The Number of Double Triangles in Random Planar Maps

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Abstract

The purpose of this paper is to provide a central limit theorem for the number of occurrences of double triangles in random planar maps. This is the first result of this kind that goes beyond face counts of given valency. The method is based on generating functions, an involved combinatorial decomposition scheme that leads to a system of catalytic functional equations and an analytic extension of the *Quadratic Method* to systems of equations.

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1 Introduction

A *planar map* is a connected planar graph (loops and multiple edges are allowed) embedded into the plane up to homeomorphism. A map is *rooted* if a vertex v is chosen from the map and a half-edge e is chosen from all the edges incident to v, and called the *root vertex* and *root edge*, respectively. Moreover, a planar map separates the plane into several connected regions called *faces*. The *root face* in a rooted map is the face which is on the left side of e (sometimes the root face is defined as the right side of e, but this does not make a principle difference). Without loss of generality we may assume that the root face is the infinite (or outer) face, in particular the root edge e is then adjacent to the outside face. In this paper, all maps we consider are rooted and planar. By convenience we also include the trivial map that consists just of one vertex and one face (which are also rooted). It is well known that there are precisely $M_n = \dfrac{2 \cdot 3^n \binom{2n}{n}}{n(n+1)}$ different rooted planar maps with n edges [12]. In what

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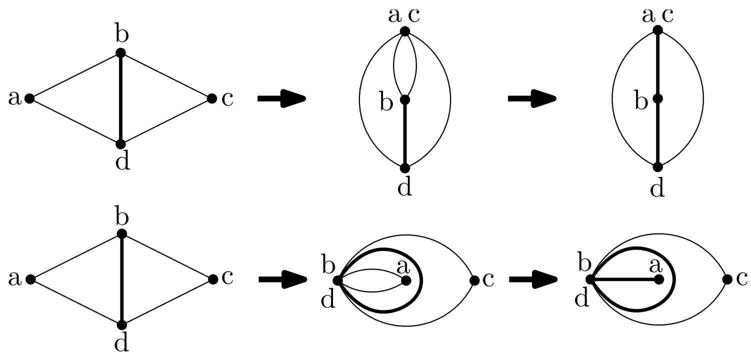

Figure 1 Degenerate cases of double triangles that are represented as bold edges.

follows we assume that for any fixed n every map with n edges is equally likely. Hence every parameter of rooted planar maps can be considered as a random variable related to random planar maps with n edges.

The main goal of this paper is to prove the following theorem:

▶ **Theorem 1.** *The number X_n of edges with valency 3 faces on both sides in a random planar map with n edges satisfies a central limit law, i.e.,*

$$\frac{X_n - \mathbb{E}[X_n]}{\mathrm{Var}[X_n]^{1/2}} \to \mathcal{N}(0,1), \tag{1}$$

where $\mathbb{E}[X_n] = \mu n + O(1)$ and $\mathrm{Var}[X_n] = \sigma^2 n + O(1)$, and μ, σ are positive constants.

▶ Remark. We cannot derive a simple analytic expression μ and σ since our analysis is implicitly based on an infinite system of equations. So they are definitely hard to compute, even in an approximate sense.

In an slight abuse of notation we will call the occurrence of an edge with valency 3 faces on both sides a *double triangle*. Namely there are some degenerate cases as Figure 1 shows (in the first case we identify vertices a and c and then also the edges ab and bc so that we havee two double triangles between two triangles; in the second case, we identify vertices b and d and then the edges ab and ad so that a bridge represents a double triangle).

The background of this result is a widely believed conjecture that the number of pattern occurrences in planar maps (and many related graph classes) obeys a central limit theorem. For (general) planar maps there are only very few results in this direction, see [7, 4] for the number of faces of given valency or [9] for triangulation patterns in 2-connects triangulations and quadrangulations patterns in simple quadrangulations. We also want to mention that the expected number of occurrences of a given pattern in a random planar map with n edges is asymptotically linear: $\mathbb{E}\, X_n \sim cn$ for some constant $c > 0$. This follows from the fact that random planar maps have a Benjamini-Schramm limit, see [8, 1, 10, 11]. As mentioned before it is expected that X_n satisfies a central limit theorem in all cases. However, it seems that this is out of reach at the moment. Even the simplest case beyond face-pattern that is considered in this paper requires a thorough and delicate analysis for the combinatorial part as well as for the analytic part. We use an approach that is in principle close to that of [7], namely we use generating functions, set up a system of catalytic functional equations (Section 2) and finally provide a proper analytic extension of the classical *Quadratic Method* [3, 12] (Section 3).

2 Combinatorics

Our goal is to set up a recursive structure of planar maps that is suitable to take occurrence of double triangles into account. For this purpose we distinguish between three different cases: the initial case (a map without any edge, denoted by $\bullet$), the bridgeable case (maps, where the root edge is a bridge, denoted by $\mathcal{D}^{(b)}$) and the non-bridgeable case (maps, where the root edge is not a bridge, denoted by $\mathcal{D}^{(n)}$):

$$\mathcal{D} = \bullet + \mathcal{D}^{(b)} + \mathcal{D}^{(n)}.$$

We let $D(z,u,w)$ be the ordinary generating function

$$D(z,u,w) = \sum_{n,k,\ell \geq 0} d_{n,k,\ell} z^n u^k w^\ell,$$

where $d_{n,k,\ell}$ is the number of planar maps with n edges, valency k on its root face and ℓ edges that represent double triangles, where edges on the root face are not considered. For the sake of brevity, we denote $D(z,u,w)$ by D and $D(z,1,w)$ by $D(1)$. (The same *rule* will be applied to other generating functions.)

Clearly, the initial case corresponds to the generating function 1 and the bridgeable case to zu^2D^2. The non-bridgeable case is split into two different classes: $\mathcal{D}_{\not\triangleright}$ denotes the class where the *second face* (the face on the right side of the root edge) has valency not equal to 3 and $\mathcal{D}_{\triangleright}$ denotes the class where the second face has valency 3. This means that we have $D = 1 + zu^2D^2 + D_{\not\triangleright} + D_{\triangleright}$, where $D_{\not\triangleright}$ and $D_{\triangleright}$ are the corresponding generating functions of $\mathcal{D}_{\not\triangleright}$ and $\mathcal{D}_{\triangleright}$, respectively.

▶ **Lemma 2.** *The generating functions $D = D(z,u,w)$, $D_{\not\triangleright} = D_{\not\triangleright}(z,u,w)$, and $D_{\triangleright} = D_{\triangleright}(z,u,w)$ satisfy the following system of equations:*

$$\begin{aligned}
D &= 1 + zu^2D^2 + D_{\not\triangleright} + D_{\triangleright}, \\
D_{\not\triangleright} &= zu\frac{D(1)-uD}{1-u} - zu^{-1}\left(D - 1 - u[u^1]D\right), \qquad (2)\\
D_{\triangleright} &= zu^{-1}\left(D - 1 - u[u^1]D\right) \\
&\quad + (w-1)\cdot\Bigg[z^2uD + (w+1)\left(zu^{-1}D_{\triangleright} - z[u^1]D_{\triangleright}\right) - z^2u(w-1)DD_{\triangleright} \\
&\quad - (w-1)\left(z^2\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - z^2D_{\triangleright}(1) - z^2u^{-2}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right)\Bigg].
\end{aligned}$$

▶ Remark. If $w = 1$ the system collapses to the well-known catalytic equation for the generating function $M(z,u) = D(z,u,1)$ of planar maps:

$$M(z,u) = 1 + zu^2M(z,u)^2 + zu\frac{M(z,1)-uM(z,u)}{1-u}. \qquad (3)$$

Proof. We have already discussed the first equation of (2). Thus, we can concentrate on the non-bridgeable case. Here we relate the original map with the resulting map, where we have removed the root edge. Actually it is more transparent to consider the reverse process of adding a new root edge that *cuts across* the root face. This operation separates the root face into two faces. For instance, there are five possible situations of cutting across a root face of valency 4 as Figure 2 shows, and which have the following effect to the variable u:

$$u^4 \mapsto z(u^5 + u^4 + u^3 + u^2 + u).$$

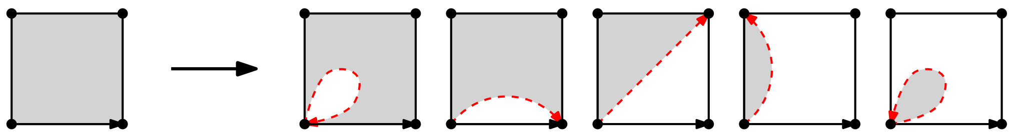

Figure 2 Cutting-across-process.

