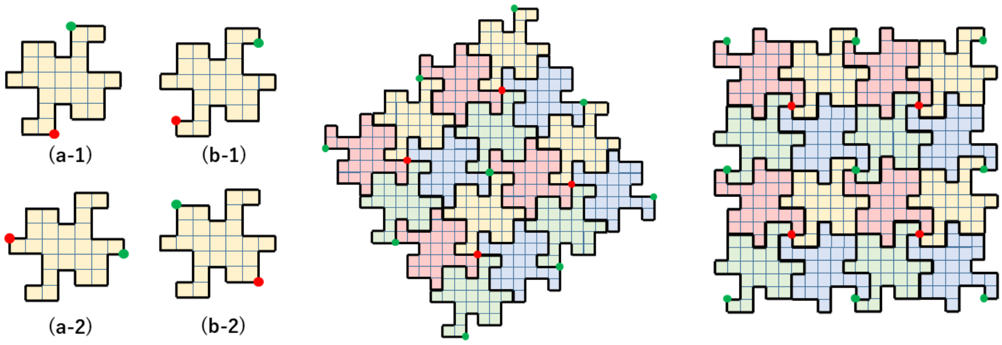

Figure 3 A 25-omino having p4-tilings with centers $(1, 7)$ and $(5, 5)$. It has four pairs of rotation centers. The rotation centers (a-1) and (a-2) admit the left tiling, and (b-1) and (b-2) admit the right tiling. Note that this is the only 25-omino having (at least) four pairs of rotation centers found through our experiments.

Hence, for example, $n = 5^{2k-1}$ has $k = \Theta(\log n)$ solutions.

Figure 4 shows a 1300-omino that has three p4-tilings with centers $(x, y) = (10, 50), (22, 46)$ and $(34, 38)$. Note that $325 = (5^2 + 25^2)/2 = (11^2 + 23^2)/2 = (17^2 + 19^2)/2$ is the smallest integer having three solutions to Eq. (1) (up to the order of x and y), but we have not succeeded to find a 325-omino that has p4-tilings for these three distances. Note also that we found the 1300-omino shown in Figure 4 by using a SAT solver [12].

Now the following questions become interesting: what is the maximum number of p4-tilings that a single polyomino can have? Is it bounded, or is there a polyomino having an unbounded number of p4-tilings?

1.1 Our Contributions

The contribution of this paper is to show that the number of p4-tilings by an n-omino is bounded by a constant (Theorem 1). This is true even for n having an unbounded number of solutions to Eq. (1). It is in sharp contrast to the tiling by *translations*; a $1 \times n$ rectangle has $\Theta(n)$ translation tilings.

In order to show the upper bound on the number of p4-tilings, we use an equivalence between p4-tilings and factorizations of the boundary word of a polyomino into some specific form. Then, we show that the number of such factorizations is bounded for every polyomino based on the analysis of words having such factorizations.

1.2 Related Works

There are plenty of works dealing with polyomino tilings since Golomb [9] initiated the work in 60s. We listed here only those closely related to our work.

Fukuda et al. [6, 7] enumerated n-ominoes that have a p4-tiling for $n \leq 10$. Horiyama and Samejima [13] gave an algorithm for generating many n-ominoes for p4-tilings.

Winslow [18] gave a linear time algorithm for deciding whether a given polyomino has an isohedral tiling by *translations*. This improved the earlier works of Beauquier and Nivat [1], Gambini and Vuillon [8] and Provençal [16] in terms of its running time, and generalized the works of Brlek et al. [2, 3] that dealt with some restricted cases. In [18], Winslow also

