Periods in Subtraction Games

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Abstract

We discuss the structure of periods in subtraction games. In particular, we discuss ways that a computational approach yields insights to the periods that emerge in the asymptotic structure of these combinatorial games.

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1 Overview

Subtraction games are one of the most fundamental combinatorial games. In the document *Unsolved Problems in Combinatorial Games* [4], maintained by Richard J. Nowakowski, the structure of combinatorial games is the first open problem that is discussed. Such games are so fundamental because the underlying premise is the same as Nim: there are several piles of beans, and on a player's turn, he/she can remove beans from exactly one pile. As in many areas of mathematics, this simple concept gives rise to much deeper mathematical structure. In the case of subtraction games, an even richer structure emerges because the moves of a player are limited. For instance, in the three-dimensional version of subtraction games with subtraction set $\{s_1, s_2, s_3\}$, the number of beans that can be removed from a heap during a player's turn is limited to one of these three possibilities. In other words, a player can only remove either s_1, s_2, or s_3 beans.

The problem of understanding the associated Nim values of a subtraction game is sufficiently challenging and useful that a table of values for small s_1, s_2, s_3 is given in the 4-volume set of books called *Winning Ways for your Mathematical Plays* [2].

The problem of understanding the asymptotic periodicities of subtraction games with a subtraction set of size three has been open for more than 40 years; see [1] for early analysis.

Mark Paulhus and Alex Fink have derived values of the periods in two cases, for subtraction sets of size 3, namely, in the case where $s_1 = 1$ and s_2, s_3 are arbitrary, and in the case where $s_1 < s_2 < s_3 < 32$ (see [4]). Achim Flammenkamp [3] has made conjectures about the types of periodicities that arise, based on calculations with all s_j's bounded above by 256.

We organized a team of colleagues to work on this problem at the American Institute of Mathematics (AIM), under the auspices of the Research Experiences for Undergraduate Faculty (REUF) workshops, starting in July 2016. (Ward had already been working on a computational attack for this problem in his spare moments, for more than a decade.) Our REUF team relies on a data-driven approach. We have computed the Nim values and the resulting (asymptotic) periodicity of the games for s_j's bounded above by 16384. The computational aspects of this problem are nontrivial. Each time the size of the parameters grows by a factor of 2, the computational time required for the resulting computations grows by a factor of (roughly) 17. Therefore, our most recent computation took a full 37 years of CPU time. It was accomplished by running a massive parallel computation on three of the computational clusters at Purdue University (using thousands of computational cores). After all, we made $\binom{16384}{3} = 732{,}873{,}539{,}584$ distinct computations altogether. We have generated terabytes of data about this combinatorial problem.

We will present our computational approach to determining the combinatorial structure of the asymptotic periods that arise in these subtraction games. Importantly, we emphasize that our algorithms allow us to know the asymptotic periods, without resorting at all to the traditional approach (which relies on minimal excluded numbers). Instead, we have obtained structural insights about this problem. These results should continue to be useful for revealing completely new viewpoints about the structure of combinatorial games.

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