Pumping Lemmas for Weighted Automata

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Abstract

We present three pumping lemmas for three classes of functions definable by fragments of weighted automata over the min-plus semiring and the semiring of natural numbers. As a corollary we show that the hierarchy of functions definable by unambiguous, finitely-ambiguous, polynomially-ambiguous weighted automata, and the full class of weighted automata is strict for the min-plus semiring.

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1 Introduction

Weighted automata (WA) are an expressible extension of finite state automata for computing functions over words. They have been extensively studied since Schützenberger [28], and its decidability problems [18, 1], extensions [9], logic characterization [9, 17], and applications [22, 7] have been deeply investigated.

The class of functions defined by WA has several equivalent representations in terms of computational models or logics. Recently Alur et al. introduced the computational model of cost register automata (CRA) [2, 3], an alternative model for computing functions over words, which are currently extensively studied [20, 21, 8]. The idea of this model is to enhance deterministic finite automata with registers that can be combined by using operations over a fixed semiring. In [2], it was shown that CRA are strictly more expressive than WA. Interestingly, it was also shown that a natural fragment of CRA is equally expressive to WA, which gives a new representation to understand this class of functions.

Regarding the logical representation of WA, Droste and Gastin introduced in [9] the so-called Weighted Logics (WL), a natural extension of monadic second order logics (MSO) from the boolean semiring to any commutative semiring. The semantics of this logics maps any formula in MSO over strings to one or zero in the semiring, depending whether the input satisfies the formula or not. Furthermore, WL includes sum and product quantifiers that allow to aggregate the output of boolean formulas producing an output value in the semiring. Although WL is far more expressive than WA, it was shown in [9] that a natural syntactic restriction of WL is equally expressive to WA, giving the first logical characterization of WA.

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Weighted logics or, more generally, quantitative logics have found many applications in understanding WA [10, 17], verification [5] and computational complexity [4].

The complexities of decision problems for WA have also been investigated, unfortunately often with undecidability results [18, 1]. For this reason various fragments of WA over different semirings have been studied. Recently, over one-letter alphabets, where WA are equivalent to linear recurrences, some new decidability results were shown for limited fragments [23, 24]. Other restrictions of WA involve bounding their numbers of runs. Among them most studied classes are *unambiguous automata*, *finitely-ambiguous automata*, and *polynomially-ambiguous automata*, where the numbers of accepting runs is bounded by 1, a constant, a polynomial in the size of input, respectively [29, 16, 15]. These are robust subclasses of functions inside WA that also have found several characterization in terms of cost register automata [2] and weighted logics [17].

Although functions defined by WA and its subclasses have been studied in terms of representations and decidability, little is known about its expressibility. Indeed, we are not aware of any general techniques to show if a function is definable or not by WA or any of its subclasses. Results related to the inexpressibility of WA usually require sophisticated arguments for each particular function [16, 20] and there is no clear path to generalize these techniques. As a matter of fact, the strict inclusions between unambiguous, finitely-ambiguous, polynomially-ambiguous, and the full class of WA are "well-known" to the community, but it is hard to find references to formal proofs (see related work below). In contrast, for regular languages or first order logics there exist elegant and useful techniques for showing inexpressibility like, for example, the standard pumping lemma for regular languages [13] or Ehrenfeucht-Fraïssé games for first-order logics [12, 11, 19]. One would like to have similar techniques in the quantitative world that simplifies inexpressibility arguments of WA, cost register automata, or even weighted logics to a small number of lines. Such techniques help to understand the inner structure of these functions and unveil their limits of expressibility.

In this paper, we embark in the work of loading the expressibility toolbox of weighted automata with pumping lemmas. We present three pumping lemmas, each of them for a different class or subclass of functions defined by WA over the min-plus semiring or the semiring of natural numbers. For every pumping lemma we show examples of functions that do not satisfy the lemma, giving very short inexpressibility proofs. Our results do not attempt to fully characterize the class or subclasses of weighted automata in terms of pumping properties, nor to provide conditions that can be verified by a computer. Our goal is to give the first tools for expressibility of weighted automata and to provide researchers with simple arguments for showing that functions do not belong to a given class.

Related work. In [14] it is shown that over the min-plus semiring polynomially-ambiguous automata are strictly more expressive than finite-ambiguous automata. In [16] strict inclusions between unambiguous automata, finitely-ambiguous automata, and the full class of WA are shown over the max-plus semiring. In both papers the strict inclusions are shown by analyzing particular functions. Using results in [6] one can deduce that unambiguous automata are strictly included in the other classes over the min-plus and max-plus semirings. Gathering these results we obtain strict inclusions between unambiguous automata, finitely-ambiguous automata, and the full class of WA over the min-plus semiring. However, to our knowledge, there is no reference for a strict inclusion between polynomially-ambiguous automata and the full class of WA.

Organization. In Section 2 we introduce weighted automata and some basic definitions. In Section 3 and Section 4 we present and prove pumping lemmas for weighted automata over the semiring of natural numbers and its extension using the operation min. In Section 5 we show the pumping lemma for polynomially-ambiguous automata over the min-plus semiring.

Some concluding remarks can be found in Section 6.

2 Preliminaries

In this section, we recall the definitions of weighted automata (WA). We start with the definitions that are standard in this area. A monoid $\mathbb{M} = (M, \otimes, \nVdash)$ is a set M with an associative operation $\otimes$ and a neutral element $\nVdash$. Standard examples of monoids are: the set of words Σ^* with concatenation and empty word; or the set of matrices with multiplication and the identity matrix. A semiring is a structure $\mathbb{S} = (S, \oplus, \odot, \nvdash, \nVdash)$, where $(S, \oplus, \nvdash)$ is a commutative monoid, $(S - \{\nvdash\}, \odot, \nVdash)$ is a monoid, multiplication distributes over addition, and $\nvdash \odot s = s \odot \nvdash = \nvdash$ for each $s \in S$. If the multiplication is commutative, we say that $\mathbb{S}$ is commutative. In this paper, we always assume that $\mathbb{S}$ is commutative. We usually denote S or M by the name of the semiring or monoid $\mathbb{S}$ or $\mathbb{M}$. In this paper, we are interested in the *min-plus semiring* $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and the *semiring of natural numbers with* ∞ $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ where we assume that $\infty + n = \infty$ for every $n \in \mathbb{N} \cup \{\infty\}$ and $\infty \cdot n = \infty$ if $n \neq 0$ and 0 otherwise. We denote the former by $\mathbb{N}_{\min,+}$ and the later by $\mathbb{N}_{+,\times}$. Note that $\mathbb{N}_{+,\times}$ is an extension of the standard semiring of natural numbers $\mathbb{N}$ and all our results for $\mathbb{N}_{+,\times}$ also hold for $\mathbb{N}$. We use this extended version of $\mathbb{N}$ to easily apply some results from $\mathbb{N}_{+,\times}$ to $\mathbb{N}_{\min,+}$ (see Section 4). Given a finite set Q, we denote by $\mathbb{S}^{Q \times Q}$ ($\mathbb{S}^Q$) the set of square matrices (vectors resp.) over $\mathbb{S}$ indexed by Q. The algebra induced by $\mathbb{S}$ over $\mathbb{S}^{Q \times Q}$ and $\mathbb{S}^Q$ is defined as usual.

We also consider two finite semirings that will be useful during proofs. We consider the boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ and the extended boolean semiring $\mathbb{B}_\infty = (\{0, 1, \infty\}, \vee, \wedge, 0, 1)$ such that $\infty \vee n = \infty$ for every $n \in \{0, 1, \infty\}$, $\infty \wedge 0 = 0$, and $\infty \wedge n = \infty$ if $n \in \{1, \infty\}$. Both finite semirings will be used as *abstractions* of $\mathbb{N}_{\min,+}$ and $\mathbb{N}_{+,\times}$, respectively.

