Hanani–Tutte for Approximating Maps of Graphs

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Abstract

We resolve in the affirmative conjectures of A. Skopenkov and Repovš (1998), and M. Skopenkov (2003) generalizing the classical Hanani–Tutte theorem to the setting of approximating maps of graphs on 2-dimensional surfaces by embeddings. Our proof of this result is constructive and almost immediately implies an efficient algorithm for testing whether a given piecewise linear map of a graph in a surface is approximable by an embedding. More precisely, an instance of this problem consists of (i) a graph G whose vertices are partitioned into clusters and whose inter-cluster edges are partitioned into bundles, and (ii) a region R of a 2-dimensional compact surface M given as the union of a set of pairwise disjoint discs corresponding to the clusters and a set of pairwise disjoint "pipes" corresponding to the bundles, connecting certain pairs of these discs. We are to decide whether G can be embedded inside M so that the vertices in every cluster are drawn in the corresponding disc, the edges in every bundle pass only through its corresponding pipe, and every edge crosses the boundary of each disc at most once.

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1 Introduction

The Hanani–Tutte theorem is a classical result [20, 32] stating that a graph G is planar if it can be drawn in the plane so that every pair of edges not sharing a vertex cross an

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even number of times. According to Schaefer [29, Remark 3.6], *"The planarity criterion of Hanani–Tutte brings together computational, algebraic and combinatorial aspects of the planarity problem."* Perhaps the most remarkable algorithmic aspect of this theorem is that it implies the existence of a polynomial-time algorithm for planarity testing [28, Section 1.4.2]. In particular, the Hanani–Tutte theorem reduces planarity testing to solving a system of linear equations over $\mathbb{Z}_2$.

Seeing a graph G as a 1-dimensional topological space, an embeddability-testing algorithm decides whether there exists an injective continuous map, also called an *embedding*, $\psi : G \to M$, where M is a given triangulated compact 2-dimensional manifold without boundary. It is a classical result of Hopcroft and Tarjan that graph embeddability in the plane can be tested in linear time [22], and a linear-time algorithm is also known for testing whether G can be embedded into an arbitrary compact 2-dimensional manifold M [24], though computing the orientable genus (as well as Euler genus and non-orientable genus) of a graph is NP-hard [31]. We study a variant of this algorithmic problem in which we are given a piecewise linear continuous map $\varphi : G \to M$, which is typically not an embedding, and we are to decide whether for every $\varepsilon > 0$ there exists an embedding $\psi : G \to M$ such that $\|\psi - \varphi\| < \varepsilon$, where $\|.\|$ is the supremum norm. Such a map ψ is called an *ε-approximation* of φ, and in this case we say that φ is *approximable by an embedding*; or as in [2], a *weak embedding*. If φ is a constant map, the problem is clearly equivalent to the classical planarity testing. Obviously, an instance of our problem is negative if there exists a pair of edges e and g in G such that the curves $\varphi(e)$ and $\varphi(g)$ induced by φ properly cross. Hence, in a typical instance of our problem the map φ somewhat resembles an embedding except that we allow a pair of edges to overlap and an edge to be mapped onto a single point.

Building upon the work of Minc [23], M. Skopenkov [30] gave an algebraic characterization via van Kampen obstructions of maps φ approximable by an embedding in the plane in the case when G is a cycle or when G is subcubic and the image of φ is a simple closed curve. This implies a polynomial-time algorithm for the decision problem in the corresponding cases and can be seen as a variant of the characterization of planar graphs due to Hanani and Tutte. The aim of this work is to prove a conjecture of M. Skopenkov [30, Conjecture 1.6] generalizing his results along with its algorithmic consequences to arbitrary graphs. Independently of the aforementioned developments, a series of recent papers [1, 7, 9] on weakly simple embeddings shows that the problem of deciding the approximability of φ by an embedding is tractable and can be carried out in $O(|\varphi| \log |\varphi|)$ time, where $|\varphi|$ is the number of line segments specifying φ.

In spite of the analytic definition, the algorithmic problem of deciding whether φ is approximable by an embedding admits a polynomially equivalent reformulation that is of combinatorial flavor and that better captures the essence of the problem. Therefore we state our results in terms of the reformulation, whose planar case is a fairly general restricted version of the c-planarity problem [10, 11] of Feng, Cohen and Eades introduced by Cortese et al. [9]. The computational complexity of c-planarity testing is a well-known notoriously difficult open problem in the area of graph visualization [8]. To illustrate this state of affairs we mention that Angelini and Da Lozzo [5] have recently studied our restricted variant (as well as its generalizations) under the name of *c-planarity with embedded pipes* and provided an FPT algorithm for it [5, Corollary 18].

Roughly speaking, in the *clustered planarity* problem, shortly *c-planarity*, we are given a planar graph G equipped with a hierarchical structure of subsets of its vertex set. The subsets are called *clusters*, and two clusters are either disjoint or one contains the other. The question is whether a planar embedding of G with the following property exists: the vertices in each cluster are drawn inside a disc corresponding to the cluster so that the boundaries of the discs do not intersect, the discs respect the hierarchy of the clusters, and every edge in the embedding crosses the boundary of each disc at most once.

Notation. Let us introduce the notation necessary for precisely stating the problem that we study. Let $G = (V, E)$ be a multigraph without loops. If we treat G as a topological space, then a *drawing* ψ of G is a piecewise linear map from G into a triangulated 2-dimensional manifold M where every vertex in $V(G)$ is mapped to a unique point and every edge $e \in E$ joining u and v is mapped bijectively to a simple arc joining $\psi(u)$ and $\psi(v)$. We understand E as a multiset, and by a slight abuse of notation we refer to an edge e joining u and v as uv even though there might be other edges joining the same pair of vertices. Multiple edges are mapped to distinct arcs meeting at their endpoints. Given a map m we denote by $m|_X$, where X is a subset of the domain of m, the function obtained from m by restricting its domain to X. If H is a graph equipped with an embedding, we denote by $H|_X$, where X is a subgraph of H, the graph X with the embedding inherited from H.

If it leads to no confusion, we do not distinguish between a vertex or an edge and its image in the drawing and we use the words "vertex" and "edge" in both contexts. Also when talking about a drawing we often mean its image.

We assume that drawings satisfy the following standard general position conditions. No edge passes through a vertex, every pair of edges intersect in finitely many points, no three edges intersect at the same inner point, and every intersection point between a pair of edges is realized either by a proper crossing or a common endpoint. Here, by a *proper crossing* we mean a transversal intersection that is a single point.

