Tree Drawings Revisited

Timothy M. Chan

Department of Computer Science, University of Illinois at Urbana-Champaign, USA
tmc@illinois.edu

──── **Abstract** ────────────────────

We make progress on a number of open problems concerning the area requirement for drawing trees on a grid. We prove that

1. every tree of size n (with arbitrarily large degree) has a straight-line drawing with area $n2^{O(\sqrt{\log\log n \log\log\log n})}$, improving the longstanding $O(n \log n)$ bound;
2. every tree of size n (with arbitrarily large degree) has a straight-line upward drawing with area $n\sqrt{\log n}(\log\log n)^{O(1)}$, improving the longstanding $O(n \log n)$ bound;
3. every binary tree of size n has a straight-line orthogonal drawing with area $n2^{O(\log^* n)}$, improving the previous $O(n \log\log n)$ bound by Shin, Kim, and Chwa (1996) and Chan, Goodrich, Kosaraju, and Tamassia (1996);
4. every binary tree of size n has a straight-line order-preserving drawing with area $n2^{O(\log^* n)}$, improving the previous $O(n \log\log n)$ bound by Garg and Rusu (2003);
5. every binary tree of size n has a straight-line orthogonal order-preserving drawing with area $n2^{O(\sqrt{\log n})}$, improving the $O(n^{3/2})$ previous bound by Frati (2007).

1 Introduction

Drawing graphs with small area has been a subject of intense study in combinatorial and computational geometry for more than two decades [11, 12]. The goal is to determine worst-case bounds on the area needed to draw any n-vertex graph in a given class, subject to certain drawing criteria, where vertices are mapped to points on an integer grid $\{1, \dots, W\} \times \{1, \dots, H\}$, and the *area* of the drawing is defined to be the width W times the height H. All drawings in this paper are required to be *planar*, where edge crossings are not allowed. All our results will be about *straight-line* drawings, where edges are drawn as straight line segments, although *poly-line* drawings that allow bends along the edges have also received considerable attention.

It is well known [10, 23] that every planar graph of size n has a straight-line drawing with area $O(n^2)$ (with width and height $O(n)$), and this bound is asymptotically tight in the worst case. Much research is devoted to understanding which subclasses of planar graphs admit subquadratic-area drawings, and obtaining tight area bounds for such classes.

Drawing arbitrary trees. Among the simplest is the class of all trees. As hierarchical structures occur naturally in many areas (from VLSI design to phylogeny), visualization of trees is of particular interest. Although there have been numerous papers on tree drawings
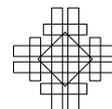
■ **Figure 1** The "standard" algorithm to produce a straight-line upward drawing of any tree of size n, with width at most n and height at most $\lceil \log n \rceil$: reorder the subtrees so that T_d is the largest, then recursively draw $T_1, \ldots, T_d$.

(e.g., [2, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19, 20, 22, 25, 26, 24, 27, 28]), the most basic question of determining the worst-case area needed to draw arbitrary trees, without any additional criteria other than being planar and straight-line, is surprisingly still open.

An $O(n \log n)$ area upper bound is folklore and can be obtained by a straightforward recursive algorithm, as described in Figure 1, which we will refer to as the *standard* algorithm (the earliest reference was perhaps Shiloach's 1976 thesis [24, page 94]; see also Crescenzi, Di Battista, and Piperno [8] for the same algorithm for binary trees). The algorithm gives linear width and logarithmic height. An analogous algorithm, with x and y coordinates swapped, gives logarithmic width and linear height.

However, no single improvement to the $O(n \log n)$ bound has been found for general trees. No improvement is known even if drawings are relaxed to be poly-line!

In an early SoCG'93 paper by Garg, Goodrich, and Tamassia [15], it was shown that linear area is attainable for poly-line drawings of trees with degree bounded by $O(n^{1-\varepsilon})$ for any constant $\varepsilon > 0$. Later, Garg and Rusu [18, 17] obtained a similar result for straight-line drawings for degree up to $O(n^{1/2-\varepsilon})$.[1] These approaches do not give good bounds when the maximum degree is linear.

To understand why unbounded degree can pose extra challenges, consider the extreme case when the tree is a star of size n, and we want to draw it on an $O(\sqrt{n}) \times O(\sqrt{n})$ grid. A solution is not difficult if we use the fact that relatively prime pairs are abundant, but most tree drawing algorithms use geometric divide-and-conquer strategies that do not seem compatible with such number-theoretic ideas.

New results. Our first main result is the first $o(n \log n)$ area upper bound for straight-line drawings of arbitrary trees: the bound is $n2^{O(\sqrt{\log \log n \log \log \log n})}$, which in particular is better than $O(n \log^{\varepsilon} n)$ for any constant $\varepsilon > 0$.

Even to those who care less about refining logarithmic factors, our method has one notable advantage: it can give drawings achieving a full range of width–height tradeoffs (in other words, a full range of aspect ratios). For example, we can simultaneously obtain width and height $\sqrt{n}2^{O(\sqrt{\log n \log \log n})}$. Although the extra factor is now superpolylogarithmic, the result is still new. In contrast, the standard algorithm (Figure 1) produces only narrow drawings, whereas the previous approaches of Garg et al. [15, 18] provided width–height tradeoffs but inherently cannot give near $\sqrt{n}$ perimeter if degree exceeds $\sqrt{n}$.

For rooted trees, it is natural to consider *upward* drawings, where the y-coordinate of each node is greater than or equal to the y-coordinate of each child. The drawing obtained by the standard algorithm is upward. We obtain the first $o(n \log n)$ area bound for straight-line upward drawings of arbitrary trees as well: the bound is near $O(n\sqrt{\log n})$, ignoring small

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[1] It is not clear to this author if their analysis assumed a much stronger property, that every subtree of size m has degree at most $O(m^{1/2-\varepsilon})$.

