Evolution of Impossible Objects

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Abstract

Impossible objects – 3D objects that can create a visual effect that seems impossible – can be classified by generation based on the order in which they were discovered or produced. The first generation consists of objects whose appearance when observed from a certain viewpoint matches a picture of an impossible object. Many such objects can be created, as there are multiple 3D objects that will project the same two-dimensional picture, including shapes that the human vision system is unable to perceive. The gap between the mathematical and the psychological can also create other types of "impossible" visual effects. Impossible objects are here classified into seven groups.

2012 ACM Subject Classification Applied computing → Computer-aided design

Keywords and phrases Ambiguous cylinder, anomalous picture, impossible motion, impossible object, optical illusion

Digital Object Identifier 10.4230/LIPIcs.FUN.2018.2

Category Invited Paper

Acknowledgements This work is supported by the Grant-in-Aid for Basic Scientific Research (A) No. 16H01728 of MEXT.

1 Introduction

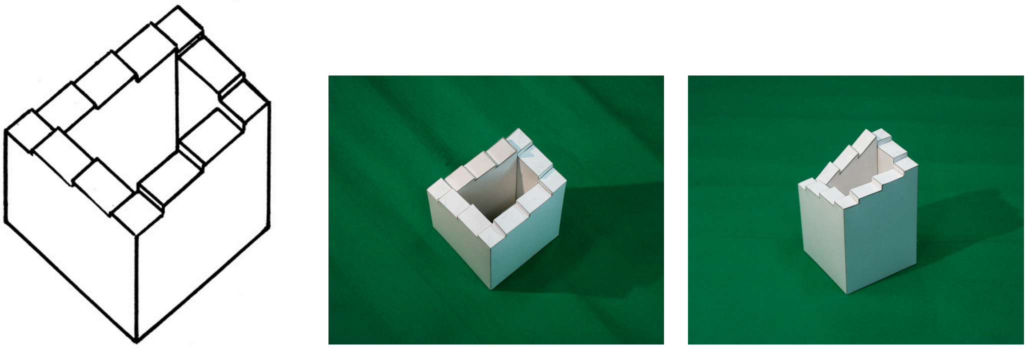
"Anomalous pictures" or "pictures of impossible objects" are a class of pictures that give viewers the impression of a 3D structure that is perceived to be impossible [2, 16]. A typical example of such a picture is the endless loop of stairs proposed by Penrose and Penrose [6] and which appears in Escher's "Ascending and Descending" [4].

It was once thought that impossible objects exist only in the mind and that they could not be constructed as actual 3D structures. However, several tricks were soon found for creating 3D structures that could reproduce pictures of impossible objects. One such trick is to make a discontinuous structure appear to be continuous when seen from a particular viewpoint [5, 7]. A second trick is to use curved surfaces instead of planar surfaces [1, 3].

Sugihara describes a third trick [8, 9] in which the creator uses angles other than 90 degrees to produce a rectangular look. He called this the "non-rectangularity trick". He extended the trick in various directions and proposed new types of impossible objects, including "impossible motion objects" in which the inserted motions appear to be impossible; "ambiguous cylinders", whose mirror images appear to be impossible; and "partly invisible objects", parts of which disappear when reflected in a mirror.

In this presentation, we classify Sugihara's impossible objects according to their generation and present typical examples. We also touch on some of the underlying mathematics.

9th International Conference on Fun with Algorithms (FUN 2018).
Editors: Hiro Ito, Stefano Leonardi, Linda Pagli, and Giuseppe Prencipe; Article No. 2; pp. 2:1–2:8
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



(a) (b) (c)

Figure 1 First-generation "anomalous object": (a) picture of an endless loop of stairs; (b) solid object producing the same picture; (c) another view of the object.

