

How Bad is the Freedom to Flood-It?

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Abstract

FIXED-FLOOD-IT and FREE-FLOOD-IT are combinatorial problems on graphs that generalize a very popular puzzle called *Flood-It*. Both problems consist of recoloring moves whose goal is to produce a monochromatic (“flooded”) graph as quickly as possible. Their difference is that in FREE-FLOOD-IT the player has the additional freedom of choosing the vertex to play in each move. In this paper, we investigate how this freedom affects the complexity of the problem. It turns out that the freedom is *bad* in some sense. We show that some cases trivially solvable for FIXED-FLOOD-IT become intractable for FREE-FLOOD-IT. We also show that some tractable cases for FIXED-FLOOD-IT are still tractable for FREE-FLOOD-IT but need considerably more involved arguments. We finally present some combinatorial properties connecting or separating the two problems. In particular, we show that the length of an optimal solution for FIXED-FLOOD-IT is always at most twice that of FREE-FLOOD-IT, and this is tight.

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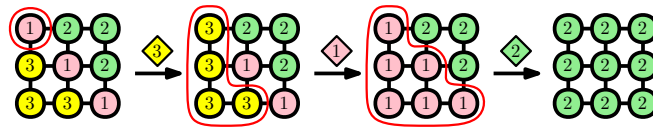
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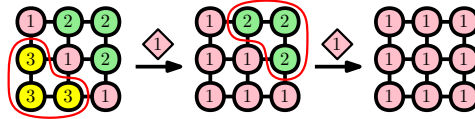


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■ **Figure 1** A flooding sequence on a 3×3 grid.



■ **Figure 2** A flooding sequence with no restriction on selected monochromatic components.

1 Introduction

Flood-It is a popular puzzle, originally released as a computer game in 2006 by LabPixies (see [1]). In this game, the player is presented with (what can be thought of as) a vertex-colored grid graph, with a designated special *pivot* vertex, usually the top-left corner of the grid. In each move, the player has the right to change the color of all vertices contained in the same monochromatic component as the pivot to a different color of her choosing. Doing this judiciously gradually increases the size of the pivot's monochromatic component, until the whole graph is *flooded* with one color. The goal is to achieve this flooding with the minimum number of moves. See Figure 1 for an example.

Following the description above, *Flood-It* immediately gives rise to a natural optimization problem: given a vertex-colored graph, determine the shortest sequence of flooding moves that wins the game. This problem has been extensively studied in the last few years (e.g. [10, 12, 14, 13, 7, 3, 17, 4, 15, 9]; a more detailed summary of known results is given below), both because of the game's popularity (and addictiveness!), but also because the computational complexity questions associated with this problem have turned out to be surprisingly deep, and the problem has turned out to be surprisingly intractable.

The goal of this paper is to add to our understanding of this interesting, puzzle-inspired, optimization problem, by taking a closer look at the importance of the *pivot vertex*. As explained above, the classical version of the game only allows the player to change the color of a special vertex and its component and has been studied under the name FIXED-FLOOD-IT [12, 14, 13] (or FLOOD-IT in some papers [1, 17, 3, 4, 9]). However, it is extremely natural to also consider a version where the player is also allowed to play a different vertex of her choosing in each turn. This has also been well-studied under the name FREE-FLOOD-IT [1, 10, 12, 14, 13, 3, 17]. See Figure 2.

Since both versions of this problem have been studied before, the question of the impact of the pivot vertex on the problem's structure has (at least implicitly) been considered. Intuitively, one would expect FREE-FLOOD-IT to be a harder problem; after all, the player has to choose a color to play *and* a vertex to play it on, and is hence presented with a larger set of possible moves. The state of the art seems to confirm this intuition, as only some of the positive algorithmic results known for FIXED-FLOOD-IT are known also for FREE-FLOOD-IT, while there do exist some isolated cases where FIXED-FLOOD-IT is tractable and FREE-FLOOD-IT is hard, for example co-comparability graphs [5, 7] and grids of height 2 [1, 13]. Nevertheless, these results do not completely pinpoint the added complexity brought by the task of selecting a vertex to play, as the mentioned algorithms for FIXED-FLOOD-IT are already non-trivial, and hence the jump in complexity is likely to be the result of *the combination* of the tasks of picking a color and a vertex. More broadly, [3] presented a

generic reduction from FIXED-FLOOD-IT to FREE-FLOOD-IT that preserves a number of crucial parameters (number of colors, optimal value, etc.) and gives convincing evidence that FREE-FLOOD-IT is always at least as hard as FIXED-FLOOD-IT, but not necessarily harder.

Our Results: We investigate the complexity of FREE-FLOOD-IT, mostly from the point of view of parameterized complexity,¹ as well as the impact on the combinatorics of the game of allowing moves outside the pivot. Due to space constraints, some proofs are omitted and marked with ★.

Our first result is to show that FREE-FLOOD-IT is W[2]-hard parameterized by the number of moves in an optimal solution. We recall that for FIXED-FLOOD-IT this parameterization is trivially fixed-parameter tractable: when a player has only k moves available, then we can safely assume that the graph uses at most (roughly) k colors, hence one can easily consider all possible solutions in FPT time. The interest of our result is, therefore, to demonstrate that the task of deciding which vertex to play next is sufficient to make FREE-FLOOD-IT significantly harder than FIXED-FLOOD-IT. Indeed, the W[2]-hardness reduction we give, implies also that FREE-FLOOD-IT is not solvable in $n^{o(k)}$ time under the ETH. This tightly matches the complexity of a trivial algorithm which considers all possible vertices and colors to be played. This is the first concrete example showing a case where FIXED-FLOOD-IT is essentially trivial, but FREE-FLOOD-IT is intractable.

