A Muffin-Theorem Generator

Guangqi Cui
Montgomery Blair High School
bestwillcui@gmail.com

John Dickerson
Department of Computer Science and UMIACS, Univ of MD at College Park
john@cs.umd.edu

Naveen Durvasula
Montgomery Blair High School
140.naveen.d@gmail.com

William Gasarch
Department of Computer Science, Univ of MD at College Park
gasarch@cs.umd.edu

Erik Metz
Department of Mathematics, Univ of MD at College Park (ugrad)
emetz1618@gmail.com

Jacob Prinz
Department of Physics, Univ of MD at College Park (ugrad)
jacobeliasprinz@gmail.com

Naveen Raman
Richard Montgomery High School
nav.j.raman@gmail.com

Daniel Smolyak
Department of Computer Science (ugrad). Univ of MD at College Park
dsmolyak@gmail.com

Sung Hyun Yoo
Bergen County Academies (a High School)
sunnyyoo812@gmail.com

Abstract

Consider the following FUN problem. Given m, s you want to divide m muffins among s students so that everyone gets $\frac{m}{s}$ muffins; however, you want to maximize the minimum piece so that nobody gets crumbs. Let $f(m, s)$ be the size of the smallest piece in an optimal procedure.

We study the case where $\lceil \frac{2m}{s} \rceil = 3$ because (1) many of our hardest open problems were of this form until we found this method, (2) we have used the technique to *generate muffin-theorems*, and (3) we conjecture this can be used to solve the general case. We give (1) an algorithm to find an upper bound for $f(m, s)$ when $\lceil \frac{2m}{s} \rceil = 3$ (and some ways to speed up that algorithm if certain conjectures are true), (2) an algorithm that uses the information from (1) to try to find a lower bound on $f(m, s)$ (a procedure) which matches the upper bound, (3) an algorithm that uses the information from (1) to generate muffin-theorems, and (4) an algorithm that we think works well in practice to find $f(m, s)$ for any m, s.

2012 ACM Subject Classification Mathematics of computing → Combinatorial optimization

Keywords and phrases Fair Division, Theorem Generation

9th International Conference on Fun with Algorithms (FUN 2018).
Editors: Hiro Ito, Stefano Leonardi, Linda Pagli, and Giuseppe Prencipe; Article No. 15; pp. 15:1–15:19
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Digital Object Identifier 10.4230/LIPIcs.FUN.2018.15

Related Version https://arxiv.org/abs/1709.02452

Acknowledgements We thank Nancy Blachman who compiled the list of problems that introduced us to this problem, Alan Frank who first came up with the problem. We would also like to thank Alan Frank, James Propp, and Sam Zbarsky for stimulating discussions of the topic.

1 Introduction

Consider the following FUN problem. Given m, s you want to divide m muffins among s students so that everyone gets $\frac{m}{s}$ muffins; however, you want to maximize the minimum piece so that nobody gets crumbs. Let $f(m, s)$ be the size of the smallest piece in an optimal procedure.

We give an example:
You have 47 muffins and 36 students. You want to divide the muffins evenly, but no student wants a small piece. Find a protocol that maximizes the smallest piece. We show in Section 5 that there is a procedure for this with smallest piece $\frac{31}{90}$ and that this is optimal. Hence $f(47, 36) = \frac{31}{90}$.

Convention. When discussing a muffin being cut we refer to *pieces*. When discussing a student receiving we refer to *shares*. They are the same; however, it will be good to have different terminologies to focus on what's important. We treat a piece, a share, and its value as the same thing. So we may say *let $x \geq \frac{1}{3}$ be given to a student.*

▶ **Definition 1.** Let $m, s \in \mathbb{N}$. An (m, s)*-protocol* is a protocol to cut m muffins into pieces and then distribute them to the s students so that each student gets $\frac{m}{s}$ muffins. An (m, s)-protocol is *optimal* if it has the largest smallest piece of any protocol. $f(m, s)$ is the size of the smallest piece in an optimal (m, s)-protocol.

Clearly, for all $a \in \mathbb{N}$, $f(am, as) \geq f(m, s)$. All of our theorems indicate that $f(am, as) = f(m, s)$. We have not been able to prove this; however, we will only consider the cases where m, s are relatively prime.

We came upon this problem in a pamphlet *Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles* compiled by Nancy Blachman. On Page 2 was *The Muffin Puzzle* which asked about the problem for several particular cases. Nancy Blachman attributes the problem to Alan Frank and points out that it was described by Jeremy Copeland [3]. We are the first ones to consider this problem seriously for general m, s with one caveat: There was some discussion of this problem in the math-fun email list in 2009. We have obtained a copy of their arxives and discovered that they already had Theorem 3 and 11. We will credit the individuals when we get to those theorems.

Given m, s how hard is it to compute $f(m, s)$? Computing $f(m, s)$ can be rephrased as a mixed integer program on $O(ms)$ variables (the proof is in the Section A). Since the input is of size $O(\log m + \log s)$ this result does not even put the problem into NP. One of the upshots of this paper will be a procedure that we conjecture puts the computation of $f(m, s)$ into P.

We study the case where $\lceil \frac{2m}{s} \rceil = 3$ because (1) many of our hardest open problems were of this form until we found this method, (2) we have used the technique to *generate muffin-theorems*, (3) we conjecture this can be used to solve the general case.

We have a long paper [2] and some programs [1] for computing $f(m,s)$. For $1 \le s \le 50$, $1 \le m \le 60$ we have computed $f(m,s)$. In this paper we focus on a subset of the material that lends itself to generating theorems about muffins via an algorithm.

2 Summary of Results

In Sections 3,4 we give basic theorems and definitions used throughout the paper. In Section 5 we illustrate the *Buddy-Match techniques* by proving $f(47,36) \le \frac{31}{90}$. In Section 6 we illustrate how to obtain lower bounds and present the result $f(47,36) \ge \frac{31}{90}$.

In Sections 7 we discuss how to generate theorems from the Buddy-Match Technique. These theorems are of the form:

If $d \in \mathbb{N}$ and $1 \le a \le 3d-1$, a,d relatively primes, then

$$(\forall k \ge 1)\left[f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}\right]$$

where X is a constant which can depend on a,d but not on k. In Section 8 we discuss how to generate theorems that are more general. Here is an example:

If $1 \le a \le \frac{5d}{7}$ and $a \ne \frac{2d}{3}$ then $f(3dk+a+d, 3dk+a) \le \frac{X}{360}$ where $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

In Sections 10, 11 we show how, assuming certain conjectures, one can speed up the Buddy-Match Technique. In Section 12 we give an algorithm that we conjecture puts $f(m,s)$ into P. In Section 13 we speculate about that algorithm and other muffin-issues.

In the appendix we state and sometimes prove theorems that are needed to fill in some of the gaps in our narrative. We also give some examples of the theorems we generated.

3 Basic Theorems

In this section we prove two theorems that will enable us, for the rest of the paper, to only consider m,s and protocols such that (1) $m > s \ge 3$, (2) s does not divide m, and (4) every muffin is cut into exactly two pieces.

The following theorem takes care of the cases $s=1$ and $s=2$. The proofs are easy and left to the reader.

▶ **Theorem 2.**
1. $(\forall m)[f(m,1)=1]$
2. $(\forall m)[m \equiv 0 \pmod 2 \rightarrow f(m,2)=1]$
3. $(\forall m)[m \equiv 1 \pmod 2 \rightarrow f(m,2)=\frac{1}{2}]$
4. $(\forall m,s)[s \textit{ divides } m \rightarrow f(m,s)=1]$.

The following theorem shows that if you know $f(m,s)$ then you know $f(s,m)$. Combined with Theorem 2 we need only consider $m > s \ge 3$. This theorem was independently discovered by Erich Friedman, within the math-fun email list, in 2009.