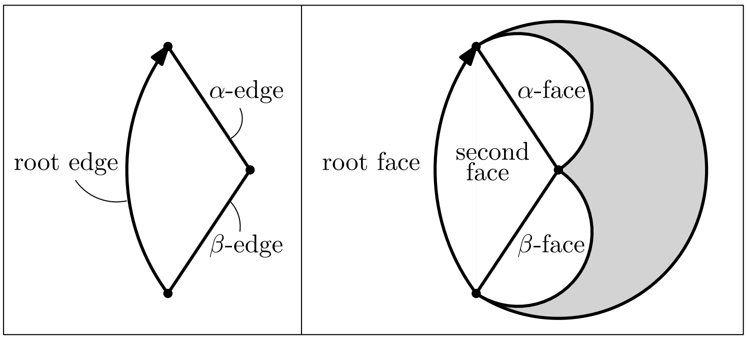

Figure 3 Definition of the α and β-edge and face.

When we consider $\mathcal{D}_{\ntriangleright}$, we have to discount the case where the second face has valency 3. In the cutting-across-process, we take out the situation that the new-appearing second face has valency 3. The corresponding effect with the root face of valency r is

$$u^r \mapsto z(u^{r+1} + u^r + \cdots + u^2 + u^1) + \begin{cases} -zu^{r-1} & \text{, if } r \geq 2 \\ 0 & \text{, if } r = 0 \text{ or } 1. \end{cases}$$

So the corresponding generating function of $\mathcal{D}_{\ntriangleright}$ is given by

$$D_{\ntriangleright} = zu\frac{D(1) - uD}{1 - u} - zu^{-1}\left(D - 1 - [u^1]D\right).$$

Next, we consider maps whose second face is of valency 3 and whose generating function is $D_{\triangleright}$. We introduce some notations. When the second face has valency 3, the edges following the root edge in clockwise order are called the *α-edge* and the *β-edge*. One side of the α-edge is the second face, we call the face on the other side the *α-face*. Similar to the α-face, the *β-face* is the face incident to the β-edge. Note that the α-face and the β-face might coincide (see Figure 3).

For describing the class $\mathcal{D}_{\triangleright}$, we consider four different cases: both the α-face and the β-edge are equal to the root face (denoted by $\mathcal{D}_{\triangleright}^{\alpha,\beta}$), only the α-face is equal to the root face (denoted by $\mathcal{D}_{\triangleright}^{\alpha}$), only the β-face is equal to the root face (denoted by $\mathcal{D}_{\triangleright}^{\beta}$) and neither the α-face nor the β-face is equal to the root face (denoted by $\mathcal{D}_{\underline{\triangleright}}$) (see Figure 4). Thus, we have $D_{\triangleright} = D_{\triangleright}^{\alpha,\beta} + D_{\triangleright}^{\alpha} + D_{\triangleright}^{\beta} + D_{\underline{\triangleright}}$.

The maps corresponding to the class $\mathcal{D}_{\triangleright}^{\alpha,\beta}$ can be divided into a triangle and three maps. Thus, we have $D_{\triangleright}^{\alpha,\beta} = z^3u^3D^3$.

The maps corresponding to the class $\mathcal{D}_{\triangleright}^{\alpha}$ and $\mathcal{D}_{\triangleright}^{\beta}$ can be divided into a map and a map stuck together with a triangle attached to an edge (see the left part of Figure 5). The structure of a map stuck together with a triangle attached to an edge has either the property that this edge corresponds to a double triangle or not (see the right of Figure 5).

If this edge is (resp. is not) a double triangle, we can think of it as adding two edges to a map which belong to $\mathcal{D}_{\triangleright}$ (resp. $\mathcal{D}_{\ntriangleright}$). The effect of these two additional edges is that the number of edges increased by 2 and the valency of the outside (root) face increased by 1. Hence, $D_{\triangleright}^{\alpha} = D_{\triangleright}^{\beta} = z^2u\left(wD_{\triangleright} + D_{\ntriangleright}\right)D$.

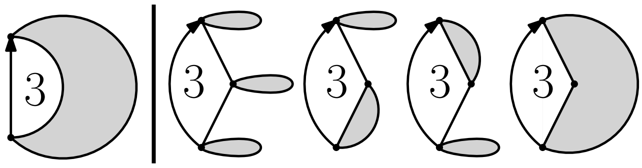

Figure 4 Four different cases of $\mathcal{D}_{\triangleright}$.

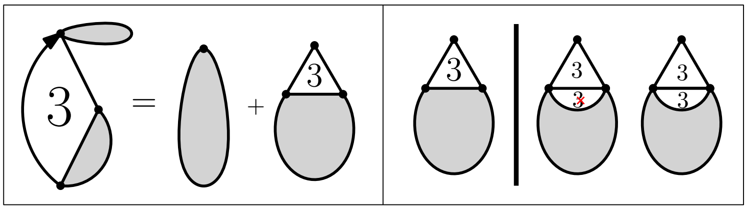

Figure 5 A map in $\mathcal{D}^{\alpha}_{\triangleright}$ or $\mathcal{D}^{\beta}_{\triangleright}$ can be divided into a map and a map stuck together with a triangle at an edge. This edge corresponds to a double triangle or not.

For the fourth class $\mathcal{D}_{\trianglerighteq}$ we need to consider three different cases. The first one is when the α-edge is different from the β-edge but the α-face equals to the β-face (denoted by $\mathcal{D}^{\Cup}_{\trianglerighteq}$). The second one is when the α-edge is different from the β-edge and the α-face is different from the β-face (denoted by $\mathcal{D}^{\varpi}_{\trianglerighteq}$). The third one is when the α-edge equals to the β-edge. In this case, both the α-face and the β-face are equal to the second face (denoted by $\mathcal{D}^{\psi}_{\trianglerighteq}$) (see Figure 6). By definition we have $D_{\trianglerighteq} = D^{\Cup}_{\trianglerighteq} + D^{\varpi}_{\trianglerighteq} + D^{\psi}_{\trianglerighteq}$.

When we deal with the maps in $\mathcal{D}^{\Cup}_{\trianglerighteq}$, the α-face coincides with the β-face if both of them have valency 3, in particular, both the α-edge and the β-edge represent double triangles. Therefore, we have to take care of the valency of the α-face and of the β-face. For this purpose we consider the so-called "*border-(α,β)-path*", that starts from the α-edge, goes clockwise along the border of the α-face and finishes at the β-edge but dones not include the α-edge nor the β-edge. We distinguish between three different cases by considering the length of the border-(α,β)-path (denoted by $|(\alpha,\beta)|$): $|(\alpha,\beta)| = 0$, $|(\alpha,\beta)| = 1$ and $|(\alpha,\beta)| \geq 2$. The corresponding sets of maps are denoted by $\mathcal{D}^{\Cup_0}_{\trianglerighteq}$, $\mathcal{D}^{\Cup_1}_{\trianglerighteq}$, and $\mathcal{D}^{\Cup_{\geq 2}}_{\trianglerighteq}$ respectively, see Figure 7. From the above relation, we have $D^{\Cup}_{\trianglerighteq} = D^{\Cup_0}_{\trianglerighteq} + D^{\Cup_1}_{\trianglerighteq} + D^{\Cup_{\geq 2}}_{\trianglerighteq}$ and (similar to the above considerations) they can be further decomposed which leads to the following relations:

$$\begin{aligned} D^{\Cup_0}_{\trianglerighteq} &= z^3uD\left[w^2\left(w[u^1]D_{\triangleright} + [u^1]D_{\not\triangleright}\right) + \left(D(1) - [u^1]D\right)\right], \\ D^{\Cup_1}_{\trianglerighteq} &= z^3w^2(wD_{\triangleright} + D_{\not\triangleright} + zu^2D^2) + z^3\left(D(1) - 1\right)(D-1), \\ D^{\Cup_{\geq 2}}_{\trianglerighteq} &= z^2D(1)\left(zu\frac{D(1) - uD}{1-u} - z(D-1) - zuD\right), \end{aligned} \tag{4}$$

The proof is given in the Appendix A.1.

Next, $\mathcal{D}^{\psi}_{\trianglerighteq}$ is the class of maps that combines maps and an edge inside a loop. The edge inside the loop is a double triangle. Thus, we have $D^{\psi}_{\trianglerighteq} = z^2uwD$.