Motivated by this negative result we consider several other parameterizations of the problem. We show that FREE-FLOOD-IT is fixed-parameter tractable when parameterized by the number of possible moves and the clique-width. This result is tight in the sense that the problem is hard when parameterized by only one of these parameters. It also implies the fixed-parameter tractability of the problem parameterized by the number of colors and the modular-width. In a similar vein, we present a polynomial kernel when FREE-FLOOD-IT is parameterized by the input graph's neighborhood diversity and number of colors. An analogous result was shown for FIXED-FLOOD-IT in [4], but because of the freedom to select vertices, several of the tricks used there do not apply to FREE-FLOOD-IT, and our proofs are slightly more involved. Our previously mentioned reduction also implies that FREE-FLOOD-IT does not admit a polynomial kernel parameterized by vertex cover, under standard assumptions. This result was also shown for FIXED-FLOOD-IT in [4], but it does not follow immediately for FREE-FLOOD-IT, as the reduction of [3] does not preserve the graph's vertex cover.

Motivated by the above results, which indicate that the complexity of the problem can be seriously affected if one allows non-pivot moves, we also study some more purely combinatorial questions with algorithmic applications. The main question we pose here is the following. It is obvious that for all instances the optimal number of moves for FREE-FLOOD-IT is upper-bounded by the optimal number of moves for FIXED-FLOOD-IT (since the player has strictly more choices), and it is not hard to construct instances where FIXED-FLOOD-IT needs strictly more moves. Can we bound the optimal number of FIXED-FLOOD-IT moves needed as a function of the optimal number of FIXED-FLOOD-IT moves? Somewhat surprisingly, this extremely natural question does not seem to have been explicitly considered in the literature before. Here, we completely resolve it by showing that the two optimal values cannot be more than a factor of 2 apart, and constructing a family of simple instances where they are exactly a factor of 2 apart. As an immediate application, this gives a 2-approximation for FREE-FLOOD-IT for every case where FIXED-FLOOD-IT is known to be tractable.

¹ For readers unfamiliar with the basic notions of this field, we refer to standard textbooks [2, 6].

We also consider the problem's monotonicity: FIXED-FLOOD-IT has the nice property that even an adversary that selects a single bad move cannot increase the optimal (that is, in the worst case a bad move is a wasted move). We construct minimal examples which show that FREE-FLOOD-IT does not have this nice monotonicity property, even for extremely simple graphs, that is, making a bad move may not only waste a move but also make the instance strictly worse. Such a difference was not explicitly stated in the literature, while the monotonicity of FIXED-FLOOD-IT was seem to be known or at least assumed. The only result we are aware of is the monotonicity of FREE-FLOOD-IT on paths shown by Meeks and Scott [12].

Known results: In 2009, the NP-hardness of FIXED-FLOOD-IT with six colors was sketched by Elad Verbin as a comment to a blog post by Sarel Har-Peled [16]. Independently to the blog comment, Clifford et al. [1] and Fleischer and Woeginger [5] started investigations of the complexity of the problem, and published the conference versions of their papers at FUN 2010. Below we mostly summarize some of the known results on FREE-FLOOD-IT. For more complete lists of previous result, see e.g. [7, 10, 4].

FREE-FLOOD-IT is NP-hard if the number of colors is at least 3 [1] even for trees with only one vertex of degree more than 2 [10, 3], while it is polynomial-time solvable for general graphs if the number of colors is at most 2 [1, 12, 10]. Moreover, it is NP-hard even for height-3 grids with four colors [12]. Note that this result implies that FREE-FLOOD-IT with a constant number colors is NP-hard even for graphs of bounded bandwidth. If the number of colors is unbounded, then it is NP-hard for height-2 grids [13], trees of radius 2 [3], and, proper interval graphs and caterpillars [7]. Also, it is known that there is no constant-factor approximation with a factor independent of the number of colors unless $P = NP$ [1].

There are a few positive results on FREE-FLOOD-IT. Meeks and Scott [14] showed that every colored graph has a spanning tree with the same coloring such that the minimum number of moves coincides in the graph and the spanning tree. Using this property, they showed that if a graph has only a polynomial number of vertex subsets that induce connected subgraphs, then FREE-FLOOD-IT (and FIXED-FLOOD-IT) on the graph can be solved in polynomial time. This in particular implies the polynomial-time solvability on subdivisions of a fixed graph. It is also known that FREE-FLOOD-IT for interval graphs and split graphs is fixed-parameter tractable when parameterized by the number of colors [7].

2 Preliminaries

For a positive integer k , we use $[k]$ to denote the set $\{1, \dots, k\}$. Given a graph $G = (V, E)$, a coloring function $\mathbf{col}: V \rightarrow [c_{\max}]$, where c_{\max} is a positive integer, and $u \in V$, we denote by $\mathbf{Comp}(\mathbf{col}, u)$ the maximal set of vertices S such that for all $v \in S$, $\mathbf{col}(u) = \mathbf{col}(v)$ and there exists a path from u to v such that for all its internal vertices w we have $\mathbf{col}(w) = \mathbf{col}(u)$. In other words, $\mathbf{Comp}(\mathbf{col}, u)$ is the monochromatic connected component that contains u under the coloring function \mathbf{col} .