■ **Table 1** Worst-case area bounds for straight-line drawings of *arbitrary trees*. (In all tables, c denotes some constant, and Θ denotes tight results that have matching lower bounds.)

	non-order-preserving		order-preserving
non-upward	$O(n \log n)$ $O(nc^{\sqrt{\log\log n \log\log\log n}})$	by standard alg'm **new**	$O(n \log n)$ by Garg–Rusu'03 [16]
upward	$O(n \log n)$ $O(n\sqrt{\log n} \log^c \log n)$	by standard alg'm **new**	$O(nc^{\sqrt{\log n}})$ by Chan'99 [6]
strictly upward	$\Theta(n \log n)$ by standard alg'm [8]		$O(nc^{\sqrt{\log n}})$ by Chan'99 [6]

■ **Table 2** Worst-case area bounds for straight-line drawings of *binary trees*.

	non-order-preserving	order-preserving	
non-upward	$\Theta(n)$ by Garg–Rusu'04 [18]	$O(n \log\log n)$ by Garg–Rusu'03 [16] $O(nc^{\log^* n})$	**new**
upward	$O(n \log\log n)$ by Shin–Kim–Chwa'96 [25]	$O(n^{1.48})$ by Chan'99 [6] $O(nc^{\sqrt{\log n}})$ by Chan'99 [6] $O(n \log n)$ by Garg–Rusu'03 [16]	
strictly upward	$\Theta(n \log n)$ by standard alg'm [8]	$O(n^{1.48})$ by Chan'99 [6] $O(nc^{\sqrt{\log n}})$ by Chan'99 [6] $\Theta(n \log n)$ by Garg–Rusu'03 [16]	

$\log\log$ factors. (See Table 1.)

These results represent significant progress towards Open Problems 5, 6, 17, and 18 listed in Di Battista and Frati's recent survey [12].

We will describe the near-$O(n\sqrt{\log n})$ upward algorithm first, in Section 2, which prepares us for the more involved $n2^{O(\sqrt{\log\log n \log\log\log n})}$ non-upward algorithm in Section 3.

Drawing binary trees.   Next we turn to drawings of binary trees, where there has been a large body of existing work, due to the many combinations of aesthetic criteria that may be imposed. We may consider

- *upward* drawings, as defined earlier;
- *strictly upward* drawings, where the y-coordinate of each node is strictly greater the y-coordinate of each child;
- *order-preserving* drawings, where the order of children of each node v is preserved, i.e., the parent, the left child, and the right child of v appear in counterclockwise order around v;
- *orthogonal* drawings, where all edges are drawn with horizontal or vertical line segments.

Tables 2–3 summarize the dizzying array of known results on straight-line drawings. (To keep the table size down, we omit numerous other results on poly-line drawings, and on special subclasses of balanced trees. See Di Battista and Frati's survey [12] for more.)

■ **Table 3** Worst-case area bounds for straight-line *orthogonal* drawings of *binary trees*. (Strictly upward drawings are not possible here.)

	non-order-preserving		order-preserving
non-upward	$O(n \log \log n)$ $O(nc^{\log^* n})$ **new**	by Chan–Goodrich–Kosaraju–Tamassia'96 [7] & Shin–Kim–Chwa'96 [25]	$O(n^{3/2})$ Frati'07 [13] $O(nc^{\sqrt{\log n}})$ **new**
upward	$\Theta(n \log n)$ by standard alg'm [8]		$\Theta(n^2)$

New results.   In this paper, we concentrate on two of the previous $O(n \log \log n)$ entries in the table. In 1996, Shin, Kim, and Chwa [25] and Chan et al. [7] independently obtained $O(n \log \log n)$-area algorithms for straight-line orthogonal drawings of binary trees; a few years later, Garg and Rusu [16] adapted their technique to obtain similar results for straight-line (non-orthogonal) order-preserving drawings. We improve the area bound for both types of drawings to *almost* linear: $n2^{O(\log^* n)}$, where $\log^*$ denotes the iterated logarithm. (We can also obtain width–height tradeoffs for these drawings.)

Although improving $\log \log n$ to iterated logarithm may not come as a total surprise, the problem for straight-line orthogonal drawings has resisted attack for 20 years. (Besides, improvement should not be taken for granted, since there is at least one class of drawings for which $\Theta(n \log \log n)$ turns out to be tight: poly-line upward orthogonal drawings of binary trees [15].)

We have additionally one more result on straight-line orthogonal order-preserving drawings of binary trees: in 2007, Frati [13] presented an $O(n^{3/2})$-area algorithm. We improve the bound to $n2^{O(\sqrt{\log n})}$, which in particular is better than $O(n^{1+\varepsilon})$ for any constant $\varepsilon > 0$.

These results represent significant progress towards Open Problems 9, 12, and 14 listed in Di Battista and Frati's survey [12].

(The author has obtained still more new results, on a special class of so-called *LR drawings* of binary trees [6, 14], making progress on Open Problem 10 in the survey, which will be reported later elsewhere.)

We will describe the $n2^{O(\log^* n)}$ algorithm for orthogonal drawings first, in Section 4; the algorithm for non-orthogonal order-preserving drawings is similar, and is described in the full paper. The $n2^{O(\sqrt{\log n})}$ algorithm for orthogonal order-preserving drawings is different and is also deferred to the full paper.

Techniques.   Various tree-drawing techniques have been identified in the large body of previous work, and we will certainly draw upon some of these existing techniques in our new algorithms—in particular, the use of "skewed" centroids for divide-and-conquer in trees (see Section 2 for the definition), and height–width tradeoffs to obtain better area bounds.

However, as the unusual bounds would suggest, our $n2^{O(\sqrt{\log \log n \log \log \log n})}$ and our $n2^{O(\log^* n)}$ algorithms will require new forms of recursion and bootstrapping.

Our $n2^{O(\sqrt{\log \log n \log \log \log n})}$ result for arbitrary trees requires novelty not just in fancier recurrences, but also in geometric insights. All existing divide-and-conquer algorithms for tree drawings divide a given tree into subtrees and recursively draw different subtrees in different, disjoint axis-aligned bounding boxes. We will depart from tradition and draw some parts of the tree in distorted grids inside narrow sectors, which are remapped to regular grids through

affine transformations every time we bootstrap. The key is a geometric observation that any two-dimensional convex set (however narrow) containing a large number of integer points must contain a large subset of integer points forming a grid after affine transformation (with unspecified aspect ratio). The proof of the observation follows from well known facts about lattices and basis reduction (by Gauss)—a touch of elementary number theory suffices. We are not aware of previous applications of this geometric observation, which seems potentially useful for graph drawing on grids in general.