An *embedding* of a graph G is a drawing of G in M without crossings. The *rotation* at a vertex v in a drawing of G is the clockwise cyclic order of the edges incident to v in a small neighborhood of v in the drawing w.r.t a chosen orientation at the vertex. The *rotation system* of a drawing of G is the set of rotations of all the vertices in the drawing. The embedding of G is combinatorially determined by the rotation system and consistently chosen orientations at the vertices if M is orientable. If M is non-orientable we need to additionally specify the signs of the edges as follows. We assume that M is constructed from a punctured 2-sphere by turning all the holes into cross-caps, i.e., by identifying the pairs of opposite points on every hole. A *sign* on an edge is positive if overall the edge passes an even number of times through the cross-caps, and negative otherwise.

Refer to Figure 1. We refer the reader to the monograph by Mohar and Thomassen [25] for a detailed introduction into surfaces and graph embeddings. Let $\varphi : G \to M$ be a piecewise linear map with finitely many linear pieces. Suppose that φ is free of edge crossings, and in φ, edges do not pass through vertices. As we will see later, the image of φ can be naturally identified with a graph H embedded in M. Throughout the paper we denote both vertices and edges of H by Greek letters. Let the *thickening* $\mathcal{H}$ of H be a 2-dimensional surface with boundary obtained as a quotient space of a union of pairwise disjoint topological discs as follows. We take a union of pairwise disjoint closed discs $\mathcal{D}(\nu)$, called *clusters*, for all $\nu \in V(H)$ and closed rectangles $\mathcal{P}(\rho)$, called *pipes*, for all $\rho \in E(H)$. We connect every pair of discs $\mathcal{D}(\nu)$ and $\mathcal{D}(\mu)$, such that $\rho = \nu\mu \in E(H)$, by $\mathcal{P}(\rho)$ in correspondence with the rotations at vertices of the given embedding of H as described next. Let ∂X denote the boundary of X. We consider a subset of $\partial\mathcal{D}(\nu)$, for every $\nu \in V(H)$, consisting of $\deg(\nu)$ pairwise disjoint closed (non-trivial) arcs $\mathcal{A}(\nu, \mu)$, one for every $\nu\mu \in E(H)$, appearing along $\partial\mathcal{D}(\nu)$ in correspondence with the rotation of ν. For every $\mathcal{D}(\nu)$, we fix an orientation of $\partial\mathcal{D}(\nu)$ and $\partial\mathcal{P}(\nu\mu)$.

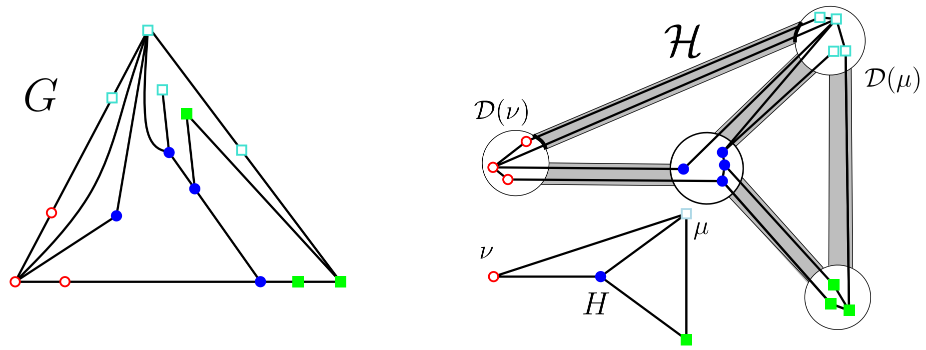

Figure 1 An instance (G, H, φ) and its approximation by an embedding contained in the thickening $\mathcal{H}$ of H. The valves of the pipe of $\rho = \nu\mu$ at $\mathcal{D}(\nu)$ and $\mathcal{D}(\mu)$ are highlighted by bold arcs.

If M is orientable, for every $\mathcal{P}(\nu\mu)$, we identify by an orientation reversing homeomorphism its opposite sides with $\mathcal{A}(\nu, \mu)$ and $\mathcal{A}(\mu, \nu)$ w.r.t. the chosen orientations of $\partial\mathcal{D}(\nu)$, $\partial\mathcal{D}(\mu)$, and $\partial\mathcal{P}(\nu\mu)$. If M is non-orientable, for every $\mathcal{P}(\nu\mu)$ with the positive sign we proceed as in the case when M is orientable and for every every $\mathcal{P}(\nu\mu)$ with the negative sign, we identify by an orientation preserving homeomorphism its opposite sides with $\mathcal{A}(\nu, \mu)$ and $\mathcal{A}(\mu, \nu)$ w.r.t. the chosen orientations of $\partial\mathcal{D}(\nu)$ and $\partial\mathcal{P}(\nu\mu)$, and the reversed orientation of $\partial\mathcal{D}(\mu)$. We call the intersection of $\partial\mathcal{D}(\nu) \cap \partial\mathcal{P}(\nu\mu)$ a *valve* of $\nu\mu$.

Instance. An instance of the problem that we study is defined as follows. The instance is a triple (G, H, φ) of an (abstract) graph G, a graph H embedded in a closed 2-dimensional manifold M, and a map $\varphi : V(G) \to V(H)$ such that every pair of vertices joined by an edge in G are mapped either to a pair of vertices joined by an edge in H or to the same vertex of H. We naturally extend the definition of φ to each subset U of $V(G)$ by putting $\varphi(U) = \{\varphi(u)|\ u \in U\}$, and to each subgraph G_0 of G by putting $\varphi(G_0) = (\varphi(V(G_0)), \{\varphi(e)|\ e \in E(G_0), |\varphi(e)| = 2\})$. The map φ induces a partition of the vertex set of G into *clusters* V_ν, where $V_\nu = \varphi^{-1}[\nu]$.

Question. Decide whether there exists an embedding ψ of G in the interior of a thickening $\mathcal{H}$ of H so that the following hold.

(A) Every vertex $v \in V_\nu$ is drawn in the interior of $\mathcal{D}(\nu)$, i.e., $\psi(v) \in \mathrm{int}(\mathcal{D}(\nu))$.

(B) For every $\nu \in V(H)$, every edge $e \in E(G)$ intersecting $\partial\mathcal{D}(\nu)$ does so in a single proper crossing, i.e., $|\psi(e) \cap \partial\mathcal{D}(\nu)| \leq 1$.

Note that conditions (A) and (B) imply that every edge of G is allowed to pass through at most one pipe as long as G is drawn in $\mathcal{H}$. The instance is *positive* if an embedding ψ of G satisfying (A) and (B) exists and *negative* otherwise. If (G, H, φ) is a positive instance we say that (G, H, φ) is *approximable by the embedding* ψ, shortly *approximable*. We call ψ the *approximation* of (G, H, φ). When the instance (G, H, φ) is clear from the context, we call ψ the *approximation* of φ.

The instance (G, H, φ), or shortly φ, is *locally injective* if for every vertex $v \in V(G)$, the restriction of φ to the union of v and the set of its neighbors is injective, or equivalently, no two vertices that are adjacent or have a common neighbor in G are mapped by φ to the same vertex in H. An edge of G is a *pipe edge* if it is mapped by φ to an edge of H. When talking about pipe edges, we have a particular instance in mind, which is clear from the context.