In this paper, we study the specification of functions from words to values, namely, from Σ^* to $\mathbb{S}$. We say that a function $f : \Sigma^* \to \mathbb{S}$ is definable by a computational system $\mathcal{A}$ (e.g. by WA) if $f(w) = [\![\mathcal{A}]\!](w)$ for any $w \in \Sigma^*$, where $[\![\mathcal{A}]\!]$ is the semantics of $\mathcal{A}$ over words.

2.1 Weighted automata

Fix a finite alphabet Σ and a commutative semiring $\mathbb{S}$. A *weighted automaton* (WA) over Σ and $\mathbb{S}$ is a tuple $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$ where Q is a finite set of states, $\{M_a\}_{a\in\Sigma}$ is a set of matrices such that $M_a \in \mathbb{S}^{Q\times Q}$ and $I, F \in \mathbb{S}^Q$ are the initial and the final vectors, respectively [27, 10]. We say that a state q is initial if $I(q) \neq \nvdash$ and accepting if $F(q) \neq \nvdash$. We usually say that an entry $M_a(p, q) = s$ is a transition and write $p \xrightarrow{a/s} q$. Furthermore, we say that a run ρ of $\mathcal{A}$ over a word $w = a_1 \dots a_n$ is a sequence of transitions: $\rho = q_0 \xrightarrow{a_1/s_1} q_1 \xrightarrow{a_2/s_2} \cdots \xrightarrow{a_n/s_n} q_n$, where $s_i \neq \nvdash$ for all $1 \leq i \leq n$ and $I(q_0) \neq \nvdash$. We refer to q_i as the i-th state of the run ρ. The run ρ is accepting if $F(q_n) \neq \nvdash$, and the weight of an accepting run ρ is defined by $|\rho| = I(q_0) \odot (\bigodot_{i=1}^n s_i) \odot F(q_n)$. We define $\mathrm{Run}_{\mathcal{A}}(w)$ as the set of all accepting runs of $\mathcal{A}$ over w. Finally, the output of $\mathcal{A}$ over a word w is defined by $[\![\mathcal{A}]\!](w) = I^t \cdot M_{a_1} \cdot \ldots \cdot M_{a_n} \cdot F = \bigoplus_{\rho\in\mathrm{Run}_{\mathcal{A}}(w)} |\rho|$ where I^t is the transpose of I and the second sum is equal to $\nvdash$ if $\mathrm{Run}_{\mathcal{A}}(w)$ is empty. For a word $w = a_1 \dots a_n$ we usually denote $M_w = M_{a_1} \cdot \ldots \cdot M_{a_n}$ and then $[\![\mathcal{A}]\!](w) = I^t \cdot M_w \cdot F$. Note that $M_w(p, q)$ provides the cost of moving from state p to state q reading the word w.

A weighted automaton $\mathcal{A}$ is called *unambiguous* (U-WA) if $|\mathrm{Run}_{\mathcal{A}}(w)| \leq 1$ for every $w \in \Sigma^*$; and $\mathcal{A}$ is called *finitely-ambiguous* (FA-WA) if there exists a uniform bound N such that $|\mathrm{Run}_{\mathcal{A}}(w)| \leq N$ for every $w \in \Sigma^*$ [29, 16]. Furthermore, $\mathcal{A}$ is called *polynomially-ambiguous* (PA-WA) if the function $|\mathrm{Run}_{\mathcal{A}}(w)|$ is bounded by a polynomial in the length of w [15]. We call classes of functions definable by such automata *unambiguous regular*, *finitely-ambiguous regular* and *polynomially-ambiguous regular* functions. The class of functions defined by weighted automata are called regular functions.

Note that every unambiguous WA over $\mathbb{N}_{\min,+}$ can be defined by a polynomially-ambiguous WA over $\mathbb{N}_{+,\times}$ [16, 2] (recall that ∞ is in $\mathbb{N}_{+,\times}$). Therefore, the class of unambiguous regular functions over $\mathbb{N}_{\min,+}$ is included in the class of regular functions over $\mathbb{N}_{+,\times}$ (see Example 1). This inclusion is strict since regular functions over $\mathbb{N}_{\min,+}$ are always bounded by a linear function in the size of the word, and it is easy to define the function $f(w) = 2^{|w|}$ over $\mathbb{N}_{+,\times}$. Below, we give several examples of functions defined by WA over $\mathbb{N}_{+,\times}$ and $\mathbb{N}_{\min,+}$ that will be used in paper. Recall that in the latter semiring $\not\vdash = \infty$ and $\odot = +$. Transitions $p \xrightarrow{a/s} q$, where $s = \not\vdash$ are omitted.

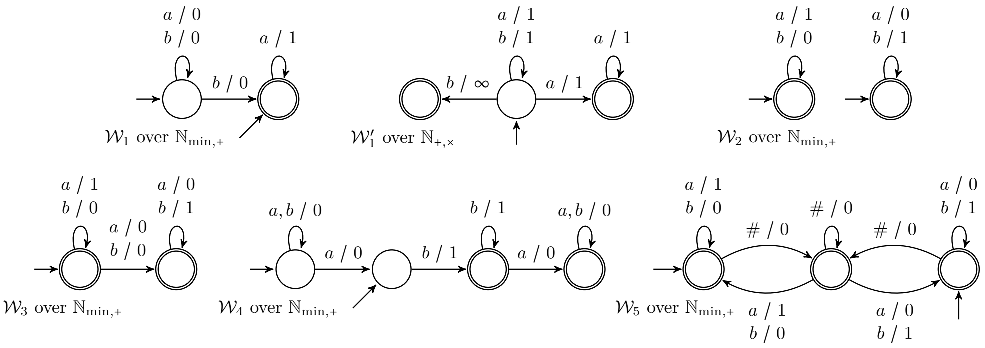

Figure 1 Examples of weighted automata. For WA over $\mathbb{N}_{\min,+}$ the initial and accepting states are labeled by 0 in the corresponding vector, and ∞ otherwise. Similarly, for WA over $\mathbb{N}_{+,\times}$ the initial and accepting states are labeled by 1 in the corresponding vector, and 0 otherwise.

▶ **Example 1.** Let $\Sigma = \{a, b\}$. Consider the function f_1 that for given word $w \in \Sigma^*$ outputs the length of the biggest suffix of a's (and ∞ if the word ends in b). This is defined by $\mathcal{W}_1$ over $\mathbb{N}_{\min,+}$ in Figure 1. One can easily check that $\mathcal{W}_1$ is unambiguous, hence f_1 belongs to unambiguous regular functions over $\mathbb{N}_{\min,+}$. In Figure 1, $\mathcal{W}_1'$ over $\mathbb{N}_{+,\times}$ also defines f_1.

▶ **Example 2.** Let $\Sigma = \{a, b\}$. Consider the function f_2 that for given word $w \in \Sigma^*$ outputs $\min\{|w|_a, |w|_b\}$, namely, counts the number of each letter and returns the minimum. This is defined by $\mathcal{W}_2$ in Figure 1. The WA $\mathcal{W}_2$ is finitel-ambiguous, hence f_2 belongs to finitely-ambiguous regular functions.

▶ **Example 3.** Let $\Sigma = \{a, b\}$. Consider the function f_3 that for a given word $w = a_1 \dots a_n \in \Sigma^*$ outputs $\min_{0 \le i \le n}\{|a_1 \dots a_i|_a + |a_{i+1} \dots a_n|_b\}$. This is defined by $\mathcal{W}_3$ in Figure 1. The WA is polynomially-ambiguous, hence f_3 belongs to polynomially-ambiguous functions.

