The Dominating Set Problem in Geometric Intersection Graphs*†

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Abstract

We study the parameterized complexity of dominating sets in geometric intersection graphs.

- In one dimension, we investigate intersection graphs induced by translates of a fixed pattern Q that consists of a finite number of intervals and a finite number of isolated points. We prove that Dominating Set on such intersection graphs is polynomially solvable whenever Q contains at least one interval, and whenever Q contains no intervals and for any two point pairs in Q the distance ratio is rational. The remaining case where Q contains no intervals but does contain an irrational distance ratio is shown to be NP- complete and contained in FPT (when parameterized by the solution size).
- In two and higher dimensions, we prove that Dominating Set is contained in W[1] for intersection graphs of semi-algebraic sets with constant description complexity. This generalizes known results from the literature. Finally, we establish W[1]-hardness for a large class of intersection graphs.

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1 Introduction

A *dominating set* in a graph $G = (V, E)$ is a subset $D \subseteq V$ of vertices such that every node in V is either contained in D or has some neighbor in D. The decision version of the dominating set problem asks for a given graph G and a given integer k, whether G admits a dominating set of size at most k. Dominating set is a popular and classic problem in algorithmic graph theory. It has been studied extensively for various graph classes; we only mention that it is polynomially solvable on interval graphs, strongly chordal graphs, permutation graphs and co-comparability graphs and that it is NP-complete on bipartite graphs, comparability graphs, and split graphs. We refer the reader to the book [9] by Hales, Hedetniemi and Slater for lots of comprehensive information on dominating sets.

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Dominating set is also a model problem in parameterized complexity, as it is one of the few natural problems known to be W[2]-complete (with the solution size k as natural parameterization); see [5]. In the parameterized setting, dominating set on a concrete graph class typically is either in P, FPT, W[1]-complete, or W[2]-complete. (Note that the problem cannot be on higher levels of the W-hierarchy, as it is W[2]-complete on general graphs.)

In this paper we study the dominating set problem on geometric intersection graphs: Every vertex in V corresponds to a geometric object in $\mathbb{R}^d$, and there is an edge between two vertices if and only if the corresponding objects intersect. Well-known graph classes that fit into this model are interval graphs and unit disk graphs. In $\mathbb{R}^1$, Chang [3] has given a polynomial time algorithm for dominating set in interval graphs and Fellows, Hermelin, Rosamond and Vialette [6] have proven W[1]-completeness for 2-interval graphs (where the geometric objects are pairs of intervals). In $\mathbb{R}^2$, Marx [10] has shown that dominating set is W[1]-hard for unit disk graphs as well as for unit square graphs. For unit square graphs the problem is furthermore known to be contained in W[1] [10], whereas for unit disk graphs this was previously not known.

Our contribution

We investigate the dominating set problem on intersection graphs of 1- and 2-dimensional objects, thereby shedding more light on the borderlines between P and FPT and W[1] and W[2].

For 1-dimensional intersection graphs, we consider the following setting. There is a fixed *pattern* Q, which consists of a finite number of points and a finite number of closed intervals (specified by their endpoints). The objects corresponding to the vertices in the intersection graph simply are a finite number of translates of this fixed pattern Q. More formally, for a real number x we define $Q(x) := x + Q$ to be the pattern Q translated by x, and for the input $\{x_1, \dots, x_n\}$, we consider the intersection graph defined by the objects $\{x_1 + Q, \dots, x_n + Q\}$. The class of unit interval graphs arises by choosing $Q = [0, 1]$. Our model of computation is the word RAM model, where real numbers are restricted to a field K which is a finite extension of the rationals.

▶ Remark (Machine representation of numbers). As finite extensions of $\mathbb{Q}$ are finite dimensional vector spaces over $\mathbb{Q}$, there exists a basis $b_1, \dots, b_k$ with $k = [K : \mathbb{Q}]$, so that any real $x \in K$ is representable in the form $x = q_1 b_1 + q_2 b_2 + \dots + q_k b_k$ for some $q_1, \dots, q_k \in \mathbb{Q}$. As k is fixed, any arithmetic operation that takes $O(1)$ steps on the rationals will also take $O(1)$ steps on elements of K.

We define the *distance ratio* of two point pairs $(x_1, x_2), (x_3, x_4) \in \mathbb{R} \times \mathbb{R}$ as $\frac{|x_1 - x_2|}{|x_3 - x_4|}$. We derive the following complexity classification for Q-Intersection Dominating Set.

▶ **Theorem 1.** Q-Intersection Dominating Set *has the following complexity:*
- **(i)** *It is in* P *if the pattern Q contains at least one interval.*
- **(ii)** *It is in* P *if the pattern Q does not contain any intervals, and if for any two point pairs in Q the distance ratio is rational.*
- **(iii)** *It is* NP*-complete and in* FPT *if pattern Q is a finite point set which has at least one irrational distance ratio.*

In the final version we show that any graph can be obtained as a 1-dimensional pattern intersection graph for a suitable choice of pattern Q. Consequently Q-Intersection Dominating Set is W[2]-complete if the pattern Q is part of the input.