Finally, we discuss the class $\mathcal{D}^{\varpi}_{\trianglerighteq}$. By distinguishing whether the α-face and the β-face have valency 3 we have to consider four different situations: neither the α-face nor the β-face has valency 3 (denoted by $\mathcal{D}^{\otimes}_{\trianglerighteq}$), only the β-face has valency 3 (denoted by $\mathcal{D}^{\beta}_{\trianglerighteq}$), only the α-face has valency 3 (denoted by $\mathcal{D}^{\alpha}_{\trianglerighteq}$) and both the α-face and the β-face have valency 3 (denoted by $\mathcal{D}^{\alpha,\beta}_{\trianglerighteq}$): $D^{\varpi}_{\trianglerighteq} = D^{\otimes}_{\trianglerighteq} + D^{\beta}_{\trianglerighteq} + D^{\alpha}_{\trianglerighteq} + D^{\alpha,\beta}_{\trianglerighteq}$.

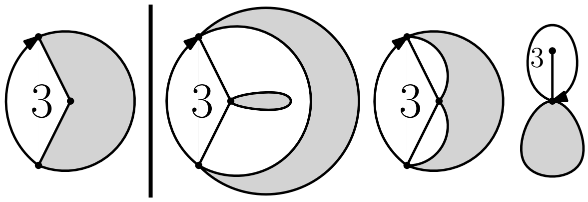

Figure 6 Three different cases of $\mathcal{D}_{\trianglerighteq}$.

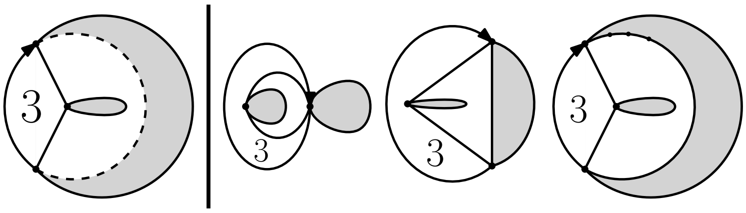

Figure 7 Three different cases of the length of the border-(α,β)-path of the maps in $\mathcal{D}^{\Cup}_{\trianglerighteq}$.

When we study the class $\mathcal{D}^{\otimes}_{\trianglerighteq}$, we need to build up maps, where the second face has valency 3 and neither the α-face nor the β-face has valency 3. We start with $\mathcal{D}_{\not\triangleright}$ and do the cutting across process that adds an edge starting from the end point of the root edge of the map. In cutting across process (see Figure 2) we always keep the second face valency different from 3 and the outside face valency greater than 1 (in order to make sure that the new α-edge and the new β-edge exist). In a second step we add an edge to complete the construction (see the left of Figure 8).

We have to be careful in the cutting across process. For example, if the root face valency equals r before we start the process, we have to avoid the case, where the root face valency would get $r-1$ in cutting across process. This means that the cases $r = 0, 1, 2$ have to be considered separately. If $r = 0$ or $r = 1$ the root face valency $r-1$ in cutting across process can not appear, and when $r = 2$ the resulting root face of valency $r-1 = 1$ is also excluded. The effect on the variable u is therefore

$$u^r \mapsto z(u^{r+1} + u^r + \cdots + u^2 + u^1) - zu^1 + \begin{cases} -zu^{r-1} & \text{, if } r \geq 3 \\ 0 & \text{, if } r = 0 \text{ or } 1 \text{ or } 2. \end{cases}$$

After adding an edge in second step we obtain the following relations for the corresponding generating function: $\mathcal{D}^{\otimes}_{\trianglerighteq}$ is

$$D^{\otimes}_{\trianglerighteq} = zu^{-1}\left(zu\frac{D_{\not\triangleright}(1) - uD_{\not\triangleright}}{1-u} - zuD_{\not\triangleright}(1) - zu^{-1}\left(D_{\not\triangleright} - u[u^1]D_{\not\triangleright} - u^2[u^2]D_{\not\triangleright}\right)\right).$$

By using similar ideas (by using $\mathcal{D}_{\triangleright}$ instead) and by observing that the new β-edge will be a double triangle (see the right of Figure 8) we obtain

$$D^{\beta}_{\trianglerighteq} = zu^{-1}w\left(zu\frac{D_{\triangleright}(1) - uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right).$$

By symmetry we have $D^{\alpha}_{\trianglerighteq} = D^{\beta}_{\trianglerighteq}$.

In order to describe the class $\mathcal{D}^{\alpha,\beta}_{\trianglerighteq}$ we need to *adjust* both the root face and the face (we call this face *clockwise-face*) on the right of the clockwise-edge have valency 3 (see Figure 9), where the *clockwise-edge* is the edge in clockwise direction of the root edge on the outside face. Suppose that $\mathcal{D}_{\mathfrak{B}}$ is the class of maps, where both the root face and the clockwise-face have valency 3, we have $D^{\alpha,\beta}_{\trianglerighteq} = zu^{-1}w^2D_{\mathfrak{B}}$.

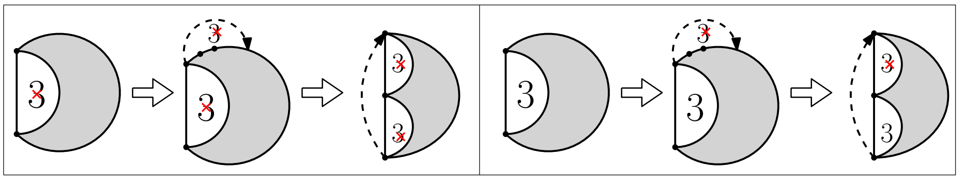

Figure 8 Construction of a map contained in $\mathcal{D}_{\trianglerighteq}^{\otimes}$ and $\mathcal{D}_{\trianglerighteq}^{\beta}$, respectively.

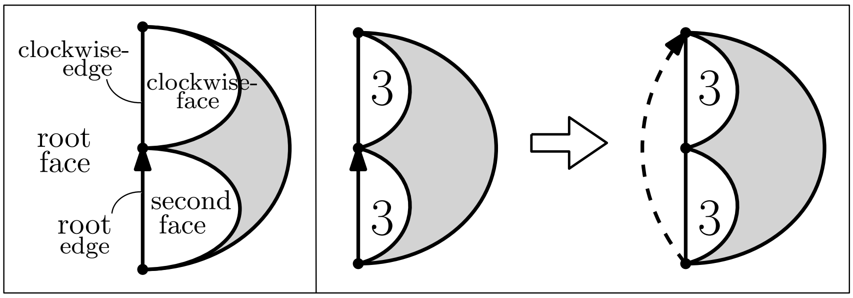

Figure 9 Definition of clockwise-edge and clockwise-face. Relation between $\mathcal{D}_{\mathfrak{B}}$ and $\mathcal{D}_{\trianglerighteq}^{\alpha,\beta}$.

The class $\mathcal{D}_{\mathfrak{B}}$ is a subclass of $\mathcal{D}_{\triangleright}$. Hence, we can get $D_{\mathfrak{B}}$ by eliminating some cases of $\mathcal{D}_{\triangleright}$. When we consider the clockwise-edge and the clockwise-face of $\mathcal{D}_{\triangleright}$, we have five different cases. The first three cases where the clockwise-edge is not a bridge, and first, where the clockwise-face has valency 3, second, where the clockwise-face has valency not equal to 3 and third, where the clockwise-face is equal to the second face. In the fourth case the clockwise-edge is a bridge and in the last the clockwise-edge does not exist (see Figure 10).

The first case of $\mathcal{D}_{\triangleright}$ is precisely $\mathcal{D}_{\mathfrak{B}}$.

The second case of $\mathcal{D}_{\triangleright}$ (clockwise-face has valency not equal to 3) corresponds precisely to the first step of the construction of $\mathcal{D}_{\trianglerighteq}^{\beta}$ in Figure 8. Hence, the corresponding generating function is given by

$$zu\frac{D_{\triangleright}(1) - uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right).$$

The only difference to $D_{\trianglerighteq}^{\beta}$ is the factor $zu^{-1}w$.

In the third case of $\mathcal{D}_{\triangleright}$ we have to consider several subcases that lead to the following generating function:

$$z^2u^2wD[u^1]D_{\triangleright} + z^2u^2D[u^1]D_{\not\triangleright} + z^2uwD_{\triangleright} + z^2uD_{\not\triangleright} + z^3u^3D^2.$$

In the fourth case of $\mathcal{D}_{\triangleright}$ the second face has valency 3 and the clockwise-edge is a bridge. Thus, it corresponds to the generating function $zu^2DD_{\triangleright}$.

Finally, in the last case the root face valency equals 1 and the second face has valency 3. Consequently its corresponding generating function is $u[u^1]D_{\triangleright}$.

