Balanced Judicious Bipartition is Fixed-Paramater Tractable*

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Abstract

The family of judicious partitioning problems, introduced by Bollobás and Scott to the field of extremal combinatorics, has been extensively studied from a structural point of view for over two decades. This rich realm of problems aims to counterbalance the objectives of classical partitioning problems such as Min Cut, Min Bisection and Max Cut. While these classical problems focus solely on the minimization/maximization of the number of edges crossing the cut, judicious (bi)partitioning problems ask the natural question of the minimization/maximization of the number of edges lying in the (two) sides of the cut. In particular, Judicious Bipartition (JB) seeks a bipartition that is "judicious" in the sense that neither side is burdened by too many edges, and Balanced JB also requires that the sizes of the sides themselves are "balanced" in the sense that neither of them is too large. Both of these problems were defined in the work by Bollobás and Scott, and have received notable scientific attention since then. In this paper, we shed light on the study of judicious partitioning problems from the viewpoint of algorithm design. Specifically, we prove that BJB is FPT (which also proves that JB is FPT).

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1 Introduction

More than twenty years ago, Bollobás and Scott [3] defined the notion of *judicious partitioning problems*. Since then, the family of judicious partitioning problems has been extensively studied in the field of Extremal Combinatorics, as can be evidenced by the abundance of structural results described in surveys such as [7, 35]. This rich realm of problems aims to counterbalance the objectives of classical partitioning problems such as Min Cut, Min Bisection, Max Cut and Max Bisection. While these classical problems focus

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solely on the minimization/maximization of the number of edges crossing the cut, judicious (bi)partitioning problems ask the natural questions of the minimization/maximization of the number of edges lying in the (two) sides of the cut. Another significant feature of judicious partitioning problems that also distinguishes them from other classical partitioning problems is that they inherently and naturally encompass several objectives, aiming to minimize (or maximize) the number of edges in several sets *simultaneously.*

In this paper, we shed light on properties of judicious partitioning problems from the viewpoint of the design of algorithms. Up until now, the study of such problems has essentially been overlooked at the algorithmic front, where one of the underlying reasons for this discrepancy might be that standard machinery does not seem to handle them effectively. Specifically, we focus on the Judicious Bipartition problem, where we seek a bipartition that is "judicious” in the sense that neither side is burdened by too many edges, and on the Balanced Judicious Bipartition problem, where we also require that the sizes of the sides themselves are “balanced” in the sense that neither of them is too large. Both of these problems were defined in the work by Bollobás and Scott, and have received notable scientific attention since then. Formally, Balanced Judicious Partition is defined as follows.

Balanced Judicious Bipartition (BJB) **Parameter:** $k_1 + k_2$
Input: A multi-graph G, and integers μ, k_1 and k_2
Question: Does there exist a partition (V_1, V_2) of $V(G)$ such that $|V_1| = \mu$ and for all $i \in \{1, 2\}$, it holds that $|E(G[V_i])| \leq k_i$?

We note that in the literature, the term BJB refers to the case where $\mu = \lceil \frac{|V(G)|}{2} \rceil$, and hence it is more restricted then the definition above. By dropping the requirement that $|V_1| = \mu$, we get the Judicious Bipartition (JB) problem. By using new crucial insights into these problems on top of the most advanced machinery in Parameterized Complexity to handle partitioning problems,[1] we are able to resolve the question of the Parameterized Complexity of BJB (and hence also of JB). In particular, we prove the following theorem.

▶ **Theorem 1.** BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot |V(G)|^{\mathcal{O}(1)}$.

Structural Results

Denote $n = |V(G)|$ and $m = |E(G)|$. To survey several structural results about judicious partitioning problems, we first define the notions of *t-cut* and *max (min) t-judicious partitioning.* Given a partition of $V(G)$ into t parts, a t-cut is the number of edges going across the parts, while a max (min) judicious t-partitioning is the maximum (minimum) number of edges in any of the parts. When $t = 2$, we use the standard terms *bipartite-cut* and *judicious bipartitioning*, respectively. Furthermore, by t-judicious partitioning we mean max t-judicious partitioning. As stated earlier, Bollobás and Scott [3] defined the notion of *judicious partitioning problems* in 1993. In that paper, they showed that for any positive integer t and graph G, we can partition $V(G)$ into t sets, $V_1, \dots, V_t$, so that $|E(G[V_i])| \leq \frac{t}{t+1}m$ for all $i \in \{1, \dots, t\}$. Bollobás and Scott also studied this problem on graphs of maximum degree Δ, and showed that there exists a partition of $V(G)$ into t sets $V_1, \dots, V_t$ so that it simultaneously satisfies an upper bound and a lower bound on the number of edges in each part as well as on edges between every pair of parts. Later,

[1] To the best of our knowledge, up until now, this machinery has actually only been proven useful to solve one natural problem which could not have been tackled using earlier tools.

Bollobás and Scott [7] gave several new results, leaving open other new questions around judicious partitioning. In [8] they showed an optimal bound for judicious partitioning on bounded-degree graphs. These problems have also been studied on general hypergraphs [4], uniform hypergraphs [23], 3-uniform hypergraphs [6] and directed graphs [25].

The special cases of judicious partitioning problems called judicious bipartitioning and balanced judicious bipartitioning problems have also been studied intensively. Bollobás and Scott [5] proved an upper bound on judicious bipartitioning and proved that every graph that achieves the essentially best known lower bound on bipartite-cut, given by Edwards in [17] and [18], also achieves this upper bound for judicious bipartitioning. In fact, they showed that this is exact for complete graphs of odd order, which are the only extremal graphs without isolated vertices. Alon et al. [1] gave a non-trivial connection between the size of a bipartite-cut in a graph and judicious partitioning into two sets. In particular, they showed that if a graph has a bipartite-cut of size at least $\frac{m}{2} + \delta$ where $\delta \leq m/30$, then there exists a bipartition (V_1, V_2) of $V(G)$ such that $|E(G[V_i])| \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}$ for $i \in \{1,2\}$. They complemented these results by showing an upper bound on the number of edges in each part when $\delta > m/30$. Bollobás and Scott [9] studied similar relations between t-cuts and t-judicious partitionings for $t \geq 3$. Recently, these results were further refined [38, 28]. Xu et al. [37] and Xu and Yu [39] studied balanced judicious bipartitioning where both parts are of almost equal size (that is, one of the sizes is $\lceil \frac{n}{2} \rceil$). Both of these papers concern the following conjecture of Bollobás and Scott [7]: if G is a graph with minimum degree of at least 2, then $V(G)$ admits a balanced bipartition (V_1, V_2) such that for each $i \in \{1,2\}$, $|E(G[V_i])| \leq \frac{m}{3}$. For further results on judicious partitioning, we refer to the surveys [7, 35].

