A New Approach to Multi-Party Peer-to-Peer Communication Complexity

Adi Rosén[1]
CNRS and Université Paris Diderot
adiro@irif.fr

Florent Urrutia[2]
Université Paris Diderot
urrutia@irif.fr

Abstract

We introduce new models and new information theoretic measures for the study of communication complexity in the natural peer-to-peer, multi-party, number-in-hand setting. We prove a number of properties of our new models and measures, and then, in order to exemplify their effectiveness, we use them to prove two lower bounds. The more elaborate one is a tight lower bound of $\Omega(kn)$ on the multi-party peer-to-peer randomized communication complexity of the k-player, n-bit function Disjointness, Disj_k^n. The other one is a tight lower bound of $\Omega(kn)$ on the multi-party peer-to-peer randomized communication complexity of the k-player, n-bit bitwise parity function, Par_k^n. Both lower bounds hold when $n = \Omega(k)$. The lower bound for Disj_k^n improves over the lower bound that can be inferred from the result of Braverman et al. (FOCS 2013), which was proved in the coordinator model and can yield a lower bound of $\Omega(kn/\log k)$ in the peer-to-peer model.

To the best of our knowledge, our lower bounds are the first tight (non-trivial) lower bounds on communication complexity in the natural *peer-to-peer* multi-party setting.

In addition to the above results for communication complexity, we also prove, using the same tools, an $\Omega(n)$ lower bound on the number of random bits necessary for the (information theoretic) private computation of the function Disj_k^n.

2012 ACM Subject Classification Theory of computation → Communication complexity, Mathematics of computing → Information theory, Theory of computation → Cryptographic protocols

Keywords and phrases communication complexity, multi-party communication complexity, peer-to-peer communication complexity, information complexity, private computation

Digital Object Identifier 10.4230/LIPIcs.ITCS.2019.64

Related Version A full version of the paper is available at `http://arxiv.org/abs/1901.01178`.

Acknowledgements We thank Iordanis Kerenidis and Rotem Oshman for very useful discussions.

1 Introduction

Communication complexity, first introduced by Yao [44], has become a major topic of research in Theoretical Computer Science, both for its own sake, and as a tool which has yielded important results (mostly lower bounds) in various theoretical computer science fields such as circuit complexity, streaming algorithms, or data structures (e.g., [34, 36, 24, 39, 23]). Communication complexity is a measure for the amount of communication needed in order to

[1] Research supported in part by ANR project RDAM.
[2] Research supported in part by ERC QCC and by ANR project RDAM.

solve a problem whose input is distributed among several players. The two-party case, where two players, usually called Alice and Bob, cooperate in order to compute a function of their respective inputs, has been widely studied with many important results; yet major questions in this area are still open today (e.g., the log-rank conjecture, see [34]). The multi-party case, where $k \geq 3$ players cooperate in order to compute a function of their inputs, is much less understood.

A number of variants have been proposed in the literature to extend the two-party setting into the multi-party one. In this paper we consider the more natural *number-in-hand* (NIH) setting, where each player has its own input, as opposed to the so-called *number-on-forehead* (NOF) setting, where each player knows all pieces of the input except one, its own. Moreover, also the communication structure between the players in the multi-party setting was considered in the literature under a number of variants. For example, in the *blackboard* (or *broadcast*) model the communication between the players is achieved by each player writing, in turn, a message on the board, to be read by all other players. In the *coordinator* model, introduced in [20], there is an additional entity, the coordinator, and all players communicate back and forth only with the coordinator. The most natural setting is, however, the *peer-to-peer message-passing* model, where each pair of players is connected by a communication link, and each player can send a separate message to any other player. This latter setting has been studied, in the context of communication complexity, even less than the other multi-party settings, probably due to the difficulty in tracking the distributed communication patterns that occur during a run of a protocol in that setting. This setting is, however, not only the most natural one, and the one that occurs the most in real systems, but is the setting studied widely in the distributed algorithms and distributed computation communities, for complexity measures which are usually other than communication complexity.

In the present paper we attempt to fill this gap in the study of peer-to-peer communication complexity, and, further, to create a more solid bridge between the research field of communication complexity and the research field of distributed computation. We propose a computation model, together with an information theoretic complexity measure, for the analysis of the communication complexity of protocols in the asynchronous multi-party peer-to-peer (number-in-hand) setting. We argue that our model is, on the one hand, only a slight restriction over the asynchronous model usually used in the distributed computation literature, and, on the other hand, stronger than the models that have been previously suggested in order to study communication complexity in the peer-to-peer setting common in the distributed computation literature (e.g., [20, 42]). Furthermore, our model lends itself to the analysis of communication complexity, most notably using information theoretic tools.

Indeed, after defining our model and our information theoretic measure, that we call *Multi-party Information Cost* (MIC), we prove a number of properties of that measure, and then prove a number of fundamental properties of protocols in our model. We then exemplify the effectiveness of our model and information theoretic measure by proving two tight lower bounds. The more elaborate one is a tight lower bound of $\Omega(kn)$, when $n = \Omega(k)$, on the peer-to-peer randomized communication complexity of the function `set-disjointness` (Disj_k^n). This function is a basic, important function, which has been the subject of a large number of studies in communication complexity, and is often seen as a test for our ability to give lower bounds in a given model (cf. [16]). We note that the communication complexity of Disjointness in the two-party case is well understood [29, 38, 3, 7, 9]. From a quantitative point of view, our result for peer-to-peer multi-party Disjointness improves by a $\log k$ factor the lower bound that could be deduced for the peer-to-peer model from the lower bound on the communication complexity of Disjointness in the coordinator model [8]. The second lower bound that we prove is a tight lower bound of $\Omega(kn)$, when $n = \Omega(k)$, on the peer-to-peer randomized communication complexity of the bitwise parity function Par_k^n. Both our lower bounds are obtained by giving a lower bound on the MIC of the function at hand, which yields the lower bound on the communication complexity of that function. We believe that our lower bounds are the first tight (non-trivial) lower bound on communication complexity in a peer-to-peer multi-party setting.[3]

It is important to note that, to the best of our knowledge, there is no known method to obtain tight lower bounds on multi-party communication complexity in a peer-to-peer setting via lower bounds in other known multi-party settings. Lower bounds obtained in the coordinator model can be transferred to the peer-to-peer model at the cost of a $\log k$ factor, where k is the number of players, because any peer-to-peer protocol can be simulated in the coordinator model by having the players attach to every message the identity of the destination of that message [37, 21]. The loss of this factor in the lower bounds is unavoidable when the communication protocols can exploit a flexible communication pattern, since there are examples of functions where this factor in the communication complexity is necessary, while others, e.g., the parity function of single-bit inputs, have the same communication complexity in the coordinator and peer-to-peer settings (see a more detailed discussion on this point in Section 2.2). Therefore, one cannot prove tight lower bounds in the peer-to-peer setting by proving corresponding results in the coordinator model. Note that flexible communication configurations arise naturally for mobile communicating devices, for example, when these devices exchange information with the nearby devices. Constructions based on the pointer jumping problem also seem to be harder in the coordinator model, as solving the problem usually requires exchanging information in a specific order determined by the inputs of the players. It is thus important to develop lower bound techniques which apply directly in the peer-to-peer model, as we do in the present paper. Information theoretic tools seem, as we show, most suitable for this task.