Summing up, the generating function of $\mathcal{D}_{\mathfrak{B}}$ is given by

$$\begin{aligned}D_{\mathfrak{B}} &= D_{\triangleright} - zu^2DD_{\triangleright} - u[u^1]D_{\triangleright}\\ &\quad - \left(zu\frac{D_{\triangleright}(1) - uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right)\\ &\quad - \left(z^2u^2D[u^1]D_{\not\triangleright} + z^2u^2wD[u^1]D_{\triangleright} + z^2uD_{\not\triangleright} + z^2uwD_{\triangleright} + z^3u^3D^2\right).\end{aligned}$$

By collecting all these parts and by applying some simplifications (that are described in the Appendix A.2) we obtain the third equation of the system (2). ◀

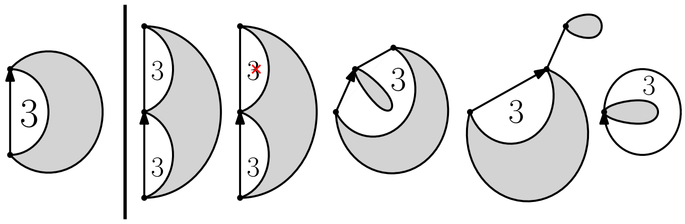

Figure 10 Five different cases of $\mathcal{D}_{\triangleright}$.

3 Asymptotic analysis

In order to analyze the system of equations (2) we apply a 2-step procedure that is in principle close to that of [7]. In the first step we eliminate the terms $[u^1]D$, $[u^1]D_{\triangleright}$, and $[u^2]D_{\triangleright}$ so that the system (2) is transferred into a catalytic system of equations that will be solved then in a second step.

▶ **Lemma 3.** *Suppose that $D = D(z,u,w)$, $D_{\not\triangleright} = D_{\not\triangleright}(z,u,w)$, and $D_{\triangleright} = D_{\triangleright}(z,u,w)$ are the solution functions of the system (2). Then there exist analytic functions $K_{ij}(z,w,x_0,x_1,x_2)$ (for $|z| < \frac{1}{2}$, $|x_0| < 2$, $|x_1| < 2$, $|x_2| < 2$, and $|w-1| < \eta$ for some sufficiently small $\eta > 0$), $i \in \{0,1,2\}$, $j \in \{1,2\}$ such that for $j \in \{1,2\}$*

$$\begin{aligned}
[u^j]D(z,u,w) &= K_{0,j}\left(z,w,D(z,1,w),D_{\not\triangleright}(z,1,w),D_{\triangleright}(z,1,w)\right),\\
[u^j]D_{\not\triangleright}(z,u,w) &= K_{1,j}\left(z,w,D(z,1,w),D_{\not\triangleright}(z,1,w),D_{\triangleright}(z,1,w)\right),\\
[u^j]D_{\triangleright}(z,u,w) &= K_{2,j}\left(z,w,D(z,1,w),D_{\not\triangleright}(z,1,w),D_{\triangleright}(z,1,w)\right).
\end{aligned}$$

Proof. We rewrite the system (2) into an equivalent one. We substitute in all instances $D_{\not\triangleright} = D_1 - zu^{-1}(D - 1 - u[u^1]D)$ and $D_{\triangleright} = D_2 + zu^{-1}(D - 1 - u[u^1]D)$ so that we obtain a system of the form

$$D = 1 + zu^2D^2 + D_1 + D_2, \quad D_1 = zu\frac{D(1) - uD}{1-u}, \quad D_2 = (w-1)H, \tag{5}$$

where H is equal to

$$\begin{aligned}
&z^2uD + (w+1)\Big(zu^{-1}\left(D_2 - u[u^1]D_2\right) + z^2u^{-2}\left(D - 1 - u[u^1]D - u^2[u^2]D\right)\Big)\\
&+ (w-1)\Big(-z^2uDD_2 - z^3D(D - 1 - u[u^1]D) - z^2u\frac{D_2(1) - D_2}{1-u} - z^3\frac{uD(1) - D}{1-u} - z^3\\
&+ z^2u^{-2}\left(D_2 - u[u^1]D_2 - u^2[u^2]D_2\right) + z^3u^{-3}\left(D - 1 - u[u^1]D - u^2[u^2]D - u^3[u^3]D\right)\Big).
\end{aligned}$$

Next we consider the functions D, D_1, D_2 as power series in u:

$$D = 1 + \sum_{\ell\ge1} d_\ell u^\ell, \quad D_1 = \sum_{\ell\ge1} d_{1,\ell}u^\ell, \quad D_2 = \sum_{\ell\ge1} d_{2,\ell}u^\ell,$$

and rewrite the system (5) into an infinite system of equations:

$$\begin{aligned}
d_\ell &= z\sum_{j=0}^{\ell-2} d_jd_{\ell-2-j} + zD(1) - z\sum_{j=0}^{\ell-2} d_j + (w-1)[u^\ell]H,\\
d_{1,\ell} &= zD(1) - z\sum_{j=0}^{\ell-2} d_j,\\
d_{2,\ell} &= (w-1)[u^\ell]H,
\end{aligned} \tag{6}$$

where $\ell \ge 1$, $d_0 := 1$, and $[u^\ell]H$ is equal to

$$z^2 d_{\ell-1} + (w+1)(zd_{2,\ell+1} + z^2 d_{\ell+2}) + (w-1)\Bigg[-z^2 \sum_{i=0}^{\ell-2} d_i d_{2,\ell-1-i} - z^3 \sum_{i=0}^{\ell-2} d_i d_{\ell-i} \\ - z^2 \left(D_2(1) - \sum_{i=0}^{\ell-1} d_{2,i} \right) - z^3 \left(D(1) - \sum_{i=0}^{\ell} d_i \right) + z^2 d_{2,\ell+2} + z^3 d_{\ell+3} \Bigg].$$

Note that we have not substituted $D(1)$, $D_1(1)$, and $D_2(1)$. In a final step we use the substitutions $y_{0,\ell} = d_\ell v^\ell$, $y_{1,\ell} = d_{1,\ell} v^\ell$, $y_{2,\ell} = d_{2,\ell} v^\ell$, $\ell = 1, 2, \ldots$ (and $y_{0,0} = 1$) for some parameter $v > 0$ to rewrite (6) to

$$y_{0,\ell} = zv^2 \sum_{j=0}^{\ell-2} y_{0,j} y_{0,\ell-2-j} + zD(1)v^\ell - zv^2 \sum_{j=0}^{\ell-2} y_{0,j} v^{\ell-2-j} + (w-1)H_\ell, \tag{7}$$

$$y_{1,\ell} = zD(1)v^\ell - zv^2 \sum_{j=0}^{\ell-2} y_{0,j} v^{\ell-2-j}, \qquad y_{2,\ell} = (w-1)H_\ell,$$

where

$$H_\ell = z^2 v y_{0,\ell-1} + (w+1)(zv^{-1} y_{2,\ell+1} + z^2 v^{-2} y_{0,\ell+2}) \\ + (w-1)\Bigg[-z^2 v \sum_{i=0}^{\ell-2} y_{0,i} y_{2,\ell-1-i} - z^3 \sum_{i=0}^{\ell-2} y_{0,i} y_{0,\ell-i} + z^2 v^{-2} y_{2,\ell+2} + z^3 v^{-3} y_{0,\ell+3} \\ - z^2 \left(v^\ell D_2(1) - v \sum_{i=0}^{\ell-1} y_{2,i} v^{\ell-1-i} \right) - z^3 \left(v^\ell D(1) - \sum_{i=0}^{\ell} y_{0,i} v^{\ell-i} \right) \Bigg].$$

Now we consider $D(1)$, $D_1(1)$, and $D_2(1)$ as new variables x_0, x_1, and x_2 and rewrite the system (7) into a new system

$$y_{0,\ell} = zv^2 \sum_{j=0}^{\ell-2} y_{0,j} y_{0,\ell-2-j} + zx_0 v^\ell - zv^2 \sum_{j=0}^{\ell-2} y_{0,j} v^{\ell-2-j} + (w-1)\tilde{H}_\ell, \tag{8}$$

$$y_{1,\ell} = zx_0 v^\ell - zv^2 \sum_{j=0}^{\ell-2} y_{0,j} v^{\ell-2-j}, \qquad y_{2,\ell} = (w-1)\tilde{H}_\ell,$$

where $\tilde{H}_\ell$ results from H_ℓ by this substitution. The solution functions $y_{i,\ell} = y_{i,\ell}(z, w, x_0, x_1, x_2)$ are now considered as functions in z, w, x_0, x_1, x_2 and in a next step we will show that these functions are actually analytic in these variables (in a certain range). Of course, if we have proved this assertion then we can obtain, for example,

$$d_\ell = d_\ell(z,w) = y_{0,\ell}(z, w, D(z,1,w), D_1(z,1,w), D_2(z,1,w))v^{-\ell}$$

as an analytic function in $z, w, D(z,1,w), D_1(z,1,w), D_2(z,1,w)$. This also proves the lemma after re-substituting $D_{\not\triangleright}$ and $D_\triangleright$ in terms of D, D_1, and D_2.