Algorithmic Results

While classical partitioning problems such as Min Cut, Min Bisection, Max Cut and Max Bisection have been studied extensively algorithmically, the same is not true about judicious partitioning problems. Apart from Min Cut, all the above mentioned partitioning problems are NP-complete. These NP-complete partitioning problems were investigated by all algorithmic paradigms meant for coping with NP-complete, including approximation algorithms and parameterized complexity. In what follows, we discuss known results related to these problems in the realm of parameterized complexity.

First, note that for every graph G, there always exists a bipartition of the vertex set into two parts (in fact equal parts [21, Corollory 1]) such that at least $m/2$ edges are going across. This immediately implies that Max Cut and Max Bisection are FPT when parameterized by the cut size (the number of edges going across the partition). This led Mahajan and Raman [29] to introduce the notion of above-guarantee parameterization. In particular, they showed that one can decide whether a graph has a bipartite-cut of size $\frac{m}{2} + k$ in time $\mathcal{O}(m + n + k4^k)$. However, Edwards [17] showed that every connected graph G has a bipartite-cut of size $\frac{m}{2} + \frac{n-1}{4}$. Thus, a more interesting question asks whether finding a bipartite-cut of size at least $\frac{m}{2} + \frac{n-1}{4} + k$ is FPT. Crowston et al. [15] showed that indeed this is the case as they design an algorithm with running time $\mathcal{O}(8^k n^4)$. Recently, Etscheid and Mnich [19] discovered a kernel with a linear number of vertices (improving upon a kernel by Crowston et al. [14]), and the aforementioned algorithm was sped-up to run in time $\mathcal{O}(8^k m)$ [19]. Gutin and Yeo studied an above-guarantee version of Max Bisection [21], proving that finding a balanced bipartition such that it has at least $\frac{m}{2} + k$ edges is

FPT (also see [33]).[2] In this context MAX BISECTION, it is also relevant to mention the $(k, n-k)$-MAX CUT, which asks for a bipartite-cut of size at least p where one of the sides is of size exactly k. Parameterized by k, this problems is W[1]-hard [11], but parameterized by p, this problem is solvable in time $\mathcal{O}^*(2^p)$ [34] (this result improved upon algorithms given in [10, 36]).

Until recently, the parameterized complexity of MIN BISECTION was open. Approaches to tackle this problem materialized when the parameterized complexity of ℓ-WAY CUT was resolved. Here, given a graph G and positive integers k and ℓ, the objective is to delete at most k edges from G such that it has at least ℓ components. Kawarabayashi and Thorup [24] showed that this problem is FPT. Later, Chitnis et al. [13] developed a completely new tool based on this, called *randomized contractions*, to deal with plethora of cut problems. Other cut problems that have been shown to be FPT include the generalization of MIN CUT to MULTIWAY CUT and MULTICUT [12, 31, 32]. Eventually, Cygan et al. [16], combining ideas underlying the algorithms developed for MULTIWAY CUT, MULTICUT, ℓ-WAY CUT and randomized contractions together with a new kind of decomposition, showed MIN BISECTION to be FPT. Finally, let us also mention the min c-judicious partitioning (which is a maximization problem), called c-LOAD COLORING, where given a graph G and a positive integer k, the goal is to decide whether $V(G)$ can be partitioned into c parts so that each part has at least k edges. Barbero et al. [2] showed that this problem is FPT (also see [20]).

Despite the abundance of work described above, the parameterized complexity of JB and BJB has not yet considered. We fill this gap in our studies by showing that both of these problems are FPT. It is noteworthy to remark that one can show that the generalization of MIN BISECTION to c-MIN BISECTION, where the objective is to find a partition into c-parts such that each part are almost equal and there are at most k edges going across different parts, is FPT [16]. However, such a generalization is not possible for either JB or BJB. Indeed, even the existence of an algorithm with running time $n^{f(k)}$, for any arbitrary function f, would imply a polynomial-time algorithm for 3-COLORING, where k is set to 0.

Our Approach

For the sake of readability, our strategy of presentation of our proof consists of the definition of a series of problems, each more "specialized" (in some sense) than the previous one, where each section shows that to eventually solve BJB, it is sufficient to focus on some such problem rather than the previous one. We start by showing that we can focus on the solution of the case of BJB where the input graph is bipartite at the cost of the addition of annotations. For this purpose, we present a (not complicated) Turing reduction that employes a known algorithm for the ODD CYCLE TRANSVERSAL problem (see Section 3). The usefulness of the ability to assume that the input graph is bipartite is a key insight in our approach. In particular, the technical parts of our proof crucially rely on the observation that a connected bipartite graph has only two bipartitions (here, we consider bipartitions as ordered pairs). Keeping this intuition in mind, our next step is to reduce the current annotated problem to one where the input graph is also assumed to be connected (this specific argument relies on a simple application of dynamic programming).