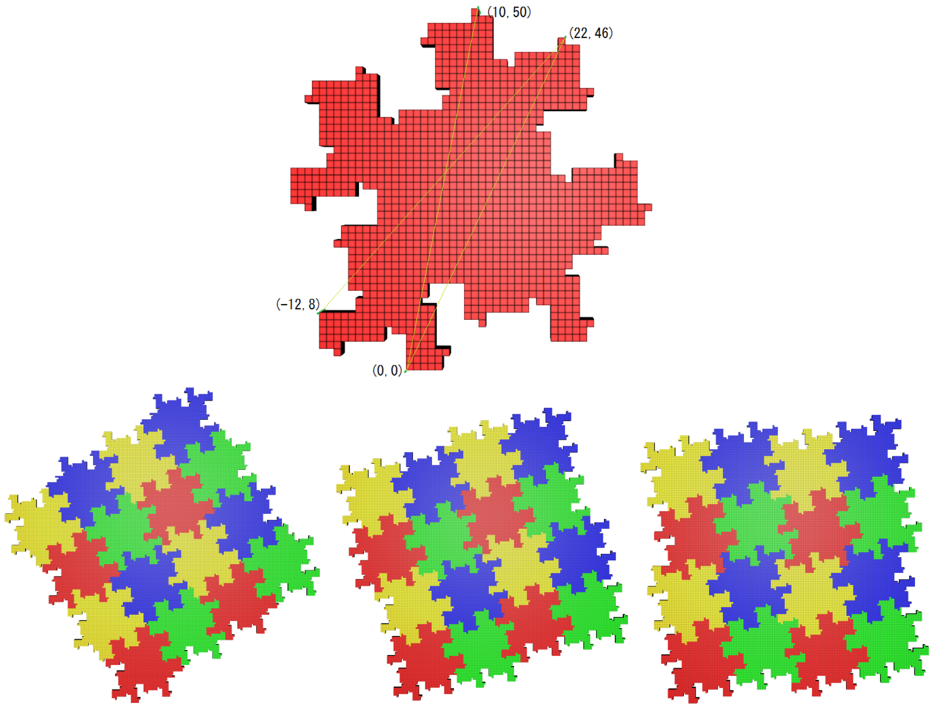

Figure 4 Top: A 1300-omino having p4-tilings for three pairs of rotation centers with distinct relative distances. Bottom: A p4-tiling by rotation centers $(0,0)$ and $(10,50)$ (left), $(0,0)$ and $(22,46)$ (center), and $(-12,8)$ and $(22,46)$ (right).

showed that every n-omino has $O(n)$ translation tilings. Recently, Langerman and Winslow [14] extended this work to give quasi-linear time algorithms for all of seven types of isohedral tilings. Their results include a linear time algorithm for deciding whether a polyomino has a p4-tiling. We note here that our proof borrows many useful analyses on words that appeared in their work [14].

2 Notations and Definitions

The notations and definitions used in this paper are similar to those of Langerman and Winslow [14].

A *polyomino* is a two-dimensional shape formed by connecting one or more unit squares edge to edge. A polyomino consisting of n unit squares is called *n-omino*. A polyomino having a p4-tiling never includes a hole, and so its boundary is naturally represented by a four-letter word. A *letter* is a symbol $x \in \Sigma = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, where $\mathbf{0}, \mathbf{1}, \mathbf{2}$ and $\mathbf{3}$ represent the directions up, right, down and left, respectively. If no confusion arises, we identify the letters $\mathbf{0}, \mathbf{1}, \mathbf{2}$ and $\mathbf{3}$ with the integers 0, 1, 2 and 3, respectively. This is convenient when we consider a rotation. For $\Theta \in \{0, 90, 180, 270\}$, the $\Theta°$*-rotation* of a letter x, denoted $t_\Theta(x)$, is the letter obtained by rotating x clockwise by $\Theta°$, i.e., $t_{90k}(x) = x + k \pmod 4$ for $k \in \{0, 1, 2, 3\}$[2].

[2] Remark that our definition of the function $t_\Theta(\cdot)$ is different from the one used in [14].

A *word* is a sequence of letters and the *length* of a word W, denoted by $|W|$, is the number of letters in W. For an integer $1 \le i \le |W|$, the i-th letter of a word W is denoted by $W[i]$, and the i-th from the last letter of W is denoted by $W[-i]$.

A *factor* (or *subword*) of W is a contiguous sequence X of letters in W, written $X \preceq W$. A factor $X \preceq W$ is a *prefix* if X starts at $W[1]$, written $X \preceq_{\mathrm{pre}} W$. Similarly, $X \preceq W$ is a *suffix* if X ends at $W[-1]$, written $X \preceq_{\mathrm{suf}} W$. For $1 \le i \le j \le |W|$, a factor of W that starts at $W[i]$ and ends at $W[j]$ is denoted by $W[i : j]$. We say that a word W has a *period* p if $W[i] = W[i + p]$ for every $1 \le i \le |W| - p$.

The *reverse* of a word W, denoted by $\overleftarrow{W}$, is the word obtained by reading W in reverse order. The Θ-*rotation* of a word W, denoted by $t_\Theta(W)$, is the word obtained by replacing each letter of W by its Θ-rotation. A word W is called a *palindrome* if $W = \overleftarrow{W}$, and is called a Θ-*drome* if $W = X t_\Theta(\overleftarrow{X})$ for some word X. For example, $W = \mathbf{010121}$ is a 90-drome with $X = \mathbf{010}$. Note that, in this paper, we only deal with a palindrome of even length.

The boundary of a polyomino can naturally be represented by a *circular* word on Σ, in which a first letter is not fixed. The *boundary word* of a polyomino P, denoted by $\mathcal{B}(P)$, is the circular word on Σ coding the boundary of P in a clockwise manner. For example, the boundary word of the pentomino P_5 shown in Figure 2 is $\mathcal{B}(P_5) = \mathbf{300110122332}$. A Θ-drome (or a palindrome for which $\Theta := 0$) factor X of a circular word W is said to be *admissible* if $W = XU$ satisfies $U[-1] \ne t_\Theta(U)[1]$, which intuitively says that X is maximal in a natural sense.

3 p4-tilings with Multiple Rotation Centers

This section is devoted to prove the main theorem of this paper.

▶ **Theorem 1.** *Every polyomino has $O(1)$ p4-tilings.*

In Section 3.1, we describe a number of lemmas which will be used in the proof of Theorem 1. Many of them are taken from Langerman and Winslow [14]. Here and hereafter, when we refer [14] the numbering of the lemmas or theorems is according to its full version (appeared in `arXiv:1507.02762v2`). The main body of the proof of Theorem 1 is described in Section 3.2.