▶ **Theorem 3.** *Let $m,s \in \mathbb{N}$. Then $f(s,m) = \frac{s}{m}f(m,s)$.*

Proof. Assume $f(m,s) \ge \alpha$. We show $f(s,m) \ge \frac{s}{m}\alpha$. Let $M_1,\ldots,M_m$ be the muffins. Let $S_1,\ldots,S_s$ be the students. The protocol that achieves $f(m,s) \ge \alpha$ must be of the following form:
1. For each $1 \le i \le m$ divide M_i into pieces $(a_{i1}, a_{i2}, \ldots, a_{im_i})$ where $\sum_{j=1}^{m_i} a_{ij} = 1$.
2. For each $1 \le j \le s$ give S_j the shares $[b_{1j}, b_{2j}, \ldots, b_{s_j j}]$ where $\sum_{i=1}^{s_j} b_{ij} = \frac{m}{s}$.

The following hold:

- $\bigcup_{i=1}^{m}\bigcup_{j=1}^{m_i}\{a_{ij}\} = \bigcup_{j=1}^{s}\bigcup_{i=1}^{s_j}\{b_{ij}\}$
- The min over all of the a_{ij} is α.

The following protocol shows that $f(s,m) \ge \frac{s}{m}\alpha$. Let $M_1',\ldots,M_s'$ be the muffins. Let $S_1',\ldots,S_m'$ be the students.

1. For each $1 \le j \le s$ divide M_j' into $(\frac{s}{m}b_{1j}, \frac{s}{m}b_{2j},\ldots,\frac{s}{m}b_{s_jj})$. Note that $\sum_{i=1}^{s_j}\frac{s}{m}b_{ij} = \frac{s}{m}\sum_{i=1}^{s_j}b_{ij} = \frac{s}{m}\times\frac{m}{s} = 1$.
2. For each $1 \le i \le m$ give S_j' $[\frac{s}{m}a_{i1}, \frac{s}{m}a_{ij},\ldots,\frac{s}{m}a_{im_i}]$. Note that $\sum_{j=1}^{m_i}\frac{s}{m}a_{ij} = \frac{s}{m}\sum_{j=1}^{m_i}a_{ij} = \frac{s}{m}\times 1 = \frac{s}{m}$.

Clearly this is a correct protocol and the minimum piece is of size $\frac{s}{m}\alpha$.

We now show that $f(s,m) = \frac{s}{m}f(m,s)$. By the above we have both (1) $f(s,m) \ge \frac{s}{m}f(m,s)$, and (2) $f(m,s) \ge \frac{m}{s}f(s,m)$. Hence

$$f(s,m) \ge \frac{s}{m}f(m,s) \ge \frac{s}{m}\frac{m}{s}f(s,m) = f(s,m).$$

Therefore $f(s,m) = \frac{s}{m}f(m,s)$. ◀

▶ **Theorem 4.** *Let $m,s \in \mathbb{N}$.*

1. *If $f(m,s) \ge \alpha$ and $\alpha > \frac{1}{3}$ via protocol P then protocol P cuts every muffin into 1 or 2 pieces.*
2. *$f(m,s) \ge \alpha$ and $\alpha \le \frac{1}{2}$ via protocol P then there is a protocol P' such that (1) P' also yields $f(m,s) \ge \alpha$, and (2) P' cuts every muffin into 2 or more pieces.*

Proof.

a) If any muffin is cut into ≥ 3 pieces then there is a piece $\le \frac{1}{3} < \alpha$.

b) If any muffin is uncut and given to (say) Alice then we can add a step where we cut the muffin into $(\frac{1}{2},\frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to Alice. Since $\alpha \le \frac{1}{2}$ adding in some pieces of size $\frac{1}{2}$ does not affect the smallest piece. ◀

By Theorem 4 we have the following convention.

Convention: When trying to show that $f(m,s) \le \alpha$ where $\frac{1}{3} < \alpha < \frac{1}{2}$ we will assume, by way of contradiction, that there is a protocol showing $f(m,s) > \alpha$ where every muffin is cut into exactly 2 pieces.

4 Basic Definitions

▶ **Definition 5.** Let $m,s \in \mathbb{N}$. Assume there is an (m,s)-protocol.

1. The two pieces that come from the same muffin are called *buddies*. $B(x)$ is the buddy of x. Note that $B(x) = 1 - x$.
2. A student that gets A shares is an *A-student*. A share given to an A-student is an *A-share*.
3. 2-Shares that are given to the same 2-student are *matched*. $M(x)$ is the match of 2-share x. Note that $M(x) = \frac{m}{s} - x$.
4. If x is a share given to a 3-student then $M_S(x)$ is the smallest share (not including x) that the student has, and $M_L(x)$ is the largest. Note that $M_S(x) \le \frac{(m/s)-x}{2}$. Hence $B(M_S(x)) \ge 1 - \frac{(m/s)-x}{2}$.

Notation: (a,b) will mean the set of shares that have size strictly between a and b. Hence $|(a,b)|$ will be the number of such shares. We use similar notation for $[a,b]$.

5 An Example is Worth A Thousand Theorems: 43 muffins, 39 Students

The method we demonstrate in this section is called *The Buddy-Match Method*.

▶ **Theorem 6.** $f(47,36) \le \frac{31}{90} = \frac{124}{360}$.

Proof. To make the notation easier we write all fractions as having denominator 360.

Assume there is an $(47,36)$-procedure. We show that there is a piece $\le \frac{124}{360}$. Note that $\frac{47}{36} = \frac{470}{360}$.

Case 1: Some student gets ≥ 4 shares. Then some students has a share $\le \frac{47}{36\times 4} < \frac{124}{360}$.

Case 2: Some student gets ≤ 1 share. $1 < \frac{47}{36}$, so this is impossible.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 94. Let s_2 (s_3) be the number of 2-students (3-students).

$$\begin{aligned} 2s_2 + 3s_3 &= 94 \\ s_2 + s_3 &= 36 \end{aligned}$$

So $s_2 = 14$ and $s_3 = 22$.

Case 3.1: There is a 2-share $x \le \frac{234}{360}$. $M(x) \ge \frac{470}{360} - \frac{234}{360} = \frac{236}{360}$ so $B(M(x)) \le 1 - \frac{236}{360} = \frac{124}{360}$

Case 3.2: There is a 3-share $x \ge \frac{222}{360}$. $B(M_S(x)) \le 1 - \frac{\frac{470}{360} - \frac{222}{360}}{2} = \frac{124}{360}$.

Case 3.3: There is a 2-share $x \ge \frac{236}{360}$. $B(x) \le 1 - \frac{236}{360} = \frac{124}{360}$

Case 3.4: There is a 3-share $x \le \frac{124}{360}$. This one is self-explanatory.

Case 3.5: All 3-shares are in $(\frac{124}{360}, \frac{222}{360})$ and all 2-shares are in $(\frac{234}{360}, \frac{236}{360})$.

The following picture captures what we know so far.

(	$---$	)[	$---$	](	$---$	)
$\frac{124}{360}$	3-shs	$\frac{222}{360}$	No shs	$\frac{234}{360}$	2-shs	$\frac{236}{360}$

Since there are no shares in $[\frac{222}{360}, \frac{234}{360}]$, there are no shares in $B([\frac{222}{360}, \frac{234}{360}]) = [\frac{126}{360}, \frac{138}{360}]$ The following picture captures what we know so far.

(	$---$	)[	$---$	](	$---$	)[	$---$	](	$---$	)
$\frac{124}{360}$	S3-shs	$\frac{126}{360}$	No shs	$\frac{138}{360}$	L3-shs	$\frac{222}{360}$	No shs	$\frac{234}{360}$	2-shs	$\frac{236}{360}$

S3-shs stands for *short 3-shares* and L3-shs stands for *large 3-shares*. There are $2s_2 = 28$ 2-shares so there are 28 S3-shares (B is a bijection between 2-shares and S3-shares). Since there are $3s_3 = 66$ 3-shares total that leaves 38 S3 shares.

Since the midpoint of L3-shs is $\frac{360}{2}$, the Buddy function is a bijection from $(\frac{138}{360}, \frac{180}{360})$ to $(\frac{180}{360}, \frac{222}{360})$, Hence these two intervals have the same number of shares.