Having at hand an (annotated) problem where the input graph is assumed to be a connected bipartite graph, we proceed to the technical part of our proof, which employs the (heavy) machinery developed by Cygan et al. [16]. While this machinery primarily

[2] We refer to surveys [30, 22] for details regarding above-guarantee parameterizations.

aims to tackle problems where one seeks small cuts in addition to some size constraint, our problem involves a priori seemingly different type of constraints. Nevertheless, we observe that once we handle a connected graph, the removal of any set of k edges (to deal with the size constraint and annotations) would not break the graph to more than $k+1$ connected components, and each of these components would clearly be a bipartite graph. Hence, we can view (in some sense) our problem as a cut problem. In practice, the relation between our problem and a cut problem is quite more intricate, and to realize our idea, we crucially rely on the fact that the connected components are bipartite graphs, which allows us to "guess" a binary vector specifying the biparition of their vertex sets in the final solution. This operation entitles the employment of coloring functions (employing $k+1$ colors) and their translation into bipartitions (which at a certain point in our paper, we would start viewing as colorings employing two colors). Let us remark that the machinery introduced by [16] is the computation of a special type of tree decomposition. Accordingly, our approach would eventually involve the introduction of a specialization of BJB that aims to capture the work to perform when handling a bag of the tree decomposition. The definition of this specific problem is very technical, and hence we defer the description of related intuitive explanations to the appropriate locations in Section 5, where we have already set up the required notations to discuss it.

Note that the proofs of statements marked by ($\star$) are omitted due to space constraints and can be found in the full version of the paper [27].

2 Preliminaries

Let $f : A \to B$ be some function. Given $A' \subseteq A$, the notation $f(A') = b$ indicates that for all $a \in A'$, it holds that $f(a) = b$. An *extension* f' of the function f is a function whose domain A' is a superset of A and whose range is B, such that for all $a \in A$, it holds that $f'(a) = f(a)$. For any $A' \subseteq A$, the *restriction* $f|_{A'}$ of f is a function from A' to B such that for any $a \in A'$, $f|_{A'}(a) = f(a)$. Bold face lower-case letters are used to denote tuples (vectors). For any tuple $\mathbf{v}$, we let $\mathbf{v}[i]$ denote the ith coordinate of $\mathbf{v}$. Given some condition ψ, we define $[\psi] = 1$ if ψ is true and $[\psi] = 0$ otherwise. For any positive integer x, we denote by $[x]$ the set $\{1, 2, \ldots, x\}$ and by $[x]_0$ the set $\{0, 1, \ldots, x\}$.

Given a graph G, we let $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of G, respectively. For a subset $A \subseteq V(G)$, we denote by $\delta(A)$ the set of boundary vertices of A, that is, $\delta(A) = \{v \in A : \text{there exists } u \in V(G) \setminus A \text{ such that } \{u, v\} \in E(G)\}$. We let $G \setminus A$ denote the subgraph of G induced by $V(G) \setminus A$. A *bipartite graph* is a graph G such that there exists a bipartition (X, Y) of $V(G)$ where X and Y are independent sets. In this paper, we treat such bipartitions as *ordered pairs*. That is, if (X, Y) is a bipartition of some bipartite graph G, then (Y, X) is assumed to be a *different* bipartition of the graph G. For *connected* bipartite graphs, we have the following simple yet powerful insight.

▶ **Proposition 1** (Folklore). *Any connected bipartite graph G has exactly 2 bipartitions, (X, Y) and (Y, X).*

The *treewidth* of a graph aims to measure how close the graph is to a tree. Formally, this notion is defined as follows. A *tree decomposition* of a graph G is a pair (T, β) such that T is a rooted tree, $\beta : V(T) \to 2^{V(G)}$, and the following conditions are satisfied.

1. For all $\{u, v\} \in E(G)$, there exists $t \in V(T)$ such that $u, v \in \beta(t)$.
2. For all $v \in V(G)$, the subgraph of T induced by $X_v = \{t : v \in \beta(t)\}$ is a (connected) subtree of T on at least one node.

Given $t, \widehat{t} \in V(G)$, the notation $\widehat{t} \preceq t$ indicates that $\widehat{t}$ is a descendant of t in T. Note that t is a descendant of itself. For any $t \in V(T)$, let t' denote the unique parent of t in T. We also need the standard notations $\sigma(t) = \beta(t) \cap \beta(t')$ and $\gamma(t) = \bigcup_{\widehat{t} \preceq t} \beta(\widehat{t})$.

▶ **Proposition 2** (Folklore). *Let (T, β) be a tree decomposition of a graph G. Given a node $t \in V(T)$, let $t_1, \ldots, t_s$ denote the children of t in T, and for all $i \in [s]$, define $V_{t_i} = \gamma(t_i) \setminus \beta(t)$. Let $V_{t'} = V(G) \setminus (\beta(t) \cup \bigcup_{i=1}^{s} V_{t_i})$. Then, the vertex-set of each connected component of $G \setminus \beta(t)$ is a subset of one of $V_{t_1}, \ldots, V_{t_s}, V_{t'}$.*

Let H be some hypergraph. A s *panning forest* of H is a subset $E' \subseteq E(H)$ of minimum size such that the set containing all endpoints of the hyperedges in E' is equal to $V(H)$. In this paper, we implicitly assume that hypergraphs contain no isolated vertices.

A *separation* of a graph G is a pair (X, Y) such that $X, Y \subseteq V(G)$ and $X \cup Y = V(G)$. The *order of a separation* (X, Y) is equal to $|X \cap Y|$. Let G be a graph, $A \subseteq V(G)$, and $q, k \in \mathbb{N}$. The set A is said to be *(q, k)-unbreakable* in G if for *every* separation (X, Y) of G of order at most k, either $|(X \setminus Y) \cap A| \leq q$ or $|(Y \setminus X) \cap A| \leq q$. We also define a notion of unbreakability in the context of functions. A function $g : U \to [k]_0$ is called *(q, k)-unbreakable* if there exists $i \in [k]_0$ such that $\sum_{j \in [k]_0 \setminus \{i\}} |g^{-1}(j)| \leq q$. Let us now claim that there do no exist "too many" (q, k)-unbreakable functions.