The characterization of a polyomino that admits a p4-tiling in terms of its boundary word is formalized as follows (See e.g., [14, Section 3].)

▶ **Theorem 2.** *A polyomino P has a p4-tiling if and only if its circular boundary word $\mathcal{B}(P)$ can be factorized as $\mathcal{B}(P) = ABC$ where A is a palindrome and B, C are 90-dromes. In addition, all of A, B and C are admissible, and A can be empty but B and C are non-empty.* ◀

The admissibility is shown in [14, Lemma 5.8], and the non-emptiness of 90-dromes is verified as follows: Suppose to the contrary that a polyomino P with $\mathcal{B}(P) = AB$, where $|B| \ne 0$, has a p4-tiling. One of the rotation centers should be just before $B[1]$ or just after $B[-1]$ (the other center is between $B[|B|/2]$ and $B[|B|/2 + 1]$). Suppose without loss of generality that $B[1] = \mathbf{0}$ and $B[-1] = \mathbf{1}$. Then, $A[1] (= A[-1]) \in \{\mathbf{2}, \mathbf{3}\}$ is prohibited since it contradicts that $\mathcal{B}(P)$ is the boundary word of a polyomino, and $A[1] (= A[-1]) \in \{\mathbf{0}, \mathbf{1}\}$ is prohibited since it contradicts that exactly one of four cells surrounding a rotation center should be occupied by P.

Recall that two tilings are said to be equivalent if they can be mapped onto each other by combination of rotations, translations and reflections. Obviously, the number of non-equivalent p4-tilings by a polyomino P is upper bounded by the number of factorizations of the boundary word $\mathcal{B}(P)$ into a form given by Theorem 2. For example, the boundary word of the pentomino P_5 shown in Figure 2 has four factorizations;

$$\begin{aligned} \mathcal{B}(P_5) &= \mathbf{300})(\mathbf{110122})(\mathbf{332} && (3)\\ &= (\mathbf{300110})(\mathbf{12})(\mathbf{2332}) && (4)\\ &= (\mathbf{300110})(\mathbf{122332}) && (5)\\ &= (\mathbf{30})(\mathbf{0110})(\mathbf{122332}) && (6) \end{aligned}$$

where the factorizations (3), (4), (5) and (6) correspond to (a), (b), (c) and (d) in Figure 2, respectively.

In the factorizations (3) and (5), a palindrome A is empty. The rotation centers are designated by the centers of each of two 90-dromes. As shown in Figure 2, the factorizations (3) and (5) (or (4) and (6)) admit an equivalent p4-tiling, and hence the number of factorizations and the number of non-equivalent p4-tilings are not necessarily identical.

3.1 Miscellaneous Lemmas

We start with the following lemmas on words.

▶ **Lemma 3.** *([14, Lemma 4.1]) Let $W = PX$ with P, W palindromes and $0 < |P| < |W|$. Then W has a period $|X|$.* ◀

▶ **Lemma 4.** *(Fine and Wilf's theorem [5], or see [14, Lemma 4.2]) Let W be a word with periods p and q. If $p + q \leq |W|$, then W also has a period GCD(p, q).* ◀

▶ **Lemma 5.** *Let k be an integer such that $k \geq 2$. Let $X_1, \ldots, X_k$ be palindromes such that $X_{i+1} \preceq_{\text{pre}} X_i$ for every $1 \leq i \leq k-1$ and $|X_1| > |X_2| > \cdots > |X_k| \geq (2/3)|X_1|$. Then X_1 has a period p such that $p \leq (|X_1| - |X_k|)/(k-1)$.*

Proof. For each $2 \leq i \leq k$, let $a_k := |X_{k-1}| - |X_k|$. Let $p := \text{GCD}(a_2, \ldots, a_k)$. We will show that X_1 has a period p, which implies Lemma 5 by $\sum_{i=2}^{k} a_k = |X_1| - |X_k|$.

By Lemma 3, X_1 has a period a_2. This implies that X_k also has a period a_2 since $X_k \preceq_{\text{pre}} X_1$. Similarly, for every $2 \leq j \leq k-1$, X_j has a period a_{j+1}. These imply that X_k has periods $a_2, \ldots, a_k$. Since $|X_k| \geq \sum_{i=2}^{k} a_k$, Lemma 4 implies that X_k has a period p. Since $X_k \preceq_{\text{pre}} X_1$, $X_1[i] = X_1[i+p]$ for every $1 \leq i \leq |X_k| - p$. Since X_1 is a palindrome and $p \leq |X_1|/3$, we have $X_1[(2/3)|X_1| - p + \alpha] = X_1[|X_1|/3 + p - \alpha + 1] = X_1[|X_1|/3 - \alpha + 1] = X_1[(2/3)|X_1| + \alpha]$ for every $1 \leq \alpha \leq |X_1|/3$, which completes the proof. ◀

We will use the following lemma extensively in Section 3.2.

▶ **Lemma 6.** *([14, Lemma 5.3]) Let P, Q, W be 90-dromes with $P, Q \preceq_{\text{pre}} W$ and $|P| < |Q| < |W|$. Then $|P| < (2/3)|W|$, or equivalently $|W| > (3/2)|P|$.* ◀

An analogous argument to the proof of Lemma 6 gives a suffix version of the lemma.