2 First-Generation "Anomalous Objects"

Not all of the 3D objects represented in anomalous pictures are impossible; in some cases, they can be constructed as 3D solid objects. An example is shown in Fig. 1, where panel (a) shows a picture of an endless loop of stairs and panel (b) shows an object made from paper whose appearance from above coincides with the picture. The stairs at the top of the four walls form a loop, suggesting that if we continue to ascend the stairs, we will eventually come back to the starting point, which is impossible since ascending the stairs should bring us always to a higher position. Fig. 1 (c) shows another view of the object, from which we can see that the stairs on the left rear wall are not normal. Note that in this realization, all faces are planar and the structures that look connected are actually connected.

This kind of realization can be found mathematically in the following way. As illustrated in Fig. 2, let us fix an (x, y, z) Cartesian coordinate system, place a picture on the $z = 1$ plane, and fix the viewpoint at the origin. We are interested in judging whether the object represented by the picture is realizable. Let $(x_i, y_i, 1)$ be the coordinates of the i-th vertex in the picture. The associated vertex in 3D space, if it exists, should have coordinates $(x_i/t_i, y_i/t_i, 1/t_i)$, where t_i is an unknown variable, because the vertex should be on the half-line emanating from the origin and passing through the vertex $(x_i, y_i, 1)$. For the j-th face, let

$$a_j x + b_j y + c_j z + 1 = 0 \tag{1}$$

be the equation of the plane containing the face. All the coefficients a_j, b_j, c_j are unknowns.

Suppose that the i-th vertex is on the j-th face. Then we can substitute the coordinates of the vertex into the face equation and obtain

$$a_j x_i + b_j y_i + c_j + t_i = 0,$$

which is linear in the unknowns t_i, a_j, b_j, c_j. We obtain a similar equation for each such vertex and face pair. Collecting them all, we get a system of linear equations

$$Aw = 0, \tag{2}$$

where A is a constant matrix and w is the vector of unknown variables.

Pictures also have relative depth information. As shown in Fig. 3, let l be an edge separating the j-th face and the k-th face. Suppose that l is a convex edge and the k-th face

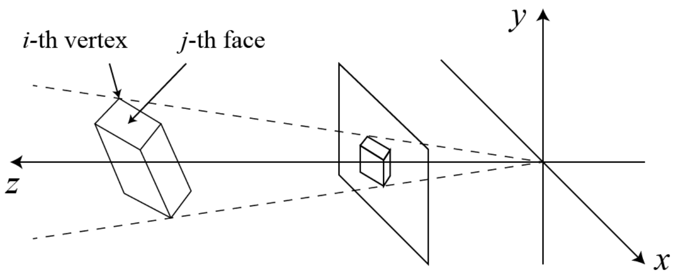


Figure 2 Object and its projection.

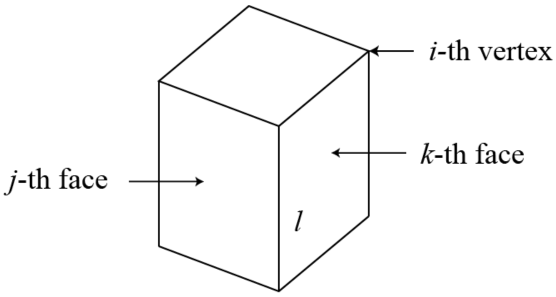


Figure 3 Relative depth.

contains the i-th vertex. Then the plane containing the j-th face passes between the i-th vertex and the viewpoint, which is represented by