For 2-dimensional intersection graphs, our results are inspired by a question that was not resolved in [10]: *"Is dominating set on unit disk graphs contained in* W[1]*?"* We answer this question affirmatively (and thereby fully settle the complexity status of this problem). Our result is in fact far more general: We show that dominating set is contained in W[1] whenever the geometric objects in the intersection graph come from a family of semi-algebraic sets that can be described by a constant number of parameters. We also show that this restriction to shapes of constant-complexity is crucial, as dominating set is W[2]-hard on intersection graphs of convex polygons with a polynomial number of vertices. On the negative side, we generalize the W[1]-hardness result of Marx [10] by showing that for any non-trivial simple polygonal pattern Q, the corresponding version of dominating set is W[1]-hard.

The full version of this paper is available as a preprint [4].

2 1-dimensional patterns

In this section, we study the Q-Intersection Dominating Set problem in $\mathbb{R}^1$. If Q contains an unbounded interval, then all translates are intersecting; the intersection graph is a clique and the minimum dominating set is a single vertex. In what follows, we assume that all intervals in Q are bounded. We define the *span* of Q to be the distance between its leftmost and rightmost point. We prove Theorem 1 by studying each claim separately.

▶ **Lemma 2.** *Q-Intersection Dominating Set can be solved in $O(n^{6w+4})$ time if Q contains at least one interval, where w is the ratio of the span of Q and the length of the longest interval in Q.*

Note that since Q is a fixed pattern, the value of w does not depend on the input size and so Lemma 2 implies Theorem 1(i). We translate Q so that its leftmost endpoint lies at the origin, and we rescale Q so that its longest interval has length 1. Consider an intersection graph $\mathcal{G}$ of a set of translates of Q. The vertices of $\mathcal{G}$ are $Q(x_i)$ for the given values x_i. We call x_i the *left endpoint* of Q_i. Let $+$ also denote the Minkowski sum of sets: $A+B = \{a+b \mid a \in A, b \in B\}$. If A or B is a singleton, then we omit the braces, i.e., we let $a + B$ denote $\{a\} + B$. In order to prove Lemma 2, we need the following lemma first.

▶ **Lemma 3.** *Let $D \subseteq V(\mathcal{G})$ be a minimum dominating set and let $X(D)$ be the set of left endpoints corresponding to the patterns in D. Then for all $y \in \mathbb{R}$ it holds that $|X(D) \cap [y, y+w]| \leqslant 3w$.*

Proof. We prove this lemma first for unit interval graphs (where Q consists of a single interval). The following observation is easy to prove.

▶ **Observation 4.** *In any unit interval graph there is a minimum dominating set whose intervals do not overlap.*

Notice that the lemma immediately follows from this claim in case of unit interval graphs since then $|X(D) \cap [y, y+1]| \leqslant 1 < 3 = 3w$. Let Q be any other pattern, and suppose that $|X(D) \cap [y, y+w]| \geqslant 3w+1$. The patterns starting in $[y, y+w]$ can only dominate patterns with a left endpoint in $[y-w, y+2w]$, a window of width $3w$. Let H be the set of patterns starting in $[y-w, y+2w]$ (see Figure 1. Let I be a unit interval of Q, and let U the set of unit intervals that are the translates of I in the patterns of H. Notice that $X(U)$ is a point set that is also in a window of length $3w$. By the claim above, we know that the interval graph $\mathcal{G}(U)$ defined by U has a dominating set that contains non-overlapping intervals, in particular, a dominating set D_U of size at most $3w$. Since $\mathcal{G}(U)$ corresponds to a spanning

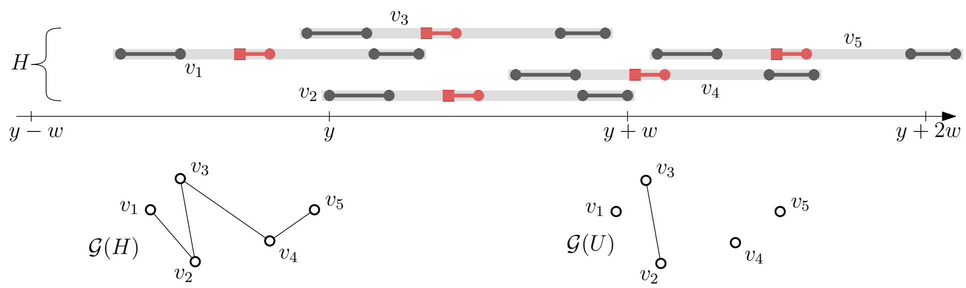

Figure 1 Patterns in a window $[y-w, y+2w]$. Intervals of U are red.

subgraph of $\mathcal{G}(H)$, the patterns D_U^H corresponding to D_U in H form a dominating set of $\mathcal{G}(H)$. Thus, $(D \setminus H) \cup D_U^H$ is a dominating set of our original graph that is smaller than D, which contradicts the minimality of D. ◀

We can now move on to the proof of Lemma 2.

Proof. We give a dynamic programming algorithm. We translate our input so that the left endpoint of the leftmost pattern is 0. Moreover, we can assume that the graph induced by our pattern is connected, since we can apply the algorithm to each connected component separately. The connectivity implies that the left endpoint of the rightmost pattern is at most $(n-1)w$. Let $0 < k \leqslant n$ be an integer and let $\mathcal{G}(k)$ be the intersection graph induced by the patterns with left endpoints in $[0, kw]$. Let $\mathcal{I}(k)$ be the set of input patterns with left endpoints in $[(k-1)w, kw)$ and let $S \subseteq \mathcal{I}(k)$. Let $A(k,S)$ be the size of a minimum dominating set D of $\mathcal{G}(k)$ for which $D \cap \mathcal{I}(k) = S$. By Lemma 3 it follows that $|S| \leqslant 3w$.