The idea of solving (8) is to consider it as a fixed point equation in a complete metric space and to solve it with the help of Banach's fixed point theorem. For this purpose we have to adjust the parameter $v > 0$ so that the right hand side of (8) is a contraction. More precisely we set $\mathbf{y}_0 = (y_{0,\ell})_{\ell \ge 1}$, $\mathbf{y}_1 = (y_{1,\ell})_{\ell \ge 1}$, $\mathbf{y}_2 = (y_{2,\ell})_{\ell \ge 1}$, and $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)$ and consider the ℓ^1 norm $\|\mathbf{y}\|_1 = \|\mathbf{y}_0\|_1 + \|\mathbf{y}_1\|_1 + \|\mathbf{y}_2\|_1$, where

$$\|\mathbf{y}_j\|_1 = \sum_{\ell \ge 1} |y_{j,\ell}|, \qquad j \in \{0, 1, 2\}.$$

Furthermore we define the mapping $\mathbf{T}:\ell^1(\mathbb{C})^3\to\ell^1(\mathbb{C})^3$ by $\mathbf{T}(\mathbf{y})=(\mathbf{T}_0(\mathbf{y}),\mathbf{T}_1(\mathbf{y}),\mathbf{T}_2(\mathbf{y}))$, where

$$\mathbf{T}_0(\mathbf{y})=\left(zv^2\sum_{j=0}^{\ell-2}y_{0,j}y_{0,\ell-2-j}+zx_0v^\ell-zv^2\sum_{j=0}^{\ell-2}y_{0,j}v^{\ell-2-j}+(w-1)\tilde{H}_\ell\right)_{\ell\ge 1},$$

$$\mathbf{T}_1(\mathbf{y})=\left(zx_0v^\ell-zv^2\sum_{j=0}^{\ell-2}y_{0,j}v^{\ell-2-j}\right)_{\ell\ge 1},\qquad \mathbf{T}_2(\mathbf{y})=\big((w-1)\tilde{H}_\ell\big)_{\ell\ge1},$$

where z,w are considered as complex parameters and $v>0$ will be chosen in a proper way. Clearly, a fixed point of $\mathbf{T}$ is a solution of (8).

By definition it follows that

$$\begin{aligned}
\|\mathbf{T}_0(\mathbf{y})\|_1 &\le v^2|z|\left(1+\|\mathbf{y}_0\|_1\right)^2+\frac{v}{1-v}|zx_0|+\frac{v^2}{1-v}|z|\left(1+\|\mathbf{y}_0\|_1\right)\\
&\quad+\frac{|w-1|}{v^2}P_0\left(|z|,|x_0|,|x_1|,|x_2|,\|\mathbf{y}_0\|_1,\|\mathbf{y}_1\|_1,\|\mathbf{y}_2\|_1,v,\frac{1}{1-v}\right),\\
\|\mathbf{T}_1(\mathbf{y})\|_1 &\le \frac{v}{1-v}|zx_0|+\frac{v^2}{1-v}|z|\left(1+\|\mathbf{y}_0\|_1\right),\\
\|\mathbf{T}_2(\mathbf{y})\|_1 &\le \frac{|w-1|}{v^2}P_0\left(|z|,|x_0|,|x_1|,|x_2|,\|\mathbf{y}_0\|_1,\|\mathbf{y}_1\|_1,\|\mathbf{y}_2\|_1,v,\frac{1}{1-v}\right),
\end{aligned}$$

where P_0 is some polynomial with non-negative coefficients. Similarly we get

$$\begin{aligned}
\|\mathbf{T}_0(\mathbf{y})-\mathbf{T}_0(\mathbf{z})\|_1 &\le \left(v^2|z|\left(2+\|\mathbf{y}_0\|_1+\|\mathbf{z}_0\|_1\right)+\frac{v^2|z|}{1-v}\right)\|\mathbf{y}_0-\mathbf{z}_0\|_1\\
&\quad+\frac{|w-1|}{v^2}\tilde{P}_0\left(|z|,|x_0|,|x_1|,|x_2|,\|\mathbf{y}_0\|_1,\|\mathbf{y}_1\|_1,\|\mathbf{y}_2\|_1,v,\frac{1}{1-v}\right)\|\mathbf{y}-\mathbf{z}\|_1,\\
\|\mathbf{T}_1(\mathbf{y})-\mathbf{T}_1(\mathbf{z})\|_1 &\le \frac{v^2|z|}{1-v}\|\mathbf{y}_0-\mathbf{z}_0\|_1,\\
\|\mathbf{T}_2(\mathbf{y})-\mathbf{T}_2(\mathbf{z})\|_1 &\le\\
&\quad\frac{|w-1|}{v^2}\tilde{P}_0\left(|z|,|x_0|,|x_1|,|x_2|,\|\mathbf{y}_0\|_1,\|\mathbf{y}_1\|_1,\|\mathbf{y}_2\|_1,v,\frac{1}{1-v}\right)\|\mathbf{y}-\mathbf{z}\|_1,
\end{aligned}$$

where $\tilde{P}_0$ is another polynomial with non-negative coefficients.

Thus, given upper bounds Z, X_0, X_1, X_2, and Y for $|z|$, $|x_0|$, $|x_1|$, $|x_2|$, and $\|\mathbf{y}\|_1$ it is easy to choose $v>0$ and $\eta>0$ such that for $|w-1|\le\eta$ the mapping $\mathbf{T}$ maps the set $\{\mathbf{y}\in\ell^1(\mathbb{C})^3:\|\mathbf{y}\|_1\le Y\}$ into itself and is a contraction, too. This shows that (8) has a unique solution that can be obtained as the uniform limit of the iterations $\mathbf{T}^k(\mathbf{0})$. By definition it is clear that all components of $\mathbf{T}^k(\mathbf{0})$ are analytic functions in z,w,x_0,x_1,x_2. Hence, the limits are analytic, too. This completes the proof of the lemma. ◀

We now go back to the original system (2) and substitute $[u^1]D$, $[u^1]D_{\triangleright}$, and $[u^2]D_{\triangleright}$ by the analytic functions K_{ij} given by Lemma 3 so that it can be rewritten as

$$\begin{aligned}
D &= 1+zu^2D^2+D_{\not\triangleright}+D_{\triangleright},\\
D_{\not\triangleright} &= Q_1(z,u,w,D,D(1),D_{\not\triangleright},D_{\not\triangleright}(1),D_{\triangleright},D_{\triangleright}(1)),\\
D_{\triangleright} &= Q_2(z,u,w,D,D(1),D_{\not\triangleright},D_{\not\triangleright}(1),D_{\triangleright},D_{\triangleright}(1))
\end{aligned}$$

with proper functions Q_1, Q_2. This is a catalytic system of three equations. In order to make our analysis a little bit easier we eliminate D and $D(1)$ by using the first equation. By substituting D (and similarly $D(1)$) by

$$D = \frac{1 - \sqrt{1 - 4zu^2(1 + D_{\not\triangleright} + D_{\triangleright})}}{2zu^2}$$

in the second and the third equation we finally obtain a system of two equations that we represent in the form

$$P_1(z, u, w, D_{\not\triangleright}, D_{\not\triangleright}(1), D_{\triangleright}, D_{\triangleright}(1)) = 0, \quad P_2(z, u, w, D_{\not\triangleright}, D_{\not\triangleright}(1), D_{\triangleright}, D_{\triangleright}(1)) = 0$$

for proper functions P_1, P_2 (that are by the way non-linear in $D_{\not\triangleright}, D_{\not\triangleright}(1), D_{\triangleright}, D_{\triangleright}(1)$).

We recall now a method by Bousquet-Mélou and Jehanne [2] on catalytic equations of the form

$$P(z, u, M(z, u), M_1(z)) = 0,$$

where $M_1(z)$ is usually $M(z, 1)$ or $M(z, 0)$ and $P = P(z, u, x_0, x_1)$ is usually a polynomial (however, the method also works with proper regularity conditions for P). The first step is to find functions $u(z)$, $y(z)$, and $f(z)$ that satisfy the system of equations

$$\begin{aligned} P(z, u(z), y(z), f(z)) &= 0, \\ P_u(z, u(z), y(z), f(z)) &= 0, \\ P_{x_0}(z, u(z), y(z), f(z)) &= 0, \end{aligned} \tag{9}$$

where P_u and P_{x_0} denote the partial derivatives $\frac{\partial P}{\partial u}$ and $\frac{\partial P}{\partial x_0}$, respectively. Then we can set $M_1(z) = f(z)$ and can recover $M(z, u)$ – if necessary – from the equation

$$P(z, u, M(z, u), f(z)) = 0. \tag{10}$$

This method generalizes the classical *Quadratic Method* and can be extended in various ways. It is also possible to guarantee unique power series solutions etc., for details we refer to [2].