Information theoretic complexity measures. As indicated above, our work makes use of information theoretic tools. Based on information theory, developed by Shannon [40], *Information Complexity* (IC), originally defined in [2, 14], is a powerful tool for the study of two-party communication protocols. Information complexity is a measure of how much *information*, about each other's input, the players must learn during the course of the protocol, if that protocol must compute the function correctly. Since IC can be shown to provide a lower bound on the communication complexity, this measure has proven to be a strong and useful tool for obtaining lower bounds on two-party communication complexity in a sequence of papers (e.g., [3, 4, 11, 7]). However, information complexity cannot be extended in a straightforward manner to the multi-party setting. This is because with three players or more, any function can be computed privately (cf. [5, 19]), i.e., in a way such that the players learn nothing but the value of the function to compute. This implies that the information complexity of any function is too low to provide a meaningful lower bound on the communication complexity in the natural peer-to-peer multi-party setting. Therefore, before the present paper, information complexity and its variants have been used to obtain lower bounds on multi-party communication complexity only in settings which do not allow for private protocols (and most notably not in the natural peer-to-peer setting), with the single

[3] Lower bounds in a seemingly peer-to-peer setting were given in [42]. However, in the model of that paper, the communication pattern is determined by an external view of the transcript, which makes the model equivalent to the coordinator model.

exception of [30]. For example, a number of lower bounds have been obtained via information complexity for a promise version of set-disjointness in the broadcast model [3, 13, 26] (also cf. [28]), and external information complexity was used in [10] for a lower bound on the general disjointness function, also in the broadcast model. In the coordinator model, lower bounds on the communication complexity of set-disjointness were given via variants of information complexity [8]. The latter result was extended in [15] to the function *Tribes*. A notion of external information cost in the coordinator model was introduced in [27] to study maximum matching in a distributed setting. We note that the study of communication complexity in number-in-hand multi-party settings via techniques other than those based on information theory is limited to very few papers. One such example is the technique of *symmetrization* that was introduced for the coordinator model in [37], and was shown to be useful to study functions such as the bitwise AND. That technique was further developed along with other reduction techniques in [41, 42, 43]. Another example is the notion of strong fooling sets, introduced in [12] to study deterministic communication complexity of discreet protocols, also defined in [12].

Private computation. It is well known that in the multi-party number-in-hand peer-to-peer setting, unlike in the two-party case, *any* function can be privately computed [5, 19]. The model that we define in the present paper does allow for (information theoretic) private computation of any function [5, 19, 1]. The minimum amount of private randomness needed in order to compute privately a given function is often referred to in this context as the *randomness complexity* of that function. Randomness complexity (in private computation) is of interest because true randomness is considered a costly resource, and since randomness complexity in private computation has been shown to be related to other complexity measures, such as the circuit size of the function or its sensitivity. For example, it has been shown [35] that a boolean function f has a linear size circuit if and only if f has constant randomness complexity. A small number of works [6, 33, 25, 30] prove lower bounds on the randomness complexity of the parity function. The parity and other modulo-sum functions are, to the best of our knowledge, the only functions for which randomness complexity lower bounds are known. Using the information theoretic results that we obtain in the present paper for the set-disjointness function, we are able to give a lower bound of $\Omega(n)$ on the randomness complexity of Disj_k^n. The significance of this result lies in that it is the first such lower bound that grows with the size of the input (which is kn), while the output remains a single bit, contrary to the sum function (see [6]) or the bitwise parity function (see [30]).

1.1 Our techniques and contributions

Our contribution in the present paper is twofold.

First, on the conceptual, modeling and definitions side we lay the foundations for proving lower bounds on (randomized) communication complexity in the natural peer-to-peer multi-party setting. Specifically, we propose a model that, on the one hand, is a very natural peer-to-peer model, and very close to the model used in the distributed computation literature, and, at the same time, does have properties that allow one to analyze protocols in terms of their information complexity and communication complexity. While at first sight the elaboration of such model does not seem to be a difficult task, many technical, as well as fundamental, issues render this task non-trivial. For example, one would like to define a notion of "transcript" that would guarantee both a relation between the length of the transcript and the communication complexity, and at the same time will contain *all* the information that the players get and use while running the protocol. The difficulty in elaborating such model may be the reason for which, prior to the present paper, hardly any work studied communication complexity directly in a *peer-to-peer*, multi-party setting (cf. [21]), leaving the field with only the results that can be inferred from other models, hence suffering the appropriate loss in the obtained bounds. We propose our model (see Section 2.1) and prove a number of fundamental properties that allow one to analyze protocols in that model (see Section 3.2), as well as prove the accurate relationship between the entropy of the transcript and the communication complexity of the protocol (Proposition 2.4).

We then define our new information theoretic measure, that we call "Multi-party Information Cost" (MIC), intended to be applied to peer-to-peer multi-party protocols, and prove that it provides, for any (possibly randomized) protocol, a lower bound on the communication complexity of that protocol (Lemma 3.4). We further show that MIC has certain properties such as a certain direct-sum property (Theorem 3.5). We thus introduce a framework as well as tools for proving lower bounds on communication complexity in a peer-to-peer multi-party setting.

Second, we exemplify the effectiveness of our conceptual contributions by proving, using the new tools that we define, two tight lower bounds on the randomized communication complexity of certain functions in the peer-to-peer multi-party setting. Both these lower bounds are proved by giving a lower bound on the Multi-party Information Complexity of the function at hand. The more elaborate lower bound is a tight lower bound of $\Omega(nk)$ on the randomized communication complexity of the function Disj_k^n (under the condition that $n = \Omega(k)$). The function Disjointness is a well studied function in communication complexity and is often seen as a test-case of one's ability to give lower bounds in a given model (cf. [16]). While the general structure of the proof of this lower bound does have similarities to the proof of a lower bound for Disjointness in the coordinator model [8],[4] we do, even in the parts that bear similarities, have to overcome a number of technical difficulties that require new ideas and new proofs. For example, the very basic *rectangularity* property of communication protocols is, in the multi-party (peer-to-peer) setting, very sensitive to the details of the definition of the model and the notion of a transcript. We therefore need first to give a proof of this property in the peer-to-peer model (Lemma 3.6 and Lemma 3.7). We then use a distribution of the input which is a modification over the distributions used in [8, 15] (see Section 5). Our proof proceeds, as in [8], by proving a lower bound for the function AND, on a certain information theoretic measure that, in our proof, is called SMIC (for Switched Multi-party Information Cost), and then, by using a direct-sum-like lemma, to infer a lower bound on SMIC for Disjointness (we note that SMIC is an adaptation to the peer-to-peer model of a similar measure used in [8]). However, the lack of a "coordinator" in a peer-to-peer setting necessitates a definition of a more elaborate reduction protocol, and a more complicated proof for the direct-sum argument, inspired by classic secret-sharing primitives. See Lemma 6.1 for our construction and proof. We then show that SMIC provides a lower bound on MIC, which yields our lower bound on the communication complexity of Disjointness.

We further give a tight lower bound of $\Omega(nk)$ on the randomized communication complexity of the function Par_k^n (bitwise parity) in the peer-to-peer multi-party setting (under the condition that $n = \Omega(k)$). This proof proceeds by first giving a lower bound on MIC for the parity function Par_k^1, and then using a direct-sum property of MIC to get a lower bound on MIC for Par_k^n. The latter yields the lower bound of $\Omega(nk)$ on the communication complexity of Par_k^n.

[4] The lower bound in [8] would yield an $\Omega(\frac{1}{\log k} \cdot nk)$ lower bound in the peer-to-peer setting.

To the best of our knowledge, our lower bounds are the first tight (non-trivial) lower bound on communication complexity in a peer-to-peer multi-party setting.

In addition to our results on communication complexity, we analyze the number of random bits necessary for private computations [5, 19], making use of the model, tools and techniques we develop in the present paper. It has been shown [30] that the *public information cost* (defined also in [30]) can be used to derive a lower bound on the randomness complexity of private computations. In the present paper we give a lower bound on the public information cost of any synchronous protocol computing the Disjointness function by relating it to its Switched Multi-party Information Cost, which yields the lower bound on the randomness complexity of Disjointness.