▶ **Example 4.** Let $\Sigma = \{a, b\}$. Consider the function f_4 that for a given word $w \in \Sigma^*$ computes the shortest subword of b's (if there is none it outputs ∞). This is defined by $\mathcal{W}_4$ in Figure 1. The WA is polynomially-ambiguous, hence f_4 belongs to polynomially-ambiguous functions.

▶ **Example 5.** Let $\Sigma = \{a, b, \#\}$. Consider the function f_5 such that, for any $w \in \Sigma^*$ of the form $w_0 \# w_1 \# \dots \# w_n$ with $w_i \in \{a, b\}^*$, it computes the minimum number of a's or b's for each subword w_i (i.e. $\min\{|w_i|_a, |w_i|_b\}$) and then it sums these values over all subwords w_i, that is, $f_5(w) = \sum_{i=0}^{n} \min\{|w_i|_a, |w_i|_b\}$. This is defined by $\mathcal{W}_5$ in Figure 1. Given that the WA has an exponential number of runs, the function f_5 is a regular function but not necessarily a polynomially-ambiguous regular function.

We assume that our weighted automata are always trim, namely, all their states are reachable from some initial state (i.e., they are accessible) and they can reach some final state (i.e., they are co-accessible). Verifying if a state is accessible or co-accessible is reduced to a reachability test in the transition graph [25] and this can be done in NLogSpace. Thus, we can assume without loss of generality that all our automata are trimmed.

2.2 Finite monoids and idempotents

We say that a monoid is finite if the set of its elements is finite. Let $\mathbb{M} = (M, \otimes, \mathbb{1})$ be a finite monoid. We say that $\iota \in \mathbb{M}$ is an idempotent if $\iota \otimes \iota = \iota$. The following lemma is a standard result for finite monoids and idempotents (e.g. see Theorem 6.37 in [26]).

▶ **Lemma 6.** *Let $\mathbb{M}$ be a finite monoid. There exists $N > 0$ such that for every sequence $m_1 \otimes \ldots \otimes m_n$ with $m_i \in \mathbb{M}$ and $n \geq N$, there exist a factorization:*

$$(m_1 \otimes \ldots \otimes m_i) \otimes (m_{i+1} \otimes \ldots \otimes m_j) \otimes (m_{j+1} \ldots \otimes m_n),$$

where $i < j \leq n$ and $(m_{i+1} \otimes \ldots \otimes m_j)$ is an idempotent.

We will work with the finite monoid of matrices $\mathbb{B}^{Q\times Q}$ or $\mathbb{B}_\infty^{Q\times Q}$. For this, we define abstractions, i.e., homomorphisms of $\mathbb{N}_{\min,+}^{Q\times Q}$ to $\mathbb{B}^{Q\times Q}$ and $\mathbb{N}_{+,\times}^{Q\times Q}$ to $\mathbb{B}_\infty^{Q\times Q}$. These are given by the homomorphisms defined on elements of the matrices $h_1 : \mathbb{N}_{\min,+} \to \mathbb{B}$ and $h_2 : \mathbb{N}_{+,\times} \to \mathbb{B}_\infty$, defined: $h_1(m) = 0$ iff $m = \infty$; and $h_2(m) = 0$ if $m = 0$, $h_2(m) = \infty$ if $m = \infty$ and $h_2(m) = 1$ otherwise. For matrices $M \in \mathbb{N}_{\min,+}^{Q\times Q}$ or $N \in \mathbb{N}_{+,\times}^{Q\times Q}$ we denote by $\bar{M} = h_1(M)$ or $\bar{N} = h_2(N)$ their abstractions in $\mathbb{B}^{Q\times Q}$ or $\mathbb{B}_\infty^{Q\times Q}$, respectively.

3 Regular functions without min

In this section we consider regular functions over $\mathbb{N}_{+,\times}$. As a corollary of the pumping lemma in this section we show that FA-WA are strictly more expressive than U-WA over $\mathbb{N}_{\min,+}$ (Example 8). Moreover, we show that there are finitely-ambiguous regular functions over $\mathbb{N}_{\min,+}$ that cannot be defined by any regular function over $\mathbb{N}_{+,\times}$.

We introduce some notation to simplify the presentation. Given $u \cdot v \cdot w = \hat{u} \cdot \hat{v} \cdot \hat{w}$, where $u, v, w, \hat{u}, \hat{v}, \hat{w} \in \Sigma^*$, we say that $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ is a *refinement* of $u \cdot \underline{v} \cdot w$ if there exist u', w' such that $u \cdot u' = \hat{u}$, $w' \cdot w = \hat{w}$, $u' \cdot \hat{v} \cdot w' = v$, and $\hat{v} \neq \epsilon$. We underline the infixes v and $\hat{v}$ to emphasize the refined part.

▶ **Theorem 7** (Pumping Lemma for regular functions over $\mathbb{N}_{+,\times}$). *Let $f : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ be a regular function over $\mathbb{N}_{+,\times}$. There exists N such that for all words of the form $u \cdot v \cdot w \in \Sigma^*$ with $|v| \geq N$, there exists a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ of $u \cdot \underline{v} \cdot w$ such that at least one of the following two conditions holds:*

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \;=\; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ *for every $i \geq N$.*
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \;<\; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ *for every $i \geq N$.*

Before going into the details of the proof let us show how to use the lemma.

▶ **Example 8.** We show that f_2 from Example 2 is not definable by any WA over $\mathbb{N}_{+,\times}$. Indeed, suppose it is definable and fix N from Theorem 7. Consider the word $w = a^{(N+1)^2}\underline{b^N}$ and notice that $f_2(w) = N$. By refining w we get $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = a^{(N+1)^2} b^n \underline{b^m} b^l$ for some n, m, l such that $1 \leq m \leq N$ and $n + m + l = N$. Since $n + m \cdot N + l < n + m \cdot (N+1) + l < (N+1)^2$ it must be the case that $f_2(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) < f_2(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for all $i \geq N$. However, $f_2(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = (N+1)^2$ for i sufficiently large, which is a contradiction.

▶ **Example 9.** On the other hand, the function f_1 from Example 1 satisfies Theorem 7. Consider a word $u \cdot \underline{v} \cdot w \in \Sigma^*$ and its refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$. If $\hat{w}$ or $\hat{v}$ contain b then $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ because the suffix of a's remains the same. Otherwise, $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) < f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ since the suffix of a's increases when pumping. Moreover, it is straightforward to generalize this argument and prove Theorem 7 for all U-WA over $\mathbb{N}_{\min,+}$.

To prove Theorem 7 we use the following definitions. For a matrix $M \in \mathbb{N}_{+,\times}^{Q\times Q}$ recall that $\bar{M}$ is its homomorphic image in $\mathbb{B}_{\infty}^{Q\times Q}$ (see Section 2.2). We write that M and N in $\mathbb{N}_{+,\times}^{Q\times Q}$ are equivalent, denoted $M \equiv_{\mathbb{B}_\infty} N$, iff $\bar{M} = \bar{N}$. We also extend the homomorphic image and equivalence relation from matrices to vectors. We say that $D \in \mathbb{N}_{+,\times}^{Q\times Q}$ is an *idempotent* if $\bar{D}$ is an idempotent in the finite monoid $\mathbb{B}_{\infty}^{Q\times Q}$.