We emphasize here some further extensions. First we can directly add a parameter w or several parameters $\mathbf{w} = (w_1, \ldots, w_k)$ into the equation without any change of the method. From $P(z, u, \mathbf{w}, M(z, u, \mathbf{w}), M_1(z, \mathbf{w})) = 0$ we, thus, obtain the solutions $M_1(z, \mathbf{w})$ and $M(z, u, \mathbf{w})$.

It was shown in [7] and [6] that the solution function $M_1(z)$ of a catalytic equation (10) that is singular at $z = \rho$ has usually a singularity of the form

$$M_1(z) = g(z) + h(z)\left(1 - \frac{z}{\rho}\right)^{3/2}, \tag{11}$$

where $g(z)$ and $h(z)$ are analytic at $z = \rho$. This is in particular true for the generating function $M(z, 1)$ that counts planar maps and is the solution of the catalytic equation (3):

$$M(z, 1) = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2}.$$

Here $\rho = 1/12$ is the radius of convergence of $M(z, 1)$. Since $M(z, 1) = D(z, 1, 1)$ it also follows that $D(z, 1, 1)$ and consequently the functions $D_{\not\triangleright}(z, 1, 1)$ and $D_{\triangleright}(z, 1, 1)$ have the same kind of singularity at $z = 1/12$. What we show next (and which is actually the main property that will be used to prove the central limit theorem) is that we have the same kind of singular behavior if we add some parameters. In particular we will show that $D(z,1,w)$ can be represented as

$$D(z,1,w) = g_D(z,w) + h_D(z,w)\left(1 - \frac{z}{\rho(w)}\right)^{3/2}, \qquad (12)$$

where g_D, h_D and ρ are analytic at $z = 1/12$ and $w = 1$.

We will first consider one catalytic equation and will then generalize it to a system.

▶ **Lemma 4.** *Suppose that $M(z,u,\mathbf{w})$ and $M_1(z,\mathbf{w})$ are the solutions of the catalytic equation $P(z,u,\mathbf{w},M(z,u,\mathbf{w}),M_1(z,\mathbf{w})) = 0$, where the function $P(z,u,\mathbf{w},x_0,x_1)$ is analytic and $M_1(z,\mathbf{1})$ has a singularity at $z = \rho_0$ of form (11) with $g(\rho_0) \neq 0$, $h(\rho_0) \neq 0$ such that for $z = \rho_0$, $u = u_0$, $x_0 = M(\rho_0,u_0,\mathbf{1})$, $x_1 = M_1(\rho_0,\mathbf{1})$, and $\mathbf{w} = \mathbf{1}$ we have*[3]

$$P = 0, \quad P_u = 0, \quad P_{x_0} = 0, \quad P_{x_1} \neq 0, \quad P_{x_0x_0}P_{uu} = P_{x_0u}^2.$$

Furthermore, let $z = \rho(\mathbf{w})$, $u = u_0(\mathbf{w})$, $x_0 = x_0(\mathbf{w})$, $x_1 = x_1(\mathbf{w})$ for $\mathbf{w}$ close to $\mathbf{1}$ be defined by $\rho(\mathbf{1}) = \rho_0$, $u_0(\mathbf{1}) = u_0$, $x_0(\mathbf{1}) = M(\rho_0,u_0,\mathbf{1})$, $x_1(\mathbf{1}) = M_1(\rho_0,\mathbf{1})$ and by the system

$$P = 0, \quad P_u = 0, \quad P_{x_0} = 0, \quad P_{x_0x_0}P_{uu} = P_{x_0u}^2.$$

Then for $\mathbf{w}$ close to $\mathbf{1}$ the function $M_1(z,\mathbf{w})$ has a local singular representation of the form

$$M_1(z,\mathbf{w}) = \overline{g}(z,\mathbf{w}) + \overline{h}(z,\mathbf{w})\left(1 - \frac{z}{\rho(\mathbf{w})}\right)^{3/2}, \qquad (13)$$

where $\overline{g}(z,\mathbf{w})$, $\overline{h}(z,\mathbf{w})$ are analytic at $z = \rho_0$ and $\mathbf{w} = \mathbf{1}$ and satisfy $\overline{g}(\rho_0,\mathbf{1}) = g(\rho_0) \neq 0$, $\overline{h}(\rho_0,\mathbf{1}) = h(\rho_0) \neq 0$.

The **Proof** is an adaption of the methods of [7]. The essential step is to represent (with the help of the Weierstrass preparation theorem) the function P locally around $z = \rho_0$, $u = u_0$, $x_0 = M(\rho_0,u_0,\mathbf{1})$, $x_1 = M_1(\rho_0,\mathbf{1})$, and $\mathbf{w} = \mathbf{1}$ by

$$P(z,u,\mathbf{w},x_0,x_1) = K(z,u,\mathbf{w},x_0,x_1)\left((x_0 - G(z,u,\mathbf{w},x_1))^2 - H(z,u,\mathbf{w},x_1)\right),$$

where all appearing functions are analytic and we have $K(\rho_0,u_0,\mathbf{1},M(\rho_0,u_0,\mathbf{1}),M_1(\rho_0,\mathbf{1})) \neq 0$, $G(\rho_0,u_0,\mathbf{1},M_1(\rho_0,\mathbf{1})) = M(\rho_0,u_0,\mathbf{1})$ and $H(\rho_0,u_0,\mathbf{1},M_1(\rho_0,\mathbf{1})) = 0$. The system (9) translates into a smaller system of the form $H(z,\mathbf{w},u(z,\mathbf{w}),f(z,\mathbf{w})) = 0$, $H_u(z,\mathbf{w},u(z,\mathbf{w}), f(z,\mathbf{w})) = 0$ which is suitable to extract the singular behavior of the form (13). In particular the condition $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ is equivalent to $H_{uu} = 0$. Now we proceed as in [7], observe the singular expansion for $M_1(z,\mathbf{w})$ of the form (13) and by comparing it with (11) we also get the properties $\overline{g}(\rho_0,\mathbf{1}) = g(\rho_0) \neq 0$, $\overline{h}(\rho_0,\mathbf{1}) = h(\rho_0) \neq 0$.

In the case of a system of two catalytic equations $P_1 = 0$, $P_2 = 0$ (in unknown functions $M(z,u,\mathbf{w})$, $M_1(z,\mathbf{w})$, $N(z,u,\mathbf{w})$, $N_1(z,\mathbf{w})$) we apply an elimination procedure to reduce it to a single catalytic equation so that Lemma 4 can be applied. We consider first the second equation and replace $M(z,u,\mathbf{w})$, $M_1(z,\mathbf{w})$ by two new variables v_0, v_1:

$$P_2(z,u,\mathbf{w},v_0,v_1,N,N_1) = 0$$

[3] The notation P_x denotes the partial derivative with respect to x – and similarly for partial derivatives with respect to other variables or for higher order derivatives.

and solve this catalytic equation in order to obtain solution functions $N = \overline{N}(z, u, \mathbf{w}, v_0, v_1)$ and $N_1 = \overline{N}_1(z, \mathbf{w}, v_0, v_1)$. Then we substitute these solutions into the first equation and obtain a single catalytic equation for $M = M(z, u, \mathbf{w})$, $M_1 = M_1(z, \mathbf{w})$:

$$P_1(z, u, \mathbf{w}, M, M_1, \overline{N}(z, u, \mathbf{w}, M, M_1), \overline{N}_1(z, \mathbf{w}, M, M_1)) = 0.$$

Finally we apply Lemma 4 and obtain the proposed singular representation. The only thing that has to be checked is that $P_{2,NN}P_{2,uu} \neq P_{2,Nu}^2$ and $P_{2,N_1} \neq 0$ so that the functions $N = \overline{N}(z, u, \mathbf{w}, v_0, v_1)$ and $N_1 = \overline{N}_1(z, \mathbf{w}, v_0, v_1)$ are analytic in the region of interest. In our special situation this is easy to check. With this method we obtain singular representations for $D_{\not\triangleright}(z, 1, w)$ and $D_{\triangleright}(z, 1, w)$ and consequently (12) for $D(z, 1, w)$.