▶ **Lemma 2** (⋆). *For all $q, k \in \mathbb{N}$, the number of (q, k)-unbreakable functions from a universe U to $[k]_0$ is upper bounded by $\sum_{l=0}^{q} \binom{|U|}{l} \cdot q^k \cdot (k + 1)$.*

3 Solving Balanced Judicious Bipartition

In this section, we prove Theorem 1 under the assumption that we are given an algorithm for an annotated, yet restricted, variant of BJB. Throughout this section, an instance of BJB is denoted by BJB(G, μ, k_1, k_2), and we define $k = k_1 + k_2$. Given a partition (V_1, V_2) that witnesses that an instance BJB(G, μ, k_1, k_2) is a YES-instance, we think of the vertices in V_1 as colored 1 and the vertices in V_2 as colored 2; hence, we call such a partition a *witnessing coloring* of BJB(G, μ, k_1, k_2). To prove Theorem 1, we first define the Odd Cycle Transversal problem. Here, given a graph G, a set $S \subseteq V(G)$ is called an *odd cycle transversal* if $G \setminus S$ is a bipartite graph.

Odd Cycle Transversal (OCT) **Parameter:** k
Input: An undirected multi-graph graph G, and an integer k.
Question: Does G have an odd cycle transversal of size at most k?

An instance of Odd Cycle Transversal is denoted by OCT(G, k). The algorithm given by the result below shall be a central component in the design of our algorithm for BJB.

▶ **Proposition 3** ([26]). Odd Cycle Transversal *can be solved in time $2.3146^k n^{\mathcal{O}(1)}$.*

Apart from OCT, we also need to define an auxiliary problem that we call Annotated Bipartite-BJB (AB-BJB). As we proceed with our proofs, we shall continue defining auxiliary problems, where each problem captures a task more specific and technically more challenging than the previous one. The choice of this structure aims to ease the readability of our paper. Intuitively, AB-BJB is basically the BJB problem on bipartite graphs, with

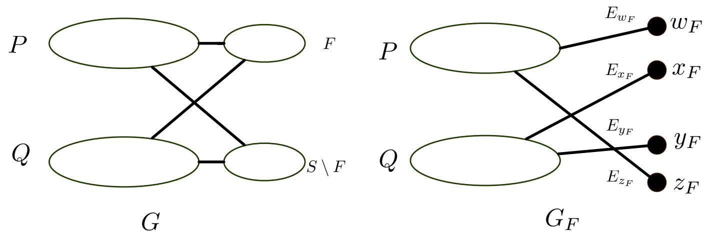

Figure 1 The construction in the proof of Theorem 1.

an extra constraint that demands that certain vertices are assigned a particular color by the witnessing coloring. We remark that the necessity of the reduction to bipartite graphs stems from the fact that we would like to employ Proposition 1 later. The formal definition of AB-BJB is given below.

Annotated Bipartite-BJB (AB-BJB) **Parameter:** $k_1 + k_2$
Input: A bipartite multi-graph G with bipartition (P, Q), $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$, and integers μ, k_1 and k_2.
Question: Does there exist a partition (V_1, V_2) of $V(G)$ such that $A \subseteq V_1$, $B \subseteq V_2$, $|V_1| = \mu$ and for $i \in \{1, 2\}$, $|E(G[V_i])| \leq k_i$?

An instance of AB-BJB is denoted by AB-BJB(G, A, B, μ, k_1, k_2). A partition (V_1, V_2) satisfying the above properties is called a *witnessing coloring* of AB-BJB(G, A, B, μ, k_1, k_2). Furthermore, we need the following theorem, proven later in this paper.

▶ **Theorem 3.** AB-BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$.

Let us now turn to focus on the proof of Theorem 1.

Proof of Theorem 1. Given an instance BJB(G, μ, k_1, k_2), call the algorithm given by Proposition 3 with the instance OCT(G, k) as input.

▶ **Claim 1.** *If* OCT(G, k) *is a* NO*-instance, then* BJB(G, μ, k_1, k_2) *is a* NO*-instance.*

Proof. Suppose BJB(G, μ, k_1, k_2) is a YES-instance. Let (V_1, V_2) be a witnessing coloring for this instance. Let $E' = E(G[V_1]) \cup E(G[V_2])$. Then, observe that $G \setminus E'$ is a bipartite graph. Let V' be a set of vertices of minimum size such that every edge in E' has at least one endpoint in V'. Since $|E'| \leq k$, it holds that $|V'| \leq k$. Moreover, $G \setminus V'$ is bipartite. Therefore, V' is an odd cycle transversal of G of size at most k. Thus, OCT(G, k) is a YES-instance. ◀

Henceforth, let S be an odd cycle transversal of G of size at most k. Then, $G \setminus S$ is a bipartite graph. Fix some bipartition (P, Q) of $G \setminus S$. Let $\mathcal{F}$ be the family of all subsets of S, that is, $\mathcal{F} = 2^S$. For any $F \in \mathcal{F}$, denote $l_1^F = |E(G[F])|$ and $l_2^F = |E(G[S \setminus F])|$, and let G_F be the graph constructed as follows (see Fig. 1).

- $V(G_F) = V(G \setminus S) \cup \{w_F, x_F, y_F, z_F\}$, where w_F, x_F, y_F, z_F are new distinct vertices.
- $E(G_F) = E(G \setminus S) \cup E_{w_F} \cup E_{x_F} \cup E_{y_F} \cup E_{z_F}$, where the *multisets* $E_{w_F}, E_{x_F}, E_{y_F}$ and E_{z_F} are defined as follows.
  - $E_{w_F} = \{e_u = (w_F, u) : u \in P,$ and there exists $v \in F$ such that $(u, v) \in E(G)\}$,
  - $E_{x_F} = \{e_u(x_F, u) : u \in Q,$ and there exists $v \in F$ such that $(u, v) \in E(G)\}$,
  - $E_{y_F} = \{e_u = (y_F, u) : u \in Q,$ and there exists $v \in S \setminus F$ such that $(u, v) \in E(G)\}$,
  - $E_{z_F} = \{e_u = (z_F, u) : u \in P,$ and there exists $v \in S \setminus F$ such that $(u, v) \in E(G)\}$.