▶ **Lemma 10.** *If $M \equiv_{\mathbb{B}_\infty} N$, then $x^T \cdot M \cdot y > 0$ if and only if $x^T \cdot N \cdot y > 0$ for every $x, y \in \mathbb{N}_{+,\times}^{Q}$.*

Proof. Suppose that $x^T \cdot M \cdot y > 0$. By definition $x^T \cdot M \cdot y = \sum_{p,q} x(p) \cdot M(p,q) \cdot y(q)$. Then there exist $p, q \in Q$ such that $x(p) \cdot M(p,q) \cdot y(q) > 0$ and, in particular, $M(p,q) > 0$. Given that $M \equiv_{\mathbb{B}_\infty} N$ we conclude $N(p,q) > 0$ and $x(p) \cdot N(p,q) \cdot y(q) > 0$, which proves $x^T \cdot N \cdot y > 0$. ◀

Proof of Theorem 7. Let $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$ be a WA over $\mathbb{N}_{+,\times}$ such that $f = [\![\mathcal{A}]\!]$. Without loss of generality, we assume that $I(q) \neq \infty$ and $M_a(p,q) \neq \infty$ for every $p, q \in Q$ and $a \in \Sigma$, namely, ∞ can only appear in the final vector F. Indeed, if ∞ is used in I or some M_a, we can construct two weighted automata $\mathcal{A}', \mathcal{A}^\infty$ such that $\mathcal{A}'$ is the same as $\mathcal{A}$ but each ∞-initial state or each ∞-transition is replaced with 0, and $\mathcal{A}^\infty$ outputs ∞ if there exists some run in $\mathcal{A}$ that outputs ∞ and 0 otherwise. Note that $\mathcal{A}'$ has no ∞-transition or ∞-initial state and $\mathcal{A}^\infty$ can be constructed in such a way that only the final vector contains ∞-values. The disjoint union of $\mathcal{A}'$ and $\mathcal{A}^\infty$ is equivalent to $\mathcal{A}$.

Let $N = \max\{|Q|, K\}$ where K is the constant from Lemma 6 for the finite monoid $\mathbb{B}_{\infty}^{Q\times Q}$. For every word $u \cdot v \cdot w \in \Sigma^*$ such that $v = a_1 \dots a_n$ with $n \geq N$, consider the output $I^T \cdot M_u \cdot M_v \cdot M_w \cdot F$ of $\mathcal{A}$ over $u \cdot v \cdot w$. By Lemma 6, there exists a factorization of the form:

$$M_v = (M_{a_1} \cdot \ldots \cdot M_{a_i}) \cdot (M_{a_{i+1}} \cdot \dots \cdot M_{a_j}) \cdot (M_{a_{j+1}} \cdot \ldots \cdot M_{a_n})$$

for some $i < j$ where $M_{a_{i+1}} \cdot \ldots \cdot M_{a_j}$ is an idempotent (i.e., $\bar{M}_{a_{i+1}} \cdot \ldots \cdot \bar{M}_{a_j}$ is an idempotent). We define the refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ of $u \cdot \underline{v} \cdot w$ such that $\hat{u} = u \cdot (a_1 \dots a_i)$, $\hat{v} = a_{i+1} \dots a_j$, and $\hat{w} = (a_{j+1} \dots a_n) \cdot w$. Furthermore, define $x = I \cdot M_u \cdot M_{a_1} \cdot \ldots \cdot M_{a_i}$, $D = M_{a_{i+1}} \cdot \ldots \cdot M_{a_j}$, and $y = M_{a_{j+1}} \cdot \ldots \cdot M_{a_n} \cdot M_w \cdot F$. Note that $f(\hat{u} \cdot \hat{v}^i \cdot \hat{w}) = x^T \cdot D^i \cdot y$ for every $i \geq 0$ and D is an idempotent (i.e. $\bar{D}$ is an idempotent). It remains to show the following lemma.

▶ **Lemma 11.** *For every idempotent $D \in \mathbb{N}_{+,\times}^{Q\times Q}$ and $x, y \in \mathbb{N}_{+,\times}^{Q}$ where D and x do not contain ∞-values, one of the conditions holds:*

$$x^T \cdot D^i \cdot y = x^T \cdot D^{i+1} \cdot y \quad \text{for every } i \geq |Q|, \quad \text{or} \tag{1}$$

$$x^T \cdot D^i \cdot y < x^T \cdot D^{i+1} \cdot y \quad \text{for every } i \geq |Q|. \tag{2}$$

We start showing that Lemma 11 holds when $y = e_p$ for some $p \in Q$, where $e_p(q) = 1$ if $q = p$ and 0 otherwise. Note that $z = \sum_{p\in Q} z(p) \cdot e_p$ for every vector z.

We say that p is *D-stable* (or just *stable*) if $D(p,p) > 0$. Note that if p is stable, then $D^i(p,p) > 0$ for every $i > 0$ (recall that D is idempotent). Furthermore, $D \cdot e_p = e_p + z$ for some $z \in \mathbb{N}_{+,\times}^{Q}$. Suppose that p is stable and $D \cdot e_p = e_p + z$ for some vector z. Then for $i > 0$:

$$x^T \cdot D^{i+1} \cdot e_p = x^T \cdot D^i \cdot (e_p + z) = x^T \cdot D^i \cdot e_p + x^T \cdot D^i \cdot z$$

Given that D is idempotent and $D^i \equiv_{\mathbb{B}_\infty} D$, by Lemma 10 we have that $x^T \cdot D^i \cdot z > 0$ if, and only if, $x^T \cdot D \cdot z > 0$. Therefore, if $x^T \cdot D \cdot z > 0$, we get that $x^T \cdot D^i \cdot e_p < x^T \cdot D^{i+1} \cdot e_p$ for every $i > 0$, in particular, for every $i \geq |Q|$. Otherwise, $x^T \cdot D \cdot z = 0$ and $x^T \cdot D^i \cdot e_p = x^T \cdot D^{i+1} \cdot e_p$ for every $i > 0$, in particular, for every $i \geq |Q|$.

Let $P \subseteq Q$ be the set of all non-stable states in D. Consider the relation $\preceq_D \subseteq P \times P$ such that $p \preceq_D q$ if $p = q$ or $D(p,q) > 0$. One can easily check that $\preceq_D$ forms a partial order over P, namely, that $\preceq_D$ is reflexive, antisymmetric, and transitive. Indeed, transitivity holds because D is idempotent. To prove antisymmetry, note that for every non-stable states p and q, if $p \preceq_D q$, $q \preceq_D p$ and $p \neq q$ hold, then $D(p,p) > 0$. This is a contradiction since p is non-stable.

Since $\preceq_D$ is a partial order, we prove the lemma for $y = e_p$ by induction over $\preceq_D$. Formally, we strengthen the inductive hypothesis such that conditions (1) and (2) hold for every $i \geq N_q$, where $N_q = |\{q' \in P \mid q' \preceq_D q\}|$ (notice that $N_q \leq |Q|$ for every q). The base case is for $N_p = 0$, which means that p is stable. In the inductive case $N_p > 0$ the state p is non-stable. Then

$$x^T \cdot D^{i+1} \cdot e_p \;=\; x^T \cdot D^i \cdot (c_1 \cdot e_{q_1} + \ldots + c_k \cdot e_{q_k}) \;=\; c_1(x^T \cdot D^i \cdot e_{q_1}) + \ldots + c_k(x^T \cdot D^i \cdot e_{q_k})$$

for pairwise different states $q_1, \ldots, q_k$ and positive values $c_1, \ldots, c_k \in \mathbb{N}$ such that q_j is either stable or $q_j \prec_D p$. Thus all states $q_1, \ldots, q_k$ satisfy our inductive hypothesis.