The **Proof of Theorem 1** is now almost immediate. Let Y_n denote the number of edges in a random planar map with n edges that represent double triangles but are not on the root face. Then we have

$$D(z, 1, w) = \sum_{n \geq 0} M_n \mathbb{E}[w^{Y_n}]\, z^n,$$

where $M_n = [z^n]M(z, 1)$ denotes the number of planar maps with n edges. By a direct application of [5, Theorem 2.35] it follows that Y_n satisfies a central limit theorem of the form (1) with expected value and variance asymptotically proportional to n. The only difference between X_n and Y_n is the number of edges on the root face that represent a double triangle. However, if X_n and Y_n are different then the root face has valency 3 which means that the difference between X_n and Y_n is at most 3. Hence, the central limit theorem (as well as asymptotics for expected value and variance) of Y_n transfers directly into a corresponding central limit theorem for X_n which completes the proof of Theorem 1.

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A Appendix

A.1 Proof of the relations (4)

In the first part of the Appendix, we present a proper decomposition of the sets $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_0}$, $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_1}$, and $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_{\geq 2}}$ that translate into the system (4).

In order to represent $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_0}$ (and consequently the generating function of $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_0}$) which corresponds to the case $|(\alpha,\beta)| = 0$, the main argument will focus on the valency of the α-face (that equals to that of the β-face) which depends on the outside (root) face valency of the map between (or inside) the α-edge and the β-edge. If this map has root face valency 1, then the α-face has valency 3 which means the α-edge and the β-edge are both double triangles. Moreover, in case this map has root face valency 1, if this map belongs to $\mathcal{D}_{\triangleright}$ (the second face has valency 3), then the root edge of this map will become a double triangle after putting this map into the chink between the α-edge and the β-edge and vice versa. Therefore, we have $z^3uDw^2\left(w[u^1]D_{\triangleright} + [u^1]D_{\ntriangleright}\right)$. Contrarily, if this map has root face valency not equal to 1, then the valency of the α-face is not equal to 3, it corresponds to $z^3uD\left(D(1) - [u^1]D\right)$. Thus, the corresponding generating function is $D_{\trianglerighteq}^{\mathbb{W}_0} = z^3uD\left[w^2\left(w[u^1]D_{\triangleright} + [u^1]D_{\ntriangleright}\right) + \left(D(1) - [u^1]D\right)\right]$.

If $|(\alpha,\beta)| = 1$ which corresponds to the class $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_1}$ the border-(α,β)-path is just an edge and the valency of the α-face (and of the β-face) is three (because of the α-edge, the β-edge and the border-(α,β)-path) plus the outside (root) face valency of the map inside this triangle.

If the map inside the triangle has no edge (which means the corresponding generating function of the map is 1), then the α-face has valency 3 which means that both the α-edge and the β-edge represent double triangles. And whether the edge that equals to the border-(α,β)-path corresponds to a double triangle or not depends on the other incident face of this edge. The face on the other side may or may not have valency 3 and also may equal to the outside face (see the above case of Figure 11). Hence this part corresponds to $z^3w^2(wD_{\triangleright} + D_{\ntriangleright} + zu^2D^2)$. If the map (inside the triangle) has some edges (corresponding to the generating function $D(1) - 1$), then the valency of α-face is not equal to 3 which means that neither the α-edge nor the β-edge correspond to a double triangle, and the edge that equals to the border-(α,β)-path must not correspond to a double triangle. We also have to distinguish between three different coases fo the other incident face (see the below case of Figure 11). This part corresponds to $z^3\left(D(1) - 1\right)\left(D_{\triangleright} + D_{\ntriangleright} + zu^2D^2\right)$ which can be simplified to $z^3\left(D(1) - 1\right)\left(D - 1\right)$ by the first equation of (2). Summing up we get the corresponding generating function of $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_1}$ as follows:

$$D_{\trianglerighteq}^{\mathbb{W}_1} = z^3w^2(wD_{\triangleright} + D_{\ntriangleright} + zu^2D^2) + z^3\left(D(1) - 1\right)\left(D - 1\right).$$

Finally $\mathcal{D}_{\trianglerighteq}^{\mathbb{W}_{\geq 2}}$ is easier to describe, since the α-face (that is equal to the β-face) has valency not equal to 3. So we do not have to care about whether the α-edge and the β-edge are double triangles. We can directly decompose the map into two parts: one is a map with second face valency greater than 3 (the length of the border-(α,β)-path greater than 2 and plus the root edge), the other one is a map with plus edges (see Figure 12).

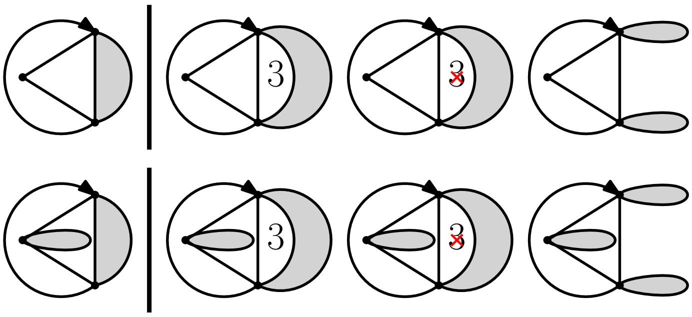

Figure 11 Decomposition of $\mathcal{D}_{\trianglerighteq}^{\mathbb{U}1}$: In the first (upper) case the the map (inside the triangle) has no edge, whereas in the second (below) case this map is non-trivial. In both case we have the right side face of the border-(α,β)-path is different to the root face and its valency is either equal to 3 or not, or it equals the root face.

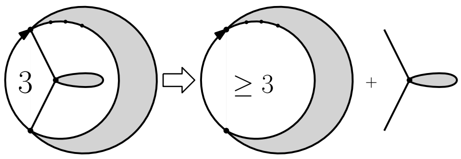

Figure 12 Decompose a map that belongs to $\mathcal{D}_{\trianglerighteq}^{\mathbb{U}\geq 2}$ into two parts.

The first map class can be counted with the help of a cutting across process (see Figure 2) where we have take out the situation where the new-appearing second face has valency 1 or 2. The corresponding effect to u^r is

$$u^r \mapsto z(u^{r+1} + u^r + \cdots + u^2 + u^1) - z(u^{r+1} + u^r) + \begin{cases} 0 & \text{, if } r \geq 1 \\ zu^0 & \text{, if } r = 0 \end{cases}$$

which leads to $zu\frac{D(1)-uD}{1-u} - z(D-1) - zuD$. After combining this with a map plus two edges (which is counted by $z^2D(1)$) we have,

$$D_{\trianglerighteq}^{\mathbb{U}\geq 2} = z^2 D(1)\left(zu\frac{D(1)-uD}{1-u} - z(D-1) - zuD\right)$$

which completes the proof of (4).

A.2 Simplification of the representation of $D_{\triangleright}$

In the second part of the Appendix we prove that $D_{\triangleright}$ can be simplified into the form that is given in (2).

After collecting all parts of $D_{\triangleright}$ that are described in the Proof of Lemma 2 we obtain

$$\begin{aligned}
D_{\triangleright} &= z^3u^3D^3 + 2z^2uD\left(wD_{\triangleright} + D_{\ntriangleright}\right)\\
&\quad + z^3uD\left[w^2\left(w[u^1]D_{\triangleright} + [u^1]D_{\ntriangleright}\right) + \left(D(1) - [u^1]D\right)\right]\\
&\quad + z^3w^2(wD_{\triangleright} + D_{\ntriangleright} + zu^2D^2) + z^3\left(D(1)-1\right)(D-1)\\
&\quad + z^2D(1)\left(zu\frac{D(1)-uD}{1-u} - z(D-1) - zuD\right) + z^2uwD\\
&\quad + zu^{-1}\left(zu\frac{D_{\ntriangleright}(1)-uD_{\ntriangleright}}{1-u} - zuD_{\ntriangleright}(1) - zu^{-1}\left(D_{\ntriangleright} - u[u^1]D_{\ntriangleright} - u^2[u^2]D_{\ntriangleright}\right)\right)\\
&\quad + 2zu^{-1}w\left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right)\\
&\quad + zu^{-1}w^2D_{\triangleright} - z^2uw^2DD_{\triangleright} - zw^2[u^1]D_{\triangleright}\\
&\quad - zu^{-1}w^2\left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right)\\
&\quad - zw^2\left(z^2uD[u^1]D_{\ntriangleright} + z^2uwD[u^1]D_{\triangleright} + z^2D_{\ntriangleright} + z^2wD_{\triangleright} + z^3u^2D^2\right).
\end{aligned}$$