Observe that G_F is a bipartite graph with $(P \cup \{x_F, y_F\}, Q \cup \{w_F, z_F\})$ as a bipartition.

▶ **Claim 2.** BJB(G, μ, k_1, k_2) *is a* YES-*instance if and only if there exists* $F \in \mathcal{F}$ *such that* AB-BJB$(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$ *is a* YES-*instance.*

Proof. In the forward direction, suppose that BJB(G, μ, k_1, k_2) is a YES-instance, and let (V_1, V_2) be a witnessing coloring for BJB(G, μ, k_1, k_2). Moreover, let $F = V_1 \cap S$. Now, we define a partition (V_1', V_2') of $V(G_F)$ as follows: $V_1' = (V_1 \setminus S) \cup \{x_F, y_F\}$ and $V_2' = (V_2 \setminus S) \cup \{w_F, z_F\}$. Let us now argue that (V_1', V_2') is a witnessing coloring for AB-BJB$(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$. First, by the construction of (V_1', V_2'), we have that $\{x_F, y_F\} \subseteq V_1'$ and $\{w_F, z_F\} \subseteq V_2'$. Second, as $V_1' = (V_1 \setminus S) \cup \{x_F, y_F\}$, we also have that $|V_1'| = |V_1| - |F| + 2 = \mu + |F| + 2$. Third, observe that for any $i \in \{1, 2\}$, $|E(G[V_i'])| = |E(G[V_i])| - |E(G[F])|$. Thus, $|E(G[V_i])| \leq k_i - l_i^F$.

In the backward direction, suppose that there exists an $F \in \mathcal{F}$ such that AB-BJB$(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$ is a YES-instance, and let (V_1', V_2') be a witnessing coloring for this instance. We now define a partition (V_1, V_2) of $V(G)$ as follows: $V_1 = (V_1' \cap V(G)) \cup F$ and $V_2 = (V_2' \cap V(G)) \cup (S \setminus F)$. Let us now argue that (V_1, V_2) is a witnessing coloring for BJB(G, μ, k_1, k_2). From the definition of V_1, and since $V(G) = (V(G_F) \setminus \{w_F, x_F, y_F, z_F\}) \cup F$ and $F \cap V(G_F) = \emptyset$, we have that $|V_1| = |V_1'| - |\{x_F, y_F\}| + |F| = \mu - |F| + 2 - 2 + |F| = \mu$. Moreover, observe that $|E(G[V_1])| = |E(G[V_1'])| + |E(G[F])| \leq k_1 + l_1^F$ and $|E(G[V_2])| = |E(G[V_2'])| + |E(G[S \setminus F])| \leq k_2 + l_2^F$. This concludes the proof of the claim. ◀

Thus, to solve an instance of BJB, it is enough to solve $2^{|S|} \leq 2^k$ instances of AB-BJB. Hence, by Theorem 3, BJB can be solved in time $2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$. ◀

4 Solving Annotated Bipartite-BJB

Recall the problem definition of Annotated Bipartite-BJB (AB-BJB) from Section 3. In this section, we prove Theorem 3. For this purpose, let us define another auxiliary problem, which we call Annotated Bipartite Connected-BJB (ABC-BJB). Intuitively, ABC-BJB is exactly the same problem as AB-BJB where we are interested in an answer for every choice of $\mu \in [n]_0$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, and additionally we demand the input graph to be connected.

Annotated Bipartite Connected-BJB (ABC-BJB) **Parameter:** $k_1 + k_2$
Input: A connected bipartite multi-graph $G = (P, Q)$, $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$, and integers k_1 and k_2.
Output: For all $\mu \in [n]_0$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, output a binary value, $\mathtt{aJP}[\mu, l_1, l_2]$, which is 1 if and only if there exists a partition (V_1, V_2) of $V(G)$ such that

- $A \subseteq V_1$ and $B \subseteq V_2$,
- $|V_1| = \mu$, and
- for $i \in \{1, 2\}$, $|E(G[V_i])| \leq l_i$.

For any $\mu \in [n]_0$, $l_1 \in [k_1]_0$, $l_2 \in [k_2]_0$, a partition witnessing that $\mathtt{aJP}[\mu, l_1, l_2] = 1$ is called a *witnessing coloring* for $\mathtt{aJP}[\mu, l_1, l_2] = 1$. Moreover, an instance of ABC-BJB is denoted by ABC-BJB(G, A, B, k_1, k_2). In the rest of this paper, we prove the following theorem.

▶ **Theorem 4.** ABC-BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$.

Having Theorem 4 at hand, a simple application of the method of dynamic programming results in the proof of Theorem 3 (see full version of the paper [27]).

5 Solving Annotated Bipartite Connected-BJB

Recall the problem definition of ABC-BJB from Section 5. In this section, we prove Theorem 4. Let us start by stating a known result that is a crucial component of our proof. By this result, we would have an algorithm that efficiently computes a special type of tree decomposition, that we call a *highly connected tree decomposition*, where every bag is "highly-connected" rather than "small" as in the case of standard tree decompositions. While this property is the main feature of this decomposition, it is also equipped with other beneficial properties, such as a (non-trivial) upper bound on the size of its adhesions, which are *all* exploited by our algorithm.

▶ **Theorem 5** ([16]). *There exists an $2^{\mathcal{O}(k^2)}n^2m$-time algorithm that, given a* connected *graph G together with an integer k, computes a tree decomposition (T, β) of G with at most n nodes such that the following conditions hold, where $\eta = 2^{\mathcal{O}(k)}$.*

1. *For each $t \in V(T)$, the graph $G[\gamma(t) \setminus \sigma(t)]$ is connected and $N(\gamma(t) \setminus \sigma(t)) = \sigma(t)$.*
2. *For each $t \in V(T)$, the set $\beta(t)$ is (η, k)-unbreakable in $G[\gamma(t)]$.*
3. *For each non-root $t \in V(T)$, we have that $|\sigma(t)| \leq \eta$ and $\sigma(t)$ is $(2k, k)$-unbreakable in $G[\gamma(parent(t))]$.*

In order to process such a tree decomposition in a bottom-up fashion, relying on the method of dynamic programming, we need to address a specific problem associated with every bag, called Hypergraph Painting (HP). We chose the name HP to be consistent with the choice of problem name in [16], yet we stress that our problem is more general than the one in [16] (since the handling of a bag in our case is more intricate than the one in [16]).