The following recursion holds for $A(k,S)$ if we define $A(0,S) := 0$:

$$A(k,S) = \min \left\{ A(k-1,S') + |S| \;\middle|\; S' \subset \mathcal{I}(k-1),\ |S'| \leqslant 3w,\ S \cup S' \text{ dominates } \mathcal{I}(k) \right\}.$$

The inequality "$\leqslant$" is easy to see, we are only minimizing over the sizes of feasible dominating sets of $\mathcal{G}(k)$. For the other direction ("$\geqslant$"), Lemma 3 implies that there is a minimum dominating set containing at most $3w$ left endpoints from both $\mathcal{I}(k-1)$ and $\mathcal{I}(k)$, therefore its size is $A(k-1,S')+|S|$ for some $S' \subset \mathcal{I}(k-1)$, $|S'| \leqslant 3w$ that together with S dominates $\mathcal{I}(k)$. The number of subproblems for a fixed value of k is $\sum_{j=0}^{3w} \binom{n}{j} = O(n^{3w})$; thus the number of subproblems is $O(n^{3w+1})$. Computing the value of a subproblem requires looking at $O(n^{3w+1})$ potential subsets S', and $O(n^2)$ time is sufficient to check whether $S \cup S'$ dominates $\mathcal{I}(k)$. Overall, the running time of our algorithm is $O(n^{6w+4})$. ◀

▶ **Lemma 5.** *If Q is a point pattern so that the distance ratios of any two point pairs of Q are rational, then Q-*Intersection Dominating Set *can be solved in polynomial time.*

Proof. By shifting and rescaling, we may assume without loss of generality that the leftmost point in Q is in the origin and that all points in Q have integer coordinates. (Note that this could not be done if the pattern contained an irrational distance ratio.) We define a new pattern Q' that results from Q by replacing point 0 by the interval $[0, 1/3]$.

Now consider an intersection graph whose vertices are associated with $x_i + Q$ where $x_1 \leq x_2 \leq \cdots \leq x_n$. We assume without loss of generality that the graph is connected and that all x_i are integers. It can be seen that the intersection graph does not change, if every object $x_i + Q$ is replaced by the object $x_i + Q'$. Since pattern Q' contains the interval $[0, 1/3]$, we may simply apply Lemma 2 to compute the optimal dominating set in polynomial time. ◀

▶ **Lemma 6.** *If Q is a point pattern that contains two point pairs with an irrational distance ratio, then Q-Intersection Dominating Set is* NP*-complete.*

Proof. The containment in NP is trivial; we show the hardness by reducing from dominating set on induced triangular grid graphs. (These are finite induced subgraphs of the triangular grid, which is the graph with vertex set $V = \mathbb{Z}^2$ and edge set $E = \big\{\big((a,b),(a+\alpha,b+\beta)\big) : |\alpha| \leqslant 1, |\beta| \leqslant 1, \alpha \neq \beta\big\}$.) The NP-hardness of dominating set in induced triangular grid graphs is proven in the final version. Note that the dominating set problem is known to be NP-hard on induced grid graphs, but this does not imply the hardness on triangular grids, because triangular grid graphs are not a superclass of grid graphs.

We show that the infinite triangular grid can be realized as a Q-intersection graph, where the Q-translates are in a bijection with the vertices of the triangular grid. Therefore, any induced triangular grid graph is realized as the intersection graph of the Q-translates corresponding to its vertices.

Rescale Q so that it has span 1. It cannot happen that all the points are rational, because it would make all distance ratios rational as well. Let $x^* \in Q$ be the smallest irrational point. Let $a \in \mathbb{Z}$, and consider the intersection of the translate $ax^* + Q$ with the set $\mathbb{Z} + Q$. We claim that this intersection is non-empty only for a finite number of values $a \in \mathbb{Z}$. Suppose the opposite. Since Q is a finite pattern, there must be a pair $z, z' \in Q$ such that $ax^* + z = b + z'$ has infinitely many solutions $(a, b) \in \mathbb{Z}^2$. In particular, there are two solutions (a_1, b_1) and (a_2, b_2) such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Subtracting the two equations we get $(a_1 - a_2)x^* = b_1 - b_2$, which implies $x^* = \frac{b_1 - b_2}{a_1 - a_2}$. This is a contradiction since x is irrational.

Let $y^* = a'x^*$, where a' is the largest value a for which $ax^* + Q$ intersects $\mathbb{Z} + Q$. It follows that $\big\{j \in \mathbb{Z} \,\big|\, (jy^* + Q) \cap (\mathbb{Z} + Q) \neq \emptyset\big\} = \big\{-1, 0, 1\big\}$.