Consider the partition of $q_1, \ldots, q_k$ into sets $C_=$ and $C_<$ such that $C_=$ and $C_<$ satisfy condition (1) and (2), respectively. If $C_< = \varnothing$, then for every $i \geq N_p$ we have:

$$\begin{aligned} x^T \cdot D^{i+1} \cdot e_p &= c_1(x^T \cdot D^i \cdot e_{q_1}) + \ldots + c_k(x^T \cdot D^i \cdot e_{q_k}) \\ &= c_1(x^T \cdot D^{i-1} \cdot e_{q_1}) + \ldots + c_k(x^T \cdot D^{i-1} \cdot e_{q_k}) \\ &= x^T \cdot D^i \cdot e_p. \end{aligned} \tag{3}$$

Note that $x^T \cdot D^i \cdot e_{q_j} = x^T \cdot D^{i-1} \cdot e_{q_j}$ holds by the inductive hypothesis and because $N_p > N_{q_j}$ for every q_j. Suppose otherwise, that $C_< \neq \varnothing$ and there exists a state q_j that satisfies $x^T \cdot D^i \cdot e_{q_j} < x^T \cdot D^{i+1} \cdot e_{q_j}$ for every $i \geq N_{q_j}$. Then it is straightforward that equality (3) becomes a strict inequality and condition (2) holds.

We have shown that either (1) or (2) holds for $y = e_p$. It remains to extend this to any vector $y \in \mathbb{N}^Q_{+,\times}$ (possibly with ∞). Note that

$$x^T \cdot D^{i+1} \cdot y \;=\; y(q_1) \cdot (x^T \cdot D^{i+1} \cdot e_{q_1}) + \ldots + y(q_k) \cdot (x^T \cdot D^{i+1} \cdot e_{q_k})$$

for some states $q_1, \ldots, q_k$ such that $y(q_j) > 0$ for every $j \leq k$. We consider two cases. First, if there exists j such that $y(q_j) = \infty$ and $x^T \cdot D^i \cdot e_{q_j} > 0$ for $i \geq N$, then $x^T \cdot D^i \cdot y = \infty$ for every $i \geq 0$. Thus, $x^T \cdot D^i \cdot y$ satisfies condition (1). Second, suppose that for every j we have $y(q_j) \neq \infty$ or $x^T \cdot D^i \cdot e_{q_j} = 0$ for $i \geq N$. It suffices to consider the case when $y(q_j) \neq \infty$ for all j. Then if some $x^T \cdot D^i \cdot e_{q_j}$ satisfies condition (2) we have that $x^T \cdot D^i \cdot y$ satisfies condition (2). Conversely, if every $x^T \cdot D^i \cdot e_{q_j}$ satisfies condition (1) we have that $x^T \cdot D^i \cdot y$ satisfies condition (1). ◀

One could try to simplify Theorem 7 changing the condition $i \geq N$ to $i \geq 0$. Unfortunately, we do not know if the theorem would remain true. A naive approach would be to use a generalization of Lemma 6, but intuitively, the behavior of non-stable registers is problematic. Examples of this behavior are very technical and we leave this for future work. We conclude with the following remarks, straightforward from the proof. We will use them in Section 4.

▶ **Remark 12.** *Changing y to y' such that $y \equiv_{\mathbb{B}_\infty} y'$ does not influence whether condition* (1) *or condition* (2) *holds in Lemma 11 (notice that here we need that the abstractions have values in $\mathbb{B}_\infty$ not in $\mathbb{B}$). Similarly, changing x to x' such that $x \equiv_{\mathbb{B}_\infty} x'$ does not influence whether condition* (1) *or* (2) *holds.*

▶ **Remark 13.** *The constant N and the refinement of w depend only on the finite monoid $\mathbb{B}_{\infty}^{Q\times Q}$. In particular they are independent from the initial vectors I and F.*

4 Finite-min regular functions

In this section we focus on regular functions over $\mathbb{N}_{+,\times}$ with some min allowed. Formally, we say that $f : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ is a finite-min regular function, if there exist regular functions $f_1, \ldots, f_m$ over $\mathbb{N}_{+,\times}$ such that $f(w) = \min\{f_1(w), \ldots, f_m(w)\}$. It is known that FA-WA are equivalent to a finite sum of U-WA [29], hence functions defined by FA-WA over $\mathbb{N}_{\min,+}$ are included in the class of finite-min regular functions. As a corollary of the pumping lemma in this section we show that PA-WA are strictly more expressive than FA-WA over $\mathbb{N}_{\min,+}$ (Example 15 and Example 16).

We start by introducing some notation to ease the presentation. For every word w we define an n-pumping representation

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where $w = u_0 \cdot v_1 \cdot u_1 \cdot v_2 \cdot \ldots v_n \cdot u_n$ and $v_k \neq \epsilon$ for all k. We define a refinement of an n-pumping representation as

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

if $v_k = x_k \cdot y_k \cdot z_k$, $u'_k = z_k \cdot u_k \cdot x_{k+1}$; where $z_0 = x_{n+1} = \epsilon$ and $y_k \neq \epsilon$ for every k. Let $S \subseteq \{1, \ldots, n\}$ such that $S \neq \emptyset$. Let $\underline{v_k}$ be a fragment of an n-pumping representation w. By $\underline{v_k}(S, i)$ we denote the word v_k^i if $k \in S$ and v_k otherwise. By $w(S, i)$ we denote the word

$$w = u_0 \cdot \underline{v_1}(S, i) \cdot u_1 \cdot \underline{v_2}(S, i) \cdot \ldots u_{n-1} \cdot \underline{v_n}(S, i) \cdot u_n.$$

In other words we pump the fragments v_k for all $k \in S$.

▶ **Theorem 14** (Pumping Lemma for finite-min regular functions)**.** *Let $f : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ be a finite-min regular function. There exists N such that for all n-pumping representations*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where $n \geq N$ and $|v_i| \geq N$ for all i, there exists a refinement

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every sequence of nonempty pairwise different subsets $S_1, \ldots, S_k \subseteq \{1 \ldots n\}$ with $k \geq N$ at least one of the following holds:

- *there exists j such that $f(w(S_j, i)) < f(w(S_j, i+1))$ for i sufficiently large;*
- *there exists $j_1 \neq j_2$ such that $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$ for i sufficiently large.*

Before proving Theorem 14, we show how to use it with two examples.

▶ **Example 15.** We show that f_3 from Example 3 is not definable by finite-min regular functions. Indeed, fix N from Theorem 14 and consider the n-pumping representation $w = (\underline{b^N} \cdot \underline{a^N})^N$. We index each pumping fragment with a pair (s, j), where $j \leq N$ denotes the block and $s \leq 2$ denotes the fragment in the block. First, notice that $f_3(w) = N \cdot (N-1)$ because runs minimizing the value for $\mathcal{W}_3$ change the state after reading the last b in one of the blocks. We define the sets $S_j = \{(1, j), (2, j)\}$ for $j \in \{1, \ldots, N\}$. Clearly $f_3(w(S_j, i)) = N \cdot (N-1)$ for any j and i, because the run minimizing the value changes the state after the last b in the j-th block. On the other hand $f_3(w(S_{j_1} \cup S_{j_2}, i)) < f_3(w(S_{j_1} \cup S_{j_2}, i+1))$ for all i and $j_1 \neq j_2$. Hence f_3 does not satisfy the pumping lemma for finite-min regular functions.

▶ **Example 16.** We show that f_4 from Example 4 is not definable by finite-min regular functions. Indeed, fix N from Theorem 14. Consider the N-pumping representation $w = (\underline{b^N}a)^N$. Then by definition $f_4(w) = N$. In the refinement all pumping parts will be of the form b^n for $1 \le n \le N$. We define the sets $S_j = \{1,\dots,N\} \setminus \{j\}$ for all $1 \le i \le N$. Clearly $f_4(w(S_j,i)) = N$ for any j and any i. On the other hand $f_4(w(S_{j_1} \cup S_{j_2}, i)) < f_4(w(S_{j_1} \cup S_{j_2}, i+1))$ for all i and $j_1 \ne j_2$. Hence f_4 does not satisfy the pumping lemma for finite-min regular functions.

