

A Schur Complement Cheeger Inequality

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Abstract

Cheeger’s inequality shows that any undirected graph G with minimum normalized Laplacian eigenvalue λ_G has a cut with conductance at most $O(\sqrt{\lambda_G})$. Qualitatively, Cheeger’s inequality says that if the mixing time of a graph is high, there is a cut that certifies this. However, this relationship is not tight, as some graphs (like cycles) do not have cuts with conductance $o(\sqrt{\lambda_G})$.

To better approximate the mixing time of a graph, we consider a more general object. Specifically, instead of bounding the mixing time with cuts, we bound it with cuts in graphs obtained by Schur complementing out vertices from the graph G . Combinatorially, these Schur complements describe random walks in G restricted to a subset of its vertices. As a result, all Schur complement cuts have conductance at least $\Omega(\lambda_G)$. We show that unlike with cuts, this inequality is tight up to a constant factor. Specifically, there is a Schur complement cut with conductance at most $O(\lambda_G)$.

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1 Introduction

In this paper, we give a constant-factor approximation for the minimum ratio of electrical conductance to volume of any pair of sets S_1, S_2 . When S_1 is the complement of S_2 , for example, this quantity is what is classically called the *conductance* of the set S_1 , so the ratio that we consider is less than conductance of the graph G . We obtain this constant-factor approximation by showing that the minimum electrical conductance-to-volume ratio is approximated within a constant factor by λ_G . Thus, our quantity closes the classical $\sqrt{\lambda_G}$ gap present between the upper and lower bounds in Cheeger’s inequality.

In particular, we prove the following partitioning result, which relates $1/\lambda_G$ to effective resistances between sets in the graph G . Think of the weighted graph G as an electrical network, where each edge represents a conductor with electrical conductance equal to its weight. For two sets of vertices S_1 and S_2 , obtain a graph H by contracting all vertices in S_1 to a single vertex s_1 and all vertices in S_2 to a single vertex s_2 . Let $\text{Reff}_G(S_1, S_2)$ denote the effective resistance between the vertices s_1 and s_2 in the graph H . Then, we show the following in Appendix A:

► **Theorem 1.** *In any weighted graph G , there are two sets of vertices S_1 and S_2 for which $\text{Reff}_G(S_1, S_2) \geq 1/(25600\lambda_G \min(\text{vol}_G(S_1), \text{vol}_G(S_2)))$. Furthermore, for any pair of sets S'_1, S'_2 , $\text{Reff}_G(S'_1, S'_2) \leq 2/(\lambda_G \min(\text{vol}_G(S'_1), \text{vol}_G(S'_2)))$.*

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1.1 Relationship to Cheeger's Inequality

For a set of vertices S , let ϕ_S denote the total weight of edges leaving S divided by the total degree of the vertices in S . Throughout the literature, this quantity is often called the conductance of S . To avoid confusion with electrical conductance, we call this quantity the *fractional conductance* of S . Let ϕ_G denote the minimum fractional conductance of any set S with at most half of the volume (total vertex degree). Cheeger's inequality for graphs [3, 2] is as follows:

► **Theorem 2 (Cheeger's Inequality).** *For any weighted graph G , $\lambda_G/2 \leq \phi_G \leq \sqrt{2\lambda_G}$.*

Cheeger's inequality was originally introduced in the context of manifolds [6]. It is a fundamental primitive in graph partitioning [20, 15] and for upper bounding the mixing time of Markov chains [19]. Motivated by spectral partitioning, much work has been done on higher-order generalizations of Cheeger's inequality [13, 14]. The myriad of applications for Cheeger's inequality and generalizations of it [4, 21], along with the $\sqrt{\lambda_G}$ gap between the upper and lower bounds, have led to a long line of work that seeks to improve the quality of the partition found when the spectrum has certain properties (for example, bounded eigenvalue gap [11] or when the graph has special structure [10].)

Here, we get rid of the $\sqrt{\lambda_G}$ gap by taking a different approach. Instead of assuming special combinatorial or spectral structure of the input graph to obtain a tighter relationship between conductance and λ_G , we introduce a more general object than graph cuts that enable a tighter approximation to λ_G . Instead of just considering cuts in the given graph G , we consider cuts obtained by picking two disjoint sets of vertices S_1 and S_2 , computing the Schur complement of G onto $S_1 \cup S_2$, and looking at the cut consisting of all edges between S_1 and S_2 in that Schur complement. Let ρ_G be the minimum conductance of any such cut (defined formally in Section 2). We show that the minimum conductance of any such cut is a constant factor approximation to λ_G :

► **Theorem 3.** *Let G be a weighted graph. Then*

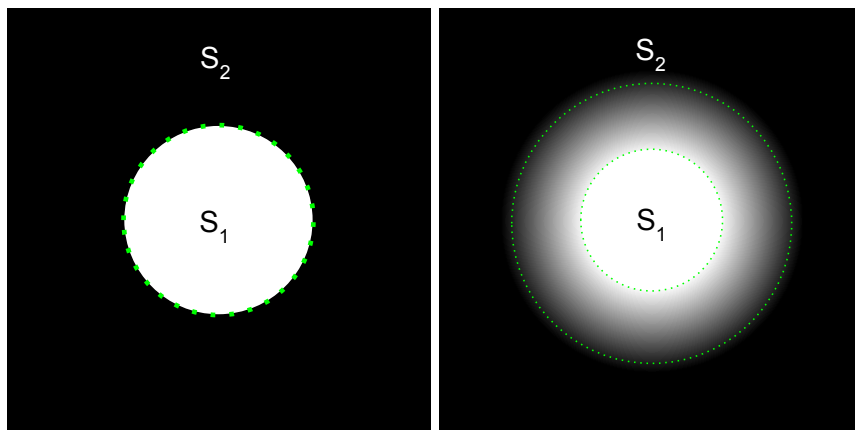
$$\lambda_G/2 \leq \rho_G \leq 25600\lambda_G$$