Our $n2^{O(\log^* n)}$ result is noteworthy, because occurrences of iterated logarithm are rare in graph drawing (to be fair, we should mention that it has appeared before in one work by Shin et al. [26], on poly-line orthogonal drawings of binary trees with $O(1)$ bends per edge). We realize that more can be gained from the recursion in the previous $O(n \log \log n)$ algorithm, by bootstrapping. This requires a careful setup of the recursive subproblems, and constant switching of x and y (width and height) every time we bootstrap. (The author is reminded of an algorithm by Matoušek [21] on a completely different problem, Hopcroft's problem, where iterated logarithm arose due to constant switching of points and lines by duality at each level of recursion.)

Our $n2^{O(\sqrt{\log n})}$ result for orthogonal order-preserving drawings has the largest quantitative improvement compared to previous results, but actually requires the least originality in techniques. We use the exact same form of recursion as in an earlier algorithm of Chan [6] for non-orthogonal upward order-preserving drawings, although the new algorithm requires trickier details.

2 Straight-line upward drawings of arbitrary trees

In this section, we consider arbitrary (rooted) trees and describe our first algorithm to produce straight-line upward drawings with $o(n \log n)$ area. It serves as a warm-up to the further improved algorithm in Section 3 when upwardness is dropped.

2.1 Preliminaries

We begin with some basic number-theoretic and tree-drawing facts. The first, on the denseness of relatively prime pairs, is well known:

▶ **Fact 1.** *There are $\Omega(AB)$ relatively prime pairs in $\{1, \ldots, A\} \times \{\lfloor B/2 \rfloor + 1, \ldots, B\}$.*

Next, we consider drawing trees not on the integer grid but on a user-specified set of points. We note that any point set of near linear size that is not too degenerate is "universal", in the sense that it can be used to draw any tree.

▶ **Fact 2.** *Let P be a set of $(\ell - 1)n - \ell + 2$ points in the plane, with no ℓ points lying on a common line. Let T be a tree of size n. Then T has a straight-line upward drawing where all vertices are drawn in P.*

Proof. We describe a straightforward recursive algorithm: Let $n_1, \ldots, n_d$ be the sizes of the subtrees $T_1, \ldots, T_d$ at the children of the root v_0, with $\sum_{i=1}^{d} n_i = n - 1$. Place v_0 at the highest point p_0 of P (in case of ties, prefer the leftmost highest point). Form d disjoint sectors with apex at p_0, so that the i-th sector S_i contains between $(\ell - 1)n_i - \ell + 2$ and $(\ell - 1)n_i$ points of $P - \{p_0\}$. This is possible since any line through p_0 contains at most $\ell - 2$ points of $P - \{p_0\}$, and $\sum_{i=1}^{d}(\ell - 1)n_i = (\ell - 1)(n - 1) = |P - \{p_0\}|$. For each $i = 1, \ldots, d$, recursively draw T_i using $(\ell - 1)n_i - \ell + 2$ points of $P \cap S_i$. Lastly, draw the edges from
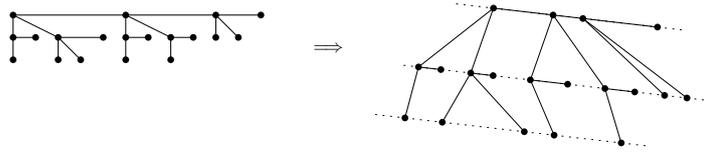
Figure 2 The drawing in Fact 3.

v_0 to the roots of the T_i's (these edges create no crossings since the roots are drawn at the highest points of P in their respective sectors). The base case $n = 1$ is trivial. ◄

The following is a slight generalization of the standard algorithm (mentioned in the introduction) for straight-line upward drawings of general trees with width $O(n)$ and height $O(\log n)$. We note that the algorithm can draw any tree on any point set that "behaves" like an $n \times \lceil \log n \rceil$ grid.

▶ **Fact 3.** *Let G be a set of $\lceil \log n \rceil$ parallel (non-vertical) line segments in the plane. Let P be a set of $n \lceil \log n \rceil$ points, with n points lying on each of the $\lceil \log n \rceil$ line segments in G. Let T be a tree of size n. Then T has a straight-line drawing where all vertices are drawn in P, and the root is drawn on the segment of G whose line has the highest y-intercept.*

Furthermore, if the segments of G are horizontally separated (i.e., the y-projections are disjoint), the drawing is upward.

Proof. Without loss of generality, assume that the segments have negative slope, and arrange the segments of G in decreasing order of y-intercepts. Apply the standard algorithm to get a straight-line upward grid drawing of T with width at most n and height at most $\lceil \log n \rceil$. Map the vertices on the i-th topmost row of the grid drawing to the points on the i-th segment of G, while preserving the left-to-right ordering of the vertices. (See Figure 2.) The resulting drawing is planar (since each edge is drawn either on a segment or in the region between two consecutive segments, and there are no crossings in the region between two consecutive segments). Note that the drawing is upward if the segments of G are horizontally separated. ◄

2.2 The augmented-star algorithm

The main difficulty of drawing arbitrary trees is due to the presence of vertices of large degree. In the extreme case when the tree is a star of size n, we can produce a straight-line drawing of width $O(A)$ and $O(n/A)$ for any given $1 \leq A \leq n$, by placing the root at the origin and placing the remaining vertices at points with co-prime x- and y-coordinates, using Fact 1.

We first study a slightly more general special case which we call *augmented stars*, where the input tree is modified from a star by attaching to each leaf a small subtree of size $\leq s$.