Consider the intersection graph induced by the sets $\big\{jy^* + k + Q \,\big|\, (j,k) \in \mathbb{Z}^2\big\}$. The above shows that a fixed translate $jy^* + k + Q$ is not intersected by the translates $(j+\alpha)y^* + (k+\beta) + Q$ if $|\alpha| \geq 2$. It is easy to see that $|\beta| \geq 2$ does not lead to an intersection either. Also note that $\alpha = \beta = \pm 1$ does not give an intersection; however all the remaining cases are intersecting, i.e., if

$$(\alpha, \beta) \in \big\{(-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0)\big\}$$

then $(j+\alpha)y^* + (k+\beta) + Q$ intersects $jy^* + k + Q$. Thus, the intersection graph induced by $\big\{jy^* + k + Q \,\big|\, (j,k) \in \mathbb{Z}^2\big\}$ is a triangular grid. ◀

▶ **Lemma 7.** *If Q is a point pattern that has point pairs with an irrational distance ratio, then Q-Intersection Dominating Set has an* FPT *algorithm parameterized by solution size.*

Proof. In polynomial time, we can remove all duplicate translates, since a minimum dominating set contains at most one of these objects, and any minimum dominating set of the resulting graph is a dominating set of the original graph. Suppose our pattern consists of t points. In the duplicate-free graph, point i of the pattern translate may intersect point j of another translate, for some $i \neq j$, so the maximum degree is $t^2 - t$. Therefore we are looking for a dominating set in a graph of bounded degree. Hence, a straightforward branching approach gives an FPT algorithm: choose any undominated vertex v; either v or one of its at most $t^2 - t$ neighbors is in the dominating set, so we can branch $t^2 - t + 1$ ways. If all vertices are dominated after choosing k vertices, then we have found a solution. This branching algorithm has depth k, with linear time required at each branching, so the total running time is $O\big(t^{2k}(|V| + |E|)\big)$. ◀

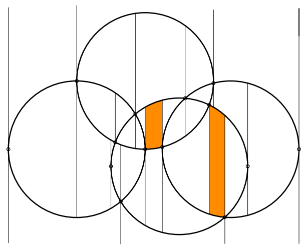

Figure 2 Two faces of a vertical decomposition.

▶ Remark. In our handling of the problem, the pattern was part of the problem definition. Making the pattern part of the input leads to an NP-complete problem: Lemma 6 can be adapted to this scenario. If we also allow the size of the pattern to depend on the input, then the problem is W[2] complete (when parameterized by solution size): see the final version, where we show that for any graph G there is a finite pattern whose translates can produce G as an intersection graph.

We propose the following problem for further study, where the pattern depends on the input, but has fixed size.

Open question. Let Q be the pattern defined by two unit intervals on a line at distance ℓ. Is there an FPT algorithm (either with parameter k or $k+\ell$) on intersection graphs defined by translates of Q, that can decide if such a graph has a dominating set of size k? It can be shown that this problem is NP-complete, and Theorem 10 below shows that it is contained in W[1].

3 Higher dimensional shapes: W[1] vs. W[2]

In this section we show that dominating set on intersection graphs of 2-dimensional objects is contained in W[1] if the shapes have a constant size description. First, we demonstrate the method on unit disk graphs, and later we state a much more general version where the shapes are semi-algebraic sets. In order to show containment, it is sufficient to give a non-deterministic algorithm that has an FPT time deterministic preprocessing, then a nondeterministic phase where the number of steps is only dependent on the parameter. More precisely, we use the following theorem.

▶ **Theorem 8** ([7]). *A parameterized problem is in* W[1] *if and only if it can be computed by a nondeterministic RAM program accepting the input that*

1. *performs at most $f(k)p(n)$ deterministic steps;*
2. *uses at most $f(k)p(n)$ registers;*
3. *contains numbers smaller than $f(k)p(n)$ in any register at any time;*
4. *for any run on any input, the nondeterministic steps are among the last $g(k)$ steps.*

Here n is the size of the input, k is the parameter, p is a polynomial and f, g are computable functions. The non-deterministic instruction is defined as guessing a natural number between 0 and the value stored in the first register, and storing it in the first register. Acceptance of an input is defined as having a computation path that accepts.

▶ **Theorem 9.** *The dominating set problem on unit disk graphs is contained in* W[1].

Proof. Let P be the set of centers of the unit disks that form the input instance. For a subset $D \subseteq P$, let $\mathcal{C}_2(D)$ and $\mathcal{D}_2(D)$ be the set of circles and disks of radius 2, respectively, centered at the points of D. (Note that D is a dominating set if and only if $\bigcup \mathcal{D}_2(D)$, the union of the disks in $\mathcal{D}_2(D)$, covers all points in P.) Shoot a vertical ray up and down from each of the $O(k^2)$ intersection points between the circles of $\mathcal{C}_2(D)$, and also from the leftmost and rightmost point of each circle. Each ray is continued until it hits a circle (or to infinity). The arrangement we get is a *vertical decomposition* [2] (see Fig. 2). Each face of this decomposition is defined by at most four circles. This is not only true for the 2-dimensional faces, but also for the 1-dimensional faces (the edges of the arrangement) and 0-dimensional faces (the vertices). We consider the faces to be relatively open, so that they are pairwise disjoint.

In our preprocessing phase, we compute all potential faces of a vertical decomposition of any subset $D \subseteq P$ by looking at all 4-tuples of circles from $\mathcal{C}_2(P)$. We create a lookup table that contains the number of input points covered by each potential face in $O(n^4)$ time.