Since the midpoint of 2-shs is $\frac{470}{2}$, the Match function is a bijection from $(\frac{234}{360}, \frac{235}{360})$ to $(\frac{235}{360}, \frac{236}{360})$. Hence these two intervals have the same number of shares. Applying the Buddy function to both these intervals we obtain that $(\frac{124}{360}, \frac{125}{360})$ and $(\frac{125}{360}, \frac{126}{360})$ have the same number of shares.

In the scenarios above there are an even number of shares of size the midpoint. We arbitrarily assign half to the left and half to the right.

We define the following intervals.

▶ **Definition 7.**
1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ ($|I_1| = |I_2|$, $|I_1 \cup I_2| = 28$)
3. $I_3 = (\frac{138}{360}, \frac{180}{360})$
4. $I_4 = (\frac{180}{360}, \frac{222}{360})$ ($|I_3| = |I_4|$, $|I_3 \cup I_4| = 38$)

Henceforth all of the students considered will be 3-students. We now look at the students in a more detailed way than 2-students and 3-students.

▶ **Definition 8.** Let $1 \le i_1 \le \cdots \le i_3 \le 4$. An $e(i_1, i_2, i_3)$-student is a student who has, for each $1 \le j \le 3$, a share in I_{i_j}. For example, an $e(1,1,4)$-students has two shares in I_1 and one share in I_4.

▶ **Claim 1.**
1. *The only possible students are:*
   a. $e(1,1,4)$
   b. $e(1,2,4)$
   c. $e(1,3,3)$
   d. $e(1,3,4)$
   e. $e(2,2,4)$
   f. $e(2,3,3)$
   g. $e(2,3,4)$
   h. $e(3,3,3)$
   i. $e(3,3,4)$
2. *There are no shares in* $[\frac{208}{360}, \frac{218}{360}]$
3. *There are no shares in* $[\frac{142}{360}, \frac{152}{360}]$ *(this follows from the prior part and buddying).*

Proof of Claim 1.
1) We establish that some students are impossible.
   A $e(1,4,4)$-student has more than $\frac{124}{360} + 2 \times \frac{180}{360} = \frac{484}{360}$
   A $e(2,2,3)$-student has less than $2 \times \frac{126}{360} + \frac{180}{360} = \frac{432}{360}$
   The result follows from these two statements, though the proof is tedious.
2) We look at which I_4-shares are used
   A $e(1,1,4)$ student uses I_4-share $> \frac{470}{360} - 2 \times \frac{125}{360} = \frac{220}{360}$
   A $e(1,2,4)$ student uses I_4-shares $> \frac{470}{360} - \frac{125}{360} - \frac{126}{360} = \frac{219}{360}$
   A $e(1,3,4)$ student uses I_4-shares $< \frac{470}{360} - \frac{124}{360} - \frac{138}{360} = \frac{208}{360}$
   A $e(2,2,4)$ student uses I_4-shares $> \frac{470}{360} - 2 \times \frac{126}{360} = \frac{218}{360}$
   A $e(2,3,4)$ student uses I_4-shares $< \frac{470}{360} - \frac{125}{360} - \frac{138}{360} = \frac{207}{360}$
   A $e(3,3,4)$ student uses I_4-shares $< \frac{470}{360} - 2 \times \frac{138}{360} = \frac{194}{360}$
   Hence the only shares in I_4 that can be used are those $< \frac{208}{360}$ or $> \frac{218}{360}$. The result follows. ◀

We redefine the intervals.

▶ **Definition 9.**
1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ ($|I_1| = |I_2|$), $|I_1 \cup I_2| = 28$)
3. $I_3 = (\frac{138}{360}, \frac{142}{360})$
4. $I_4 = (\frac{152}{360}, \frac{180}{360})$
5. $I_5 = (\frac{180}{360}, \frac{208}{360})$ ($|I_4| = |I_5|$)
6. $I_6 = (\frac{218}{360}, \frac{222}{360})$ ($|I_3| = |I_6|$, $|I_3 \cup I_4 \cup I_5 \cup I_6| = 38$)

By a proof similar to that of Claim 1 we obtain the following:

▶ **Claim 2.**
1. *The only possible students are:* $e(1,1,6)$, $e(1,2,6)$, $e(1,3,5)$, $e(1,4,4)$, $e(1,4,5)$, $e(2,2,6)$, $e(2,3,5)$, $e(2,4,4)$, $e(2,4,5)$, $e(3,3,5)$, $e(3,4,4)$, *and* $e(4,4,4)$.
2. *There are no shares in* $[\frac{194}{360}, \frac{202}{360}]$
3. *There are no shares in* $[\frac{158}{360}, \frac{166}{360}]$ *(this follows from the prior part and buddying).*

We define the following intervals.

▶ **Definition 10.**
1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ $(|I_1| = |I_2|, |I_1 \cup I_2| = 28)$
3. $I_3 = (\frac{138}{360}, \frac{142}{360})$
4. $I_4 = (\frac{152}{360}, \frac{158}{360})$
5. $I_5 = (\frac{166}{360}, \frac{180}{360})$
6. $I_6 = (\frac{180}{360}, \frac{194}{360})$ $(|I_5| = |I_6|)$
7. $I_7 = (\frac{202}{360}, \frac{208}{360})$ $(|I_4| = |I_7|)$
8. $I_8 = (\frac{218}{360}, \frac{222}{360})$ $(|I_3| = |I_8|, |I_3 \cup \cdots \cup I_8| = 38)$

By a proof similar to that of Claim 1 we obtain:

▶ **Claim 3.** *The only possible students are:* $e(1,1,8)$, $e(1,2,8)$, $e(1,3,7)$, $e(1,4,6)$, $e(1,5,5)$, $e(2,2,8)$, $e(2,3,7)$, $e(2,4,6)$, $e(2,5,5)$, $e(3,3,6)$, *and* $e(4,4,4)$.

Let
1. $|e(1,1,8)| = a$
2. $|e(1,2,8)| = b$
3. $|e(1,3,7)| = c$
4. $|e(1,4,6)| = d$
5. $|e(1,5,5)| = e$
6. $|e(2,2,8)| = f$
7. $|e(2,3,7)| = g$
8. $|e(2,4,6)| = h$
9. $|e(2,5,5)| = i$
10. $|e(3,3,6)| = j$
11. $|e(4,4,4)| = k$

Since $|I_1| = |I_2|$, $2a + b + c + d + e = b + 2f + g + h + i$, *so* $2a + c + d + e = 2f + g + h + i$
Since $|I_3| = |I_8|$, $c + g + 2j = a + b + f$
Since $|I_4| = |I_7|$, $d + h + 3k = c + g$
Since $|I_5| = |I_6|$, $2e + 2i = d + h + j$
Since $|I_1 \cup I_2| = 28$, $2a + 2b + c + d + e + 2f + g + h + i = 28$
Since there are 22 3-students, $a + b + c + d + e + f + g + h + i + k = 22$
From the last two equations we obtain $a + b + f = 6$
We combine I_1 *and* I_2 *into a single interval. This reduces the system to 6 variables, resulting in the equation*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 22 \\ 28 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

However, one can check that eliminating the bottom 3 rows requires the top 2 rows to be in the ratio 7 : 9. 22 : 28 $\neq$ 7 : 9, so there is no solution. ◀

The above proof used that $\lceil \frac{2m}{s} \rceil = 3$ since that is the condition that leads to having 2-shares and 3-shares. This is usually important since it gives us symmetry from matches, not just from buddying; however, in this case we just so happened to not need that symmetry.

6 Finding a Procedure

We now describe the program that finds the procedure showing $f(47, 36) \geq \frac{124}{360}$. We *guess* that all shares are of the form $\frac{x}{360}$ where $124 \leq x \leq 236$. But we can cut down those variables *a lot* based on the proof. For example, by modifying the proof slightly, we can deduce that there are no share of size $\frac{127}{360}, \frac{128}{360}, \ldots, \frac{137}{360}$. This is a key factor in speeding up the program. We can also use the symmetries of where shares can be.