▶ **Lemma 4.** *Let T be a tree of size n such that the subtree at each child of the root has size at most s. For any given $n \geq A \geq 1$, T has a straight-line upward drawing with width $O(A \log s)$ and height $O((n/A) \cdot s \log^2 s)$, where the root is placed at the top left corner of the bounding box, and the left side of the box contains no other vertices.*

Proof. Let $\ell = s \lceil \log s \rceil$. Let $B = \lceil c\ell n/A \rceil$ for some constant c. Let $P = \{(x,y) \in \{1, \ldots, A\} \times \{-B, \ldots, -\lfloor B/2 \rfloor -1\} : x$ and y are relatively prime$\}$. By Fact 1, $|P| = \Omega(AB)$, and so $|P| \geq \ell n$ by making c sufficiently large.

Let $n_1, \ldots, n_d$ be the sizes of the subtrees $T_1, \ldots, T_d$ at the children of the root v_0, with $\sum_{i=1}^{d} n_i = n - 1$ and $n_i \leq s$ for each i. Place v_0 at the origin. Form d disjoint sectors, where

the i-th sector S_i contains exactly ℓn_i points of P. This is possible, since any line through the origin contains at most one point of P and $\sum_{i=1}^{d} \ell n_i < \ell n \le |P|$. We will draw T using not just the points of P, but also scaled copies of these points, up to scaling factor $t := \lceil \log s \rceil$.

For each i, consider two cases, depending on how degenerate $S_i \cap P$ is:

- CASE 1: S_i does not contain ℓ points of P on a common line. Here, we can draw T_i using the $\ell n_i > (\ell - 1)n_i - \ell + 2$ points of $S_i \cap P$ by Fact 2.

- CASE 2: S_i contains ℓ points of P on a common line L. (Note that L does not pass through the origin, by definition of P.) Let σ be a horizontal slab of height $B/(2t)$ that contains at least $\ell/t = s$ points of $L \cap S_i \cap P$. Let $\overline{L} = L \cap S_i \cap \sigma$. Let G be the set of t line segments $\overline{L}, 2\overline{L}, \ldots, t\overline{L}$, where $\alpha \overline{L}$ denotes the scaled copy of $\overline{L}$ by factor α (with respect to the origin). Each of the $t = \lceil \log s \rceil$ segments of G contain s integer points inside S_i, and the segments are horizontally separated. Thus, we can draw T_i using the integer points on G by Fact 3.

Lastly, draw the edges from v_0 to the roots of the T_i's. The total width is $O(tA) = O(A \log s)$ and the height is $O(tB) = O((n/A) \cdot s \log^2 s)$. ◀

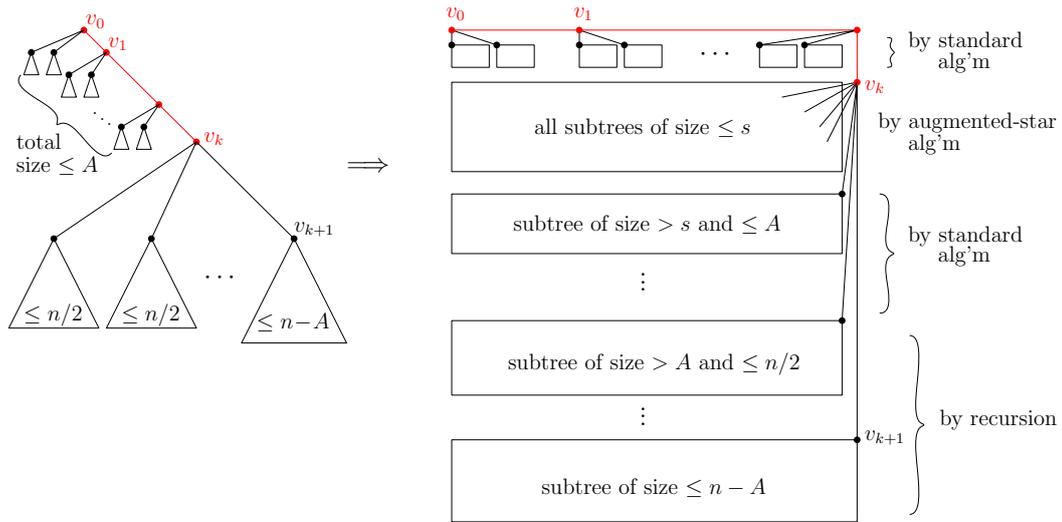## 2.3 The general algorithm

We are now ready to present the algorithm for the general case, using the augmented-star algorithm as a subroutine:

▶ **Theorem 5.** *For any given $n \ge A \ge 1$, every tree T of size n has a straight-line upward drawing with width $O(A + \log n)$ and height $O((n/\sqrt{A}) \log^2 A)$, where the root is placed at the top left corner of the bounding box.*

Proof. We describe a recursive algorithm to draw T: Let s be a fixed parameter with $A \ge \log s$. Let v_0 be the root of T, and define v_{i+1} to be the child of v_i whose subtree is the largest (the resulting root-to-leaf path $v_0 v_1 v_2 \cdots$ is called the *heavy path* of T). Let k be the largest index such that the subtree at v_k has size more than $n - A$ (we will call the node v_k the *A-skewed centroid*). Then the total size of the subtrees at the siblings of $v_1, \ldots, v_k$ is at most A, the subtree at v_{k+1} has size at most $n - A$, and the subtree at each sibling of v_{k+1} has size at most $\min\{n - A, n/2\}$.

The drawing of T, depicted in Figure 3, is constructed as follows (which includes multiple applications of the standard algorithm in steps 1 and 3, one application of the augmented-star algorithm in step 2, and recursive calls in step 4):

1. Draw the subtrees at the siblings of $v_1, \ldots, v_k$ by the standard algorithm. Stack these drawings horizontally. Since these subtrees have total size at most A, the drawing so far has total width $O(A)$ and height $O(\log A)$.