$$a_j x_i + b_j y_i + c_j + t_i < 0. \tag{3}$$

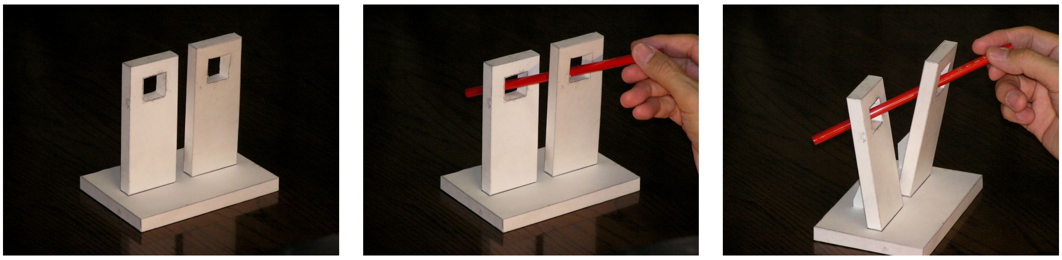
Collecting all similar inequalities, we get the system of linear inequalities

$$Bw > 0. \tag{4}$$

We can then prove that the picture represents a polyhedral object if and only if the system of equations (2) and inequalities (4) has solutions [8]. Thus, we can determine the realizability of impossible objects. The object shown in Fig. 1 corresponds to a solution of the system of equations and inequalities. In this way, we can construct a potentially large number of 3D objects whose projections coincide with anomalous pictures. We classify this type of impossible object as first generation and call the objects “anomalous objects”.

3 Second-Generation “Impossible Motion Objects”

If a given picture is correct, then the associated system of equations (2) and inequalities (4) has solutions but they are not unique. We can utilize this property to construct another type of impossibility. Let D be a picture of an ordinary 3D object. The system of (2) and (4) contains a solution corresponding to the original object, but it contains many other solutions. Each solution corresponds to an object whose appearance is the same as the original object but whose actual shape is different. For example, a picture of a slope ascending rightward also contains a solution corresponding to a slope ascending leftward. By choosing an appropriate



(a) (b) (c)

Figure 4 Second-generation "impossible motion object": (a) two walls with holes; (b) a rod penetrating two windows; (c) another view.

solution, we can create the impression of impossible motion such as a ball climbing a slope against gravity [10].

An example is shown in Fig. 4, where panel (a) shows an object composed of two walls with rectangular holes, and panel (b) shows the motion of a rigid straight rod penetrating through two windows. Fig. 4(c) shows another view of the object, from which we can see that the actual shape of the object is different from what we perceive when we see the image in Fig. 4(a).

We classify this type of impossible object as second generation and name the objects "impossible motion objects".

4 Cylinder-Type Impossible Objects

Anomalous objects and impossible motion objects create an "impossible" visual effect when they are seen from a single special viewpoint. Another way to generate a sense of impossibility is to observe an object from two or more viewpoints. If the appearance of the object is so different when seen from the different viewpoints that the viewer is unable to believe that it is the same object, then we may well have found a new impossible object. On the basis of this idea, we can construct several additional classes of impossible objects.

4.1 Third-Generation "Ambiguous Cylinders"

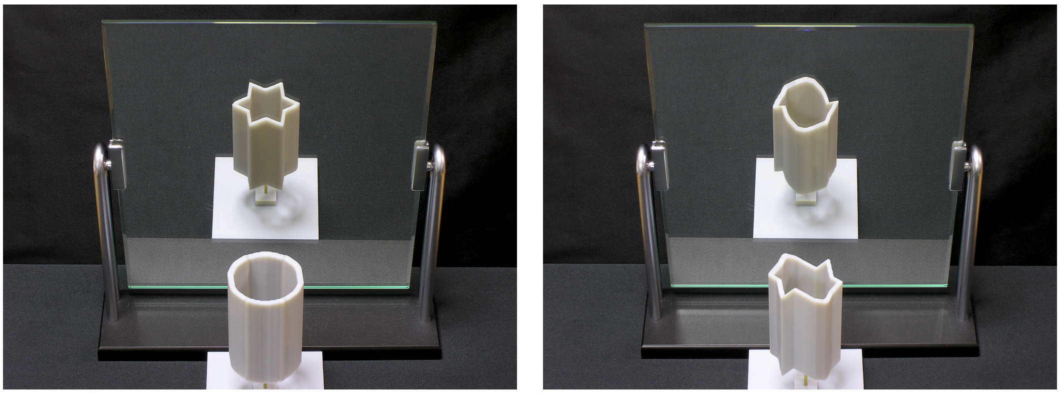
The system of linear equations and equalities described in (2) and (4) has infinitely many solutions if the associated picture is correct. This implies that the same 2D appearance can be realized by many different 3D shapes, and, consequently, we may construct a 3D object that projects two desired appearances when the object is seen from two special viewpoints. This kind of object can be found by solving two systems of (2) and (4) corresponding to two pictures with respect to two viewpoints.