Next, using nondeterminism we guess k integers, representing the points of our solution; let D be this point set. The rest of the algorithm deterministically checks if D is dominating. We need to compute the vertical decomposition of $\mathcal{C}_2(D)$; this can be done in $O(k^2)$ time [2]. Finally, for each of the $O(k^2)$ resulting faces of $\bigcup \mathcal{D}_2(D)$, we can get the number of input points covered from the lookup table in constant time. We accept if these numbers sum to n. By Theorem 8 we can thus conclude that dominating set on unit disk graphs is in W[1]. ◀

In order to state the general version of this theorem, we introduce semi-algebraic sets. A *semi-algebraic set* is a subset of $\mathbb{R}^d$ obtained from a finite number of sets of the form $\{x \in \mathbb{R}^d \mid g(x) \geqslant 0\}$, where g is a d-variate polynomial with integer coefficients, by Boolean operations (unions, intersections, and complementations). Let $\Gamma_{d,\Delta,s}$ denote the family of all semi-algebraic sets in $\mathbb{R}^d$ defined by at most s polynomial inequalities of degree at most Δ each. If d, Δ, s are all constants, we refer to the sets in $\Gamma_{d,\Delta,s}$ as constant-complexity semi-algebraic sets.

Let $\mathcal{F}$ be a family of constant complexity semi-algebraic sets in $\mathbb{R}^d$ that can be specified using t parameters $a_1, \dots, a_t$. If the expressions defining $\mathcal{F}$ are also polynomials in terms of the parameters, then we call $\mathcal{F}$ a *t-parameterized family of semi- algebraic sets*. For example, the family of all balls in the $\mathbb{R}^3$ is a 4-parameterized family of semi- algebraic sets, since any ball can be specified using an inequality of the form $(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2-a_4^2 \leqslant 0$. As another example, the family of all triangles in the plane is a 6-parameterized algebraic set, since any triangle is the intersection of three half-planes, and any half- plane can be specified using two parameters.

We only give a sketch of the proof, the complete proof can be found in the final version.

▶ **Theorem 10.** *Let $\mathcal{F}$ be a t-parameterized family of semi-algebraic sets, for some constant t. Then dominating set is in* W[1] *for intersection graphs defined by $\mathcal{F}$.*

Proof sketch of Theorem 10. By definition, any set $S \in \mathcal{F}$ can be specified using t parameters $a_1, \dots, a_t$. Thus we can represent S by the point $\mathbf{p}(S) := (a_1, \dots, a_t)$ in $\mathbb{R}^t$. Conversely, for a point $(a_1, \dots, a_t) \in \mathbb{R}^t$, let $S(a_1, \dots, a_t)$ be the corresponding semi-algebraic set. Now we define, for any set $S \in \mathcal{F}$, a region $\mathbf{R}(S)$ as follows:

$$\mathbf{R}(S) := \{(a_1, \dots, a_t) \in \mathbb{R}^t : S(a_1, \dots, a_t) \cap S \neq \emptyset\}.$$

Thus for any two sets $S_1, S_2 \in \mathcal{F}$ we have that $S_1 \cap S_2 \neq \emptyset$ if and only if $\mathbf{p}(S_1) \in \mathbf{R}(S_2)$.

Now consider a set $\mathcal{S} \subset \mathcal{F}$ of n sets from the family $\mathcal{F}$. We proceed in a similar way as in the proof of Theorem 9, where the sets $\mathbf{R}(S)$ for $S \in \mathcal{S}$ play the same role as the radius-2 disks in that proof. Consider any subset $\mathcal{D} \subseteq \mathcal{S}$, and note that $\mathcal{D}$ is a dominating set if and only if $\bigcup_{S\in\mathcal{D}} \mathbf{R}(S)$ contains the point set $\{\mathbf{p}(S)|S \in \mathcal{S}\}$.

Now we can decompose the arrangement defined by $\{\mathbf{R}(S) : S \in \mathcal{D}\}$ into polynomially many cells using a so-called *cylindrical decomposition* [1]; note that such a decomposition is made possible by the fact that the regions $\mathbf{R}(S)$ are semi-algebraic. (This decomposition plays the role of the vertical decomposition in the proof for unit disks.) Each cell of the cylindrical decomposition is defined by at most t' regions $\mathbf{R}(S)$, for some $t' = O(1)$. Thus, for each subset of at most t' regions $\mathbf{R}(S)$, we compute all cells that arise in the cylindrical decomposition of the subset. The number of possible cells is polynomial in n.

In the preprocessing phase, we compute for each possible cell the number of points $\mathbf{p}(S)$ contained in it, and store the results in a lookup table. The next phase of the algorithm is the same as for unit disks: we guess a solution, compute the cells in the cylindrical decomposition of the corresponding arrangement, and check using the lookup table if the guessed solution is a dominating set. ◀

W[1]-hardness for simple polygon translates

We generalize a proof by Marx [10] for the W[1]-hardness of dominating set in unit square/unit disk graphs. Our result is based on the observation that many 2-dimensional shapes share the crucial properties of unit squares when it comes to the type of intersections needed for this specific construction. We prove the following theorem.