2. Draw the subtrees at the children of v_k that have *size $\le s$*, together with the edges from v_k to the roots of these subtrees, by the augmented-star algorithm in Lemma 4 with parameter $\tilde{A} = \lceil A/\log s \rceil$. By reflection, make v_k lie on the top-right corner of its corresponding bounding box. Place the drawing below the drawings from step 1. This part has width $O(\tilde{A} \log s) = O(A)$ and height $O((n'/\tilde{A}) \cdot s \log^2 s) = O((n'/A) \cdot s \log^3 s)$ where n' is the total size of these subtrees.

   (Note that if $n' \le A$, we can just use the standard algorithm with width $O(A)$ and height $O(\log A)$ for this step.)

3. Draw the subtrees at the children of v_k that have *size $> s$ and $\le A$*, by the standard algorithm. By reflection, make the roots lie on the top-right corners of their respective bounding boxes. Stack these drawings vertically, underneath the drawing from step 2. This

Figure 3 The general algorithm in Theorem 5.

part has width $O(A)$ and height $O((\text{number of these subtrees}) \cdot \log A) \leq O((n''/s) \cdot \log A)$, where n'' is the total size of these subtrees.

4. Recursively draw the subtrees at the children of v_k that have *size* $> A$. By reflection, make the roots lie on the top-right corners of their respective bounding boxes. Stack these drawings vertically, underneath the drawings from step 3. Put the drawing of the subtree at v_{k+1} at the bottom.

The special case $k = 1$ is similar, except that we place v_k on the left, and so do not reflect in steps 2–4. The special case $k = 0$ is also similar, but bypassing step 1.

The overall width satisfies the following recurrence $W(n) \leq \max\{O(A),\ W(n/2) + 1,\ W(n - A)\}$, which solves to $W(n) = O(A + \log n)$.

The overall height satisfies the following recurrence

$$H(n) \leq \sum_{i=1}^{m} H(n_i) + c(\log A + (n'/A)s\log^3 s + (n''/s)\log A)$$

for some $n', n'', m, n_1, \ldots, n_m$ with $n' + n'' + \sum_i n_i \leq n$, $n_i \leq n - A$, and $n_i \geq A$, for some constant c. It is straightforward to verify by induction that $H(n) \leq c((2n/A - 1)\log A + (n/A)s\log^3 s + (n/s)\log A)$. (The constraint $n_i \leq n - A$ is needed in the $m = 1$ case.) Choosing $s = \Theta(\sqrt{A}/\log A)$ to balance the last two terms gives the height bound in the theorem. ◀

Finally, choosing $A = \lceil \log n \rceil$ gives area $O(n\sqrt{\log n}\log^2 \log n)$.

3 Straight-line drawings of arbitrary trees

To obtain still better area bounds for straight-line non-upward drawings of arbitrary trees, the idea is to bootstrap: we show how to use a given general algorithm to obtain an improved augmented-star algorithm, which in turn is used to obtain an improved general algorithm. In order to bootstrap, we need to identify large grid substructures inside each sector in the augmented-star algorithm. This requires an interesting geometric observation about lattices, described in the following subsection.
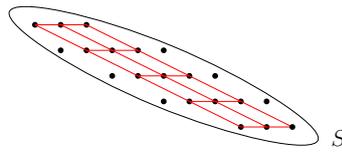
Figure 4 Observation 6: A convex set that contains many lattice points must contain a large affine grid in the lattice.

3.1 An observation about lattices

A *two-dimensional lattice* is a set of the form $\Lambda = \{i\mathbf{u} + j\mathbf{v} : i, j \in \mathbb{Z}\}$ for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. The vector pair $\{\mathbf{u}, \mathbf{v}\}$ is called a *basis* of Λ.

In this paper, we use the term $a \times b$ *affine grid* to refer to a set of the form $\{i\mathbf{u} + j\mathbf{v} : i \in \{x_0 + 1, \ldots, x_0 + a\}, j \in \{y_0 + 1, \ldots, y_0 + b\}\}$ for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and some $x_0, y_0 \in \mathbb{R}$. In other words, it is a set that is equivalent to the regular $a \times b$ grid $\{1, \ldots, a\} \times \{1, \ldots, b\}$ after applying some affine transformation.

The following observation is the key (see Figure 4). The author is not aware of any references of this specific statement (but would not be surprised if this was known before).

▶ **Observation 6.** *If a convex set S in the plane contains n points from a lattice Λ, then $S \cap \Lambda$ contains an $a \times b$ affine grid for some a and b with $ab = \Omega(n)$.*

Proof. First, apply an affine transformation to make S *fat*, i.e., $D^- \subset S \subset D^+$ for some disks D^- and D^+ with $\text{diam}(D^-) = \Omega(\text{diam}(D^+))$. (This follows immediately from well-known properties of the *Löwner–John ellipsoid*; or see [1, 3] for simple, direct algorithms.)

After the transformation, Λ is still a lattice. It is well known that there exists a basis $\{\mathbf{u}, \mathbf{v}\}$ for Λ satisfying $60° \leq \angle(\mathbf{u}, \mathbf{v}) \leq 120°$. (A *Gauss-reduced basis* satisfies this property; for example, see [29, Section 27.2].)

Let R^+ be the smallest rhombus containing D^+, with sides parallel to $\mathbf{u}$ and $\mathbf{v}$. Let R^- be the largest rhombus R^- contained in D^-, with sides parallel to $\mathbf{u}$ and $\mathbf{v}$. Then R^+ and R^- have side lengths $r^+ = O(\text{diam}(D^+))$ and $r^- = \Omega(\text{diam}(D^-))$ respectively, since $\angle(\mathbf{u}, \mathbf{v})$ is bounded away from $0°$ or $180°$. It follows that $r^- = \Omega(r^+)$.

Now, $S \cap \Lambda \subset R^+ \cap \Lambda$ is contained in an $\lceil r^+/\|\mathbf{u}\| \rceil \times \lceil r^+/\|\mathbf{v}\| \rceil$ affine grid. Thus, $n \leq \lceil r^+/\|\mathbf{u}\| \rceil \cdot \lceil r^+/\|\mathbf{v}\| \rceil$.