Roughly speaking, an input of HP would consist of the following components. First, we are given "budget" parameters k_1 and k_2 as in an instance of ABC-BJB. Second, we are given an argument b which would simply be n (to upper bound $|\gamma(t)|$) when we construct an instance of HP while processing some node t in the tree decomposition. Third, we are given a hypergraph H which would essentially be the graph $G[\beta(t)]$ to which we add hyperedges that are supposed to represent the sets $\sigma(\widehat{t})$ for the children $\widehat{t}$ of t. Fourth, we are given an integer q whose purpose is clarified in the discussion below the definition of HP (in Definition 8). Finally, for every hyperedge F, we are given a function $f_F : [k]_0^F \times [b]_0 \times [k_1]_0 \times [k_2]_0 \to \{0, 1\}$. To roughly understand the meaning of this function, first recall that F is supposed to represent $\sigma(\widehat{t})$ for some child $\widehat{t}$ of t. Now, the function f_F aims to capture all information obtained while we processed the child $\widehat{t}$ of t that might be relevant to the node t. In particular, let us give an informal, intuitive interpretation of an element (Γ, μ, l_1, l_2) in the domain of f_F. For this purpose, note that when we remove at most k edges from the (connected) graph $G[\gamma(\widehat{t})]$, we obtain at most $k + 1$ connected components. The function Γ can be thought of as a method to assign to each vertex in $\sigma(\widehat{t})$ the connected component in which it should lie. Such information is extremely useful since each such connected component is in particular a bipartite graph, and hence by relying on Proposition 1 and an exhaustive search, we would be able to use it to extract a witnessing coloring for an instance of ABC-BJB. The arguments μ, l_1 and l_2 can be thought of as those in the definition of an output of ABC-BJB. Now, the value $f_F(\Gamma, \mu, l_1, l_2)$ aims to indicate whether Γ, μ, l_1 and l_2 are "realizable" in the context of the child $\widehat{t}$ (the precise meaning of this value would become clearer later, once we establish additional necessary definitions.)

Let us now give the formal definition of HP. In this definition, we denote $k = k_1 + k_2$.

Hypergraph Painting (HP)
Input: Integers k_1, k_2, b, d and q, a multi-hypergraph H with hyperedges of size at most d, and for all $F \in E(H)$, a function $f_F : [k]_0^F \times [b]_0 \times [k_1]_0 \times [k_2]_0 \rightarrow \{0,1\}$.
Output: For all $0 \leq \mu \leq b$, $0 \leq l_1 \leq k_1$, $0 \leq l_2 \leq k_2$, output the binary value

$$\texttt{aHP}[\mu, l_1, l_2] = \bigvee_{\Upsilon : V(H) \rightarrow [k]_0} \bigvee_{\substack{\{\mu^F\}|_{F \in E(H)} \\ \{l_1^F\}|_{F \in E(H)} \\ \{l_2^F\}|_{F \in E(H)}}} \bigwedge_{F \in E(H)} f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F),$$

where $\mu = \sum_{F \in E(H)} \mu^F$, $\sum_{F \in E(H)} l_1^F \leq l_1$, $\sum_{F \in E(H)} l_2^F \leq l_2$ and each of μ^F, l_1^F and l_2^F is a non-negative integer.

For a particular choice of μ, l_1 and l_2, a function Υ witnessing that $\texttt{aHP}[\mu, l_1, l_2] = 1$ is called a witnessing coloring for $\texttt{aHP}[\mu, l_1, l_2]$. An instance of Hypergraph Painting is denoted by $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$.

Although we are not able to tackle HP efficiently at its full generality, we are still able to solve those instances that are constructed when we would like to "handle" a single bag in a highly connected tree decomposition. For the sake of clarity, let us now address the beneficial properties that these instances satisfy individually, where each of them ultimately aims to ease our search for a witnessing coloring. The first property, called local unbreakability, unconditionally restricts the way a function $\Gamma : F \rightarrow [k]_0$, to be thought of as a restriction of the witnessing coloring we seek, can color a hyperedge F so that the value of f_F is 1.[3]

▶ **Definition 6** (Local Unbreakability). An instance $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$ is *locally unbreakable* if every $F \in E(H)$ satisfies the following property: for any $\Gamma : F \rightarrow [k]_0$ that is not $(3k^2, k)$-unbreakable, $f_F(\Gamma, \mu, l_1, l_2) = 0$ for all $0 \leq \mu \leq b$, $0 \leq l_1 \leq k_1$ and $0 \leq l_2 \leq k_2$.

The second property, called connectivity, implies that if we would like to use a function $\Gamma : F \rightarrow [k]_0$ to color a hyperedge (as a restriction of a witnessing coloring) with more than one color, then we would have to "pay" at least 1 from our budget $l_1 + l_2$.

▶ **Definition 7** (Connectivity). An instance $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$ is *connected* if every $F \in E(H)$ satisfies the following property: for any $\Gamma : F \rightarrow [k]_0$ for which there exist distinct $i, j \in [k]_0$ such that $|\Gamma^{-1}(i)|, |\Gamma^{-1}(j)| > 0$, it holds that $f_F(\Gamma, \mu, l_1, l_2) = 1$ only if $l_1 + l_2 \geq 1$.

The third property, called global unbreakability, directly restricts our "solution space" by implying that we only need to determine whether there exists a (q, k)-unbreakable witnessing coloring.