The *pipe degree*, $pdeg(C)$, of a subgraph C of $G[V_\nu]$ is the number of edges ρ of H for which there exists a pipe edge e with one vertex in C such that $\varphi(e) = \rho$.

1.1 The result

An edge in a drawing is *even* if it crosses every other edge an even number of times. A vertex in a drawing is *even* if every pair of its incident edges cross an even number of times. An edge in a drawing is *independently even* if it crosses every other non-adjacent edge an even number of times. A drawing of a graph is *(independently) even* if all edges are (independently) even. Note that every embedding is an even drawing.

We formulate our main theorem in terms of a relaxation of the notion of an approximable instance (G, H, φ). An instance (G, H, φ) is $\mathbb{Z}_2$*-approximable* if there exists an independently even drawing of G in the interior of $\mathcal{H}$ satisfying (A) and (B). We call such a drawing a $\mathbb{Z}_2$*-approximation* of (G, H, φ). The proof of the Hanani–Tutte theorem from [26] proves that given an independently even drawing of a graph in the plane, there exists an embedding of the graph in which the rotations at even vertices are preserved, that is, they are the same as in the original independently even drawing. We refer to this statement as to the *unified Hanani–Tutte theorem* [14]. Our result can be thought of as a generalization of this theorem, which also motivates the following definition. A drawing ψ of G is *compatible* with a drawing ψ_0 of G if every even vertex in ψ_0 is also even in ψ and has the same rotation in both drawings ψ_0 and ψ.[3]

It is known that $\mathbb{Z}_2$-approximability of (G, H, φ) does not have to imply its approximability by an embedding [27, Figure 1(a)]. Our main result characterizes the instances (G, H, φ) for which such implication holds. The characterization is formulated in terms of the derivative of (G, H, φ), whose formal definition is postponed to Section 2, since its definition relies on additional concepts that we need to introduce, which would unnecessarily further delay stating of our main result.

▶ **Theorem 1.** *If an instance (G, H, φ) is $\mathbb{Z}_2$-approximable by an independently even drawing ψ_0 then either (G, H, φ) is approximable by an embedding ψ compatible with ψ_0, or it is not approximable by an embedding and in the ith derivative $(G^{(i)}, H^{(i)}, \varphi^{(i)})$, for some $i \in \{1, 2, \ldots, 2|E(G)|\}$, there exists a connected component $C \subseteq G^{(i)}$ such that C is a cycle, $\varphi^{(i)}$ is locally injective and $(C, H^{(i)}|_{\varphi^{(i)}(C)}, \varphi^{(i)}|_C)$ is not approximable by an embedding.*

The obstruction $(C, H^{(i)}|_{\varphi^{(i)}(C)}, \varphi^{(i)}|_C)$ from the statement of the theorem has the form of the "standard winding example" [27, Figure 1(a)], in which the cycle C is forced by $\varphi^{(i)}$ to wind around a point inside a face of H more than once (and an odd number of times, since it has a $\mathbb{Z}_2$-approximation). Our main result implies the following.

▶ **Corollary 2.** *If G is a forest, the $\mathbb{Z}_2$-approximability implies approximability by an embedding.*

Theorem 1 confirms a conjecture of M. Skopenkov [30, Conjecture 1.6], since our definition of the derivative agrees with his definition in the case when G is a cycle and since for every cycle C in $G^{(i)}$ there exists a cycle D in G such that $(D^{(i)}, H^{(i)}|_{\varphi(D^{(i)})}, \varphi^{(i)}|_{D^{(i)}}) = (C, H^{(i)}|_{\varphi^{(i)}(C)}, \varphi^{(i)}|_C)$. Corollary 2 confirms a conjecture of Repovš and A. Skopenkov [27, Conjecture 1.8]. The main consequence of Theorem 1 is the following.

[3] Since in general we work also with non-orientable surfaces the rotation is determined only up to the choice of an orientation at each vertex for non-orientable surfaces.

▶ **Theorem 3.** *We can test in $O(|\varphi|^{2\omega})$, where $O(n^\omega)$ is the complexity of multiplication*[4] *of square $n \times n$ matrices, whether (G, H, φ) is approximable by an embedding.*

Theorem 3 implies tractability of c-planarity with embedded pipes [9] and therefore solves a related open problem of Chang, Erickson and Xu [7, Section 8.2] and Akitaya et al. [1]. The theorem also implies that strip planarity introduced by Angelini et al. [3] is tractable, and hence, solves the main problem asked therein. The theorem generalizes results of [13] and [17], and implies that c-planarity [10, 11] for flat clustered graphs is tractable for instances with three clusters, which has been open, to the best of our knowledge. We remark that only solutions to the problem for two clusters were given so far [6, 19, 21]. Nevertheless, after the completion of this work our running time was improved to $O(|\varphi|^2 \log |\varphi|)$ [2]. The improvement on the running time was achieved by using a similar strategy as in the present work, while eliminating the need to solve the linear system and employing a very careful running time analysis. Previously, polynomial running time $O(|\varphi|^4)$ was obtained by the first author [12] for graphs with fixed combinatorial embedding.

We mention that Theorem 1 and Theorem 3 easily generalize to the setting when clusters are homeomorphic to cylinders and $\mathcal{H}$ is homeomorphic to a torus or a cylinder, which extends some recent work [4, 15, 16]. It is an interesting open problem to find out if the technique of [2] generalizes to this setting as well.

Our proof of Theorem 1 extends the technique of Minc [23] and M. Skopenkov [30]. In particular, we extend the definition of the derivative for maps of graphs to instances in a certain normal form which can be assumed without loss of generality. Roughly, the derivative is an operator that takes an input instance (G, H, φ) and produces a simpler instance (G', H', φ') that is positive if and only (G, H, φ) is. We remark that the operations of *cluster expansion* and *pipe contraction* of Cortese et al. [9] bear many similarities with the derivative, and can be considered as local analogs of the derivative. One of the reasons for introducing the normal form is to impose on the instance conditions analogous to the properties of a *contractible base* [9], or a *safe pipe* [2], which make the derivative reversible.

Organization. In Section 2, we define the normal form of instances and the operation of the derivative for instances in the normal form. Furthermore, we state a claim (Claim 8) saying that for any instance admitting a $\mathbb{Z}_2$-approximation there exists, in some sense, an equivalent instance in the normal form. Thus, it is enough to define the derivative only for instances in the normal form. In Section 3, we prove Theorem 1.