1.2 Graph Partitioning

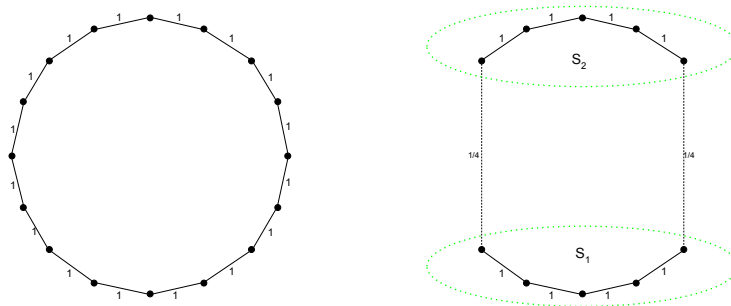
Effective resistance in spectral graph theory has been used several times recently (for example [16, 1]) to obtain improved graph partitioning results. $1/\lambda_G$ may not yield a good approximation to the effective resistance between pairs of vertices [5]. For example, on an n -vertex grid graph G , all effective resistances are between $\Omega(1)$ and $O(\log n)$, but $\lambda_G = \Theta(1/n)$. Theorem 1 closes this gap by considering pairs of *sets* of vertices, not just pairs of vertices.

Cheeger's inequality is the starting point for analysis of spectral partitioning. In some partitioning tasks, cutting the graph does not make sense. For example, spectral partitioning is an important tool in image segmentation [18, 17]. Graph partitioning makes the most sense in image segmentation when one wants to find an object with a sharp boundary. However, in many images, like the one in Figure 1 on the right, objects may have fuzzy boundaries. In these cases, it is not clear which cut an image segmentation algorithm should return.

Considering cuts in Schur complements circumvents this ambiguity. Think of an image as a graph by making a vertex for each pixel and making an edge between adjacent pixels, where the weight on an edge is inversely related to the disparity between the colors of the endpoint pixels for the edge. An optimal segmentation in our setting would consist of the



■ **Figure 1** Spectral partitioning finds the S_1 - S_2 cut in the left image, but may not in the right due to the presence of many equal weight cuts. The minimum fractional conductance Schur complement cut is displayed in both images.



■ **Figure 2** A tight example for the upper bound in Cheeger's inequality. The minimum fractional conductance of any cut in this graph is $1/8$, while the fractional conductance of the illustrated Schur complement cut on the right is $2(1/4)/(2(1/4) + 8(1)) = 1/17 < 1/8$.

two sets S_1 and S_2 corresponding to pixels on either side of the fuzzy boundary. Computing the Schur complement of the graph onto $S_1 \cup S_2$ eliminates all vertices corresponding to pixels in the boundary.

Some examples in which Cheeger's inequality is not tight illustrate a similar phenomenon in which there are many equally good cuts. For example, let G be an unweighted n -vertex cycle. This is a tight example for the upper bound in Cheeger's inequality, as no cut has fractional conductance smaller than $O(1/n)$ despite the fact that $\lambda_G = \Theta(1/n^2)$. Instead, divide the cycle into four equal-sized quarters and let S_1 and S_2 be two opposing quarters. The Schur complement cut between S_1 and S_2 has fractional conductance at most $O(1/n^2)$, which matches λ_G up to a constant factor.

1.3 Mixing Times of Markov Chains

Cheeger's inequality and variants of it can be used to upper bound the relaxation time of reversible Markov chains. The *relaxation time* of a reversible Markov chain is defined to be $1/\lambda_G$ and approximates the mixing time of a chain up to a $\log(1/\pi_{\min})$ factor, where π_{\min} is the minimum stationary distribution probability. To do this, one finds a fractional multicommodity flow that routes the demand $\text{degree}(u)\text{degree}(v)$ for every pair of vertices

$u, v \in V(G)$. The relevant quantity is the *congestion* of this flow. Specifically, view a multicommodity flow as a collection of paths \mathcal{P} , where a flow path has flow value f_p . The congestion of the flow is defined to be

$$\max_{e \in E(G)} \sum_{p \in \mathcal{P}: e \in p} |f_p|$$

The fractional conductance of any cut can be lower bounded using the the congestion of the flow. By the upper bound in Theorem 2, we get a lower bound on λ_G as well, which in turn yields an upper bound on the relaxation time of G . However, the congestion of the optimal fractional multicommodity flow is not a close approximation to the relaxation time of the graph, due to the loss in both Cheeger's inequality and in the gap between fractional multicommodity flows and sparsest cuts. One can improve upon this by considering a length-weighted form of congestion instead:

$$\max_{e \in E(G)} \sum_{p \in \mathcal{P}: e \in p} |f_p| |p|$$

The length-weighted congestion of any multicommodity flow in G routing the right demand is an upper bound on the relaxation time of G . For a precise statement of this, see Theorem 4.6 of [8]. This overcomes the square root present in the upper bound in Cheeger's inequality. However, this approach does not overcome the multicommodity flow-cut gap. In particular, any multicommodity flow in a constant-degree random Erdos-Renyi graph has length-weighted congestion at least $\Omega(\log^2 n)$ (see, for example, Section 4.1 of [9]), despite the fact that these graphs have constant relaxation time.