Organization. Due to space limitation *all* proofs are deferred to the full version of the paper. Section 2 introduces our model. In Section 3 we define our new information theoretic measure, MIC, give some of its properties, and give a number of fundamental properties of protocols in our model. In Section 4 we give the lower bound for the bitwise parity function. In Section 5 we prove a lower bound on the switched multi-party information cost of AND_k, and in Section 6, we prove, using the results of Section 5, the lower bound on the communication complexity of Disj_k^n. In Section 7 we apply our information theoretic lower bounds in order to give a lower bound on the number of random bits necessary for the private computation of Disj_k^n. Last, in Section 8 we discuss some open questions.

2 Multi-party communication protocols

We start with our model, and, to this end, give a number of notations.

Notations. We denote by k the number of players. We often use n to denote the size (in bits) of the input to each player. Calligraphic letters will be used to denote sets. Upper case letters will be used to denote random variables, and given two random variables A and B, we will denote by AB the joint random variable (A, B). Given a string (of bits) s, $|s|$ denotes the length of s. Using parentheses we denote an ordered set (family) of items, e.g., (Y_i). Given a family (Y_i), Y_{-i} denotes the sub-family which is the family (Y_i) *without* the element Y_i. The letter X will usually denote the input to the players, and we thus use the shortened notation X for (X_i), i.e., the input to all players. A protocol will usually be denoted by π.

We now define a natural communication model which is a slight restriction of the general asynchronous peer-to-peer model. The restriction of our model compared to the general asynchronous peer-to-peer model is that for a given player at a given time, the set of players from which that player waits for a message before sending any message of its own is determined by that player's own local view, i.e., from that player's input and the messages it has read so far, as well as its private randomness, and the public randomness. This allows us to define information theoretic tools that pertain to the transcripts of the protocols, and at the same time to use these tools as lower bounds for communication complexity. This restriction however does not exclude the existence of private protocols, as other special cases of the general asynchronous model do. We observe that practically all multi-party protocols in the literature are implicitly defined in our model, and that without such restriction, one bit of communication can bring $\log k$ bits of information, because not only the content of the message, but also the identity of the sender may reveal information. To exemplify why the general asynchronous model is problematic consider the following simple example (that we borrow from our work in [30]).

▶ **Example 2.1.** There are 4 players A, B and C, D. The protocol allows A to transmit to B its input bit x. But all messages sent in the protocol are the bit 0, and the protocol generates only a single transcript over all possible inputs. The protocol works as follows:

A: If $x = 0$ send 0 to C; after receiving 0 from C, send 0 to D.
If $x = 1$ send 0 to D; after receiving 0 from D, send 0 to C

B: After receiving 0 from a player, send 0 back to that player.

C,D: After receiving 0 from A send 0 to B. After receiving 0 from B send 0 to A.

It is easy to see that B learns the value of x from the order of the messages it gets.

In what follows we formally define our model, compare it to the general one and to other restricted ones, and explain the usefulness and logic of our specific model.

2.1 Definition of the model

We work in a *multi-party, number-in-hand, peer-to-peer* setting. Each player $1 \leq i \leq k$ has unbounded local computation power and, in addition to its input X_i, has access to a source of private randomness R_i. We will use the notation R for (R_i), i.e., the private randomness of all players. A source of public randomness R^p is also available to all players. We will call a protocol with no private randomness a public-coins protocol. The system consists of k players and a family of k functions $f = (f_i)_{i \in [\![1,k]\!]}$, with $\forall\ i \in [\![1,k]\!]$, $f_i : \Pi_{\ell=1}^{k} \mathcal{X}_\ell \to \mathcal{Y}_i$, where $\mathcal{X}_\ell$ denotes the set of possible inputs of player ℓ, and $\mathcal{Y}_i$ denotes the set of possible outputs of player i. The players are given some input $x = (x_i) \in \Pi_{i=1}^{k} \mathcal{X}_i$, and for every i, player i has to compute $f_i(x)$.

We define the communication model as follows, which is the asynchronous setting, with some restrictions. To make the discussion simpler we assume a global time which is *unknown* to the players. Every pair of players is connected by a bidirectional communication link that allows them to send messages to each other. There is no bound on the delivery time of a message, but every message is delivered in finite time, and the communication link maintains FIFO order in each of the two directions. Given a specific time we define the *view* of player i as the input of this player, X_i, its private randomness, R_i, the public randomness, R^p, and the messages read so far by player i. After the protocol has started, each player runs the protocol in *local rounds*. In each round, player i sends messages to some subset of the other players. The identity of these players, as well as the content of these messages, depend on the current view of player i. The player also decides whether it should stop, and output (or "return") the result of the function f_i. Then (if player i did not stop and return the output), the player waits for messages from a certain subset of the other players, this subset being also determined by the current view of the player. Then the (local) round of player i terminates.[5] To make it possible for the player to identify the arrival of the *complete* message that it waits for, we require that each message sent by a player in the protocol is self-delimiting.

Denote by $\mathcal{D}_i^\ell$ the set of possible views of player i at the end of local round ℓ, $\ell \geq 0$, where the beginning of the protocol is considered round 0.

Formally, a **protocol** π is defined by a set of local programs, one for each player i, where the local program of player i is defined by a sequence of functions, parametrized by the index of the *local* round ℓ, $\ell \geq 1$:

[5] The fact that the receiving of the incoming messages comes as the last step of the (local) round comes only to emphasize that the sending of the messages and the output are a function of only the messages received in previous (local) rounds.

- $S_i^{\ell,s} : \mathcal{D}_i^{\ell-1} \to 2^{\{1,\dots,k\}\setminus\{i\}}$, defining the set of players to which player i *sends* the messages.
- $m_{i,j}^{\ell} : \mathcal{D}_i^{\ell-1} \to \{0,1\}^*$, such that for any $D_i^{\ell-1} \in \mathcal{D}_i^{\ell-1}$, if $j \in S_i^{\ell,s}(D_i^{\ell-1})$, then $m_{i,j}^{\ell}(D_i^{\ell-1})$ is the content of the message player i sends to player j. Each such message is self-delimiting.
- $O_i^{\ell} : \mathcal{D}_i^{\ell-1} \to \{0,1\}^* \cup \{\perp\}$, defining whether or not the local program of player i stops and the player returns its output, and what is that output. If the value is $\perp$ then no output occurs. If the value is $y \in \{0,1\}^*$, then the local program stops and the player returns the value y.
- $S_i^{\ell,r} : \mathcal{D}_i^{\ell-1} \to 2^{\{1,\dots,k\}\setminus\{i\}}$, defining the set of players from which player i waits to *receive* a message.

To define the **transcript** of a protocol we proceed as follows. We first define $k(k-1)$ basic transcripts $\Pi_{i,j}^r$, denoting the transcript of the messages read by player i from its link from player j, and another $k(k-1)$ basic transcripts $\Pi_{i,j}^s$, denoting the transcript of the messages sent by player i on its link to player j.
We then define the transcript of player i, Π_i, as the $2(k-1)$-tuple of the $2(k-1)$ basic transcripts $\Pi_{i,j}^r, \Pi_{i,j}^s$, $j \in [\![1,k]\!] \setminus \{i\}$. The transcript of the whole protocol Π is defined as the k-tuple of the k player transcripts Π_i, $i \in [\![1,k]\!]$. We denote by $\Pi_i(x,r)$ the transcript of player i when protocol π is run on input x and on randomness (public and private of all players) r. By $\Pi_i^{\ell}(x,r)$ we denote $\Pi_i(x,r)$ modified such that all the messages that player i sends in local rounds $\ell' > \ell$, and all the messages that player i reads in local rounds $\ell' > \ell$ are eliminated from the transcript. Observe that while $\Pi_{i,j}^r$ is always a prefix of $\Pi_{j,i}^s$, the definition of a protocol does not imply that they are equal. Further observe that each bit sent in π appears in Π at most twice.