An example is shown in Fig. 5. As shown in panel (a), when the cylinder is viewed directly, it has the shape of a full moon, but when it is reflected in the vertical mirror behind, it has the shape of a star.

We classify these types of impossible objects as third generation and call them "ambiguous cylinders" [11].

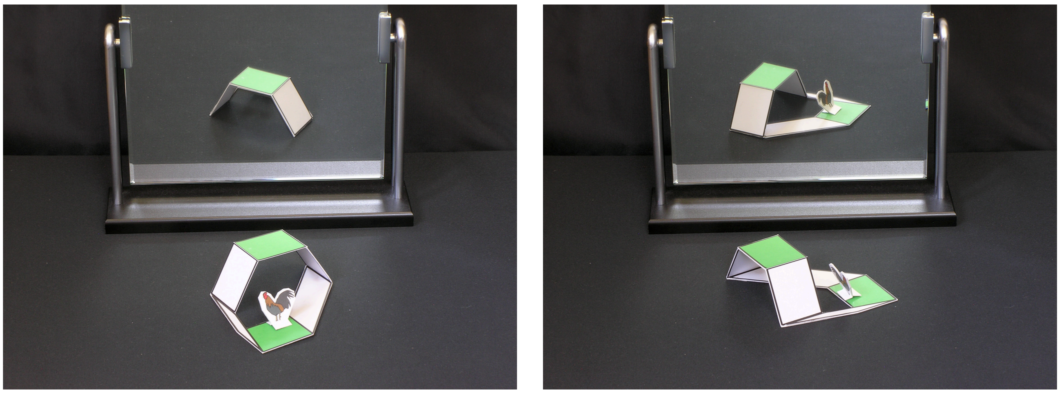
4.2 Fourth-Generation "Partly invisible Objects"

The design method used for ambiguous cylinders can also be used to create another visual illusion in which part of an object disappears when it is seen from a second viewpoint. To



(a) (b)

Figure 5 Third-generation "ambiguous cylinder": (a) full moon and star; (b) another view.



(a) (b)

Figure 6 Fourth-generation "partly invisible object": (a) object VH; (b) another view.

understand how this might work, consider an object composed of two parts, A and B. We can construct an ambiguous cylinder in such a way that part A appears as it is from both viewpoints, whereas part B appears as A when seen from the second viewpoint. The resulting object is such that, when we see the object from the second viewpoint, parts A and B overlap and, as a consequence, one is hidden by the other.

An example is given in Fig. 6. In panel (a), the object appears to be a regular hexagonal cylinder on its side, while the lower half disappears in the mirror. As shown in panel (b), the lower half is actually horizontal and gets hidden behind the upper half when seen from the second viewpoint.

We classify these types of impossible objects as fourth generation and name them "partly invisible objects" [14].