For every way to split a muffin we have a variable for how many muffins are split that way, as follows: $(\frac{124}{360}, \frac{236}{360})$ is associated to the variables $y_{124,236}$, $(\frac{125}{360}, \frac{235}{360})$ is associated with the variable $y_{125,235}$, etc. This variable is the *number of muffins* that are split that way.

For every way to give muffin shares to a student we have a variable for how many students get that set of shares, as follows: $[\frac{87}{360}, \frac{79}{360}, \frac{69}{360}]$ is associated to the variable $z_{87,79,69}$, $[\frac{118}{360}, \frac{117}{360}]$ is associated to the variables $z_{118,117}$, etc. This variable is the *number of students* who get that share-size.

For each size we express how many pieces are of that size in two ways.

- The number of pieces of that size based on the muffins. For example, the number of pieces of size $\frac{131}{360}$ is $y_{131,256}$. The number of pieces of size $\frac{180}{360}$ is $2 \times y_{180,180}$.
- The number of shares of that size based on the students. For example, the number of shares of size $\frac{131}{360}$ is

$$z_{124,131,215} + \cdots + z_{130,131,209} + 2z_{131,131,208} + z_{132,131,207} + \cdots + z_{215,131,124}$$

For each size we get an equation by equating the muffin-based and student-based expressions. We have more equations based on the number of pieces and the number in each interval which falls out of the proof of the upper bound. This leads to a set of linear equations whose solution leads to a procedure.

Here is the procedure for $f(47, 36) \geq \frac{124}{360} = \frac{117}{180}$ we obtained with this method:

1. Divide 1 muffin $(\frac{90}{180}, \frac{90}{180})$
2. Divide 2 muffins $(\frac{93}{180}, \frac{87}{180})$
3. Divide 2 muffins $(\frac{101}{180}, \frac{79}{180})$
4. Divide 2 muffins $(\frac{104}{180}, \frac{76}{180})$
5. Divide 6 muffins $(\frac{109}{180}, \frac{71}{180})$
6. Divide 6 muffins $(\frac{111}{180}, \frac{69}{180})$
7. Divide 14 muffins $(\frac{117}{180}, \frac{63}{180})$
8. Divide 14 muffins $(\frac{118}{180}, \frac{62}{180})$
9. Give 2 students $[\frac{87}{180} \frac{79}{180} \frac{69}{180}]$
10. Give 2 students $[\frac{90}{180} \frac{76}{180} \frac{69}{180}]$
11. Give 2 students $[\frac{93}{180} \frac{71}{180} \frac{71}{180}]$
12. Give 2 students $[\frac{101}{180} \frac{71}{180} \frac{63}{180}]$
13. Give 2 students $[\frac{104}{180} \frac{69}{180} \frac{62}{180}]$
14. Give 6 students $[\frac{109}{180} \frac{63}{180} \frac{63}{180}]$
15. Give 6 students $[\frac{111}{180} \frac{62}{180} \frac{62}{180}]$
16. Give 14 students $[\frac{118}{180} \frac{117}{180}]$

The reader should be able to see how to generalize the method outlined above.

What is described above is not quite what we have coded up (though we will). The Interval Method (see Section B) is another method to find lower bounds that gives information that can be used to cut down the time to find a procedure. We have coded up a version of what is outlined above with the interval method.

We denote the algorithm given above (the one using Buddy-Match) VLOWER(m, s, α) where one finds a procedure showing $f(m, s) \geq \alpha$, hence verifying that $f(m, s) \geq \alpha$.

7 The Proof that $f(47, 36) \leq \frac{31}{90}$ Reveals Much More

The proof that $f(47, 36) \leq \frac{31}{90}$ can be modified very slightly (just notation) to obtain the following result (which we write in a strange way for later exposition):

$$(\forall k \geq 1)\left[f(3 \times 11 \times k + 11 + 3, 33k + 3) \leq \frac{11k + \frac{7}{5}}{3 \times 11 \times 3k + 3}\right]$$

More generally the following seems to be true empirically:
for all d (d stands for difference and is $m - s$), for all $1 \leq a \leq 3d - 1$ (a, d relatively primes), there exists X:

$$(\forall k \geq 1)\left[f(3dk + d + a, 3dk + a) \leq \frac{dk + X}{3dk + a}\right]$$

For $d = 1$ to 8, for all relevant a, we have found X. In many concrete cases we have shown that it is also an upper bound. In Section C we present the results for the $d = 7$ case.

Note that we need $k \geq 1$ since if $k = 0$ then we no longer have $\lceil \frac{2m}{s} \rceil = 3$.

8 Generating More General Theorems

The techniques discussed in Section 7 generate theorems of the form

$$(\forall k \geq 1)\left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a}\right].$$

However, the program can be modified to obtain more general theorems. As noted in Section 7 our program finds interesting values of X. That is, the program may find that (say) if $X \leq \frac{7}{6}$ then there are no $e(1, 3, 4)$-students. What is it about $X \leq \frac{7}{6}$ that makes this happen? It may be that (say) $1 \leq a \leq \frac{5d}{7}$ and $a \neq \frac{2d}{3}$ makes this work, and it may be that $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

We have taken the results from the program and, with the help of additional programs and our own ingenuity generated many theorems (we hope to fully automate it soon). These theorems are a great time saver since often the result we want falls out of them directly. We present a sample of such theorems in the Section D.

9 How to find X

The proof of Theorem 6 can be summarizes as follows: The assumption $f(47, 36) > \frac{31}{90}$ implies that a certain system of linear equations have a solution where all of the variables are natural numbers between 0 and $s_3 = 22$. The system had no such solution, hence a contradiction.

Imagine that we want an upper bound on $f(47,36)$ but do not know what it is ahead of time. Following the line of reasoning in Section 7 we seek X such that

$$f(33+3+11, 33+3) \le \frac{11+X}{33+3}.$$

We use a program to simulate the proof of Theorem 6 but with X instead of the actual numbers. This program will produce many values of X where something interesting happens, such as a type of student no longer being allowed. The program looks at the (finite) set of interesting values of X and finds the least one that causes the resulting system of linear equations to be unsolvable using natural numbers between 0 and 22. Hence we have a value of X. We then use $\mathrm{VLOWER}(47, 36, \frac{11+X}{36})$ to find the matching lower bound (if this does not work then the algorithm failed to find $f(m,s)$).

For the values $47, 36$ it was easy to find the value of X. For larger m, s it may be that verifying $f(m,s) \le \alpha$ is faster than finding the α. In the next two sections we examine how to speed up finding X.

We leave it to the reader to generalize the algorithm to any m, s where $\lceil \frac{2m}{s} \rceil = 3$; however, we give the following picture which represents intervals where 3-shares can be. In the picture each nonempty interval has the number of 3-shares in it (though y is not known) and a label such as I_1 so we can refer to it. This picture is the result of many buddy-match sequences.

$$\underset{\frac{dk+X}{3dk+a}}{(} \quad a+d \quad (I_1) \quad \underset{\frac{dk+\frac{a}{2}}{3dk+a}}{|} \quad a+d \quad (I_2) \quad \underset{\frac{dk+a-X}{3dk+a}}{)[} \quad 0 \quad \underset{\frac{dk+2X}{3dk+a}}{]}$$

$$\underset{\frac{dk+2X}{3dk+a}}{(} \quad y \quad (I_3) \quad \underset{\frac{dk+a+d-3X}{3dk+a}}{)[} \quad 0 \quad \underset{\frac{dk+d-a+2X}{3dk+a}}{]}$$

$$\underset{\frac{dk+d-a+2X}{3dk+a}}{(} \quad 2d-a-y \ (I_4) \quad \underset{\frac{dk+\frac{a+d}{2}}{3dk+a}}{|} \quad 2d-a-y \ (I_5) \quad \underset{\frac{dk+2a-2X}{3dk+a}}{)}$$

$$\underset{\frac{dk+2a-2X}{3dk+a}}{)[} \quad 0 \quad \underset{\frac{dk+3X}{3dk+a}}{](} \quad y \ (I_6) \quad \underset{\frac{dk+a+d-2X}{3dk+a}}{)}$$

Facts and Caveats:

1. $|I_1| = |I_2|$
2. $|I_4| = |I_5|$
3. In the picture it is unclear if the endpoint of I_1 is included in I_1. We do not include it; however, we take the even number of shares that are at that endpoint and arbitrary assign half to I_1 and half to I_2.
4. There is a similar comment for I_2, I_4, and I_5.