We note that while seemingly the model that we introduce here is the same as the one used in [30], there are important differences between the models, and that these differences are crucial for the properties that we prove in the present paper to hold. See Section 2.2 for a comparison.

For a k-party protocol π we denote the set of possible inputs as $\mathcal{X}$, and denote the projection of this set on the i'th coordinate (i.e., the set of possible inputs for player i) by $\mathcal{X}_i$. Thus $\mathcal{X} \subseteq \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$. The set of possible transcripts for a protocol is denoted $\mathcal{T}$, and the projection of this set on the i'th coordinate (i.e., the set of possible transcripts of player i) is denoted $\mathcal{T}_i$. Observe that $\mathcal{T} \subseteq \mathcal{T}_1 \times \cdots \times \mathcal{T}_k$.

Furthermore, in the course of the proofs, we sometimes consider a protocol that does *not* have access to public randomness (but may have private randomness). We call such protocol a *private-coins protocol.*

We now formally define the notion of a protocol computing a given function with certain bounded error. We will give most of the following definitions for the case where all functions f_i are the same function, that we denote by f. The definitions in the case of family of functions are similar.

► **Definition 2.2.** For a given $0 \le \epsilon < 1$, a protocol π *ϵ-computes* a function f if for all $x \in \Pi_{i=1}^k \mathcal{X}_i$:

- For all possible assignments for the random sources R_i, $1 \le i \le k$, and R^p, every player eventually stops and returns an output.
- With probability at least $1-\epsilon$ (over all random sources) the following event occurs: each player i outputs the value $f(x)$, i.e., the correct value of the function.

We also consider the notion of *external computation.*

▶ **Definition 2.3.** For a given $0 \leq \epsilon < 1$, a protocol π *externally* ϵ-computes f if there exists a deterministic function θ taking as input the possible transcripts of π and verifying $\forall\, x \in \mathcal{X},\ \Pr[\theta(\Pi(x)) = f(x)] \geq 1 - \epsilon$.

The **communication complexity** of a protocol is defined as the worst case, over the possible inputs and the possible randomness, of the number of bits sent by all players. For a protocol π we denote its communication complexity by $\mathsf{CC}(\pi)$. For a given function f and a given $0 \leq \epsilon < 1$, we denote by $\mathsf{CC}^\epsilon(f)$ the ϵ*-error communication complexity of* f, i.e., $\mathsf{CC}^\epsilon(f) = \inf_{\pi\ \epsilon\text{-computing}\ f} \mathsf{CC}(\pi)$.

Finally, we give a proposition that relates the communication complexity of a k-party protocol π to the entropy of the transcripts of the protocol π.

▶ **Proposition 2.4.** *Let the input to a k-party protocol π be distributed according to an arbitrary distribution. Then, $\sum_{i=1}^{k} H(\Pi_i) \leq 4 \cdot \mathsf{CC}(\pi) + 4k^2$, where the entropy is according to the input distribution and the randomization of protocol π.*

2.2 Comparison to other models

The somewhat restricted model (compared to the general asynchronous model) that we work with allows us to use information theoretic tools for the study of protocols in this model, and in particular to give lower bounds on the multi-party communication complexity. Notice that the general asynchronous model is problematic in this respect since one bit of communication can bring $\log k$ bits of information, because not only the content of the message, but also the identity of the sender may reveal information. Thus, information cannot be used as a lower bound on communication. In our case, the sets $S_i^{l,r}$ and $S_i^{l,s}$ are determined by the current view of the player, Π contains only the content of the messages, and thus the desirable relation between the communication and the information is maintained. On the other hand, our restriction is natural, does not seem to be very restrictive (practically all protocols in the literature adhere to our model), and does not exclude the existence of private protocols. To exemplify why the general asynchronous model is problematic see Example 2.1.

While the model that we introduce in the preset paper bears some similarities to the model used in [30], there are a number of important differences between them. First, the definition of the transcript is different, resulting in a different relation between the entropy of the transcript and the communication complexity. More important is the natural property of the model in the present paper that the local program of a protocol in a given node ends its execution when it locally gives its output. It turns out that the very basic rectangularity property of protocols, used in many papers, holds in this case (and when the transcript is defied as we define in the present paper), while if the local protocol may continue to operate after output, there are examples where this property does not hold. Thus, we view the introduction of the present model also as a contribution towards identifying the necessary features of a peer-to-peer model so that basic and useful properties of protocols hold in the peer-to-peer setting.

There has been a long series of works about multi-party communication protocols in different variants of models, for example [20, 13, 26, 28, 37, 17, 18] (see [21] for a comparison of a few of these models). In the *coordinator model* (cf. [20, 37, 8]), an additional player (the coordinator) with no input can communicate privately with each player, and the players can only communicate with the coordinator. We first note that the coordinator model does not yield exact bounds for the multi-party communication complexity in the peer-to-peer setting (neither in our model nor in the most general one). Namely, any protocol in the peer-to-peer model can be transformed into a protocol in the coordinator model with an

$O(\log k)$ multiplicative factor in the communication complexity, by sending each message to the coordinator with an $O(\log k)$-bit label indicating its destination. This factor is sometimes necessary, e.g., for the `permutation` functional defined as follows: Given a permutation $\sigma : [\![1, k]\!] \to [\![1, k]\!]$, each player i has as input a bit b_i and $\sigma^{-1}(\sigma(i)-1)$ and $\sigma^{-1}(\sigma(i)+1)$ (i.e., each player has as input the indexes of the players before and after itself in the permutation).[6] For player i the function f_i is defined as $f_i = b_{\sigma^{-1}(\sigma(i)+1)}$ (i.e., the value of the input bit of the next player in the permutation σ). Clearly in our model the communication complexity of this function is k (each player sends its input bit to the correct player), and the natural protocol is valid in our model. On the other hand, in the coordinator model $\Omega(k \log k)$ bits of communication are necessary. But this multiplicative factor between the complexities in the two models is not always necessary: the communication complexity of the parity function Par is $\Theta(k)$ both in the peer-to-peer model and in the coordinator model.

Moreover, when studying private protocols in the multi-party setting, the coordinator model does not offer any insight. In the coordinator model, described in [20] and used for instance in [8], if one does not impose any privacy requirement with respect to the coordinator, it is trivial to have a private protocol by all players sending their input to the coordinator, and the coordinator returning the results to the players. If there is a privacy requirement with respect to the coordinator, then if there is a random source shared by all the players (but not the coordinator), privacy is always possible using the protocol of [22]. If no such source exists, privacy is impossible in general. This follows from the results of Braverman et al. [8] who show a non-zero lower bound on the total internal information complexity of all parties (including the coordinator) for the function *Disjointness* in that model. Our model, on the other hand, does allow for the private computation of any function [5, 19, 1].

It is worthwhile to contrast our model, and the communication complexity measure that we are concerned with, with work in the so-call congested-clique model that has gained increasing attention in the distributed computation literature (cf. [31, 32]). While both models are based on a communication network in the form of a complete graph (i.e., every player can send messages to any other player, and these messages can be different) there are two significant differences between them. Most of the works in the congested clique model deal with graph-theoretic problems and the input to each player is related to the adjacency list of a node (identified with that player) in the input graph, while in our model the input is not associated in any way with the communication graph. More importantly, the congested clique model is a *synchronous* model while ours is an asynchronous one. This brings about a major difference between the complexity measures studied in each of the models. Work in the congested clique model is concerned with giving bounds on the number of rounds necessary to fulfill a certain task under the condition that in each round each player can send to any other player a limited number of bits (usually $O(\log k)$ bits). The measure of communication complexity, that is of interest to us in the present paper, deals with the total number of communication bits necessary to fulfill a certain task in an asynchronous setting without any notion of global rounds.[7]

[6] All additions are modulo k. This is a promise problem.