Given G, \mathbf{col} , a *move* is defined as a pair (u, i) where $u \in V$, $i \in [c_{\max}]$. The *result* of the move (u, c) is a new coloring function \mathbf{col}' defined as follows: $\mathbf{col}'(v) = c$ for all $v \in \mathbf{Comp}(\mathbf{col}, u)$; $\mathbf{col}'(v) = \mathbf{col}(v)$ for all other vertices. In words, a move consists of changing the color of u , and of all vertices in the same monochromatic component as u , to c . Given the above definition we can also define the result of a sequence of moves $(u_1, c_1), (u_2, c_2), \dots, (u_k, c_k)$ on a colored graph with initial coloring function \mathbf{col}_0 in the natural way, that is, for each $i \in [k]$, \mathbf{col}_i is the result of move (u_i, c_i) on \mathbf{col}_{i-1} .

The FREE-FLOOD-IT problem is defined as follows: given a graph $G = (V, E)$, an integer k , and an initial coloring function \mathbf{col}_0 , decide if there exists a sequence of k moves $(u_1, c_1), (u_2, c_2), \dots, (u_k, c_k)$ such that the result \mathbf{col}_k obtained by applying this sequence of moves on \mathbf{col}_0 is a constant function (that is, $\forall u, v \in V$ we have $\mathbf{col}_k(u) = \mathbf{col}_k(v)$).

In the FIXED-FLOOD-IT problem we are given the same input as in the FREE-FLOOD-IT problem, as well as a designated vertex $p \in V$ (the pivot). The question is again if there exists a sequence of moves such that \mathbf{col}_k is monochromatic, with the added constraint that we must have $u_i = p$ for all $i \in [k]$.

We denote by $\text{OPT}_{\text{Free}}(G, \mathbf{col})$, $\text{OPT}_{\text{Fixed}}(G, \mathbf{col}, p)$ the minimum k such that for the input (G, \mathbf{col}) (or (G, \mathbf{col}, p) respectively) the FREE-FLOOD-IT problem (respectively the FIXED-FLOOD-IT problem) admits a solution.

Graph parameters: The graph parameters considered in this paper are the *vertex cover number* $\text{vc}(G)$, the *neighborhood diversity* $\text{nd}(G)$, the *modular-width* $\text{mw}(G)$, and the *clique-width* $\text{cw}(G)$. It is known that $\text{cw}(G) \leq \text{mw}(G) \leq \text{nd}(G) \leq 2^{\text{vc}(G)} + \text{vc}(G)$ for every graph G [8, 11]. (See [11, 8, 2] for definitions.)

3 W[2]-hardness of Free-Flood-It

The main result of this section is that FREE-FLOOD-IT is W[2]-hard when parameterized by the minimum length of any valid solution (the natural parameter). The proof consists of a reduction from SET COVER, a canonical W[2]-complete problem.

Before presenting the construction, we recall two basic observations by Meeks and Vu [15], both of which rest on the fact that any single move can (at most) eliminate a single color from the graph, and this can only happen if a color induces a single component.

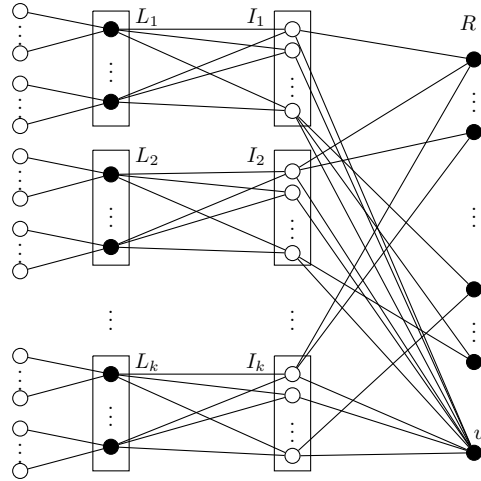
► **Lemma 3.1** ([15]). *For any graph $G = (V, E)$, and coloring function \mathbf{col} that uses c_{\max} distinct colors, we have $\text{OPT}_{\text{Free}}(G, \mathbf{col}) \geq c_{\max} - 1$.*

► **Lemma 3.2** ([15]). *For any graph $G = (V, E)$, and coloring function \mathbf{col} that uses c_{\max} distinct colors, such that for all $c \in [c_{\max}]$, $G[\mathbf{col}^{-1}(c)]$ is a disconnected graph, we have $\text{OPT}_{\text{Free}}(G, \mathbf{col}) \geq c_{\max}$.*

The proof of Theorem 3.6 relies on a reduction from a special form of SET COVER, which we call MULTI-COLORED SET COVER (MCSC for short). MCSC is defined as follows:

► **Definition 3.3.** In MULTI-COLORED SET COVER (MCSC) we are given as input a set of elements R and k collections of subsets of R , $\mathcal{S}_1, \dots, \mathcal{S}_k$. We are asked if there exist k sets S_1, \dots, S_k such that for all $i \in [k]$, $S_i \in \mathcal{S}_i$, and $\cup_{i \in [k]} S_i = R$.

Observe that MCSC is just a version of SET COVER where the collection of sets is given to us pre-partitioned into k parts and we are asked to select one set from each part to form a set cover of the universe. It is not hard to see that any SET COVER instance (\mathcal{S}, R) where we are asked if there exists a set cover of size k can easily be transformed to an equivalent MCSC instance simply by setting $\mathcal{S}_i = \mathcal{S}$ for all $i \in [k]$, since the definition of MCSC does not require that the sub-collections \mathcal{S}_i be disjoint. We conclude that known hardness results for SET COVER immediately transfer to MCSC, and in particular MCSC is W[2]-hard when parameterized by k .