▶ **Theorem 11.** *The dominating set problem is* W[1]*-hard for intersection graphs of the translates of a simple polygon in* $\mathbb{R}^2$.

Our proof uses the same global strategy as Marx's proof [10] for the W[1]-hardness of dominating set for intersection graphs of squares. (We give an overview of the proof in the final version.) To apply this proof strategy, all we need to prove is that the family of shapes for which we want to prove W[1]-hardness has a certain property, as defined next.

We say that a shape $S \subseteq \mathbb{R}^2$ is *square-like* if there are two base vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ and for any n there are two small offset vectors $\mathbf{u}_1 = \mathbf{u}_1(n)$ and $\mathbf{u}_2 = \mathbf{u}_2(n)$ with the following properties. Define $S(i,j) := S + i\mathbf{u}_1 + j\mathbf{u}_2$ for all $-n^2 \leqslant i,j \leqslant n^2$, and consider the set $\mathcal{K} := \{S(i,j) : -n^2 \leqslant i,j \leqslant n^2\}$. Note that $\mathcal{K}$ consists of $(2n^2+1)^2$ translated copies of S whose reference points from a $(2n^2+1) \times (2n^2+1)$ grid. Also note that $S = S(0,0)$. For the shape S to be square-like, we require the following properties:

- $\mathcal{K}$ is a clique in the intersection graph, i.e.,
  for all $-n^2 \leqslant i,j \leqslant n^2$ we have: $S \cap S(i,j) \neq \emptyset$.
- "Horizontal" neighbors intersect only when close:
  for all $-n^2 \leqslant j \leqslant n^2$ we have: $S \cap (\mathbf{b}_1 + S(i,j)) \neq \emptyset \iff i \leqslant 0$.
- "Vertical" neighbors intersect only when close:
  for all $-n^2 \leqslant i \leqslant n^2$ we have: $S \cap (\mathbf{b}_2 + S(i,j)) \neq \emptyset \iff j \leqslant 0$.
- Distant copies of $\mathcal{K}$ are disjoint:
  for all $-n^2 \leqslant i,j,i',j' \leqslant n^2$ we have: $|k|+|\ell| \geqslant 2 \Rightarrow S(i,j) \cap (k\mathbf{b}_1 + \ell\mathbf{b}_2 + S(i',j')) = \emptyset$.

Moreover, we require that each of the vectors can be represented on $O(\log n)$ bits. It is helpful to visualize a square grid, with unit side lengths $\mathbf{b}_1$ and $\mathbf{b}_2$, where we place the centers of unit squares with small offsets compared to the grid points. We are requiring a very similar intersection structure here. See Figure 5 for an example of a good choice of vectors.

Since the above properties are sufficient for the construction given by Marx [10], we only need to prove the following theorem.

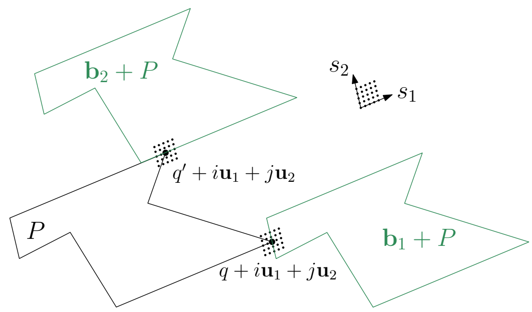

Figure 3 A good choice of $\mathbf{b}_1, \mathbf{b}_2$ and offsets.

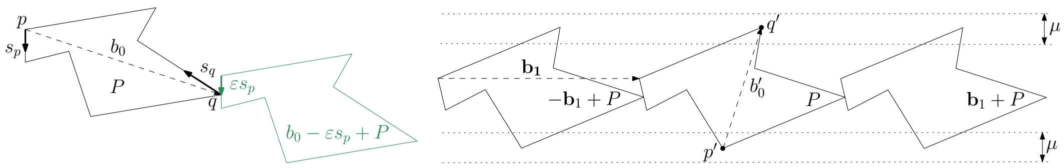

Figure 4 Left: Defining $\mathbf{b}_1$. Right: Defining $\mathbf{b}_2$.

▶ **Theorem 12.** *Every simple polygon is square-like.*

Before giving a formal proof, we give a short overview. First, we would like to define a "horizontal" direction, i.e., a good vector $\mathbf{b}_1$. A natural choice would be to select a diameter of the polygon (see b_0 on the left of Figure 4), however that would result in S and $\mathbf{b}_1 + S$ intersecting each other at vertices. That would pose a severe restriction on the offset vectors; therefore, we use a perturbed version of a diameter, making sure that the intersection of S and $\mathbf{b}_1 + S$ is realized by a polygon side from at least one party. The direction of this polygon side also defines a suitable direction of the offset vector $\mathbf{u}_2$: because of the second property, choosing $\mathbf{u}_2$ to be parallel to this direction ensures the independence with respect to the choice of j.

Next, we define the other base vector $\mathbf{b}_2$. This definition is based on laying out an infinite sequence of translates horizontally next to each other (right side of Figure 4). We want a translate of this sequence to touch the original sequence in a "non-intrusive" way: small perturbations of $\mathbf{b}_2 + S$ should only intersect S, but stay disjoint from $\mathbf{b}_1 + S$ or $-\mathbf{b}_1 + S$. This is fairly easy to achieve; again with a small perturbation of our first candidate vector we can also ensure that the intersection between $\mathbf{b}_2 + S$ and S is not a vertex-vertex intersection. Finally, a suitable direction for the offset vector $\mathbf{u}_1$ is given by the polygon side taking part in the intersection between $\mathbf{b}_2 + S$ and S.