We denote the version where you do not already have upper bound to check $\mathrm{BUDMAT}(m,s)$ and the version where you do $\mathrm{BUDMAT}(m,s,\alpha)$ where α is the bound. We will avoid using $\mathrm{BUDMAT}(m,s)$ unless m, s are small since it may be slow.

10 How to find X Cheating a Little

Say you want to find $f(213, 200)$. Since $\lceil \frac{2\times 213}{200} \rceil = 3$ you could run $\mathrm{BUDMAT}(213, 200)$. But the numbers are large! Following the line of reasoning in Section 7 we note that $d = 213 - 200 = 13$ and generalize the problem to finding an X such that

$$f(39k+5+13, 39k+5) \le \frac{13k+X}{39k+5}.$$

Lets look at the $k = 1$ case: $f(57, 44)$. Since $\lceil \frac{2\times 57}{44} \rceil = 3$ you could run BUDMAT(57, 44). But the numbers are small! Oh, thats a good thing! Lets say the answer is α. Run VLOWER$(57, 44, \alpha)$ to verify that its a lower bound. If it is then solve $\alpha = \frac{13+X}{39+5}$ to find X. The proof you did for $f(57, 44) \le \frac{13+X}{39+5}$ can be modified to show $(\forall k \ge 1)[f(39k + 5 + 13, 39k + 5) \le \frac{13k+X}{39k+5}]$. In particular $f(213, 200) \le \frac{13\times 5+X}{39\times 5+5} = \beta$. Run VLOWER$(213, 300, \beta)$ to verify the lower bound (if this does not work then the algorithm failed to find $f(57, 44)$).

This is cheating a little since we don't really know that the such an X exists. But it has so far. And we do verify in the end.

We leave it to the reader to generalize this procedure. We denote this algorithm CHEATALITTLE(m, s).

11 How to find X Cheating a Lot

Say you want to find $f(1717, 1650)$. Since $\lceil \frac{2\times 1717}{1650} \rceil = 3$ you could run BUDMAT(1717, 1650). But the numbers are really large! Following the line of reasoning in Section 7 we note that $d = 1717 - 1650 = 67$ and generalize the problem to finding an X such that

$$f(201k + 42 + 67, 201k + 42) \le \frac{67k + X}{201k + 42}.$$

Lets look at the $k = 1$ case: $f(310, 243)$. These numbers are still big!

Lets look at the $k = 0$ case: $f(109, 42)$. These numbers are small! Since $\lceil \frac{2\times 109}{42} \rceil \ge 4$ you cannot run BUDMAT(109, 42)). But the situation is worse than that. Even if we bound $f(109, 42)$ the proof will not use BUDMAT and hence cannot be modified to get an upper bound for $f(201k + 42 + 67, 201k + 42)$. In fact, the answer for $f(109, 42)$ should have no bearing on our problem.

Except for one thing. Empirically it does. In all cases that we looked at the X obtained from knowing an upper bound on the $k = 0$ case of $f(3dk + a + d, 3dk + a)$ was the correct X for $k \ge 1$. We proceed as if this is always true.

We cannot use BUDMAT(109, 42); however, there are other techniques that to find an upper bound on $f(m, s)$. They summarized in Section B. Use them. Lets say the answer is α. Run VLOWER$(109, 42, \alpha)$ to verify that its a lower bound. If it is then solve $\alpha = \frac{X}{42}$ to find X. The proof you did for $f(109, 42) \le \frac{X}{42}$ *cannot* be modified to show $(\forall k \ge 1)[f(201k + 42 + 67, 201k + 42) \le \frac{67k+X}{201+42}]$. But you have a very good conjecture. Run BUDMAT$(109, 42, \frac{67+X}{201+42})$. If it returns YES and a proof then modify the proof to obtain $(\forall k \ge 1)[f(201k + 42 + 67, 201k + 42) \le \frac{67k+X}{201+42}]$ (if this does not work then the algorithm failed to find $f(1717, 1650)$). In particular $f(1717, 1658) \le \frac{67\times 5+X}{201\times 5+5} = \beta$. Run VLOWER$(1717, 1658, \beta)$ to verify the lower bound (if this does not work then the algorithm failed to find $f(1717, 1650)$).

This is cheating a lot since we don't really know that the $k = 0$ case has any bearing on the $k \ge 1$ case. But it has so far, and we verify in the end.

We leave it to the reader to generalize this procedure. We denote this algorithm CHEATALOT(m, s).

12 A General Algorithm

We present an algorithm that we conjecture always finds $f(m, s)$ and operates in polynomial time.

The reader should read Section B since we will be using FC, INT, and BUD which are explained there. They are other methods to find or verify upper bounds on $f(m, s)$.

1. Input(m, s).
2. If $m = s$ output 1. If $gcd(m, s) = d \geq 1$ then call the algorithm recursively with $f(m/d, s/d)$. If $s = 2$ then output $\frac{1}{2}$. If $m < s$ then call the algorithm recursively to find $f(s, m)$ and output $\frac{m}{s} f(s, m)$.
3. Compute $\alpha = \mathrm{FC}(m, s)$. Compute VLOWER$(m, s, \alpha)$ to see if α is a matching lower bound. If it is then output α and stop.
4. Compute $\alpha = \mathrm{INT}(m, s)$. Compute VLOWER$(m, s, \alpha)$ to see if α is a matching lower bound. If it is then output α and stop.
5. If $\lceil \frac{2m}{s} \rceil = 3$ then:
   a. Compute $\alpha = \mathrm{CHEATALOT}(m, s)$. Compute VLOWER$(m, s, \alpha)$ to see if α is a matching lower bound. If it is then output α and stop. (This might fail if the methods of Section B do not work on the input they are given.)
   b. Compute $\alpha = \mathrm{CHEATALITTLE}(m, s)$. Compute VLOWER$(m, s, \alpha)$ to see if α is a matching lower bound. If it is then output α and stop.
6. If $\lceil \frac{2m}{s} \rceil \geq 4$ then let $a = s$ and $d = m - a$. We seek $f(3d \times 0 + a + d, 3d \times 0 + a)$. Recursively call $f(3d + a + d, 3d + a)$ (we could tell it to not bother with CHEATALOT(m, s) since that just asks to compute $f(a + d, a)$ using FC and INT). If the computation succeeds and returns α then run BUD(m, s, α) to verify that $f(m, s) \leq \alpha$. If this is verified then compute VLOWER(m, s, α) to see if α is a matching lower bound. If it is then output α and stop.
7. If nothing above works then output **FAILED!**

This can be sped up by, upon first seeing m, s, see if any of the general theorems such as those in Sections C and D apply to get an upper bound α and then run VLOWER(m, s, α).

13 Open Problems and Speculation

We would like to think that the algorithm in the last section will always work and hence computing $f(m, s)$ is in P. But we've been down this road before where we think we can compute all $f(m, s)$ only to come to a troublesome case which leads to a new technique and more co-authors. The following are possible outcomes: (1) we prove that the algorithm always works, (2) we keep running the algorithm and it always works but when the numbers get too big we can't tell, (3) we come across a value the algorithm does not work on and this leads to a a new technique and more co-authors.

We believe that computing $f(m, s)$ is in P. One piece of evidence for this is that for all s, for all $m \geq s^3$, $f(m, s) = \mathrm{FC}(m, s)$. Hence if you fix s then for large enough s the problem is *very easy*. One might call this Fixed Parameter *very tractable*.

We believe that $f(m, s)$ only depends on $\frac{m}{s}$. This seems provable.

References

1 Guangiqi Cui, John Dickerson, Naveen Durvasula, William Gasarch, Erik Metz, Jacob Prinz, Naveen Raman, Daniel Smolyak, and Sung Hyun Yoo. Code for muffin problems, 2017. https://github.com/jeprinz/MuffinProblem.