Proof of Theorem 14. Let $f_1,\dots,f_m$ be regular functions over $\mathbb{N}_{+,\times}$ such that $f(w) = \min\{f_1(w),\dots,f_m(w)\}$ for every w. Furthermore, consider $\mathcal{A}_j = (Q_j, \Sigma, \{M_{j,a}\}_{a\in\Sigma}, I_j, F_j)$ the corresponding WA for f_j. Let $Q = \bigcup_j Q_j$ (we assume that $Q_1,\dots,Q_m$ are pairwise disjoint) and consider the set of matrices $\{U_a\}_{a\in\Sigma}$ where $U_a \in \mathbb{N}_{+,\times}^{Q\times Q}$ such that $U_a(p,q) = M_{j,a}(p,q)$ whenever $p,q \in Q_j$ and 0 otherwise. Then $f_j(w) = (I'_j)^t \cdot U_w \cdot F'_j$ for every j and $w \in \Sigma^*$ where I'_j and F'_j are the extensions of I_j and F_j from Q_j into Q such that $I'_j(q) = I_j(q)$ and $F'_j(q) = F_j(q)$ whenever $q \in Q_j$ and 0 otherwise. Notice that $\{U_a\}_{a\in\Sigma}$ synchronize the behavior of $f_1,\dots,f_m$ in a single set of matrices and project the output of f_j with I'_j and F'_j. Let $N = \max\{K, m+1\}$ such that K is the constant from Lemma 6 applied to $\mathbb{B}_\infty^{Q\times Q}$. Let $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$. For every v_i we use Theorem 7 over $u_{\le i} \cdot v_i \cdot s_{\ge i}$, where $u_{\le i} = u_0 \cdot v_1 \cdot \ldots u_{i-1}$ and $s_{\ge i} = u_i \cdot v_{i+1} \cdot \ldots u_n$ obtaining a refinement

$$w \;=\; u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

where each y_i comes from Theorem 7 applied to $\{U_a\}_{a\in\Sigma}$. Recall that the refinement of $u_{\le i} \cdot v_i \cdot s_{\ge i}$ depends only on $\{U_a\}_{a\in\Sigma}$ and not on the initial final vector (Remark 13). In particular, the refinement is the same for each function f_j. Then

$$f_j(w) \;=\; (I'_j)^t \cdot U_{u'_0} \cdot D_1 \cdot \ldots \cdot U_{u'_{n-1}} \cdot D_n \cdot U_{u'_n} \cdot F'_j$$

where $D_i = U_{y_i}$ are idempotents.

▶ **Lemma 17.** *Let $S \subseteq \{1,\dots,n\}$ be a nonempty set and fix one function f_j. Then $f_j(w(S,i)) < f_j(w(S,i+1))$ for every $i \ge N$ iff there exists $k \in S$ such that $f_j(w(\{k\},i)) < f_j(w(\{k\},i+1))$ for every $i \ge N$.*

Proof. By definition $f_j(w(S,i)) = (I'_j)^t \cdot U_{u'_0} \cdot D_1^{s_1} \cdot \ldots \cdot U_{u'_{n-1}} \cdot D_n^{s_n} \cdot U_{u'_n} \cdot F'_j$ where $s_k = i$ if $k \in S$ and $s_k = 1$ otherwise. Since all D_i are idempotents then for all k the fragments before and after $D_k^{s_k}$ are $\equiv_{\mathbb{B}_\infty}$ equivalent, i.e.,

$$\begin{aligned}(I'_j)^t \cdot U_{u'_0} \cdot D_1^{s_1} \cdot \ldots \cdot D_{k-1}^{s_{k-1}} \cdot U_{u'_{k-1}} &\equiv_{\mathbb{B}_\infty} (I'_j)^t \cdot U_{u'_0} \cdot D_1 \cdot \ldots \cdot D_{k-1} \cdot U_{u'_{k-1}} \\ U_{u'_k} \cdot D_{k+1}^{s_{k+1}} \cdot \ldots \cdot D_n^{s_n} \cdot U_{u'_n} \cdot F'_j &\equiv_{\mathbb{B}_\infty} U_{u'_k} \cdot D_{k+1} \cdot \ldots \cdot D_n \cdot U_{u'_n} \cdot F'_j.\end{aligned}$$

Hence, the lemma follows from Remark 12. ◀

To finish the proof we analyze $f(w(S,i)) = \min\{f_1(w(S,i)),\dots,f_m(w(S,i))\}$. Consider a sequence of subsets $S_1,\dots,S_k$ with $k \ge N$. Suppose there is a set S_l for some l such that for every $j \le m$ there exists $k \in S_l$ such that $f_j(w(\{k\},i)) < f_j(w(\{k\},i+1))$ for every $i \ge N$. It follows from Lemma 17 that $f(w(S_l,i)) < f(w(S_l,i+1))$ for all $i \ge N$, namely, the first condition of the theorem holds. Suppose otherwise, and for every S_l let $X_l \subseteq \{1,\dots,m\}$ be the set of functions such that $f_j(w(S_l,i)) = f_j(w(S_l,i+1))$ for all $j \in X_l$ and $i \ge N$. Since $k \ge N > m$ there exists l_1, l_2 such that $X_{l_1} \cap X_{l_2} \ne \emptyset$. From Lemma 17 it follows that for $i \ge N$ holds: $f_j(w(S_{l_1} \cup S_{l_2}, i)) = f_j(w(S_{l_1} \cup S_{l_2}, i+1))$ for all $j \in X_{l_1} \cap X_{l_2}$; and $f_j(w(S_{l_1} \cup S_{l_2}, i)) < f_j(w(S_{l_1} \cup S_{l_2}, i+1))$ for all $j \in \{1,\dots,m\} \setminus (X_{l_1} \cap X_{l_2})$. Hence for i sufficiently large $f(w(S_{l_1} \cup S_{l_2}, i)) = \min_{j\in X_{l_1}\cap X_{l_2}}(f_j(w(S_{l_1} \cup S_{l_2}, i)))$, which concludes the proof. ◀

5 Poly-ambiguous regular functions over the min-plus semiring

In this section we focus on polynomially-ambiguous regular functions over $\mathbb{N}_{\min,+}$. We expect that there is a wider class of functions, definable like in the previous section, where Theorem 18 holds but it is left for future work. A corollary from the pumping lemma in this section is that WA are strictly more expressive than PA-WA (Example 19 and 20).

We will use the notation of n-pumping representations from Section 4. As usual, a sequence of non-empty sets $S_1, \dots, S_m$ over $\{1, \dots, n\}$ is a *partition* if they are pairwise disjoint and $\bigcup S_i = \{1, \dots, n\}$. Furthermore, we say that $S \subseteq \{1, \dots, n\}$ is a *selection set* of $S_1, \dots, S_m$ if $|S \cap S_i| = 1$ for every i.

▶ **Theorem 18** (Pumping Lemma for polynomially-ambiguous automata). *Let $f : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ be a polynomially-ambiguous regular function over $\mathbb{N}_{\min,+}$. There exists N and a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all n-pumping representations:*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where $|v_i| \geq N$ for every $i \leq n$, there exists a refinement:

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every partition $\pi = S_1, \dots, S_m$ of $\{1, \dots, n\}$ with $m \geq \varphi(\max_i(|S_i|))$, at least one of the following holds:
- *there exists j such that $f(w(S_j, i)) = f(w(S_j, i+1))$ for i sufficiently large;*
- *there exists a selection set S of π such that $f(w(S, i)) < f(w(S, i+1))$ for i sufficiently large.*

▶ **Example 19.** We show that f_5 from Example 5 is not definable by PA-WA. Indeed, let N and φ be the constant and the function from Theorem 18. Consider the following n-pumping representation: $w = (\underline{a^N} \cdot \underline{b^N}\#)^m$ where $m \geq \varphi(2)$ (here $\max_i(|S_i|) = 2$). We index each pumping fragment with a pair (s, j), where $j \leq m$ denotes the block and $s \leq 2$ denotes the fragment in the block. We define the subsets $S_1 \dots S_m$ as follows: $S_j = \{(1, j), (2, j)\}$. Clearly for all j we have $f_5(w(S_j, i)) < f_5(w(S_j, i+1))$. On the other hand for every selection set S we have $f_5(w(S, i)) = f_5(w(S, i+1))$. Hence f_5 does not satisfy the Pumping Lemma above.