[7] Any function can be computed in the congested clique model with $O(k)$ communication complexity (at a cost of having many rounds) by each player, having input x, sending a single bit to player 1 only at round number x. On the other hand, in the asynchronous model any function can be computed in a single "round" (at a cost of high communication complexity) by each player sending its whole input to player 1.

3 Tools for the study of multi-party communication protocols

In this section we consider two important tools for the study of peer-to-peer multi-party communication protocols. First, we define and introduce an information theoretic measure that we call *Multi-party Information Cost* (MIC); we later use it to prove our lower bounds. Then, we prove, in the peer-to-peer multi-party model that we define, the so-called rectangularity property of communication protocols, that we also use in our proofs.

3.1 Multi-party Information Cost

We now introduce an information theoretic measure for multi-party peer-to-peer protocols that we later show to be useful for proving lower bounds on the communication complexity of multi-party peer-to-peer protocols. We note that a somewhat similar measure was proposed in [8] for the coordinator model, but, to the best of our knowledge, never found an application as a tool in a proof of a lower bound.

▶ **Definition 3.1.** For any k-player protocol π and any input distribution μ, we define the *multi-party information cost* of π:

$$\mathsf{MIC}_\mu(\pi) = \sum_{i=1}^{k} \left(I(X_{-i}; \Pi_i \mid X_i R_i) + I(X_i; \Pi_i \mid X_{-i} R_{-i}) \right) \ .$$

Observe that the second part of each of the k summands can be interpreted as the information that player i "leaks" to the other players on its input. While the "usual" intuitive interpretation of two-party IC is "what Alice learns on Bob's input plus what Bob learns on Alice's input", one can also interpret two-party IC as "what Alice learns on Bob's input plus what Alice leaks on her input". Thus, MIC can be interpreted as summing over all players i of "what player i learns on the other players' inputs, plus what player i leaks on its input." Indeed, the expression defining MIC is equal to the sum, over all players i, of the two-party IC for the two-party protocol that results from collapsing all players, except i, into one virtual player. Thus, for number of players $k = 2$, $\mathsf{MIC} = 2 \cdot \mathsf{IC}$. We note that defining our measure without the private randomness in the condition of the mutual information expressions would yield the exact same measure (as is the case for 2-party IC); we prefer however to define MIC with the randomness in the conditions, as we believe that it allows one to give shorter, but still clear and accurate, proofs.

On the other hand observe that the second of the two mutual information expressions has X_{-i} in the condition, contrary to a seemingly similar measure used in [8] (Definition 3 in [8]). Our measure is thus "internal" in nature, while the one of [8] has an "external" component. The fact that MIC is "internal" allows us to give lower bounds on MIC, and thus to use it for lower bounds on the communication complexity, contrary to the measure of [8].

Further observe that the summation, over all players, of each one of the two mutual information expressions alone would not yield a measure useful for proving lower bounds on the communication complexity of functions. The first mutual information expression would yield a measure for functions that would never be higher than the entropy of the function at hand, due to the existence of private protocols for all functions [5, 19]. For the second mutual information expression there are functions for which that measure would be far too low compared to the communication complexity: e.g., the function $f = x_1$, $x \in \{0,1\}^n$ (i.e., the value of the function is the input of player 1); in that case the measure would equal only n, while the communication complexity of that function is $\Omega(kn)$.

We now define the multi-party information complexity of a function.

▶ **Definition 3.2.** For any function f, any input distribution μ, and any $0 \le \epsilon \le 1$, we define the quantity

$$\mathsf{MIC}^{\epsilon}_{\mu}(f) = \inf_{\pi\ \epsilon\text{-computing } f} \mathsf{MIC}_{\mu}(\pi) \ .$$

▶ **Definition 3.3.** For any f, and any $0 \le \epsilon \le 1$, we define the quantity

$$\mathsf{MIC}^{\epsilon}(f) = \inf_{\pi\ \epsilon\text{-computing } f} \sup_{\mu} \ \mathsf{MIC}_{\mu}(\pi) \ .$$

We now claim that the multi-party information cost and the communication complexity of a protocol are related, as formalized by the following lemma.

▶ **Lemma 3.4.** *For any k-player protocol π, and for any input distribution μ,*

$$\mathsf{CC}(\pi) \ge \frac{1}{8}\mathsf{MIC}_{\mu}(\pi) - k^2 \ .$$

We now show that the multi-party information cost satisfies a direct sum property for product distributions. In what follows, the notation $f^{\otimes n}$ denotes the task of computing n instances of f, where the requirement from an ϵ-computing protocol is that each instance is computed correctly with probability at least $1-\epsilon$ (as opposed to the stronger requirement that the whole vector of instances is computed correctly with probability at least $1-\epsilon$).

▶ **Theorem 3.5.** *For any protocol π (externally) ϵ-computing a function $f^{\otimes n}$, there exists a protocol π' (externally) ϵ-computing f such that, for any product distribution μ for the input, it holds that*

$$\mathsf{MIC}_{\mu^n}(\pi) \ge n \cdot \mathsf{MIC}_{\mu}(\pi') \ .$$

3.2 The rectangularity property

The *rectangularity property* (or *Markov property*) is one of the key properties that follow from the structure and definition of (some) protocols. For randomized protocols it was introduced in the two-party setting and in the multi-party blackboard model in [3], and in the coordinator model in [8]. We prove a similar rectangularity property in the peer-to-peer model that we consider in the present paper.

We note that the proof of this property in the peer-to-peer model makes explicit use of the specific properties of the model we defined: the proof that follows explicitly uses the definition of the transcript on an edge by edge basis as in our model, as well as the fact that a player returns and stops as one operation. One can build examples where if any of these two properties does not hold, then the rectangularity property of protools does not hold. Thus we view the following proof of rectangularity in our model also as an identification of model properties needed for the useful rectangularity property of multiparty peer-to-peer protocols to hold.

To define this property, for any transcript $\overline{\tau} \in \mathcal{T}_i$, let $\mathcal{A}_i(\overline{\tau}) = \{(x,r) \mid \Pi_i(x,r) = \overline{\tau}\}$ (i.e., the set of input, randomness pairs that lead to transcript $\overline{\tau}$), and define the projection of $\mathcal{A}_i(\overline{\tau})$ on coordinate i as $\mathcal{I}_i(\overline{\tau}) = \{(x',r'), \exists\ (x,r) \in \mathcal{A}_i(\overline{\tau}), x' = x_i \ \&\ r' = r_i\}$, and the projection of $\mathcal{A}_i(\overline{\tau})$ on the complement of coordinate i as $\mathcal{J}_i(\overline{\tau}) = \{(x',r'), \exists\ (x,r) \in \mathcal{A}_i(\overline{\tau}), x' = x_{-i} \ \&\ r' = r_{-i}\}$. Similarly, for any transcript $\tau \in \mathcal{T}$, let $\mathcal{B}(\tau) = \{(x,r) \mid \Pi(x,r) = \tau)\}$, and for any player i, let $\mathcal{H}_i(\tau) = \{(x',r'), \exists\ (x,r) \in \mathcal{B}(\tau), x' = x_{-i} \ \&\ r' = r_{-i}\}$.

We start by proving a combinatorial property of transcripts of communication protocols, which intuitively follows from the fact that each player has access to only its own input and private randomness. The proof of this property is technically more involved compared to the analogous property in other settings, since the structure of protocols and the manifestation of the transcripts in the peer-to-peer setting are more flexible than in the other settings.