Theorem 3 yields a upper lower bound on the relaxation time of a reversible Markov chain. Unlike the bounds discussed in the previous paragraph, it is tight up to a constant factor on all graphs. We do not know of a lower bound for Schur complement cut fractional conductance analogous to multicommodity flows for standard cuts.

2 Preliminaries

Graph theory. Consider an undirected, connected graph H with edge weights $\{c_e^H\}_{e \in E(H)}$, m edges, and n vertices. Let $V(H)$ and $E(H)$ denote the vertex and edge sets of H respectively. For two sets of vertices $A, B \subseteq V(H)$, let $E_H(A, B)$ denote the set of edges in H incident with one vertex in A and one vertex in B and let $c^H(A, B) := \sum_{e \in E_H(A, B)} c_e^H$. For a set of edges $F \subseteq E(H)$, let $c^H(F) := \sum_{e \in F} c_e^H$. For a set of vertices $A \subseteq V(H)$, let $\partial_H A := E_H(A, V(H) \setminus A)$. For a vertex $v \in V(H)$, let $\partial_H v := \partial_H \{v\}$ denote the edges incident with v in H and let $c_v^H := \sum_{e \in \partial_H v} c_e^H$. For a set of vertices $A \subseteq V(H)$, let $\text{vol}_H(A) := \sum_{v \in A} c_v^H$. When A and B are disjoint, let $H/(A, B)$ denote the graph with all vertices in A identified to one vertex a and all vertices in B identified to one vertex b . Formally, let $H/(A, B)$ be the graph with $V(H/(A, B)) = (V(H) \setminus (A \cup B)) \cup \{a, b\}$, embedding $f : V(H) \rightarrow V(H/(A, B))$ with $f(u) := a$ if $u \in A$, $f(u) := b$ if $u \in B$, and $f(u) := u$ otherwise, and edges $\{f(u), f(v)\}$ for all $\{u, v\} \in E(H)$. Let $H/A := H/(A, \emptyset)$.

Laplacians. Let D_H be the $n \times n$ diagonal matrix with rows and columns indexed by vertices in H and $D_H(v, v) = c_v^H$ for all $v \in V(H)$. Let A_H be the adjacency matrix of H ; that is the matrix with $A_H(u, v) = c_{uv}^H$ for all $u, v \in V(H)$. Let $L_H := D_H - A_H$ be the Laplacian matrix of H . Let $N_H := D_H^{-1/2} L_H D_H^{-1/2}$ denote the normalized Laplacian matrix

of H . For a matrix M , let M^\dagger denote the Moore-Penrose pseudoinverse of M . For subsets A and B of rows and columns of M respectively, let $M[A, B]$ denote the $|A| \times |B|$ submatrix of M restricted to those rows and columns. For a set of vertices $S \subseteq V(H)$, let $\mathbf{1}_S$ denote the indicator vector for the set S . For two vertices $u, v \in \mathbb{R}^n$, let $\chi_{uv} := \mathbf{1}_{\{u\}} - \mathbf{1}_{\{v\}}$. When the graph is clear from context, we omit H from all of the subscripts and superscripts of H . For a vector $x \in \mathbb{R}^n$, let $x_S \in \mathbb{R}^S$ denote the restriction of x to the coordinates in S .

Let λ_H denote the smallest nonzero eigenvalue of N_H . Equivalently,

$$\lambda_H := \min_{x \in \mathbb{R}^n: x^T D_H^{1/2} \mathbf{1}_{V(H)} = 0} \frac{x^T N_H x}{x^T x}$$

For any set of vertices $X \subseteq V(H)$, let

$$L_{\text{Schur}(H, X)} := L_H[X, X] - L_H[X, V(H) \setminus X] L_H[V(H) \setminus X, V(H) \setminus X]^{-1} L_H[V(H) \setminus X, X]$$

where brackets denote submatrices with the indexed rows and columns. The following fact applies specifically to Laplacian matrices:

► **Remark (Fact 2.3.6 of [12]).** For any graph H and any $X \subseteq V(H)$, $L_{\text{Schur}(H, X)}$ is the Laplacian matrix of an undirected graph.

Let $\text{Schur}(H, X)$ denote the graph referred to in Remark 2. Schur complementation commutes with edge contraction and deletion and is associative:

► **Theorem 4** (Lemma 4.1 of [7], statement from [12]). *Given H , $S \subseteq V(H)$, and any edge e with both endpoints in S ,*

$$\text{Schur}(H \setminus e, S) = \text{Schur}(H, S) \setminus e$$

and, for any pair of vertices $x, y \in S$,

$$\text{Schur}(H/\{x, y\}, S) = \text{Schur}(H, S)/\{x, y\}$$

► **Theorem 5.** *Given H and two sets of vertices $X \subseteq Y \subseteq V(H)$, $\text{Schur}(\text{Schur}(H, Y), X) = \text{Schur}(H, X)$.*

The following property follows from the definition of Schur complements:

► **Remark.** Let H be a graph and $S \subseteq V(H)$. Let $I := \text{Schur}(H, S)$. For any $x \in \mathbb{R}^{V(H)}$ that is supported on S with $x^T \mathbf{1}_{V(H)} = 0$,

$$x^T L_H^\dagger x = x_S^T L_I^\dagger x_S$$

The weight of edges in this graph can be computed using the following folklore fact, which we prove for completeness:

► **Theorem 6.** *For two disjoint sets $C, D \subseteq V(H)$, let $I := \text{Schur}(H, C \cup D)$. Then*

$$c^I(C, D) = \frac{1}{\chi_{cd}^T L_{H/(C, D)}^\dagger \chi_{cd}}$$

Proof. By definition, $c^I(C, D) = c^{I/(C, D)}(\{c\}, \{d\})$. By Theorem 4, $I/(C, D) = \text{Schur}(H/(C, D), \{c, d\})$. By Remark 2, $c^{\text{Schur}(H/(C, D), \{c, d\})}(\{c\}, \{d\}) = \frac{1}{\chi_{cd}^T L_{H/(C, D)}^\dagger \chi_{cd}}$. Combining these equalities gives the desired result. ◀

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We also use the following folklore fact about electrical flows, which we prove for the sake of completeness:

► **Theorem 7.** For two vertices $s, t \in V(H)$,

$$\chi_{st}^T L_H^\dagger \chi_{st} = \frac{1}{\min_{p \in \mathbb{R}^{V(H)}: p_s \leq 0, p_t \geq 1} p^T L_H p}$$

Proof. We first show that

$$\chi_{st}^T L_H^\dagger \chi_{st} = \frac{1}{\min_{p \in \mathbb{R}^{V(H)}: p_s = 0, p_t = 1} p^T L_H p}$$

Taking the gradient of the objective $p^T L_H p$ shows that that the optimal p are the potentials for an electrical flow with flow conservation at all vertices besides s and t . Therefore, p is proportional to $L_H^\dagger \chi_{st} + \gamma \mathbf{1}$ for some $\gamma \in \mathbb{R}$. The constant of proportionality is $\chi_{st}^T L_H^\dagger \chi_{st}$ since the s - t potential drop in p is 1. Therefore,

$$\begin{aligned} \min_{p \in \mathbb{R}^{V(H)}: p_s = 0, p_t = 1} p^T L_H p &= \left(\frac{L_H^\dagger \chi_{st}}{\chi_{st}^T L_H^\dagger \chi_{st}} \right)^T L_H \left(\frac{L_H^\dagger \chi_{st}}{\chi_{st}^T L_H^\dagger \chi_{st}} \right) \\ &= \frac{1}{\chi_{st}^T L_H^\dagger \chi_{st}} \end{aligned}$$

The desired result follows from the fact that in the optimal p , all potentials are between 0 and 1 inclusive. ◀

Notions of fractional conductance. For a set of vertices $A \subseteq V(H)$, let

$$\phi_A^H := \frac{c^H(\partial_H(A))}{\min(\text{vol}_H(A), \text{vol}_H(V(H) \setminus A))}$$

be the *fractional conductance* of A . Let

$$\phi_H := \min_{A \subseteq V(H): A \neq \emptyset} \phi_A^H$$

be the *fractional conductance* of H .

For two disjoint sets of vertices $A, B \subseteq V(H)$, let $I := \text{Schur}(H, A \cup B)$ and

$$\rho_{A,B}^H := \frac{c^I(A, B)}{\min(\text{vol}_I(A), \text{vol}_I(B))}$$

be the *Schur complement fractional conductance* of the pair of sets (A, B) . Define the *Schur complement fractional conductance* of the graph H to be

$$\rho_H := \min_{A, B \subseteq V(H): A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset} \rho_{A,B}^H$$

It will be helpful to deal with the quantities

$$\sigma_{A,B}^H := \frac{c^I(A, B)}{\min(\text{vol}_H(A), \text{vol}_H(B))}$$

and

$$\sigma_H := \min_{A, B \subseteq V(H): A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset} \sigma_{A,B}^H$$

as well, which we call the *mixed fractional conductances* of (A, B) and H respectively.

The following will be useful in relating $\rho_{A,B}^H$ to $\sigma_{A,B}^H$:

► **Proposition 8.** *For any two sets $X \subseteq Y \subseteq V(H)$, let $I := \text{Schur}(H, Y)$. Then,*

$$\text{vol}_I(X) \leq \text{vol}_H(X)$$

Proof. It suffices to show this result when $|X| = 1$ because vol is a sum of volumes (degrees) of vertices in the set. Furthermore, by Theorem 5, it suffices to show the result when $|Y| = |V(H)| - 1$. Let v be the unique vertex in H outside of Y and let u be the unique vertex in X . Then, by definition of the Schur complement,

$$\begin{aligned} \text{vol}_I(X) &= c_u^I \\ &= \sum_{w \in V(I)} c_{uw}^I \\ &= \sum_{w \in V(I)} \left(c_{uw}^H + \frac{c_{uv}^H c_{vw}^H}{c_v^H} \right) \\ &= \left(\sum_{w \in V(I)} c_{uw}^H \right) + \frac{c_{uv}^H}{c_v^H} \left(\sum_{w \in V(I)} c_{vw}^H \right) \\ &\leq \left(\sum_{w \in V(I)} c_{uw}^H \right) + c_{uv}^H \\ &= c_u^H \\ &= \text{vol}_H(X) \end{aligned}$$

as desired. ◀

To prove the upper bound, we given an algorithm for constructing a low fractional conductance Schur complement cut. The following result is helpful for making this algorithm take near-linear time:

► **Theorem 9** (Theorem 8.2 of [23]). *Given a graph H , there is a $\tilde{O}(m)$ -time algorithm that produces a vector $x \leftarrow \text{ApxFiedler}(H) \in \mathbb{R}^{V(H)}$ with $x^T D_H^{1/2} \mathbf{1}_{V(H)} = 0$ for which*

$$x^T N_H x \leq 2\lambda_H x^T x$$

3 Lower bound

We now show the first inequality in Theorem 3, which follows from the following lemma by Proposition 8, which implies that $\sigma_G \leq \rho_G$.