On the other hand, $S \cap \Lambda \supset R^- \cap \Lambda$ contains an $\lfloor r^-/\|\mathbf{u}\| \rfloor \times \lfloor r^-/\|\mathbf{v}\| \rfloor$ affine grid, with $\lfloor r^-/\|\mathbf{u}\| \rfloor \times \lfloor r^-/\|\mathbf{v}\| \rfloor = \Omega(\lceil r^+/\|\mathbf{u}\| \rceil \cdot \lceil r^+/\|\mathbf{v}\| \rceil) = \Omega(n)$ points, assuming that $\|\mathbf{u}\|, \|\mathbf{v}\| \leq r^-$.

This almost completes the proof. It remains to address the special case when $\|\mathbf{u}\| > r^-$ (the case $\|\mathbf{v}\| > r^-$ is similar). Here, $S \cap \Lambda \subset R^+ \cap \Lambda$ is contained in an $O(1) \times \lceil r^+/\|\mathbf{v}\| \rceil$ affine grid. Some row of the grid must contain $\Omega(n)$ points of $S \cap \Lambda$. The row is a $1 \times \Omega(n)$ affine grid. ◀

3.2 Improved augmented-star algorithm

We first show how to use a given general algorithm $\mathcal{G}_0$ to obtain an improved algorithm for the augmented-star case:

▶ **Lemma 7.** *Suppose we are given a* general *algorithm $\mathcal{G}_0$ that takes as input any $n \geq A \geq g_0(n)$ and any tree of size n, and outputs a straight-line drawing of width at most A and height at most $(n/A)f_0(A)$, where the root is drawn at the top left corner of the bounding box. Here, f_0 and g_0 are some increasing functions satisfying $f_0(n) \geq g_0(n)$.*

Then we can obtain an improved augmented-star algorithm *that takes as input any* $n \geq A \geq 1$ *and a tree of size n such that the subtree at each child of the root has size at most s, and outputs a straight-line drawing with width $O(A \log s)$ and height $O((n/A) \cdot f_0(s) \log s)$, where the root is placed at the top left corner of the bounding box, and the left side of the box contains no other vertices.*

Proof. Let $\ell = c f_0(s)$ for some constant c. Let $B = \lceil c\ell n/A \rceil$. Let $P = \{(x, y) \in \{1, \ldots, A\} \times \{-B, \ldots, -1\} : x$ and y are relatively prime$\}$. By Fact 1, $|P| = \Omega(AB)$, and so $|P| \geq \ell n$ by making c sufficiently large.

Let $n_1, \ldots, n_d$ be the sizes of the subtrees $T_1, \ldots, T_d$ at the children of the root v_0, with $\sum_{i=1}^{d} n_i = n - 1$ and $n_i \leq s$ for each i. Place v_0 at the origin. Form d disjoint sectors, where the i-th sector S_i contains exactly ℓn_i points of P. This is possible, since any line through the origin contains at most one point of P and $\sum_{i=1}^{d} \ell n_i < \ell n \leq |P|$.

Take a fixed i. Applying Observation 6 to the convex set $S_i \cap ((0, A] \times [-B, 0))$, we see that $S_i \cap (\{1, \ldots, A\} \times \{-B, \ldots, -1\})$ must contain an $a \times b$ affine grid for some a and b with $ab = \Omega(\ell n_i)$. Note that $b \geq (n_i/a) f_0(s)$ by making c sufficiently large. Consider two cases:

- CASE 1: $g_0(n_i) \leq a \leq n_i$. Here, we can draw T_i in the $a \times b$ affine grid by the given algorithm $\mathcal{G}_0$, after applying an affine transformation to convert to a standard integer $a \times b$ grid. Note that planarity and straightness are preserved under the transformation (but not upwardness). The root of T_i can be placed at the highest corner of the grid.
- CASE 2: $a > n_i$ or $a < g_0(n_i)$. Note that in the latter subcase, $b \geq (n_i/a) f_0(s) \geq (n_i/a) g_0(n_i) \geq n_i$. In either subcase, S_i contains n_i points of P on a common line L. (Note that L does not pass through the origin, by definition of P.) Let $t = \lceil \log s \rceil$ and $\overline{L} = L \cap S_i$. Let G be the t line segments $\overline{L}, 2\overline{L}, \ldots, t\overline{L}$. Then each of the $t = \lceil \log s \rceil$ segments of G contain n_i integer points inside S_i. Thus, we can draw T_i using the integer points on G by Fact 3. The root is placed on the highest segment of G.

Lastly, draw the edges from v_0 to the roots of the T_i's. The total width is $O(tA) = O(A \log s)$ and height is $O(tB) = O((n/A) \cdot f_0(s) \log s)$. ◀

3.3 Improved general algorithm

Using the improved augmented-star algorithm, we can then obtain an improved general algorithm, by following the same approach as in the proof of Theorem 5, except with Lemma 4 replaced by the improved Lemma 7 in step 2. The same analysis shows the following:

▶ **Theorem 8.** *Suppose we are given a* general algorithm $\mathcal{G}_0$ *that takes as input any $n \geq A \geq g_0(n)$ and any tree of size n, and outputs a straight-line drawing of width at most A and height at most $(n/A) f_0(A)$, where the root is drawn at the top left corner of the bounding box. Here, f_0 and g_0 are some increasing functions satisfying $f_0(n) \geq g_0(n)$.*

Then we can obtain an improved general algorithm *that takes as input any $n \geq A \geq \log s$ and any tree of size n, and outputs a straight-line upward drawing with width $O(A + \log n)$ and height $O((n/A) \log A + (n/A) f_0(s) \log^2 s + (n/s) \log A)$, where the root is placed at the top left corner of the bounding box.*

Assume inductively that there is a general algorithm $\mathcal{G}_0$ satisfying the assumption of the above theorem with $f_0(A) = C_j A^{1/j} \log^j A$ and $g_0(n) = c_0 \log n$ for some C_j and c_0. For $j = 1$, this follows from the standard algorithm, which has logarithmic width and linear height after swapping x and y, with $C_1, c_0 = O(1)$.