■ **Figure 3** The graph $G = (V, E)$ of FREE-FLOOD-IT constructed from the given MCSC instance. All the vertices in each I_i have color i and all black vertices have color $k + 1$. Boxes containing black vertices have size $3k$. Also each vertex in L_i has k neighbors with degree 1 colored $1, \dots, k$.

Construction

We are now ready to describe our reduction which, given a MCSC instance with universe R and k collections of sets $\mathcal{S}_i, i \in [k]$, produces an equivalent instance of FREE-FLOOD-IT, that is, a graph $G = (V, E)$ and a coloring function \mathbf{col} on V . We construct this graph as follows:

- for every set $S \in \mathcal{S}_i$, construct a vertex in V . The set of vertices in V corresponding to sets of \mathcal{S}_i is denoted by I_i and $\mathbf{col}(v) = i$ for each $v \in I_i$. $I_1 \cup \dots \cup I_k$ induces an independent set colored $\{1, \dots, k\}$.
- for each $i \in [k]$, construct $3k$ new vertices, denoted by L_i and connect all of them to all vertices of I_i such that $L_i \cup I_i$ induces a complete bipartite graph of size $3k \times |I_i|$. Then set $\mathbf{col}(v) = k + 1$ for each $v \in L_i$, for all $i \in [k]$.
- for each vertex $v \in L_i$ for $1 \leq i \leq k$, construct k new leaf vertices connected to v with distinct colors $1, \dots, k$.
- for each element $e \in R$, construct a vertex e . For each $S \in \mathcal{S}_i$ such that $e \in S$ we connect e to the vertex of I_i that represents S .
- add a special vertex u with $\mathbf{col}(u) = k + 1$ which is connected it to all vertices in I_i for $i \in [k]$.

An illustration of G is shown in Fig.3. In the following we will show that (G, \mathbf{col}) as an instance of FREE-FLOOD-IT is solvable with at most $2k$ moves if and only if the given MCSC instance has a set cover of size k which contains one set of each \mathcal{S}_i .

► **Lemma 3.4.** *If $(\mathcal{S}_1, \dots, \mathcal{S}_k, R)$ is a YES instance of MCSC then $\text{OPT}_{\text{Free}}(G, \mathbf{col}) \leq 2k$.*

Proof. Suppose that there is a solution S_1, \dots, S_k of the given MCSC instance, with $S_i \in \mathcal{S}_i$, for $i \in [k]$ and $\cup_{i \in [k]} S_i = R$. Recall that for each S_i there is a vertex in I_i in the constructed graph representing S_i . Our first k moves consist of changing the color of each of these k vertices to $k + 1$ in some arbitrary order.

Observe that in the graph resulting after these k moves the vertices with color $k + 1$ form a single connected component: because $\cup S_i$ is a set cover, all vertices of R have a neighbor with color $k + 1$; all vertices with color $k + 1$ in some I_i are in the same component as u ; and all vertices of $\cup_{i \in [k]} L_i$ are connected to one of the vertices we played. Furthermore, observe

that this component dominates the graph: all remaining vertices of $\cup I_i$, as well as all leaves attached to vertices of $\cup_{i \in [k]} L_i$ are dominated by the vertices of $\cup_{i \in [k]} L_i$. Hence, we can select an arbitrary vertex with color $k + 1$, say u , and cycle through the colors $1, \dots, k$ on this vertex to make the graph monochromatic. \blacktriangleleft

Now we establish the converse of Lemma 3.4.

► **Lemma 3.5.** *If $\text{OPT}_{\text{Free}}(G, \text{col}) \leq 2k$, then $(\mathcal{S}_1, \dots, \mathcal{S}_k, R)$ is a YES instance of MCSC.*

Proof. Suppose that there exists a sequence of at most $2k$ moves solving (G, col) . We can assume without loss of generality that the sequence has length exactly $2k$, since performing a move on a monochromatic graph keeps the graph monochromatic. Let $(u_1, c_1), \dots, (u_{2k}, c_{2k})$ be a solution, let $\text{col}_0 = \text{col}$, and let col_i denote the coloring of G obtained after the first i moves. The key observation that we will rely on is the following:

(i) For all $i \in [k]$, there exist $j \in [k], v \in I_i$ such that $\text{col}_j(v) = k + 1$.

In other words, we claim that for each group I_i there exists a vertex that received color $k + 1$ at some point during the first k moves. Before proceeding, let us prove this claim. Suppose for contradiction that the claim is false. Then, there exists a group I_i such that no vertex in that group has color $k + 1$ in any of the colorings $\text{col}_0, \dots, \text{col}_k$. We now consider the vertices of L_i and their attached leaves. Since L_i contains $3k > k + 2$ vertices, there exist two vertices v_1, v_2 of L_i such that $\{u_1, \dots, u_k\}$ contains neither v_1, v_2 , nor any of their attached leaves. In other words, there exist two vertices of L_i on which the winning sequence does not change colors by playing them or their private neighborhood directly. However, since v_1, v_2 only have neighbors in I_1 (except for their attached leaves), and no vertex of I_1 received color $k + 1$, we conclude that $\text{col}_k(v_1) = \text{col}_k(v_2) = k + 1$, that is, the colors of these two vertices have remained unchanged, and the same is true for their attached leaves. Consider now the graph G with coloring col_k : we observe that this coloring uses $k + 1$ distinct colors, and that each color induces a disconnected graph. This is true for colors $1, \dots, k$ because of the leaves attached to v_1, v_2 , and true of color $k + 1$ because of v_1, v_2 and the fact that no vertex of I_i has color $k + 1$. We conclude that $\text{OPT}_{\text{Free}}(G, \text{col}_k) \geq k + 1$ by Lemma 3.2, which is a contradiction, because the whole sequence has length $2k$.