► **Lemma 10.**

$$\lambda_G \leq 2\sigma_G$$

Proof. We lower bound the Schur complement fractional conductance of any pair of disjoint sets $A, B \subseteq V(G)$. Let $I := \text{Schur}(G, A \cup B)$. Let P be the $(A \cup B) \times (A \cup B)$ diagonal matrix with $P(u, u) = c_u^G$ for each $u \in A \cup B$. We start by lower bounding the minimum nonzero eigenvalue λ of the matrix $P^{-1/2} L_I P^{-1/2}$. Let $\lambda_{\max}(M)$ denote the maximum eigenvalue of a symmetric matrix M . By definition of the Moore-Penrose pseudoinverse,

$$1/\lambda = \lambda_{\max}(P^{1/2} L_I^\dagger P^{1/2})$$

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By Remark 2,

$$\lambda_{\max}(P^{1/2}L_I^\dagger P^{1/2}) \leq \lambda_{\max}(N_G^\dagger) = 1/\lambda_G$$

Therefore, $\lambda \geq \lambda_G$. We now plug in a test vector. Let

$$z := P^{1/2} \left(\frac{\mathbf{1}_A}{\text{vol}_G(A)} - \frac{\mathbf{1}_B}{\text{vol}_G(B)} \right)$$

$z^T(P^{1/2}\mathbf{1}_{V(I)}) = 0$, so

$$\begin{aligned} \lambda_G &\leq \lambda \\ &= \min_{x \in \mathbb{R}^{A \cup B}: x^T P^{1/2} \mathbf{1}_{V(I)} = 0} \frac{x^T (P^{-1/2} L_I P^{-1/2}) x}{x^T x} \\ &\leq \frac{z^T (P^{-1/2} L_I P^{-1/2}) z}{z^T z} \\ &= \frac{c^I(A, B) ((1/\text{vol}_G(A)) + (1/\text{vol}_G(B)))^2}{(\text{vol}_G(A)/\text{vol}_G(A)^2) + (\text{vol}_G(B)/\text{vol}_G(B)^2)} \\ &= \frac{c^I(A, B) \text{vol}_G(A \cup B)}{\text{vol}_G(A) \text{vol}_G(B)} \\ &\leq 2\sigma_{A, B}^G \end{aligned}$$

4 Upper bound

We now show the second inequality in Theorem 3:

► **Lemma 11.**

$$\rho_G \leq 25600\lambda_G$$

To prove this lemma, we need to find a pair of sets A and B with low Schur complement fractional conductance:

► **Lemma 12.** *There is a near-linear time algorithm $\text{SweepCut}(G)$ that takes in a graph G with $\lambda_G \leq 1/25600$ and outputs a pair of nonempty sets A and B with the following properties:*

- (Low Schur complement fractional conductance) $\sigma_{A, B}^G \leq 640\lambda_G$
- (Large interior) $\phi_A^G \leq 1/4$ and $\phi_B^G \leq 1/4$

We now prove Lemma 11 given Lemma 12:

Proof of Lemma 11 given Lemma 12. Let $I := \text{Schur}(G, A \cup B)$. For any two vertices $u, v \in A \cup B$, $c_{uv}^I \geq c_{uv}^G$. Therefore, $\text{vol}_I(A) \geq 2 \sum_{u, v \in A} c_{uv}^G$ and $\text{vol}_I(B) \geq 2 \sum_{u, v \in B} c_{uv}^G$. By the ‘‘Large interior’’ guarantee of Lemma 12, $2 \sum_{u, v \in A} c_{uv}^G \geq (3/4)\text{vol}_G(A)$ and $2 \sum_{u, v \in B} c_{uv}^G \geq (3/4)\text{vol}_G(B)$. Therefore,

$$\rho_{A, B}^G \leq 4/3\sigma_{A, B}^G \leq 1280\lambda_G$$

by the ‘‘Low Schur complement weight’’ guarantee when $\lambda_G \leq 1/25600$, as desired. When $\lambda_G > 1/25600$, the lemma is trivially true, as desired. ◀

Now, we implement **SweepCut**. The standard Cheeger sweep examines all thresholds $q \in \mathbb{R}$ and for each threshold, computes the fractional conductance of the cut $\partial S_{\leq q}$ of edges from vertices with eigenvector coordinate at most q to ones greater than q . Instead, the algorithm **SweepCut** examines all thresholds $q \in \mathbb{R}$ and computes an upper bound (a proxy) for the $\sigma_{S_{\leq q/2}, S_{\geq q}}^G$ for each positive q and $\sigma_{S_{\leq q}, S_{\geq q/2}}^G$ for each negative q . Let $I_q := \text{Schur}(G, S_{\geq q} \cup S_{\leq q/2})$ for $q > 0$ and $I_q := \text{Schur}(G, S_{\leq q} \cup S_{\geq q/2})$. Let $\kappa_q(y) := \min(q, \max(q/2, y))$ for $q > 0$ and $\kappa_q(y) = \min(q/2, \max(q, y))$ for $q \leq 0$. The proxy is the following quantity, which is defined for a specific shift of the Rayleigh quotient minimizer $y \in \mathbb{R}^{V(G)}$.