2 Normal form and derivative

Normal form. We define the normal form of an instance (G, H, φ) to which we can apply the derivative. In order to keep the definition more compact we define the normal form via its topologically equivalent subdivided variant; see Figure 2 for an illustration. This variant also facilitates the definition of the derivative and those are its only purposes in this work. Roughly speaking, (G, H, φ) is in the subdivided normal form if there exists an independent set $V_s \subset V(G)$ without degree-1 vertices such that every connected component C of $G[V \setminus V_s]$ is mapped by φ to an edge $\varphi(C) = \nu\mu$ of H and both its parts mapped to ν and μ are forests.

[4] The best current algorithms for matrix multiplication give $\omega < 2.3729$ [18, 34]. Since a linear system appearing in the solution is sparse, it is also possible to use Wiedemann's randomized algorithm [33], with expected running time $O(|\varphi|^4 \log |\varphi|^2)$ in our case.

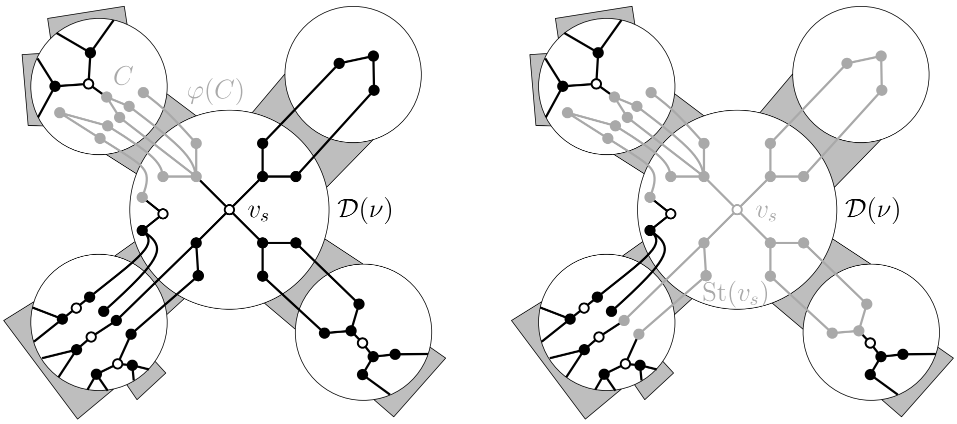

Figure 2 A part of (G, H, φ) in the subdivided normal form illustrated by its approximation in $\mathcal{H}$. A connected component C of $G[V \setminus V_s]$ (left) and $\mathrm{St}(v_s)$ (right), both colored gray. The empty vertices belong to the independent set $V_s \subset V(G)$.

We call vertices in V_s *central vertices*, which conveys an intuition that every vertex of V_s constitutes in some sense a center of a connected component induced by a cluster.

The normal form is obtained from the subdivided normal form by *suppressing* in V_s any vertices of degree 2, i.e., by replacing each such vertex v_s and both its incident edges by a single edge, and performing the same replacement for $\varphi(v_s)$ which must be also of degree 2; see Figure 3 left for an illustration.

▶ **Definition 4.** The instance (G, H, φ) is in the *subdivided normal form* if there exists an independent set $V_s \subset V(G)$ with the following properties.
For every connected component C of $G[V \setminus V_s]$ the image $\varphi(C)$ is an edge of $E(H)$; and $\varphi^{-1}|_\nu[\varphi(C)]$ is a forest for both vertices $\nu \in \varphi(C)$. For every connected component C of $G[V_\nu]$, for every $\nu \in V(H)$ the following holds. If $pdeg(C) \geq 2$ then $|V(C) \cap V_s| = 1$, and $V(C) \cap V_s = \emptyset$ otherwise; for every $v_s \in V_s \cap V(C)$ we have that $\deg(v_s) = pdeg(C)$; and no two edges incident to v_s join v_s with connected components C_1 and C_2 of $G[V \setminus V_s]$ such that $\varphi(C_1) = \varphi(C_2)$.

The instance obtained from an instance (G, H, φ) in the subdivided normal form by suppressing all degree-2 vertices in V_s is in the *normal form.* Such an instance in the normal form *corresponds* to the original instance (G, H, φ) in the subdivided normal form, and vice-versa.

Derivative. The rest of the section is inspired by the work of M. Skopenkov [30] and also Minc [23]. In particular, the notion of the derivative in the context of map approximations was introduced by Minc and adapted to the setting of $\mathbb{Z}_2$-approximations by M. Skopenkov for instances (G, H, φ) where G is subcubic and H is a cycle. We extend his definition to instances (G, H, φ) in the normal form. (A somewhat simplified extension was already used in [12].) Thus, by derivating (G, H, φ) we, in fact, mean bringing the instance (G, H, φ) into the normal form and then derivating the instance in the normal form. Given the instance (G, H, φ) in the normal form or the subdivided normal form already, the operation of the derivative outputs an instance (G', H', φ'), where $G' = G$ and H' is defined next.

In order to keep the definition more compact we formulate it first for the instances in the subdivided normal form. Thus, in the following we assume (G, H, φ) to be in the subdivided normal form.

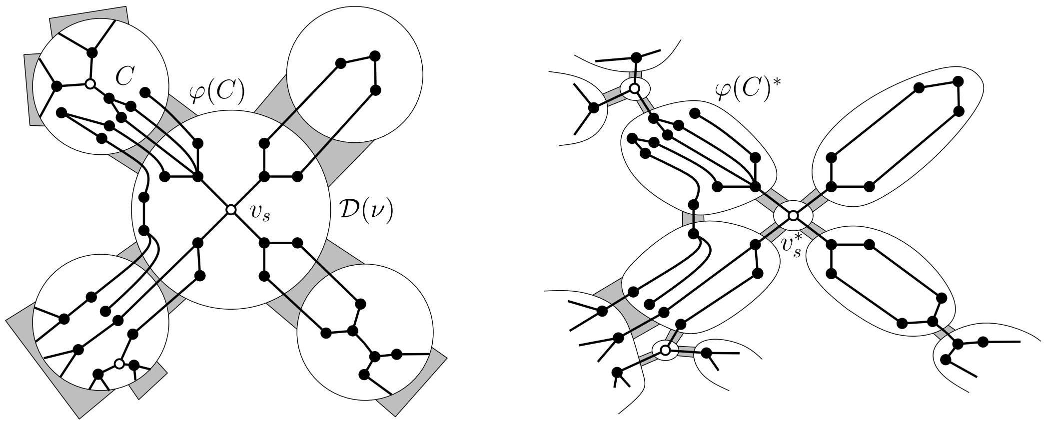

Figure 3 A part of (G, H, φ) in the normal form (left) illustrated by its approximation in $\mathcal{H}$, and its derivative (right). The empty vertices belong to the independent set $V_s \subset V(G)$.

Let V_s be the set of central vertices in G. Let $\mathcal{G}$ be a bipartite graph with the vertex set $V_s \cup \mathcal{C}$, where $\mathcal{C} = \{C \mid C$ is a connected component of $G[V \setminus V_s]\}$, in which $v_s \in V_s$ and $C \in \mathcal{C}$ are joined by an edge if and only if v_s is joined by an edge with a vertex of C in G.