▶ **Definition 8** (Global Unbreakability). An instance $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$ is *globally unbreakable* if for all $0 \leq \mu \leq b$, $0 \leq l_1 \leq k_1$, $0 \leq l_2 \leq k_2$: if $\texttt{aHP}[\mu, l_1, l_2] = 1$, then there exists a witnessing coloring $\Upsilon : V(H) \rightarrow [k]_0$ that is (q, k)-unbreakable.

An instance $HP(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$ is called a *favorable instance of* HP if it is locally unbreakable, connected and globally unbreakable. For such instances we have the following theorem.

[3] In this context, it may be insightful to recall Lemma 2.

▶ **Theorem 9** ($\star$).
HP *on favorable instances is solvable in time* $2^{\mathcal{O}(\min(k,q)\log(k+q))}d^{\mathcal{O}(k^2)}m^{\mathcal{O}(1)}$.

The proof of this theorem is very technical, involving non-trivial analysis of a very "messy" picture obtained by guessing part of a hypothetical witnessing coloring via the method of color coding. Due to space constraints, the details are omitted here but can we found in the full version of the paper [27].

From now onwards, to simplify the presentation of arguments ahead with respect to ABC-BJB, we would abuse notation and directly define a witnessing coloring as a function rather than a partition. More precisely, the term witnessing coloring for $\texttt{aJP}[\mu, l_1, l_2] = 1$ would refer to a function $col : V(G) \to \{V_1, V_2\}$ such that $A \subseteq V_1$, $B \subseteq V_2$, $|V_1| = \mu$ and for $i \in \{1,2\}$, $|E(G[V_i])| \leq l_i$. To proceed to our proof of Theorem 4, we first need to introduce an additional notation. Roughly speaking, this notation translates a coloring Υ of the form that witnesses some $\texttt{aHP}[\mu, l_1, l_2] = 1$ to a coloring of the form that witnesses $\texttt{aJP}[\mu, l_1, l_2] = 1$ via some tuple $\mathbf{v} \in \{0,1\}^{k+1}$. Formally,

▶ **Definition 10.** For a tuple $\mathbf{v} \in \{0,1\}^{k+1}$, bipartite graph G with bipartition (P, Q), $A \subseteq V(G)$ and $\Upsilon : A \to [k]_0$, define $\widehat{\Upsilon}_{\mathbf{v}} : A \to \{V_1, V_2\}$ as follows.

- For all $v \in P$, $\widehat{\Upsilon}_{\mathbf{v}}(v) = V_1$ if and only if $\mathbf{v}[\Upsilon(v)] = 0$.
- For all $v \in Q$, $\widehat{\Upsilon}_{\mathbf{v}}(v) = V_1$ if and only if $\mathbf{v}[\Upsilon(v)] = 1$.

Suppose we are given an instance ABC-BJB(G, A, B, k_1, k_2). Fix some bipartition (P, Q) of G. Let (T, β) be the highly connected tree decomposition computed by the algorithm of Theorem 5, and let r be the root of T. In what follows, $\eta = 2^{\mathcal{O}(k)}$ as in Theorem 5, and $q = (\eta + k)k$. We now proceed to define a binary variable that is supposed to represent the answer we would like to compute when we process the bag of a specific node of the tree. Hence, one of the arguments is a node t, and three additional arguments are $\mu \in [n]_0, l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$. However, we cannot be satisfied with one answer, but need an answer for every possible "interaction" between the bag of t and the bag of its parent t'. Thus, the definition also includes a coloring of $\sigma(t)$. The tuple $\mathbf{v} \in \{0,1\}^{k+1}$ is necessary for the translation process described in Definition 10 (the way in which we shall obtain such a "right" tuple later in the proof would essentially rely on brute-force).

▶ **Definition 11.** Given $t \in V(T)$, a $(3k^2, k)$-unbreakable function $\Upsilon^\sigma : \sigma(t) \to [k]_0$, a tuple $\mathbf{v} \in \{0,1\}^{k+1}$, and integers $\mu \in [n]_0, l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, the binary variable $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ is 1 if and only if there exists $\Upsilon : \gamma(t) \to [k]_0$ extending Υ^σ such that

1. The translation $\widehat{\Upsilon}_{\mathbf{v}}$ maps to V_1 exactly μ vertices, that is, $|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)| = \mu$.
2. The translation $\widehat{\Upsilon}_{\mathbf{v}}$ maps $A \cap \gamma(t)$ to V_1 and $B \cap \gamma(t)$ to V_2, that is, $A \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)$ and $B \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)$.
3. For all $i \in \{1,2\}$, it holds that $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_i)])| \leq l_i$.
4. The set of edges between vertices receiving different colors by Υ is exactly the set of edges between vertices that are mapped to the same side by the translation $\widehat{\Upsilon}_{\mathbf{v}}$, that is,

$$\bigcup_{i,j \in [k]_0, i \neq j} E(\Upsilon^{-1}(i), \Upsilon^{-1}(j)) = E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)]).$$

A function Υ as above is called a *witnessing coloring* for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$.

▶ **Lemma 12** ($\star$). *For any* $\mu \in [n]_0$, $l_1 \in [k_1]_0$ *and* $l_2 \in [k_2]_0$, $\mathit{aJP}[\mu, l_1, l_2] = 1$ *if and only if there exists* $\mathbf{v} \in \{0,1\}^{k+1}$ *such that* $y[r, \phi, \mathbf{v}, \mu, l_1, l_2] = 1$.