We use the first two terms of the 2$^{\text{nd}}$ line and the first three terms of the 3$^{\text{rd}}$ line to cancel the last line. We also cancel the third term of the 2$^{\text{nd}}$ line and the third term of the 4$^{\text{th}}$ line. Moreover, we cancel part of the last term of the 3$^{\text{rd}}$ line and the second term of the 4$^{\text{th}}$ line.

$$\begin{aligned}
D_{\triangleright} &= z^3u^3D^3 + z^2uD\left(wD_{\triangleright} + D_{\ntriangleright}\right) + z^2uD\left(wD_{\triangleright} + D_{\ntriangleright}\right)\\
&\quad - z^3uD[u^1]D - z^3(D-1) + z^2D(1)\left(zu\frac{D(1)-uD}{1-u}\right) + z^2uwD\\
&\quad + zu^{-1}\left(zu\frac{D_{\ntriangleright}(1)-uD_{\ntriangleright}}{1-u} - zuD_{\ntriangleright}(1) - zu^{-1}\left(D_{\ntriangleright} - u[u^1]D_{\ntriangleright} - u^2[u^2]D_{\ntriangleright}\right)\right)\\
&\quad + 2zu^{-1}w\left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right)\\
&\quad + zu^{-1}w^2D_{\triangleright} - z^2uw^2DD_{\triangleright} - zw^2[u^1]D_{\triangleright}\\
&\quad - zu^{-1}w^2\left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right).
\end{aligned}$$

We now rewrite $D_{\triangleright}$ according to the appearing power of w and separate as follows:

$$\begin{aligned}
D_{\triangleright} &= A_0 + wA_1 + w^2A_2\\
&= A_0 + A_1 + A_2 + (w-1)A_1 + (w^2-1)A_2\\
&= (A_0 + A_1 + A_2) + (w-1)\left(A_1 + (w+1)A_2\right)
\end{aligned}$$

where A_0, A_1, A_2 are explicit functions in $z, u, D, D_{\ntriangleright}, D_{\triangleright}, [u^1]D, [u^2]D, [u^1]D_{\ntriangleright}, [u^2]D_{\ntriangleright}$, $[u^1]D_{\triangleright}, [u^2]D_{\triangleright}$.

In order to show that this representation can be simplified to the form in (2) we first have to show that $A_0 + A_1 + A_2 = zu^{-1}\left(D - 1 - u[u^1]D\right)$. By summing up the expressions of $A_0 + A_1 + A_2$ (and cancelling already two terms) we get

$$\begin{aligned}
&z^3u^3D^3 + z^2uD\left(D_{\triangleright} + D_{\not\triangleright}\right) + z^2uD\left(D_{\triangleright} + D_{\not\triangleright}\right) \\
&- z^3uD[u^1]D - z^3(D-1) + z^2D(1)\left(zu\frac{D(1)-uD}{1-u}\right) + z^2uD \\
&+ zu^{-1}\left(zu\frac{D_{\not\triangleright}(1)-uD_{\not\triangleright}}{1-u} - zuD_{\not\triangleright}(1) - zu^{-1}\left(D_{\not\triangleright} - u[u^1]D_{\not\triangleright} - u^2[u^2]D_{\not\triangleright}\right)\right) \\
&+ zu^{-1}\left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right) \\
&+ zu^{-1}D_{\triangleright} - z^2uDD_{\triangleright} - z[u^1]D_{\triangleright}.
\end{aligned}$$

Now by using the relation $D_{\triangleright} + D_{\not\triangleright} = D - 1 - zu^2D^2$, we can deduce two properties:

$$\begin{aligned}
F_1: &\quad [u^1]D_{\triangleright} + [u^1]D_{\not\triangleright} = [u^1](D - 1 - zu^2D^2) = [u^1]D, \\
F_2: &\quad [u^2]D_{\triangleright} + [u^2]D_{\not\triangleright} = [u^2](D - 1 - zu^2D^2) = [u^2]D - z.
\end{aligned}$$

We combine the 3rd and 4th line by applying F_1 and F_2 and use the last term of it to cancel z^3 in the 2nd line. Then, applying the relation $D = 1 + zu^2D^2 + D_{\not\triangleright} + D_{\triangleright}$ in the 1st line, we obtain

$$\begin{aligned}
&z^2uD(D-1) + z^2uD\left(D_{\triangleright} + D_{\not\triangleright}\right) \\
&- z^3uD[u^1]D - z^3D + z^2D(1)\left(zu\frac{D(1)-uD}{1-u}\right) + z^2uD \\
&+ z^2\frac{uD(1) - uD + zu^3D^2 - zuD(1)^2}{1-u} - z^2u^{-2}\left(D - 1 - zu^2D^2 - u[u^1]D - u^2[u^2]D\right) \\
&+ zu^{-1}D_{\triangleright} - z^2uDD_{\triangleright} - z[u^1]D_{\triangleright}.
\end{aligned}$$

We cancel some terms from 1st, 2nd, and 4th. Next, We introduce the notation $K := zu\frac{D(1)-uD}{1-u}$ and use it in the 2nd and 3rd line:

$$\begin{aligned}
&z^2uD^2 + z^2uDD_{\not\triangleright} - z^3uD[u^1]D - z^3D + z^2D(1)K \\
&- z^2D(1) + zu^{-1}K - z^2(uD + D(1))K - z^2u^{-2}\left(D - 1 - zu^2D^2 - u[u^1]D - u^2[u^2]D\right) \\
&+ zu^{-1}D_{\triangleright} - z[u^1]D_{\triangleright}.
\end{aligned}$$

After canceling some terms from the first two lines and applying $D_{\triangleright} = D - 1 - zu^2D^2 - D_{\not\triangleright}$ in the 3rd line we obtain

$$\begin{aligned}
&z^2uD^2 + z^2uDD_{\not\triangleright} - z^3uD[u^1]D - z^3D \\
&- z^2D(1) + zu^{-1}K - z^2uDK - z^2u^{-2}\left(D - 1 - zu^2D^2 - u[u^1]D - u^2[u^2]D\right) \\
&+ zu^{-1}(D - 1 - zu^2D^2 - D_{\not\triangleright}) - z([u^1]D - [u^1]D_{\not\triangleright}).
\end{aligned}$$

We apply $D_{\not\triangleright} = K - zu^{-1}(D - 1 - u[u^1]D)$ in the 1st and 3rd line and apply $[u^1]D_{\not\triangleright} = [u^1]K - [u^1]zu^{-1}\left(D - 1 - u[u^1]D\right) = [u^1]K - z[u^2]D$ in the 3rd. After simplifying we obtain

$$z^2uD^2 - z^2D(1) + zu^{-1}K + zu^{-1}(D - 1 - zu^2D^2 - K) - z([u^1]D - [u^1]K).$$

We replace now K by $D - 1 - zu^2D^2$ and apply $[u^1]K = [u^1]D$ so that we have

$$z^2uD^2 - z^2D(1) + zu^{-1}\left(D - 1 - zu^2D^2\right)$$

which reduces to $zu^{-1}(D-1-zuD(1))$.

Finally we observe that we have relation

$$[u^1]D = [u^1](1+zu^2D^2+zu\frac{D(1)-uD}{1-u}) = z[u^0]\frac{D(1)-uD}{1-u} = zD(1)$$

which implies that we actually end up with

$$A_0+A_1+A_2 = zu^{-1}(D-1-zuD(1)) = zu^{-1}(D-1-u[u^1]D)$$

as proposed.

Finally we apply some simplifications to A_1 and A_2. Recall that the second term of $D_{\triangleright}$ is $(w-1)\left(A_1+(w+1)A_2\right)$. It is clear that

$$A_1 = 2z^2uDD_{\triangleright} + z^2uD + 2zu^{-1}P(D_{\triangleright})$$
$$A_2 = zu^{-1}D_{\triangleright} - z^2uDD_{\triangleright} - z[u^1]D_{\triangleright} - zu^{-1}P(D_{\triangleright})$$

where

$$P(D_{\triangleright}) = \left(zu\frac{D_{\triangleright}(1)-uD_{\triangleright}}{1-u} - zuD_{\triangleright}(1) - zu^{-1}\left(D_{\triangleright} - u[u^1]D_{\triangleright} - u^2[u^2]D_{\triangleright}\right)\right).$$

After canceling some terms we finally get that $A_1+(w+1)A_2$ is equal to

$$z^2uD + (w+1)\left(zu^{-1}D_{\triangleright} - z[u^1]D_{\triangleright}\right) - z^2u(w-1)DD_{\triangleright} - zu^{-1}(w-1)P(D_{\triangleright})$$

which completes the proof.