Let the *star* of v_s, $\mathrm{St}(v_s)$, with $v_s \in V_s$ be the subgraph of G induced by $\{v_s\} \cup \bigcup_{C \in V(\mathcal{G}):\ v_s C \in E(\mathcal{G})} V(C)$; see Figure 2 (right) for an illustration. The vertices of H' are in the bijection with the union of V_s with the set of the edges of H. This bijection is denoted by superscript $*$. The graph H' is a bipartite graph with the vertex set $V(H') = \{\rho^* | \rho \in E(H)\} \cup \{v_s^* |\ v_s \in V_s\}$. We have $\rho^* v_s^* \in E(H')$ if and only if $\rho \in E(\varphi(\mathrm{St}(v_s)))$. We use the convention of denoting a vertex in $V(H')$ whose corresponding edge in $E(H)$ is $\rho = \nu\mu$ by both ρ^* or $(\nu\mu)^*$. An embedding of H' in M and signs on the edges of H', if M is non-orientable, are naturally inherited from those of H. Figure 3 (right) illustrates the restriction of the embedding of H' to its subgraph "stemming" from ν.

▶ **Definition 5.** Let (G, H, φ) be $\mathbb{Z}_2$-approximable and in the subdivided normal form. The *derivative* $(G, H, \varphi)'$ of (G, H, φ) is the instance (G', H', φ') such that $\varphi'(v_s) = v_s^*$, for $v_s \in V_s$, and $\varphi'(v) = \varphi(C)^*$, for every $v \in V(C)$, where C is a connected component of $G[V \setminus V_s]$. (Hence, $\varphi(C)$ is an edge of H by the definition of the subdivided normal form.)

The *derivative* $(G, H, \varphi)'$ of (G, H, φ), where (G, H, φ) is $\mathbb{Z}_2$-approximable and in the normal form, is the instance obtained from the derivative of the corresponding instance in the subdivided normal form by suppressing every vertex v_s of degree 2 in V_s and its image $\varphi(v_s)$ in H', and eliminating multiple edges in H'.

Remark. Since (G', H', φ') is defined only for instances in the (subdivided) normal form, by derivating an instance, which is not in the normal form, we will mean an operation that, first, brings the instance into the normal form, and second, applies the derivative to the instance. We take the liberty of denoting by G', H' and φ' an object that does not depend only on G, H and φ, respectively, but on the whole instance (G, H, φ).

The proof of Theorem 1 proceeds by induction on the *potential* $p(G, H, \varphi)$, which is always non-negative and is defined as follows. Let $E_p(G)$ be the set of pipe edges in G, we put $p(G, H, \varphi) = |E_p(G)| - |E(H)|$. We will show that an application of the derivative decreases the potential unless the instance is locally injective. The latter can be thought of as the base case of the induction. In order to prove that the inductive step goes through we will need the following three claims.

We show that if (G,H,φ) in the normal form is $\mathbb{Z}_2$-approximable then (G',H',φ') is $\mathbb{Z}_2$-approximable as well. More precisely, we prove the following claim.

▶ **Claim 6.** *If an instance (G,H,φ) in the normal form is $\mathbb{Z}_2$-approximable by a drawing ψ_0 then $(G'=G,H',\varphi')$ is $\mathbb{Z}_2$-approximable by a drawing ψ_0' such that ψ_0 is compatible with ψ_0'. Moreover, if ψ_0 is crossing free so is ψ_0'.*

The previous claim implies that if (G,H,φ) is approximable by an embedding the same holds for (G',H',φ'), which we use in the proof of Theorem 1 to conclude that if (G',H',φ') is not approximable by an embedding the same holds for (G,H,φ). However, we need also the converse of this to hold, which is indeed the case.

▶ **Claim 7.** *If the instance (G',H',φ') is approximable by an embedding ψ' compatible with ψ_0' from Claim 6 then (G,H,φ) is approximable by an embedding compatible with ψ_0.*

We say that $(\hat{G},\hat{H},\hat{\varphi},\hat{\psi}_0)$ is a *clone* of (G,H,φ,ψ_0) if the following holds. If $(\hat{G},\hat{H},\hat{\varphi})$ is approximable by an embedding compatible with $\hat{\psi}_0$ then (G,H,φ) is approximable by an embedding compatible with ψ_0; and if (G,H,φ) is approximable by an embedding then $(\hat{G},\hat{H},\hat{\varphi})$ is approximable by an embedding. Note that being a clone is a transitive relation. However, the relation is not symmetric, and thus, it is not an equivalence relation.

Due to the following claim, it is indeed enough to work with instances in the normal form in Claim 6.

▶ **Claim 8.** *Given a $\mathbb{Z}_2$-approximation ψ_0 of (G,H,φ) there exist an instance $(\hat{G},\hat{H},\hat{\varphi})$ in the normal form that is $\mathbb{Z}_2$-approximable, such that $p(G,H,\varphi)=p(\hat{G},\hat{H},\hat{\varphi})$; and (2) a $\mathbb{Z}_2$-approximation $\hat{\psi}_0$ of $(\hat{G},\hat{H},\hat{\varphi})$ such that $(\hat{G},\hat{H},\hat{\varphi},\hat{\psi}_0)$ is a clone of (G,H,φ,ψ_0).*

3 Proof of Theorem 1

Let (G,H,φ) be an instance that is $\mathbb{Z}_2$-approximable by an independently even drawing ψ_0. We start with a claim that helps us to identify instances that cannot be further simplified by derivating. We show that by successively applying the derivative we eventually obtain an instance such that φ is locally injective.

▶ **Claim 9.** *If (G,H,φ) is in the normal form then $p(G',H',\varphi')\le p(G,H,\varphi)$. If additionally φ is not locally injective after suppressing in G all degree-2 vertices incident to an edge induced by a cluster, the inequality is strict; that is, $p(G',H',\varphi')<p(G,H,\varphi)$.*

Furthermore, if G is connected and every connected component C induced by V_ν, for all $\nu\in V(H)$, has pipe-degree at most 2, then $|E_p(G')|\le|E_p(G)|$.

Proof. We prove the first part of the claim and along the way establish the second part. Let ψ_0 be a $\mathbb{Z}_2$-approximation of (G,H,φ). Note that in G' the pipe edges incident to a central vertex $v_s\in V_s$ (every such v_s has degree at least 3) and edges in H' incident to $\varphi'(v_s)$ contribute together zero towards $p(G',H',\varphi')$. Let $H_0'=H'\setminus\{v_s^*|\ v_s\in V_s\}$.

(∗) The number of edges in H_0' is at least $|V(H_0')|-c=|E(H)|-c$,
where c is the number of connected components of H_0' that are trees.