▶ **Corollary 7.** *Let P, Q, W be 90-dromes with $P, Q \preceq_{\text{suf}} W$ and $|P| < |Q| < |W|$. Then $|P| < (2/3)|W|$, or equivalently $|W| > (3/2)|P|$.* ◀

The following lemma, which will also be used in Section 3.2, can easily be derived from Lemma 6 (and Corollary 7).

▶ **Lemma 8.** *There is no word that can be factorized into two 90-dromes in more than 8 ways.*

Proof. Suppose to the contrary that, W can be factorized into X_iY_i where X_i and Y_i are 90-dromes for $1 \leq i \leq 9$. Without loss of generality, we assume that $|X_1| < |X_2| < \cdots < |X_9|$, and also that $|X_5| \geq |W|/2$ (by interchanging X's and Y's and will use Corollary 7 instead of Lemma 6 if necessary). Since $X_i \preceq_{\text{pre}} X_{i+1}$ for every $1 \leq i \leq 8$, Lemma 6 implies $|X_9| > (3/2)|X_7| > (9/4)|X_5| > |W|$, which is a contradiction. ◀

▶ **Lemma 9.** *([14, Lemma 5.4]) Let W be a boundary word. There exist $O(1)$ admissible 90-drome factors of W with length at least $|W|/3$.* ◀

The following lemma, whose proof is elementary but a bit long, is also useful in the proof of Theorem 1.

▶ **Lemma 10.** *Let W be a word. The number of factorizations of W in such a way that $W = XY$ with X a non-empty 90-drome and Y a palindrome is at most two. The same holds for $W = YX$ with X a non-empty 90-drome and Y a palindrome.*

Proof. We only show the first case of the lemma (the proof of the second case is completely analogous.). Suppose to the contrary that $W = X_1Y_1 = X_2Y_2 = X_3Y_3$ where X_1, X_2, X_3 are 90-dromes with $0 < |X_1| < |X_2| < |X_3|$ and Y_1, Y_2, Y_3 are palindromes. Put $p_i := |X_i|/2$ for $1 \leq i \leq 3$. Note that $0 < p_1 < p_2 < p_3$. For $1 \leq i \leq 3$, let L_i and R_i denote the first and second half of X_i, respectively.

We divide the proof into several cases depending on the values of p_1, p_2 and p_3.

Case 1: $p_1 + p_2 > p_3$.

Note that, in this case, we have $p_1 \geq 2$ since $p_3 - p_2 \geq 1$. Let $v_0 := p_1 + p_2 - p_3$ and suppose without loss of generality that $W[v_0] = \mathbf{0}$. Since $1 \leq v_0 \leq p_1 - 1$, $W[v_0] \in L_1$. So the reflection of $W[v_0]$ w.r.t. the center of X_1 is the $v_1 := (p_1 - p_2 + p_3 + 1)$-th letter in W and $W[v_1] = W[v_0] + 1 = \mathbf{1}$ since X_1 is a 90-drome. Alternatively, we can write this as

$$W[v_1] \quad = \quad X_1[-v_0] = X_1[v_0] + 1 = W[v_0] + 1 = \mathbf{1}.$$

Since $v_1 \leq p_3$, we have $W[v_1] \in L_3$. Hence the reflection of $W[v_1]$ w.r.t. the center of X_3 is the $v_2 := (-p_1 + p_2 + p_3)$-th letter in W and $W[v_2] = W[v_1] + 1 = \mathbf{2}$. By continuing this argument to X_2, X_1, X_3, X_2 in this order, we have the chain of implications:

$$\begin{aligned} W[v_0] = \mathbf{0} \quad &\Rightarrow \quad W[v_1 := p_1 - p_2 + p_3 + 1] = \mathbf{1} \qquad (\because W[v_0] \in L_1) \\ &\Rightarrow \quad W[v_2 := -p_1 + p_2 + p_3] = \mathbf{2} \qquad (\because W[v_1] \in L_3) \\ &\Rightarrow \quad W[v_3 := p_1 + p_2 - p_3 + 1] = \mathbf{1} \qquad (\because W[v_2] \in R_2) \\ &\Rightarrow \quad W[v_4 := p_1 - p_2 + p_3] = \mathbf{2} \qquad (\because W[v_3] \in L_1) \\ &\Rightarrow \quad W[v_5 := -p_1 + p_2 + p_3 + 1] = \mathbf{3} \qquad (\because W[v_4] \in L_3) \\ &\Rightarrow \quad W[v_0] = \mathbf{2} \qquad (\because W[v_5] \in R_2), \end{aligned}$$

which is a contradiction.

Case 2: $p_1 + p_2 \leq p_3$.

Case 2.1: $p_2 \geq 2p_1$.

Suppose $W[1] = \mathbf{0}$. By considering the reflection of $W[1]$ w.r.t. the center of X_1, we have $W[2p_1] = X_1[-1] = X_1[1] + 1 = \mathbf{1}$. Then, by considering the reflection of these two letters w.r.t. the center of X_2, we have $W[2p_2] = X_2[-1] = X_2[1] + 1 = W[1] + 1 = \mathbf{1}$, and $W[2p_2 - 2p_1 + 1] = X_2[-2p_1] = X_2[2p_1] + 1 = W[2p_1] + 1 = \mathbf{2}$.