$$\widehat{c}^{I_q}(S_{\geq q}, S_{\leq q/2}) := \frac{4}{q^2} \sum_{e=uv \in E(G)} c_e^G (\kappa_q(y_u) - \kappa_q(y_v))^2$$

for $q > 0$ and

$$\widehat{c}^{I_q}(S_{\leq q}, S_{\geq q/2}) := \frac{4}{q^2} \sum_{e=uv \in E(G)} c_e^G (\kappa_q(y_u) - \kappa_q(y_v))^2$$

for $q \leq 0$. We now show that this is indeed an upper bound:

► **Proposition 13.** *For all $q > 0$,*

$$c^{I_q}(S_{\leq q/2}, S_{\geq q}) \leq \widehat{c}^{I_q}(S_{\leq q/2}, S_{\geq q})$$

For all $q \leq 0$,

$$c^{I_q}(S_{\leq q}, S_{\geq q/2}) \leq \widehat{c}^{I_q}(S_{\leq q}, S_{\geq q/2})$$

Proof. We focus on the $q > 0$, as the reasoning for the $q \leq 0$ case is the same. By Theorems 6 and 7,

$$c^{I_q}(S_{\leq q/2}, S_{\geq q}) = \min_{p \in \mathbb{R}^{V(G)}: p_a \leq 0 \forall a \in S_{\leq q/2}, p_a \geq 1 \forall a \in S_{\geq q}} p^T L_G p$$

The vector p with $p_a := \frac{2}{q} \kappa_q(y_a) - 1$ for all vertices $a \in V(G)$ is a feasible solution to the above optimization problem with objective value $\widehat{c}^{I_q}(S_{\leq q/2}, S_{\geq q})$. This is the desired result. ◀

This proxy allows us to relate Schur complement conductances together across different thresholds q in a similar proof to the proof of the upper bound of Cheeger's inequality given in [22]. One complication in our case is that Schur complements for different values of q overlap in their eliminated vertices. Our choice of $\leq q/2, \geq q$ plays a key role here (as opposed to $\leq 0, \geq q$, for example) in ensuring that the overlap is small. We now give the algorithm **SweepCut**:

Algorithm 1: SweepCut(G).

Input: A graph G with $\lambda_G \leq 1/25600$
Output: Two sets of vertices A and B satisfying the guarantees of Lemma 12

- 1 $z \leftarrow$ vector with $z^T N_G z \leq 2\lambda_G z^T z$ and $z^T (D_G^{1/2} \mathbf{1}_{V(G)}) = 0$
- 2 $x \leftarrow D_G^{-1/2} z$
- 3 $y \leftarrow x - \alpha \mathbf{1}_{V(G)}$ for a value α such that $\text{vol}_G(\{v : y_v \leq 0\}) \geq \text{vol}_G(V(G))/2$ and $\text{vol}_G(\{v : y_v \geq 0\}) \geq \text{vol}_G(V(G))/2$
- 4 **foreach** $q \in \mathbb{R}$ **do**
- 5 $S_{\geq q} \leftarrow$ vertices with $y_v \geq q$
- 6 $S_{\leq q} \leftarrow$ vertices with $y_v \leq q$
- 7 **end**
- 8 **foreach** $q > 0$ **do**
- 9 **if** (1) $\widehat{c}^{I_q}(S_{\leq q/2}, S_{\geq q}) \leq 640\lambda_G \min(\text{vol}_G(S_{\leq q/2}), \text{vol}_G(S_{\geq q}))$, (2) $c^G(\partial S_{\geq q/2}) \leq 1/4 \text{vol}_G(S_{\geq q})$, and (3) $\phi_{S_{\geq q}} \leq 1/4$ **then**
- 10 **return** $(S_{\leq q/2}, S_{\geq q})$
- 11 **end**
- 12 **end**
- 13 **foreach** $q \leq 0$ **do**
- 14 **if** (1) $\widehat{c}^{I_q}(S_{\geq q/2}, S_{\leq q}) \leq 640\lambda_G \min(\text{vol}_G(S_{\geq q/2}), \text{vol}_G(S_{\leq q}))$, (2) $c^G(\partial S_{\geq q/2}) \leq 1/4 \text{vol}_G(S_{\leq q})$, and (3) $\phi_{S_{\leq q}} \leq 1/4$ **then**
- 15 **return** $(S_{\leq q}, S_{\geq q/2})$
- 16 **end**
- 17 **end**

Our analysis relies on the following key technical result, which we prove in Appendix B:

► **Proposition 14.** For any $a, b \in \mathbb{R}$,

$$\int_0^\infty \frac{(\kappa_q(a) - \kappa_q(b))^2}{q} dq \leq 10(a - b)^2$$

Proof of Lemma 12.

Algorithm well-definedness. We start by showing that SweepCut returns a pair of sets.