4.3 Fifth-Generation "Topology-Disturbing Objects"

We can apply the design method for ambiguous cylinders in still another way, one whereby we create objects whose topology changes in the mirror. An example is shown in Fig. 7. As shown in panel (a), the object appears in both views to consist of two circular cylinders; however, the cylinders are separated in the direct view, while they appear to be mutually

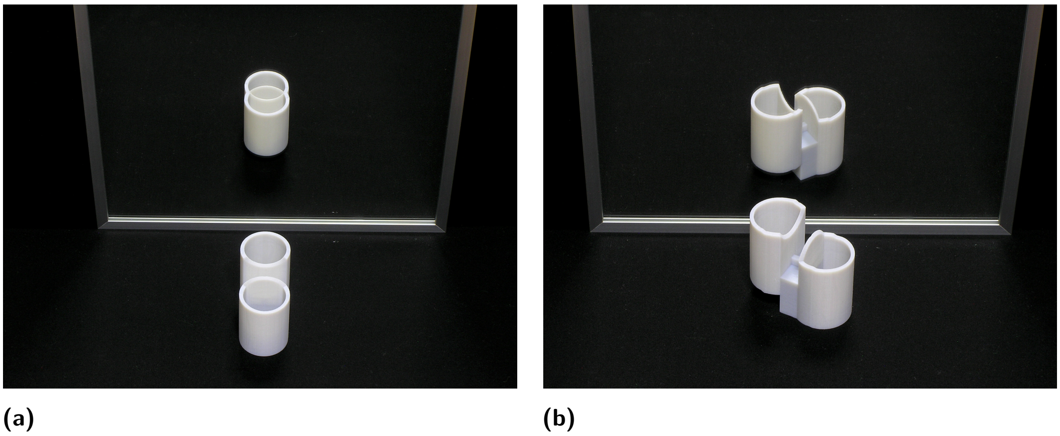


Figure 7 Fifth-generation "topology-disturbing objects": (a) two vertically aligned cylinders; (b) another view.

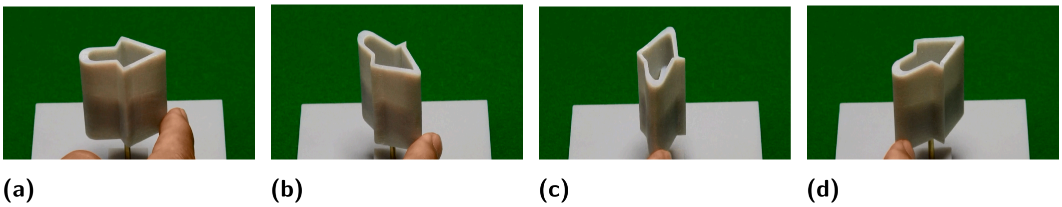


Figure 8 Sixth-generation "deformable objects": (a) arrow that likes to face rightward; (b), (c), (d) sequence of other views.

intersecting in the mirror. In other words, the shape of each part does not change, but their topology is disturbed in the mirror. The actual shape of the object can be understood when we see it from another direction, as in panel (b).

We classify these types of impossible objects as fifth generation and name them "topology-disturbing objects" [15].

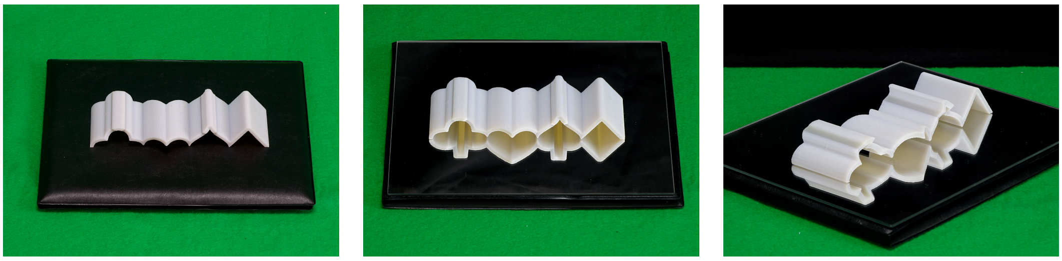
4.4 Sixth-Generation "Deformable Objects"

Some ambiguous cylinders create another interesting visual effect in the sense that the rotation of the object around a vertical axis generates the impression of a dynamic change of shape. An example is shown in Fig. 8, where panels (a), (b), (c), and (d) show pictures of the object being rotated around a vertical axis by approximately 0, 40, 100, and 150 degrees. The object appears to be an arrow facing rightward; however, if we rotate it by 180 degrees around the vertical axis, it again faces rightward. Moreover, during the rotation, the object appears to be deforming continuously [13].