By Lemma 12, it is sufficient to compute $y[r,\phi,\mathbf{v},\mu,l_1,l_2]$ for all $\mu\in[n]$, $l_1\in[k_1]_0$ and $l_2\in[k_2]_0$. To this end, we need to compute $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]$ for every node $t\in V(T)$, function $\Upsilon^\sigma:\sigma(t)\to[k]_0$ that is $(3k^2,k)$-unbreakable, tuple $\mathbf{v}\in\{0,1\}^{k+1}$, and integers $\mu\in[n]_0, l_1\in[k_1]_0$ and $l_2\in[k_2]_0$. Here, we employ bottom-up dynamic programming over the tree decomposition (T,β). Let us now zoom into the computation of $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]$ for all $\mu\in[n]$, $l_1\in[k_1]_0$ and $l_2\in[k_2]_0$, for some specific t,Υ^σ and $\mathbf{v}$. Note that we now assume that values corresponding to the children of t (if such children exist) have been already computed correctly. Moreover, note that $|\sigma(t)|\le\eta$, the number of $(3k^2,k)$-unbreakable functions $\Upsilon^\sigma:\sigma(t)\to[k]_0$ is at most $|\eta|^{k^{\mathcal{O}(1)}}=2^{k^{\mathcal{O}(1)}}$ (by Lemma 2), and the number of binary vectors of size $k+1$ is at most 2^{k+1}. Thus, the total running time would consist of the computation time of (T,β), and $n\cdot q^{\mathcal{O}(k)}\cdot 2^{k+1}$ times the computation time for a set of values as the one we examine now. Hence, it remains to show how to compute the current set of values in time $2^{k^{\mathcal{O}(1)}}$.

To compute our current set of values, let us construct an instance $\mathrm{HP}(k_1,k_2,n,\eta,q,H,\{f_F\}|_{F\in E(H)})$ of HP where $V(H)=\beta(t)$, and $E(H)$ and $\{f_F\}|_{F\in E(H)}$ are defined as follows.

1. **Type-1 Hyperedges.** For all $v\in\beta(t)$, insert $F=\{v\}$ into $E(H)$. Define $f_F:[k]_0^F\times[n]_0\times[k_1]_0\times[k_2]_0\to\{0,1\}$ as

$$f_F(\Gamma,\mu,l_1,l_2)=\begin{cases}0, & \text{if } v\in\sigma(t) \text{ and } \Gamma(v)\neq\Upsilon^\sigma(v)\\ 1, & \text{if } v\in A,\ \widehat{\Gamma}_{\mathbf{v}}(F)=V_1,\ l_1=l_2=0 \text{ and } \mu=1\\ 1, & \text{if } v\in B,\ \widehat{\Gamma}_{\mathbf{v}}(F)=V_2,\ l_1=l_2=0 \text{ and } \mu=0\\ 1, & \text{if } v\notin A\cup B,\ l_1=l_2=0 \text{ and } \mu=[\widehat{\Gamma}_{\mathbf{v}}(F)=V_1]\\ 0, & \text{otherwise}\end{cases}$$

2. **Type-2 Hyperedges.** For all $(u,v)\in E(G[\beta(t)])$, add $F=\{u,v\}$ in $E(H)$. Define $f_F:[k]_0^F\times[n]_0\times[k_1]_0\times[k_2]_0\to\{0,1\}$ as

$$f_F(\Gamma,\mu,l_1,l_2)=\begin{cases}0, & \text{if } \mu\neq 0\\ 1, & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u)\neq\widehat{\Gamma}_{\mathbf{v}}(v) \text{ and } \Gamma(u)=\Gamma(v)\\ 1, & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u)=\widehat{\Gamma}_{\mathbf{v}}(v)=V_1 \text{ and } l_1\ge 1\\ 1, & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u)=\widehat{\Gamma}_{\mathbf{v}}(v)=V_2 \text{ and } l_2\ge 1\\ 0, & \text{otherwise}\end{cases}$$

3. **Type-3 Hyperedges.** For all $\widehat{t}\in V(T)$ that is a child of t in the tree T, insert $F=\sigma(\widehat{t})$ into $E(H)$. Define $f_F:[k]_0^F\times[n]_0\times[k_1]_0\times[k_2]_0\to\{0,1\}$ as

$$f_F(\Gamma,\mu,l_1,l_2)=\begin{cases}0, & \text{if } \Gamma \text{ is not } (3k^2,k)\text{-unbreakable or } y[\widehat{t},\Gamma,\mu+\mu',l_1+l_1',l_2+l_2']=0\\ 1, & \text{otherwise}\end{cases}$$

where $\mu'=|\widehat{\Gamma}_{\mathbf{v}}^{-1}(V_1)|$, and $l_i'=|\{\{u,v\}\in E(G[\sigma(\widehat{t})]):\widehat{\Gamma}_{\mathbf{v}}(u)=\widehat{\Gamma}_{\mathbf{v}}(v)=V_i\}|$ for $i\in[2]$.

Let us first claim that witnessing colorings related to $\mathrm{HP}(k_1,k_2,n,\eta,q,H,\ \{f_F\}|_{F\in E(H)})$ are useful in the sense that they can be extended to witnessing colorings for the binary values in which we are interested.

▶ **Lemma 13** (⋆). *For all $\mu\in[n]$, $l_1\in[k_1]_0$, $l_2\in[k_2]_0$, if $\mathit{aHP}[\mu,l_1,\ l_2]=1$, then $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$. In fact, for any witness $\Upsilon:\beta(t)\to[k]_0$ of $\mathit{aHP}[\mu,l_1,l_2]=1$, there exists a function $\Upsilon':\gamma(t)\to[k]_0$ that extends Υ and witnesses $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$.*

In light of Lemma 13, we now turn to verify that $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ is of the form that we are actually able to solve.

▶ **Lemma 14** (⋆). $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ *is a favorable instance of* HP.

Finally, we turn to address the statement complementary to the one of Lemma 13.

▶ **Lemma 15** (⋆). *For all $\mu \in [n]$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, if $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$, then* $\mathit{aHP}[\mu, l_1, l_2] = 1$.

Recall that we have argued that to prove Theorem 9, it is sufficient to show that the current set of values $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ can be computed in time $2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$. Here, n refers to $|V(G)|$. By Lemmas 13 and 15, this set of values can be derived from the solution of $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$. Since $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ is a favorable instance of HP (by Lemma 14), the algorithm given by Theorem 9 solves it in time $2^{\mathcal{O}(\min(k,q)\log(k+q))} d^{\mathcal{O}(k^2)} |E(H)|^{\mathcal{O}(1)} = 2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$. This concludes the proof of Theorem 9.

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