▶ **Example 20.** The function f_5 in Example 5 is essentially the function f_2 from Example 2 applied to the subwords between the symbols $\#$, where the outputs are aggregated with $+$. In a similar way one can define a min-plus automaton recognizing $f_6(w) = \sum_i f_4(w_i)$ for any $w \in \Sigma^*$ of the form $w_0\#w_1\# \dots \#w_n$ with $w_i \in \{a, b\}^*$, where f_4 is the function computing the minimal block of b's from Example 4. We show that f_6 is not definable by PA-WA over $\mathbb{N}_{\min,+}$. Consider the following n-pumping representation: $w = (\underline{b^N} \cdot a \cdot \underline{b^N}\#)^m$ where $m \geq \varphi(2)$ (here $\max_i(|S_i|) = 2$). We index each pumping fragment with a pair (s, j) like in Example 19 and we define the subsets $S_1 \dots S_m$ as follows: $S_j = \{(1, j), (2, j)\}$. Clearly for all j we have $f_6(w(S_j, i)) < f_6(w(S_j, i+1))$. On the other hand for every selection set S we have $f_6(w(S, i)) = f_6(w(S, i+1))$.

Consider the set of matrices $\mathbb{N}_{\min,+}^{Q\times Q}$ over the min-plus semiring. Recall that here $\oplus = \min$, $\odot = +$, $\mathbb{0} = \infty$, $\mathbb{1} = 0$, and the product of matrices $M, N \in \mathbb{N}_{\min,+}^{Q\times Q}$ is defined by $M \cdot N(p, q) = \min_r(M(p, r) + N(r, q))$. Also, recall that for any $M \in \mathbb{N}_{\min,+}^{Q\times Q}$ we denote by $\bar{M}$ the homomorphic image of M into the finite monoid $\mathbb{B}^{Q\times Q}$ (see Section 2.2). Similar as in Section 3 and Section 4, we say that $D \in \mathbb{N}_{\min,+}^{Q\times Q}$ is an idempotent if $\bar{D}$ is an idempotent in the finite monoid $\mathbb{B}^{Q\times Q}$.

The following lemma is a special property of polynomially-ambiguous automata that we exploit in the proof of Theorem 18. The proof is omitted here due to lack of space.

▶ **Lemma 21.** *Let $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$ be a polynomially-ambiguous weighted automaton over the min-plus semiring. For every idempotent $D \in \{M_w \mid w \in \Sigma^*\}$ and for every $p, q \in Q$, there exist constants $c, d \in \mathbb{N}_{\min,+}$ and $b \in \mathbb{N}$ such that $D^{b+i}(p,q) = c \cdot i + d$ for all $i \geq 0$.*

Proof of Theorem 18. Consider a polynomially-ambiguous WA $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a\in\Sigma}, I, F)$ over $\mathbb{N}_{\min,+}$ such that $f = [\![\mathcal{A}]\!]$. We take as N the constant from Lemma 6 for the finite monoid $\mathbb{B}^{Q\times Q}$. The function $\varphi : \mathbb{N} \to \mathbb{N}$ will be determined later in the proof. Consider an n-pumping representation w like in the statement of the lemma. Recall that the output for the word w is defined as $I \cdot M_w \cdot F$. By Lemma 6, for every v_k there exists a factorization $v_k = x_k y_k z_k$ such that M_{y_k} is an idempotent and $|y_k| \leq N$. We denote $D_k = M_{y_k}$ and define:

$$w \;=\; u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$$

such that each word y_k is the infix of v_k corresponding to the idempotent D_k. For the rest of the proof we denote $w_{\leq k} = u'_0 \cdot y_1 \cdot \ldots u'_{k-1}$. For every $S \subseteq \{1 \ldots n\}$ we denote by $w_{\leq k}(S, i)$ the word $w_{\leq k}$ with all y_j pumped i times for all $j < k$ such that $j \in S$.

Recall that $\mathrm{Run}_{\mathcal{A}}(w)$ is the set of all accepting runs and let $\rho \in \mathrm{Run}_{\mathcal{A}}(w)$. Every run induces two states for each $1 \leq k \leq n$: states preceding and following each word y_k. In the rest of the proof these will be the most important parts of a run. To work with them, we define the abstraction of ρ, denoted by $\bar{\rho} : \{1 \ldots n\} \to Q \times Q$, such that $\bar{\rho}(k) = (p, q)$ where p and q are the $|w_{\leq k}|$-th and $|w_{\leq k} \cdot y_k|$-th states of ρ, respectively. Similarly, for $S \subseteq \{1 \ldots n\}$, $i \geq 1$, and $\rho \in \mathrm{Run}_{\mathcal{A}}(w(S, i))$ we define $\bar{\rho} : \{1 \ldots n\} \to Q \times Q$ such that $\bar{\rho}(k) = (p, q)$ where p and q are the $|w_{\leq k}(S, i)|$-th and $|w_{\leq k}(S, i) \cdot y_k(S, i)|$-th states of ρ, respectively. We denote by $\overline{\mathrm{Run}_{\mathcal{A}}}(w)$ the set of all abstraction of runs in $\mathrm{Run}_{\mathcal{A}}(w)$. Observe that since all D_k are idempotents, $\overline{\mathrm{Run}_{\mathcal{A}}}(w(S, i)) = \overline{\mathrm{Run}_{\mathcal{A}}}(w)$ for all subsets S and $i \geq 1$.

The next step is to prove that there exists a polynomial function $p(x)$, depending only on $\mathcal{A}$, such that $|\overline{\mathrm{Run}_{\mathcal{A}}}(w)| \leq p(n)$. Let w' be the word obtained from w were each u'_i is replaced with a word u''_i of length at most $|\mathbb{B}^{Q\times Q}|$ such that $\overline{M_{u'_i}} = \overline{M_{u''_i}}$ (it is straightforward to prove that u''_i exists by pigeonhole principle). Then $|\mathrm{Run}_{\mathcal{A}}(w')| \geq |\overline{\mathrm{Run}_{\mathcal{A}}}(w)|$. Recall that $|y_i| \leq N$ and that N depends only on $|\mathbb{B}^{Q\times Q}|$. Then by definition $|w'| \leq (N + |\mathbb{B}^{Q\times Q}|) \cdot (n+1)$ and thus $|\mathrm{Run}_{\mathcal{A}}(w')| \leq r((N + |\mathbb{B}^{Q\times Q}|) \cdot (n+1))$, where r is the polynomial bounding the number of runs in $\mathcal{A}$. The claim follows for $p(n) = r((N + |\mathbb{B}^{Q\times Q}|) \cdot (n+1))$.