We further divide this case into two subcases.

Case 2.1.1: $p_3 \geq 2p_2 - 2p_1 + 1$.

By considering the reflection of $W[1]$ and $W[2p_2 - 2p_1 + 1]$ w.r.t. the center of X_3, we have $W[2p_3] = X_3[-1] = X_3[1] + 1 = W[1] + 1 = \mathbf{1}$, and $W[2p_3 - 2p_2 + 2p_1] =$

$X_3[-(2p_2 - 2p_1 + 1)] = X_3[2p_2 - 2p_1 + 1] + 1 = W[2p_2 - 2p_1 + 1] + 1 = \mathbf{3}$ (*). Here we use the condition $p_3 \geq 2p_2 - 2p_1 + 1$ to derive the second equality. However, we have

$$\begin{aligned}\mathbf{1} &= W[2p_3] = Y_2[2p_3 - 2p_2] = Y_2[-(2p_3 - 2p_2)] \qquad (\because Y_2 \text{ is a palindrome.}) \\ &= W[-(2p_3 - 2p_2)] = Y_1[-(2p_3 - 2p_2)] = Y_1[(2p_3 - 2p_2)] \quad (\because Y_1 \text{ is a palindrome.}) \\ &= W[2p_3 - 2p_2 + 2p_1],\end{aligned}$$

which contradicts (*).

We can show the remaining three cases (Cases 2.1.2, 2.2.1 and 2.2.2) similarly.

Case 2.1.2: $p_3 \leq 2p_2 - 2p_1$.

By considering the reflection of $W[2p_1](= \mathbf{1})$ and $W[2p_2](= \mathbf{1})$ w.r.t. the center of X_3, we have $W[2p_3 - 2p_1 + 1] = X_3[-2p_1] = X_3[2p_1] + 1 = W[2p_1] + 1 = \mathbf{2}$, and $W[2p_3 - 2p_2 + 1] = X_3[-2p_2] = X_3[2p_2] - 1 = W[2p_2] - 1 = \mathbf{0}$ (**), where the second equality follows from $2p_2 > p_3$ in this case. On the other hand, we have

$$\begin{aligned}\mathbf{2} &= W[2p_3 - 2p_1 + 1] \\ &= Y_2[2p_3 - 2p_2 - 2p_1 + 1] \qquad (\because 2p_3 - 2p_1 + 1 \geq p_2) \\ &= Y_2[-(2p_3 - 2p_2 - 2p_1 + 1)] \qquad (\because Y_2 \text{ is a palindrome.}) \\ &= W[-(2p_3 - 2p_2 - 2p_1 + 1)] \\ &= Y_1[-(2p_3 - 2p_2 - 2p_1 + 1)] \\ &= Y_1[2p_3 - 2p_2 - 2p_1 + 1] \qquad (\because Y_1 \text{ is a palindrome.}) \\ &= W[2p_3 - 2p_2 + 1],\end{aligned}$$

which contradicts (**).

Case 2.2: $p_2 < 2p_1$.

Case 2.2.1: $p_3 < 2p_2$.

Suppose that $W[1] = \mathbf{0}$. By considering the reflection of $W[1]$ w.r.t. the center of X_1 and X_2, we get $W[2p_1] = \mathbf{1}$ and $W[2p_2] = \mathbf{1}$, respectively. Then, by considering the reflection of $W[2p_2]$ w.r.t. the center of X_3, we have $W[2p_3 - 2p_2 + 1] = X_3[-2p_2] = X_3[2p_2] - 1 = W[2p_2] - 1 = \mathbf{0}$, and $W[2p_3 - 2p_1 + 1] = X_3[-2p_1] = X_3[2p_1] + 1 = W[2p_1] + 1 = \mathbf{2}$ (*3). Here we use $2p_2 > p_3$ and $2p_1 \leq p_3$. However, we have

$$\begin{aligned}\mathbf{0} &= W[2p_3 - 2p_2 + 1] \\ &= Y_1[2p_3 - 2p_2 - 2p_1 + 1] \qquad (\because 2p_3 - 2p_2 + 1 > 2p_1 = |X_1|) \\ &= Y_1[-(2p_3 - 2p_2 - 2p_1 + 1)] \qquad (\because Y_1 \text{ is a palindrome.}) \\ &= W[-(2p_3 - 2p_2 - 2p_1 + 1)] \\ &= Y_2[-(2p_3 - 2p_2 - 2p_1 + 1)] \qquad (\because (2p_3 - 2p_2 - 2p_1 + 1) + |X_2| \leq 2p_3 \leq |W|) \\ &= Y_2[2p_3 - 2p_2 - 2p_1 + 1] \qquad (\because Y_2 \text{ is a palindrome.}) \\ &= W[2p_3 - 2p_1 + 1],\end{aligned}$$

which contradicts (*3).

Case 2.2.2: $p_3 \geq 2p_2$.