2 Guangiqi Cui, John Dickerson, Naveen Durvasula, William Gasarch, Erik Metz, Jacob Prinz, Naveen Raman, Daniel Smolyak, and Sung Hyun Yoo. The muffin problem, 2017. https://arxiv.org/abs/1709.02452.

3 Alan Frank. The muffin problem, 2013. Described to Jeremy Copeland and in the New York Times Numberplay Online Blog wordplay.blogs.nytimes.com/2013/08/19/cake.

A A Mixed Integer Program for $f(m, s)$

The following theorem shows that $f(m, s)$ always exists (as opposed to having better and better algorithms), is rational, and is computable. This theorem was independently discovered by Veit Elser, within the math-fun email list, in 2009.

▶ **Theorem 11.** *Let* $m, s \geq 1$.

1. *There is a mixed integer program with* $O(ms)$ *binary variables,* $O(ms)$ *real variables,* $O(ms)$ *constraints, and all coefficients integers of absolute value* $\leq \max\{m, s\}$ *such that, from the solution, one can extract* $f(m, s)$ *and a protocol that achieves this bound. This MIP can easily be obtained given* m, s.
2. $f(m, s)$ *is always rational. This follows from part 1.*
3. *In every optimal protocol for* m *muffins and* s *students all of the pieces are of rational size. This follows from part 1.*
4. *The problem of, given* m, s, *determine* $f(m, s)$, *is decidable. This follows from part 1.*

Proof. Consider the following (failed) attempt to solve the problem using linear programming.

1. The variables are x_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq s$. The intent is that x_{ij} is the fraction of muffin i that student j gets.
2. For all $1 \leq i \leq m$, $1 \leq j \leq s$, $0 \leq x_{ij} \leq 1$.
3. For each $1 \leq i \leq m$, $\sum_{j=1}^{s} x_{ij} = 1$.
   This says that the amount of muffin i that student 1 gets, students 2 gets, …, student s gets all adds up to 1.
4. For each $1 \leq j \leq s$, $\sum_{i=1}^{m} x_{ij} = \frac{m}{s}$.
   This says that the amount that student j gets from muffin 1, muffin 2, …, muffin m all adds up to $\frac{m}{s}$.
5. For all $1 \leq i \leq m$, $1 \leq j \leq s$, $x_{ij} \geq z$.
6. Maximize z.

This does not work. The problem is that (say) x_{13} could be 0. In fact it is likely that some x_{ij} is 0. This makes $z = 0$. What we really want is

$$x_{ij} \neq 0 \implies x_{ij} \geq z$$

It is easy to show that $f(m, s) \geq \frac{1}{s}$. Hence every nonzero x_{ij} is $\geq \frac{1}{s}$. We will use this in our proof.

For $1 \leq i \leq m$, $1 \leq j \leq s$ modify the linear program above as follows.

1. Add variable y_{ij} which is in $\{0, 1\}$.
2. Add the constraint $x_{ij} + y_{ij} \leq 1$. Note that
   - $x_{ij} = 0 \implies x_{ij} + y_{ij} \leq 1$, so the constraint imposes no condition on y_{ij}.
   - $x_{ij} > 0 \implies y_{ij} < 1 \implies y_{ij} = 0 \implies x_{ij} + y_{ij} = x_{ij}$.
3. Add the constraint $x_{ij} + y_{ij} \geq \frac{1}{s}$. Note that
   - $x_{ij} = 0 \implies y_{ij} \geq \frac{1}{s} \implies y_{ij} = 1 \implies x_{ij} + y_{ij} = 1$
   - $x_{ij} > 0 \implies x_{ij} \geq \frac{1}{s}$ (since we know all non-zero pieces are $\geq \frac{1}{s}$) $\implies x_{ij} + y_{ij} \geq \frac{1}{s}$, so the constraint imposes no condition on y_{ij}.
4. Replace the constraint $z \leq x_{ij}$ with $z \leq x_{ij} + y_{ij}$.

If $x_{ij} = 0$ then the constraint

$$z \leq x_{ij} + y_{ij} = 1$$

is always met and hence is (as it should be) irrelevant. If $x_{ij} > 0$ then the constraint

$$z \le x_{ij} + y_{ij} = x_{ij}$$

is the constraint we want.

Solve the resulting mixed integer program. Since all of the coefficients are rational the answer will be rational. ◀

B Other Methods

We discuss three methods for finding an upper bound on $f(m, s)$.

The method from the following theorem is called *The Floor Ceiling Method* or just FC-method. Note that it is very fast and gives you the upper bound.

▶ **Theorem 12.** *Assume that $m, s \in \mathbb{N}$ and $\frac{m}{s} \notin \mathbb{N}$.*

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Proof. Assume we have an optimal (m, s) protocol. Since $\frac{m}{s} \notin \mathbb{N}$ we can assume every muffin is cut into at least 2 pieces.

Case 1: Some muffin is cut into $u \ge 3$ pieces. Then some piece is $\le \frac{1}{3}$.

Case 2: All muffins are cut into 2 pieces.

Since there are $2m$ shares and s students both of the following happen:

- Some student gets $t \ge \lceil 2m/s\rceil$ shares, so some share is $\le \frac{m}{s\lceil 2m/s\rceil}$.
- Some student gets $t \le \lfloor 2m/s\rfloor$ shares, so some share x is $\ge \frac{m}{s\lfloor 2m/s\rfloor}$. $B(x)) \le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

Putting together Cases 1 and 2 yields the theorem. ◀

We denote the function from Theorem 12 FC(m, s).

The other two methods are to long to describe fully here so we just sketch.

The *Interval Method* is a primitive version of the Buddy-Match method where we do not use symmetry and (since we have shares other than 2-shares and 3-shares) cannot use the Match in Buddy-Match. This method is fast and can be used to derive the answer. We denote the result INT(m, s).

The *Buddy Method* is like the Buddy-Match Method only we do not use the Match part since we have shares other than 2-shares and 3-shares. And like the Buddy-Match Method this one is faster if you already have the answer. We denote the version where you do not already an upper bound to check BUD(m, s) and the version where you do BUD(m, s, α) where α is the bound.

C Everything You Ever Wanted to Know About $f(s + 7, s)$

By either cheating a little (Section 10) or cheating a lot (Section 11) we have obtained formulas for $f(3dk+a+d, 3dk+a)$ for $1 \le d \le 50$ and $1 \le a \le 3d-1$ (a, d relatively primes). We present the results for $d = 7$. Note that for most of the formulas the formula which is supposed to only hold for $k \ge 1$ also holds for $k = 0$ (with a different proof).

▶ **Theorem 13.**

1. **a.** *$f(8, 1) = 1$. For all $k \ge 1$, $f(21k + 8, 21k + 1) \le \frac{7k+X}{21k+1}$ where $X = \frac{1}{2}$.*
   b. *For all $k \ge 0$, $f(21k + 9, 21k + 2) \le \frac{7k+X}{21k+2}$ where $X = 1$.*

2. *For all* $k \geq 0$, $f(21k+10, 21k+3) \leq \frac{7k+X}{21k+3}$ *where* $X = \frac{4}{3}$.
3. *For all* $k \geq 0$, $f(21k+11, 21k+4) = \frac{7k+X}{21k+4}$ *where* $X = \frac{9}{5}$.
4. *For all* $k \geq 0$, $f(21k+12, 21k+5) \leq \frac{7k+X}{21k+5}$ *where* $X = 2$.
5. *For all* $k \geq 0$, $f(21k+13, 21k+6) \leq \frac{7k+X}{21k+6}$ *where* $X = \frac{13}{5}$.
6. *For all* $k \geq 0$, $f(21k+15, 21k+8) \leq \frac{7k+X}{21k+8}$ *where* $X = 3$.
7. *For all* $k \geq 0$, $f(21k+16, 21k+9) \leq \frac{7k+X}{21k+9}$ *where* $X = \frac{11}{3}$.
8. *For all* $k \geq 0$, $f(21k+17, 21k+10) \leq \frac{7k+X}{21k+10}$ *where* $X = 4$.
9. *For all* $k \geq 0$ $f(21k+18, 21k+11) \leq \frac{7k+X}{21k+11}$ *where* $X = \frac{9}{2}$.
10. *For all* $k \geq 0$ $f(21k+19, 21k+12) \leq \frac{7k+X}{2ak+12}$ *where* $X = \frac{19}{4}$.
11. *For all* $k \geq 0$ $f(21k+20, 21k+13) \leq \frac{7k+X}{21k+13}$ *where* $X = 5$.
12. *For all* $k \geq 0$: $f(21k+22, 21k+15) = \frac{1}{3}$,
13. *For all* $k \geq 0$: $f(21k+23, 21k+16) = \frac{1}{3}$,
14. *For all* $k \geq 0$: $f(21k+24, 21k+17) = \frac{1}{3}$,
15. *For all* $k \geq 0$: $f(21k+25, 21k+18) = \frac{1}{3}$,
16. *For all* $k \geq 0$: $f(21k+26, 21k+19) = \frac{1}{3}$,
17. *For all* $k \geq 0$: $f(21k+27, 21k+20) = \frac{1}{3}$.