We call this type of impossible object sixth generation and name the objects "deformable objects".

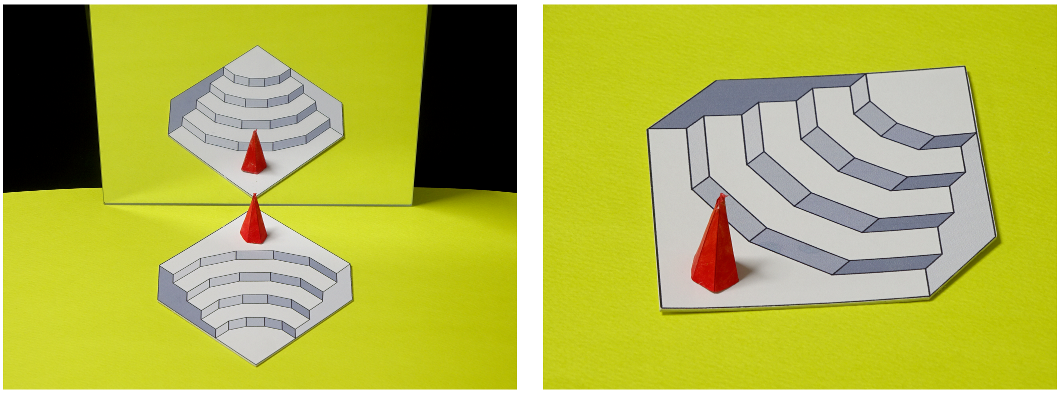
4.5 Eighth-Generation "Reflexively Fused Objects"

Another application of the ambiguous cylinder is to make part of the goal shape as a solid object and to provide the remaining part by its mirror image. An example is shown in Fig. 9, where panel (a) shows an object, panel (b) shows the same object placed on a horizontally



(a) (b) (c)

Figure 9 Eighth-generation "reflexively fused object": (a) object alone; (b) object on a horizontally oriented mirror; (c) another view.



(a) (b)

Figure 10 Seventh-generation "height-reversal object": (a) amphitheater and hill; (b) another view.

oriented mirror so that the object and its mirror image are fused, and panel (c) shows another view of the object on the mirror. The object itself is meaningless, but the object together with its mirror image gives a meaningful shape. This type of an object can be constructed in the following way. We first decompose a goal shape into an upper part and a lower part, next transform the lower part into its height-reversed version, and finally apply the ambiguous cylinder method to this pair of the shapes.

We call this type of impossible object eighth generation and name the objects "reflexively fused objects".

5 Seventh-Generation "Height Reversal Objects"

A picture placed on a horizontal plane can sometimes generate two interpretations of a 3D object whose height is reversed when the object is seen from mutually opposite sides with the same slant angle [12]. If we add to such a picture a 3D object showing the direction of gravity and place a vertical mirror behind it, then the direct view and the mirror image give quite different impressions of the 3D surfaces. An example is shown in Fig. 10. The direct view appears to be an amphitheater with the stage at the bottom, whereas the mirror image appears to be a hill.

We classify these types of impossible objects as seventh generation and name them "height-reversal objects" [12].

6 Concluding Remarks

We have classified objects that generate the impression of impossibility into several generations and shown an example object for each generation, where all of objects involve visual illusions.

The mathematics behind the illusions is based on the principle that a single image does not have depth information and hence there are many possible 3D shapes that give the same 2D appearance. By combining this mathematical property with the psychological preferences of human vision systems, we can effectively create many new visual effects, with potential applications to toys, tourism, magic, and so on. Our next goal is to realize these various applications.

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