Assume, for the sake of contradiction, that SweepCut does not return a pair of sets. Let $I_q := \text{Schur}(G, S_{\geq q} \cup S_{\leq q/2})$ for $q > 0$ and $I_q := \text{Schur}(G, S_{\leq q} \cup S_{\geq q/2})$ for $q \leq 0$. By the contradiction assumption, for all $q > 0$,

$$\text{vol}_G(S_{\geq q}) \leq \frac{\widehat{c}^{I_q}(S_{\geq q}, S_{\leq q/2})}{640\lambda_G} + 4c^G(\partial S_{\geq q}) + 4c^G(\partial S_{\leq q/2})$$

and for all $q < 0$,

$$\text{vol}_G(S_{\leq q}) \leq \frac{\widehat{c}^{I_q}(S_{\leq q}, S_{\geq q/2})}{640\lambda_G} + 4c^G(\partial S_{\leq q}) + 4c^G(\partial S_{\geq q/2})$$

Since $\sum_{v \in V(G)} c_v^G x_v = 0$,

$$\sum_{v \in V(G)} c_v^G x_v^2 \leq \sum_{v \in V(G)} c_v^G y_v^2$$

Now, we bound the positive y_v and negative y_v parts of this sum separately. Negating y shows that it suffices to bound the positive part. Order the vertices in $S_{\geq 0}$ in decreasing order by y_v value. Let v_i be the i th vertex in this ordering, let $k := |S_{\geq 0}|$, $y_{k+1} := 0$, $y_i := y_{v_i}$, $c_i := c_{v_i}^G$, and $S_i := \{v_1, v_2, \dots, v_i\}$ for each integer $i \in [k]$. Then

$$\begin{aligned} \sum_{v \in S_{\geq 0}} c_v^G y_v^2 &= \sum_{i=1}^k c_i y_i^2 \\ &= \sum_{i=1}^k (\text{vol}_G(S_i) - \text{vol}_G(S_{i-1})) y_i^2 \\ &= \sum_{i=1}^k \text{vol}_G(S_i) (y_i^2 - y_{i+1}^2) \\ &= 2 \int_0^\infty \text{vol}_G(S_{\geq q}) q dq \end{aligned}$$

By our volume upper bound from up above,

$$\begin{aligned} 2 \int_0^\infty \text{vol}_G(S_{\geq q}) q dq &\leq 2 \int_0^\infty \frac{\widehat{c}^I q(S_{\geq q}, S_{\leq q/2})}{640 \lambda_G} q dq + 8 \int_0^\infty c^G(\partial S_{\geq q}) q dq + 8 \int_0^\infty c^G(\partial S_{\leq q/2}) q dq \\ &= 2 \int_0^\infty \frac{\widehat{c}^I q(S_{\geq q}, S_{\leq q/2})}{640 \lambda_G} q dq + 8 \int_0^\infty c^G(\partial S_{\geq q}) q dq + 8 \int_0^\infty c^G(\partial S_{> q/2}) q dq \\ &= 2 \int_0^\infty \frac{\widehat{c}^I q(S_{\geq q}, S_{\leq q/2})}{640 \lambda_G} q dq + 40 \int_0^\infty c^G(\partial S_{\geq q}) q dq \end{aligned}$$

Substitution and Proposition 14 show that

$$\begin{aligned} 2 \int_0^\infty \text{vol}_G(S_{\geq q}) q dq &\leq 8 \sum_{e=uv \in E(G)} c_e^G \int_0^\infty \left(\frac{(\kappa_q(y_u) - \kappa_q(y_v))^2}{640 \lambda_G q} + 5 \mathbb{1}_{q \in [y_u, y_v]} q \right) dq \\ &\leq 8 \sum_{e=uv \in E(G)} c_e^G \left(\frac{10}{640 \lambda_G} (y_u - y_v)^2 + 5 |y_u^2 - y_v^2| \right) \end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned} 8 \sum_{e=uv \in E(G)} c_e^G \left(\frac{10}{640 \lambda_G} (y_u - y_v)^2 + 5 |y_u^2 - y_v^2| \right) &\leq \frac{1}{8 \lambda_G} \sum_{e=uv \in E(G)} c_e^G (y_u - y_v)^2 \\ &\quad + 40 \sqrt{\sum_{e=uv \in E(G)} c_e^G (y_u - y_v)^2} \sqrt{\sum_{e=uv \in E(G)} c_e^G (y_u + y_v)^2} \\ &\leq \frac{1}{4} \sum_{v \in V(G)} c_v^G x_v^2 \\ &\quad + 80 \sqrt{\lambda_G} \sqrt{\sum_{v \in V(G)} c_v^G x_v^2} \sqrt{\sum_{v \in V(G)} c_v^G y_v^2} \end{aligned}$$

But since $\sum_{v \in V(G)} c_v^G x_v^2 \leq \sum_{v \in V(G)} c_v^G y_v^2$ and $\lambda_G < 1/25600$,

$$\frac{1}{4} \sum_{v \in V(G)} c_v^G x_v^2 + 80\sqrt{\lambda_G} \sqrt{\sum_{v \in V(G)} c_v^G x_v^2} \sqrt{\sum_{v \in V(G)} c_v^G y_v^2} < \frac{1}{2} \sum_{v \in V(G)} c_v^G y_v^2.$$

Negating y shows that $\sum_{v \in S_{\leq 0}} c_v^G y_v^2 < 1/2 \sum_{v \in V(G)} c_v^G y_v^2$ as well. But these statements cannot both hold; a contradiction. Therefore, **SweepCut** must output a pair of sets.

Runtime. Computing z takes $\tilde{O}(m)$ time by Theorem 9. Therefore, it suffices to show that the foreach loops can each be implemented in $O(m)$ time. This implementation is similar to the $O(m)$ -time implementation of the Cheeger sweep.

We focus on the first foreach loop, as the second is the same with q negated. First, note that the functions $\phi_{S_{\geq q}}$, $c^G(\partial S_{\geq q/2})$, and $\text{vol}_G(S_{\geq q})$ of q are piecewise constant, with breakpoints at $q = y_u$ and $q = 2y_u$ for each $u \in V(G)$. Furthermore, these functions can be computed for all values in $O(m)$ time using an $O(m)$ -time Cheeger sweep for each function.