Choosing $s = \lceil A^{j/(j+1)}/\log^j A \rceil$ to balance the last two terms in the above theorem gives a width bound of $O(A + \log n)$ and height bound of

$$O((n/A)\log A + (n/A)C_j s^{1/j} \log^{j+2} s + (n/s)\log A) = O(C_j(n/A)A^{1/(j+1)} \log^{j+1} A).$$

By setting $\tilde{A} = c_0 A$ and $C_{j+1} = O(1) \cdot C_j$, with a sufficiently large absolute constant c_0, the width is at most $\tilde{A}$ and the height is at most $C_{j+1}(n/\tilde{A})\tilde{A}^{1/(j+1)} \log^{j+1} \tilde{A}$ for any $n \geq \tilde{A} \geq c_0 \log n$. We have thus obtained a new general algorithm with $f_0(\tilde{A}) = C_{j+1}\tilde{A}^{1/(j+1)} \log^{j+1} \tilde{A}$ and $g_0(n) = c_0 \log n$.

Note that $C_j = 2^{O(j)}$. For the best bound, we choose a nonconstant $j = \Theta(\sqrt{\log A/\log\log A})$ so that $f_0(A) = 2^{O(j)}A^{1/j} \log^j A = 2^{O((\log A)/j + j\log\log A)} = 2^{O(\sqrt{\log A \log\log A})}$, yielding:

▶ **Corollary 9.** *For any given $n \geq A \geq \log n$, every tree of size n has a straight-line drawing with width $O(A)$ and height $(n/A)2^{O(\sqrt{\log A \log\log A})}$.*

Finally, choosing $A = \lceil \log n \rceil$ gives area $n2^{O(\sqrt{\log\log n \log\log\log n})}$.

▶ **Remark.** It is straightforward to implement the algorithms in Section 2 and this section in polynomial time. One open question is whether the improved bound holds for upward drawings. Another open question is whether further improvements are possible if we allow poly-line drawings.

4 Straight-line orthogonal drawings of binary trees

In this section, we consider binary trees and describe algorithms to produce straight-line orthogonal (non-upward) drawings. We improve previous algorithms with $O(n\log\log n)$ area by Shin, Kim, and Chwa [25] and Chan et al. [7]. The idea is (again) to bootstrap.

Given a binary tree T and two distinct vertices u and v, such that v is a descendant of u but not an immediate child of v, the *chain* from u to v is defined to be the subtree at u minus the subtree at v. (To explain the terminology, note that the chain consists of the path from u to the parent of v, together with a sequence of subtrees attached to the nodes of this path.) We show how to use a given algorithm for drawing chains to obtain a general algorithm for drawing trees, which together with the given chain algorithm is used to obtain an improved chain algorithm.

4.1 The general algorithm

Given a chain algorithm $\mathcal{C}_0$, we can naively use it to draw the entire tree, since a tree can be viewed as a chain from the root to an artificially created leaf. We first show how to use a given chain algorithm $\mathcal{C}_0$ to obtain a general algorithm that achieves *arbitrary width–height tradeoffs*. This is done by adapting previous algorithms [25, 7].

▶ **Lemma 10.** *Suppose we are given a chain algorithm $\mathcal{C}_0$ that takes as input any binary tree and a chain from v_0 to v_k where the size of the chain is n, and outputs a straight-line orthogonal drawing of the chain with width at most $W_0(n)$ and height at most $H_0(n)$, where v_0 is placed at the top left corner of the bounding box, and the parent of v_k is placed at the bottom left corner of the box. Here, $W_0(n)$ and $H_0(n)$ are increasing functions.*

Then we can obtain a general algorithm that takes as input $n \geq A \geq 1$ and any binary tree T of size n, and outputs a straight-line orthogonal drawing with width $O(W_0(A) + \log n)$ and height $O((n/A)H_0(A))$, where the root is placed at the top left corner of the bounding box.
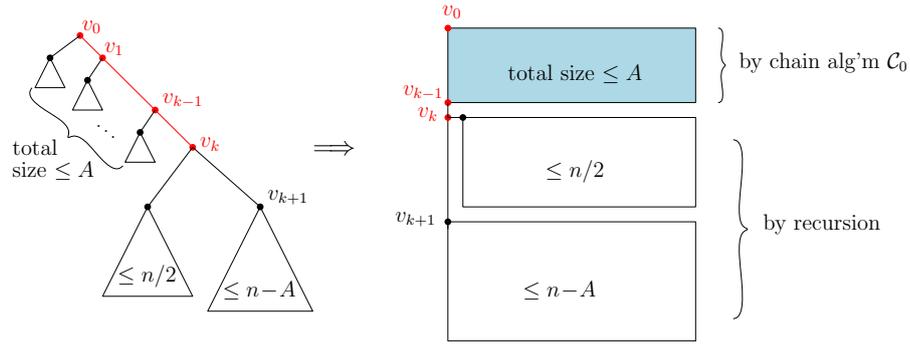
Figure 5 The general algorithm in Lemma 10 for orthogonal drawings.
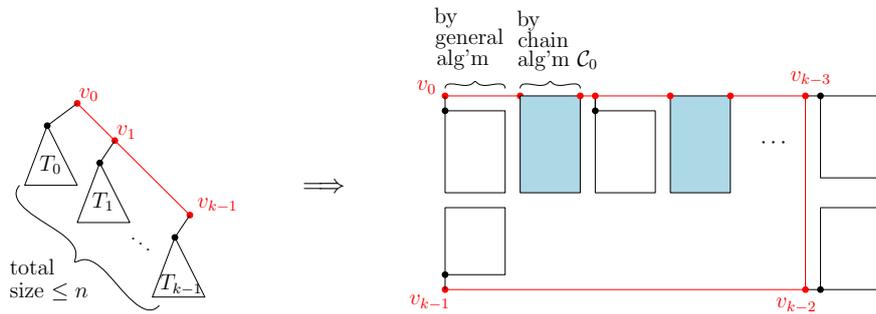
Figure 6 The improved chain algorithm in Theorem 11 for orthogonal drawings.