Proof of Theorem 12. Let P be a simple polygon, and let p and q be two endpoints of a diameter of P. Let $b_0 = q - p$. Since P is a polygon, both p and q are vertices. Let s_p and s_q be unit vectors in the direction of the side of P that follows vertex p and q in the counter-clockwise order. Let $\varepsilon > 0$ be a small number to be specified later. Consider the intersection of P and the translate $b_0 + \varepsilon s_q + P$. If ε is small enough, then depending on the angle of s_p and s_q, this intersection is either the point $b_0 + \varepsilon s_q$, or part of the side with direction s_q, or it is an intersection of positive area. The left of Figure 4 illustrates the third case. In the first case, let $\mathbf{b}_1 = b_0 + \varepsilon s_p$; in the second and third case, let $\mathbf{b}_1 = b_0 - \varepsilon s_q$. Furthermore, let $s_1 = \mathbf{b}_1 - b_0$. We will later use s_1 to define the offset vector $\mathbf{u}_2$.

Imagine that $\mathbf{b}_1$ is the horizontal direction, and consider the set $P_\infty = \{k\mathbf{b}_1 + P \mid k \in \mathbb{Z}\}$ (right side of Figure 4). Its top and bottom boundary are infinite periodic polylines, with

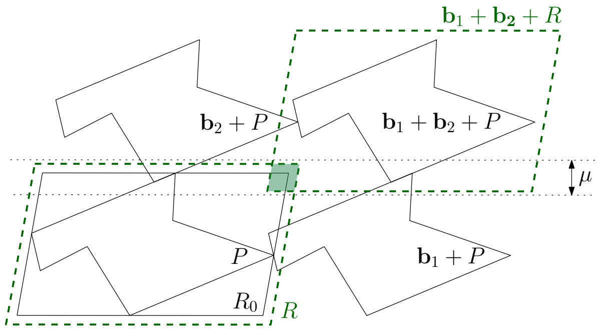

Figure 5 Part of the grid $k\mathbf{b}_1 + \ell\mathbf{b}_2 + P$.

period length $|\mathbf{b}_1|$. Take a pair of horizontal lines that touch the top and bottom boundary. By manipulating ε in the definition of $\mathbf{b}_1$, we can achieve a general position in the sense that both of these lines touch the respective boundaries exactly once in each period, moreover, there is a value μ, such that there are no vertices other than the touching points in the $\frac{\mu}{2}$-neighborhood of the touching lines. Let p' and q' be vertices touched by the bottom and top lines inside P, and let $b_0' = q' - p'$. Similarly as before, the direction of the sides following p' and q' counter-clockwise are denoted by $s_{p'}$ and $s_{q'}$. If the intersection of P and the translate $b_0' + \varepsilon s_{q'} + P$ has zero area, then let $\mathbf{b}_2 = b_0' - \varepsilon s_{q'}$; otherwise, (if the area of the intersection is positive), let $\mathbf{b}_2 = b_0' + \varepsilon s_{p'}$. We denote by s_2 the difference $\mathbf{b}_2 - b_0'$. If s_2 and s_1 are parallel, then we can define s_2 similarly, by replacing the sides $s_{p'}$ and $s_{q'}$ with the sides that follow p' and q' in clockwise direction. The new direction of s_2 will not be parallel to the old one, therefore it will not be parallel to s_1.

We need to choose the values of $\mathbf{u}_1$ and $\mathbf{u}_2$. Let $\mathbf{u}_1 = \frac{\varepsilon}{2n^2} s_2$ and let $\mathbf{u}_2 = \frac{\varepsilon}{2n^2} s_1$. We claim that if ε is small enough, then P is square-like for the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{u}_1, \mathbf{u}_2$. It is easy to check that for a small enough value of ε, the first condition is satisfied, namely that $P \cap P(i,j) \neq \emptyset$ for all $-n^2 \leqslant i,j \leqslant n^2$.

Next, we show that for $i \leqslant 0$, the intersection of P and $\mathbf{b}_1 + P(i,j)$ is non-empty. Consider the small grid of points $q - i\mathbf{u}_1 - j\mathbf{u}_2$, $-n^2 \leqslant i,j \leqslant n^2$ (see Figure 3). This grid fits into a parallelogram whose sides are parallel to s_2 and s_1. Notice that if ε is small enough, then $q - i\mathbf{u}_1 - j\mathbf{u}_2$ for all $-n^2 \leqslant i < 0$ and $-n^2 \leqslant j \leqslant n^2$ is contained in $\mathbf{b}_1 + P$, thus the intersection $P \cap (\mathbf{b}_1 + i\mathbf{u}_1 + j\mathbf{u}_2 + P)$ is non-empty if $i \leqslant 0$. Moreover, (if ε is small enough), then no other type of intersection can happen by moving $\mathbf{b}_1 + P$ slightly: the only sides that can intersect $\mathbf{b}_1 + P(i,j)$ from P are adjacent to q. Therefore, if q is outside $\mathbf{b}_1 + P(i,j)$, then the intersection is empty – which is true for $i > 0$. A similar argument works for the intersection of P and $\mathbf{b}_2 + P(i,j)$.