Because of claim (i) we can now conclude that for all $i \in [k]$ there exists a $j \in [k]$ such that $\text{col}_{j-1}(u_j) = i$. In other words, for each color i there exists a move among the first k moves of the solution that played a vertex which at that point had color i . To see that this is true consider again for contradiction the case that for some $i \in [k]$ this statement does not hold: this implies that vertices with color i in col_0 still have color i in $\text{col}_1, \dots, \text{col}_k$, which means that no vertex of I_i has received color $k + 1$ in the first k moves, contradicting (i).

As a result of the above, we therefore claim that for all $j \in [k]$, we have $\text{col}_{j-1}(u_j) \neq k + 1$. In other words, we claim that none of the first k moves changes the color of a vertex that at that point had color $k + 1$. This is because, as argued, for each of the other k colors, there is a move among the first k moves that changes a vertex of that color. We therefore conclude that for all vertices v for which $\text{col}_0(v) = k + 1$ we have $\text{col}_j(v) = k + 1$ for all $j \in [k]$. In addition, because in col_0 all colors induce independent sets, each of the first k moves changes the color of a single vertex. Because of claim (i), this means that for each $i \in [k]$ one of the first k moves changes the color of a single vertex from I_i to $k + 1$. We select the corresponding set of \mathcal{S}_i in our MCSC solution.

We now observe that, since all vertices of $\cup_{i \in [k]} L_i$ retain color $k + 1$ throughout the first k moves, col_k is a coloring function that uses $k + 1$ distinct colors, and colors $1, \dots, k$ induce disconnected graphs (because of the leaves attached to the vertices of each L_i). Thanks to

Lemma 3.2, this means that $\text{col}_k^{-1}(k+1)$ must induce a connected graph. Hence, all vertices of R have a neighbor with color $k+1$ in col_k , which must be one of the k vertices played in the first k moves; hence the corresponding element is dominated by our solution and we have a valid set cover selecting one set from each \mathcal{S}_i . \blacktriangleleft

We are now ready to combine Lemmas 3.4 and 3.5 to obtain the main result of this section.

► **Theorem 3.6.** *FREE-FLOOD-IT is $W[2]$ -hard parameterized by OPT_{Free} , that is, parameterized by the length of the optimal solution. Furthermore, if there is an algorithm that decides if a FREE-FLOOD-IT instance has a solution of length k in time $n^{o(k)}$, then the ETH is false.*

Proof. The described construction, as well as Lemmas 3.4 and 3.5 give a reduction from MCSC, which is $W[2]$ -hard parameterized by k , to an instance of FREE-FLOOD-IT with $k+1$ colors, where the question is to decide if $OPT_{\text{Free}}(G, \text{col}) \leq 2k$. Furthermore, it is known that MCSC generalizes DOMINATING SET, which does not admit an algorithm running in time $n^{o(k)}$, under the ETH [2]. Since our reduction only modifies k by a constant, we obtain the same result for FREE-FLOOD-IT. \blacktriangleleft

We note that because of Lemma 3.1 we can always assume that the number of colors of a given instance is not much higher than the length of the optimal solution. As a result, FREE-FLOOD-IT parameterized by OPT_{Free} is equivalent to the parameterization of FREE-FLOOD-IT by $OPT_{\text{Free}} + c_{\max}$ and the result of Theorem 3.6 also applies to this parameterization. Also, as a byproduct of the reduction above, we can show a kernel lower bound for FREE-FLOOD-IT parameterized by the vertex cover number.

► **Theorem 3.7 (★).** *FREE-FLOOD-IT parameterized by the vertex cover number admits no polynomial kernel unless $\text{PH} = \Sigma_3^{\text{P}}$.*

4 Clique-width and neighborhood diversity

In this section, we consider as a combined parameter for FREE-FLOOD-IT the length of an optimal solution and the clique-width. We show that this case is indeed fixed-parameter tractable by using the theory of the monadic second-order logic on graphs. As an application of this result, we also show that combined parameterization by the number of colors and the modular-width is fixed-parameter tractable.

► **Theorem 4.1 (★).** *Given an instance (G, col) of FREE-FLOOD-IT such that G has n vertices and clique-width at most w , it can be decided in time $O(f(k, w) \cdot n^3)$ whether $OPT_{\text{Free}}(G, \text{col}) \leq k$, where f is some computable function.*

► **Corollary 4.2 (★).** *Given an integer k and an instance (G, col) of FREE-FLOOD-IT such that G has n vertices and modular-width at most w , it can be decided in time $O(f(c_{\max}, w) \cdot n^3)$ whether $OPT_{\text{Free}}(G, \text{col}) \leq k$, where f is some computable function.*

Since the modular-width of a graph is upper bounded by its neighborhood diversity, the corollary above implies that FREE-FLOOD-IT is fixed-parameter tractable when parameterized by both the neighborhood diversity and the number of colors. Here we show that FREE-FLOOD-IT admits a polynomial kernel with the same parameterization.