Proof. We describe a recursive algorithm to draw T: Let $v_0 v_1 v_2 \cdots$ be the heavy path, and v_k be the A-skewed centroid, as in the proof of Theorem 5. Then the chain from v_0 to v_k has size at most A, the subtree at v_{k+1} has size at most $n - A$, and the subtree at the sibling of v_{k+1} has size at most $\min\{n - A, n/2\}$.

The drawing of T, depicted in Figure 5, is constructed as follows:

1. Draw the chain from v_0 to v_k by the given algorithm $\mathcal{C}_0$, with width at most $W_0(A)$ and height at most $H_0(A)$.
2. Recursively draw the subtrees at the two children of v_k. Stack the two drawings vertically, underneath the drawing from step 1. Put the drawing of the subtree at v_{k+1} at the bottom. (Note that if any of these subtrees has size at most A, we can just use algorithm $\mathcal{C}_0$ with width at most $W_0(A)$ and height at most $H_0(A)$.)

The special case $k = 1$ is similar, except that in step 1 we can just apply algorithm $\mathcal{C}_0$ to draw the subtree at the sibling of v_1, and connect v_0 to v_k directly. The special case $k = 0$ is also similar, but bypassing step 1.

The overall width satisfies the recurrence $W(n) \leq \max\{O(W_0(A)), W(n/2) + 1, W(n - A)\}$, which solves to $W(n) = O(W_0(A) + \log n)$.

The overall height satisfies the recurrence $H(n) \leq \sum_{i=1}^{m} H(n_i) + cH_0(A)$ for some $m, n_1, \ldots, n_m$ with $m \leq 2$, $\sum_i n_i \leq n$, $n_i \leq n - A$, and $n_i \geq A$, for some constant c. The recurrence solves to $H(n) \leq c(2n/A - 1)H_0(A)$ (similarly to the proof of Theorem 5). ◀

4.2 The improved chain algorithm

Using both the general algorithm from Lemma 10 and the given chain algorithm $\mathcal{C}_0$, we describe an improved chain algorithm:

▶ **Theorem 11.** *Suppose we are given a* chain algorithm C_0 *that takes as input any binary tree and a chain from v_0 to v_k where the size of the chain is n, and outputs a straight-line orthogonal drawing of the chain with width at most $W_0(n)$ and height at most $H_0(n)$, where v_0 is placed at the top left corner of the bounding box, and the parent of v_k is placed at the bottom left corner of the box. Here, $W_0(n)$ and $H_0(n)$ are increasing functions.*

Then we can obtain an improved chain algorithm *that takes as input any $n \geq A \geq 1$ and any binary tree and a chain from v_0 to v_k where the size of the chain is n, and outputs a straight-line orthogonal drawing of the chain with width $O((n/A)H_0(A))$ and height $O(W_0(A) + \log n)$, where v_0 is placed at the top left corner of the bounding box, and the parent of v_k is placed at the bottom left corner.*

Proof. Let $v_0 v_1 \cdots v_k$ denote the path from v_0 to v_k. Let T_i denote the subtree at the sibling of v_{i+1}. Let n_i be the size of T_i plus 1.

Divide the sequence $v_0 v_1 \cdots v_{k-4}$ into subsequences, where each subsequence is either (i) a *singleton* v_i, or (ii) a contiguous *block* $v_i v_{i+1} \cdots v_\ell$ of length at least 2 with $n_i + n_{i+1} + \cdots + n_\ell \leq A$. By making the blocks maximal, we can ensure that the number of singletons and blocks is $O(n/A)$. We add $v_{k-3}, \ldots, v_{k-1}$ as 3 extra singletons.

- For each singleton v_i, draw T_i by the general algorithm in Lemma 10 if $n_i \geq A$, or directly by the given algorithm C_0 if $n_i < A$. By swapping x and y, the width is $O((n_i/A + 1)H_0(A))$ and the height is $O(W_0(A) + \log n)$.
- For each block $v_i v_{i+1} \cdots v_\ell$, draw the subchain from v_i to $v_{\ell+1}$, which has size at most A, by the given algorithm C_0. By swapping x and y, the width is $O(H_0(A))$ and the height is $O(W_0(A))$.

All these drawings are stacked horizontally as shown in Figure 6, except for T_{k-2} and T_{k-1}, which are placed below and flipped upside-down.

The special cases with $k \leq 3$ are simpler: just stack the $O(1)$ drawings vertically, with the bottom drawing of T_{k-1} flipped upside-down.

The total width due to singletons is $O(\sum_i (n_i/A + 1)H_0(A)) = O((n/A)H_0(A))$, and the total width due to blocks is also $O((n/A)H_0(A))$, because the number of singletons and blocks is $O(n/A)$. The overall height is $O(W_0(A) + \log n)$.   ◀

Assume inductively that there is a chain algorithm C_0 satisfying the assumption of Theorem 11 with $W_0(n) = C_j(n/\log n)\log^{(j)} n$ and $H_0(n) = C_j \log n$ for some C_j, where $\log^{(j)}$ denotes the j-th iterated logarithm. For $j = 1$, this follows by simply applying the standard algorithm to draw the subtrees T_i in the proof of Theorem 11, with $C_1 = O(1)$.

Choosing $A = \left\lceil \log n \log \log n / \log^{(j+1)} n \right\rceil$ in Theorem 11 gives a width bound of $O((n/A)H_0(A)) = O((n/A)C_j \log A) = O(C_j(n/\log n)\log^{(j+1)} n)$ and a height bound of $O(W_0(A) + \log n) = O(C_j(A/\log A)\log^{(j)} A + \log n) = O(C_j \log n)$. By setting $C_{j+1} = O(1) \cdot C_j$, we have thus obtained a new chain algorithm with $W_0(n) = C_{j+1}(n/\log n)\log^{(j+1)} n$ and $H_0(n) = C_{j+1} \log n$.

Note that $C_j = 2^{O(j)}$. For the best bound, we choose a nonconstant $j = \log^* n$, yielding:

▶ **Corollary 12.** *Every binary tree of size n has a straight-line orthogonal drawing with area $n2^{O(\log^* n)}$.*

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