▶ **Lemma 3.6.** *Let π be a k-player private-coins protocol with inputs from $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$. Let $\mathcal{T}$ denote the set of possible transcripts of π, and for $i \in [\![1,k]\!]$ let $\mathcal{T}_i$ denote the set of possible transcript observed by player i, so that $\mathcal{T} \subseteq \mathcal{T}_1 \times \cdots \times \mathcal{T}_k$. Then, $\forall\, i \in [\![1,k]\!]$:*

- $\forall\, \overline{\tau} \in \mathcal{T}_i, \quad \mathcal{A}_i(\overline{\tau}) = \mathcal{I}_i(\overline{\tau}) \times \mathcal{J}_i(\overline{\tau})$.
- $\forall\, \tau \in \mathcal{T}, \quad \mathcal{B}(\tau) = \mathcal{I}_i(\tau_i) \times \mathcal{H}_i(\tau)$.

We now prove the *rectangularity property of randomized protocols* in the peer-to-peer setting.

▶ **Lemma 3.7.** *Let π be a k-player private-coins protocol with inputs from $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$. Let $\mathcal{T}$ denote the set of possible transcripts of π, and for $i \in [\![1,k]\!]$ let $\mathcal{T}_i$ denote the set of possible transcript observed by player i, so that $\mathcal{T} \subseteq \mathcal{T}_1 \times \cdots \times \mathcal{T}_k$. Then for every $i \in [\![1,k]\!]$, there exist functions $q_i : \mathcal{X}_i \times \mathcal{T}_i \to [0,1]$, $q_{-i} : \mathcal{X}_{-i} \times \mathcal{T}_i \to [0,1]$ and $p_{-i} : \mathcal{X}_{-i} \times \mathcal{T} \to [0,1]$ such that $\forall\ x \in \mathcal{X}, \forall\ \tau = (\tau_1, \ldots, \tau_k) \in \mathcal{T}, \Pr[\Pi_i(x) = \tau_i] = q_i(x_i, \tau_i) q_{-i}(x_{-i}, \tau_i)$, and $\forall\ x \in \mathcal{X}, \forall\ \tau = (\tau_1, \ldots, \tau_k) \in \mathcal{T}, \Pr[\Pi(x) = \tau] = q_i(x_i, \tau_i) p_{-i}(x_{-i}, \tau)$.*

4 The function parity

We now prove a lower bound on the multi-party peer-to-peer randomized communication complexity of the k-party n-bit parity function Par_k^n, defined as follows: each player i receives n bits $(x_i^p)_{p \in [\![1,n]\!]}$ and player 1 has to output the bitwise sum modulo 2 of the inputs, i.e., $\mathsf{Par}_k^n(x) = \left(\oplus_{i=1}^k x_i^1, \oplus_{i=1}^k x_i^2, \ldots, \oplus_{i=1}^k x_i^n\right)$ (the case where all k players compute the function is trivial). To start, we prove a lower bound on the multi-party information complexity of the parity function, where each player has a single input bit. For simplicity we denote this function Par_k, rather than Par_k^1.

▶ **Theorem 4.1.** *Let μ be the uniform distribution on $\{0,1\}^k$. Given any fixed $0 \le \epsilon < \frac{1}{2}$, for any protocol π ϵ-computing Par_k, it holds that $\mathsf{MIC}_\mu(\pi) = \Omega(k)$.*

The next theorem follows immediately from Theorem 4.1 and Theorem 3.5.

▶ **Theorem 4.2.** *Let μ be the uniform distribution on $\{0,1\}^k$. Given any fixed $0 \le \epsilon < \frac{1}{2}$, for any protocol π ϵ-computing Par_k^n, it holds that $\mathsf{MIC}_{\mu^n}(\pi) = \Omega(kn)$.*

We can now prove a lower bound on the communication complexity of Par_k^n. Note that the lower bound for Par_k^n given in [30] is valid only for a restricted class of protocols, called “oblivious” in [30].

▶ **Theorem 4.3.** *Given any fixed $0 \le \epsilon < \frac{1}{2}$, there is a constant α such that for $n \ge \frac{1}{\alpha} k$,*

$$\mathsf{CC}^\epsilon(\mathsf{Par}_k^n) = \Omega(kn) \ .$$

5 The function AND

In this section we consider an arbitrary k-party protocol, π, where each player has an input bit x_i, and where π has to compute the AND of all the input bits. We prove a lower bound on a certain information theoretic measure (that we define below) for π. The proof makes use of the following input distribution.

Input distribution. Consider the distribution μ defined as follows. Draw a bit $M \sim \mathbf{Ber}(\frac{2}{3}, \frac{1}{3})$, and a uniformly random index $Z \in [\![1, k]\!]$. Assign 0 to X_Z. If $M = 0$, sample X_{-Z} uniformly in $\{0,1\}^{k-1}$; if $M = 1$, assign 1^{k-1} to X_{-Z}. We will also work with the product distribution μ^n. Our distribution is similar to the ones of [8, 15] in that it leads to a high information cost (or similar measures) for the function AND_k. The distribution that we use has the property that the AND of any input in the support of μ is 0. This allows us to prove lower bounds for the Disjointness function without the constraint that $k = \Omega(\log n)$ which was necessary in [8] (but not in [15]).

5.1 Switched multi-party information cost of AND_k

We propose the following definition, which is an adaptation of the switched information cost of [8]. We call it Switched Multi-party Information Cost (SMIC).

▶ **Definition 5.1.** For a k-player protocol π with inputs drawn from μ^n let

$$\mathsf{SMIC}_{\mu^n}(\pi) = \sum_{i=1}^{k} \left(I(X_i; \Pi_i \mid MZ) + I(M; \Pi_i \mid X_i Z) \right) .$$

Note that the notion of SMIC is only defined with respect to the distribution μ^n that we defined, and we may thus omit the distribution from the notation. We note that in order to simplify the expressions we often consider the public randomness as implicit in the information theoretic expressions we use below. It can be materialized either as part of the transcript or in the conditioning of the information theoretic expressions.

We can now prove the main result of this section.

▶ **Theorem 5.2.** *For any fixed $0 \le \epsilon < \frac{1}{2}$, for any protocol π externally ϵ-computing AND_k,*

$$\mathsf{SMIC}_{\mu}(\pi) = \Omega(k) .$$

6 The function Disjointness

In the k players n-bit disjointness function Disj_k^n, every player $i \in [\![1, k]\!]$ has an n-bit string $(x_i^\ell)_{\ell \in [\![1,n]\!]}$, and the players have to output 1 if and only if there exists a coordinate ℓ where all players have the bit 1. Formally, $\mathsf{Disj}_k^n(x) = \bigvee_{\ell=1}^{n} \bigwedge_{i=1}^{k} x_i^\ell$.

6.1 Switched multi-party information cost of Disj_k^n

We first prove a direct-sum-type property which allows us to make the link between the functions AND_k and Disj_k^n. A similar property was proved in [8] in the coordinator model; our peer-to-peer model requires a different, more involved, construction, since we do not have the coordinator, and moreover no player can act as the coordinator since it would get too much information. Since Disj_k^n is the disjunction of n AND_k functions, we analyze the switched multi-party information cost of Disj_k^n using the distribution μ^n.