Therefore, it suffices to compute the value of $\tilde{c}^{Iq}(S_{\leq q/2}, S_{\geq q})$ for all $q \geq 0$ that are local minima in $O(m)$ time. Let $h(q) := \tilde{c}^{Iq}(S_{\leq q/2}, S_{\geq q})$. Notice that the functions $h(q)$ and $h'(q)$ are piecewise quadratic and linear functions of q respectively, with breakpoints at $q = y_u$ and $q = 2y_u$. Using five $O(m)$ -time Cheeger sweeps, one can compute the q^2, q and 1 coefficients of $h(q)$ and the q and 1 coefficients of $h'(q)$ between all pairs of consecutive breakpoints. After computing these coefficients, one can compute the value of each function at a point q in $O(1)$ time. Furthermore, given two consecutive breakpoints a and b , one can find all points $q \in (a, b)$ with $h'(q) = 0$ in $O(1)$ time. Each local minimum for h is either a breakpoint or a point with $h'(q) = 0$. Since h and h' have $O(n)$ breakpoints, all local minima can be computed in $O(n)$ time. h can be evaluated at all of these points in $O(n)$ time. Therefore, all local minima of h can be computed in $O(m)$ time. Since the algorithm does return a q , some local minimum for h also suffices, so this implementation produces the desired result in $O(m)$ time.

Low Schur complement fractional conductance. By Proposition 13,

$$c^{Iq}(S_{\geq q}, S_{\leq q/2}) \leq \tilde{c}^{Iq}(S_{\geq q}, S_{\leq q/2})$$

Therefore, $c^{Iq}(S_{\geq q}, S_{\leq q/2}) \leq 640\lambda_G \min(\text{vol}_G(S_{\geq q}), \text{vol}_G(S_{\leq q/2}))$ for $q \geq 0$ by the foreach loop if condition. Repeating this reasoning for $q < 0$ yields the desired result.

Large interior. By definition of α , $\text{vol}_G(S_{\geq q}) \leq \text{vol}_G(S_{\leq q/2})$ for $q > 0$. Since $c^G(\partial S_{\leq q/2}) \leq 1/4\text{vol}_G(S_{\geq q})$, $\phi_{S_{\leq q/2}} \leq 1/4$, as desired. \blacktriangleleft

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A Proof of Theorem 1

Proof of Theorem 1. For any two sets of vertices S_1, S_2 in a graph G ,

$$\text{Reff}_G(S_1, S_2) \min(\text{vol}_G(S_1), \text{vol}_G(S_2)) = \frac{1}{\sigma_{S_1, S_2}^G}$$

Therefore, the desired result follows from Lemmas 11 and 10. ◀

B Proof of Proposition 14

Proof of Proposition 14. Without loss of generality, suppose that $a \leq b$. We break the analysis up into cases:

Case 1: $a \leq 0$. In this case, $\kappa_q(a) = q/2$ for all $q \geq 0$, so

$$\begin{aligned} \int_0^\infty \frac{(\kappa_q(a) - \kappa_q(b))^2}{q} dq &= \int_0^b \frac{(q/2 - q)^2}{q} dq \\ &\quad + \int_b^{2b} \frac{(q/2 - b)^2}{q} dq \\ &\quad + \int_{2b}^\infty \frac{(q/2 - q/2)^2}{q} dq \\ &= \frac{b^2}{8} + \int_b^{2b} (q/4 - b + b^2/q) dq \\ &= \frac{b^2}{2} - b^2 + b^2(\ln 2) \\ &\leq 10(a - b)^2 \end{aligned}$$

as desired.

Case 2: $a > 0$ and $b \leq 2a$. In this case,

$$\begin{aligned}
\int_0^\infty \frac{(\kappa_q(a) - \kappa_q(b))^2}{q} dq &= \int_0^a \frac{(q-q)^2}{q} dq \\
&+ \int_a^b \frac{(a-q)^2}{q} dq \\
&+ \int_b^{2a} \frac{(a-b)^2}{q} dq \\
&+ \int_{2a}^{2b} \frac{(q/2-b)^2}{q} dq \\
&+ \int_{2b}^\infty \frac{(q/2-q/2)^2}{q} dq \\
&\leq \int_a^{2b} \frac{(a-b)^2}{q} dq \\
&= (a-b)^2 \ln(2b/a) \\
&\leq (a-b)^2 \ln 4 \leq 10(a-b)^2
\end{aligned}$$

as desired.

Case 3: $a > 0$ and $b > 2a$. In this case,

$$\begin{aligned}
\int_0^\infty \frac{(\kappa_q(a) - \kappa_q(b))^2}{q} dq &= \int_0^a \frac{(q-q)^2}{q} dq \\
&+ \int_a^{2a} \frac{(a-q)^2}{q} dq \\
&+ \int_{2a}^b \frac{(q/2-q)^2}{q} dq \\
&+ \int_b^{2b} \frac{(q/2-b)^2}{q} dq \\
&+ \int_{2b}^\infty \frac{(q/2-q/2)^2}{q} dq \\
&\leq \int_a^{2b} \frac{(q/2-q)^2}{q} dq \\
&\leq b^2/2 \\
&\leq 2(a-b)^2 \leq 10(a-b)^2
\end{aligned}$$

as desired. ◀