In this case, we first suppose that $W[p_1] = X_1[p_1] = \mathbf{0}$. We have $W[p_1 + 1] = X_1[p_1 + 1] = X_1[p_1] + 1 = \mathbf{1}$ since X_1 is a 90-drome, and this implies $W[2p_2 - p_1] = X_2[-(p_1 + 1)] = X_2[p_1 + 1] + 1 = W[p_1 + 1] + 1 = \mathbf{2}$ since X_2 is a 90-drome. Since X_3 is a 90-drome, we have $W[2p_3 - p_1 + 1] = X_3[-p_1] = X_3[p_1] + 1 = W[p_1] + 1 = \mathbf{1}$, and $W[2p_3 - 2p_2 + p_1 + 1] = X_3[-(2p_2 - p_1)] = X_3[2p_2 - p_1] + 1 = W[2p_2 - p_1] + 1 = \mathbf{3}$ (*4).

Here we use $2p_2 - p_1 \le p_3$ to derive the second equality. However, we have

$$\begin{aligned}
\mathbf{1} &= W[2p_3 - p_1 + 1] \\
&= Y_2[2p_3 - 2p_2 - p_1 + 1] \qquad (\because 2p_3 - p_1 + 1 > 2p_2 = |Y_2|) \\
&= Y_2[-(2p_3 - 2p_2 - p_1 + 1)] \qquad (\because Y_2 \text{ is a palindrome.}) \\
&= W[-(2p_3 - 2p_2 - p_1 + 1)] \\
&= Y_1[-(2p_3 - 2p_2 - p_1 + 1)] \\
&= Y_1[2p_3 - 2p_2 - p_1 + 1] \qquad (\because Y_1 \text{ is a palindrome.}) \\
&= W[2p_3 - 2p_2 + p_1 + 1],
\end{aligned}$$

which contradicts (*4). This completes the proof of Lemma 10. ◀

The following two lemmas are also from [14].

▶ **Lemma 11.** *([14, Lemma 5.1]) Let W be a word with a period p, and X a 90-drome subword of W. Then $|X| \le p$.* ◀

▶ **Lemma 12.** *([14, Lemma 5.7]) Let W be a word. There exists an $O(1)$-sized set $\mathcal{F}$ of factors W such that every admissible palindrome factor with length at least $|W|/3$ is an affix (i.e., a prefix or suffix) factor of an element of $\mathcal{F}$.* ◀

The final statement in this subsection is a famous theorem on integer sequences due to Erdős and Szekeres.

▶ **Theorem 13.** *([4]) Any sequence of $n^2 + 1$ distinct integers has either an increasing or a decreasing subsequence of length $n + 1$.* ◀

3.2 Proof of Theorem 1

Proof of Theorem 1. Given an arbitrary polyomino P, we will show that the number of ways such that $\mathcal{B}(P)$ is factorized into ABC with A a palindrome and B, C 90-dromes as described in Theorem 2 is $O(1)$, which is sufficient to prove Theorem 1. Let $n := |\mathcal{B}(P)|$.

Below we show this for each of two cases (i) $|B|$ or $|C|$ is at least $n/3$, and (ii) $|A|$ is at least $n/3$. The first case follows easily from the lemmas shown in Section 3.1.

Case 1: $|B| \ge n/3$ or $|C| \ge n/3$.

Suppose that $|B| \ge n/3$. (The case $|C| \ge n/3$ is analogous.) Lemma 9 says that there are $O(1)$ possibilities of B.

Fix a 90-drome B. Lemma 10 implies that the number of factorizations of $\mathcal{B}(P)$ including B as an admissible 90-drome factor is at most two. This completes the proof of Case 1.

Case 2: $|A| \ge n/3$.

This case covers the complement of Case 1. Lemma 12 implies that there are $O(1)$ possibilities of the position of the first or the last letter of A in $\mathcal{B}(P)$.

Fix a position of the first letter of A in $\mathcal{B}(P)$. The case for fixing the last letter of A is analogous. Let W be a non-circular word obtained from $\mathcal{B}(P)$ by applying a circular shift if necessary such that A starts at $W[1]$. Let c be a sufficiently large constant whose value will be determined at the end of the proof. Suppose to the contrary that W has at least c factorizations ABC with A a palindrome and B and C 90-dromes as described in Theorem 2. By Lemma 8, for every fixed A, there are at most eight such factorizations. Hence, we can assume that there are at least $c_0 := c/8$ factorizations $W = A_i B_i C_i$ ($1 \le i \le c_0$) such that A_i starts at $W[1]$ for every $1 \le i \le c_0$, and that all the $|A_i|$s are distinct. Moreover, all the B_is and the C_is are non-empty.

We label these factorizations so that $|A_1| > |A_2| > \cdots > |A_{c_0}|$. Let S_0, S_1 and S_2 be the partition of the indices $\{1, \ldots, c_0\}$ such that $S_0 := \{i : (2/3)n \leq |A_i| \leq n\}$, $S_1 := \{i : (4/9)n \leq |A_i| < (2/3)n\}$ and $S_2 := \{i : n/3 \leq |A_i| < (4/9)n\}$. Choose the largest set among S_0, S_1 and S_2, and relabel the indices in the chosen set as $1, \ldots, c_1$, where $c_1 \geq c_0/3$. Note that this ensures that $|A_{c_1}| \geq (2/3)|A_1|$, which will be needed when we apply Lemma 5.