We use this fact together with a simple charging scheme in terms of an injective mapping ζ defined in the next paragraph to prove the claim.

Suppose for a while that H_0' is connected. The set of pipe edges of G' not incident to any $v_s\in V_s$ forms a matching M' in G' by Definitions 4 and 5. Let $D(v)$, $v\in V(G')$, be the connected component of $G' \setminus E_p(G')$ containing the vertex v. By the first property of the components $G[V \setminus V_s]$ in Definition 4, for every $v \in V(M')$, the component $D(v)$ contains at least one former pipe edge, i.e., a pipe edge in (G, H, φ). Let V_p be the set of vertices in G' incident to these former pipe edges.

We construct an injective mapping ζ from the set $V(M')$ to V_p. The mapping ζ maps a vertex $v \in V(M')$ to a closest vertex (in terms of the graph theoretical distance in $D(v)$) in $V_p \cap V(D(v))$. The mapping ζ is injective by the fact, that in the corresponding subdivided normal form, every connected component of $G[V_\nu]$ of pipe-degree 2, for $\nu \in V(H)$, contains at most one central vertex. Indeed, recall that this central vertex is suppressed in the normal form and the edge thereby created becomes a pipe edge e in H', and thus, belongs to M'. Each end vertex v of e is mapped by ζ to a vertex u such that $\varphi(u) = \varphi(v)$. Thus, the injectivity could be violated only by the end vertices of e. However, this cannot happen since every connected component of $G[V_\nu]$ is a tree. The injectivity of ζ implies that $2|M'|$ is upper bounded by $2|E_p(G)|$, and therefore $|M'|$ is upper bounded by $|E_p(G)|$, which proves the second part of the claim. Furthermore, $|E_p(G)| = |M'|$ only if after suppressing all vertices of degree 2 incident to an edge induced by a cluster, φ is locally injective, and H'_0 contains a cycle.

If H'_0 is disconnected, then we have $|M'| \le |E_p(G)| - c$, where the inequality is strict if φ is not locally injective after suppressing all degree-2 vertices incident to an edge induced by a cluster. Indeed, if $|M'| = |E_p(G)| - c$, then there exist exactly $2c$ vertices v, $v \in e \in E_p(G)$, that are not in the image of the map ζ. However, there are at least $2c$ vertices in G' each of which is mapped by φ' to a vertex of degree at most 1 in H'_0. This follows since a connected component of H'_0, that is an isolated vertex ν, contributes at least two end vertices of an edge $e \in E_p(G)$ such that $\varphi'(e) = \nu$; and a connected component of H'_0, that is a tree, has at least two leaves each of which contributes by at least one end vertex of an edge in $E_p(G)$ mapped to it by φ'. Hence, if $|M'| = |E_p(G)| - c$ then all the vertices that are not contained in the image of ζ, are accounted for by these $2c$ vertices.

Putting it together, we have $|M'| \le |E_p(G)| - c$ and $(*)$ $\;|E(H)| - c \le |E(H'_0)|$, where the first inequality is strict if φ is not locally injective after suppressing all degree-2 vertices incident to an edge induced by a cluster. Since the remaining pipe edges of G' and edges of H' contribute together zero towards $p(G', H', \varphi')$, summing up the inequalities concludes the proof. ◀

▶ **Claim 10.** *Suppose that φ is locally injective after suppressing all degree-2 vertices incident to an edge induced by a cluster. Applying the derivative $|E_p(G)|$ many times yields an instance in which no connected component of G is a path.*

Proof of Theorem 1. We assume that every edge of H is in the image of φ and proceed by induction on $p(G, H, \varphi)$. First, we discuss the inductive step. By Claim 8, we assume that (G, H, φ) is in the normal form, which leaves $p(G, H, \varphi)$ unchanged. Suppose that φ is not locally injective after suppressing degree-2 vertices incident to an edge induced by a cluster. Derivating (G, H, φ) decreases $p(G, H, \varphi)$ by Claim 9. By Claims 6 and 7, $(G', H', \varphi', \psi'_0)$ is a clone of (G, H, φ, ψ_0), where ψ'_0 is obtained by Claim 6. Hence, in this case we are done by induction. Thus, we assume that φ is locally injective, which includes the case when $p(G, H, \varphi) = 0$. This means that either we reduced G to an empty graph, or every connected component C of $G[V_\nu]$, for every $\nu \in V(H)$, is a single vertex. We suppose that G is not a trivial graph, since otherwise we are done. The proof will split into two cases, the **acyclic** and the **cyclic** case below, but first we introduce some tools from [13] that we use extensively in the argument.

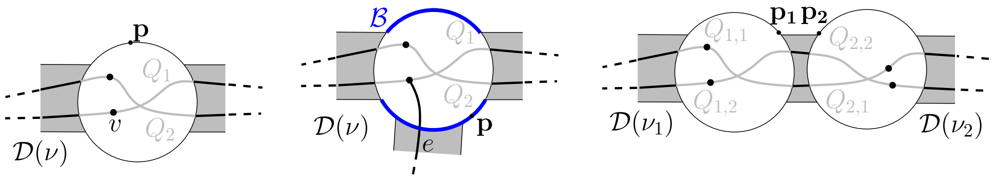

Figure 4 A pair of ν-diagonals Q_1 and Q_2, and a vertex v such that $v <_{\mathbf{p}} Q_1$ and $Q_2 <_{\mathbf{p}} Q_1$ (left). The edge e forcing $Q_1 <_{\mathbf{p}} Q_2$ (middle). The relation $Q_{1,1} <_{\mathbf{p_1}} Q_{2,1}$ forces $Q_{1,2} <_{\mathbf{p_2}} Q_{2,2}$ (right).

We will work with a $\mathbb{Z}_2$-approximation ψ_0 of (G, H, φ) unless specified otherwise. Let P be a path of length 2 in G. Let the internal vertex u of P belong to $G[V_\nu]$, for some $\nu \in V(H)$. The curve obtained by intersecting the disc $\mathcal{D}(\nu)$ with P is a *ν-diagonal supported* by u. By a slight abuse of notation we denote in different drawings ν-diagonals with the same supporting vertex and joining the same pair of valves by the same symbol. Let Q be a ν-diagonal supported by a vertex u of $G[V_\nu]$. Since φ is locally injective, Q must connect a pair of distinct valves. Let $\mathbf{p}$ be a point on the boundary of the disc $\mathcal{D}(\nu)$ of ν such that $\mathbf{p}$ is not contained in any valve.

▶ **Definition 11.** (See Figure 4 (left) for an illustration.) For a vertex $v \in V_\nu$, $v \neq u$, we define $v <_{\mathbf{p}} Q$ if in the two-coloring of the complement of Q (such that connected regions sharing a non-trivial part of the boundary receive different colors) in the disc $\mathcal{D}(\nu)$, v receives the same color as the component having $\mathbf{p}$ on the boundary. Let Q_1 and Q_2 be a pair of ν-diagonals connecting the same pair of valves. We define $Q_1 <_{\mathbf{p}} Q_2$ if for the vertex v supporting Q_1 we have $v <_{\mathbf{p}} Q_2$. Analogously we define the relation $>_{\mathbf{p}}$.