Let R_0 be a minimum area parallelogram containing P whose sides are parallel to $\mathbf{b}_1$ and $\mathbf{b}_2$ (see Figure 5). Notice that the side lengths of this parallelogram are at most $|\mathbf{b}_1| + \varepsilon$ and $|\mathbf{b}_2| + \varepsilon$ respectively. Let $\bar{P} = \bigcup \mathcal{K} = \bigcup_{-n^2 \leqslant i,j \leqslant n^2} P(i,j)$. Notice that $\bar{P}$ is contained in the slightly larger rectangle R that we get by extending all sides of R_0 by 2ε.

Now consider the rectangle translates $k\mathbf{b}_1 + \ell\mathbf{b}_2 + R$. Since ε is small enough, if either k or ℓ is at least two then $R \cap (k\mathbf{b}_1 + \ell\mathbf{b}_2 + R) = \emptyset$, so specifically, $\bar{P}$ is disjoint from $k\mathbf{b}_1 + \ell\mathbf{b}_2 + \bar{P}$. It remains to show that $\bar{P}$ is disjoint from $k\mathbf{b}_1 + \ell\mathbf{b}_2 + \bar{P}$ if $|k| = |\ell| = 1$. Consider $\bar{P}$ and $\mathbf{b}_1 + \mathbf{b}_2 + \bar{P}$ for example. They could only intersect inside $R \cap (\mathbf{b}_1 + \mathbf{b}_2 + R)$; however, if $\varepsilon < \frac{\mu}{4}$, then this is contained in the μ wide horizontal strip defined earlier. By the definition of this strip, it also means that there is an intersection point q that is within distance $O(\varepsilon)$ from both q' and $\mathbf{b}_1 + \mathbf{b}_2 + p'$. This would mean that $|\mathbf{b}_1| = O(\varepsilon)$, and thus it can be avoided by choosing a small enough ε.

Finally, we note that all restrictions on the value of ε are dependent on the polygon P itself, thus the length of the short vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ is $\Omega(n^{-2})$, and a precision of $O(n^{-2})$ is sufficient for all the vectors, thus the vectors can be represented on $O(\log n)$ bits. ◀

We remark that it is fairly easy to further generalize the above theorem to other families of objects, we can allow objects with certain curved boundaries for example. A simple example of an object that is not square-like is a pair of perpendicular disjoint unit segments: for any choice of offset vectors, the set $\mathcal{K}$ does not form a clique (as required by the first property of square-like objects).

W[2]-hardness for convex polygons

We conclude with the following hardness result; the reduction uses a basic geometric idea that has been used for hardness proofs before [8, 11]. Note the crucial difference between the setting in this theorem, where the polygons defining the intersection graph can be different and have description complexity dependent on n, versus the previous settings (where we had constant description complexity and some uniformity among the object descriptions).

▶ **Theorem 13.** *The dominating set problem is* W[2]*-hard for intersection graphs of convex polygons.*

Proof. A *split graph* is a graph that has a vertex set which can be partitioned into a clique C and an independent set I. It was shown by Raman and Saurabh [12] that dominating set is W[2]-hard on split graphs. Thus it is sufficient to show that any split graph can be represented as the intersection graph of convex polygons.

Let $G = (C \cup I, E)$ be an arbitrary split graph. Let Q' be a regular $2|I|$-gon and let Q be the regular I-gon defined by every second vertex of Q'. Notice that $Q' \setminus Q$ consists of small triangles, any subset of which together with Q forms a convex polygon.

The polygons corresponding to I are small equilateral triangles, placed in the interior of each small triangle of $Q' \setminus Q$. The polygon corresponding to a vertex $v \in C$ whose neighborhood in I is $N_I(v)$ is the union of Q and the small triangles corresponding to the vertices of $N_I(v)$.

In this construction, the polygons corresponding to C all intersect (they all contain Q), and the polygons corresponding to I are all disjoint. Finally, for any pair of vertices $u \in C$ and $v \in I$ the polygon of u contains the polygon of v if and only if $uv \in E$. ◀

4 Conclusion

We have classified the parameterized complexity of dominating set in intersection graphs defined by sets of various types in $\mathbb{R}^1$ and $\mathbb{R}^2$. More precisely, in $\mathbb{R}^1$, we gave a classification for the case when the intersection graph is defined by the translates of a fixed pattern that consists of points and intervals that is independent of the input. In $\mathbb{R}^2$, we have identified a fairly large class of W[1]-complete instances, namely, if our intersection graph is defined by a subset of a constant description complexity family of semi-algebraic sets. Even though our results hold for a large class of geometric intersection graphs, there are still some open problems. In particular, the complexity of dominating set on the following types of intersections graphs is unknown.

- translates of a 1-dimensional pattern that contains two unit intervals at some distance ℓ (given by the input) (FPT vs. W[1]?)

- translates of a 2-dimensional pattern that contains two disjoint perpendicular unit intervals (FPT vs. W[1]?)
- n translates of a regular n-gon (W[1] vs. W[2]?)

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