Note that the last few answers were $\frac{1}{3}$ and there is an equality. The $\frac{1}{3}$ follows from Theorem 14. The equality holds since we have proven that, for all $m > s$, $f(m, s) \geq \frac{1}{3}$.

D A Sample of General Theorems

In all cases a, d are relatively prime.

▶ **Theorem 14.** *If* $a \in \{2d+1, \dots, 3d-1\}$ *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \frac{a}{3}$, *so* $f(3dk+a+d, 3dk+a) \leq \frac{1}{3}$.

▶ **Theorem 15.** *If* $a \in \{1, \dots, 3d-1\}$, $a \neq d$, *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{a}{3}, \frac{a+d}{5}, \frac{2a-d}{3}\}$.

▶ **Theorem 16.** *If* $1 \leq a \leq 3d-1$ *and* $5a \neq 7d$ *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{a}{3}, \frac{a+d}{5}, \frac{a+2d}{6}, \frac{3a-2d}{4}\}$.

▶ **Theorem 17.** *If* $1 \leq a \leq \frac{5d}{7}$ *and* $a \neq \frac{2d}{3}$ *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

▶ **Theorem 18.** *If* $\frac{5d}{7} \leq a \leq d-1$ *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{2a}{5}, \frac{3a-d}{4}\}$.

▶ **Theorem 19.** *If* $\frac{5d}{13} \leq a \leq \frac{13d}{29}$ *and* $a \neq \frac{2}{5}d$ *then* $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{5a-d}{6}, \frac{a+d}{8}, \frac{3a}{7}\}$.

E If $m \geq s$ then $f(m, s) \geq 1/3$

Before showing the general technique we give an example.

▶ **Example.** $f(19, 17) \geq \frac{1}{3}$.

We express $\frac{19}{17}$ as $\frac{57}{51}$ since other fractions will have a denominator of 51.

We initially divide the 19 muffins $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. There are now 57 pieces $\frac{1}{3}$-shares. We initially give 11 students 3 $\frac{1}{3}$-shares and 6 students 4 $\frac{1}{3}$-shares. (In the proof below $W = 3$,

$s_W = s_3 = 11$, and $s_{W+1} = s_4 = 6$.) A student who gets 3 (4) shares is called a *3-student (4-student)*.

We describe a process whereby students give pieces of muffins, called gifts, to other students so that, in the end, all students have $\frac{57}{51}$. Each gift leads to a change in how the muffins are cut in the first place; however, there will never be a muffin of size $< \frac{1}{3}$.

Each 4-student has $\frac{4}{3} = \frac{68}{51}$ and hence has to give (perhaps in several increments) $\frac{68}{51} - \frac{57}{51} = \frac{11}{51}$ to get *down to* $\frac{57}{51}$. Realize that if a 4-student gives $\frac{11}{51}$ to a 3-student, then the 3-student now has $\frac{51}{51} + \frac{11}{51} = \frac{62}{51} > \frac{57}{51}$.

Each 3-student has $\frac{51}{51}$ and hence has to receive $\frac{57}{51} - \frac{51}{51} = \frac{6}{51}$ to get *up to* $\frac{57}{51}$.

Call the 11 3-students $g_1, \ldots, g_{11}$.

Call the 6 4-students $f_1, \ldots, f_6$.

We use a notation that we just give an example of:

f_1 gives x to g_1 by taking two $\frac{1}{3}$-pieces, combining them, cutting off a piece of size x, giving it to g_1 while keeping the rest. g_1 takes the piece given to him and combines it with a $\frac{1}{3}$ piece. Notice that in terms of pieces we are taking three pieces of size $\frac{1}{3}$ (2 from f_1 and 1 from g_1) and turning them into 1 piece of size $\frac{2}{3} - x$ and one of size $\frac{1}{3} + x$. Hence we can easily rearrange how the muffins are cut.

$x(f_1 \rightarrow g_1)$

We need to make sure this procedure never results in a piece that is $< \frac{1}{3}$. In the above example (1) f_1 now has a piece of size $\frac{2}{3} - x$, hence we need $x \leq \frac{1}{3}$, (2) g_1 now has a piece of size $\frac{1}{3} + x$, which is clearly $\geq \frac{1}{3}$. Hence the only restriction is $x \leq \frac{1}{3}$.

1. $\frac{11}{51}(f_1 \rightarrow g_1)$. Now f_1 has $\frac{57}{51}$. YEAH. However, g_1 has $\frac{62}{51}$.
2. $\frac{5}{51}(g_1 \rightarrow g_2)$. Now g_1 has $\frac{62}{51} - \frac{5}{51} = \frac{57}{51}$. YEAH. However, g_2 has $\frac{51}{51} + \frac{5}{51} = \frac{56}{51}$.
3. $\frac{1}{51}(f_2 \rightarrow g_2)$. Now g_2 has $\frac{57}{51}$. YEAH. However, f_2 has $\frac{67}{51}$.
4. $\frac{10}{51}(f_2 \rightarrow g_3)$. Now f_2 has $\frac{57}{51}$. YEAH. However, g_3 has $\frac{61}{51}$.
5. $\frac{4}{51}(g_3 \rightarrow g_4)$. Now g_3 has $\frac{57}{51}$. YEAH. However, g_4 has $\frac{55}{51}$.
6. $\frac{2}{51}(f_3 \rightarrow g_4)$. Now g_4 has $\frac{57}{51}$. YEAH. However, f_3 has $\frac{66}{51}$.
7. $\frac{9}{51}(f_3 \rightarrow g_5)$. Now f_3 has $\frac{57}{51}$. YEAH. However, g_5 has $\frac{60}{51}$.
8. $\frac{3}{51}(g_5 \rightarrow g_6)$. Now g_5 has $\frac{57}{51}$. YEAH. However, g_6 has $\frac{54}{51}$.
9. $\frac{3}{51}(f_4 \rightarrow g_6)$. Now g_6 has $\frac{57}{51}$. YEAH. However, f_4 has $\frac{65}{51}$.
10. $\frac{8}{51}(f_4 \rightarrow g_7)$. Now f_4 has $\frac{57}{51}$. YEAH. However, g_7 has $\frac{59}{51}$.
11. $\frac{2}{51}(g_7 \rightarrow g_8)$. Now g_7 has $\frac{57}{51}$. YEAH. However, g_8 has $\frac{53}{51}$.
12. $\frac{4}{51}(f_5 \rightarrow g_8)$. Now g_8 has $\frac{57}{51}$. YEAH. However, f_5 has $\frac{64}{51}$.
13. $\frac{7}{51}(f_5 \rightarrow g_9)$. Now f_5 has $\frac{57}{51}$. YEAH. However, g_9 has $\frac{58}{51}$.
14. $\frac{1}{51}(g_9 \rightarrow g_{10})$. Now g_9 has $\frac{58}{51}$. YEAH. However, g_{10} has $\frac{52}{51}$.
15. $\frac{5}{51}(f_6 \rightarrow g_{10})$. Now g_{10} has $\frac{57}{51}$. YEAH. However, f_6 has $\frac{63}{51}$.
16. $\frac{6}{51}(f_6 \rightarrow g_{11})$. Now f_6 has $\frac{57}{51}$. YEAH. However, g_{11} has $\frac{57}{51}$. OH. thats a good thing!