▶ **Lemma 6.1.** *Let $k > 3$. For any protocol π externally ϵ-computing Disj_k^n, there exists a protocol π' externally ϵ-computing AND_k such that*

$$\mathsf{SMIC}_{\mu^n}(\pi) \ge n \cdot \mathsf{SMIC}_{\mu}(\pi') .$$

Coupled with the lower bound on $\mathsf{SMIC}(\pi')$ for any protocol π' that computes AND_k (Section 5), the above lemma gives us a lower bound on $\mathsf{SMIC}(\pi)$ for any protocol that computes the function Disj_k^n :

▶ **Theorem 6.2.** *Let $k > 3$. Given any fixed $0 \leq \epsilon < \frac{1}{2}$, for any protocol π externally ϵ-computing Disj_k^n it holds that*

$$\mathsf{SMIC}_{\mu^n}(\pi) = \Omega(kn) \ .$$

6.2 Multi-party information complexity and communication complexity of Disj_k^n

The next lemma is key to our argument. The theorem that follows is a consequence of it and of Theorem 6.2.

▶ **Lemma 6.3.** *For any k-player protocol π, $\mathsf{SMIC}_{\mu^n}(\pi) \leq \mathsf{MIC}_{\mu^n}(\pi)$.*

The next theorem follows immediately from Theorem 6.2 and Lemma 6.3.

▶ **Theorem 6.4.** *Let $k > 3$. Given any fixed $0 \leq \epsilon < \frac{1}{2}$, for any protocol π externally ϵ-computing Disj_k^n, it holds that*

$$\mathsf{MIC}_{\mu^n}(\pi) = \Omega(kn) \ .$$

We now conclude with a lower bound on the randomized communication complexity of the disjointness function.

▶ **Theorem 6.5.** *Given any fixed $0 \leq \epsilon < \frac{1}{2}$, there is a constant α such that for $n \geq \frac{1}{\alpha}k$,*

$$\mathsf{CC}^{\epsilon}(\mathsf{Disj}_k^n) = \Omega(kn) \ .$$

We note that our tight lower bound holds also for protocols where only one player is required to output the value of the function.

7 Randomness complexity of private protocols

A protocol π is said to *privately* compute a function f if, at the end of the execution of the protocol, the players have learned nothing but the value of that function. We now prove that the (information theoretic) private computation of Disj_k^n requires $\Omega(n)$ random bits. We prove this result using the information theoretic results for Disj_k^n of the previous sections. The definitions and the details of the proof are deferred to the full version of the paper.

▶ **Theorem 7.1.** *Let $k > 3$. Then $\mathcal{R}(\mathsf{Disj}_k^n) = \Omega(n)$, where $\mathcal{R}(f)$ is the minimum number of random bits necessary for a protocol to privately compute f.*

8 Conclusions and open problems

We introduce new models and new information theoretic tools for the study of communication complexity, and other complexity measures, in the natural peer-to-peer, multi-party, number-in-hand setting. We prove a number of properties of our new models and measures, and exemplify their effectiveness by proving two lower bounds on communication complexity, as well as a lower bound on the amount of randomness necessary for certain private computations.

To the best of our knowledge, our lower bounds on communication complexity are the first tight (non-trivial) lower bounds on communication complexity in the natural *peer-to-peer* multi-party setting, and our lower bound on the randomness complexity of private computations is the first that grows with the size of the input, while the computed function is a boolean one (i.e., the size of the output does not grow with the size of the input).

We believe that our models and tools may find additional applications and may open the way to further study of the natural peer-to-peer setting and to the building of a more solid bridge between the the fields of communication complexity and of distributed computation.

Our work raises a number of questions. First, how can one relax the restrictions that we impose on the general asynchronous model and still prove communication complexity lower bounds in a peer-to-peer setting? Our work seems to suggest that novel techniques and ideas, possibly not based on information theory, are necessary for this task, and it would be most interesting to find those. Second, it would be interesting to identify the necessary and sufficient conditions that guarantee the "rectangularity" property of communication protocols in a peer-to-peer setting. While this property is fundamental to the analysis of two-party protocols, it turns out that once one turns to the multi-party peer-to-peer setting, not only does this property become subtle to prove, but also this property does not always hold. Given the central (and sometimes implicit) role of the rectangularity property in the literature, it would be interesting to identify when it holds in the multi-party peer-to-peer number-in-hand setting.

References

1 Gilad Asharov and Yehuda Lindell. A Full Proof of the BGW Protocol for Perfectly Secure Multiparty Computation. *J. Cryptology*, 30(1):58–151, 2017. doi:10.1007/s00145-015-9214-4.

2 Reuven Bar-Yehuda, Benny Chor, Eyal Kushilevitz, and Alon Orlitsky. Privacy, additional information and communication. *IEEE Transactions on Information Theory*, 39(6):1930–1943, 1993. doi:10.1109/18.265501.

3 Ziv Bar-Yossef, T. S. Jayram, Ravi Kumar, and D. Sivakumar. An information statistics approach to data stream and communication complexity. *J. Comput. Syst. Sci.*, 68(4):702–732, 2004. doi:10.1016/j.jcss.2003.11.006.

4 Boaz Barak, Mark Braverman, Xi Chen, and Anup Rao. How to compress interactive communication. In *Proceedings of the 42nd ACM symposium on Theory of computing*, STOC '10, pages 67–76, New York, NY, USA, 2010. ACM. doi:10.1145/1806689.1806701.

5 Michael Ben-Or, Shafi Goldwasser, and Avi Wigderson. Completeness theorems for non-cryptographic fault-tolerant distributed computation. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, STOC '88, pages 1–10, New York, NY, USA, 1988. ACM. doi:10.1145/62212.62213.

6 C. Blundo, A. De Santis, G. Persiano, and U. Vaccaro. Randomness complexity of private computation. *computational complexity*, 8(2):145–168, 1999. doi:10.1007/s000370050025.

7 Mark Braverman. Interactive Information Complexity. *SIAM J. Comput.*, 44(6):1698–1739, 2015. doi:10.1137/130938517.

8 Mark Braverman, Faith Ellen, Rotem Oshman, Toniann Pitassi, and Vinod Vaikuntanathan. A Tight Bound for Set Disjointness in the Message-Passing Model. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 668–677. IEEE Computer Society, 2013. doi:10.1109/FOCS.2013.77.

9 Mark Braverman, Ankit Garg, Denis Pankratov, and Omri Weinstein. From information to exact communication. In *Proceedings of the 45th annual ACM symposium on Symposium on theory of computing*, STOC '13, pages 151–160, New York, NY, USA, 2013. ACM. doi:10.1145/2488608.2488628.

10 Mark Braverman and Rotem Oshman. On Information Complexity in the Broadcast Model. In Chryssis Georgiou and Paul G. Spirakis, editors, *Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, PODC 2015, Donostia-San Sebastián, Spain, July 21 - 23, 2015*, pages 355–364. ACM, 2015. doi:10.1145/2767386.2767425.

11 Mark Braverman and Anup Rao. Information Equals Amortized Communication. *IEEE Trans. Information Theory*, 60(10):6058–6069, 2014. doi:10.1109/TIT.2014.2347282.

12 Amit Chakrabarti and Sagar Kale. Strong Fooling Sets for Multi-player Communication with Applications to Deterministic Estimation of Stream Statistics. In Irit Dinur, editor, *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, New Brunswick, New Jersey, USA*, pages 41–50. IEEE Computer Society, 2016. doi:10.1109/FOCS.2016.14.

13 Amit Chakrabarti, Subhash Khot, and Xiaodong Sun. Near-optimal lower bounds on the multi-party communication complexity of set disjointness. In *In IEEE Conference on Computational Complexity*, pages 107–117, 2003.

14 Amit Chakrabarti, Yaoyun Shi, Anthony Wirth, and Andrew Chi-Chih Yao. Informational Complexity and the Direct Sum Problem for Simultaneous Message Complexity. In *FOCS*, pages 270–278, 2001. doi:10.1109/SFCS.2001.959901.