Let $d_i := |A_i| + |B_i|/2$ which represents the position of the center of B_i in W. We now focus on the d_is. We can assume that the d_is are all different. This is because $d_i = d_j$ with $i < j$ implies that B_i is a subword of B_j having the same center, and hence B_i is not admissible. Now the theorem of Erdős-Szekeres (Theorem 13) guarantees that we can pick a sequence of indices $1 \leq i_1 < i_2 < \cdots < i_{c_2} \leq c_1$ with $c_2 := \lceil \sqrt{c_1} \rceil$ such that $d_{i_1}, d_{i_2}, \ldots, d_{i_{c_2}}$ are sorted in increasing or decreasing order.

Now we divide Case 2 into two subcases depending on the order of $d_{i_1}, d_{i_2}, \ldots, d_{i_{c_2}}$. In what follows, we write $A_{i_k}B_{i_k}C_{i_k}$ as $A_kB_kC_k$ for simplicity (to avoid a double subscript). For $1 \leq k \leq c_2$, let $b_k := |A_k| + |B_k| - |A_1|$, which represents the position of $B_k[-1]$ in W where we count the position of $B_1[1]$ in W as 1.

Case 2.1: $d_1 < d_2 < \cdots < d_{c_2}$.

Notice that $|B_1| = b_1 > b_2 > \cdots > b_{c_2}$. For each $1 \leq k \leq c_2$, let X_k be a 90-drome subword of B_k obtained from B_k by truncating a same number of letters from both sides of B_k so that X_k starts at $W[|A_1|+1]$. Equivalently, $X_k := W[|A_1|+1 : 2|A_k|+|B_k|-|A_1|]$. Note that $X_1 = B_1$ and $X_k \preceq_{\text{pre}} X_{k+1}$ for every $1 \leq k \leq c_2 - 1$.

Suppose that $c_2 \geq 17$. Put $m := |B_1C_1|(= n - |A_1|)$. By Lemma 6, we have $|X_{k+2}| > (3/2)|X_k|$ for every $k \geq 1$. Thus, we can assume that $|B_1| = |X_1| < m/10$. This is because otherwise $|X_{17}| > (3/2)^8(1/10)m > m$ which is impossible. Hence, the center of C_1 is located at the position between $|A_1| + m/2$ and $|A_1| + (m/2 + m/20)$ in W (*5). By noticing that $C_{k+1} \preceq_{\text{suf}} C_k$ for every k, we can apply Corollary 7 to get $|C_9| < (2/3)^4|C_1| \leq (2/3)^4m < m/5$ which implies $b_9 > (4/5)m$. We also have $|X_9| < m/5$ because otherwise $|X_{17}| > (1/5)(3/2)^4m > m$ by Lemma 6.

Notice that $A_{i+1} \preceq_{\text{pre}} A_i$ for every $1 \leq i \leq 8$ and $|A_9| \geq (2/3)|A_1|$. By Lemma 5, A_1 has a period at most $(|A_1|-|A_9|)/8$. We have $|A_1|-|A_9| \leq m$ since otherwise $d_9 > d_1 > |A_1|$ implies $b_9 > m$ which is impossible. Hence, A_1 has a period at most $m/8$, and this implies $W[|A_1| + |X_9| + 1 : |A_9| + |B_9|]$ (i.e., a suffix of B_9 succeeding X_9) also has a period at most $m/8$. However, we can pick a 90-drome factor of length greater than $m/8$ inside this interval as a subword of C_1 (and hence a subword of W) by truncating a same number of letters from both side of C_1 since (*5) holds. This contradicts Lemma 11 which completes the proof of Case 2.1.

Case 2.2: $d_1 > d_2 > \cdots > d_{c_2}$.

Notice that, in this case, $d_i > d_j$ *does not* imply $b_i > b_j$. So we first apply Lemma 10 and Theorem 13 to the sequence $b_1, \ldots, b_{c_2}$ to obtain a subsequence of length $c_3 := \lceil \sqrt{c_2/2} \rceil$ such that the selected b_is are sorted in increasing or decreasing order. (Here we first pick $c_2/2$ indices such that the selected b_is are all distinct (the existence of such a set is guaranteed by Lemma 10), and then apply Theorem 13 to get a desired subsequence.) For simplicity, we write the indices of the selected b_is as $1, 2, \ldots, c_3$. That is, $W = A_kB_kC_k$ with A_k a palindrome and B_k and C_k non-empty admissible 90-dromes for $1 \leq k \leq c_3$. Moreover, $b_1, b_2, \ldots, b_{c_3}$ are increasing or decreasing. As to Case 2.1, we put $m := |B_1C_1|(= n - |A_1|)$, and put $b_k := |A_k| + |B_k| - |A_1|$ for $1 \leq k \leq c_3$.