► **Theorem 4.3.** *FREE-FLOOD-IT admits a kernel of $\text{nd}(G) \cdot c_{\max} \cdot (\text{nd}(G) + c_{\max} - 1)$ vertices.*

Our reduction rules are as follows:

- Rule *TT*: Let u and v be true twins of the same color in (G, \mathbf{col}) . Remove v .
- Rule *FT*: Let F be a set of false-twin vertices of the same color in (G, \mathbf{col}) such that $|F| = \text{nd}(G) + c_{\max}$. Remove arbitrary one vertex in F .

Observe that after applying *TT* and *FT* exhaustively in polynomial time, the obtained graph can have at most $\text{nd}(G) \cdot c_{\max} \cdot (\text{nd}(G) + c_{\max} - 1)$ vertices. This is because each set of twin vertices can contain at most $\text{nd}(G) + c_{\max} - 1$ vertices. Hence, to prove Theorem 4.3, it suffices to show the safeness of the rules.

► **Lemma 4.4 (★)**. *The rules *TT* and *FT* are safe.*

5 Relation Between Fixed and Free Flood-It

The main theorem of this section is the following:

► **Theorem 5.1**. *For any graph $G = (V, E)$, coloring function \mathbf{col} on G , and $p \in V$ we have*

$$OPT_{Free}(G, \mathbf{col}) \leq OPT_{Fixed}(G, \mathbf{col}, p) \leq 2OPT_{Free}(G, \mathbf{col}).$$

Theorem 5.1 states that the optimal solutions for *FREE-FLOOD-IT* and *FIXED-FLOOD-IT* can never be more than a factor of 2 apart. It is worthy of note that we could not hope to obtain a constant smaller than 2 in such a theorem, and hence the theorem is tight.

► **Theorem 5.2**. *There exist instances of *FIXED-FLOOD-IT* such that $OPT_{Fixed}(G, \mathbf{col}, p) = 2OPT_{Free}(G, \mathbf{col})$*

Proof. Consider a path on $2n + 1$ vertices properly colored with colors 1, 2. If we set the pivot to be one of the endpoints then $OPT_{Free} = 2n$. However, it is not hard to obtain a *FREE-FLOOD-IT* solution with n moves by playing every vertex at odd distance from the pivot. ◀

Before we proceed to give the proof of Theorem 5.1, let us give a high-level description of our proof strategy and some general intuition. The first inequality is of course trivial, so we focus on the second part. We will establish it by induction on the number of non-pivot moves performed by an optimal *FREE-FLOOD-IT* solution. The main inductive argument is based on observing that a valid *FREE-FLOOD-IT* solution will either at some point play a neighbor u of the component of p to give it the same color as p , or if not, it will at some point play p to give it the same color as one of its neighbors. The latter case is intuitively easier to handle, since then we argue that the move that changed p 's color can be performed first, and if the first move is a pivot move we can easily fall back on the inductive hypothesis. The former case, which is the more interesting one, can be handled by replacing the single move that gives u the same color as p , with two moves: one that gives p the same color as u , and one that flips p back to its previous color. Intuitively, this basic step is the reason we obtain a factor of 2 in the relationship between the two versions of the game.

The inductive strategy described above faces some complications due to the fact that rearranging moves in this way may unintentionally re-color some vertices, which makes it harder to continue the rest of the solution as before. To avoid this we define a somewhat generalized version of *FREE-FLOOD-IT*, called *SUBSET-FREE-FLOOD-IT*.

► **Definition 5.3**. Given $G = (V, E)$, a coloring function \mathbf{col} on G , and a pivot $p \in V$, a *set-move* is a pair (S, c) , with $S \subseteq V$ and $S = \mathbf{Comp}(\mathbf{col}, u)$ for some $u \in V$, or $\{p\} \subseteq S \subseteq \mathbf{Comp}(\mathbf{col}, p)$. The result of (S, c) is the coloring \mathbf{col}' that sets $\mathbf{col}'(v) = c$ for $v \in S$; and $\mathbf{col}'(v) = \mathbf{col}(v)$ otherwise.

We define SUBSET-FREE-FLOOD-IT as the problem of determining the minimum number of set-moves required to make a graph monochromatic, and SUBSET-FIXED-FLOOD-IT as the same problem when we impose the restriction that every move must change the color of p , and denote as $\text{OPT}_{\text{S-Free}}$, $\text{OPT}_{\text{S-Fixed}}$ the corresponding optimum values.

Informally, a set-move is the same as a normal move in FREE-FLOOD-IT, except that we are also allowed to select an arbitrary connected monochromatic set S that contains p (even if S is not maximal) and change its color. Intuitively, one would expect moves that set S to be a proper subset of $\mathbf{Comp}(\mathbf{col}, p)$ to be counter-productive, since such moves split a monochromatic component into two pieces. Indeed, we prove below in Lemma 5.4 that the optimal solutions to FIXED-FLOOD-IT and SUBSET-FIXED-FLOOD-IT coincide, and hence such moves do not help. The reason we define this version of the game is that it gives us more freedom to define a solution that avoids unintentionally recoloring vertices as we transform a given FREE-FLOOD-IT solution to a FIXED-FLOOD-IT solution.

► **Lemma 5.4.** *For any graph $G = (V, E)$, coloring function \mathbf{col} on G , and pivot $p \in V$ we have $\text{OPT}_{\text{Fixed}}(G, \mathbf{col}, p) = \text{OPT}_{\text{S-Fixed}}(G, \mathbf{col}, p)$.*

Proof. First, observe that $\text{OPT}_{\text{S-Fixed}}(G, \mathbf{col}, p) \leq \text{OPT}_{\text{Fixed}}(G, \mathbf{col}, p)$ is trivial, as any solution of FIXED-FLOOD-IT is a solution to SUBSET-FIXED-FLOOD-IT by playing the same sequence of colors and always selecting all of the connected monochromatic component of p .