15 Arkadev Chattopadhyay and Sagnik Mukhopadhyay. Tribes Is Hard in the Message Passing Model. In Ernst W. Mayr and Nicolas Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, volume 30 of *LIPIcs*, pages 224–237. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. doi:10.4230/LIPIcs.STACS.2015.224.

16 Arkadev Chattopadhyay and Toniann Pitassi. The Story of Set Disjointness. *SIGACT News*, 41(3):59–85, September 2010. doi:10.1145/1855118.1855133.

17 Arkadev Chattopadhyay, Jaikumar Radhakrishnan, and Atri Rudra. Topology Matters in Communication. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 631–640, 2014. doi:10.1109/FOCS.2014.73.

18 Arkadev Chattopadhyay and Atri Rudra. The Range of Topological Effects on Communication. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II*, volume 9135 of *Lecture Notes in Computer Science*, pages 540–551. Springer, 2015. doi:10.1007/978-3-662-47666-6_43.

19 David Chaum, Claude Crépeau, and Ivan Damgard. Multiparty unconditionally secure protocols. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, STOC '88, pages 11–19, New York, NY, USA, 1988. ACM. doi:10.1145/62212.62214.

20 Danny Dolev and Tomás Feder. Multiparty Communication Complexity. In *30th Annual Symposium on Foundations of Computer Science, Research Triangle Park, North Carolina, USA, 30 October - 1 November 1989*, pages 428–433. IEEE Computer Society, 1989. doi:10.1109/SFCS.1989.63514.

21 Faith Ellen, Rotem Oshman, Toniann Pitassi, and Vinod Vaikuntanathan. Brief Announcement: Private Channel Models in Multi-party Communication Complexity. In *27th International Symposium on Distributed Computing (DISC), Jerusalem, Israel*, pages 575–576, 2013.

22 Uri Feige, Joe Killian, and Moni Naor. A Minimal Model for Secure Computation (Extended Abstract). In *Proceedings of the Twenty-sixth Annual ACM Symposium on Theory of Computing*, STOC '94, pages 554–563, New York, NY, USA, 1994. ACM. doi:10.1145/195058.195408.

23 Silvio Frischknecht, Stephan Holzer, and Roger Wattenhofer. Networks cannot compute their diameter in sublinear time. In Yuval Rabani, editor, *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, Kyoto, Japan, January 17-19, 2012*, pages 1150–1162. SIAM, 2012. doi:10.1137/1.9781611973099.

24 Anna Gál and Parikshit Gopalan. Lower Bounds on Streaming Algorithms for Approximating the Length of the Longest Increasing Subsequence. *SIAM J. Comput.*, 39(8):3463–3479, 2010. doi:10.1137/090770801.

25 Anna Gál and Adi Rosén. Omega(log n) Lower Bounds on the Amount of Randomness in 2-Private Computation. *SIAM J. Comput.*, 34(4):946–959, 2005. doi:10.1137/S0097539703432785.

26 Andre Gronemeier. Asymptotically Optimal Lower Bounds on the NIH-Multi-Party Information Complexity of the AND-Function and Disjointness. In Susanne Albers and Jean-Yves Marion, editors, *26th International Symposium on Theoretical Aspects of Computer Science, STACS 2009, February 26-28, 2009, Freiburg, Germany, Proceedings*, volume 3 of *LIPIcs*, pages 505–516. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2009. doi:10.4230/LIPIcs.STACS.2009.1846.

27 Zengfeng Huang, Bozidar Radunovic, Milan Vojnovic, and Qin Zhang. Communication Complexity of Approximate Matching in Distributed Graphs. In Ernst W. Mayr and Nicolas Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, volume 30 of *LIPIcs*, pages 460–473. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. doi:10.4230/LIPIcs.STACS.2015.460.

28 T. S. Jayram. Hellinger Strikes Back: A Note on the Multi-party Information Complexity of AND. In *Proceedings of the 12th International Workshop and 13th International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, APPROX '09 / RANDOM '09, pages 562–573, Berlin, Heidelberg, 2009. Springer-Verlag. doi:10.1007/978-3-642-03685-9_42.

29 Bala Kalyanasundaram and Georg Schintger. The Probabilistic Communication Complexity of Set Intersection. *SIAM J. Discret. Math.*, 5(4):545–557, November 1992. doi:10.1137/0405044.

30 Iordanis Kerenidis, Adi Rosén, and Florent Urrutia. Multi-Party Protocols, Information Complexity and Privacy. In Piotr Faliszewski, Anca Muscholl, and Rolf Niedermeier, editors, *41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 - Kraków, Poland*, volume 58 of *LIPIcs*, pages 57:1–57:16. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPIcs.MFCS.2016.57.

31 Janne H. Korhonen and Jukka Suomela. Brief Announcement: Towards a Complexity Theory for the Congested Clique. In Andréa W. Richa, editor, *31st International Symposium on Distributed Computing, DISC 2017, October 16-20, 2017, Vienna, Austria*, volume 91 of *LIPIcs*, pages 55:1–55:3. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017. doi:10.4230/LIPIcs.DISC.2017.55.

32 Janne H. Korhonen and Jukka Suomela. Towards a complexity theory for the congested clique. *CoRR*, abs/1705.03284, 2017. arXiv:1705.03284.

33 Eyal Kushilevitz and Yishay Mansour. Randomness in Private Computations. *SIAM J. Discrete Math.*, 10(4):647–661, 1997. doi:10.1137/S0895480196306130.

34 Eyal Kushilevitz and Noam Nisan. *Communication complexity.* Cambridge University Press, 1997.

35 Eyal Kushilevitz, Rafail Ostrovsky, and Adi Rosén. Characterizing Linear Size Circuits in Terms of Pricacy. *J. Comput. Syst. Sci.*, 58(1):129–136, 1999. doi:10.1006/jcss.1997.1544.

36 Peter Bro Miltersen, Noam Nisan, Shmuel Safra, and Avi Wigderson. On Data Structures and Asymmetric Communication Complexity. *J. Comput. Syst. Sci.*, 57(1):37–49, 1998. doi:10.1006/jcss.1998.1577.

37 Jeff M. Phillips, Elad Verbin, and Qin Zhang. Lower Bounds for Number-in-Hand Multiparty Communication Complexity, Made Easy. *SIAM J. Comput.*, 45(1):174–196, 2016. doi:10.1137/15M1007525.

38 A. A. Razborov. On the Distributional Complexity of Disjointness. *Theor. Comput. Sci.*, 106(2):385–390, December 1992. doi:10.1016/0304-3975(92)90260-M.

39 Atish Das Sarma, Stephan Holzer, Liah Kor, Amos Korman, Danupon Nanongkai, Gopal Pandurangan, David Peleg, and Roger Wattenhofer. Distributed Verification and Hardness of Distributed Approximation. *CoRR*, abs/1011.3049, 2010. arXiv:1011.3049.

40 C. E. Shannon. A mathematical theory of communication. *Bell system technical journal*, 27, 1948.

41 David P. Woodruff and Qin Zhang. Tight Bounds for Distributed Functional Monitoring. *CoRR*, abs/1112.5153, 2011. arXiv:1112.5153.

42 David P. Woodruff and Qin Zhang. When Distributed Computation does not Help. *CoRR*, abs/1304.4636, 2013. arXiv:1304.4636.

43 David P. Woodruff and Qin Zhang. An Optimal Lower Bound for Distinct Elements in the Message Passing Model. In *Proceedings of the Twenty-fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '14, pages 718–733, Philadelphia, PA, USA, 2014. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=2634074.2634128.

44 Andrew Chi-Chih Yao. Protocols for Secure Computations (Extended Abstract). In *23rd Annual Symposium on Foundations of Computer Science, Chicago, Illinois, USA, 3-5 November 1982*, pages 160–164. IEEE Computer Society, 1982. doi:10.1109/SFCS.1982.38.