YEAH- we are done.

Note that the first x was $\frac{11}{51} \leq \frac{1}{3}$ and the remaining x were all $\leq \frac{11}{51} \leq \frac{1}{3}$. Hence all pieces in the final protocol are $\geq \frac{1}{3}$.

▶ **Theorem 20.** *For all $m \geq s$, $f(m,s) \geq \frac{1}{3}$.*

Proof. Divide all the muffins into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Initially distribute them as evenly as possible among the students. There will be a number W such that some students get W shares and some get $(W+1)$-shares. Let s_W (s_{W+1}) be the number of students who get W $(W+1)$ shares.

We do not need the following but are noting it anyway. If s does not divide $3m$ then $W = \frac{3m}{s}$ and s_W, s_{W+1} are unique and determined by:

$$\begin{aligned} Ws_W + (W+1)s_{W+1} &= 3m \\ s_W + s_{W+1} &= s \end{aligned}$$

(Technically, if $s \mid 3m$ there are two possible values of W.)

A student who gets W $(W+1)$ shares we call a W-student ($(W+1)$-student). All W-students get $\frac{W}{3}$. All $(W+1)$-students get $\frac{W+1}{3}$.

A W-student must get $< \frac{m}{s}$: if a W-student got $> \frac{m}{s}$ then all students would get $> \frac{m}{s}$ and hence there would be $> s\frac{m}{s} = m$ muffins total. A $(W+1)$-student must get $> \frac{m}{s}$: if a $(W+1)$-student got $< \frac{m}{s}$ then all students would get $< \frac{m}{s}$ and hence there would be $< s\frac{m}{s} = m$ muffins total.

Hence we have:

$$\frac{m}{s} - \frac{W}{3} \le \frac{1}{3} \tag{1}$$

$$\frac{W+1}{3} - \frac{m}{s} \le \frac{1}{3} \tag{2}$$

Now we will need to smooth out the distribution so that everyone receives $\frac{m}{s}$. We will do this by doing a sequence of moves of the form $x(f_i \to g_j)$ or $x(g_i \to g_j)$. as defined in the example.

We will assume s_{W+1} and s_W are relatively prime (this only comes up in Claim 3 below). This is fine because if they have a common factor d, we can just use the procedure for the $\frac{s_{W+1}}{d}$, $\frac{s_W}{d}$ case repeated d times.

▶ **Claim 1.**
1. *If* $s_{W+1} < s_W$ *then* $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.
2. *If* $s_W < s_{W+1}$ *then* $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

Proof of Claim 1.

$$s_{W+1} \times \frac{W+1}{3} + s_W \times \frac{W}{3} = m$$

$$s_{W+1} \times \left(\frac{m}{s} + \frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{m}{s} + \frac{W}{3} - \frac{m}{s}\right) = m.$$

$$\left(s_{W+1} + s_W\right)\frac{m}{s} + s_{W+1}\left(\frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$s \times \frac{m}{s} + s_{W+1}\left(\frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$\frac{W+1}{3} - \frac{m}{s} = \frac{s_W}{s_{W+1}}\left(\frac{m}{s} - \frac{W}{3}\right)$$

Both parts follow. ◀

We give the procedure to obtain $f(m,s) \le \frac{1}{3}$. There are two cases.

Case 1: $s_{W+1} < s_W$. Hence by Claim 1 $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.
Call the s_W W-students $g_1, \ldots, g_{s_W}$.
Call the s_{W+1} $(W+1)$-students $f_1, \ldots, f_{s_{W+1}}$.

1. Let $x = \frac{W+1}{3} - \frac{m}{s}$. Note that $x \le \frac{1}{3}$. Do $x(f_1 \rightarrow g_1)$. Now f_1 has $\frac{m}{s}$. YEAH. However, g_1 has $\frac{W}{3} + \frac{W+1}{3} - \frac{m}{s} > \frac{m}{s}$. (This is where we use $s_{W+1} < s_W$, or more accurately the consequence of that from Claim 1.)
2. Let $x = \frac{2W+1}{3} - 2\frac{m}{s}$. Do $x(g_1 \rightarrow g_2)$. Now g_1 has $\frac{m}{s}$. YEAH.
3. If g_2 has $> \frac{m}{s}$ then g_2 gives enough to g_3 so that g_2 has $\frac{m}{s}$. Keep up this chain of $g_1, g_2, g_3, \ldots$ until there is a g_i such that g_i end up with $< \frac{m}{s}$ (though more than the $\frac{W}{3}$ that g_i had originally).
4. Do $x(f_2 \rightarrow g_i)$ where x is such that g_i will now have $\frac{m}{s}$.
5. Do $x(f_2 \rightarrow g_{i+1})$ where x is such that f_2 will now have $\frac{m}{s}$. Repeat the same chain of g_i's as in step 3.
6. Repeat the above steps until you are done.

We need to show that (1) there is never a piece of size $< \frac{1}{3}$, and (2) the process ends with every student getting $\frac{m}{s}$.

▶ **Claim 2.** *The first gift is $\le \frac{1}{3}$ and no gift is larger.*

Proof of Claim 2. Let $C = \frac{W+1}{3} - \frac{m}{s}$ which is the size of the first gift. By equation (2) $C \le \frac{1}{3}$.

Assume that all gifts so far have been $\le C$. We analyze the three kinds of gifts and show that in all cases the gift is $\le C$.

- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift f_i has $\frac{m}{s}$. When this occurs it is f_i's first or second gift giving. (This happens in steps 1 and 5 above, and later as well.) Before the gift f_i has at least $\frac{m}{s}$ but at most $\frac{W+1}{3}$, so this gift has size at most $\frac{W+1}{3} - \frac{m}{s} = C$.
- $x(g_i \rightarrow g_{i+1})$ where (1) initially g_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_i has $\frac{m}{s}$. When this occurs g_i has received a gift once and this is g_i's first time giving. (This happens in steps 2 and in the chain referred to in step 5.) Since g_i just received a gift of size $\le C$ she has $\le \frac{W}{3} + C$. Hence the gift is $\le \frac{W}{3} - \frac{m}{s} + C \le C$.
- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_j has $\frac{m}{s}$. This will be f_i's first time giving. (This happens in step 4 above.) Before the gift f_i has at least $\frac{W}{3}$ but at most $\frac{m}{s}$, so this gift has size at most $\frac{m}{s} - \frac{W}{3} \le C$ (by Claim 1). ◀

▶ **Claim 3.** *If s_W and s_{W+1} are relatively prime then the process terminates with all students having $\frac{m}{s}$.*

Proof of Claim 3. In each step all of the f_i have at least $\frac{m}{s}$. In each step the number of students who have the correct amount of muffin goes up. One may be worried that at some point we will try to do step 4 (for example) of the procedure and there will be no g_i left who need more muffin. But this is not possible because until the process terminates the f's always have more muffin than they need, so there is always a g with insufficient muffin.

One may also be worried that eventually we will get all of the f's to have $\frac{m}{s}$, but the g's will not all have $\frac{m}{s}$. This is not possible either, because whenever we only make gifts from f to g when there is no g with more than $\frac{m}{s}$.

Finally, if s_W and s_{W+1} are not relatively prime, it is possible that the procedure will terminate early because in step 5 the size of the donation x is 0. If this occurred it would mean that there is some subset of F f's and G g's each of which having exactly $\frac{m}{s}$, who only made donations amongst themselves. But then $\frac{F}{G} = \frac{s_{W+1}}{s_W}$, a contradiction. ◀

Case 2: $s_W < s_{W+1}$. This is similar to Case 1 except that instead of f_1 giving g_1 so that f_1 has $\frac{m}{s}$, f_1 gives to g_1 so that g_1 has $\frac{m}{s}$. Hence we have a chain of f_i's instead of a chain of g_i's. ◀