Recall that H contains no multiple edges, since we do not introduce them by derivating. Thus, the upcoming Claims 12 and 13 follow easily by the same argument as (1) and (2) in [13, Theorem 13].

▶ **Claim 12.** *The relation $<_{\mathbf{p}}$ is anti-symmetric: If for a pair of ν-diagonals Q_1, Q_2 of $G[V_\nu]$ we have $Q_1 <_{\mathbf{p}} Q_2$ then $Q_1 \not>_{\mathbf{p}} Q_2$.*

By Claim 12, the relation $<_{\mathbf{p}}$ defines a tournament, that is, a complete oriented graph, on ν-diagonals joining the same pair of valves. A pair of a ν_1-diagonal Q_1 and a ν_2-diagonal Q_2 of G is *neighboring* if Q_1 and Q_2 have endpoints on the same (pipe) edge.

Let $Q_{1,i}$ and $Q_{2,i}$ be a neighboring pair of a ν_1-diagonal and a ν_2-diagonal sharing a pipe edge e_i, for $i = 1, 2$, such that $\varphi(e_1) = \varphi(e_2) = \rho = \nu_1\nu_2$. Let $\mathbf{p_1}$ and $\mathbf{p_2}$ be on the boundary of $\mathcal{D}(\nu_1)$ and $\mathcal{D}(\nu_2)$, respectively, very close to the same side of the pipe of ρ.

▶ **Claim 13.** *If $Q_{1,1} <_{\mathbf{p_1}} Q_{1,2}$ then $Q_{2,1} <_{\mathbf{p_2}} Q_{2,2}$; see Figure 4 (right) for an illustration.*

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be a set of ν_1-diagonals and ν_2-diagonals, respectively, in G of the same cardinality such that every ν_1-diagonal in $\mathcal{D}_1$ ends on the valve of ρ and forms a neighboring pair with a ν_2-diagonal in $\mathcal{D}_2$. We require that all the diagonals in $\mathcal{D}_1$ join the same pair of valves. Let $\overrightarrow{G(\mathcal{D}_i)}$, for $i = 1, 2$, be the tournament with vertex set $\mathcal{D}_i$ defined by the relation $<_{\mathbf{p_i}}$. An oriented graph $\overrightarrow{D}$ is *strongly connected* if there exists a directed path in $\overrightarrow{D}$ from u to v for every ordered pair of vertices u and v in $V(\overrightarrow{D})$. The following claim follows from Claim 13.

▶ **Claim 14.** *If $\overrightarrow{G(\mathcal{D}_1)}$ is strongly connected then all the diagonals in $\overrightarrow{G(\mathcal{D}_2)}$ join the same pair of valves, and the oriented graph $\overrightarrow{G(\mathcal{D}_2)}$ is strongly connected.*

The part of G inside $\mathcal{D}(\nu)$ is the union of all ν-diagonals. The part of G inside $\mathcal{D}(\nu)$ is *embedded* if ψ_0 does not contain any edge crossing in $\mathcal{D}(\nu)$.

Acyclic case. In this case, we assume that for every $\nu \in V(H)$ and every $\mathbf{p} \in \partial\mathcal{D}(\nu)$ not contained in any valve, the relation $<_{\mathbf{p}}$ induces an acyclic tournament on every set of pairwise vertex-disjoint ν-diagonals joining the same pair of valves.

We show that we can embed the part of G inside every disc $\mathcal{D}(\nu)$ while respecting the relation $<_{\mathbf{p}}$ defined according to ψ_0. In other words, in every cluster we embed connected components (now just vertices) induced by V_ν together with parts of their incident pipe edges ending on valves so that the relations $Q_1 <_{\mathbf{p}} Q_2$ are preserved for every pair of ν-diagonals Q_1 and Q_2 joining the same pair of valves. Then by reconnecting the parts G inside $\mathcal{D}(\nu)$'s we obtain a required embedding of G which will conclude the proof.

By an easy application of the unified Hanani–Tutte theorem we obtain an embedding of the part of G inside $\mathcal{D}(\nu)$. We apply the theorem to an independently even drawing of an auxiliary graph $G_{aux}(\nu)$ in $\mathcal{D}(\nu)$, where the drawing is obtained as the union of the part of G inside $\mathcal{D}(\nu)$ and $\partial\mathcal{D}(\nu)$. By subdividing edges in $G_{aux}(\nu)$ we achieve that the vertices drawn in $\partial\mathcal{D}(\nu)$ are even and therefore we indeed obtain an embedding of the part of G inside $\mathcal{D}(\nu)$ as required. Suppose that in the embedding of the part of G inside $\mathcal{D}(\nu)$ we have $Q_1 >_{\mathbf{p}} Q_2$ while in the drawing ψ_0 we have $Q_1 <_{\mathbf{p}} Q_2$. For the sake of contradiction we consider the embedding with the smallest number of such pairs, and consider such pair Q_1 and Q_2 whose end points are closest to each other along the valve that contains them.

First, we assume that both Q_1 and Q_2 pass through a connected component (a single vertex) of $G[V_\nu]$ of pipe degree 2. The endpoints of Q_1 and Q_2 are consecutive along valves, since $<_{\mathbf{p}}$ is acyclic. Thus, we just exchange them thereby contradicting the choice of the embedding. Second, we show that if Q_1 passes through a connected component C_1 of $G[V_\nu]$ of pipe degree at least 3 and Q_2 passes through a component C_2 of pipe degree 2, then the relation $Q_1 >_{\mathbf{p}} Q_2$ in the drawing of ψ_0 leads to contradiction as well. Let ρ be an edge of H such that there exists an edge incident to C_1 mapped to ρ by φ and there does not exist such an edge incident to C_2, see Figure 4 (middle) for an illustration. Let $\mathcal{B}$ be the complement of the union of the valves containing the endpoints of Q_1 or Q_2 in the boundary of $\mathcal{D}(\nu)$. Suppose that the valve of ρ and $\mathbf{p}$ are contained in the same connected component of $\mathcal{B}$. It must be that $Q_1 <_{\mathbf{p}} Q_2$ in every $\mathbb{Z}_2$-approximation of (G, H, φ). If the valve of ρ and $\mathbf{p}$ are contained in the different connected components of $\mathcal{B}$, it must be that $Q_1 >_{\mathbf{p}} Q_2$ in every $\mathbb{Z}_2$-approximation of (G, H, φ), in particular also in an approximation.