Fix a nonempty set $S \subseteq \{1, \ldots, n\}$ and $\rho \in \mathrm{Run}_{\mathcal{A}}(w)$. For every $k \in S$ let $b^k_{\bar{\rho}(k)}$, $c^k_{\bar{\rho}(k)}$ and $d^k_{\bar{\rho}(k)}$ be the constants from Lemma 21 such that $D_k^{b^k_{\bar{\rho}(k)}+i}[\bar{\rho}(k)] = c^k_{\bar{\rho}(k)} \cdot i + d^k_{\bar{\rho}(k)}$ for i sufficiently large. Since ρ is accepting then $c^k_{\bar{\rho}(k)}, d^k_{\bar{\rho}(k)} < +\infty$. We show that:

1. $[\![\mathcal{A}]\!](w(S, i)) = [\![\mathcal{A}]\!](w(S, i+1))$ for i sufficiently large iff there exists a run $\rho \in \mathrm{Run}_{\mathcal{A}}(w)$ such that $c^k_{\bar{\rho}(k)} = 0$ for every $k \in S$;
2. $[\![\mathcal{A}]\!](w(S, i)) < [\![\mathcal{A}]\!](w(S, i+1))$ for i sufficiently large iff for every run $\rho \in \mathrm{Run}_{\mathcal{A}}(w)$ there exists k such that $c^k_{\bar{\rho}(k)} > 0$.

Let $\rho \in \mathrm{Run}_{\mathcal{A}}(w(S, i+1))$ be a run realizing the minimum value for $i \geq i_0$. Given that D_k are idempotents one can always find a run $\rho' \in \mathrm{Run}_{\mathcal{A}}(w(S, i))$ such that $\bar{\rho}' = \bar{\rho}$ by removing one part on each y_k. In particular $|\rho'| \leq |\rho|$, which proves $[\![\mathcal{A}]\!](w(S, i)) \leq [\![\mathcal{A}]\!](w(S, i+1))$. It follows that if we prove (1) then (2) also holds. To prove (1) suppose first $[\![\mathcal{A}]\!](w(S, i)) = [\![\mathcal{A}]\!](w(S, i+1))$ for i sufficiently large. Let $\rho \in \mathcal{A}(w(S, i+1))$ and $\rho' \in \mathcal{A}(w(S, i))$ be the previous runs realizing the minimum and its shortening, respectively.

Since $|y_k| \leq N$ we assume a universal bound i_0 such that $b^k_{\bar{\rho}(k)} = i_0$ for all k in Lemma 21. By Lemma 21 $D_k^{i_0+i+1}[\bar{\rho}(k)] = c^k_{\bar{\rho}(k)} \cdot (i+1) + d^k_{\bar{\rho}(k)}$. If $c^k_{\bar{\rho}(k)} > 0$ for some k then the inequality $[\![\mathcal{A}]\!](w(S, i_0+i)) \leq [\![\mathcal{A}]\!](w(S, i_0+i+1))$ would be sharp, which is a contradiction. For the other direction suppose there exists a run $\rho \in \mathrm{Run}_{\mathcal{A}}(w)$ such that $c^k_{\bar{\rho}(k)} = 0$ for every $k \in S$. Then for every $i \geq 0$ there exists a run $\rho_i \in \mathrm{Run}_{\mathcal{A}}(w(S, i_0+i))$ such that $|\rho_i| \leq |\rho| + \sum_k d^k_{\bar{\rho}(k)}$. Since $[\![\mathcal{A}]\!](w(S, i_0+i)) \leq [\![\mathcal{A}]\!](w(S, i_0+i+1)) \leq |\rho| + \sum_k d^k_{\bar{\rho}(k)}$ it follows that $[\![\mathcal{A}]\!](w(S, i_0+i)) = [\![\mathcal{A}]\!](w(S, i_0+i+1))$ for i sufficiently large.

Given the previous discussion, let $\bar{R}_k = \{\bar{\rho} \in \overline{\mathrm{Run}}_{\mathcal{A}}(w) \mid c^k_{\bar{\rho}(k)} > 0\}$ for every $k \in \{1, \ldots, n\}$. The set $\bar{R}_k$ represents indirectly the runs that will grow when pumping $w(\{k\}, i)$. Then, we can restate (2) as: $[\![\mathcal{A}]\!](w(S, i)) < [\![\mathcal{A}]\!](w(S, i+1))$ for i sufficiently large iff $\bigcup_{k \in S} \bar{R}_k = \overline{\mathrm{Run}}_{\mathcal{A}}(w)$.

We are ready to prove the theorem. Fix a partition $S_1, \ldots, S_m$ for some $m \geq \varphi(\max |S_l|)$. Suppose the first condition is not true, namely, for all j there exists arbitrarily big values i such that $f(w(S_j, i)) \neq f(w(S_j, (i+1)))$. From (2) it follows that $f(w(S_j, i)) < f(w(S_j, i+1))$ for i sufficiently large and $\bigcup_{k \in S_j} \bar{R}_k = \overline{\mathrm{Run}}_{\mathcal{A}}(w)$ for every $j \leq m$. Let $L = \max |S_l|$. We assume that $L > 1$, otherwise every selection S contains a whole set S_k for some k and we are done by (2). To construct the set $S = \{k_1, \ldots, k_m\}$ we define by induction the sets G_j. Let $G_0 = \overline{\mathrm{Run}}_{\mathcal{A}}(w)$ and for every $j \in \{1, \ldots, m\}$ let $G_j = \overline{\mathrm{Run}}_{\mathcal{A}}(w) \setminus \bigcup_{l \leq j} \bar{R}_{k_l}$. Intuitively, G_j correspond to runs that are not covered by the set $\{k_1, \ldots, k_j\}$. For the inductive case, suppose that $G_j \neq \emptyset$. Since $\bigcup_{k \in S_{j+1}} \bar{R}_k = \overline{\mathrm{Run}}_{\mathcal{A}}(w)$, by the pigeonhole principle there exist $k_{j+1} \in S_{j+1}$ such that $|\bar{R}_{k_{j+1}} \cap G_j| \geq |G_j|/|S_{j+1}|$. We add k_{j+1} to S and so $|G_{j+1}| \leq |G_j| - |G_j|/|S_{j+1}| = |G_j| \cdot (|S_{j+1}| - 1)/|S_{j+1}| \leq |G_j| \cdot (L-1)/L$. Suppose this procedure continues until $j = m$ and $G_m \neq \emptyset$. Then $1 \leq |\overline{\mathrm{Run}}_{\mathcal{A}}(w))| \cdot ((L-1)/L)^m$, and $|\overline{\mathrm{Run}}_{\mathcal{A}}(w))| \geq (L/(L-1))^m$. However, we know that $|\overline{\mathrm{Run}}_{\mathcal{A}}(w))|$ is bounded by a polynomial function $p(n)$ depending on $|\mathcal{A}|$. Thus, it suffices to choose φ such that $m \geq \varphi(L)$ implies $(L/(L-1))^m > p(L \cdot m) \geq p(n) \geq |\overline{\mathrm{Run}}_{\mathcal{A}}(w))|$ (recall that $S_1, \ldots, S_m$ is a partition of $\{1, \ldots, n\}$ and $L \cdot m \geq n$). Therefore, $G_m = \emptyset$ and thus $\bigcup_{k \in S} \bar{R}_k = \overline{\mathrm{Run}}_{\mathcal{A}}(w)$, which concludes the proof. ◀

6 Conclusions

We have shown three pumping lemmas for three different classes of functions. We believe that the last pumping lemma in Section 5 could be proved for a wider class of functions that would contain the class $\mathbb{N}_{+,\times}$, but this is left for future work. As a corollary of our results, we showed that regular functions over $\mathbb{N}_{\min,+}$ form a strict hierarchy, namely:

$$\text{U-WA} \subsetneq \text{FA-WA} \subsetneq \text{PA-WA} \subsetneq \text{WA}.$$

All strict inclusions, except for PA-WA $\subsetneq$ WA, could be extracted from the analysis of examples in [16]. However, our results provide a general machinery to prove such results.

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