Let us also establish the converse inequality. Consider a solution $(S_1, c_1), \dots, (S_k, c_k)$ of SUBSET-FIXED-FLOOD-IT, where by definition we have $p \in S_i$ for all $i \in [k]$. We would like to prove that $(p, c_1), (p, c_2), \dots, (p, c_k)$ is a valid solution for FIXED-FLOOD-IT. Let \mathbf{col}_i be the result of the first i set-moves of the former solution, and \mathbf{col}'_i be the result of the first i moves of the latter solution. We will establish by induction the following:

1. For all $i \in [k]$ we have $\mathbf{Comp}(\mathbf{col}_i, p) \subseteq \mathbf{Comp}(\mathbf{col}'_i, p)$.
2. For all $i \in [k], u \in V \setminus \mathbf{Comp}(\mathbf{col}'_i, p)$ we have $\mathbf{col}_i(u) = \mathbf{col}'_i(u)$.

The statements are true for $i = 0$. Suppose that the two statements are true after $i - 1$ moves. The first solution now performs the set-move (S_i, c_i) with $S_i \subseteq \mathbf{Comp}(\mathbf{col}_{i-1}, p) \subseteq \mathbf{Comp}(\mathbf{col}'_{i-1}, p)$. We now have that $\mathbf{Comp}(\mathbf{col}_i, p)$ contains S_i plus the neighbors of S_i which have color c_i in \mathbf{col}_{i-1} . Such vertices either also have color c_i in \mathbf{col}'_{i-1} , or are contained in $\mathbf{Comp}(\mathbf{col}'_{i-1}, p)$; in both cases they are included in $\mathbf{Comp}(\mathbf{col}'_i, p)$, which establishes the first condition. To see that the second condition continues to hold observe that every vertex for which $\mathbf{col}_{i-1}(u) \neq \mathbf{col}_i(u)$ or $\mathbf{col}'_{i-1}(u) \neq \mathbf{col}'_i(u)$ belongs in $\mathbf{Comp}(\mathbf{col}'_i, p)$; the colors of other vertices remain unchanged. Since in the end $\mathbf{Comp}(\mathbf{col}_k, p) = V$ the first condition ensures that $\mathbf{Comp}(\mathbf{col}'_k, p) = V$. ◀

We are now ready to state the proof of Theorem 5.1.

Proof of Theorem 5.1. As mentioned, we focus on proving the second inequality as the first inequality follows trivially from the definition of the problems. Given a graph $G = (V, E)$, an initial coloring function $\mathbf{col} = \mathbf{col}_0$, and a pivot $p \in V$, we suppose we have a solution to FREE-FLOOD-IT $(u_1, c_1), (u_2, c_2), \dots, (u_k, c_k)$. In the remainder, we denote by \mathbf{col}_i the coloring that results after the moves $(u_1, c_1), \dots, (u_i, c_i)$. We can immediately construct an equivalent solution to SUBSET-FREE-FLOOD-IT from this, producing the same sequence of colorings: $(\mathbf{Comp}(\mathbf{col}_0, u_1), c_1), (\mathbf{Comp}(\mathbf{col}_1, u_2), c_2), \dots, (\mathbf{Comp}(\mathbf{col}_{k-1}, u_k), c_k)$. We will transform this solution to a solution of SUBSET-FIXED-FLOOD-IT of length at most $2k$, and then invoke Lemma 5.4 to obtain a solution for FIXED-FLOOD-IT of length at most $2k$. More precisely, we will show that for any G, \mathbf{col}, p we have $\text{OPT}_{\text{S-Fixed}}(G, \mathbf{col}, p) \leq 2\text{OPT}_{\text{S-Free}}(G, \mathbf{col}, p)$.

For a solution $\mathcal{S} = (S_1, c_1), (S_2, c_2), \dots, (S_k, c_k)$ to SUBSET-FREE-FLOOD-IT we define the number of bad moves of \mathcal{S} as $b(\mathcal{S}) = |\{(S_i, c_i) \mid p \notin S_i\}|$. We will somewhat more strongly prove the following statement for all G, \mathbf{col}, p : for any valid SUBSET-FREE-FLOOD-IT solution \mathcal{S} , we have

$$\text{OPT}_{\text{S-Fixed}}(G, \mathbf{col}, p) \leq |\mathcal{S}| + b(\mathcal{S}).$$

Since $|\mathcal{S}| + b(\mathcal{S}) \leq 2|\mathcal{S}|$, the above statement will imply the promised inequality and the theorem.

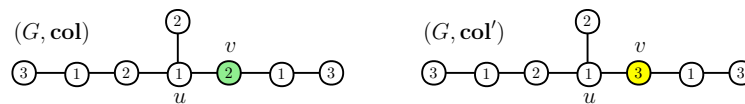
We prove the statement by induction on $|\mathcal{S}| + 2b(\mathcal{S})$. If $|\mathcal{S}| + 2b(\mathcal{S}) \leq 2$ then \mathcal{S} is already a SUBSET-FIXED-FLOOD-IT solution, so the statement is trivial. Suppose then that the statement holds when $|\mathcal{S}| + 2b(\mathcal{S}) \leq n$ and we have a solution \mathcal{S} with $|\mathcal{S}| + 2b(\mathcal{S}) = n + 1$. We consider the following cases:

- The first move (S_1, c_1) has $p \in S_1$. By the inductive hypothesis there is a SUBSET-FIXED-FLOOD-IT solution of length at most $|\mathcal{S}| + b(\mathcal{S}) - 1$ for (G, \mathbf{col}_1, p) . We build a solution for SUBSET-FIXED-FLOOD-IT by appending this solution to the move (S_1, c_1) , since this is a valid move for SUBSET-FIXED-FLOOD-IT.
- There exists a move (S_i, c_i) with $S_i = \mathbf{Comp}(\mathbf{col}_{i-1}, u)$, for some vertex u in $N(\mathbf{Comp}(\mathbf{col}_{i-1}, p)) \setminus \mathbf{Comp}(\mathbf{col}_{i-1}, p)$ such that $c_i = \mathbf{col}_{i-1}(p)$. That is, there exists a move that plays a vertex u that currently has a different color than p , and as a result of this move the component of u and p merge, because u receives the same color as p and u has a neighbor in the component of p .

Consider the first such move. We build a solution \mathcal{S}' as follows: we keep moves $(S_1, c_1) \dots (S_{i-1}, c_{i-1})$; we add the moves $(\mathbf{Comp}(\mathbf{col}_{i-1}, p), \mathbf{col}_{i-1}(u))$, $(\mathbf{Comp}(\mathbf{col}_{i-1}, p) \cup \mathbf{Comp}(\mathbf{col}_{i-1}, u), \mathbf{col}_{i-1}(p))$; we append the rest of the previous solution $(S_{i+1}, c_{i+1}), \dots$. To see that \mathcal{S}' is still a valid solution we observe that $\mathbf{Comp}(\mathbf{col}_{i-1}, p) \cup \mathbf{Comp}(\mathbf{col}_{i-1}, u)$ is monochromatic and connected when we play it, and that the result of the first $i - 1$ moves, plus the two new moves is exactly \mathbf{col}_i . We also note that $\mathcal{S}' + b(\mathcal{S}') = \mathcal{S} + b(\mathcal{S})$ because we replaced one bad move with two good moves. However, $\mathcal{S}' + 2b(\mathcal{S}') < \mathcal{S} + 2b(\mathcal{S})$, hence by the inductive hypothesis there exists a SUBSET-FIXED-FLOOD-IT solution of the desired length.

- There does not exist a move as specified in the previous case. We then show that this reduces to the first case. If no move as described in the previous case exists and the initial coloring is not already constant, \mathcal{S} must have a move (S_i, c_i) where $\{p\} \subseteq S_i \subseteq \mathbf{Comp}(\mathbf{col}_0, p)$ and $c_i = \mathbf{col}_{i-1}(u)$ for $u \in N(\mathbf{Comp}(\mathbf{col}_0, p)) \setminus \mathbf{Comp}(\mathbf{col}_0, p)$. In other words, this is a good move (it changes the color of p), that adds a new vertex u to the connected monochromatic component of p . Such a move must exist, since if the initial coloring is not constant, the initial component of p must be extended, and we assumed that no move that extends it by recoloring one of its neighbors exists.

Consider the first such good move (S_i, c_i) as described above. We build a solution \mathcal{S}' as follows: the first move is $(\mathbf{Comp}(\mathbf{col}_0, p), \mathbf{col}_0(u))$, where u is, as described above, the neighbor of $\mathbf{Comp}(\mathbf{col}_0, p)$ with $\mathbf{col}_{i-1}(u) = c_i$. For $j \in [i - 1]$ we add the move (S_j, c_j) if $u \notin S_j$, or the move $(\mathbf{Comp}(\mathbf{col}_{j-1}, u) \cup \mathbf{Comp}(\mathbf{col}_0, p), c_j)$ if $u \in S_j$. In other words, we keep other moves unchanged if they do not affect u , otherwise we add to them $\mathbf{Comp}(\mathbf{col}_0, p)$. We observe that these moves are valid since we maintain the invariant that $\mathbf{Comp}(\mathbf{col}_0, p)$ and u have the same color and since none of the first $i - 1$ moves of \mathcal{S} changes the color of p (since we selected the first such move). The result of these i moves is exactly \mathbf{col}_i . We now append the remaining move $(S_{i+1}, c_{i+1}), \dots$, and we have a solution that starts with a good move, has the same length and the same (or smaller) number of bad moves as \mathcal{S} and is still valid. We have therefore reduced this to the first case. ◀



■ **Figure 4** Non-monotonicity of FREE-FLOOD-IT.

6 Non-monotonicity of Free-Flood-It

As a final remark, we consider the (non-)monotonicity of the problem. A game has the *monotonicity property* if no legal move makes the situation worse. That is, if FIXED-FLOOD-IT (or FREE-FLOOD-IT) has the monotonicity property, then no single move increases the minimum number of steps to make the input graph monotone. We believe that the monotonicity of FIXED-FLOOD-IT was known as folklore and used implicitly in the literature. On the other hand, we are not sure that the non-monotonicity of FREE-FLOOD-IT was widely known. The only result we are aware of is by Meeks and Scott [12] who showed that on paths FREE-FLOOD-IT has the monotonicity property. Figure 4 shows that FREE-FLOOD-IT loses its monotonicity property as soon as the underlying graph becomes a path with one attached vertex. The instance (G, \mathbf{col}') is obtained from (G, \mathbf{col}) by playing the move $(v, 3)$. We can show that $\text{OPT}_{\text{Free}}(G, \mathbf{col}) < \text{OPT}_{\text{Free}}(G, \mathbf{col}')$.

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