Finally, we assume that Q_1 and Q_2 pass through a connected component C_1 and C_2, respectively, of $G[V_\nu]$ of pipe degree at least 3. Similarly as above, let ρ_1 and ρ_2 be edges of H such that there exists an edge incident to C_1 and C_2, respectively, mapped to ρ_1 and ρ_2 by φ, and neither Q_1 nor Q_2 ends on its valve. By applying the unified Hanani–Tutte theorem to $G_{aux}(\nu)$ as above, we have $\rho_1 \neq \rho_2$. By the same token, the valve of ρ_1 and ρ_2 are not contained in the same connected component $\mathcal{B}$. Thus, by the same argument as in the previous case it must be that either in every $\mathbb{Z}_2$-approximation of (G, H, φ) we have $Q_1 <_{\mathbf{p}} Q_2$ or in every $\mathbb{Z}_2$-approximation of (G, H, φ) we have $Q_1 >_{\mathbf{p}} Q_2$.

In order to finish the proof in the acyclic case, we would like to reconnect neighboring pairs of diagonals by curves inside the pipes without creating a crossing. Let $Q_{1,i}$ and $Q_{2,i}$, for $i = 1, 2$, be a pair of a neighboring ν_1-diagonal and ν_2-diagonal sharing a pipe edge e_i, for $i = 1, 2$, such that $\varphi(e_1) = \varphi(e_2) = \rho$. We would like the endpoints of $Q_{1,1}$ and $Q_{1,2}$ to be ordered along the valve of ρ consistently with the endpoints of $Q_{2,1}$ and $Q_{2,2}$ along the other valve of ρ. We are done if Claim 13 applies to $Q_{i,j}$'s. However, this does not have to be the case if, let's say $Q_{1,1}$ and $Q_{2,1}$, does not join the same pair of valves. Nevertheless, by treating all the valves distinct from the valve of ρ at $\mathcal{D}(\nu_1)$ as a single valve, we see that both Claim 12 and Claim 13, in fact, apply to $<_{\mathbf{p_1}}$ and $<_{\mathbf{p_2}}$.

Cyclic case. In this case, we assume that for a vertex $\nu \in V(H)$ and $\mathbf{p} \in \partial\mathcal{D}(\nu)$ not contained in any valve, the relation $<_{\mathbf{p}}$ induces an acyclic tournament on a set of pairwise vertex-disjoint ν-diagonals joining the same pair of valves.

We consider at least three ν-diagonals $Q_1, \ldots, Q_l$ inducing a strongly connected component in the tournament defined by $<_{\mathbf{p}}$. Let $\mathbf{p_k}$ and $\mathbf{q_k}$ be endpoints of Q_k, for $k = 1, \ldots, l$. We assume that $\mathbf{p_k}$, for $k = 1, \ldots, l$, are contained in the same valve, and therefore the same holds for $\mathbf{q_k}$. By Claim 10, we assume that no connected component in G is a path. Hence, by Claim 14 every Q_k is contained in a (drawing of a) connected component of G which is a cycle. Indeed, a vertex of degree at least 3 in a connected component of G, whose vertex supports Q_k, would inevitably lead to a pair of independent edges crossing oddly in ψ_0, since we assume that φ is locally injective. Thus, by a simple inductive argument using Claim 14 and the fact that no two distinct strongly connected components in an oriented graph share a vertex we obtain the following property of $Q_1, \ldots, Q_l$.

Every endpoint $\mathbf{p_k}$ is joined by a curve in the closure of $\psi_0(G) \setminus \bigcup_{l'=1}^{l} Q_{l'}$ with an endpoint $\mathbf{q_{k'}}$. This defines a permutation π of the set $\{Q_1, \ldots, Q_l\}$, where $\pi(Q_k) = Q_{k'}$. On the one hand, each orbit in the permutation π must obviously consist of ν-diagonals supported by vertices in the same connected component of G, which is a cycle as we discussed in the previous paragraph. On the other hand, every pair of diagonals belonging to different orbits is supported by vertices in different cycles in G. Hence, the orbits of π are in a one-to-one correspondence with a subset of connected components in G all of which are cycles. Let $C_1 \ldots C_o$, $o \le l$, denote such cycles. By a simple inductive argument which uses Claim 14, we have that every $\varphi(C_k) = W_k, \ldots, W_k$ with W_k being repeated o_k-times, where W_k is a closed walk of H and o_k is the size of the orbit corresponding to C_k. By the hypothesis of Theorem 1 we assume that $(C_k, H|_{\varphi(C_k)}, \varphi|_{C_k})$, for $k = 1, \ldots, o$, is a positive instance.

By the previous assumption, if the number of negative signs on the edges in W_k (counted with multiplicities) is even then $o_k = 1$. Indeed, a closed neighborhood of an approximation ψ_{C_k} (which is an embedding) of $(C_k, H|_{\varphi(C_k)}, \varphi|_{C_k})$ is the annulus, in which (the image of) ψ_{C_k} is a non-self intersecting closed piecewise linear curve. Analogously, we show that if the number of negative signs on the edges in W_k (counted with multiplicities) is odd then $o_k \le 2$, and $o_k = 1$ for at most a single value of k, i.e., if $o_{k_1} = o_{k_2} = 1$ then $k_1 = k_2$. Suppose that the previous claim holds for every $k = 1, \ldots, o$. Since $l \ge 3$ and $o_k \le 2$ for $k = 1, \ldots, o$, we have that $o \ge 2$. We assume that $o_2 \le o_1$. We remove the cycle C_1 from G and apply induction. Let ψ be an approximation of $(G \setminus C_1, H, \varphi|_{G\setminus C_1})$ that we obtain by the induction hypothesis. We construct the desired approximation of (G, H, φ) by extending ψ to G as follows. We embed C_1 alongside C_2 while satisfying (A) and (B) for (G, H, φ), which is possible since $1 \le o_2 \le o_1 \le 2$.

It remains to show the claim. For the sake of contradiction we assume that $o_{k_1} = o_{k_2} = 1$, for $k_1 \ne k_2$. The curves $\psi_0(C_{k_1})$ and $\psi_0(C_{k_2})$ are one-sided and homotopic, and therefore they must cross an odd number of times in $\psi_0(G)$ (contradiction). Finally, for the sake of contradiction suppose that for some k, we have $o_k \ge 3$ and that there exists an approximation ψ_{C_k} of $(C_k, H|_{\varphi(C_k)}, \varphi|_{C_k})$ (which is an embedding). If o_k is odd we replace (the image of) ψ_{C_k} by the boundary of its small closed neighborhood, which is connected. Thus, we can and shall assume that o_k is even and still bigger than 2. A closed neighborhood of ψ_{C_k} is the Möbius band. By lifting $\psi(C_k)$ to the annulus via the double cover of the Möbius band, we obtain a piecewise linear closed non-self intersecting curve winding $o_k/2 > 1$ times around its center (contradiction). ◀

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