Case 2.2.1: $d_1 > d_2 > \cdots > d_{c_3}$ and $b_1 < b_2 < \cdots < b_{c_3}$

Suppose that $c_3 \geq 13$. Note that $(3/2)^6 > 10$. By Corollary 7 and $|C_1| \leq m$, we have $|C_{13}| < (2/3)^6m < m/10$, and equivalently $b_{13} > (9/10)m$. As to Case 2.1, for

$1 \le k \le c_3$, let $X_k := W[|A_1|+1 : 2|A_k|+|B_k|-|A_1|]$. In other words, X_k is a 90-drome subword of B_k sharing the center with B_k that starts at $W[|A_1|+1]$. If $d_k \le |A_1|$, $X_k := \emptyset$. We have $|X_{13}| < m/10$ because otherwise Lemma 6 implies $|B_1| = |X_1| > (3/2)^6(1/10)m > m$, which is impossible.

Now we focus on the A_is. Recall that $A_{k+1} \preceq_{\text{pre}} A_k$ for every $1 \le k < c_3 - 1$. Then by Lemma 5, A_1 has a period at most $p := (|A_1| - |A_{13}|)/12$. We have $d_{13} \ge |A_1| - p/2$ since $d_{13} < |A_1| - p/2$ implies that we can pick a 90-drome factor of length greater than p which shares the center with B_{13} within $W[1 : |A_1|]$ and this contradicts Lemma 11. Then, we have

$$\frac{|B_{13}|}{2} = d_{13} - |A_{13}| \ge -\frac{p}{2} + (|A_1| - |A_{13}|) = -\frac{p}{2} + 12p \ge \frac{23}{2}p,$$

and hence $m \ge b_{13} = d_{13} + |B_{13}|/2 - |A_1| \ge 11p$, which implies $p \le m/11$. Since B_k is a 90-drome, $W[|A_1|+|X_{13}|+1 : |A_{13}|+|B_{13}|]$ has a period at most $m/11$.

By recalling that $b_{13} \ge (9/10)m$, we have $|X_{13}| < m/10$. Assume that $|B_1| \le m/2$. Then the center of C_1 is located left to the $|A_1| + (3/4)m$-th letter in W. This guarantees that we can pick a 90-drome factor of length greater than $m/10$ whose center is common to C_1 within $W[|A_1|+|X_{13}|+1 : |A_{13}|+|B_{13}|]$, which contradicts Lemma 11.

Now we can assume that $|B_1| > m/2$. Then the position of the center of B_1 is located right to the $|A_1| + (1/4)m$-th letter in W. However, in this case, we can pick a 90-drome factor of W sharing the center with B_1 whose length is greater than $m/10$. This contradicts Lemma 11, and thus completes the proof of Case 2.2.1.

Case 2.2.2: $d_1 > d_2 > \cdots > d_{c_3}$ and $b_1 > b_2 > \cdots > b_{c_3}$.

Suppose that $c_3 \ge 19$. By the same argument to the second paragraph of Case 2.2.1, we can show that $|A_1|$ and also $W[|A_1|+|X_{13}|+1 : |A_{13}|+|B_{13}|]$ has a period at most $m/11$ and $b_{13} \ge 0$. We also have $b_{19} \ge 0$ by a similar argument to this. We have $|B_1| > (9/10)m$, or equivalently $|C_1| < m/10$, because $|C_1| \ge m/10$ implies $|C_{13}| \ge (1/10)(3/2)^6 m > m$ by Corollary 7, which contradicts $b_{13} \ge 0$. Hence the center of B_1 is located at a position between $|A_1| + (9/20)m$ and $|A_1| + m/2$ in W.

Define X_k as to Case 2.2.1. We have $|X_{13}| < m/10$, since otherwise $|B_1| = |X_1| > (1/10)(3/2)^6 m > m$ by Lemma 6. If $|A_{13}| + |B_{13}| \ge |A_1| + (7/10)m$, then there is a 90-drome factor of length greater than $m/10$ within $W[|A_1|+|X_{13}|+1 : |A_{13}|+|B_{13}|]$ which shares the center with B_1. This contradicts Lemma 11. Hence we can assume $|A_{13}|+|B_{13}| < |A_1|+(7/10)m$ which means that $|C_{13}| > (3/10)m$. However, this implies $|C_{19}| > (3/10)(3/2)^3 m > m$ (by Corollary 7), which contradicts $b_{19} \ge 0$. This completes the proof of Case 2.2.2.

By putting $c \gg 3 \cdot 8 \cdot (2 \cdot 20^2)^2$, we can satisfy all the conditions on c_0, c_1, c_2 and c_3 in the proof, which completes the proof of Theorem 1. ◀

4 Concluding Remarks

In this work, we did not make any effort to optimize the constant in Theorem 1. Currently, the upper bound is dominated by Case 2, which is roughly (the value of c in the proof) $\times$ (the size of $\mathcal{F}$ in Lemma 12) $\times$ ($2 \cdot$ (representing the choice of $A[1]$ or $A[-1]$ at the beginning of Case 2)). An inspection of the proof of Lemma 12 (in [14, Lemma 5.7]) shows that $O(1)$ in Lemma 12 can be replaced by 34, and hence our upper bound is $\sim 10^9$. Currently, we do not know a polyomino having more than three non-equivalent p4-tilings (see Figure 4 for the one having three p4-tilings). Closing the gap to determine the true value is an interesting future work.

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